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An SO(10) Grand Unified Theory of Flavor

Bhaskar Dutta[1], **Yukihiro Mimura**[2] and **R.N. Mohapatra**[2]

[1] *Department of Physics, Texas A&M University, College Station, TX 77843-4242, USA*

[2] *Maryland Center for Fundamental Physics and Department of Physics, University of Maryland, College Park, MD, 20742, USA*

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Abstract

We present a supersymmetric SO(10) grand unified theory (GUT) of flavor based on an S_4 family symmetry. It makes use of our recent proposal to use SO(10) with type II seesaw mechanism for neutrino masses combined with a simple ansatz that the dominant Yukawa matrix (the **10**-Higgs coupling to matter) has rank one. In this paper, we show how the rank one model can arise within some plausible assumptions as an effective field theory from vectorlike **16** dimensional matter fields with masses above the GUT scale. In order to obtain the desired fermion flavor texture we use S_4 flavon multiplets which acquire vevs in the ground state of the theory. By supplementing the S_4 theory with an additional discrete symmetry, we find that the flavon vacuum field alignments take a discrete set of values provided some of the higher dimensional couplings are small. Choosing a particular set of these vacuum alignments appears to lead to an unified understanding of observed quark-lepton flavor: (i) the lepton mixing matrix that is dominantly tri-bi-maximal with small corrections related to quark mixings; (ii) quark lepton mass relations at GUT scale: $m_b \simeq m_\tau$ and $m_\mu \simeq 3m_s$ and (iii) the solar to atmospheric neutrino mass ratio $m_\odot/m_{\rm atm} \simeq \theta_{\rm Cabibbo}$ in agreement with observations. The model predicts the neutrino mixing parameter, $U_{e3} \simeq \theta_{\rm Cabibbo}/(3\sqrt{2}) \sim 0.05$, which should be observable in planned long baseline experiments.

I. INTRODUCTION

A unified understanding of the diverse pattern of quark lepton masses and mixings is a fundamental challenge for physics beyond the standard model [1]. The two major elements of this flavor puzzle that any theory must explain are: (i) strong mass hierarchy in the quark and charged lepton sector and weak hierarchy for neutrinos; (ii) large lepton mixings i.e. $\theta^l_{23} \sim 45^{\mathrm{o}}$ and $\theta^l_{12} \simeq 35^{\mathrm{o}}$ as against small quark mixings $\theta^q_{23} \sim 2.5^{\mathrm{o}}$ and $\theta^q_{12} \sim 13^{\mathrm{o}}$ and apparent relation between some of the mixing angles and the fermion masses. Since grand unified theories (GUT) not only unify different gauge couplings at a high scale but also unify quarks and leptons within a single framework, they have often been thought of as an attractive venue for unraveling this puzzle. Furthermore the fact that the seesaw mechanism for understanding small neutrino masses [2] also seems to require a $B-L$ breaking scale close to the scale of coupling unification, makes this suggestion quite promising. The constraints of higher symmetry however make it highly nontrivial to understand all the details of flavor puzzle although many attempts have been made [3].

In a recent paper, we have suggested a possible way [4] to address this problem in supersymmetric SO(10) GUT models. The main assumptions of ref. [4] are: (a) all fermion masses arise from effective Yukawa couplings [5] involving **10** and **126** Higgs multiplets; (b) neutrino masses arise [6] from type II seesaw mechanism [7] and (c) the **10**-Higgs Yukawa dominates fermion masses and has rank one. We showed in ref. [4] how this program when implemented using the already mentioned Higgs content of a single **10**, **126** plus possibly another **10** or **120** Higgs fields not only explains all the qualitative features of quark and lepton flavor noted above but also makes a prediction for the lepton mixing angle U_{e3} or θ_{13}. In most models we discussed in [4], the apparent tri-bi-maximal mixing pattern [8] observed for neutrinos did not arise from any symmetry. In this note, we pursue program outlined in [4] further by using discrete family symmetries to make this ansatz more predictive. Our strategy is to use flavon fields whose vevs give the effective Yukawa couplings responsible for fermion masses at the GUT scale. We use additional discrete family symmetries whose role is to constrain the ground state of the flavon Hamiltonian such that they lead to particular textures for the fermion mass matrices within certain assumptions. We are able to isolate a set of allowed flavon vacuum states which are such that the dominant part of the lepton mixing matrix naturally has a tri-bi-maximal form, provided some of the higher dimensional terms in the flavon superpotential are small. The desired flavon vacuum alignment seems to arise naturally with an S_4 symmetry [9] which unifies all three families of fermions into a $\mathbf{3_2}$ multiplet.

