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Constitutive equations and reciprocity

Anthony F. Gangi

Department of Geology & Geophysics, Texas A & M University, College Station, TX 77843–3114, *USA. E-mail: gangi@tamu.edu*

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SUMMARY

The differential equations that must be satisfied for most fields (physical variables) in geophysics are determined primarily by conservation equations which relate the divergence of the flux of the field, the field's time rate of change, and its sources and sinks. These conservation (or equilibrium) equations do not provide sufficient constraints to determine the fluxes and fields even when boundary conditions (both in space and time) are specified. To constrain the fields completely, it is necessary to introduce the properties of the media; that is, the constitutive equations. Because the conservation equations can be determined without considering the properties of the media, these equations are valid for the most general media; that is, heterogeneous, anisotropic, time-varying, nonlinear, etc. media in which many field variables can interact or be coupled. When the fields can be described by 'self-adjoint' differential equations in space–time, these media exhibit reciprocity; that is, upon interchange of 'sources' and 'detectors', the same result is obtained. We show that viscoelastic, elastodynamic problems relating to generalized Kelvin–Voigt and generalized Maxwell media satisfy the conditions for reciprocity. In addition, we show that the introduction of tensor 'densities' (which relate the inertial-force density to the particle-acceleration, particle-velocity and particle-displacement vectors in the inertial force's constitutive equation) do not invalidate the reciprocity conditions. The two constitutive equations (the stress/strain and the inertial-force ones) lead to dispersion and attenuation in the propagation of the fields even though none of the material constants in the constitutive or conservation equations is complex (i.e. with real and imaginary parts). Complex material properties cannot exist in nature for actual materials or media; nor can the material constants or properties be functions of frequency. However, 'apparent' or 'equivalent' properties may be complex and functions of frequency if time-harmonic fields are assumed to exist.

Key words: constitutive equations, Green's functions, moduli, reciprocity, tensor densities, viscoelasticity.

INTRODUCTION

The reciprocity principle is a very useful one in dealing with boundary-value problems. It demonstrates the equality of two solutions (as quantified by reciprocity) in a medium when the source and the field detector are interchanged (Betti 1872; Lord Rayleigh 1873, 1945; Graffi 1939, 1946; Morse & Feshbach 1953; Knopoff & Gangi 1959). In making the interchange, the tensor characteristics of the source and field must be taken into account; that is, whether the fields that are to be detected and the sources that generate the fields are scalar, vector or higher-order (or rank) tensors. Another advantage of the reciprocity principle is that the representation theorem (or Green's theorem) can be readily derived from it (Gangi 1970).

In this paper we review generalized reciprocity and relate it to elastodynamic problems by introducing generalized constitutive equations. Specific constitutive equations for the inertial force (generalized) are shown to satisfy the self-adjoint property necessary for reciprocity to hold in a single medium. Then the stress/strain (viscoelastic) constitutive equations for generalized Kelvin–Voigt and Maxwell media are also shown to satisfy the self-adjoint property necessary for reciprocity to hold in a single medium.

GENERALIZED RECIPROCITY

The reciprocity principle can be demonstrated for a tensor field of any order. The condition required for reciprocity to hold is that the operator, which relates the field and source, be self-adjoint. To illustrate this fact, we first review—in outline—the generalized reciprocity principle (see, for example, Morse & Feshbach 1953, Section 7.5).

For example, we consider a tensor differential operator (be it a scalar, vector or higher-order tensor one), $\hat{\mathbf{O}}\{\mathbf{u}(\mathbf{r}, t); \mathbf{r}, t; \mathbf{C}\}$, which operates on some tensor field, $\mathbf{u}(\mathbf{r}, t)$, in the space–time domain $(\mathbf{r}, t)$ and which results from some tensor source term, $\mathbf{f}(\mathbf{r}, t)$; that is:

$$\hat{\mathbf{O}}\{\mathbf{u}(\mathbf{r}, t); \mathbf{r}, t; \mathbf{C}\} = \mathbf{f}(\mathbf{r}, t)\,, \tag{1}$$

and where $\mathbf{C}$ is some (tensor) parameter (or collection of parameters) which describes the properties of the medium contained in a region of volume V that, in turn, has a bounding surface, $S(\mathbf{r}, t)$. We also assume there are some homogeneous boundary conditions on $S(\mathbf{r}, t)$ and initial (or final) conditions in time:

$$\begin{aligned}&\{\mathbf{u}(\mathbf{r}, t), \nabla\mathbf{u}(\mathbf{r}, t)\} = 0 \text{ on } S(\mathbf{r}, t) \text{ for all } t \text{ and}\\ &\{\mathbf{u}(\mathbf{r}, t), \dot{\mathbf{u}}(\mathbf{r}, t)\} = 0 \text{ for } t \le t_0 \text{ for all } V\,,\end{aligned} \tag{2}$$

where $\{\mathbf{u}(\mathbf{r}, t), \nabla\mathbf{u}(\mathbf{r}, t)\}$ means some combination of the field and its gradient, and $\{\mathbf{u}(\mathbf{r}, t), \dot{\mathbf{u}}(\mathbf{r}, t)\}$ means some combination of the field and its time derivative. Then the adjoint problem is described in terms of the adjoint operator, $\acute{\mathbf{O}}\{\mathbf{U}(\mathbf{r}, t); \mathbf{r}, t; \mathbf{C}\}$, operating on the field $\mathbf{U}(\mathbf{r}, t)$ due to a force, $\mathbf{F}(\mathbf{r}, t)$, acting in the same medium:

$$\acute{\mathbf{O}}\{\mathbf{U}(\mathbf{r}, t); \mathbf{r}, t; \mathbf{C}\} = \mathbf{F}(\mathbf{r}, t) \tag{1$'$}$$

and a set of homogeneous boundary and time conditions (similar to those in eq. 2). The adjoint operator is defined by the following condition, which permits a (4-D) volume integral to be converted into a (4-D) surface integral:

$$\mathbf{U}\otimes\hat{\mathbf{O}} - \mathbf{u}\otimes\acute{\mathbf{O}} = \nabla\otimes\mathbf{P}(\mathbf{r}, t) + \frac{\partial\mathbf{q}(\mathbf{r}, t)}{\partial t} = \mathbf{U}\otimes\mathbf{f} - \mathbf{u}\otimes\mathbf{F}\,, \tag{3}$$

where $\mathbf{P}(\mathbf{r}, t)$ and $\mathbf{q}(\mathbf{r}, t)$ are some tensors which include the boundary and time (boundary) conditions, respectively, and $\otimes$ is some tensor-compatible multiplication, such as the dot product, cross product, double-dot product, tensor product, etc. [Note: $\mathbf{P}(\mathbf{r}, t) = \mathbf{P}(\mathbf{u}, \mathbf{U}, \mathbf{r}, t)$ and $\mathbf{q}(\mathbf{r}, t) = \mathbf{q}(\mathbf{u}, \mathbf{U}, \mathbf{r}, t)$]. Then, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_V [\mathbf{U}\otimes\hat{\mathbf{O}} - \mathbf{u}\otimes\acute{\mathbf{O}}]\,dV dt\\ &= \int_{-\infty}^{\infty}\int_V \left[\nabla\otimes\mathbf{P}(\mathbf{r}, t) + \frac{\partial\mathbf{q}(\mathbf{r}, t)}{\partial t}\right] dV dt\\ &= \int_{-\infty}^{\infty}\int_V [\mathbf{U}\otimes\mathbf{f} - \mathbf{u}\otimes\mathbf{F}]\,dV dt\,.\end{aligned} \tag{4}$$

