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The low energy limit of the $AdS_3 \times S^3 \times M_4$ spinning string

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Abstract

We derive the low-energy effective action for the spinning (GKP) string in $AdS_3 \times S^3 \times M_4$ where $M_4 = S^3 \times S^1$ or T^4. In the first case the action consists of two $O(4)$ non-linear sigma models which are coupled through their interaction with four massless Majorana fermions (plus one free decoupled scalar). While in the second case it consists of one $O(4)$ sigma model coupled to four Majorana fermions together with four free scalars from the T^4. We show that these models are classically integrable by constructing their Lax connections.

Contents

1 Introduction

String theory on $AdS_3 \times S^3 \times M_4$ with $M_4 = (S^3 \times S^1, T^4)$ preserving 16 supercharges and supported by pure Ramond-Ramond (RR) flux arises as the gravity side of the AdS_3/CFT_2-correspondence.[1] As in earlier incarnations of AdS/CFT there are strong hints of integrability, both at the classical and quantum level [2], although much less is known on the CFT side. Sparked by the discovery of the integrable structure, AdS_3/CFT_2 has recently enjoyed an increased interest in the literature. While strings on $AdS_3 \times S^3 \times M_4$ have many similarities with strings in $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ there are nevertheless crucial differences. The biggest issue seems to be the appearance of massless modes on the worldsheet which are difficult to deal with in the standard approach to integrability. Nevertheless, if one focuses on just the massive modes the Bethe equations and the exact S-matrix for these can be derived along similar lines as in earlier AdS/CFT examples, see [3] for a review. Based on the algebraic curve technique the first study of the integrable structure was initiated in [2] and later a full set of Bethe equations was conjectured in [4] and the exact S-matrix in [5]. These conjectures passed a few initial tests but the mixing of modes from the two S^3 factors turned out to be problematic [6]. This was subsequently addressed in [7, 8] and later confirmed to match with the tree-level worldsheet S-matrix in [9].[2] As in other integrable AdS/CFT examples the underlying symmetry can only determine the Bethe equations up to overall scalar phase factors. The one-loop strong coupling contribution to the phases was engineered in [11–13], see also [14, 15], and later generalized to all orders in [16]. For the massless excitations the situation is, as mentioned, more complicated and currently it is not known how to include them in the exact solution. However, by looking at the decompactifying case where one S^3 blows up, i.e. the $AdS_3 \times S^3 \times T^4$ limit, some first steps were taken in [17].

In this paper we will take a somewhat different approach to trying to understand the integrable structures of the $AdS_3 \times S^3 \times M_4$ string. We will consider the spinning or Gubser-Klebanov-Polyakov (GKP) string solution [18]. By performing a fluctuation analysis around the solution it is known that the excitations come in both massive and massless [19,20] modes. Generally, the full Lagrangian describing all excitations is fairly involved. However, by restricting to the low-energy sector, capturing the dynamics of the massless modes, the theory becomes much simpler [21,22]. Since the full sigma model is believed to be integrable beyond the classical level, the low-energy GKP string should inherit this property. This opens up new avenues for testing the quantum integrability, see for example [23–25] for recent results in AdS_4/CFT_3.

[1] For the case of both NSNS and RR flux see [1].

[2] A mismatch appears at one loop (at least in the T^4 limit). This issue was resolved for $AdS_3 \times S^3 \times T^4$ with the exact S-matrix conjectured in [10].

For the simplest case of $AdS_5 \times S^5$ the massless modes of the GKP string are described by an $O(6)$ sigma model [21] while the low-energy dynamics of the $AdS_4 \times \mathbb{CP}^3$ GKP string are captured by a $\mathbb{CP}^3$ sigma model coupled to a Dirac fermion [22]. For the $AdS_3 \times S^3 \times S^3 \times S^1$ case we will demonstrate that the low-energy dynamics of the GKP string is described by two $O(4)$ non-linear sigma models coupled to four Majorana fermions together with one decoupled scalar,[3]

$$\begin{aligned}\mathcal{L} \;=\;& \frac{1}{2}e_i{}^{\hat{a}}e^{i\hat{a}} + \frac{1}{2}e_i{}^{a'}e^{ia'} + \frac{1}{2}\partial_i y\partial^i y + \frac{i}{4}\mathrm{tr}\left(\bar{\Psi}\rho^i\partial_i\Psi\right) - \frac{1}{16}(\varepsilon^{\hat{a}\hat{b}\hat{c}}\omega_i{}^{\hat{b}\hat{c}} + 2\sqrt{\alpha}\,\delta_{ij}e^{j\hat{a}})\,\mathrm{tr}\big(\bar{\Psi}\rho^i\Psi\sigma^{\hat{a}}\big)\\ &+\frac{1}{16}(\varepsilon^{a'b'c'}\omega_i{}^{b'c'} + 2\sqrt{1-\alpha}\,\delta_{ij}e^{ja'})\mathrm{tr}\big(\bar{\Psi}\rho^i\sigma^{a'}\Psi\big)\,,\end{aligned} \tag{1.1}$$

where primed and hatted indices refer to the first and second S^3 factor respectively. Here $e^{\hat{a}}$ and $e^{a'}$ are the vielbeins of the two S^3's and $\omega^{\hat{a}\hat{b}}$, $\omega^{a'b'}$ the corresponding spin connections, while the parameter $0 \leq \alpha \leq 1$ controls the relative size of the two S^3's. The four Majorana fermions Ψ_I have been grouped into a 2×2 matrix $\Psi = \sigma^I\Psi_I$. The Pauli matrices $\sigma^{\hat{a}}$ and $\sigma^{a'}$ and the trace act in this space while the ρ^i are the 2d gamma-matrices acting on the spinor indices only. See section 4.3 for more details. Note that the presence of δ_{ij} in the coupling to the fermions explicitly breaks the 2d Lorentz invariance. This is a novel property as compared to the $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ strings whose low-energy dynamics are described by relativistic sigma models. Note also the absence of Ψ^4-terms which were present in the $AdS_4 \times \mathbb{CP}^3$ case. Despite these differences we will show that this model is also integrable, at least at the classical level.

The structure of the paper is as follows. We begin by describing the structure of the $AdS_3 \times S^3 \times S^3 \times S^1$ Green-Schwarz string action up to fourth order in fermions in section 2. We then introduce suitable coordinates and derive the low-energy effective action in sections 3 and 4. This is done by first integrating out the massive bosonic coordinate and then putting all remaining massive fields to zero by hand. We also introduce a kappa symmetry gauge-fixing which turns out to be useful. It is shown that the low-energy effective action reduces to (1.1). In section 5 we show that this model is classically integrable by constructing its Lax representation. We end the paper with some conclusions.