The new results of this paper are : (i) we show how the rank one model can arise naturally as an effective field theory from vectorlike **16** dimensional matter fields with masses above the GUT scale; and (ii) how the detailed fermion flavor textures arise from the vacuum field alignments of gauge singlet S_4 flavon fields leading to the following results naturally without adjustment of parameters: (a) the lepton mixing matrix has dominantly tri-bi-maximal form with small corrections related to quark mixings; (b) quark lepton mass relations at GUT scale: $m_b \sim m_\tau$ and $m_\mu \simeq 3m_s$ and (c) the solar to atmospheric mass ratio $m_\odot/m_{\rm atm} \simeq \theta_{\rm Cabibbo}$ in agreement with observations.

II. OVERVIEW OF THE SUSY SO(10) RANK ONE STRATEGY

We use the Higgs fields that give fermion masses to consist of two **10** dimensional multiplets (denoted by H, H') and a single $\mathbf{126} + \overline{\mathbf{126}}$ (denoted by Δ and $\overline{\Delta}$). The Yukawa superpotential for this case in a generic SO(10) model can be written as:

$$W_Y \; = \; h\,\psi\psi H + f\,\psi\psi\bar{\Delta} + h'\,\psi\psi\,H'\,, \tag{1}$$

where the symbol ψ stands for the **16** dimensional representation of SO(10) that represents the matter fields. The coupling matrices h, h' and f are symmetric. As we show later in this paper, their detailed texture will be determined by the S_4 symmetry. The representations H, H' and Δ each have two standard model (SM) doublets in them. The general way to understand how the two MSSM doublets arise from them is as follows: at the GUT scale M_U, after the GUT and the $B-L$ symmetries are broken, one linear combination of the up-type doublets and one of down-type ones remain almost massless whereas the remaining ones acquire GUT scale masses just like the color triplet and other non-MSSM multiplets. The electroweak symmetry is broken after the light MSSM doublets (to be called $H_{u,d}$) acquire vacuum expectation values (vevs) and they then generate the fermion masses. The resulting formulae for different fermion masses are given by:

$$\begin{aligned} Y_u \; &= \; h + r_2 f + r_3 h', \qquad (2) \\ Y_d \; &= \; r_1(h + f + h')\,, \\ Y_e \; &= \; r_1(h - 3f + c_e h')\,, \\ Y_{\nu^D} \; &= \; h - 3r_2 f + c_\nu h', \end{aligned}$$

where Y_a are mass matrices divided by the electro-weak vev v_{wk} and r_i and $c_{e,\nu}$ are the mixing parameters which relate the $H_{u,d}$ to the doublets in the various GUT multiplets. More precisely, the matrices h, f and h' in Y_a are multiplied by the Higgs mixing parameters when they appear in the fermion mass matrices. The definitions of the couplings and the Higgs mixing parameters are given in ref. [10]. In our particular case with a second **10**-Higgs (H'), $c_e = 1$ and $c_\nu = r_3$. Furthermore, we use the type II seesaw formula for getting neutrino masses which is possible to obtain with symmetry breaking pattern in SO(10) as given in [11].

$$\mathcal{M}_\nu \;=\; f v_L. \tag{3}$$

Note that f is the same coupling matrix that appears in the charged fermion masses in Eq. (2), up to factors from the Higgs mixings and the Clebsch-Gordan coefficients. This helps us to connect the neutrino parameters to the quark-sector parameters. The equations (2) and (3) are the key equations in our unified approach to addressing the flavor problem.

The main hypothesis of our approach in ref. [4] is that

- the fermion mass formula of Eq. (2) are dominated by the matrix h with the contributions of f and h' being small perturbations;

- the matrix h has rank one.

It follows from these assumptions that in the limit of $f, h' \to 0$, the quark and lepton mixings vanish as do the neutrino masses. Once f, h' are turned on, one can choose f to be diagonal by an appropriate choice of basis and without any loss of generality. Since the neutrino masses are diagonal in this basis, the entire Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix comes from the matrix that diagonalizes the charged lepton mass matrix and for arbitrary form of the later, the PMNS matrix will in general have large mixing angles. On the other hand, the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_{\rm CKM} = U_u^\dagger U_d$ which in the limit of $f, h' \to 0$ is equal to a unit matrix, owes the origin of quark mixings to f, h'. The quark mixings are proportional to $|f|/|h|$ and hence small as observed. It is also clear that the charged lepton and quark masses of second and first generation are also proportional to $|f|/|h|$ and thus hierarchical.

