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Geometry of The Embedding of Supergravity Scalar Manifolds in $D = 11$ and $D = 10$

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ABSTRACT

Several recent papers have made considerable progress in proving the existence of remarkable consistent Kaluza-Klein sphere reductions of $D = 10$ and $D = 11$ supergravities, to give gauged supergravities in lower dimensions. A proof of the consistency of the full gauged $SO(8)$ reduction on S^7 from $D = 11$ was given many years ago, but from a practical viewpoint a reduction to a smaller subset of the fields can be more manageable and explicit, for the purposes of lifting lower-dimensional solutions back to the higher dimension. The major complexity of the spherical reduction Ansätze comes from the spin-0 fields, and of these, it is the pseudoscalars that are the most difficult to handle. In this paper we address this problem in two cases. One arises in a truncation of $SO(8)$ gauged supergravity in four dimensions to $U(1)^4$, where there are three pairs of dilatons and axions in the scalar sector. The other example involves the truncation of $SO(6)$ gauged supergravity in $D = 5$ to a subsector containing a scalar and a pseudoscalar field, with a potential that admits a second supersymmetric vacuum aside from the maximally-supersymmetric one. We briefly discuss the use of these embedding Ansätze for the lifting of solutions back to the higher dimension.

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1 Introduction

In this paper, we discuss two examples where non-trivial subsets of the scalar sectors of gauged supergravities are obtained by spherical reduction from a higher dimension. The first example is the embedding of the scalars in the $U(1)^4$ maximal abelian truncation of $SO(8)$ gauged $N = 8$ maximal supergravity in $D = 4$, arising from $D = 11$ *via* compactification on S^7. The consistency of the full $SO(8)$ reduction on S^7 was proven in [1], although at a somewhat implicit level. The $N = 2$ truncation includes a total of six scalar fields, comprising three dilaton/axion pairs. In terms of the original $SO(8)$ representations of the full theory, where there are 35 scalars in the 35_v, and 35 pseudoscalars in the 35_c of $SO(8)$, the three dilatons come from the 35_v, and the three axions come from the 35_c. In [2], a further simplifying truncation was performed, in which the three axions were set to zero. The reduction Ansatz becomes considerably more complicated when axions are included, as was already seen in the case of the single dilaton/axion pair of the $N = 4$ gauged $SO(4)$ truncation, discussed in [3]. In the present example, the inclusion of the three axions as well as the three dilatons leads to a considerably more complicated structure in the reduction Ansatz.

The second example is a truncation of the $SO(6)$ gauged $N = 8$ maximal supergravity in $D = 5$, arising from type IIB *via* compactification on S^5. In this case there are 42 spin-0 fields in total, comprising 20 scalars in the $20'$ of $SO(6)$, 20 pseudoscalars in the 10 and $\overline{10}$, and two singlets corresponding to the original dilaton and axion of the type IIB theory. The truncation we shall consider retains two spin-0 fields, comprising one scalar from the $20'$, and one pseudoscalar from the 10 and $\overline{10}$. This particular truncation is of interest because it is large enough to include the fields that participate in two distinct supersymmetric vacua of the $D = 5$ gauged theory [4], one with maximal $N = 8$, $SO(6)$ symmetry, and the other with $N = 2$, $SU(1) \times U(1)$ symmetry. Although an explicit interpolating solution is not known it is in principle describable within the truncation we are making.

In both of our examples, we shall concentrate on elucidating the geometrical structure of the embedding in $D = 11$ or type IIB supergravity. Specifically, we shall concentrate on the Ansatz for the Kaluza-Klein reduction of the metric tensor. Strictly speaking, one can only be sure that the reduction is fully consistent with all the equations of motion of the higher-dimensional theory if one has the complete Ansatz for all the higher-dimensional fields, including the antisymmetric tensor field strengths. (Or, alternatively, if an "existence proof" for the consistency of the reduction Ansatz has independently been constructed.) Obtaining the Ansatz for the antisymmetric tensor fields is notoriously difficult, and we shall not complete this part of the analysis in this paper. In the case of our $D = 4$ example, we can appeal to the results of [1], in which a complete proof of the consistency of the S^7 reduction is exhibited. In principle it allows one to read off the Ansatz for the 4-form field strength, although only an implicit procedure for its construction is presented. On the other hand, the general Anstaz for the metric tensor *is* rather explicit, and it is by making use of this expression that we are able to obtain the $D = 4$ results in this paper. These results can be used in order to study the eleven-dimensional geometrical structure of general domain-wall solutions in $D = 4$ supported both by the three dilatonic scalars and also the three accompanying axions. Such solutions can be constructed from the purely dilatonic ones by means of $SL(2, R)$ transformations.

In $D = 5$ the situation is less clear, since no proof for the consistency of the full S^5 reduction to $SO(6)$ gauged $N = 8$ maximal supergravity currently exists. A conjecture for the metric reduction Ansatz appears in [4], which is closely analogous to the known construction in $D = 4$ given in [1], and it is this that we use in order to obtain an explicit expression for the metric embedding for our 2-scalar truncation. Again, the complexities of the antisymmetric tensor embedding have prevented us from obtaining a full non-linear result in that sector. Thus the status of our $D = 5$ embedding is that, subject to the assumption of an ultimate consistency of the S^5 reduction scheme,[1] and subject to the assumption that the conjecture for the metric Ansatz in [4] is correct, then our explicit results for the 2-scalar metric Ansatz is valid. In principle, our result can then be used to study the geometry of the RG flow describing the transition between the two supersymmetric extrema of the associated scalar potential.

2 $N = 2$ $U(1)^4$ Gauged Supergravity in $D = 4$ From $D = 11$

2.1 The Three Dilaton/Axion Pairs in $D = 4$

The $35_v + 35_c$ of spin-0 fields in $SO(8)$ gauged supergravity in $D = 4$ are described in terms of a 56-vielbein $\mathcal{V}$, with the block-diagonal form

$$\mathcal{V} = \begin{pmatrix} u_{ij}{}^{IJ} & v_{ijKL} \\ v^{k\ell IJ} & u^{k\ell}{}_{KL} \end{pmatrix}, \tag{1}$$

[1]Further evidence for the consistency of the S^5 reduction was obtained in [5], where certain scalar plus gravity truncations in Kaluza-Klein sphere reductions were proved to be consistent. Additionally, the complete consistent reductions of $D = 11$ supergravity on S^4 [6, 7] and massive type IIA supergravity on S^4 [8] have been constructed. Recently, more evidence for the consistency of the S^5 reduction was presented in [9].

which transforms under local $SU(8)$ and rigid E_7 [10, 11]. In terms of the quantities $u_{ij}{}^{IJ}$ and v_{ijKL}, it was shown in [1] (having been previously proposed in [12]) that the Ansatz for the inverse of the internal S^7 compactifying metric is

$$\hat{g}^{mn}(x,y) \equiv \hat{\Delta}^{-1}\, g^{mn}(x,y) = \tfrac{1}{2}(K^{mIJ}\, K^{nKL} + K^{nIJ}\, K^{mKL})\,(u_{ij}{}^{IJ} + v_{ijIJ})\,(u^{ij}{}_{KL} + v^{ijKL})\,, \tag{2}$$

where K^{mIJ} are the 28 Killing vectors on the round S^7, and

$$\hat{\Delta}^2 = \frac{\det(g_{mn}(x,y))}{\det(g_{mn}(y))}\,, \tag{3}$$

where $g_{mn}(x,y)$ is the inverse of $g^{mn}(x,y)$, and $g_{mn}(y)$ is $g_{mn}(x,y)$ with the scalar fields all set to zero, so that it becomes the round S^7 metric. The eleven-dimensional metric Ansatz will be given by [12, 1]

$$d\hat{s}_{11}^2 = \hat{\Delta}^{-1}\, ds_4^2 + g_{mn}(x,y)\, dy^m\, dy^n = \hat{\Delta}^{-1}\,(ds_4^2 + \hat{g}_{mn}(x,y)\, dy^m\, dy^n)\,, \tag{4}$$

where $\hat{g}_{mn}(x,y) = \hat{\Delta}\, g_{mn(x,y)}$ is the inverse of $\hat{g}^{mn}(x,y)$.[2]

We use the parameterisation of the $u_{ij}{}^{IJ}$ and v_{ijKL} matrices described in [13]. In particular, we introduce three scalars λ_i, and three associated pseudoscalars σ_i, whose kinetic Lagrangian is

$$\mathcal{L} = -\tfrac{1}{2}\sum_i \Big((\partial\lambda_i)^2 + \sinh^2\lambda_i\,(\partial\sigma_i)^2\Big)\,. \tag{5}$$

To shorten the subsequent formulae, we make the following definitions:

$$c_i \equiv \cosh\lambda_i\,, \qquad s_i \equiv \sinh\lambda_i\,. \tag{6}$$

Also, for future convenience, we introduce the "standard" dilaton/axion pairs (φ_i, χ_i), related to (λ_i, σ_i) by

$$\begin{aligned} \cosh\lambda_i &= \cosh\varphi_i + \tfrac{1}{2}\chi_i^2\, e^{\varphi_i}\,, \\ \cos\sigma_i\, \sinh\lambda_i &= \sinh\varphi_i - \tfrac{1}{2}\chi_i^2\, e^{\varphi_i}\,, \\ \sin\sigma_i\, \sinh\lambda_i &= \chi_i\, e^{\varphi_i}\,. \end{aligned} \tag{7}$$

In terms of these fields, the scalar kinetic terms are

$$\mathcal{L} = -\tfrac{1}{2}\sum_i \Big((\partial\varphi_i)^2 + e^{2\varphi_i}\,(\partial\chi_i)^2\Big)\,. \tag{8}$$

[2] For now, we shall leave out the Kaluza-Klein gauge fields from the construction of the metric. As discussed in [13, 2], the truncation to three dilaton/axion pairs is naturally accompanied by the four $U(1)$ gauge fields of the maximal abelian $U(1)^4$ subgroup of $SO(8)$. These gauge fields are easily incorporated in the Kaluza-Klein Ansatz, and we shall add them in at the end of the derivation. We shall also set the gauge coupling constant g equal to 1 for now, and restore it later.