2 The Green-Schwarz string in $AdS_3 \times S^3 \times M_4$

The type II Green-Schwarz superstring action in a general supergravity background can be expanded in the fermions as

$$S = -T\int d^2\sigma\,\mathcal{L}\,, \qquad \mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots \tag{2.1}$$

The purely bosonic Lagrangian is

$$\mathcal{L}^{(0)} = \frac{1}{2}\gamma^{ij}e_i{}^Ae_j{}^B\eta_{AB}\,, \qquad \gamma^{ij} = \sqrt{-g}\,g^{ij}\,, \tag{2.2}$$

where $e_i{}^A(X)$ $(A = 0, 1, \cdots, 9)$ are the vielbeins of the purely bosonic part of the background pulled back to the worldsheet and g_{ij} is an independent worldsheet metric with $g = \det g_{ij}$.

The terms quadratic in fermions take the form [26]

$$\mathcal{L}^{(2)} = ie_i{}^A\,\bar{\Theta}\Gamma_A K^{ij}\mathcal{D}_j\Theta\,, \qquad K^{ij} = \gamma^{ij} - \varepsilon^{ij}\Gamma_{11}\,. \tag{2.3}$$

[3] The $AdS_3 \times S^3 \times T^4$ case is obtained by taking $\alpha \to 1$ giving a single $O(4)$ sigma model coupled to fermions with four decoupled free scalars coming from the T^4.

The appearance of the matrix K^{ij} is related to kappa symmetry. The Killing spinor derivative $\mathcal{D}$ is given below.

The quartic fermion terms in the action were recently found in [27]. They take the form[4]

$$\begin{aligned}\mathcal{L}^{(4)} \quad = \quad & -\frac{1}{2}\bar{\Theta}\Gamma^{A}\mathcal{D}_{i}\Theta\,\bar{\Theta}\Gamma_{A}K^{ij}\mathcal{D}_{j}\Theta+\frac{i}{6}e_{i}{}^{A}\,\bar{\Theta}\Gamma_{A}K^{ij}\mathcal{M}\mathcal{D}_{j}\Theta+\frac{i}{48}e_{i}{}^{A}e_{j}{}^{B}\,\bar{\Theta}\Gamma_{A}K^{ij}\big(M+\tilde{M}\big)S\Gamma_{B}\Theta \\ & +\frac{1}{48}e_{i}{}^{A}e_{j}{}^{B}\,\bar{\Theta}\Gamma_{A}{}^{CD}K^{ij}\Theta\left(3\bar{\Theta}\Gamma_{B}U_{CD}\Theta-2\bar{\Theta}\Gamma_{C}U_{DB}\Theta\right) \\ & -\frac{1}{48}e_{i}{}^{A}e_{j}{}^{B}\,\bar{\Theta}\Gamma_{A}{}^{CD}\Gamma_{11}K^{ij}\Theta\left(3\bar{\Theta}\Gamma_{B}\Gamma_{11}U_{CD}\Theta+2\bar{\Theta}\Gamma_{C}\Gamma_{11}U_{DB}\Theta\right). \end{aligned} \tag{2.4}$$

The definition of $\mathcal{D}$, $\mathcal{M}$, M and U_{AB} for a general type II supergravity background can be found in [27]. Here we will only give the expressions for the case of interest here: the type IIA $AdS_3\times S^3\times S^3\times S^1$ supergravity solution with RR four-form flux. Θ is taken to be a 32-component Majorana spinor and the Killing spinor derivative is given by

$$\mathcal{D}_i\Theta=(\partial_i-\frac{1}{4}\omega_i{}^{AB}\Gamma_{AB}+\frac{1}{8}e_i{}^A\,S\Gamma_A)\Theta \quad \text{where} \qquad S=-4\Gamma^{0129}\big(1-\mathcal{P}\big)\,. \tag{2.5}$$

Here $\mathcal{P}$ is a projection matrix given by

$$\mathcal{P}=\frac{1}{2}(1+\sqrt{\alpha}\,\Gamma^{012345}+\sqrt{1-\alpha}\,\Gamma^{012678}) \tag{2.6}$$

and is in fact the projector which singles out the 16 supersymmetries preserved by the background. The AdS_3-directions are indexed by $(0,1,2)$ the first S^3 by $(3,4,5)$, the second S^3 by $(6,7,8)$ and the S^1 by (9). The parameter $0\leq\alpha\leq1$ determines the relative size of the two S^3's. In units of the AdS_3-radius the S^3 radii are

$$\hat{R}=\frac{1}{\sqrt{\alpha}}\,,\qquad R'=\frac{1}{\sqrt{1-\alpha}}\,. \tag{2.7}$$

The case $\alpha=0,1$ corresponds to $AdS_3\times S^3\times T^4$ where one of the three-spheres is decompactified. The remaining objects appearing in (2.4) reduce, in $AdS_3\times S^3\times S^3\times S^1$, to

$$\begin{aligned} U_{AB} &= \frac{1}{32}S\Gamma_{[A}S\Gamma_{B]}-\frac{1}{4}R_{AB}{}^{CD}\Gamma_{CD}\,,\\ M^\alpha{}_\beta &= \frac{i}{16}\bar{\Theta}S\Theta\,\delta^\alpha_\beta-\frac{i}{8}\Theta^\alpha\,(\bar{\Theta}S)_\beta+\frac{i}{8}(\Gamma^AS\Theta)^\alpha\,(\bar{\Theta}\Gamma_A)_\beta\,,\qquad \tilde{M}=\Gamma_{11}M\Gamma_{11}\,,\\ \mathcal{M}^\alpha{}_\beta &= M^\alpha{}_\beta+\tilde{M}^\alpha{}_\beta+\frac{i}{8}(S\Gamma^A\Theta)^\alpha\,(\bar{\Theta}\Gamma_A)_\beta-\frac{i}{16}(\Gamma^{AB}\Theta)^\alpha\,(\bar{\Theta}\Gamma_AS\Gamma_B)_\beta\,, \end{aligned} \tag{2.8}$$

where the nonzero components of the Riemann tensor of $AdS_3\times S^3\times S^3\times S^1$ are

$$R_{ab}{}^{cd}=2\delta^c_{[a}\delta^d_{b]}\,,\qquad R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}=-2\alpha\delta^{\hat{c}}_{[\hat{a}}\delta^{\hat{d}}_{\hat{b}]}\,,\qquad R_{a'b'}{}^{c'd'}=-2(1-\alpha)\delta^{c'}_{[a'}\delta^{d'}_{b']}\,, \tag{2.9}$$

where $a,\hat{a}$ and a' refer to AdS_3, the first S^3 and the second S^3 respectively.