Our procedure in this paper is as follows: we supplement the above rank one hypothesis by a discrete family symmetry S_4 so that forms of the h, f, h' are consequences of the vacuum expectation values of gauge singlet but S_4 non-singlet flavon fields, ϕ_i, thereby making the model more predictive. To implement this procedure, we first derive the GUT scale effective Lagrangian from a pre-GUT scale theory that has vectorlike **16**-dim. matter spinor with masses slightly above the GUT scale. The resulting effective theory involves non-renormalizable higher dimensional operators involving ψ, Higgs fields and the flavon fields ϕ_i whose vevs generate the flavor texture observed at GUT scale. These are then extrapolated to the weak scale to compare with observations.

	ψ	H	$\bar{\Delta}$	H'	ϕ_1	$\bar{\phi}_1$	ϕ_2	$\bar{\phi}_2$	ϕ_3	$\bar{\phi}_3$	ψ_{V1}	$\bar{\psi}_{V1}$	ψ_{V2}	$\bar{\psi}_{V2}$	s_1	s_2
SO(10)	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{126}}$	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{16}$	$\overline{\mathbf{16}}$	$\mathbf{16}$	$\overline{\mathbf{16}}$	$\mathbf{1}$	$\mathbf{1}$
S_4	$\mathbf{3_2}$	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{1_1}$	$\mathbf{3_2}$	$\mathbf{3_2}$	$\mathbf{3_1}$	$\mathbf{3_1}$	$\mathbf{3_2}$	$\mathbf{3_2}$	$\mathbf{1_1}$	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{1_2}$	$\mathbf{1_2}$	$\mathbf{1_1}$
Z_n	1	ω^{-4}	ω^{-2-a}	ω^{-1}	ω^2	ω^{-2}	ω	ω^{-1}	ω^{2+a-b}	ω^{-2-a+b}	ω^2	ω^{-2}	ω	ω^{-1}	ω^a	ω^b

TABLE I: The fields and representations to generate the desired Yukawa couplings. $\omega = e^{i\frac{2\pi}{n}}$.

III. S_4 FAMILY SYMMETRY AND MODEL OF FLAVOR

The S_4 group is a 24 element group describing permutations of four distinct objects and has five irreducible representations with dimensions $\mathbf{3_1} \oplus \mathbf{3_2} \oplus \mathbf{2} \oplus \mathbf{1_2} \oplus \mathbf{1_1}$. The distinction between the representations with subscripts 1 and 2 is that the later change sign under the transformation of group elements involving the odd number of permutations of S_4. For other details of S_4 group, see [9].

We assign the three families of $\mathbf{16}$-dim. matter fermions ψ to $\mathbf{3_2}$-dim. representation of S_4 and the Higgs field H, $\bar{\Delta}$ and H' to $\mathbf{1_1}, \mathbf{1_2}$, and $\mathbf{1_1}$ reps, respectively. We then choose three SO(10) singlet flavons ϕ_i transforming as $\mathbf{3_2}, \mathbf{3_1}, \mathbf{3_2}$ reps of S_4 and one gauge and S_4 singlet fields s_1, s_2 transforming as $\mathbf{1_2}$ and $\mathbf{1_1}$ respectively. We further assume that at a scale slightly above the GUT scale, there are two S_4 singlet vectorlike pairs of $\mathbf{16} \oplus \overline{\mathbf{16}}$ fields denoted by ψ_V and $\bar{\psi}_V$. In order to get the desired Yukawa couplings naturally from this high scale theory, we supplement the S_4 group by an Z_n group with all the above fields belonging to representations given in the Table I.

The most general high scale Yukawa superpotential involving matter fields invariant under this symmetry is given by:

$$
\begin{aligned}
W = (\phi_1\psi)\bar{\psi}_{V1} \;+\; \psi_{V1}\psi_{V1}H \;+\; M_1\bar{\psi}_{V1}\psi_{V1} \\
+(\phi_2\psi)\bar{\psi}_{V2} \;+\; \frac{1}{M_P}s_1\psi_{V2}\psi_{V2}\bar{\Delta} + \; M_2\bar{\psi}_{V2}\psi_{V2} \\
+\frac{1}{M_P^2}s_2(\phi_3\psi\psi)\bar{\Delta} + \frac{1}{M_P}(\phi_2\psi\psi)H',
\end{aligned}
\tag{4}
$$

where the brackets stand for the S_4 singlet contraction of flavor index. The singlet field s_i can have large vev as follows: consider its Z_n charge to be such that the only polynomial term involving the s_i in the superpotential has the form $s_i^{k_i}/M_P^{k_i-3}$ (in order to describe the essential potential, we ignore a possible $s_1^{\ell_1}s_2^{\ell_2}$ term). The dominant part of the potential in the presence of SUSY breaking has the form:

$$
V(s_i) \;=\; -m_{s_i}^2|s_i|^2 + k\frac{s_i^{2k_i-2}}{M_P^{2k_i-6}} + \cdots. \tag{5}
$$

Minimizing this leads to $\langle s_i \rangle \sim [m_{S_i}^2 M_P^{2k_i-6}]^{\frac{1}{2k_i-4}}$, which is above GUT scale for larger values of the integer k_i (which in turn is determined by the Z_n symmetry charge of s_i). One could also have large vevs for s_1, s_2 by using anomalous $U(1)$ charges for them using D-terms to break the $U(1)$ symmetry.