After some algebra, we find that ${u_{ij}}^{IJ}$ and v_{ijKL} are given by

$$\tfrac{1}{4}{u_{ij}}^{KL}P_{ij}\,Q_{KL} = c_1\,(P_{a_1a_2}\,Q_{a_1a_2} + P_{a_3a_4}\,Q_{a_3a_4}) \tag{9}$$
$$+c_2\,(P_{a_1a_3}\,Q_{a_1a_3} + P_{a_2a_4}\,Q_{a_2a_4}) + c_3\,(P_{a_1a_4}\,Q_{a_1a_4} + P_{a_2a_3}\,Q_{a_2a_3})$$
$$+P_{12}\,(c_1\,c_2\,c_3\,Q_{12} + c_1\,s_2\,s_3\,e^{\mathrm{i}(\sigma_2+\sigma_3)}\,Q_{34} + c_2\,s_1\,s_3\,e^{\mathrm{i}(\sigma_1+\sigma_3)}\,Q_{56} + c_3\,s_1\,s_2\,e^{\mathrm{i}(\sigma_1+\sigma_2)}\,Q_{78})$$
$$+P_{34}\,(c_1\,c_2\,c_3\,Q_{34} + c_1\,s_2\,s_3\,e^{-\mathrm{i}(\sigma_2+\sigma_3)}\,Q_{12} + c_2\,s_1\,s_3\,e^{\mathrm{i}(\sigma_1-\sigma_3)}\,Q_{78} + c_3\,s_1\,s_2\,e^{\mathrm{i}(\sigma_1-\sigma_2)}\,Q_{56})$$
$$+P_{56}\,(c_1\,c_2\,c_3\,Q_{56} + c_1\,s_2\,s_3\,e^{\mathrm{i}(\sigma_2-\sigma_3)}\,Q_{78} + c_2\,s_1\,s_3\,e^{-\mathrm{i}(\sigma_1+\sigma_3)}\,Q_{12} + c_3\,s_1\,s_2\,e^{\mathrm{i}(-\sigma_1+\sigma_2)}\,Q_{34})$$
$$+P_{78}\,(c_1\,c_2\,c_3\,Q_{78} + c_1\,s_2\,s_3\,e^{\mathrm{i}(-\sigma_2+\sigma_3)}\,Q_{56} + c_2\,s_1\,s_3\,e^{\mathrm{i}(-\sigma_1+\sigma_3)}\,Q_{34} + c_3\,s_1\,s_2\,e^{-\mathrm{i}(\sigma_1+\sigma_2)}\,Q_{12})$$
$$\tfrac{1}{4}v_{ijKL}P_{ij}\,Q_{KL} = \tag{10}$$
$$-s_1\,(e^{\mathrm{i}\sigma_1}\,\epsilon^{a_1b_1}\,\epsilon^{a_2b_2}\,P_{a_1a_2}\,Q_{b_1b_2} + e^{-\mathrm{i}\sigma_1}\,\epsilon^{a_3b_3}\,\epsilon^{a_4b_4}\,P_{a_3a_4}\,Q_{b_3b_4})$$
$$-s_2\,(e^{\mathrm{i}\sigma_2}\,\epsilon^{a_1b_1}\,\epsilon^{a_3b_3}\,P_{a_1a_3}\,Q_{b_1b_3} + e^{-\mathrm{i}\sigma_2}\,\epsilon^{a_2b_2}\,\epsilon^{a_4b_4}\,P_{a_2a_4}\,Q_{b_2b_4})$$
$$-s_3\,(e^{\mathrm{i}\sigma_3}\,\epsilon^{a_1b_1}\,\epsilon^{a_4b_4}\,P_{a_1a_4}\,Q_{b_1b_4} + e^{-\mathrm{i}\sigma_3}\,\epsilon^{a_2b_2}\,\epsilon^{a_3b_3}\,P_{a_2a_3}\,Q_{b_2b_3})$$
$$+P_{12}\,(s_1\,s_2\,s_3\,e^{\mathrm{i}(\sigma_1+\sigma_2+\sigma_3)}\,Q_{12} + s_1\,c_2\,c_3\,e^{\mathrm{i}\sigma_1}\,Q_{34} + s_2\,c_1\,c_3\,e^{\mathrm{i}\sigma_2}\,Q_{56} + s_3\,c_1\,c_2\,e^{\mathrm{i}\sigma_3}\,Q_{78})$$
$$+P_{34}\,(s_1\,s_2\,s_3\,e^{\mathrm{i}(\sigma_1-\sigma_2-\sigma_3)}\,Q_{34} + s_1\,c_2\,c_3\,e^{\mathrm{i}\sigma_1}\,Q_{12} + s_2\,c_1\,c_3\,e^{-\mathrm{i}\sigma_2}\,Q_{78} + s_3\,c_1\,c_2\,e^{-\mathrm{i}\sigma_3}\,Q_{56})$$
$$+P_{56}\,(s_1\,s_2\,s_3\,e^{\mathrm{i}(-\sigma_1+\sigma_2-\sigma_3)}\,Q_{56} + s_1\,c_2\,c_3\,e^{-\mathrm{i}\sigma_1}\,Q_{78} + s_2\,c_1\,c_3\,e^{\mathrm{i}\sigma_2}\,Q_{12} + s_3\,c_1\,c_2\,e^{-\mathrm{i}\sigma_3}\,Q_{34})$$
$$+P_{78}\,(s_1\,s_2\,s_3\,e^{\mathrm{i}(-\sigma_1-\sigma_2+\sigma_3)}\,Q_{78} + s_1\,c_2\,c_3\,e^{-\mathrm{i}\sigma_1}\,Q_{56} + s_2\,c_1\,c_3\,e^{-\mathrm{i}\sigma_2}\,Q_{34} + s_3\,c_1\,c_2\,e^{\mathrm{i}\sigma_3}\,Q_{12})\,.$$

Here, we have introduced P and Q simply as arbitrary antisymmetric tensors, in order to provide a compact way of summarising all the components of the ${u_{ij}}^{IJ}$ and v_{ijKL} matrices. The index notation is as follows. Indices with a "1" subscript, such as a_1, range over the values $(1,2)$; similarly a_2 ranges over $(3,4)$, a_3 ranges over $(5,6)$ and a_4 ranges over $(7,8)$.

Next, we substitute these results into the Ansatz (2) for the inverse S^7 metric. It is advantageous to introduce a new parameterisation for the dilaton/axion pairs, as follows:

$$Y_i \equiv e^{\frac{1}{2}\varphi_i}\,,\qquad \widetilde{Y}_i \equiv (1+\chi_i^2\,Y_i^4)^{\frac{1}{2}}\,Y_i^{-1}\,,\qquad b_i \equiv \chi_i\,Y_i^2\,, \tag{11}$$

and so

$$\begin{aligned}\cosh\lambda_i &= \tfrac{1}{2}(Y_i^2+\widetilde{Y}_i^2)\,,\\ \cos\sigma_i\,\sinh\lambda_i &= \tfrac{1}{2}(Y_i^2-\widetilde{Y}_i^2)\,,\\ \sin\sigma_i\,\sinh\lambda_i &= b_i\,.\end{aligned} \tag{12}$$

It is also advantageous to redefine the $SO(8)$ basis relative to the one we have used so far. The action of transformation, which amounts to a triality rotation under which $K_{ij} \longrightarrow \frac{1}{2}(\Gamma_{ij})^{k\ell}\,K_{k\ell}$, is given explicitly in Appendix A. After doing this, we find that the inverse internal metric (2) takes the form[3]

$$\begin{aligned}
\hat{\partial}_s^2 \equiv \hat{g}^{mn}\partial_m\,\partial_n &= Y_1^2\,(K_{13}^2+K_{14}^2+K_{23}^2+K_{24}^2)+\widetilde{Y}_1^2\,(K_{57}^2+K_{58}^2+K_{67}^2+K_{68}^2)\\
&\quad +Y_2^2\,(K_{15}^2+K_{16}^2+K_{25}^2+K_{26}^2)+\widetilde{Y}_2^2\,(K_{37}^2+K_{38}^2+K_{47}^2+K_{48}^2)\\
&\quad +Y_3^2\,(K_{17}^2+K_{18}^2+K_{27}^2+K_{28}^2)+\widetilde{Y}_3^2\,(K_{35}^2+K_{36}^2+K_{45}^2+K_{46}^2)\\
&\quad +Y_1^2\,Y_2^2\,Y_3^2\,K_{12}^2+Y_1^2\,\widetilde{Y}_2^2\,\widetilde{Y}_3^2\,K_{34}^2+\widetilde{Y}_1^2\,Y_2^2\,\widetilde{Y}_3^2\,K_{56}^2+\widetilde{Y}_1^2\,\widetilde{Y}_2^2\,Y_3^2\,K_{78}^2\\
&\quad -2b_2\,b_3\,(Y_1^2\,K_{12}\,K_{34}-\widetilde{Y}_1^2\,K_{56}\,K_{78})\\
&\quad -2b_1\,b_3\,(Y_2^2\,K_{12}\,K_{56}-\widetilde{Y}_2^2\,K_{34}\,K_{78})\\
&\quad -2b_1\,b_2\,(Y_3^2\,K_{12}\,K_{78}-\widetilde{Y}_3^2\,K_{34}\,K_{56})\,. \qquad (14)
\end{aligned}$$

In order to proceed further, it is useful to look at the geometry of the 7-sphere in some detail. Some useful results on this topic are collected in Appendix B.

2.2 The Metric Ansatz for the three dilaton/axion pairs

From the results in Appendix B, it follows that the inverse metric (14) for the system with 3 dilatons and 3 axions is a direct sum of a 4×4 part involving the ∂_{ϕ_i} basis vectors, and a 3×3 part involving the ∂_{μ_i} basis vectors (which are constrained by the fact that $\mu_i\,\mu_i=1$):

$$\hat{\partial}_s^2=\hat{\partial}_{s_4}^2+\hat{\partial}_{s_3}^2\,. \qquad (15)$$

For the 4×4 inverse metric, we find

$$\begin{aligned}
\hat{\partial}_{s_4}^2 &= \sum_i \mu_i^{-2}\,Q_i\,\partial_{\phi_i}^2-2b_2\,b_3\,(Y_1^2\,\partial_{\phi_1}\,\partial_{\phi_2}-\widetilde{Y}_1^2\,\partial_{\phi_3}\,\partial_{\phi_4})\\
&\quad -2b_1\,b_3\,(Y_2^2\,\partial_{\phi_1}\,\partial_{\phi_3}-\widetilde{Y}_2^2\,\partial_{\phi_2}\,\partial_{\phi_4})-2b_1\,b_2\,(Y_3^2\,\partial_{\phi_1}\,\partial_{\phi_4}-\widetilde{Y}_3^2\,\partial_{\phi_2}\,\partial_{\phi_3})\,, \qquad (16)
\end{aligned}$$

where

$$\begin{aligned}
Q_1 &= Y_1^2\,Y_2^2\,Y_3^2\,\mu_1^2+Y_1^2\,\mu_2^2+Y_2^2\,\mu_3^2+Y_3^2\,\mu_4^2\,,\\
Q_2 &= Y_1^2\,\widetilde{Y}_2^2\,\widetilde{Y}_3^2\,\mu_2^2+Y_1^2\,\mu_1^2+\widetilde{Y}_3^2\,\mu_3^2+\widetilde{Y}_2^2\,\mu_4^2\,,\\
Q_3 &= Y_2^2\,\widetilde{Y}_1^2\,\widetilde{Y}_3^2\,\mu_3^2+Y_2^2\,\mu_1^2+\widetilde{Y}_3^2\,\mu_2^2+\widetilde{Y}_1^2\,\mu_4^2\,,\\
Q_4 &= Y_3^2\,\widetilde{Y}_1^2\,\widetilde{Y}_2^2\,\mu_4^2+Y_3^2\,\mu_1^2+\widetilde{Y}_2^2\,\mu_2^2+\widetilde{Y}_1^2\,\mu_3^2\,. \qquad (17)
\end{aligned}$$

[3]The notation for writing the inverse metric is $\partial_s^2\equiv g^{mn}\,\partial_m\,\partial_n$. The derivatives do not act on any other objects here; it is just a convenient way of writing all the components of g^{mn} in one formula, exactly analogous to writing the downstairs metric as $ds^2=g_{mn}\,dy^m\,dy^n$. For example, the inverse of the 2-sphere metric $ds^2=d\theta^2+\sin^2\theta\,d\phi^2$ is written as

$$\partial_s^2=\partial_\theta^2+\frac{1}{\sin^2\theta}\,\partial_\phi^2\,. \qquad (13)$$

For the 3×3 part, we find

$$\hat{\partial}^2_{s_3} = Y_1^2\,(\mu_1\,\partial_{\mu_2} - \mu_2\,\partial_{\mu_1})^2 + Y_2^2\,(\mu_1\,\partial_{\mu_3} - \mu_3\,\partial_{\mu_1})^2 + Y_3^2\,(\mu_1\,\partial_{\mu_4} - \mu_4\,\partial_{\mu_1})^2 + \widetilde{Y}_1^2\,(\mu_3\,\partial_{\mu_4} - \mu_4\,\partial_{\mu_3})^2 + \widetilde{Y}_2^2\,(\mu_2\,\partial_{\mu_4} - \mu_4\,\partial_{\mu_2})^2 + \widetilde{Y}_3^2\,(\mu_2\,\partial_{\mu_3} - \mu_3\,\partial_{\mu_2})^2\,, \tag{18}$$

where $\mu_i\,\mu_i = 1$.