3 Parameterization

The metric of AdS_3 in global coordinates is

$$ds^2=-\cosh^2\rho\,dt^2+d\rho^2+\sinh^2\rho\,d\varphi^2\,. \tag{3.1}$$

[4]Our normalization of Θ differs from that of [27] by a factor of $\sqrt{2}$.

And the long spinning string solution is given by [18]

$$t = \varphi = \kappa\tau \qquad \rho = \kappa\sigma\,, \tag{3.2}$$

with κ a constant. It will be more convenient for our purposes however to use different coordinates. Defining new coordinates by

$$\begin{aligned}
\sinh 2\zeta &= -\sin(t-\varphi)\sinh 2\rho \\
e^{4iu} &= e^{2i(t+\varphi)}\frac{\cos(t-\varphi)+i\cosh 2\rho\sin(t-\varphi)}{\cos(t-\varphi)-i\cosh 2\rho\sin(t-\varphi)} \\
\sinh 2\chi &= \frac{\cos(t-\varphi)\sinh 2\rho}{\sqrt{1+\sin^2(t-\varphi)\sinh^2 2\rho}}
\end{aligned} \tag{3.3}$$

the metric takes the form [21]

$$ds^2 = -du^2 + d\chi^2 - 2\sinh 2\zeta\, du d\chi + d\zeta^2\,. \tag{3.4}$$

Defining further

$$u = z^+ + z^-\,, \qquad \chi = z^+ - z^-\,, \qquad \zeta = \log\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z} \tag{3.5}$$

the metric becomes

$$ds^2 = -4dz^+dz^- + 4\frac{z+\frac{1}{4}z^3}{\left(1-\frac{1}{4}z^2\right)^2}\Big(-(dz^+)^2+(dz^-)^2\Big) + \frac{dz^2}{\left(1-\frac{1}{4}z^2\right)^2}\,. \tag{3.6}$$

The upshot is that this metric is invariant under constant shifts of the two light-cone coordinates $z^\pm$, something which will prove convenient in solving the Virasoro constraints.

In terms of these new coordinates, the long spinning (GKP) string solution is given by

$$z^\pm = \kappa\sigma^\pm\,, \qquad \sigma^\pm = \frac{1}{2}(\tau\pm\sigma)\,. \tag{3.7}$$

We will leave the S^3 metric unspecified and work directly in terms of the spin connection and vielbeins.

4 Low-energy effective action

Having defined the AdS_3 metric it is now straightforward to expand the action around the spinning string solution (3.7). To find the spectrum we consider the quadratic action. Taking the conformal gauge $\gamma^{ij} = \eta^{ij}$ with $\eta_{+-} = 2$ and using (3.6) the bosonic action (2.2) reduces to

$$\mathcal{L}_2^{(0)} = -2\partial_- z^+\partial_+ z^- + 2\kappa z(\partial_+ z^- - \partial_- z^+) + \frac{1}{2}\partial_+ z\partial_- z + \frac{1}{2}\partial_+ y^{\hat m}\partial_- y_{\hat m} + \frac{1}{2}\partial_+ y^{m'}\partial_- y_{m'} + \frac{1}{2}\partial_+ y\partial_- y\,. \tag{4.1}$$

As we will see below solving the Virasoro constraints will eliminate $z^\pm$ from the physical spectrum and generate a mass term for the remaining AdS-coordinate z. The seven remaining scalars, three ($y^{\hat m}$) from the first S^3, three ($y^{m'}$) from the second S^3 and one (y) from the S^1 remain massless. To find the low-energy effective action we will integrate out the massive boson z leaving only the seven massless bosons.

Let us now turn to the spectrum of the fermions. Using the spinning string solution (3.7) in the quadratic fermion action (2.3) and using (2.5) we find at the quadratic level[5]

$$\mathcal{L}_2^{(2)} = i\bar{\upsilon}\Gamma_+ P\partial_-\upsilon + i\bar{\upsilon}\Gamma_- P\partial_+\upsilon + i\bar{\vartheta}\Gamma_+ P\partial_-\vartheta + i\bar{\vartheta}\Gamma_- P\partial_+\vartheta - i\kappa\bar{\vartheta}\Gamma_{29+-}P\vartheta\,, \tag{4.2}$$

where $\Gamma_\pm = \Gamma_0 \pm \Gamma_1$ and

$$P = \frac{1}{2}\big(1 + \Gamma^{01}\Gamma_{11}\big) \tag{4.3}$$

is the kappa symmetry projection matrix which ensures that only 16 of the 32 components of Θ are physical. We have used the projection operator $\mathcal{P}$ defined in (2.6) to split the fermions into $16+16$

$$\Theta = \mathcal{P}\Theta + (1-\mathcal{P})\Theta = \vartheta + \upsilon\,. \tag{4.4}$$

The 16 fermions ϑ are in one-to-one correspondence with the supersymmetries of the background. They are the fermions described by the supercoset $\frac{D(2,1;\alpha)\times D(2,1;\alpha)}{SU(1,1)\times SU(2)\times SU(2)}$ [2,6] and we refer to them as supercoset fermions. As can be seen from (4.2) it is precisely these fermions which acquire mass for the spinning string and since we are interested only in the low-energy effective action we will set them to zero in the following. The 16 non-coset fermions υ are massless and they are to be kept in the low-energy effective action. We see that it is important that we started with the full Green-Schwarz superstring action. If we had tried to use instead a partially kappa gauge-fixed version like the supercoset action we would have missed these fermions.[6]

Coordinate	Mass	Multiplicity
z	2κ	1
$y^{\hat{m}}$, $y^{m'}$, y	0	7
υ	0	4
ϑ	κ	4

Table 1: Spectrum of excitations around the GKP string.

The spectrum is summarized in table 1. To get the low-energy effective action we can simply set the massive fermions to zero but the massive boson z should be integrated out more carefully. We will now describe how to do this.