The effective theory below the scales $M_{1,2}$ and $\langle s_i \rangle$ of the vector-like pair masses and the s_i-vevs respectively is given by:

$$W = (\phi_1\psi)(\phi_1\psi)H + (\phi_2\psi)(\phi_2\psi)\bar{\Delta} + (\phi_3\psi\psi)\bar{\Delta} + (\phi_2\psi\psi)H', \tag{6}$$

where we have omitted the dimensional coupling constants to make it simple for the purpose of writing. The discrete symmetries prevent ϕ^2/M^2 corrections to these terms. So our predictions based on this effective superpotential do not receive large corrections. We note that the non-renormalizable terms in Eq.(4) can also be obtained from renormalizable couplings if we introduce further S_4-triplet vectorlike fields. Here, however we use only S_4-singlet vectorlike fields to get rank 1 contribution to h and f Yukawa couplings and that is why we need the non-renormalizable terms to be present in Eq.(4. A few comments are in order regarding the need for the extra Z_n symmetry.

- The Z_n group provides a selection rule of the flavon couplings and the charges of various fields under this are chosen so as to forbid direct renormalizable Yukawa coupling, e.g., $(\psi\psi)H$, which can lead to loss of rank one property and hence the hierarchy of fermion masses.

- The barred flavon fields $\bar{\phi}_i$ are introduced to obtain the potential of the flavons necessary for our vacuum alignment. They do not couple to matter fields.

- We note that the replacement of ϕ_1 with ϕ_3, $\bar{\phi}_3$ is forbidden if $a - b \neq 0, -4$, and similarly unwanted terms can be forbidden when n is a large number.

- The term $\phi_1\psi\psi\bar{\Delta}S_1$ is Z_n invariant, but transforms as $\mathbf{1_2}$ under S_4 because $\psi\psi$ is symmetric due to SO(10) algebra and thus it is not allowed either.

- The S_4 invariant singlet s_2 is introduced to forbid $\phi_2^2\bar{\phi}_3$, $\bar{\phi}_2^2\phi_3$ terms, which are unwanted in the flavon superpotential.

- The Z_n symmetry allows mixed higher dimensional terms of the form $\phi_i\bar{\phi}_i\phi_j\bar{\phi}_j$ terms with $i \neq j$. We assume that the couplings of these terms are small compared to other terms so that the alignment shift caused by these terms compared to that given below is small and does not affect our result.

The details of the flavon superpotential will be discussed later.

In order to get fermion masses, we have to find the alignment [12] of the vevs of the flavon fields $\phi_{1,2,3}$. We show below that the following choice of vevs are among the minima of the flavon superpotential provided the couplings of mixed terms between different ϕ_i's are small compared to other couplings:

$$\phi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{7}$$

Clearly, there are other vacua for the flavon model that we do not choose. What is however nontrivial is that the alignments are along quantized directions. This is a consequence of supersymmetry combined with discrete symmetries in the theory. Given these vev, we find from Eq. (6) that the Yukawa coupling matrices h, f, h' have the form:

$$h \propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

$$f \propto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{9}$$

$$h' \propto \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{10}$$

and the charged fermion mass matrices can then be inferred. The neutrino mass matrix in this basis has the form:

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & c & c \\ c & a & c-a \\ c & c-a & a \end{pmatrix}, \tag{11}$$

where $c/a = \lambda \ll 1$. It is diagonalized by the tri-bi-maximal matrix

$$U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \tag{12}$$

This is however not the full PMNS matrix which will receive small corrections from diagonalization of the charged lepton matrix, which not only make small contributions to the $\theta_{\rm atm}$ and $\theta_\odot$ but also generate a small θ_{13}.

The neutrino masses are given by $m_{\nu 3} = 2a - c$; $m_{\nu 2} = 2c$ and $m_{\nu 1} = -c$. To fit observations, we require $\lambda = c/a \simeq \sqrt{\Delta m^2_\odot/\Delta m^2_{\rm atm}} \sim 0.2$, which fixes the neutrino masses $m_{\nu 3} \simeq 0.05$ eV, $m_{\nu 2} \simeq 0.01$ eV, and $m_{\nu 1} \simeq 0.005$ eV. We will see below that λ is also the Cabibbo angle substantiating our claim that neutrino mass ratio and Cabibbo angle are related.