Because of the block-diagonal structure, we can invert the two parts separately. For the 4×4 part, we straightforwardly invert the inverse metric to obtain

$$d\hat{s}_4^2 = \frac{1}{\Xi}\Big[\sum_i \mu_i^2\,Z_i\,d\phi_i^2 + 2b_2\,b_3\,(\mu_1^2\,\mu_2^2\,d\phi_1\,d\phi_2 - \mu_3^2\,\mu_4^2\,d\phi_3\,d\phi_4) + 2b_1\,b_3\,(\mu_1^2\,\mu_3^2\,d\phi_1\,d\phi_3 - \mu_2^2\,\mu_4^2\,d\phi_2\,d\phi_4) + 2b_1\,b_2\,(\mu_1^2\,\mu_4^2\,d\phi_1\,d\phi_4 - \mu_2^2\,\mu_3^2\,d\phi_2\,d\phi_3)\Big]\,, \tag{19}$$

where

$$\begin{aligned}
Z_1 &= \mu_1^2 + \widetilde{Y}_2^2\,\widetilde{Y}_3^2\,\mu_2^2 + \widetilde{Y}_1^2\,\widetilde{Y}_3^2\,\mu_3^2 + \widetilde{Y}_1^2\,\widetilde{Y}_2^2\,\mu_4^2\,,\\
Z_2 &= \mu_2^2 + Y_2^2\,Y_3^2\,\mu_1^2 + \widetilde{Y}_1^2\,Y_2^2\,\mu_3^2 + \widetilde{Y}_1^2\,Y_3^2\,\mu_4^2\,,\\
Z_3 &= \mu_3^2 + Y_1^2\,Y_3^2\,\mu_1^2 + Y_1^2\,\widetilde{Y}_2^2\,\mu_2^2 + Y_3^2\,\widetilde{Y}_2^2\,\mu_4^2\,,\\
Z_4 &= \mu_4^2 + Y_1^2\,Y_2^2\,\mu_1^2 + Y_1^2\,\widetilde{Y}_3^2\,\mu_2^2 + Y_2^2\,\widetilde{Y}_3^2\,\mu_3^2\,.
\end{aligned} \tag{20}$$

The function Ξ is given by

$$\begin{aligned}
\Xi &= Y_1^2\,Y_2^2\,Y_3^2\,\mu_1^4 + Y_1^2\,\widetilde{Y}_2^2\,\widetilde{Y}_3^2\,\mu_2^4 + \widetilde{Y}_1^2\,Y_2^2\,\widetilde{Y}_3^2\,\mu_3^4 + \widetilde{Y}_1^2\,\widetilde{Y}_2^2\,Y_3^2\,\mu_4^4\\
&\quad + (Y_2^2\,\widetilde{Y}_2^2 + Y_3^2\,\widetilde{Y}_3^2)\,(Y_1^2\,\mu_1^2\,\mu_2^2 + \widetilde{Y}_1^2\,\mu_3^2\,\mu_4^2)\\
&\quad + (Y_1^2\,\widetilde{Y}_1^2 + Y_3^2\,\widetilde{Y}_3^2)\,(Y_2^2\,\mu_1^2\,\mu_3^2 + \widetilde{Y}_2^2\,\mu_2^2\,\mu_4^2)\\
&\quad + (Y_1^2\,\widetilde{Y}_1^2 + Y_2^2\,\widetilde{Y}_2^2)\,(Y_3^2\,\mu_1^2\,\mu_4^2 + \widetilde{Y}_3^2\,\mu_2^2\,\mu_3^2)\,.
\end{aligned} \tag{21}$$

There remains the problem of inverting the 3×3 part $\hat{\partial}^2_{s_3}$ of the inverse metric. Since we know the inverse metric in the form (18), expressed in terms of the four ∂_{μ_i} basis vectors formed from the the constrained μ_i, it is helpul first to solve the constraint $\mu_i\,\mu_i = 1$ explicitly, by introducing three angular coordinates as follows:

$$\mu_1 = c\,\cos\tfrac{1}{2}\theta\,,\qquad \mu_2 = c\,\sin\tfrac{1}{2}\theta\,,\qquad \mu_3 = s\,\cos\tfrac{1}{2}\tilde{\theta}\,,\qquad \mu_4 = s\,\sin\tfrac{1}{2}\tilde{\theta}\,, \tag{22}$$

where $c = \cos\xi$, $s = \sin\xi$. It then follows that

$$\begin{aligned}
\partial_\theta &= \tfrac{1}{2}(\mu_1\,\partial_{\mu_2} - \mu_2\,\partial_{\mu_1})\,,\\
\partial_{\tilde{\theta}} &= \tfrac{1}{2}(\mu_3\,\partial_{\mu_4} - \mu_4\,\partial_{\mu_3})\,,\\
\partial_\xi &= -s\,c^{-1}\,(\mu_1\,\partial_{\mu_1} + \mu_2\,\partial_{\mu_2}) + c\,s^{-1}\,(\mu_3\,\partial_{\mu_3} + \mu_4\,\partial_{\mu_4})\,.
\end{aligned} \tag{23}$$

Substituting into (18), the inverse metric is then expressed in terms of the three unconstrained basis vectors $(\partial_\xi, \partial_\theta, \partial_{\tilde\theta})$, and hence it can be straightforwardly inverted. Having done so, the downstairs metric can then be re-expressed elegantly in terms of the redundant set of four $d\mu_i$ differentials, in the form

$$\begin{aligned} d\hat s_3^2 &= \frac{1}{\Xi}\Big[\sum_i Z_i\, d\mu_i^2 + \tfrac12 b_1^2\Big((\mu_1\, d\mu_1 + \mu_2\, d\mu_2)^2 + (\mu_3\, d\mu_3 + \mu_4\, d\mu_4)^2\Big) \\ &\quad + \tfrac12 b_2^2\Big((\mu_1\, d\mu_1 + \mu_3\, d\mu_3)^2 + (\mu_2\, d\mu_2 + \mu_4\, d\mu_4)^2\Big) \\ &\quad + \tfrac12 b_3^2\Big((\mu_1\, d\mu_1 + \mu_4\, d\mu_4)^2 + (\mu_2\, d\mu_2 + \mu_3\, d\mu_3)^2\Big)\Big]\,. \end{aligned} \tag{24}$$

Finally, adding this to the 4×4 metric $d\hat s_4^2$ given in (19), we obtain the result for the downstairs 7-metric, $d\hat s_7^2 = d\hat s_4^2 + d\hat s_3^2$:

$$\begin{aligned} d\hat s_7^2 &= \frac{1}{\Xi}\Big[\sum_i Z_i\, (d\mu_i^2 + \mu_i^2\, d\phi_i^2) + 2b_2\, b_3\, (\mu_1^2\, \mu_2^2\, d\phi_1\, d\phi_2 - \mu_3^2\, \mu_4^2\, d\phi_3\, d\phi_4) \\ &\quad + 2b_1\, b_3\, (\mu_1^2\, \mu_3^2\, d\phi_1\, d\phi_3 - \mu_2^2\, \mu_4^2\, d\phi_2\, d\phi_4) + 2b_1\, b_2\, (\mu_1^2\, \mu_4^2\, d\phi_1\, d\phi_4 - \mu_2^2\, \mu_3^2\, d\phi_2\, d\phi_3) \\ &\quad + \tfrac12 b_1^2\Big((\mu_1\, d\mu_1 + \mu_2\, d\mu_2)^2 + (\mu_3\, d\mu_3 + \mu_4\, d\mu_4)^2\Big) \\ &\quad + \tfrac12 b_2^2\Big((\mu_1\, d\mu_1 + \mu_3\, d\mu_3)^2 + (\mu_2\, d\mu_2 + \mu_4\, d\mu_4)^2\Big) \\ &\quad + \tfrac12 b_3^2\Big((\mu_1\, d\mu_1 + \mu_4\, d\mu_4)^2 + (\mu_2\, d\mu_2 + \mu_3\, d\mu_3)^2\Big)\Big]\,. \end{aligned} \tag{25}$$

We can now work out the eleven-dimensional metric Ansatz, given by (4). To do this, we first note that the determinant of (25), where it is understood that the μ_i coordinates are expressed in terms of $(\xi, \theta, \tilde\theta)$ using (22), is

$$\det(\hat g_{mn}) = \Big(\frac{\mu_1^2\, \mu_2^2\, \mu_3^2\, \mu_4^2}{\Xi^2}\Big)\Big(\frac{s^2\, c^2}{16\Xi}\Big) = \frac{\mu_1^2\, \mu_2^2\, \mu_3^2\, \mu_4^2\, s^2\, c^2}{16\Xi^3}\,, \tag{26}$$

where in the first expression, the first factor is the determinant of 4×4 block involving the ϕ_i coordinates, and the second factor is from the 3×3 block involving the $(\xi, \theta, \tilde\theta)$ coordinates. From (3), it follows that

$$\hat\Delta = \Xi^{-\frac13}\,, \tag{27}$$

and hence from (4) that the Ansatz for the eleven-dimensional metric takes the following rather explicit form:

$$\begin{aligned} d\hat s_{11}^2 &= \Xi^{\frac13}\, ds_4^2 + \Xi^{\frac13}\, d\hat s_7^2 \\ &= \Xi^{\frac13}\, ds_4^2 + g^{-2}\, \Xi^{-\frac23}\Big[\sum_i Z_i\, (d\mu_i^2 + \mu_i^2\, d\phi_i^2) + 2b_2\, b_3\, (\mu_1^2\, \mu_2^2\, d\phi_1\, d\phi_2 - \mu_3^2\, \mu_4^2\, d\phi_3\, d\phi_4) \\ &\quad + 2b_1\, b_3\, (\mu_1^2\, \mu_3^2\, d\phi_1\, d\phi_3 - \mu_2^2\, \mu_4^2\, d\phi_2\, d\phi_4) + 2b_1\, b_2\, (\mu_1^2\, \mu_4^2\, d\phi_1\, d\phi_4 - \mu_2^2\, \mu_3^2\, d\phi_2\, d\phi_3) \\ &\quad + \tfrac12 b_1^2\Big((\mu_1\, d\mu_1 + \mu_2\, d\mu_2)^2 + (\mu_3\, d\mu_3 + \mu_4\, d\mu_4)^2\Big) \end{aligned}$$

$$+\tfrac{1}{2}b_2^2\Big((\mu_1\,d\mu_1+\mu_3\,d\mu_3)^2+(\mu_2\,d\mu_2+\mu_4\,d\mu_4)^2\Big)$$
$$+\tfrac{1}{2}b_3^2\Big((\mu_1\,d\mu_1+\mu_4\,d\mu_4)^2+(\mu_2\,d\mu_2+\mu_3\,d\mu_3)^2\Big)\Big]\,. \tag{28}$$

Note that we have reinstated the gauge-coupling constant g in this expression.