4.1 Integrating out the massive boson z

Since the discussion here will affect only the the AdS-coordinates $(z^\pm, z)$ we will work only with the terms in the Lagrangian involving these fields. For the low-energy effective action only the terms of mass-dimension two or less are relevant. Using the fact that $z^\pm$ have dimension zero, υ has dimension $\frac{1}{2}$ and z as it will be integrated out effectively has dimension 1 we get by expanding (2.2) and (2.3)

$$\mathcal{L}_z = -2\partial_+z^-\partial_-z^+ + 2\kappa\big(\partial_+z^- - \partial_-z^+\big)z + \frac{i}{2}(\partial_+z^- + \partial_-z^+)\,\bar{\upsilon}\Gamma_{2+-}\upsilon - i\kappa z\,\bar{\upsilon}\Gamma_{11}\Gamma_{2+-}\upsilon + \ldots \tag{4.5}$$

where we have used the expansion of the AdS vielbein and spin connection to $\mathcal{O}(z)$

$$e^+ \sim dz^+ - zdz^-\,, \qquad e^- \sim dz^- + zdz^+\,, \qquad \omega^{2-} \sim dz^+ + zdz^-\,, \qquad \omega^{2+} \sim -dz^- + zdz^+\,. \tag{4.6}$$

The quadratic fermion terms come from the spin connection inside $\mathcal{D}$ in (2.5). We also have the Virasoro constraints $G_{++} = 0 = G_{--}$ where $G_{ij} = E_i{}^A E_j{}^B \eta_{AB}$ is the induced metric on the

[5] We have rescaled the fermions by a factor $\kappa^{-1/2}$.
[6] This was true also for the $AdS_4 \times \mathbb{CP}^3$ string, see Bykov [22].

worldsheet. Using the fact that

$$E_i{}^A = e_i{}^A + i\bar{\Theta}\Gamma^A\mathcal{D}_i\Theta + \mathcal{O}(\Theta^4) \tag{4.7}$$

we find

$$\begin{aligned} 0 &= \frac{1}{4\kappa}G_{++} = -\partial_+ z^- - \kappa z + \frac{i}{4}\bar{\upsilon}\Gamma_{2+-}\upsilon + \ldots \\ 0 &= \frac{1}{4\kappa}G_{--} = -\partial_- z^+ + \kappa z + \frac{i}{4}\bar{\upsilon}\Gamma_{2+-}\upsilon + \ldots \end{aligned} \tag{4.8}$$

where we have dropped all terms of dimension greater than one. Again the fermion terms come from the AdS spin connection (4.6). Using (4.8) to solve for $\partial_\pm z^\mp$ allows us to write $\mathcal{L}_z$ as

$$\mathcal{L}_z = -2\kappa^2 z^2 - i\kappa z\,\bar{\upsilon}\Gamma_{11}\Gamma_{2+-}\upsilon - \frac{1}{8}(\bar{\upsilon}\Gamma_{2+-}\upsilon)^2 + \ldots \tag{4.9}$$

Since the kinetic term for the massive boson is $\frac{1}{2}\partial_+ z\partial_- z$ we see that z has mass 2κ. Finally, integrating out the massive boson z gives

$$\mathcal{L}_z = -\frac{1}{8}(\bar{\upsilon}\Gamma_{11}\Gamma_{2+-}\upsilon)^2 - \frac{1}{8}(\bar{\upsilon}\Gamma_{2+-}\upsilon)^2\,. \tag{4.10}$$

As it turns out, once we impose the kappa symmetry gauge-fixing this will completely vanish. Thus, in hindsight we could have put the massive bosons and fermions in the action to zero directly but only if we fix the kappa symmetry in a certain way.

4.2 z-independent part

In the previous section we took care of all terms in the low-energy effective action which involved the AdS-coordinates. The remaining terms are obtained by simply setting $(z^\pm, z)$ to zero (recall that we are also setting the massive (coset) fermions to zero). From (2.2) the purely bosonic part of the Lagrangian is simply

$$\mathcal{L}^{(0)} = \frac{1}{2}e_+{}^{\hat{a}}e_-{}^{\hat{a}} + \frac{1}{2}e_+{}^{a'}e_-{}^{a'} + \frac{1}{2}\partial_+ y\partial_- y\,, \tag{4.11}$$

where $e^{\hat{a}}$ and $e^{a'}$ are the vielbeins of the first and second S^3 respectively and y is the S^1 coordinate.

Using (2.3), (2.5) and, that to leading order, $e_+{}^+ \sim e_-{}^- \sim \omega_+{}^{2-} \sim -\omega_-{}^{2+} \sim \kappa$ the terms quadratic in fermions become

$$\begin{aligned} \mathcal{L}^{(2)} &= i\bar{\upsilon}\Gamma_+ P(\partial_- - \frac{1}{4}\omega_-^{A'B'}\Gamma_{A'B'})\upsilon + i\bar{\upsilon}\Gamma_- P(\partial_+ - \frac{1}{4}\omega_+^{A'B'}\Gamma_{A'B'})\upsilon \\ &\quad + \frac{i}{2}e_+{}^{A'}\,\bar{\upsilon}\Gamma_{A'}\Gamma_{2+}P\upsilon - \frac{i}{2}e_-{}^{A'}\,\bar{\upsilon}\Gamma_{A'}\Gamma_{2-}P\upsilon + \ldots \end{aligned} \tag{4.12}$$

where the ellipsis denotes terms of mass dimension higher than two which are to be dropped in the low-energy effective action. Here $A' = (\hat{a}, a')$ runs over the indices of the two S^3's (note that the S^1-coordinate y decouples due to the projection condition $\upsilon = (1-\mathcal{P})\upsilon$ with $\mathcal{P}$ given in (2.6)). Also note that the coupling of the vielbeins to the fermions breaks the 2d Lorentz invariance. This is in contrast to the $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ case where the low-energy effective action is Lorentz invariant.