Now, using Gauss' divergence theorem and integrating over time, the second integral becomes

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_V \left[\nabla\otimes\mathbf{P}(\mathbf{r}, t) + \frac{\partial\mathbf{q}(\mathbf{r}, t)}{\partial t}\right] dV dt\\ &= \int_{-\infty}^{\infty}\int_S \mathbf{n}\otimes\mathbf{P}(\mathbf{r}, t)\,dS dt + \int_V \mathbf{q}(\mathbf{r}, t)\Big|_{t=-\infty}^{t=\infty} dV\,.\end{aligned} \tag{5}$$

Therefore, if the boundary conditions (in space and time) are such that $\mathbf{n}\otimes\mathbf{P}(\mathbf{r}, t) = 0$ on $S(\mathbf{r}, t)$ for all time, t, and $\mathbf{q}(\mathbf{r}, t) = 0$ for $t = \pm\infty$ throughout the region V, then this integral is zero and we have the condition of generalized reciprocity:

$$\boxed{\int_{-\infty}^{\infty}\int_V [\mathbf{U}\otimes\mathbf{f} - \mathbf{u}\otimes\mathbf{F}]\,dV dt = 0\,.} \tag{6}$$

If the operator $\hat{\mathbf{O}}\{\mathbf{u}(\mathbf{r}, t); \mathbf{r}, t; \mathbf{C}\}$ is self-adjoint; that is,

$$\hat{\mathbf{O}}\{\mathbf{u}(\mathbf{r}, t); \mathbf{r}, t; \mathbf{C}\} \equiv \acute{\mathbf{O}}\{\mathbf{u}(\mathbf{r}, t); \mathbf{r}, t; \mathbf{C}\}\,, \tag{7}$$

then the governing differential equation is self-adjoint and reciprocity holds for the problem itself. However, note that the proper multiplication of the sources and fields must be used in the reciprocity relationship, and the reciprocity relationship is in terms of an integral over space–time.

ELASTODYNAMIC PROBLEMS

The governing differential equation for elastodynamic problems is Cauchy's force-equilibrium (or momentum-conservation) equation (*cf.*, for example, Malvern 1969; de Hoop 1995):

$$\begin{aligned}&\nabla\cdot\boldsymbol{\sigma}(\mathbf{r}, t) + \mathbf{f}_{\mathrm{b}}(\mathbf{r}, t) = \mathbf{f}_{\mathrm{I}}(\mathbf{r}, t)\\ &\text{with } \mathbf{u}(\mathbf{r}, t) = 0\,;\quad \dot{\mathbf{u}}(\mathbf{r}, t) = 0 \quad \text{for } t \le t_0 \text{ in } V\,,\end{aligned} \tag{8}$$

where $\boldsymbol{\sigma}(\mathbf{r}, t)$ is the second-order, symmetric stress tensor (that is, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$), $\mathbf{f}_{\mathrm{b}}(\mathbf{r}, t)$ is the vector body-force density (which usually contains the source term), $\mathbf{f}_{\mathrm{i}}(\mathbf{r}, t)$ is the vector inertial-force density, $\mathbf{u}(\mathbf{r}, t)$ is the vector displacement field, and $\dot{\mathbf{u}}(\mathbf{r}, t)$ is the particle-velocity field in some region of volume $\boldsymbol{V}$ which has a bounding surface, $S(\mathbf{r}, t)$.

The proposed reciprocal (adjoint) problem satisfies

$$\begin{aligned}&\nabla\cdot\boldsymbol{\Sigma}(\mathbf{r}, \tau) + \mathbf{F}_{\mathrm{b}}(\mathbf{r}, \tau) = \mathbf{F}_{\mathrm{i}}(\mathbf{r}, \tau)\\ &\text{with } \mathbf{U}(\mathbf{r}, \tau) = 0\,;\quad \dot{\mathbf{U}}(\mathbf{r}, \tau) = 0 \quad \text{for } \tau \le \tau_1 \text{ in } V\,,\end{aligned} \tag{8$'$}$$

where $\boldsymbol{\Sigma}(\mathbf{r}, \tau)$ is the second-order, symmetric stress tensor, $\boldsymbol{F}_{\mathrm{b}}(\mathbf{r}, \tau)$ is the vector body-force density, $\boldsymbol{F}_{\mathrm{i}}(\mathbf{r}, \tau)$ is the vector inertial-force density, $\mathbf{U}(\mathbf{r}, \tau)$ is the vector displacement field and $\dot{\mathbf{U}}(\mathbf{r}, \tau)$ is the particle-velocity field in the region V for the reciprocal problem.

Then we can form the product

$$\begin{aligned}&\mathbf{U}(\mathbf{r}, \tau)\cdot[\nabla\cdot\boldsymbol{\sigma}(\mathbf{r}, t) - \mathbf{f}_{\mathrm{i}}(\mathbf{r}, t)] - \mathbf{u}(\mathbf{r}, t)\cdot[\nabla\cdot\boldsymbol{\Sigma}(\mathbf{r}, \tau) - \mathbf{F}_{\mathrm{i}}(\mathbf{r}, \tau)]\\ &\quad = \nabla\cdot[\boldsymbol{\sigma}(\mathbf{r}, t)\cdot\mathbf{U}(\mathbf{r}, \tau) - \boldsymbol{\Sigma}(\mathbf{r}, \tau)\cdot\mathbf{u}(\mathbf{r}, t)]\\ &\qquad - [\boldsymbol{\sigma}(\mathbf{r}, t)\mathbf{:}\nabla\mathbf{U}(\mathbf{r}, \tau) - \boldsymbol{\Sigma}(\mathbf{r}, \tau)\mathbf{:}\nabla\mathbf{u}(\mathbf{r}, t)]\\ &\qquad - [\mathbf{U}(\mathbf{r}, \tau)\cdot\mathbf{f}_{\mathrm{i}}(\mathbf{r}, t) - \mathbf{u}(\mathbf{r}, t)\cdot\mathbf{F}_{\mathrm{i}}(\mathbf{r}, \tau)]\\ &\quad = -[\mathbf{U}(\mathbf{r}, \tau)\cdot\mathbf{f}_{\mathrm{b}}(\mathbf{r}, t) - \mathbf{u}(\mathbf{r}, t)\cdot\mathbf{F}_{\mathrm{b}}(\mathbf{r}, \tau)]\,.\end{aligned} \tag{9}$$

(Definition of the double-dot product: $\mathbf{C:A} = \hat{\mathbf{i}}\hat{\mathbf{j}}C_{ijkl}A_{kl} = \mathbf{B}$, where $\mathbf{C} = \hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}\hat{\mathbf{l}}C_{ijkl}$ is a fourth-order tensor, $\mathbf{A} = \hat{\mathbf{k}}\hat{\mathbf{l}}A_{kl}$ is a second-order tensor and the result, $\mathbf{B} = \hat{\mathbf{i}}\hat{\mathbf{j}}B_{ij}$, is a second-order tensor. Note that the summation convention for repeated indices is being used and that $\mathbf{A:B} = A_{ij}B_{ij}$ is a scalar.)