Having obtained the Kaluza-Klein metric Ansatz for the three dilaton/axion pairs, it is a simple matter to incorporate also the associated $U(1)^4$ gauge fields that naturally accompany this truncation of the maximal supergravity. Denoting their potentials by $A^i_{(1)}$, for $i=1,2,3,4$, we simply replace each occurrence of $d\phi_i$ in (28) by

$$d\phi_i \longrightarrow d\phi_i - g\,A^i_{(1)}\,. \tag{29}$$

Finally in this section, we may note that our result (28) is consistent with previously-obtained special cases. In particular, if we set the three axions χ_i to zero, then the function Ξ reduces to

$$\Xi=\Delta^2\,, \tag{30}$$

where

$$\Delta = Y_1\,Y_2\,Y_3\,\mu_1^2+\frac{Y_1}{Y_2\,Y_3}\,\mu_2^2+\frac{Y_2}{Y_1\,Y_3}\,\mu_3^2+\frac{Y_3}{Y_1\,Y_2}\,\mu_4^2\,. \tag{31}$$

In the absence of axions, it is natural to define

$$X_1=Y_1\,Y_2\,Y_3\,,\quad X_2=\frac{Y_1}{Y_2\,Y_3}\,,\quad X_3=\frac{Y_2}{Y_1\,Y_3}\,,\quad X_4=\frac{Y_3}{Y_1\,Y_2}\,, \tag{32}$$

implying that we shall have

$$\Delta=\sum_i X_i\,\mu_i^2\,,\qquad Z_i=\Delta\,X_i^{-1}\,. \tag{33}$$

It can be seen that the metric Ansatz (28) therefore indeed reduces to the one given in [2] if the axions are set to zero.

2.3 The Ansatz for the 4-form Field Strength

In principle, we should like to obtain also the Ansatz for the 4-form field strength $\hat F_{(4)}$ of eleven-dimensional supergravity. In spherical Kaluza-Klein reductions it is always much more difficult to obtain the Ansatz for antisymmetric tensors than for the metric, and the present case is no exception. Unfortunately, one can only obtain limited guidance from those results that are presented in [1]. In other truncations, simpler than the case in hand, it has been possible to determine the field-strength Ansatz by brute-force methods, and up to a point, this technique is still useful here. (This method was used successfully in [3], where the complete and explicit Ansätze for the S^7 reduction to the bosonic sector of

$N = 4$, $SO(4)$ gauged supergravity in $D = 4$ were obtained.) The contributions to the 4-form Ansatz can be organised into different sectors, and in all except one of these we have obtained complete results. Since these are instructive and useful in their own right, it seems to be worthwhile to present those results that we have obtained here.

We begin with a summary of the four-dimensional theory comprising gravity, the three dilaton/axion pairs, and the associated $U(1)^4$ gauge fields.

2.3.1 $D = 4$ Lagrangian

The complete Lagrangian for four-dimensional $N = 8$ $SO(8)$-gauged supergravity was obtained in [10, 11]. In [13, 2], the truncation to the $N = 2$ $U(1)^4$-gauged subsector was discussed. Adapting these results to the notation of this paper, we find that the four-dimensional bosonic Lagrangian for this $N = 2$ truncation is given by

$$\mathcal{L}_4 = R\,{*\mathbb{1}} - \tfrac{1}{2}\sum_{i=1}^{3}\Big({*d\varphi_i}\wedge d\varphi_i + e^{2\varphi_i}\,{*d\chi_i}\wedge d\chi_i\Big) - V\,{*\mathbb{1}} + \mathcal{L}_{Kin} + \mathcal{L}_{CS}\,, \tag{34}$$

where V is the potential for the scalar fields, and $\mathcal{L}_{Kin}$ and $\mathcal{L}_{CS}$ are the kinetic terms and the Chern-Simons terms for the four $U(1)$ gauge fields $F^i_{(2)} = dA^i_{(1)}$. The scalar potential is given by

$$V = -4g^2\,\sum_{i=1}^{3}(Y_i^2 + \widetilde{Y}_i^2)\,. \tag{35}$$

The kinetic terms for the gauge fields are

$$\begin{aligned}\mathcal{L}_{Kin} \;=\; -\tfrac{1}{2}|W|^{-2}\Big[&P_0\,\Big(\widetilde{Y}_1^2\,\widetilde{Y}_2^2\,\widetilde{Y}_3^2\,{*F^1_{(2)}}\wedge F^1_{(2)} + \widetilde{Y}_1^2\,Y_2^2\,Y_3^2\,{*F^2_{(2)}}\wedge F^2_{(2)}\\ &\qquad +Y_1^2\,\widetilde{Y}_2^2\,Y_3^2\,{*F^3_{(2)}}\wedge F^3_{(2)} + Y_1^2\,Y_2^2\,\widetilde{Y}_3^2\,{*F^4_{(2)}}\wedge F^4_{(2)}\Big)\\ &+2P_1\,b_2\,b_3\,(\widetilde{Y}_1^2\,{*F^1_{(2)}}\wedge F^2_{(2)} - Y_1^2\,{*F^3_{(2)}}\wedge F^4_{(2)})\\ &+2P_2\,b_1\,b_3\,(\widetilde{Y}_2^2\,{*F^1_{(2)}}\wedge F^3_{(2)} - Y_2^2\,{*F^2_{(2)}}\wedge F^4_{(2)})\\ &+2P_3\,b_1\,b_2\,(\widetilde{Y}_3^2\,{*F^1_{(2)}}\wedge F^4_{(2)} - Y_3^2\,{*F^2_{(2)}}\wedge F^3_{(2)})\Big]\,,\end{aligned} \tag{36}$$

where

$$\begin{aligned}&P_0 \equiv 1 + b_1^2 + b_2^2 + b_3^2\,,\quad W \equiv P_0 - 2\mathrm{i}\,b_1\,b_2\,b_3\,,\\ &P_1 \equiv 1 - b_1^2 + b_2^2 + b_3^2\,,\quad P_2 \equiv 1 + b_1^2 - b_2^2 + b_3^2\,,\quad P_3 \equiv 1 + b_1^2 + b_2^2 - b_3^2\,.\end{aligned} \tag{37}$$

Finally, the Chern-Simons terms for the gauge fields are

$$\begin{aligned}\mathcal{L}_{CS} \;=\; -|W|^{-2}\Big[&b_1\,b_2\,b_3\,\Big(\widetilde{Y}_1^2\,\widetilde{Y}_2^2\,\widetilde{Y}_3^2\,F^1_{(2)}\wedge F^1_{(2)} + \widetilde{Y}_1^2\,Y_2^2\,Y_3^2\,F^2_{(2)}\wedge F^2_{(2)}\\ &\qquad +Y_1^2\,\widetilde{Y}_2^2\,Y_3^2\,F^3_{(2)}\wedge F^3_{(2)} + Y_1^2\,Y_2^2\,\widetilde{Y}_3^2\,F^4_{(2)}\wedge F^4_{(2)}\Big)\end{aligned}$$

$$
\begin{aligned}
&+b_1\,(P_0+2b_2^2\,b_3^2)\,(\widetilde{Y}_1^2\,F^1_{(2)}\wedge F^2_{(2)}-Y_1^2\,F^3_{(2)}\wedge F^4_{(2)})\\
&+b_2\,(P_0+2b_1^2\,b_3^2)\,(\widetilde{Y}_2^2\,F^1_{(2)}\wedge F^3_{(2)}-Y_2^2\,F^2_{(2)}\wedge F^4_{(2)})\\
&+b_3\,(P_0+2b_1^2\,b_2^2)\,(\widetilde{Y}_3^2\,F^1_{(2)}\wedge F^4_{(2)}-Y_3^2\,F^2_{(2)}\wedge F^3_{(2)})\Big]\,. \qquad (38)
\end{aligned}
$$

From (34), we find that the equations of motion for the gauge fields are

$$
d(|W|^{-2}\,R_i)=0\,, \qquad (39)
$$

for $i=1,2,3,4$, where

$$
\begin{aligned}
R_1 &= \widetilde{Y}_1^2\,\widetilde{Y}_2^2\,\widetilde{Y}_3^2\,[P_0\,{*F^1_{(2)}}+2b_1\,b_2\,b_3\,F^1_{(2)}]+\widetilde{Y}_1^2\,[P_1\,b_2\,b_3\,{*F^2_{(2)}}+b_1\,(P_0+2b_2^2\,b_3^2)\,F^2_{(2)}] \qquad (40)\\
&\quad+\widetilde{Y}_2^2\,[P_2\,b_1\,b_3\,{*F^3_{(2)}}+b_2\,(P_0+2b_1^2\,b_3^2)\,F^3_{(2)}]+\widetilde{Y}_3^2\,[P_3\,b_1\,b_2\,{*F^4_{(2)}}+b_3\,(P_0+2b_1^2\,b_2^2)\,F^4_{(2)}]\,,\\
R_2 &= \widetilde{Y}_1^2\,Y_2^2\,Y_3^2\,[P_0\,{*F^2_{(2)}}+2b_1\,b_2\,b_3\,F^2_{(2)}]+\widetilde{Y}_1^2\,[P_1\,b_2\,b_3\,{*F^1_{(2)}}+b_1\,(P_0+2b_2^2\,b_3^2)\,F^1_{(2)}]\\
&\quad-Y_2^2\,[P_2\,b_1\,b_3\,{*F^4_{(2)}}+b_2\,(P_0+2b_1^2\,b_3^2)\,F^4_{(2)}]-Y_3^2\,[P_3\,b_1\,b_2\,{*F^3_{(2)}}+b_3\,(P_0+2b_1^2\,b_2^2)\,F^3_{(2)}]\,,\\
R_3 &= Y_1^2\,\widetilde{Y}_2^2\,Y_3^2\,[P_0\,{*F^3_{(2)}}+2b_1\,b_2\,b_3\,F^3_{(2)}]-Y_1^2\,[P_1\,b_2\,b_3\,{*F^4_{(2)}}+b_1\,(P_0+2b_2^2\,b_3^2)\,F^4_{(2)}]\\
&\quad+\widetilde{Y}_2^2\,[P_2\,b_1\,b_3\,{*F^1_{(2)}}+b_2\,(P_0+2b_1^2\,b_3^2)\,F^1_{(2)}]-Y_3^2\,[P_3\,b_1\,b_2\,{*F^2_{(2)}}+b_3\,(P_0+2b_1^2\,b_2^2)\,F^2_{(2)}]\,,\\
R_4 &= Y_1^2\,Y_2^2\,\widetilde{Y}_3^2\,[P_0\,{*F^4_{(2)}}+2b_1\,b_2\,b_3\,F^4_{(2)}]-Y_1^2\,[P_1\,b_2\,b_3\,{*F^3_{(2)}}+b_1\,(P_0+2b_2^2\,b_3^2)\,F^3_{(2)}]\\
&\quad-Y_2^2\,[P_2\,b_1\,b_3\,{*F^2_{(2)}}+b_2\,(P_0+2b_1^2\,b_3^2)\,F^2_{(2)}]+\widetilde{Y}_3^2\,[P_3\,b_1\,b_2\,{*F^1_{(2)}}+b_3\,(P_0+2b_1^2\,b_2^2)\,F^1_{(2)}]\,.
\end{aligned}
$$