The kappa-symmetry projector P is defined in (4.3). A natural choice of kappa-symmetry gauge is[7]

$$\upsilon = P\upsilon = \frac{1}{2}(1+\Gamma^{01}\Gamma_{11})\upsilon\,. \tag{4.13}$$

This gauge has the additional benefit that $\mathcal{L}_z$, the the terms in the Lagrangian resulting from integrating out z, in (4.10) vanishes. We are left with the low-energy effective action

$$\begin{aligned}\mathcal{L} \;&=\; \mathcal{L}^{(0)} + i\bar{\upsilon}\Gamma_+\partial_-\upsilon + i\bar{\upsilon}\Gamma_-\partial_+\upsilon + \frac{i}{2}e_+{}^{A'}\,\bar{\upsilon}\Gamma_{A'2+}\upsilon - \frac{i}{2}e_-{}^{A'}\,\bar{\upsilon}\Gamma_{A'2-}\upsilon \\ &\quad - \frac{i}{4}\omega_-{}^{A'B'}\,\bar{\upsilon}\Gamma_{A'B'+}\upsilon - \frac{i}{4}\omega_+{}^{A'B'}\,\bar{\upsilon}\Gamma_{A'B'-}\upsilon + \mathcal{L}^{(4)}\end{aligned} \tag{4.14}$$

where $\mathcal{L}^{(4)}$ denotes the υ^4-terms arising from (2.4). We will now show that in fact these terms give no contribution to the low-energy effective action in the present case.

The third term in (2.4) is easily seen to give no contribution at dimension two due to the projection condition $\upsilon = (1-\mathcal{P})\upsilon$, the form of S in (2.5) and the fact that $(1-\mathcal{P})\Gamma_a = \Gamma_a\mathcal{P}$ $(a=0,1,2)$. Using (2.8), the fact that the only contribution from $\mathcal{D}$ comes from the AdS spin connection and the kappa-gauge choice $\upsilon = P\upsilon$ one can show that the second term in (2.4) also gives no contribution. The remaining terms become, after some simplification,

$$\begin{aligned}\mathcal{L}^{(4)} \;&=\; \frac{1}{4}\bar{\upsilon}\Gamma^{A'}{}_{2-}\upsilon\,\bar{\upsilon}\Gamma_{A'2+}\upsilon + \frac{1}{48}\bar{\upsilon}\Gamma_+{}^{CD}\upsilon\,\big(3\bar{\upsilon}\Gamma_-(1-\Gamma_{11})U_{CD}\upsilon - 2\bar{\upsilon}\Gamma_C(1+\Gamma_{11})U_{D-}\upsilon\big) \\ &\quad + \frac{1}{48}\bar{\upsilon}\Gamma_-{}^{CD}\upsilon\,\big(3\bar{\upsilon}\Gamma_+(1+\Gamma_{11})U_{CD}\upsilon - 2\bar{\upsilon}\Gamma_C(1-\Gamma_{11})U_{D+}\upsilon\big)\,.\end{aligned} \tag{4.15}$$

Using the form of U in (2.8) the contribution from the $U_{D\pm}$-terms is easily seen to vanish due to the kappa gauge condition (4.13). For the remaining U-terms only the term involving the Riemann tensor contributes due to the fact that $\upsilon = (1-\mathcal{P})\upsilon$ and we are left with

$$\mathcal{L}^{(4)} \;=\; \frac{1}{4}\bar{\upsilon}\Gamma^{A'}{}_{2-}\upsilon\,\bar{\upsilon}\Gamma_{A'2+}\upsilon - \frac{1}{16}R_{AB}{}^{CD}\,\bar{\upsilon}\Gamma_+{}^{AB}\upsilon\,\bar{\upsilon}\Gamma_{CD-}\upsilon\,. \tag{4.16}$$

Using the relations

$$\mathcal{P}\Gamma^{a'}(1-\mathcal{P}) = \frac{1}{2}\sqrt{1-\alpha}\,\varepsilon^{a'b'c'}\,\Gamma^{012}\Gamma_{b'c'}(1-\mathcal{P})\,,\qquad \mathcal{P}\Gamma^{\hat{a}}(1-\mathcal{P}) = \frac{1}{2}\sqrt{\alpha}\,\varepsilon^{\hat{a}\hat{b}\hat{c}}\,\Gamma^{012}\Gamma_{\hat{b}\hat{c}}(1-\mathcal{P}) \tag{4.17}$$

and the form of the Riemann tensor in (2.9) we find that the two terms cancel so that there is no contribution from the quartic fermion terms to the low-energy effective action. This is in contrast to the $AdS_4\times\mathbb{CP}^3$ case where such terms were found to contribute [22].

4.3 2d fermion notation

It will be useful to write the action using a 2d notation for the fermions. Before gauge-fixing we have sixteen real massless non-coset fermions υ. Fixing the kappa-symmetry gauge (4.13) reduces these to eight real fermions. These can be combined into four two-component Majorana fermions Ψ_I $(I=1,\ldots,4)$ as described in detail in appendix A. The Lagrangian (4.14) then

[7] The more standard gauge $\Gamma^+\Theta = 0$ is clearly not a good choice here since the fermion kinetic operator degenerates in this case.

becomes

$$\begin{aligned}
\mathcal{L} = \mathcal{L}^{(0)} &+ \frac{i}{2}\bar{\Psi}_I\rho^i\partial_i\Psi_I - \frac{i}{2}(\omega_i{}^{34} + \sqrt{\alpha}\,\delta_{ij}e^{j5})\,(\bar{\Psi}_1\rho^i\Psi_2 + \bar{\Psi}_3\rho^i\Psi_4) \\
&- \frac{i}{2}(\omega_i{}^{53} + \sqrt{\alpha}\,\delta_{ij}e^{j4})\,(\bar{\Psi}_2\rho^i\Psi_4 - \bar{\Psi}_1\rho^i\Psi_3) - \frac{i}{2}(\omega_i{}^{45} + \sqrt{\alpha}\,\delta_{ij}e^{j3})\,(\bar{\Psi}_1\rho^i\Psi_4 + \bar{\Psi}_2\rho^i\Psi_3) \\
&+ \frac{i}{2}(\omega_i{}^{78} + \sqrt{1-\alpha}\,\delta_{ij}e^{j6})\,(\bar{\Psi}_1\rho^i\Psi_4 - \bar{\Psi}_2\rho^i\Psi_3) + \frac{i}{2}(\omega_i{}^{86} + \sqrt{1-\alpha}\,\delta_{ij}e^{j7})\,(\bar{\Psi}_2\rho^i\Psi_4 + \bar{\Psi}_1\rho^i\Psi_3) \\
&+ \frac{i}{2}(\omega_i{}^{67} + \sqrt{1-\alpha}\,\delta_{ij}e^{j8})\,(\bar{\Psi}_3\rho^i\Psi_4 - \bar{\Psi}_1\rho^i\Psi_2)\,. \qquad (4.18)
\end{aligned}$$