Thus, reciprocity holds if these equations equal zero upon integration over space–time. The problem will be self-adjoint if the 'virtual-strain-energy' term, $\boldsymbol{\sigma}(\mathbf{r}, t)\mathbf{:}\nabla\mathbf{U}(\mathbf{r}, \tau) - \boldsymbol{\Sigma}(\mathbf{r}, \tau)\mathbf{:}\nabla\mathbf{u}(\mathbf{r}, t)$, and the 'virtual-work' term, $\mathbf{U}(\mathbf{r}, \tau)\cdot\mathbf{f}_{\mathrm{i}}(\mathbf{r}, t) - \mathbf{u}(\mathbf{r}, t)\cdot\mathbf{F}_{\mathrm{i}}(\mathbf{r}, \tau)$, can be expressed as the time derivative of some function [which, in turn, gives a zero contribution upon integration over time over the whole region by means of initial and final time (boundary) conditions].

CONSTITUTIVE EQUATIONS

The momentum-conservation equations (eqs 8 and 8′) are valid for any medium, whether it is heterogeneous, non-linear or even time-varying. However, it is a vector equation valid in the 3-D space of the position vector **r**, and, as such, represents only three independent scalar equations.

If we assume that the body forces are known (they usually represent the sources of the field), and if we assume that the inertial force is given in terms of the particle acceleration (and other time derivatives of the displacement vector—see later), then there are nine unknowns in the above vector equation. These are the six scalar components of the symmetric, second-order stress tensor, $\boldsymbol{\sigma}$, defined in the 3-D space (of **r**), and the three scalar components of the displacement vector, **u**. The particle velocity and acceleration are also independent variables, but they have defining equations which relate them to the displacement vector, so they are not, in that sense, independent unknowns.

Therefore, to solve for these nine (scalar) unknowns we need nine independent (scalar) equations. The conservation equation gives three, so six additional equations are needed. These six equations represent the constitutive equations for the medium—or its equations of state. In a 3-D medium, these six scalar equations can be represented by a symmetric, second-order tensor (see, for example, Gangi 1981):

$$\mathbf{H}^{\mathrm{T}}=\mathbf{H}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\ddot{\boldsymbol{\sigma}},\ldots,\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\ldots\nabla\boldsymbol{\sigma},\nabla\nabla\boldsymbol{\sigma},\ldots\nabla\mathbf{u},\nabla\nabla\mathbf{u},\ldots\nabla\dot{\mathbf{u}},\nabla\dot{\boldsymbol{\sigma}},\ldots,\boldsymbol{\gamma})=0\,. \tag{10}$$

Note that this function has no explicit time or space dependence; that is, the material of the medium has the same response to the state variables no matter where it is located in physical space and no matter what the time is, provided that the state variables do not change. It can change with position and time only through the variation of the state variables in space and time; that is, implicitly only. It does depend, however, on the composition of the medium through the tensor variable $\boldsymbol{\gamma}$.

In the argument of this tensor we have introduced new state variables, namely the time and space derivatives of the primary unknowns, $\boldsymbol{\sigma}$ and **u**, and $\boldsymbol{\gamma}$, which contains the variables relating to the composition of the medium. These new state variables, except for $\boldsymbol{\gamma}$, do not introduce new unknown variables into the problem because each comes with its own defining equation. Furthermore, we will assume that the composition of the medium is constant in space and time, so we need not consider here variations in $\boldsymbol{\gamma}$. In the following we will treat the constitutive equation for the inertial force separately from that for the stress and displacement (or strain). Then, the 'stress/strain' constitutive equation becomes

$$\mathbf{H}^{\mathrm{T}}=\mathbf{H}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\nabla\boldsymbol{\sigma},\ddot{\boldsymbol{\sigma}},\nabla\nabla\boldsymbol{\sigma},\nabla\dot{\boldsymbol{\sigma}},\ldots,\mathbf{e},\dot{\mathbf{e}},\nabla\mathbf{e},\ddot{\mathbf{e}},\nabla\nabla\mathbf{e},\nabla\dot{\mathbf{e}},\ldots,\boldsymbol{\gamma})=0\,, \tag{11}$$

where $\mathbf{e}=\mathbf{e}^{\mathrm{T}}=[\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}}]/2$ is the symmetric, second-order, infinitesimal strain tensor.

Most materials are well described by only a few of the above state variables; for example, an isothermal elastic solid has only a stress and strain dependence, while isothermal visco-elastic solids depend only on the stress and strain and their time derivatives. In the following we will ignore the gradients (spatial derivatives) of the stress and strain even though there are materials that do depend on them.

The resulting constitutive equation

$$\mathbf{H}^{\mathrm{T}}=\mathbf{H}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\ddot{\boldsymbol{\sigma}},\ldots,\mathbf{e},\dot{\mathbf{e}},\ddot{\mathbf{e}},\ldots,\boldsymbol{\gamma})=\hat{\mathbf{i}}\hat{\mathbf{j}}H_{ij}=0\,, \tag{12}$$

while a second-order-tensor equation in physical space, has scalar components that represent hypersurfaces in the high-dimensional state space where, for example, the six scalar components of each of the stress and strain represent dimensions. That is, an isothermal (non-linear or linear) elastic solid has a constitutive equation in a 12-dimensional state space. The material with the above constitutive equation behaves such that it is always constrained to this hypersurface no matter what the values of the state variables are.

Therefore, assuming that the composition, i.e. $\boldsymbol{\gamma}$, does not change, and using the chain rule of differentiation (and the double-dot product), we have

$$d\mathbf{H}^{\mathrm{T}}=0=d\mathbf{H}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\ddot{\boldsymbol{\sigma}},\ldots,\mathbf{e},\dot{\mathbf{e}},\ddot{\mathbf{e}},\ldots,\boldsymbol{\gamma})=d\boldsymbol{\sigma}\!:\!\partial_{\boldsymbol{\sigma}}\mathbf{H}+d\dot{\boldsymbol{\sigma}}\!:\!\partial_{\dot{\boldsymbol{\sigma}}}\mathbf{H}+\cdots+d\mathbf{e}\!:\!\partial_{\mathbf{e}}\mathbf{H}+d\dot{\mathbf{e}}\!:\!\partial_{\dot{\mathbf{e}}}\mathbf{H}+\cdots\,, \tag{13}$$

and this differential equation also represents the constitutive equation. Note that it is a non-linear differential equation if the fourth-order coefficients [e.g. $\partial_{\boldsymbol{\sigma}}\mathbf{H}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\ddot{\boldsymbol{\sigma}},\ldots,\mathbf{e},\dot{\mathbf{e}},\ddot{\mathbf{e}},\ldots,\boldsymbol{\gamma})$] of the differentials of the state variables are functions of the state variables also. These coefficients (e.g. $\partial_{\boldsymbol{\sigma}}\mathbf{H}$) of the state-variable differentials are fourth-order tensors because the 'gradient operators' (e.g. $\partial_{\boldsymbol{\sigma}}$) are themselves second-order tensors; therefore, the 'gradients' of the second-order tensor **H** are fourth-order tensors. That is, $\partial_{\boldsymbol{\sigma}}\mathbf{H}$ is a fourth-order tensor of the form

$$\partial_{\boldsymbol{\sigma}}\mathbf{H}=\hat{\mathbf{i}}\hat{\mathbf{j}}\,\frac{\partial H_{kl}}{\partial\sigma_{ij}}\,\hat{\mathbf{k}}\hat{\mathbf{l}}\,;\quad i,j,k,l=1,2,3\,, \tag{14}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors corresponding to the stress components, while $\hat{\mathbf{k}}$ and $\hat{\mathbf{l}}$ are unit vectors corresponding to the constitutive tensor, **H**, and they all are unit vectors in the physical space containing the position vector **r** and where **H** is defined. The summation convention is used when indices are repeated.