2.3.2 The Ansatz for $\hat{F}_{(4)}$

In previous papers the Ansatz for the 4-form field strength $\hat{F}_{(4)}$ was obtained for the $U(1)^4$ truncation in absence of the three axions [2], and for the $N=4$ gauged $SO(4)$ truncation, in which there is one scalar and one axion [3]. Based on those results, it can be seen to be natural to write the Ansatz for $\hat{F}_{(4)}$ as the sum of three terms, each with its own characteristic contribution to the whole.

Thus we are led to the following construction for the 4-form field strength:

$$
\begin{aligned}
\hat{F}_{(4)} &= -2g\,U\,\epsilon_{(4)}+\hat{F}'_{(4)}+\hat{F}''_{(4)}\\
&\quad+\tfrac{1}{2g}\,(2Y_1^{-1}\,{*dY_1}-\chi_1\,Y_1^4\,{*d\chi_1})\wedge d(\mu_1^2+\mu_2^2)\\
&\quad+\tfrac{1}{2g}\,(2Y_2^{-1}\,{*dY_2}-\chi_2\,Y_2^4\,{*d\chi_2})\wedge d(\mu_1^2+\mu_3^2)\\
&\quad+\tfrac{1}{2g}\,(2Y_3^{-1}\,{*dY_3}-\chi_3\,Y_3^4\,{*d\chi_3})\wedge d(\mu_1^2+\mu_4^2)\,, \qquad (41)
\end{aligned}
$$

where

$$
U=Y_1^2\,(\mu_1^2+\mu_2^2)+\widetilde{Y}_1^2\,(\mu_3^2+\mu_4^2)+Y_2^2\,(\mu_1^2+\mu_3^2)+\widetilde{Y}_2^2\,(\mu_2^2+\mu_4^2)+Y_3^2\,(\mu_1^2+\mu_4^2)+\widetilde{Y}_3^2\,(\mu_2^2+\mu_3^2)\,, \qquad (42)
$$

and $\epsilon_{(4)}$ denotes the volume form on the four-dimensional spacetime. The term $\hat F''_{(4)}$ will be given by

$$\hat F''_{(4)} = -\frac{1}{2g^2}\,|W|^{-2}\,\sum_i d\mu_i^2\wedge(d\phi_i - g\,A^i_{(1)})\wedge R_i\,. \tag{43}$$

(We shall justify these expressions below.) The remaining term is $\hat F'_{(4)}$. This will be written in terms of a potential $\hat A'_{(3)}$, as $\hat F'_{(4)} = d\hat A'_{(3)}$. It will be the determination of $\hat A'_{(3)}$ that presents the greatest difficulty.

It will be noted that $\hat F_{(4)}$ does not identically satisfy $d\hat F_{(4)} = 0$. This feature was already seen in the truncations in [2] and [3]. It is not possible, at least within the usual second-order formulation of eleven-dimensional supergravity, to write an Ansatz for $\hat F_{(4)}$ in the S^7 reduction that identically satisfies $d\hat F_{(4)} = 0$. An implication from this is that one cannot write the Ansatz directly on the potential $\hat A_{(3)}$, which in turn means that one cannot write an Ansatz that can be substituted directly into the eleven-dimensional action. One must work at the level of the equations of motion.

In fact the requirement that $\hat F_{(4)}$ must satisfy the Bianchi identity $d\hat F_{(4)} = 0$ provides us with very important clues as to the correct form of the reduction Ansatz, and we used this in writing down our results in (41) and (43). The point is that the Bianchi identity will be satisfied by virtue of the $D = 4$ equations of motion for the scalar fields and the $U(1)$ gauge fields being satisfied. (To be precise, the scalar equations of motion in question here are those of the three dilatons φ_i, in combination with certain non-linear admixtures of the three axion equations of motion.) Of course the contribution to $\hat F_{(4)}$ from $\hat A'_{(3)}$, whose precise form we have not been able to determine, does not enter into the discussion of the Bianchi identity, since it gives a contribution $\hat F'_{(4)}$ that identically satisfies $d\hat F'_{(4)} = 0$.

To see how the Bianchi identity $d\hat F_{(4)} = 0$ implies the four-dimensional equations of motion for the scalars and the gauge fields, we note from the structure of (41) and (43) that after acting with d we shall have two distinct classes of terms. First, there will be terms of the form $d\mu_i^2\wedge\omega_{(4)}$, where $\omega_{(4)}$ is a 4-form living entirely in the four-dimensional spacetime. ($\omega_{(4)}$ will comprise terms of the form $\epsilon_{(4)}$, and of the form $d{*dY_i}$, *etc.* Of course they are all proportional to $\epsilon_{(4)}$.) The requirement of the vanishing of these terms will imply the scalar equations of motion. Secondly, there will be terms of the form $d\mu_i^2\wedge(d\phi_i - \frac12 g\,A^i_{(1)})\wedge\omega_{(3)}$ coming from the action of d on $\hat F''_{(4)}$, where $\omega_{(3)}$ is a 3-form living in the four-dimensional spacetime. The vanishing of these terms will imply the four-dimensional equations of motion for the gauge fields.

Let us consider the second type of contribution first, since it is the simpler one. The terms of this type come only from $d\hat{F}''_{(4)}$, and give

$$\sum_i d\mu_i^2 \wedge (d\phi_i - g\, A^i_{(1)}) \wedge d(|W|^{-2}\, R_i) = 0\,. \tag{44}$$

This can immediately be seen to imply precisely the equations of motion for the four $U(1)$ gauge fields, given in (39).

It remains to check that the terms of the form $d\mu_i^2 \wedge \omega_{(4)}$ coming from the Bianchi identity vanish by virtue of the four-dimensional scalar equations of motion. The kinetic terms of these scalar equations come from the action of d on the final three lines in (41). Clearly, we get the combinations of the form

$$d(2Y_1^{-1} {*dY_1} - \chi_1\, Y_1^4 {*d\chi_1})\,, \tag{45}$$

arising (with similar independent expressions involving the (Y_2, χ_2) and (Y_3, χ_3) pairs). This is a combination of the φ_1 and the χ_1 equations of motion. In fact it is

$$[d{*d\varphi_1} + e^{2\varphi_1} {*d\chi_1} \wedge d\chi_1] - \chi_1\, [d(e^{2\varphi_1} {*d\chi_1})]\,, \tag{46}$$

where the first quantity in square brackets is the dilaton equation of motion, and the second quantity in square brackets is the axion equation of motion.

This particular combination, of the dilaton equation plus an admixture of the axion equation, is an especially simple one to compare with the scalar equations of motion coming from the four-dimensional Lagrangian (34). It means that we are looking at the combination that comes from the following variation of the $D = 4$ Lagrangian:

$$\hat{\delta}\mathcal{L}_4 \equiv \frac{\delta\mathcal{L}_4}{\delta\varphi_1} - \chi_1\, \frac{\mathcal{L}_4}{\delta\chi_1}\,. \tag{47}$$

If we define a symbol $\hat{\delta}$ to denote this specific combination of field variations, *i.e.*

$$\hat{\delta} \equiv \frac{\delta}{\delta\varphi_1} - \chi_1\, \frac{\delta}{\delta\chi_1}\,, \tag{48}$$

then we find the great simplification that

$$\hat{\delta} Y_1^2 = Y_1^2\,, \qquad \hat{\delta}\widetilde{Y}_1^2 = -\widetilde{Y}_1^2\,, \qquad \hat{\delta} b_1 = 0\,. \tag{49}$$

(Of course since we are focusing on the scalars with the index $i = 1$ at the moment, all of the scalar quantities with $i = 2$ or $i = 3$ labels are invariant under this transformation.) The last equation in (49), $\hat{\delta} b_1 = 0$, leads to an *enormous* simplification when we vary $\mathcal{L}_{Kin}$ and $\mathcal{L}_{CS}$ given by (36) and (38). It means that $|W|$, the P_a, and all the b_i are invariant. We need only consider Y_1 and $\widetilde{Y}_1$, and these just vary by the very simple rules given in (49).

With these observations, it becomes a relatively straightforward matter to verify that the terms of the form $d\mu_i^2\wedge\omega_{(4)}$ that arise in the Bianchi identity for $\hat F_{(4)}$ vanish *precisely* as a consequence of the scalar equations of motion following from (34), to all orders in scalar fields and gauge field strengths. Note that the contributions to the scalar equations of motion from the potential V given in (35) arise from the action of the exterior derivative on the term $-2g\,U\,\epsilon_{(4)}$ in (41). This part of the calculation can be seen quite easily, and can be examined in isolation from the more complicated contributions from the four-dimensional gauge fields.

The contribution $\hat F'_{(4)} = \hat A'_{(3)}$ in (41) remains undetermined. We know some aspects of it structure, for example that it is of the general from

$$\hat A'_{(3)} = \sum_{i\ne j} h_{ij}\,(\mu_i^2\,d\mu_j^2 - \mu_j^2\,d\mu_i^2)\wedge(d\phi_i - g\,A^i_{(1)})\wedge(d\phi_j - g\,A^j_{(1)})\,, \tag{50}$$

where the functions h_{ij} depend on the scalars φ_i and χ_i, and the direction cosines μ_i. At leading order, these terms will give rise to the linearised Ansatz for the axions χ_i. If explicit expressions for the complete Ansatz for the $N=8$ $SO(8)$ gauged supergravity embedding were available, $A'_{(3)}$ could in principle be determined by substituting the expressions for $u_{ij}{}^{KL}$ and v_{ijKL} appearing in (9) and (10) into them. To the extent that such expressions are implicit in the work of [1], a procedure in principle exists for reading off $A'_{(3)}$. It is not clear that attempting such a substitution would be simpler than a brute-force direct attack on the problem, of the type that has proved successful in previous (simpler) cases [2, 3].