For the charged lepton, up and down quark mass matrices, we have:

$$M_\ell = \frac{r_1}{\tan\beta}\begin{pmatrix} 0 & -3m_1+\delta & -3m_1-\delta \\ -3m_1+\delta & -3m_0 & 3m_0-3m_1 \\ -3m_1-\delta & 3m_0-3m_1 & -3m_0+M \end{pmatrix}, \tag{13}$$

$$M_d = \frac{r_1}{\tan\beta}\begin{pmatrix} 0 & m_1+\delta & m_1-\delta \\ m_1+\delta & m_0 & -m_0+m_1 \\ m_1-\delta & -m_0+m_1 & m_0+M \end{pmatrix},$$

$$M_u = \begin{pmatrix} 0 & r_2m_1+r_3\delta & r_2m_1-r_3\delta \\ r_2m_1+r_3\delta & r_2m_0 & -r_2m_0+r_3m_1 \\ r_2m_1+r_3\delta & -r_2m_0+r_3m_1 & r_2m_0+M \end{pmatrix},$$

where $\tan\beta$ is a ratio of $H_{u,d}$ vevs. Note that $m_1/m_0 = \lambda \sim 0.2$ and of course $m_0 \ll M$. A quick examination of these mass matrices leads to several immediate conclusions:

1. The model predicts that at GUT scale $m_b \simeq m_\tau$.

2. Since $(M_d)_{11} \to 0$, we get $V_{us} \simeq \sqrt{m_d/m_s}$.

3. The empirically satisfied relation $m_\mu m_e \simeq m_s m_d$ can be obtained by the choice of parameters $-3m_1 + \delta = (m_1+\delta)e^{i\sigma}$, where σ is a phase. Solving this equation, we find that $\delta = m_1(1 + i\cot\sigma/2)$. We obtain $V_{us} \simeq (1 - r_3/r_2)\delta/m_0$, thereby relating Cabibbo angle to the neutrino mass ratio $m_\odot/m_{\rm atm} \simeq \lambda$.

4. $m_\mu \sim -3m_s$.

5. The leptonic mixing angle to diagonalize M_ℓ is related to quark mixing $\theta^l_{12} \sim \frac{1}{3}V_{us}$, which leads to a prediction for $\sin\theta_{13} \equiv U_{e3} \sim \frac{V_{us}}{3\sqrt{2}} \simeq 0.05$ [13].

6. $V_{cb} \sim \dfrac{m_s}{m_b}\cot\theta_{\rm atm}$.

7. The masses of up and charm quarks are given by the parameters $r_{2,3}$ and are therefore not predictions of the model.

8. CP violation in quark sector can put in by making the parameters h' complex.

9. The model predicts a small amplitude for neutrino-less double beta decay from light neutrino mass: $m_{\nu_{ee}} \sim c \sin\theta^l_{12} \simeq 0.3$ meV.

The first four relations are fairly well satisfied by observations; the fifth prediction (i.e. that for U_{e3}) can be tested in upcoming reactor and long baseline experiments. Note that the deviation from tri-bi-maximal mixing pattern coming from the charged lepton mass diagonalization could be thought of as a small perturbation of the neutrino mass matrix [14] except that we predict the form of the perturbation from symmetry considerations. The sixth prediction gives a smaller value for V_{cb} (0.02 as against observed GUT scale value of 0.03) if one uses GUT scale extrapolated value of the known b mass. However, in the MSSM there are threshold corrections to the $b-s$ quark mass mixing from gluino and wino exchange one-loop diagrams; by choosing this contribution, one could obtain the desired V_{cb}.

Note that in this model, the top quark Yukawa coupling at GUT scale arises from an effective higher dimensional operator. We have showed the effective operator in Eq.(6) by expanding ϕ/M. The more precise form for the top Yukawa coupling is $\phi^2/(M_1^2+\phi^2)h_{\psi_V\psi_V H}$, where $h_{\psi_V\psi_V H}$ is a coupling of $\psi_V\psi_V H$ term, and ϕ is the vev of ϕ_1 multiplied by $\phi_1\psi\bar{\psi}_V$ coupling. This is simply because the low energy third generation field is a linear combination of the form $\cos\alpha\,\psi_3 - \sin\alpha\,\psi_V$ with the mixing angle $\sin\alpha \simeq \phi/\sqrt{M_1^2+\phi^2}$.