For linear materials or media (with no compositional change), the fourth-order tensors are not functions of the state variables, and we can integrate the differential equation from some initial state (where, say, all the state variables are zero) to some general state to obtain

$$\boldsymbol{\sigma}\!:\!\mathbf{S}_1+\dot{\boldsymbol{\sigma}}\!:\!\mathbf{S}_2+\cdots+\mathbf{e}\!:\!\mathbf{C}_1+\dot{\mathbf{e}}\!:\!\mathbf{C}_2+\cdots=0\,. \tag{15}$$

The fourth-order tensors, $\mathbf{S}_1=\partial_{\boldsymbol{\sigma}}\mathbf{H}$, $\mathbf{C}_1=\partial_{\mathbf{e}}\mathbf{H}$, etc., are constant tensors with respect to the state variables of stress and strain and their derivatives. Note that these fourth-order tensors can be functions of position if the composition, $\boldsymbol{\gamma}$, changes from place to place but all the compositions are assumed to have no dependence on the other state variables; that is, all the compositions are assumed to be linear in the stress and strain and their time derivatives.

For the isothermal, linear-elastic case (a Hookean solid) where

$$\boldsymbol{\sigma}=\mathbf{e}\!:\!\mathbf{C}\equiv\mathbf{C}^{\mathrm{T}3412}\!:\!\mathbf{e}=e_{ij}C_{ijkl}\hat{\mathbf{k}}\hat{\mathbf{l}}=C_{klij}e_{ij}\hat{\mathbf{k}}\hat{\mathbf{l}}\,, \tag{16}$$

we have

$$\mathbf{C}=\partial_{\mathbf{e}}\mathbf{H}:(\partial_{\boldsymbol{\sigma}}\mathbf{H})^{-1}\,. \tag{17}$$

Note that if a scalar strain-energy-density function, ϕ, exists so that

$$\phi=\boldsymbol{\sigma}:\mathbf{e}=\mathbf{e}:\boldsymbol{\sigma}=\mathbf{e}:\mathbf{C}:\mathbf{e}=\mathbf{e}:\mathbf{C}^{\mathrm{T}3412}:\mathbf{e}=\sigma_{ij}e_{ij}=e_{ij}\sigma_{ij}\,, \tag{18}$$

then either $\mathbf{C}-\mathbf{C}^{\mathrm{T}3412}=0$ or $\mathbf{C}-\mathbf{C}^{\mathrm{T}3412}$ must be 'orthogonal' to $\mathbf{e}$ under the double-dot product. We choose the first alternative because $\mathbf{e}$ is arbitrary. We have $\mathbf{C}=\mathbf{C}^{\mathrm{T}2134}=\mathbf{C}^{\mathrm{T}1243}$ because of the symmetry of the strain and stress. [$\mathbf{C}^{\mathrm{T}3412}$ means interchange the first two base vectors with the second two; that is, $(\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}\hat{\mathbf{l}}C_{ijkl})^{\mathrm{T}3412}=\hat{\mathbf{k}}\hat{\mathbf{l}}\hat{\mathbf{i}}\hat{\mathbf{j}}C_{ijkl}$. See also eq. (27) for a complete definition of the generalized transpose operation.]

In the above constitutive tensor, $\mathbf{H}$, we could have introduced other state variables such as temperature (or entropy), electric and/or magnetic fields, electrochemical potentials, etc. to describe the medium. With the addition of temperature as a state variable, we would have to consider the conservation equation for heat flow and we would need to determine the temperature variation in time and space. If there is no heat flow, the moduli would be the adiabatic moduli. If we consider that there are some interactions between the temperature and the stress or strain, we would have a medium that is subjected to thermal expansion or thermal stress—alternatively, one that heats up or cools down with the application of stress or strain. If the electric and/or magnetic fields are state variables also, the material would be piezoelectric and/or magneto-elastic as well. In that case, the electromagnetic conservation equations—Maxwell's equations—would have to be included and satisfied also.

INERTIAL-FORCE CONSTITUTIVE EQUATION

First let us consider the inertial-force terms; for example, $\mathbf{f}_{\mathrm{i}}(\mathbf{r},\,t)=\,d\mathbf{p}/dt$, where $\mathbf{p}(\mathbf{r},\,t)$ is the momentum density of the medium. Usually, the inertial-force density is considered to be proportional to the acceleration, with the material density, ρ, as the constant of proportionality: $\mathbf{f}_{\mathrm{i}}(\mathbf{r},\,t)=\rho\ddot{\mathbf{u}}$. However, the inertial force need not be collinear with the acceleration—consider a single-component, spring/mass seismometer or a fluid-saturated porous medium (Biot 1956). Then the force and acceleration vectors are linearly related by, say, a second-order tensor as

$$\mathbf{f}_{\mathrm{i}}(\mathbf{r},\,t)=\mathbf{R}_{\mathrm{a}}\cdot\ddot{\mathbf{u}}(\mathbf{r},\,t)\,, \tag{19}$$

where $\mathbf{R}_{\mathrm{a}}$ is the second-order tensor. This is a 'constitutive equation' for the inertial force in a medium composed of such seismometers (or in a fluid-saturated porous medium). The reason the relationship between the force and the acceleration is tensorial is because the mass of a single-component seismometer is constrained to move in one direction only. If the acceleration is not in the allowed direction of motion of the mass—or at right angles to it—the force will not be collinear with the acceleration. Alternatively, the principal axes of the tensor $\mathbf{R}_{\mathrm{a}}$ are (1) in the direction of the mass movement, and (2) transverse to this direction, and the principal values or eigenvalues in those directions are different.

We can generalize this constitutive equation and still have a linear relationship by having the inertial force depend on the particle velocity and displacement:

$$\begin{aligned}\mathbf{f}_{\mathrm{i}}(\mathbf{r},\,t)&=\mathbf{R}_{\mathrm{a}}\cdot\ddot{\mathbf{u}}(\mathbf{r},\,t)+\mathbf{R}_{\mathrm{v}}\cdot\dot{\mathbf{u}}(\mathbf{r},\,t)+\mathbf{R}_{\mathrm{u}}\cdot\mathbf{u}(\mathbf{r},\,t)\\&=\left[\mathbf{R}_{\mathrm{a}}\,\frac{d^2}{dt^2}+\mathbf{R}_{\mathrm{v}}\,\frac{d}{dt}+\mathbf{R}_{\mathrm{u}}\right]\cdot\mathbf{u}(\mathbf{r},\,t)\,.\end{aligned} \tag{20}$$

This constitutive equation could describe the inertial force at the centre of mass, in response to motions there, for a three-component seismometer, with, in general, different characteristics for each component (see also de Hoop 1995, p. 317ff for an inertial-relaxation-function approach).

The advantage of the above formulation is that this linear system will have a frequency-dependent response which depends on the values of the second-order tensors which are assumed, here, symmetric: $\mathbf{R}_{\mathrm{s}}=(\mathbf{R}_{\mathrm{s}})^{\mathrm{T}}$, s = a, v, u (see Appendix A for a more general approach). The tensors $\mathbf{R}_{\mathrm{s}}$ themselves may be explicit functions of position and time, but we will assume that they are time-stationary and independent of the particle motions. That is, none of the scalar components of $\mathbf{R}_{\mathrm{s}}$ depends on time (or frequency), and the inertial force is a linear function of the particle motions. Nevertheless, such an inertial-force law will lead to dispersive wave propagation.