2.4 Domain wall solutions and their oxidation

The four-dimensional $U(1)^4$ Lagrangian (34) supports a four-charge AdS black hole solution [13]. In the extremal limit, the four $U(1)$ gauge fields decouple and the solution becomes AdS domain wall, supported by the scalar fields only. It is given by [14]

$$\begin{aligned} ds_4 &= (gr)^4(H_1H_2H_3H_4)^{-1/2}\,dx^\mu\,dx_\mu + (H_1H_2H_3H_4)^{1/2}\,\frac{dr^2}{g^2r^2}\,,\\ e^{\varphi_i} &= Y_1^2 = f_i\,,\qquad \chi_i = 0\,, \end{aligned}\tag{51}$$

where

$$\begin{aligned} f_1 &= \frac{(H_3H_4)^{1/2}}{(H_1H_2)^{1/2}}\,,\qquad f_2 = \frac{(H_2H_4)^{1/2}}{(H_1H_3)^{1/2}}\,,\\ f_3 &= \frac{(H_2H_3)^{1/2}}{(H_1H_4)^{1/2}}\,,\qquad H_i = 1+\frac{\ell_i^2}{r^2}\,. \end{aligned}\tag{52}$$

This solution can be oxidised back to $D=11$ [14], where it acquires the interpretation of being a continuous ellipsoidal distribution [15, 16, 17, 18, 14, 19] of M2-branes.

The scalar kinetic terms in the Lagrangian (34) are invariant under global $SL(2,R)^3$ transformations, corresponding to the usual fractional-linear group action on each of the axion/dilaton pairs. The scalar potential in (34), on the other hand, is invariant only under the $SO(2)^3$ subgroup transformations

$$\tau_i \to \tau_i' = \frac{\cos\lambda_i\,\tau + sin\lambda_i}{-\sin\lambda_i\,\tau + \cos\lambda_i}\,. \tag{53}$$

where $\tau_i \equiv= \chi_i + \mathrm{i}\, e^{-\varphi_i}$. Applying these global transformations to the original domain walls we obtain new solutions, with

$$Y_i^2 = e^{\varphi_i} = \frac{1}{f_i}(f_i^2\cos^2\lambda_i + \sin^2\lambda_i)\,, \qquad \chi_i = \frac{\frac12(f_i-1)}{f_i^2\cos^2\lambda_i + \sin^2\lambda_i}\,. \tag{54}$$

The $\widetilde{Y}_i$ are hence given by

$$\widetilde{Y}_i^2 = \frac{f^2 + \frac14(f-1)^2\sin^2(2\lambda_i)}{f_i(f_i^2\cos^2\lambda_i + \sin^2\lambda_i)}\,. \tag{55}$$

Having obtained the $SO(2)^3$ rotated domain-wall solutions, they can be oxidised back to $D=11$. The eleven-dimensional metric is given by substituting the solution into (28). These solutions with non-vanishing χ_i no longer simply describe distributed M2-branes. To see this we note from (50) that with non-vanishing axions the field strength $F_{(4)}$ will involve components lying purely in the internal S^7. By contrast, in a distributed M2-brane solution one has $F_{(4)} = d^3x\wedge dH^{-1}$, where H is the harmonic function in the transverse space. Thus for a distributed M2-brane the field strength $F_{(4)}$ always carries three world-volume indices.

3 The 2-scalar $D=5$ embedding in type IIB

In this section, we consider the embedding of the 2-scalar truncation of $D=5$ gauged supergravity discussed in the introduction, and its embedding in the type IIB theory *via* an S^5 reduction. In the early stages of the derivation, we retain all four of the scalar fields of the truncation discussed in [4].

3.1 The metric reduction Ansatz

The set of 42 spin-0 fields in the complete $SO(6)$ gauged $N=8$ supergravity in $D=5$ [20] are described by a 27-bein $\mathcal{V}$, which transforms under local $USp(8)$ and global E_6. The truncation to four spin-0 fields is described in [4], in terms of an $SL(6,R)\times SL(2,R)$ basis, for which the components of the vielbein are decomposed as $(\mathcal{V}^{IJab}, \mathcal{V}_{I\alpha}{}^{ab})$. In terms of this decomposition, the following conjecture for the inverse S^5 metric has been proposed [4]:

$$\hat{g}^{mn}(x,y) \equiv \hat{\Delta}^{-\frac23}\, g^{mn}(x,y) = 2K^m_{IJ}\,K^n_{KL}\,\widetilde{\mathcal{V}}_{IJab}\,\widetilde{\mathcal{V}}_{KLcd}\,\Omega^{ac}\,\Omega^{bd}\,, \tag{56}$$

where $\widetilde{\mathcal{V}}$ is the inverse of the vielbein $\mathcal{V}$, $\hat{\Delta}^2 = \det(g_{mn}(x,y))/\det(g_{mn}(y))$, and $g_{mn}(y)$ is the undeformed round S^5 metric where the scalar fields are set to zero. The ten-dimensional metric Ansatz will then be

$$d\hat{s}_{10}^2 = \hat{\Delta}^{-\frac{2}{3}}\, ds_5^2 + g_{mn}(x,y)\, dy^m\, dy^n = \hat{\Delta}^{-\frac{2}{3}}\, (ds_5^2 + \hat{g}_{mn}(x,y)\, dy^m\, dy^n)\,. \tag{57}$$

The process of making the 4-scalar truncation in the vielbein $\mathcal{V}$ has been described in detail in [21]. Substituting this into the metric Ansatz (56) is a mechanical exercise that is most conveniently implemented by computer. Since the final result is considerably simpler than the intermediate stages we shall, without further ado, present the final answer. We find that the inverse 5-sphere metric $\hat{\partial}_{s_5}^2 \equiv \hat{g}^{mn}\, \partial_m\, \partial_n$ is given by

$$\begin{aligned}
\hat{\partial}_{s_5}^2 \;=\;& X^{-1}\Big(\cosh 2y_2\,(\cosh 2r - \sin\theta\,\sinh 2r)\,(K_{15}^2 + K_{25}^2 + K_{35}^2 + K_{45}^2)\\
&\qquad + \cosh 2y_2\,(\cosh 2r + \sin\theta\,\sinh 2r)\,(K_{16}^2 + K_{26}^2 + K_{36}^2 + K_{46}^2)\\
&\qquad + 2\cos\theta\,\sinh 2r\,\sinh 2y_2\,(K_{26}\,K_{35} - K_{25}\,K_{36} + K_{16}\,K_{45} - K_{15}\,K_{46})\Big)\\
&+ X^2\Big(\tfrac{1}{4}(3 - \cos\theta + 2\cos^2\theta\,\cosh 4r)\,(K_{12}^2 + K_{34}^2) + (K_{14}^2 + K_{23}^2)\\
&\qquad + \cosh^2 2y_2\,(K_{13}^2 + K_{24}^2) + 2\cos^2\theta\,\sinh^2 2r\,K_{12}\,K_{34} - 2\sinh^2 2y_2\,K_{13}\,K_{24}\Big)\\
&+ X^{-4}\,K_{56}^2\,.
\end{aligned} \tag{58}$$

The scalars (X, r, y_2, θ) are related to the quantities $(\rho, \varphi_1, \varphi_2, \phi)$ appearing in [21] by

$$\rho = X^{-\frac{1}{2}}\,, \quad r = \tfrac{1}{2}(\varphi_2 - \varphi_1)\,, \quad y_2 = \tfrac{1}{2}(\varphi_1 + \varphi_2)\,, \quad \theta = 2\phi\,. \tag{59}$$

Note that the $D = 5$ scalar Lagrangian for this truncation is

$$\mathcal{L} = -2\sum_{i=1}^{3}(\partial\varphi_i)^2 - \sinh^2(\varphi_1 - \varphi_2)\,(\partial\theta)^2 - V\,, \tag{60}$$

where $X = e^{-\sqrt{6}\,\varphi_3/2}$, and the scalar potential V takes the form [4]

$$\begin{aligned}
V \;=\;& g^2\Big(X^2\,[1 - \cos^2\theta\,(\sinh^2\varphi_1 - \sinh^2\varphi_2)] + X^{-1}\,[\cosh 2\varphi_1 + \cosh 2\varphi_2] \\
&+ \tfrac{1}{16}\,X^{-4}\,[2 + 2\sin^2\theta - 2\sin^2\theta\,\cosh(2(\varphi_1 - \varphi_2)) - \cosh 4\varphi_1 - \cosh 4\varphi_2]\Big)\,.
\end{aligned} \tag{61}$$

At this stage, we impose the further truncation to the 2-scalar subsector that we really want to consider. This corresponds to setting $\theta = 0$ and $\varphi_2 = 0$ [21]. It is easily verified from (60) and (61) that this is a consistent truncation. Thus we shall have $r = -\frac{1}{2}\varphi$, and $y_2 = \frac{1}{2}\varphi$, where we now drop the "1" subscript on φ_1. The potential (61) reduces to

$$V = \cosh^2\varphi\,\Big[X^2\,(2 - \cosh^2\varphi) + 2X^{-1} - \tfrac{1}{2}X^{-4}\,\sinh^2\varphi\Big]\,. \tag{62}$$

It is convenient also at this stage to perform a labelling of indices on the Killing vectors K_{ij} in (58), under which the index values $(2,3,4)$ are cycled: $2\to 3$, $3\to 4$ and $4\to 2$.

We now adopt a description of the round 5-sphere that is precisely analogous to the one that we introduced in Appendix B for S^7. This time, we shall end up with three "direction cosines" μ_i, subject to the condition $\mu_i\,\mu_i=1$, and three azimuthal angles ϕ_i. After manipulations similar to those in section 2, we arrive at the following expression for the inverse 5-sphere metric $\hat\partial_5^2$:

$$\hat\partial^2_{s_5}=\hat\partial^2_{s_2}+\hat\partial^2_{s_3}\,, \tag{63}$$

where the 2×2 and 3×3 blocks are given by

$$\begin{aligned}
\hat\partial^2_{s_2} &= \cosh^2\varphi\,\Big(X^{-1}\,[(\mu_1\,\partial_{\mu_3}-\mu_3\,\partial_{\mu_1})^2+(\mu_2\,\partial_{\mu_3}-\mu_3\,\partial_{\mu_2})^2]+X^2\,(\mu_1\,\partial_{\mu_2}-\mu_2\,\partial_{\mu_1})^2\Big)\,,\\
\hat\partial^2_{s_3} &= \Delta\,\cosh^2\varphi\,\Big[X\,(\mu_1^{-2}\,\partial^2_{\phi_1}+\mu_2^{-2}\,\partial^2_{\phi_2})+X^{-2}\,\mu_3^{-2}\,\partial^2_{\phi_3}\Big]\\
&\quad -\sinh^2\varphi\,\Big(X\,(\partial_{\phi_1}+\partial_{\phi_2})-X^{-2}\,\partial_{\phi_3}\Big)^2\,,
\end{aligned} \tag{64}$$

and

$$\Delta\equiv(\mu_1^2+\mu_2^2)\,X+\mu_3^2\,X^{-2}\,. \tag{65}$$

Note that the 2×2 inverse metric $\hat\partial^2_{s_2}$ is just equal to the metric for the single-scalar truncation when $\varphi=0$, multiplied by a factor of $\cosh^2\varphi$. The 3×3 inverse metric is equal to $\cosh^2\varphi$ times the $\varphi=0$ metric, with the correction term appearing in its second line.