Therefore, in general, there is no gross contradiction to the fact that the top Yukawa coupling is order 1. However, in our case, if ϕ/M_1 becomes close to 1, the atmospheric mixing shifts from the maximal angle. Therefore, that needs to be addressed if the precise tri-bi-maximal mixing and $h_{\psi_V\psi_V H} \lesssim 1$ is demanded. The desired smallness of the effective f and h' couplings however are more naturally obtained due to the presence of the Planck mass in the denominator. In order to make the f-coupling dominate over the h', we have to choose a small coupling for the H' Higgs field in Eq. (4). Similarly the λ term in Eq. (9) is assumed to be small compared to the coefficient of the first matrix.

Thus within these set of assumptions, this model is in good phenomenological agreement with observations. In a more complete theory, these assumptions need to be addressed. We however find it remarkable that despite these shortcomings, the model provides a very useful unification strategy of the diverse quark-lepton mixing patterns.

IV. VACUUM ALIGNMENT

The major new point of this note is that we obtain the above fermion mass matrices from an S_4 symmetry where the minimum configuration of the flavon fields used in our analyse of fermion mixings arise from superpotential minimization with very additional assumptions.

We start our discussion by giving some simple examples and discussing the flavon alignment as a prelude to the more realistic example. First thing to note is that $\mathbf{3_1}^3$ is invariant under S_4, but $\mathbf{3_2}^3$ is not. Denoting $\phi = (x, y, z)$, we see that in the first case, the singlet of $\phi^3 = xyz$. The superpotential for a $\mathbf{3_1}$ flavon field ϕ can therefore be written as

$$W = \frac{1}{2}m\phi^2 - \lambda\phi^3 = \frac{1}{2}m(x^2 + y^2 + z^2) - \lambda xyz. \tag{14}$$

The solution of F-flat vacua ($\phi \neq 0$) are

$$\phi = \frac{m}{\lambda}\{(1, 1, 1) \text{ or } (1, -1, -1) \text{ or } (-1, 1, -1) \text{ or } (-1, -1, 1)\}. \tag{15}$$

These aligned vacua can be identified to the vertex diagonal axes of the regular hexahedron. In fact S_4 can be identified the permutation of the 4 axes of regular hexahedron. Once one of the axes is fixed, S_3-permutation is left. Therefore, the vacua break S_4 down to S_3.

On the other hand, when $\mathbf{3_2}$ flavon is used (or the cubic term is forbidden by a discrete symmetry), quartic term involving the triplet is crucial for the F-flat vacua. The invariant quartic term ϕ^4 gives two linear combinations of the form $x^4 + y^4 + z^4$ and $x^2y^2 + y^2z^2 + z^2x^2$. This is because they have to be symmetric homogenous terms and invariant under the Klein's group, which is π rotation around the x, y, z axes. Thus, the superpotential term for $\mathbf{3_2}$ field ϕ is

$$\begin{aligned} W &= \frac{1}{2}m\phi^2 - \frac{\kappa^{(1)}}{M}(\phi^4)_1 - \frac{\kappa^{(2)}}{M}(\phi^4)_2 \qquad (16)\\ &= \frac{1}{2}(x^2 + y^2 + z^2) - \frac{\kappa^{(1)}}{4M}(x^4 + y^4 + z^4) - \frac{\kappa^{(2)}}{2M}(x^2y^2 + y^2z^2 + z^2x^2). \end{aligned}$$

The nontrivial F-flat vacua ($\phi \neq 0$) are

$$\phi = \sqrt{\frac{mM}{\kappa^{(1)}}}\,\vec{a}, \quad \sqrt{\frac{mM}{\kappa^{(1)} + 2\kappa^{(2)}}}\,\vec{b}, \quad \sqrt{\frac{mM}{\kappa^{(1)} + \kappa^{(2)}}}\,\vec{c}, \tag{17}$$

where $\vec{a} = (0, 0, \pm 1)$, $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$, $\vec{b} = (\pm 1, \pm 1, \pm 1)$, and $\vec{c} = (0, \pm 1, \pm 1)$, $(\pm 1, \pm 1, 0)$, $(\pm 1, 0, \pm 1)$. We note that these vectors correspond to the axes of the regular hexahedron. The vacua break S_4 down to Z_4, Z_3, and Z_2, respectively. More importantly, the vacuum states in Eq. (7) used in the analysis of fermion masses in the previous section are a subset of the above vacua.

Note that if we add a ϕ^4 term to the superpotential involving the $\mathbf{3_1}$ flavon field, $\vec{a}$ vacuum is possible, in addition to the original $\vec{b}$ vacua. However, $\vec{c}$ vacuum is absent.