Worldsheet indices $(i, j, \ldots)$ are raised and lowered with the worldsheet metric η_{ij} with $\eta_{+-} = 2$. Note that the explicit appearance of δ_{ij} ($\delta_{++} = \delta_{--} = 2$) breaks the 2d Lorentz-invariance of the action. This somewhat complicated Lagrangian can be written in a much simpler and more illuminating form by combining the four spinors into a 2×2 matrix

$$\Psi = \sigma^I\Psi_I\,, \quad \sigma^I = \big(\sigma^1,\, \sigma^2,\, \sigma^3,\, i\mathbb{1}\big)\,, \qquad \big(\bar{\Psi} = \Psi^\dagger\rho^0 = \bar{\sigma}^I\bar{\Psi}_I\,, \quad \bar{\sigma}^I = \big(\sigma^1,\, \sigma^2,\, \sigma^3,\, -i\mathbb{1}\big)\big) \quad (4.19)$$

The Lagrangian then becomes

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{2}e_i{}^{\hat{a}}e^{i\hat{a}} + \frac{1}{2}e_i{}^{a'}e^{ia'} + \frac{1}{2}\partial_i y\partial^i y + \frac{i}{4}\mathrm{tr}\big(\bar{\Psi}\rho^i\partial_i\Psi\big) - \frac{1}{16}(\varepsilon^{\hat{a}\hat{b}\hat{c}}\omega_i{}^{\hat{b}\hat{c}} + 2\sqrt{\alpha}\,\delta_{ij}e^{j\hat{a}})\,\mathrm{tr}\big(\bar{\Psi}\rho^i\Psi\sigma^{\hat{a}}\big) \\
&+ \frac{1}{16}(\varepsilon^{a'b'c'}\omega_i{}^{b'c'} + 2\sqrt{1-\alpha}\,\delta_{ij}e^{ja'})\mathrm{tr}\big(\bar{\Psi}\rho^i\sigma^{a'}\Psi\big)\,, \qquad (4.20)
\end{aligned}$$

where $\sigma^{\hat{a}} = (\sigma^1, \sigma^2, \sigma^3)$ and similarly for $\sigma^{a'}$. This Lagrangian describes two $O(4)$ sigma models which are coupled through their interactions with the fermions. In addition there is a completely decoupled scalar y coming from the S^1. The action is invariant under the two $SO(3) \sim SU(2)$ which correspond to rotations in the first and second S^3 factor of $AdS_3 \times S^3 \times S^3 \times S^1$. The fermions transform as

$$\Psi \to U^\dagger\Psi V \qquad (\bar{\Psi} \to V^\dagger\bar{\Psi}U) \qquad \text{with} \qquad U \in SU(2)_1\,, \quad V \in SU(2)_2\,. \qquad (4.21)$$

In fact (4.20) is invariant under the full isometry group of the two S^3 i.e. $SO(4) \times SO(4)$. This can for example be seen by verifying explicitly the invariance under the (appropriately restricted) superisometry transformations given in [28]. It also follows from the Lax connection construction in the next section. Unlike the $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ case we have not found a form of the Lagrangian that makes the full $SO(4) \times SO(4)$ symmetry manifest.

As a side note, the fermion terms in the action can be written in a more compact form as

$$\frac{i}{4}\mathrm{tr}\big(\bar{\Psi}\rho^i D_i\Psi\big)\,, \qquad (4.22)$$

where the generalized "covariant" derivative is defined as

$$D_i\Psi = \partial_i\Psi + \frac{i}{4}(\varepsilon^{\hat{a}\hat{b}\hat{c}}\omega_i{}^{\hat{b}\hat{c}} + 2\sqrt{\alpha}\,\delta_{ij}e^{j\hat{a}})\Psi\sigma^{\hat{a}} - \frac{i}{4}(\varepsilon^{a'b'c'}\omega_i{}^{b'c'} + 2\sqrt{1-\alpha}\,\delta_{ij}e^{ja'})\sigma^{a'}\Psi \qquad (4.23)$$

or, equivalently,

$$D_i\Psi = \partial_i\Psi - \frac{1}{4}\omega_i'^{IJ}\sigma_{IJ}\Psi + \frac{1}{4}\hat{\omega}_i{}^{IJ}\Psi\sigma_{IJ} \qquad \sigma^{IJ} = \bar{\sigma}^{[I}\sigma^{J]} \qquad (4.24)$$

with

$$\omega_i'^{a'b'} = \omega_i{}^{a'b'}\,, \quad \omega_i'^{a'4} = \sqrt{1-\alpha}\,\delta_{ij}e^{ja'}\,, \quad \hat{\omega}_i{}^{\hat{a}\hat{b}} = \omega_i{}^{\hat{a}\hat{b}}\,, \quad \hat{\omega}_i{}^{\hat{a}4} = \sqrt{\alpha}\,\delta_{ij}e^{j\hat{a}}\,. \qquad (4.25)$$

Note however that $D_i\Psi$ is *not* 2d Lorentz-covariant.

So far what we have said refers to the $AdS_3 \times S^3 \times S^3 \times S^1$ case. The $AdS_3 \times S^3 \times T^4$ case is however easily obtained by taking the limit $\alpha \to 1$ (or, equivalently, $\alpha \to 0$) so that $e_i{}^{a'} \to \partial_i y^{a'}$ and $\omega_i{}^{a'b'} \to 0$. This leads to an $O(4)$ sigma model coupled to four Majorana fermions together with four decoupled scalars from the T^4.

5 Classical integrability

Since the low-energy effective action has no quartic fermion terms its Lax connection can be obtained from [28] by a suitable truncation. In that paper a Lax connection was written for the complete $AdS_3 \times S^3 \times S^3 \times S^1$ superstring with 32 fermions, up to quadratic order in fermions (see [29–31] for similar Lax connections for $AdS_4 \times \mathbb{CP}^3$ and $AdS_2 \times S^2 \times T^6$). Normally this Lax connection would have zero curvature, i.e.

$$dL - L \wedge L = 0 \qquad \text{or} \qquad \varepsilon^{ij}(\partial_i L_j + L_i L_j) = 0\,, \tag{5.1}$$

only modulo Θ^4-terms. Additional terms would then be needed in the Lax connection at order Θ^4 to cancel these terms. However, in the present case the low-energy effective action actually terminates at the quadratic order in fermions. Since we expect this model to be integrable we should find that the υ^4-terms coming from $L \wedge L$ actually cancel in this case. We will now show that this is precisely what happens.