The inverse of the 3×3 block $\hat\partial^2_{s_3}$ is straightforward to calculate, and we find

$$d\hat s_3^2=\frac{\mathrm{sech}^2\varphi}{\Delta}\,\Big(X^{-1}\,(\mu_1^2\,d\phi_1^2+\mu_2^2\,d\phi_2^2)+X^2\,\mu_3^2\,d\phi_3^2\Big)+\frac{\tanh^2\varphi}{\Delta^2}\,(\mu_1^2\,d\phi_1+\mu_2^2\,d\phi_2-\mu_3^2\,d\phi_3)^2\,. \tag{66}$$

Note that the determinant of $d\hat s_3^2$ is given by $(\mu_1\,\mu_2\,\mu_3)^2/(\Delta^3\,\cosh^4\varphi)$.

For the 2×2 block, the inversion gives the metric

$$d\hat s_2^2=\frac{\mathrm{sech}^2\varphi}{\Delta}\,\Big(X^{-1}\,(d\mu_1^2+d\mu_2^2)+X^2\,d\mu_3^2\Big)\,. \tag{67}$$

It is helpfull at this stage to reparameterise the direction cosines μ_i, and make redefinitions of the azimuthal angles (ϕ_1,ϕ_2) as follows:

$$\begin{aligned}
&\mu_1=\cos\xi\,\cos\tfrac12\vartheta\,,\qquad \mu_2=\cos\xi\,\sin\tfrac12\vartheta\,,\qquad \mu_3=\sin\xi\,,\\
&\phi_1=\tfrac12(\psi+\phi)\,,\qquad \phi_2=\tfrac12(\psi-\phi)\,.
\end{aligned} \tag{68}$$

In fact (ϑ,ϕ,ψ) are just the Euler angles on S^3. One can define left-invariant 1-forms σ_i, as

$$\sigma_1+\mathrm{i}\,\sigma_2=e^{-\mathrm{i}\psi}\,(d\vartheta+\mathrm{i}\,\sin\vartheta\,d\phi)\,,\qquad \sigma_3=d\psi+\cos\vartheta\,d\phi\,. \tag{69}$$

These satisfy $d\sigma_1 = -\sigma_2 \wedge \sigma_3$, and cyclically. Defining also

$$c \equiv \cos\xi\,, \qquad s \equiv \sin\xi\,, \tag{70}$$

we find that the 5-dimensional internal metric $d\hat{s}_5^2 \equiv \hat{g}_{mn}(x,y)\, dy^m\, dy^n = d\hat{s}_2^2 + d\hat{s}_3^2$ becomes

$$\begin{aligned} d\hat{s}_5^2 &= \frac{\text{sech}^2\varphi}{\Delta}\Big[X\,\Delta\, d\xi^2 + \tfrac14 X^{-1}\, c^2\,(\sigma_1^2+\sigma_2^2+\sigma_3^2) + X^2\, s^2\, d\phi_3^2\Big] \\ &\quad + \frac{\tanh^2\varphi}{4\Delta^2}\,(c^2\,\sigma_3 - 2s^2\, d\phi_3)^2\,, \end{aligned} \tag{71}$$

where

$$\Delta = X\, c^2 + X^{-2}\, s^2\,. \tag{72}$$

In the absence of the pseudoscalar field φ, this reduces to the metric Ansatz encountered in the $N=4$ gauged $SU(2)\times U(1)$ supergravity embedding obtained in [22]. In that case, the scalar field X parameterises inhomogeneous deformations of S^5 viewed as a foliation of $S^3\times S^1$ surfaces.

With the pseudoscalar φ non-vanishing, it is advantageous to rewrite the metric (71) as the sum of squares of just five quantities, by completing the square. After doing this, we obtain the result

$$d\hat{s}_5^2 = \frac{X}{\cosh^2\varphi}\, d\xi^2 + \frac{c^2\, X^{-1}}{4\Delta\,\cosh^2\varphi}\,(\sigma_1^2+\sigma_2^2) + \frac{c^2\, X}{4\Omega}\,\sigma_3^2 + \frac{s^2\,\Omega}{\Delta^2\,\cosh^2\varphi}\Big(d\phi_3 - \frac{c^2\,\sinh^2\varphi}{2\Omega}\,\sigma_3\Big)^2\,, \tag{73}$$

where

$$\Omega \equiv X^3\, c^2 + s^2\,\cosh^2\varphi\,. \tag{74}$$

This expression reduces to the one found in [22] if $\varphi = 0$. In that case, the scalar X parameterises deformations of S^5 corresponding to inhomogeneities of codimension 1 of the foliation by $S^3\times S^1$. When the pseudoscalar φ is included too, the inhomogeneities remain of codimension 1, but with a slightly more complicated structure. In addition, there is a sort of "twist" in the $S^3\times S^1$ product structure of the homogeneous foliating surfaces, as indicated by the cross-term between the interval $d\phi_3$ on S^1, and the 1-form σ_3 on S^3.

Finally, substituting our result for the internal hatted metric $d\hat{s}_5^2$ into (57), we arrive at the conjectured ten-dimensional metric Ansatz for this two-scalar truncation:

$$\begin{aligned} d\hat{s}_{10}^2 &= \Delta^{\frac12}\,\cosh\varphi\, ds_5^2 + \frac{X\,\Delta^{\frac12}}{\cosh\varphi}\, d\xi^2 + \frac{c^2\, X^{-1}}{4\Delta^{\frac12}\,\cosh\varphi}\,(\sigma_1^2+\sigma_2^2) \\ &\quad + \frac{c^2\, X\,\Delta^{\frac12}\,\cosh\varphi}{4\Omega}\,\sigma_3^2 + \frac{s^2\,\Omega}{\Delta^{\frac32}\,\cosh\varphi}\Big(d\phi_3 - \frac{c^2\,\sinh^2\varphi}{2\Omega}\,\sigma_3\Big)^2\,. \end{aligned} \tag{75}$$

3.2 The field-strength Ansätze

There does not seem to be any straightforward way to determine the Ansatz for the Kaluza-Klein reduction other fields of the ten-dimensional type IIB theory, in this two-scalar reduction. We know that when φ is taken to be zero, the Ansatz must reduce to one encompassed by the results in [22]. In particular, the remaining scalar field X enters in the Ansatz for the self-dual 5-form, whilst the dilaton, axion and 3-form field strengths of the type IIB theory vanish when $\varphi = 0$. Since it is a pseudoscalar, the field φ enters at the linearised level in the Ansatz for the NS-NS and R-R 2-form potentials $\hat{A}_{(2)} \equiv \hat{A}^{\rm NS}_{(2)}$ and $\hat{A}^{\rm RR}_{(2)}$ [23].

The relevant bosonic equations of motion of the type IIB theory are

$$\begin{aligned}
\hat{R}_{MN} &= \tfrac{1}{96}\hat{H}^2_{MN} + \tfrac14\Big((\hat{F}^1_{(3)})^2_{MN} - \tfrac{1}{12}(\hat{F}^1_{(3)})^2\hat{g}_{MN}\Big) + \tfrac14\Big((\hat{F}^2_{(3)})^2_{MN} - \tfrac{1}{12}(\hat{F}^2_{(3)})^2\hat{g}_{MN}\Big)\,,\\
d\hat{*}\hat{F}_{(3)} &= -{\rm i}\,\hat{H}_{(5)}\wedge\hat{F}_{(3)}\,, \qquad\qquad (76)\\
d\hat{H}_{(5)} &= -\tfrac{{\rm i}}{2}\,\hat{F}_{(3)}\wedge\hat{\bar{F}}_{(3)}\,,\qquad \hat{H}_{(5)} = \hat{*}\hat{H}_{(5)}\,,
\end{aligned}$$

where we have introduced the notation that

$$\hat{A}_{(2)} \equiv \hat{A}^{\rm NS}_{(2)} + {\rm i}\,\hat{A}^{\rm RR}_{(2)}\,. \qquad (77)$$

We are assuming here that the dilaton and axion of the type IIB theory vanish in the reduction. For this to be consistent with the type IIB equations of motion, it is necessary that

$$\hat{*}\hat{F}_{(3)}\wedge\hat{\bar{F}}_{(3)} = 0\,,\qquad \hat{*}\hat{F}_{(3)}\wedge\hat{F}_{(3)} = \hat{*}\hat{\bar{F}}_{(3)}\wedge\hat{\bar{F}}_{(3)}\,. \qquad (78)$$

We shall restrict our discussion from now on to the linearised level.

In the notation that we are using here, the linearised Ansatz for pseudoscalars φ will be of the form

$$\hat{A}_{(2)} = \varphi\,Y_{(2)}\,, \qquad (79)$$

where $Y_{(2)}$ is a complex 2-form spherical harmonic satisfying

$$d{*}Y_{(2)} = {\rm i}\,\lambda\,Y_{(2)} \qquad (80)$$

on the unit round 5-sphere. The Ansatz for the self-dual 5-form $\hat{H}_{(5)} \equiv \hat{G}_{(5)} + \hat{*}\hat{G}_{(5)}$ includes a Freunnd-Rubin term $\hat{G}_{(5)} = 4\epsilon_{(5)}$ (we have set the gauge coupling $g = 1$ here). Substituting into the type IIB equations of motion, one finds that the pseudoscalar φ satisfies the linearised equation of motion

$$[d{*}d\varphi + \lambda(\lambda-4)\,\varphi\,\epsilon_{(5)}]\wedge{*}Y_{(2)} = 0\,. \qquad (81)$$

A 2-form harmonic with eigenvalue λ gives a pseudoscalar φ with $m^2 = \lambda(\lambda - 4)$. We want the mass for the 10 and $\overline{10}$ members of the massless multiplet, namely $m^2 = -3$, which therefore requires $\lambda = 1$ or $\lambda = 3$. In fact, the required harmonics are those with $\lambda = 3$ (there are none with $\lambda = 1$).

There are ten such harmonics on S^5, which can be written in terms of the Killing spinors. There are Killing spinors $\eta_\pm$ satisfying $D_a \eta_\pm = \pm \frac{\mathrm{i}}{2} \Gamma_a \eta_\pm$. It turns out that the required 2-form harmonics are given by the construction

$$Y_{ab} = \bar{\eta}_- \Gamma_{ab} \eta_+ \,, \tag{82}$$

where η_- and η_+ are any two Killing spinors of the minus and plus kinds respectively. Solving for the Killing spinors, and substituting into (82), we find that one of the ten harmonics has a structure that is particularly naturally adapted to our parameterisation of the sphere, namely

$$Y_{(2)} = e^{\mathrm{i}\,\phi_3} \Big(c\, d\xi \wedge \sigma_3 + \tfrac{1}{2} s\, c^2\, \sigma_1 \wedge \sigma_2 - \mathrm{i}\, s\, c^2\, \sigma_3 \wedge d\phi_3 \Big) \,. \tag{83}$$

One may expect that this harmonic, or a closely related construction, will play a significant rôle in the construction of the reduction Ansatz at the full non-linear order, but we have not yet completed this investigation.