Turning to the model at hand, due to non-trivial Z_n charges for the flavon fields, the mass terms (bilinears) are not allowed. To solve this problem, we have included $\bar{\phi}$ fields which then lead to Dirac type mass terms. The superpotential for a single flavon field is then given $W_i = m_i\phi_i\bar{\phi}_i + \kappa_i\phi_i^2\bar{\phi}_i^2$. Denoting $\phi_i = (x_i, y_i, z_i)$ and $\bar{\phi}_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i)$. In terms of its component fields, we obtain

$$\begin{aligned} W_i \;=\; & m_i(x_i\bar{x}_i + y_i\bar{y}_i + z_i\bar{z}_i) + \frac{\kappa_i^{(1)}}{M}(x_i^2\bar{x}_i^2 + y_i^2\bar{y}_i^2 + z_i^2\bar{z}_i^2) \\ & + \frac{\kappa_i^{(2)}}{M}\left(x_i^2(\bar{y}_i^2 + \bar{z}_i^2) + y_i^2(\bar{z}_i^2 + \bar{x}_i^2) + z_i^2(\bar{x}_i^2 + \bar{y}_i^2)\right) + \frac{\kappa_i^{(3)}}{M}(x_i\bar{x}_i y_i\bar{y}_i + y_i\bar{y}_i z_i\bar{z}_i + z_i\bar{z}_i x_i\bar{x}_i). \end{aligned} \tag{18}$$

Note that there are three kinds of invariant for $\phi^2\bar{\phi}^2$. Finding the F-flat solution of this superpotential is similar to the case in Eq.(17). It is easily verified that the F-flat vacua are proportional to $\vec{a}$, $\vec{b}$, and $\vec{c}$ similarly in Eq.(17).

Several comments are now in order:

- We note that the cubic terms in the flavon superpotential, such as ϕ_2^3, $\phi_2^2\bar{\phi}_3$ are forbidden by our choice of Z_n charge of s_i since their presence will spoil a vacuum alignment of ϕ_2 in Eq.(7).

- Secondly, note that the orthogonality of the vevs of ϕ_2 and ϕ_3 is important to obtain the tri-bi-maximal mixing . One way to obtain it dynamically is to have a mixing term $\phi_2^2\phi_3^2$ such that the coupling of the mixing term is much smaller than $\phi_2^2\bar{\phi}_2^2$ and $\phi_3^2\bar{\phi}_3^2$ couplings. The invariant term $\phi_2^2\phi_3^2$ expressed in terms of components gives $x_2x_3y_2y_3 + y_2y_3z_2z_3 + z_2z_3x_2x_3$, where $\phi_2 = (x_2, y_2, z_2)$ and $\phi_3 = (x_3, y_3, z_3)$. The F-flatness condition implies that $y_2y_3 + z_2z_3 = 0$ when $x_2 = 0$ and $x_3 \neq 0$ leading to the desired orthogonality of the alignments of $\langle\phi_2\rangle$ and $\langle\phi_3\rangle$. Note that with our Z_n charge assignments, this can arise only in higher orders and its coefficients must therefore be small. The same situatrion happens also for the mixing terms of the form: $\phi_2\bar{\phi}_2\phi_3\bar{\phi}_3$[15].

- There are mixing terms between the different flavon fields in the quartic terms of the form $W_{ij} = \frac{\lambda}{M}\phi_i\bar{\phi}_i\phi_j\bar{\phi}_j$. When expressed in terms of the component fields x, y, z, they involve mixed terms like $\lambda(x_i\bar{x}_i y_j\bar{y}_j + y_i\bar{y}_i z_j\bar{z}_j + z_i\bar{z}_i x_j\bar{x}_j)$ plus similar other mixed invariants. In the previous item, we just discussed the case when $i = 2$ and $j = 3$. As for the remaining terms of this type, they will in general induce small contributions proportional to λ in the vevs in Eq.(7) where there are zeros. They will induce correction to the forms of our mass matrices. We will therefore need to assume that these λ couplings to be small, so that their effect on our mass and mixing predictions will be small.

- Depending on the values of a and b, one could in principle get very high dimensional terms of the form $s_1^x s_2^y \phi_2 \phi_2 \bar{\phi}_3$ (x, y are positive integers); however their contribution to the flavon potential is suppressed and we ignore these effects.

We therefore conclude that all the desired vacua in the SO(10) model are present. Any possible corrections to them can be made small making it possible to take a first step towards building a unified model of flavor.