The Lax connection splits into two pieces[8]

$$L = \hat{L} + L'\,, \tag{5.2}$$

coming from the two S^3's respectively. In terms of the components of the Maurer-Cartan form K on S^3 satisfying (see [28] for more details)

$$dK = K \wedge K \quad \Rightarrow [K_{\hat{a}}, K_{\hat{b}}] = \nabla_{\hat{a}} K_{\hat{b}}\,, \quad [K_{\hat{c}}, \nabla_{\hat{a}} K_{\hat{b}}] = -2\alpha\delta_{\hat{c}[\hat{a}} K_{\hat{b}]}\,, \tag{5.3}$$

and similarly for $K_{a'}$ with $\alpha \to 1-\alpha$, these Lax connection pieces are given by

$$\begin{aligned}\hat{L}_i \quad &= \quad (\alpha_1 \delta_i^j + \alpha_2 \eta_{ik}\varepsilon^{kj}) e_j{}^{\hat{a}}\, K_{\hat{a}} - \frac{\alpha_2}{8}\sqrt{\alpha}\, \varepsilon_{ij}\delta^{jk} \mathrm{tr}\big(\bar{\Psi}\rho_k \Psi \sigma^{\hat{a}}\big)\, K_{\hat{a}} \\ &\quad - \frac{\alpha_2}{16}(\alpha_2 \eta_{ij} + (1+\alpha_1)\varepsilon_{ij})\varepsilon^{\hat{a}\hat{b}\hat{c}} \mathrm{tr}\big(\bar{\Psi}\rho^j \Psi \sigma_{\hat{c}}\big)\, \nabla_{\hat{a}} K_{\hat{b}}\end{aligned} \tag{5.4}$$

$$\begin{aligned}L'_i \quad &= \quad (\alpha_1 \delta_i^j + \alpha_2 \eta_{ik}\varepsilon^{kj}) e_j{}^{a'}\, K_{a'} + \frac{\alpha_2}{8}\sqrt{1-\alpha}\, \varepsilon_{ij}\delta^{jk} \mathrm{tr}\big(\bar{\Psi}\rho_k \sigma^{a'} \Psi\big)\, K_{a'} \\ &\quad + \frac{\alpha_2}{16}(\alpha_2 \eta_{ij} + (1+\alpha_1)\varepsilon_{ij})\varepsilon^{a'b'c'} \mathrm{tr}\big(\bar{\Psi}\rho^j \sigma_{c'} \Psi\big)\, \nabla_{a'} K_{b'}\,.\end{aligned} \tag{5.5}$$

These two pieces obviously commute with each other since they are constructed using generators from different algebras. It is also worth noting that due to the explicit appearance of δ_{ij} L_i is not covariant under 2d Lorentz-transformations. This is to be expected since the low-energy effective action lacks this symmetry as we have seen. The parameters α_1 and α_2 are related by the equation

$$\alpha_2^2 = 2\alpha_1 + \alpha_1^2 \tag{5.6}$$

and can therefore be expressed in terms of a single (spectral) parameter. Let us now show that the curvature (5.1) of $\hat{L}_i$ indeed vanishes on-shell. The same is true for L'_i by an essentially

[8] We drop the additional S^1 boson y since it decouples completely. It can of course be trivially included in the Lax connection.

identical calculation. Computing the second term in (5.1) we find using (5.6)

$$\begin{aligned}\varepsilon^{ij}\hat{L}_i\hat{L}_j &= -\alpha_1\varepsilon^{ij}e_i{}^{\hat{a}}e_j{}^{\hat{b}}\,[K_{\hat{a}},K_{\hat{b}}]+\frac{\alpha_2}{16}e_i{}^{\hat{d}}\Big(2\sqrt{\alpha}\,(\alpha_1\delta^i_k-\alpha_2\varepsilon^{ij}\eta_{jk})\delta^{kl}\mathrm{tr}\big(\bar{\Psi}\rho_l\Psi\sigma^{\hat{a}}\big)\,[K_{\hat{a}},K_{\hat{d}}]\\ &\quad+(\alpha_1\delta^i_k+\alpha_2\varepsilon^{ij}\eta_{jk})\varepsilon^{\hat{a}\hat{b}\hat{c}}\mathrm{tr}\big(\bar{\Psi}\rho^k\Psi\sigma_{\hat{c}}\big)\,[K_{\hat{d}},\nabla_{\hat{a}}K_{\hat{b}}]\Big)\\ &\quad+\frac{\alpha_2^2}{256}\Big(\varepsilon^{ij}\mathrm{tr}\big(\bar{\Psi}\rho_i\Psi\sigma^{\hat{a}}\big)\mathrm{tr}\big(\bar{\Psi}\rho_j\Psi\sigma^{\hat{b}}\big)(2\alpha[K_{\hat{a}},K_{\hat{b}}]+\frac{1}{2}\varepsilon^{\hat{a}\hat{c}\hat{d}}\varepsilon^{\hat{b}\hat{e}\hat{f}}[\nabla_{\hat{e}}K_{\hat{f}},\nabla_{\hat{c}}K_{\hat{d}}])\\ &\quad-2\sqrt{\alpha}\,\delta^{ik}(\alpha_2\eta_{kj}+(1+\alpha_1)\varepsilon_{kj})\mathrm{tr}\big(\bar{\Psi}\rho_i\Psi\sigma^{\hat{c}}\big)\mathrm{tr}\big(\bar{\Psi}\rho^j\Psi\sigma_{\hat{d}}\big)\varepsilon^{\hat{a}\hat{b}\hat{d}}[K_{\hat{c}},\nabla_{\hat{a}}K_{\hat{b}}]\Big)\,. \qquad (5.7)\end{aligned}$$

Using the relations (5.3) it is easy to see that the terms quartic in Ψ indeed vanish as advertised earlier. Using these relations and the fact that $\partial_i K_{\hat{b}}=e_i{}^{\hat{a}}\nabla_{\hat{a}}K_{\hat{b}}+\omega_{i\hat{b}}{}^{\hat{c}}K_{\hat{c}}$ we get for the curvature of $\hat{L}_i$

$$\begin{aligned}\varepsilon^{ij}(\partial_i\hat{L}_j+\hat{L}_i\hat{L}_j) &= \alpha_2\big(\nabla^i e_i{}^{\hat{a}}-\frac{\sqrt{\alpha}}{4}\mathrm{tr}\big(\bar{\Psi}\rho^i\delta_{ij}D^j\Psi\sigma^{\hat{a}}\big)\big)\,K_{\hat{a}}\\ &\quad+i\frac{\alpha_2}{8}\mathrm{tr}\big(\bar{\Psi}((1+\alpha_1)+\alpha_2\rho^3)\rho^i D_i\Psi\sigma^{\hat{a}\hat{b}}\big)\,\nabla_{\hat{a}}K_{\hat{b}}\,, \qquad (5.8)\end{aligned}$$

where $\rho^3=\rho^0\rho^1$. The first term is the equation of motion for $y^{\hat{m}}$ following from the action (4.20) (modulo a term proportional to the fermionic equation of motion) and the second term is proportional to the fermionic equation of motion. Note however that the $e^{a'}$ and $\omega^{a'b'}$-terms inside D_i in (4.23) don't contribute here due to the fact that

$$\mathrm{tr}\big(\bar{\Psi}\rho^i\sigma^{a'}\Psi\sigma^{\hat{a}}\big)=0\,. \qquad (5.9)$$