3.3 Oxidation of five-dimensional solutions

Given the conjectured metric reduction Ansatz, we can oxidise the metric in any solution of the two-scalar truncation of five-dimensional maximal gauged supergravity back to a solution of type IIB supergravity in $D = 10$. In principle, one can solve the equations of motion in this two-scalar sector to obtain a supersymmetric domain wall solution, which has an interpretation as the RG-flow equations on the strongly coupled field theory side, as discussed in [21]. Unfortunately the equations seem not to allow an explicit solution in terms of elementary functions.

One simple oxidation that we *can* perform is to take the $D = 5$ solution corresponding to the second (non-trivial) supersymmetric stationary point of the potential. This corresponds to the stationary point of (62) with [4]

$$X = 2^{-\frac{1}{3}} \,, \qquad \sinh\varphi = \tfrac{1}{\sqrt{3}} \,. \tag{84}$$

(The fully-supersymmetric stationary point is at $X = 1$, $\varphi = 0$.) Substituting into (73), we find that the internal 5-sphere metric $d\hat{s}_5^2$ at this stationary point is given by

$$d\hat{s}_5^2 = \frac{3}{2^{7/3}} \Big[d\xi^2 + \frac{c^2}{2(1+s^2)} (\sigma_1^2 + \sigma_2^2) + \frac{2c^2}{3+5s^2}\, \sigma_3^2 + \frac{s^2\,(3+5s^2)}{3(1+s^2)^2} \Big(d\phi_3 - \frac{c^2}{3+5s^2}\, \sigma_3 \Big)^2 \Big] \,. \tag{85}$$

4 Conclusion

In this paper, we have obtained the metric Ansätze for two examples of Kaluza-Klein sphere reductions, both of which involve pseudoscalar as well as scalar fields. The first example is the S^7 reduction of eleven-dimensional supergravity, with a truncation from $N = 8$ to the $N = 2$ theory with $U(1)^4$ gauge fields, three dilatons and three axions. Among other uses this reduction allows one to study the eleven-dimensional geometries corresponding to the lifting of the four-dimensional BPS AdS black hole and domain-wall solutions [13] of gauged supergravity. Our results generalise those obtained previously in [2], where the problem was studied in the absence of the three axionic scalars.

Our second example is a truncation of five-dimensional maximal gauged supergravity, to a subsector in which two spin-0 fields are retained, one of which is a scalar, and the other a pseudoscalar. This truncation retains the fields necessary for describing a second supersymmetric vacuum in $D = 5$, with $N = 2$ supersymmetry and $SU(2)\times U(1)$ invariance, in addition to the maximally-supersymmetric one with $SO(6)$ invariance [4]. The metric reduction Ansatz that we obtain here allows one to study the ten-dimensional geometries corresponding to the lifting of solutions of the five-dimensional theory. In principle, this can include the renormalisation-group flow [21] associated with the second supersymmetric extremum, although the explicit form of this five-dimensional solution is not known.

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Appendix A

In this appendix, we present the explicit form of the $SO(8)$ triality rotation that we used in section 2.1 in order to simplify the Kaluza-Klein metric reduction Ansatz:

$$
\begin{aligned}
K_{12} &\longrightarrow \tfrac{1}{2}(K_{12}+K_{34}+K_{56}+K_{78})\,, & K_{13} &\longrightarrow \tfrac{1}{2}(K_{13}-K_{24}+K_{57}-K_{68})\,,\\
K_{14} &\longrightarrow \tfrac{1}{2}(K_{14}+K_{23}+K_{58}+K_{67})\,, & K_{15} &\longrightarrow \tfrac{1}{2}(K_{15}-K_{26}+K_{37}-K_{48})\,,\\
K_{16} &\longrightarrow \tfrac{1}{2}(K_{16}+K_{25}+K_{38}+K_{47})\,, & K_{17} &\longrightarrow \tfrac{1}{2}(K_{17}-K_{28}+K_{35}-K_{46})\,,\\
K_{18} &\longrightarrow \tfrac{1}{2}(K_{18}+K_{27}+K_{36}+K_{45})\,, & K_{23} &\longrightarrow \tfrac{1}{2}(K_{23}+K_{14}-K_{58}-K_{67})\,,\\
K_{24} &\longrightarrow \tfrac{1}{2}(K_{24}-K_{13}+K_{57}-K_{68})\,, & K_{25} &\longrightarrow \tfrac{1}{2}(K_{25}+K_{16}-K_{38}-K_{47})\,,\\
K_{26} &\longrightarrow \tfrac{1}{2}(K_{26}-K_{15}+K_{37}-K_{48})\,, & K_{27} &\longrightarrow \tfrac{1}{2}(K_{27}+K_{18}-K_{36}-K_{45})\,,
\end{aligned}
$$

$$
\begin{aligned}
K_{28} &\longrightarrow \tfrac{1}{2}(K_{28} - K_{17} + K_{35} - K_{46})\,, & K_{34} &\longrightarrow \tfrac{1}{2}(K_{34} + K_{12} - K_{56} - K_{78})\,,\\
K_{35} &\longrightarrow \tfrac{1}{2}(K_{35} + K_{17} + K_{28} + K_{46})\,, & K_{36} &\longrightarrow \tfrac{1}{2}(K_{36} + K_{18} - K_{27} - K_{45})\,,\\
K_{37} &\longrightarrow \tfrac{1}{2}(K_{37} + K_{15} + K_{26} + K_{48})\,, & K_{38} &\longrightarrow \tfrac{1}{2}(K_{38} + K_{16} - K_{25} - K_{47})\,,\\
K_{45} &\longrightarrow \tfrac{1}{2}(K_{45} + K_{18} - K_{27} - K_{36})\,, & K_{46} &\longrightarrow \tfrac{1}{2}(K_{46} + K_{35} - K_{17} - K_{28})\,,\\
K_{47} &\longrightarrow \tfrac{1}{2}(K_{47} + K_{16} - K_{25} - K_{38})\,, & K_{48} &\longrightarrow \tfrac{1}{2}(K_{48} + K_{37} - K_{15} - K_{26})\,,\\
K_{56} &\longrightarrow \tfrac{1}{2}(K_{56} + K_{12} - K_{34} - K_{78})\,, & K_{57} &\longrightarrow \tfrac{1}{2}(K_{57} + K_{13} + K_{24} + K_{68})\,,\\
K_{58} &\longrightarrow \tfrac{1}{2}(K_{58} + K_{14} - K_{23} - K_{67})\,, & K_{67} &\longrightarrow \tfrac{1}{2}(K_{67} + K_{14} - K_{23} - K_{58})\,,\\
K_{68} &\longrightarrow \tfrac{1}{2}(K_{68} + K_{57} - K_{13} - K_{24})\,, & K_{78} &\longrightarrow \tfrac{1}{2}(K_{78} + K_{12} - K_{34} - K_{56})\,.
\end{aligned} \tag{86}
$$

Appendix B

In this Appendix, we collect some results on the geometry of the 7-sphere. We can describe S^7 as the unit sphere in R^8, with 8 real coordinates x_I;

$$x_I\, x_I = 1\,. \tag{87}$$

As such, it has a manifest $SO(8)$ symmetry, with 28 Killing vectors K_{IJ} given by

$$K_{IJ} = x^I\, \frac{\partial}{\partial x_J} - x^J\, \frac{\partial}{\partial x_I}\,. \tag{88}$$

We can also describe S^7 as the unit sphere in C^4, with 4 complex coordinates z_i:

$$\bar z_i\, z_i = 1\,. \tag{89}$$

We can relate these complex coordinates to the previous real ones as follows:

$$z_1 = x_1 + \mathrm{i}\, x_2\,,\quad z_2 = x_3 + \mathrm{i}\, x_4\,,\quad z_3 = x_5 + \mathrm{i}\, x_6\,,\quad z_4 = x_7 + \mathrm{i}\, x_8\,. \tag{90}$$

We can parameterise these complex coordinates as

$$z_1 = \mu_1\, e^{\mathrm{i}\,\phi_1}\,,\quad z_2 = \mu_2\, e^{\mathrm{i}\,\phi_2}\,,\quad z_3 = \mu_3\, e^{\mathrm{i}\,\phi_3}\,,\quad z_4 = \mu_4\, e^{\mathrm{i}\,\phi_4}\,, \tag{91}$$

where (89) implies that

$$\sum_{i=1}^4 \mu_i^2 = 1\,. \tag{92}$$

These (μ_i, ϕ_i) coordinates are precisely the ones used for describing higher-dimensional rotating black holes in [24], and in the S^7 reduction Ansatz obtained in [2].

From the coordinate transformations above, it is straightforward to establish that the real derivatives $\partial/\partial x_I$ that appear in the Killing vectors (88) are given by

$$\frac{\partial}{\partial x_1} = \cos\phi_1\, \frac{\partial}{\partial \mu_1} - \frac{\sin\phi_1}{\mu_1}\, \frac{\partial}{\partial \phi_1}\,,\qquad \frac{\partial}{\partial x_2} = \sin\phi_1\, \frac{\partial}{\partial \mu_1} + \frac{\cos\phi_1}{\mu_1}\, \frac{\partial}{\partial \phi_1}\,, \tag{93}$$

with analogous expressions involving (μ_2, ϕ_2), (μ_3, ϕ_3) and (μ_4, ϕ_4) for the pairs (x_3, x_4), (x_5, x_6) and (x_7, x_8) respectively. It is easy to see from this that the four Killing vectors K_{12}, K_{34}, K_{56} and K_{78} are simply of the form:

$$K_{12} = \frac{\partial}{\partial \phi_1}\,, \quad K_{34} = \frac{\partial}{\partial \phi_2}\,, \quad K_{56} = \frac{\partial}{\partial \phi_3}\,, \quad K_{78} = \frac{\partial}{\partial \phi_4}\,. \tag{94}$$

These are the four commuting $U(1)$ generators. It is convenient to write them as ∂_{ϕ_1}, *etc.*

We also note that the Killing-vector bilinears in the top 3 lines in (14) are also relatively simple, when expressed in terms of the μ_i and ϕ_i coordinates. After some algebra we find, for example, that

$$K_{13}^2 + K_{14}^2 + K_{23}^2 + K_{24}^2 = (\mu_1\, \partial_{\mu_2} - \mu_2\, \partial_{\mu_1})^2 + \frac{\mu_2^2}{\mu_1^2}\, \partial_{\phi_1}^2 + \frac{\mu_1^2}{\mu_2^2}\, \partial_{\phi_2}^2 \tag{95}$$

with analogous results for the other five combinations.

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