In eqs (8′) and (9), we have used a different symbol for the time variable in the proposed reciprocal problem. The reason for this is that we wish to end up with a convolution integral in time for our reciprocity relationship so that we can compare solutions with different time variations of the source functions. Therefore, the time variable τ is set equal to $t'-t$, where t' is some arbitrary but fixed time. Then we have for the inertial force $\mathbf{F}_{\mathrm{i}}(\mathbf{r},\,\tau)$,

$$\begin{aligned}\mathbf{F}_{\mathrm{i}}(\mathbf{r},\,\tau)&=\mathbf{F}_{\mathrm{i}}(\mathbf{r},\,t'-t)\\&=\mathbf{R}_{\mathrm{a}}\cdot\ddot{\mathbf{U}}(\mathbf{r},\,t'-t)+\mathbf{R}_{\mathrm{v}}\cdot\dot{\mathbf{U}}(\mathbf{r},\,t'-t)+\mathbf{R}_{\mathrm{u}}\cdot\mathbf{U}(\mathbf{r},\,t'-t)\\&=\left[\mathbf{R}_{\mathrm{a}}\,\frac{d^2}{dt^2}-\mathbf{R}_{\mathrm{v}}\,\frac{d}{dt}+\mathbf{R}_{\mathrm{u}}\right]\cdot\mathbf{U}(\mathbf{r},\,t'-t)\,,\end{aligned} \tag{20$'$}$$

using $\tau=t'-t$ and $d/d\tau=-d/dt$, where $\dot{\mathbf{U}}(\mathbf{r},\,t'-t)=d\mathbf{U}(\mathbf{r},\,\tau)/d\tau=-d\mathbf{U}(\mathbf{r},\,t'-t)/dt$. Here we assume that the $\mathbf{R}_{\mathrm{s}}$ for the adjoint force, $\mathbf{F}$, are the same as those in the original problem (see Appendix A for a more general approach). Therefore, the scalar 'virtual-work' term, $w(\mathbf{r},\,t)$, is

$$\begin{aligned}w(\mathbf{r},\,t)&=\mathbf{U}(\mathbf{r},\,\tau)\cdot\mathbf{f}_{\mathrm{i}}(\mathbf{r},\,t)-\mathbf{u}(\mathbf{r},\,t)\cdot\mathbf{F}_{\mathrm{i}}(\mathbf{r},\,\tau)\\&=\mathbf{U}\cdot\left[\mathbf{R}_{\mathrm{a}}\,\frac{d^2}{dt^2}+\mathbf{R}_{\mathrm{v}}\,\frac{d}{dt}+\mathbf{R}_{\mathrm{u}}\right]\cdot\mathbf{u}\\&\quad-\mathbf{u}\cdot\left[\mathbf{R}_{\mathrm{a}}\,\frac{d^2}{dt^2}-\mathbf{R}_{\mathrm{v}}\,\frac{d}{dt}+\mathbf{R}_{\mathrm{u}}\right]\cdot\mathbf{U}\\&=\mathbf{R}_{\mathrm{a}}:[\mathbf{U}\ddot{\mathbf{u}}-\mathbf{u}\ddot{\mathbf{U}}]+\mathbf{R}_{\mathrm{v}}:[\mathbf{U}\dot{\mathbf{u}}-\mathbf{u}\dot{\mathbf{U}}]+\mathbf{R}_{\mathrm{u}}:[\mathbf{U}\mathbf{u}-\mathbf{u}\mathbf{U}]\,,\end{aligned} \tag{21}$$

Because $\mathbf{R}_{\mathrm{u}}=(\mathbf{R}_{\mathrm{u}})^{\mathrm{T}}$, the last term is identically zero. Because all the $\mathbf{R}_{\mathrm{s}}$ (s = a, v, u) are symmetric and independent of time, the second term becomes

$$\mathbf{R}_{\mathrm{v}}:[\mathbf{U}\dot{\mathbf{u}}-\mathbf{u}\dot{\mathbf{U}}]=\mathbf{R}_{\mathrm{v}}:[\dot{\mathbf{u}}\mathbf{U}-\mathbf{u}\dot{\mathbf{U}}]=\frac{d}{dt}\,\{\mathbf{R}_{\mathrm{v}}:[\mathbf{u}\mathbf{U}]\}\,, \tag{22}$$

and the third term becomes

$$\mathbf{R}_{\mathrm{a}}:[\mathbf{U}\ddot{\mathbf{u}}-\mathbf{u}\ddot{\mathbf{U}}]=\mathbf{R}_{\mathrm{a}}:[\ddot{\mathbf{u}}\mathbf{U}-\mathbf{u}\ddot{\mathbf{U}}]=\frac{d}{dt}\,\{\mathbf{R}_{\mathrm{a}}:[\dot{\mathbf{u}}\mathbf{U}-\mathbf{u}\dot{\mathbf{U}}]\}\,. \tag{23}$$

Therefore, the integral over time of the 'virtual-work' term (eq. 21) is zero because of the initial conditions, $\mathbf{u}=\mathbf{0}$ and $\dot{\mathbf{u}}=\mathbf{0}$ for $t\le t_0$, and the final conditions, $\mathbf{U}=\mathbf{0}$ and $\dot{\mathbf{U}}=0$ for $t\ge t_1$ (that is, for $t'-t=\tau\le\tau_1=t'-t_1$).

STRESS/STRAIN CONSTITUTIVE EQUATIONS

Now let us consider the scalar stress/strain term or 'virtual-strain-energy' term, $\psi(\mathbf{r}, t)$:

$$\begin{aligned}\psi(\mathbf{r}, t)&=\boldsymbol{\sigma}(\mathbf{r}, t)\mathbf{:}\nabla\mathbf{U}(\mathbf{r}, \tau)-\boldsymbol{\Sigma}(\mathbf{r}, \tau)\mathbf{:}\nabla\mathbf{u}(\mathbf{r}, t)\\ &=\boldsymbol{\sigma}(\mathbf{r}, t)\mathbf{:}\mathbf{E}(\mathbf{r}, \tau)-\boldsymbol{\Sigma}(\mathbf{r}, \tau)\mathbf{:}\mathbf{e}(\mathbf{r}, t)\,,\end{aligned}\tag{24}$$

where $\mathbf{E}$ and $\mathbf{e}$ are the second-order, symmetric strain tensors,

$$\mathbf{E}(\mathbf{r}, \tau)=\{\nabla\mathbf{U}(\mathbf{r}, \tau)+[\nabla\mathbf{U}(\mathbf{r}, \tau)]^{\mathrm{T}}\}/2\,;\quad\text{and}$$

$$\mathbf{e}(\mathbf{r}, t)=\{\nabla\mathbf{u}(\mathbf{r}, t)+[\nabla\mathbf{u}(\mathbf{r}, t)]^{\mathrm{T}}\}/2\,.\tag{25}$$

Kelvin-Voigt medium

The constitutive equation for a generalized, anisotropic, heterogeneous, linear, Kelvin–Voigt material (Malvern 1969, p. 313ff; see also de Hoop 1995, p. 317ff) is

$$\boldsymbol{\sigma}(\mathbf{r}, t)=\sum_{n=0}^{N}\mathbf{C}_n(\mathbf{r})\mathbf{:}\,\frac{d^n\mathbf{e}}{dt^n}\,,\tag{26}$$

where the $\mathbf{C}_{\mathrm{n}}(\mathbf{r})$ are fourth-order tensors with the following transpose-symmetry properties:

$$\mathbf{C}^{\mathrm{T}2134}=\mathbf{C}^{\mathrm{T}1243}=\mathbf{C}^{\mathrm{T}3412}=\mathbf{C}\quad\text{or}$$

$$\hat{\mathbf{j}}\hat{\mathbf{i}}\hat{\mathbf{k}}\hat{\mathbf{l}}C_{ijkl}=\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{l}}\hat{\mathbf{k}}C_{ijkl}=\hat{\mathbf{k}}\hat{\mathbf{l}}\hat{\mathbf{i}}\hat{\mathbf{j}}C_{ijkl}=\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}\hat{\mathbf{l}}C_{ijkl}=\mathbf{C}\,.\tag{27}$$

Note that the transpose operator Tijkl shows how the base vectors are to be permuted. In the above case, where all the base vectors are from same coordinate system, the transpose operation can be applied to the indices (i.e. it shows how the indices are to be permuted). This last fact is not true, however, when the base vectors come from different coordinate systems; that is, when $\mathbf{C}$ is expressed as a 'mixed tensor.'