It is clear that the curvature of $\hat{L}_i$ (5.8) vanishes on-shell. It is also clear that the flatness of $\hat{L}_i$ by itself does not imply all the equations of motion, only the equations for $y^{\hat{m}}$ and part of the fermionic equation due to the missing contributions involving $e^{a'}$ and $\omega^{a'b'}$ as mentioned above. However, together the flatness of $\hat{L}_i$ and L'_i imply all the equations of motion of (4.20) demonstrating the classical integrability of the model.

6 Conclusions

In this paper we have derived the Lagrangian that captures the low-energy dynamics of fluctuations of the $AdS_3\times S^3\times M_4$ string around the GKP vacuum [18]. The starting point of our analysis was the GS action up to quartic order in fermions, recently derived in [27]. The classical GKP solution is a spinning string in AdS_3 with a fluctuation spectrum of both massive and massless modes. While the full fluctuation Lagrangian is very involved, it simplifies drastically in the low-energy limit where only the massless modes contribute. We have found that the resulting theory consists of two $O(4)$ sigma models coupled through their coupling to four Majorana fermions. In addition there is a free boson coming from the S^1. In contrast to earlier examples in $AdS_5\times S^5$ and $AdS_4\times\mathbb{CP}^3$ we find that the model is not 2d Lorentz invariant. Furthermore, a curious fact is that the quartic fermion terms completely drop out due to a delicate cancellation. This is somewhat unexpected, since, at least for $\alpha=\frac{1}{2}$, the $AdS_4\times\mathbb{CP}^3$ and $AdS_3\times S^3\times S^3\times S^1$ strings share many similar features, see [28] for some examples. The special case where one S^3 blows up (i.e. $\alpha\to 0,1$) describes the $AdS_3\times S^3\times T^4$ GKP string and the low-energy effective action reduces to a single $O(4)$ sigma model coupled to four Majorana fermions with four free bosons coming from the T^4.

There are several interesting possible extensions of this work. It would be very interesting to perform a similar analysis as [23–25] and try to find the Bethe ansatz and exact S-matrix for this model. Since the full $AdS_3 \times S^3 \times S^3 \times S^1$ string is believed to be quantum integrable it is natural to expect that the low-energy string should inherit this integrability. For example, as in [24], one can derive the S-matrix for the low-energy excitations and match the resulting asymptotic equations with the low-energy part of the full set of conjectured equations in [16,10].

It would also be interesting to derive the low-energy limit of the GKP string for the case of $AdS_3 \times S^3 \times S^3 \times S^1$ with mixed RR and NSNS flux. Superficially this case is considerably more complicated but recent S-matrix calculations [32, 33] have shown that this case is very similar to the pure RR flux case. It would be interesting to understand what happens in the limit of pure NSNS flux since one can then connect to the RNS description of the string.

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A 2d spinors and gamma-matrices

Starting from the fermions υ with eight real components (after fixing the kappa symmetry (4.13)) we define one-component fermions by the following projections

$$\psi_{\pm\pm\pm} = \frac{1}{2}(1 \pm \Gamma^{01})\frac{1}{2}(1 \pm i\Gamma^{34})\frac{1}{2}(1 \pm i\Gamma^{67})\upsilon\,. \tag{A.1}$$

In the space of υ the gamma-matrices are effectively eight-dimensional and we take a realization such that

$$C \sim -i\sigma^2 \otimes \sigma^2 \otimes \sigma^2\,, \quad \Gamma_\pm \sim i\sigma^2(\mathbb{1} \pm \sigma^3) \otimes \mathbb{1} \otimes \mathbb{1}\,, \quad \Gamma^{45} \sim \mathbb{1} \otimes i\sigma^1 \otimes \mathbb{1}\,, \quad \Gamma^{78} \sim \mathbb{1} \otimes \mathbb{1} \otimes i\sigma^1 \tag{A.2}$$

where C is the charge-conjugation matrix. As defined the spinors ψ are not real but satisfy

$$\psi^\dagger_{\pm+-} = \psi_{\pm-+}\,, \qquad \psi^\dagger_{\pm++} = -\psi_{\pm--}\,, \tag{A.3}$$

as follows from the Majorana condition on υ. We can define real spinors as

$$\begin{aligned}
&\psi_1 = \sqrt{2}(\psi_{---} - \psi_{-++})\,, \quad &&\psi_2 = i\sqrt{2}(\psi_{---} + \psi_{-++})\,,\\
&\psi_3 = -\sqrt{2}(\psi_{-+-} + \psi_{--+})\,, \quad &&\psi_4 = i\sqrt{2}(\psi_{-+-} - \psi_{--+})\,,\\
&\chi_1 = \sqrt{2}(\psi_{+--} - \psi_{+++})\,, \quad &&\chi_2 = i\sqrt{2}(\psi_{+--} + \psi_{+++})\,,\\
&\chi_3 = -\sqrt{2}(\psi_{++-} + \psi_{+-+})\,, \quad &&\chi_4 = i\sqrt{2}(\psi_{++-} - \psi_{+-+})\,.
\end{aligned} \tag{A.4}$$

These can be combined into four 2d Majorana spinors

$$\Psi_I = \begin{pmatrix} \psi_I \\ \chi_I \end{pmatrix} \qquad (I = 1, \ldots, 4)\,. \tag{A.5}$$

We take the 2d gamma-matrices to be

$$\rho^0 = \sigma^2\,, \quad \rho^1 = i\sigma^1\,, \qquad \rho^\pm = \frac{1}{2}(\rho^0 \pm \rho^1) \tag{A.6}$$

and the conjugate spinor is defined as $\bar{\Psi} = \Psi^\dagger \rho^0$.

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