V. CONCLUSION

In summary, we have proposed a grand unified model for quark-lepton flavor starting above the GUT scale with an SO(10) theory with $S_4 \times Z_n$ discrete symmetry, S_4 non-singlet flavon fields and two vector like pairs of **16** with mass above the GUT scale and SO(10) Higgs multiplets **10** and **126** fields that give mass to fermions. The **16** matter as well as the flavon fields transform as S_4-family group triplets. The ground state of the flavon sector of the theory gives non-zero vevs to the flavon fields along specific directions due to the above discrete symmetries and when certain higher dimensional couplings between different flavon fields are assumed to be small. They fix the structure of the Yukawa couplings of **10** and **126** fields at GUT scale after the vector-like fields decouple. This leads to specific mass textures for the quarks and leptons with only a few parameters and hence the predictions for quark lepton mass relations and mixing angles in both the quark and the lepton sector. In particular, the model leads to tri-bi-maximal form for the PMNS matrix in the leading order with corrections to this coming from charged lepton fields. Using this, we predict $\theta_{13} \simeq 0.05$. The quark mass hierarchies as well as quark mixings given by the model are in agreement with observations e.g. the model predicts at GUT scale correct mass ratios for m_b/m_τ and m_s/m_μ as well as the Cabibbo angle V_{us} without any adjustment of parameters. Some assumptions are needed to get the large top quark Yukawa coupling as well as relative strengths between the various flavon couplings. Clearly, our work begins a process which seems very promising and further work is needed to improve some of the assumptions used.

Appendix : S_4 group

We briefly review the S_4 group. The group $S_4 \simeq D_2 \rtimes D_3 \simeq (Z_2 \times Z_2) \rtimes S_3$ has irreducible reps $\mathbf{1_1}$, $\mathbf{1_2}$, $\mathbf{2}$, $\mathbf{3_1}$ and $\mathbf{3_2}$ as noted. To see the detailed properties, we use the (x, y, z) coordinate for the transformation law of the three-dimensional representations of S_4. The group $Z_2 \times Z_2$ is a Klein's group $K =$

$\{\mathrm{diag}(1,1,1), \mathrm{diag}(1,-1,-1), \mathrm{diag}(-1,1,-1), \mathrm{diag}(-1,-1,1)\}$, which corresponds to π rotation around the x, y, z axes. The group S_3 is a permutations of the three axes (x, y, z): $S = \{\mathrm{diag}(1,1,1), \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$. The element of S_4 is given as $S_4 = \{(k, s) | k \in K, s \in S\}$.

The $\mathbf{3_1}$ representation ϕ (column vector) transforms by the action of S_4 as

$$\phi \to ks\phi, \tag{19}$$

while $\mathbf{3_2}$ representation ϕ' transforms as

$$\phi' \to (\det s) ks\phi'. \tag{20}$$

The singlet $\mathbf{1_2}$ transforms as

$$\mathbf{1_2} \to (\det s)\mathbf{1_2}, \tag{21}$$

and $\mathbf{1_1}$ is invariant under the action of S_4. The reps $\mathbf{1_1}$, $\mathbf{1_2}$, $\mathbf{2}$ are reps of $S_3 \simeq D_3$, and the transformation law of $\mathbf{2}$ is rotation and reflection of the regular triangle. Doublet rep (u, v) transforms as

$$\begin{pmatrix} u \\ 1 \\ v \end{pmatrix} \to U_{\mathrm{TB}}^t \, s \, U_{\mathrm{TB}} \begin{pmatrix} u \\ 1 \\ v \end{pmatrix}. \tag{22}$$

For convenience, we list the Kronecker products of the triplets:

$$\begin{aligned} &(\mathbf{3_i} \times \mathbf{3_i})_s = \mathbf{1_1} \oplus \mathbf{2} \oplus \mathbf{3_1}, \quad (\mathbf{3_i} \times \mathbf{3_i})_a = \mathbf{3_2}, \\ &\mathbf{3_1} \times \mathbf{3_2} = \mathbf{1_2} \oplus \mathbf{2} \oplus \mathbf{3_1} \oplus \mathbf{3_2}. \end{aligned}$$

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[15] An alternative way to obtain the above orthogonality is to make a different choice of reps of S_4 i.e. introduce an extra flavon field ϕ_4 whose vev is $\phi_4^t = (1,0,0)$. The rank one f coupling is obtained by $(\phi_3\phi_4\psi)\bar{\psi}_V + M_V\psi_V\bar{\psi}_V + \psi_V\psi_V\bar{\Delta}$ instead of $\phi_2\bar{\psi}_V + M_V\psi_V\bar{\psi}_V + \psi_V\psi_V\bar{\Delta}$. Suppose under S_4 ϕ_3, ϕ_4 and ψ transform as $\mathbf{3_1}$, $\mathbf{3_1}$, and $\mathbf{3_2}$, respectively. Then, if the reps of ψ_V, $\bar{\psi}_V$ are $\mathbf{1_1}$, the product of ϕ_3 and ϕ_4 should be $\mathbf{3_2}$, which is obtained by anti-symmetric product.

At that time, the orthogonal condition is satisfied automatically (i.e., $\phi_3 \times \phi_4 \perp \phi_3$).