For the (tentatively) reciprocal problem, we have

$$\boldsymbol{\Sigma}(\mathbf{r}, t'-t)=\sum_{n=0}^{N}(-1)^n\mathbf{C}_n(\mathbf{r})\mathbf{:}\,\frac{d^n\mathbf{E}}{dt^n}\,.\tag{26′}$$

Then the 'virtual strain energy' (eq. 24) becomes

$$\begin{aligned}\psi(\mathbf{r}, t)&=\boldsymbol{\sigma}(\mathbf{r}, t)\mathbf{:}\mathbf{E}(\mathbf{r}, \tau)-\boldsymbol{\Sigma}(\mathbf{r}, \tau)\mathbf{:}\mathbf{e}(\mathbf{r}, t)\\ &=\frac{d}{dt}\left[g\left(\mathbf{e}, \mathbf{E}, \frac{d\mathbf{e}}{dt}, \frac{d\mathbf{E}}{dt}, \frac{d^2\mathbf{e}}{dt^2}, \frac{d^2\mathbf{E}}{dt^2}, \quad\text{etc.}\right)\right],\end{aligned}\tag{28}$$

where $g(\mathbf{e}, \mathbf{E}, \ldots)$ is some scalar function of the strain, the strain rate, etc. (see Appendix B).

For the case $N=2$ we have

$$\begin{aligned}\psi(\mathbf{r}, t)&=\boldsymbol{\sigma}(\mathbf{r}, t)\mathbf{:}\mathbf{E}(\mathbf{r}, \tau)-\boldsymbol{\Sigma}(\mathbf{r}, \tau)\mathbf{:}\mathbf{e}(\mathbf{r}, t)\\ &=\frac{d}{dt}\{\mathbf{C}_1\,\vdots\,[\mathbf{e}\mathbf{E}]+\mathbf{C}_2\,\vdots\,[\dot{\mathbf{e}}\mathbf{E}-\mathbf{e}\dot{\mathbf{E}}]\}\,,\end{aligned}\tag{29}$$

where the symbol $\vdots$ represents the quadruple-dot product: $\mathbf{C}\,\vdots\,\mathbf{D}=C_{ijkl}D_{ijkl}$. Therefore, the time integral of eq. (28) will be zero because, over the region V, $\mathbf{e}$ and its time derivatives are zero for $t\le t_0$, and $\mathbf{E}$ and its time derivatives are zero for $t\ge t_1$ [that is, for $\tau\le\tau_1$ where τ_1 is the onset time for the source $\mathbf{F}_{\mathrm{b}}(\mathbf{r}, \tau)$ with $\tau=t'-t$ and $\tau_1=t'-t_1$].

Maxwell medium

The same result can be obtained for a generalized, anisotropic, heterogeneous, linear, Maxwell material (see, for example, Malvern 1969, p. 313ff or de Hoop 1995, p. 317ff). Its constitutive equation is

$$\mathbf{e}(\mathbf{r}, t)=\sum_{n=0}^{N}\mathbf{S}_n(\mathbf{r})\mathbf{:}\,\frac{d^n\boldsymbol{\sigma}}{dt^n}\,,\tag{30}$$

where the $\mathbf{S}_n(\mathbf{r})$ are fourth-order tensors with the same transpose-symmetry and physical-symmetry properties as the $\mathbf{C}_n(\mathbf{r})$. Then the 'virtual strain energy' (eq. 24) becomes

$$\begin{aligned}\psi(\mathbf{r}, t)&=\boldsymbol{\sigma}(\mathbf{r}, t)\mathbf{:}\mathbf{E}(\mathbf{r}, \tau)-\boldsymbol{\Sigma}(\mathbf{r}, \tau)\mathbf{:}\mathbf{e}(\mathbf{r}, t)\\ &=\frac{d}{dt}\left[G\left(\boldsymbol{\sigma}, \boldsymbol{\Sigma}, \frac{d\boldsymbol{\sigma}}{dt}, \frac{d\boldsymbol{\Sigma}}{dt}, \frac{d^2\boldsymbol{\sigma}}{dt^2}, \frac{d^2\boldsymbol{\Sigma}}{dt^2}, \quad\text{etc.}\right)\right],\end{aligned}\tag{28′}$$

where $G(\boldsymbol{\sigma}, \boldsymbol{\Sigma}, \ldots)$ is some scalar function of the strain, the strain rate, etc. (see Appendix B).

For the case $N=2$ we have

$$\begin{aligned}\psi(\mathbf{r}, t)&=\boldsymbol{\sigma}(\mathbf{r}, t)\mathbf{:}\mathbf{E}(\mathbf{r}, \tau)-\boldsymbol{\Sigma}(\mathbf{r}, \tau)\mathbf{:}\mathbf{e}(\mathbf{r}, t)\\ &=-\frac{d}{dt}\{\mathbf{S}_1\,\vdots\,[\boldsymbol{\sigma}\boldsymbol{\Sigma}]+\mathbf{S}_2\,\vdots\,[\dot{\boldsymbol{\sigma}}\boldsymbol{\Sigma}-\boldsymbol{\sigma}\dot{\boldsymbol{\Sigma}}]\}\,.\end{aligned}\tag{29′}$$

Therefore, the time integral of eq. (28′) will be zero because, over the region V, $\boldsymbol{\sigma}$ and its time derivatives are zero for $t\le t_0$, and $\boldsymbol{\Sigma}$ and its time derivatives are zero for $t\ge t_1$. Consequently, both the Kelvin–Voigt and the Maxwell materials have self-adjoint differential equations when their constitutive equations are substituted into the force-equilibrium equation and reciprocity holds.

CONCLUSIONS

The final result for reciprocity for the generalized, anisotropic, heterogeneous, but linear Kelvin–Voigt and Maxwell solids (which include the Hookean linear elastic solid as a special case) is

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_S\mathbf{n}\cdot[\boldsymbol{\sigma}(\mathbf{r}, t)\cdot\mathbf{U}(\mathbf{r}, t'-t)-\boldsymbol{\Sigma}(\mathbf{r}, t'-t)\cdot\mathbf{u}(\mathbf{r}, t)]\,dSdt\\ &\quad=\int_{-\infty}^{\infty}\int_V[\mathbf{f}_{\mathrm{b}}(\mathbf{r}, t)\cdot\mathbf{U}(\mathbf{r}, t'-t)-\mathbf{F}_{\mathrm{b}}(\mathbf{r}, t'-t)\cdot\mathbf{u}(\mathbf{r}, t)]\,dVdt\\ &\quad=0\end{aligned}\tag{31}$$

[where $\mathbf{n}=\mathbf{n}(\mathbf{r}, t)$ is the unit normal to the surface $S(\mathbf{r}, t)$ at location $\mathbf{r}$ and time t] because the first integral is zero for homogeneous boundary conditions (e.g. $\mathbf{u}=\mathbf{0}$ and $\mathbf{U}=\mathbf{0}$ or $\boldsymbol{n}\cdot\boldsymbol{\sigma}=0$ and $\boldsymbol{n}\cdot\boldsymbol{\Sigma}=0$ on $S(\mathbf{r}, t)$ for all t). The last integral is the general statement of reciprocity for all viscoelastic media of the Kelvin–Voigt and Maxwell type with tensorial 'density' terms in the inertial force and it can be written as

$$\boxed{\int_{-\infty}^{\infty}\int_V[\mathbf{f}_{\mathrm{b}}(\mathbf{r}, t'-t)\cdot\mathbf{U}(\mathbf{r}, t)-\mathbf{F}_{\mathrm{b}}(\mathbf{r}, t'-t)\cdot\mathbf{u}(\mathbf{r}, t)]\,dVdt=0\,.}\tag{32}$$

This reciprocity statement is exactly the same as that for linear-elastic media (Gangi 1970, 1980; Aki & Richards 1980, pp. 25–27; de Hoop 1995, pp. 437–441): now, however, the displacements $\mathbf{U}(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$ are the displacements in viscoelastic media (see also Gurtin & Sternberg 1962; de Hoop 1966).

The representation theorems, or Green's theorems, for viscoelastic Kelvin–Voigt and Maxwell media can readily be derived using the above reciprocity relationship (Gangi 1970; Aki & Richards 1980, pp. 28–29), provided the medium is linear.

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APPENDIX A: CONSTITUTIVE EQUATIONS FOR THE INERTIAL FORCE

The inertial-force density is usually set equal to the product of the mass density and the acceleration of the mass of a particle or of the mass in a unit volume. In most cases it is assumed—or has been shown experimentally—that the inertial force is in the same direction as the acceleration. However, this is not a necessary condition and other possibilities can be allowed. In the most general case, the inertial force can be postulated to be a non-linear function of the particle displacement (or position), its velocity (or time derivative of its displacement or position), its acceleration and even higher-order time derivatives of its position or displacement. While these conditions can be postulated, whether or not they hold must be determined either empirically or on the basis of the microstructure of the medium.

The inertial-force density is a vector and it need not be collinear with any of the other vector quantities (the displacement and its time derivatives). In the following we will assume that the inertial force depends only on displacement and its first two time derivatives; namely, the velocity and acceleration. We further assume that the inertial force is a linear function of these quantities. Therefore, it can be written as

$$\mathbf{f}(\mathbf{r}, t)=\left[\mathbf{R}_2 \frac{d^2}{dt^2}+\mathbf{R}_1 \frac{d}{dt}+\mathbf{R}_0\right]\cdot\mathbf{u}(\mathbf{r}, t) \tag{A1}$$

[using $\mathbf{f}(\mathbf{r}, t)$ and $\mathbf{F}(\mathbf{r}, t)$ for $\mathbf{f}_i(\mathbf{r}, t)$ and $\mathbf{F}_i(\mathbf{r}, t)$] for the original problem, and as

$$\mathbf{F}(\mathbf{r}, \tau)=\left[\tilde{\mathbf{R}}_2 \frac{d^2}{d\tau^2}+\tilde{\mathbf{R}}_1 \frac{d}{d\tau}+\tilde{\mathbf{R}}_0\right]\cdot\mathbf{U}(\mathbf{r}, \tau) \tag{A2}$$

for the adjoint problem. In the above, the second-order tensors $\mathbf{R}_n$ and $\tilde{\mathbf{R}}_n$ $(n=0, 1, 2)$ may be functions of position, $\mathbf{r}$, but they are stationary in time; that is, they are independent of time. Because of the second-order tensors, the inertial force will not necessarily be collinear with the displacement or its time derivatives.

With the above definitions, the 'work term' $w(\mathbf{r}, t)$ becomes

$$w(\mathbf{r}, t)=\mathbf{U}(\mathbf{r}, \tau)\cdot\mathbf{f}(\mathbf{r}, t)-\mathbf{u}(\mathbf{r}, t)\cdot\mathbf{F}(\mathbf{r}, \tau)$$

$$=\sum_{n=0}^{2}\left[\mathbf{U}\cdot\mathbf{R}_n\cdot\frac{d^n\mathbf{u}}{dt^n}-\mathbf{u}\cdot\tilde{\mathbf{R}}_n\cdot\frac{d^n\mathbf{U}}{d\tau^n}\right]=\sum_{n=0}^{2} w_n(\mathbf{r}, t) \tag{A3}$$

(recall, $\tau=t'-t$, where t' is a constant).

The $n=0$ term gives

$$w_0(\mathbf{r}, t)=\mathbf{U}\cdot\mathbf{R}_0\cdot\mathbf{u}-\mathbf{u}\cdot\tilde{\mathbf{R}}_0\cdot\mathbf{U}=(\mathbf{R}_0^{\mathrm{T}}-\tilde{\mathbf{R}}_0):(\mathbf{uU})\,, \tag{A4}$$

and this term is zero if $\mathbf{R}_0^{\mathrm{T}}=\tilde{\mathbf{R}}_0$, where $\mathbf{R}_0^{\mathrm{T}}$ is the transpose of the tensor $\mathbf{R}_0$.

The $n=1$ term gives

$$w_1(\mathbf{r}, t)=\mathbf{U}\cdot\mathbf{R}_1\cdot\frac{d\mathbf{u}}{dt}-\mathbf{u}\cdot\tilde{\mathbf{R}}_1\cdot\frac{d\mathbf{U}}{d\tau}=\mathbf{R}_1\mathbf{:U}\frac{d\mathbf{u}}{dt}-\tilde{\mathbf{R}}_1\mathbf{:u}\frac{d\mathbf{U}}{d\tau}\,. \tag{A5}$$

However, if

$$\mathbf{R}_1^{\mathrm{T}}=\tilde{\mathbf{R}}_1\,, \tag{A6}$$

then we have (with $d\mathbf{R}_1/dt = 0$ and $d/dt = -d/d\tau$)

$$w_1(\mathbf{r}, t)=\mathbf{R}_1\mathbf{:}\frac{d[\mathbf{Uu}]}{dt}=\frac{d[\mathbf{R}_1\mathbf{:Uu}]}{dt}\,. \tag{A7}$$

Therefore

$$\int_V dV\int_{-\infty}^{\infty} w_1\,dt=\int_V dV\mathbf{R}_1\mathbf{:}[\mathbf{Uu}]\big|_{t=-\infty}^{t=\infty}=0\,, \tag{A8}$$

because $\mathbf{u}=0$ for $-\infty\le t\le t_0$ and $\mathbf{U}=0$ for $\infty\ge t\ge t_1$ throughout V, where t_0 is the onset time for the source $\mathbf{f}$ and t_1 is the 'end time' of the source $\mathbf{F}(\mathbf{r}, \tau)$; that is, $\mathbf{F}=0$ for $\tau\le\tau_1=t'-t_1$ (or $t\ge t_1$).

The $n=2$ term gives

$$w_2(\mathbf{r},\,t)=\mathbf{U}\cdot\mathbf{R}_2\cdot\frac{d^2\mathbf{u}}{dt^2}-\mathbf{u}\cdot\tilde{\mathbf{R}}_2\cdot\frac{d^2\mathbf{U}}{d\tau^2}=\mathbf{R}_2\mathbf{:U}\,\frac{d^2\mathbf{u}}{dt^2}-\tilde{\mathbf{R}}_2\mathbf{:u}\,\frac{d^2\mathbf{U}}{d^2\tau}\,, \tag{A9}$$

and, if

$$\mathbf{R}_2^{\mathrm{T}}=\tilde{\mathbf{R}}_2\,, \tag{A10}$$

we have (with $d\mathbf{R}_2/dt=0$)

$$w_2(\mathbf{r},\,t)=\mathbf{R}_2\mathbf{:}\frac{d}{dt}\left[\mathbf{U}\,\frac{d\mathbf{u}}{dt}-\frac{d\mathbf{U}}{dt}\,\mathbf{u}\right]=\frac{d}{dt}\left[\mathbf{R}_2\mathbf{:}\left(\mathbf{U}\,\frac{d\mathbf{u}}{dt}-\frac{d\mathbf{U}}{dt}\,\mathbf{u}\right)\right] \tag{A11}$$

and

$$\int_V dV\int_{-\infty}^{\infty}w_2\,dt=\int_V dV\left[\mathbf{R}_2\mathbf{:}\left(\mathbf{U}\,\frac{d\mathbf{u}}{dt}-\frac{d\mathbf{U}}{dt}\,\mathbf{u}\right)\right]\Bigg|_{t=-\infty}^{t=\infty}=0\,, \tag{A12}$$

because $\mathbf{u}=0$ and $d\mathbf{u}/dt=0$ for $-\infty\le t\le t_0$, and $\mathbf{U}=0$ and $d\mathbf{U}/dt=0$ for $\infty\ge t\ge t_1$ throughout V.

In order to have a self-adjoint problem, we require that $\mathbf{R}_n=\tilde{\mathbf{R}}_n$, but we already have the conditions that $\mathbf{R}_n^{\mathrm{T}}=\tilde{\mathbf{R}}_n$ must hold if general reciprocity is to hold; that is, if there is to be zero contribution from the $w(\mathbf{r},\,t)$ term. Therefore, for self-adjoint problems, it is necessary that

$$\mathbf{R}_n=\mathbf{R}_n^{\mathrm{T}}=\tilde{\mathbf{R}}_n\,, \tag{A13}$$

or that the second-order tensors, $\mathbf{R}_n$, are symmetric if reciprocity is to hold.

Example of a non-self-adjoint case

It is known that in the case of a rotating body (say, the Earth) the inertial-force densities are given by (see, for example, Dahlen & Tromp 1998, p. 44)

$$\mathbf{f}(\mathbf{r},\,t)=\rho(\mathbf{r})[\dot{\mathbf{v}}(\mathbf{r})+\boldsymbol{\Omega}(\mathbf{r})\times\mathbf{v}(\mathbf{r})+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})]\,, \tag{A14}$$

where $\rho(\mathbf{r})$ is the density, $\dot{\mathbf{v}}(\mathbf{r})$ is the acceleration, $\mathbf{v}(\mathbf{r})$ is the velocity, $\mathbf{r}$ is the position vector and $\boldsymbol{\Omega}(\mathbf{r})$ is the angular-rotation vector. The first term is the usual acceleration term, the second is the Coriolis force, and the third is the centripetal force. Using the third-order alternating tensor of Levi–Civita, $\boldsymbol{\varepsilon}$, it can be shown that the third term is equal to

$$\mathbf{R}_0\cdot\mathbf{r}/\rho(\mathbf{r})=\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})=(\boldsymbol{\Omega}\boldsymbol{\Omega}-|\boldsymbol{\Omega}|^2\mathbf{I})\cdot\mathbf{r}\,, \tag{A15}$$

where $\mathbf{I}$ is the second-order identity tensor (it is an identity operator under the scalar or dot product), $\boldsymbol{\Omega}\boldsymbol{\Omega}=\hat{\mathbf{i}}\hat{\mathbf{j}}\,\Omega_i\Omega_j$ is the second-order tensor generated by the tensor product of the rotation vector with itself, and $|\boldsymbol{\Omega}|^2=\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}$ is the square magnitude of the angular-rotation vector.

The third-order tensor, $\boldsymbol{\varepsilon}$, is the alternating tensor of Levi–Civita, which satisfies the transposition (or symmetry) rules

$$\boldsymbol{\varepsilon}^{\mathrm{T}213}=\boldsymbol{\varepsilon}^{\mathrm{T}132}=\boldsymbol{\varepsilon}^{\mathrm{T}321}=-\boldsymbol{\varepsilon}=-\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}\,e_{ijk}\,, \tag{A16}$$

where $e_{123}=1$, the $e_{\mathrm{ijk}}=0$ or ±1 and, for example,

$$\boldsymbol{\varepsilon}^{\mathrm{T}213}=\hat{\mathbf{j}}\hat{\mathbf{i}}\hat{\mathbf{k}}\,e_{ijk}=\hat{\mathbf{2}}\hat{\mathbf{1}}\hat{\mathbf{3}}+\hat{\mathbf{3}}\hat{\mathbf{2}}\hat{\mathbf{1}}+\hat{\mathbf{1}}\hat{\mathbf{3}}\hat{\mathbf{2}}-\hat{\mathbf{1}}\hat{\mathbf{2}}\hat{\mathbf{3}}-\hat{\mathbf{2}}\hat{\mathbf{3}}\hat{\mathbf{1}}-\hat{\mathbf{3}}\hat{\mathbf{1}}\hat{\mathbf{2}}=-\boldsymbol{\varepsilon}\,. \tag{A17}$$

As usual, the summation rule is used for repeated indices. Note that the transposition rule works on the base vectors of the third-order tensor, $\boldsymbol{\varepsilon}$, and not necessarily on the indices of the scalar components. If the base vectors are all from the same coordinate system (and they need not be, but then the values of e_{ijk} are not necessarily 0 and ±1), the transposition rule also works on the indices (see, for example, Gangi 1970).

The second-order tensor, $\mathbf{R}_0$, is symmetric so that the centripetal force satisfies the conditions necessary for reciprocity to hold.

However, the Coriolis force term

$$\mathbf{R}_1\cdot\mathbf{v}/\rho(\mathbf{r})=\boldsymbol{\Omega}\times\mathbf{v}=\boldsymbol{\varepsilon}\mathbf{:}\boldsymbol{\Omega}\mathbf{v}=-(\boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega})\cdot\mathbf{v} \tag{A18}$$

has the property

$$\mathbf{R}_1=-\rho(\mathbf{r})\boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega}=-\mathbf{R}_1^{\mathrm{T}}=-\tilde{\mathbf{R}}_1\,. \tag{A19}$$

Therefore, if we wish reciprocity to hold for the adjoint problem of a rotating body, the adjoint problem must represent the same body rotating in the opposite direction (see Dahlen & Tromp 1998; p. 135). Because $\mathbf{R}_0$ depends on the square of the angular-rotation vector, it is invariant when the rotation is reversed. The third tensor, $\mathbf{R}_2$, is equal to $\rho(\mathbf{r})\mathbf{I}$, the scalar mass density times the identity tensor, which is independent of the rotation altogether.

In the following we will assume that the problems being investigated are self-adjoint and that the second-order tensors, $\mathbf{R}_n$ ($n=0$, 1, 2) are symmetric.

An example of a medium that has inertial terms depending on velocity and displacements as well as acceleration, and for which the second-order tensors are symmetric (and distinct from identity tensors) would be one made up of three component seismometers with different masses, spring constants and damping for each of the components. It is possible to conceive of crystal structures which also have these properties, but, to the best of my knowledge, no one has measured such materials yet.

APPENDIX B: CONSTITUTIVE EQUATIONS FOR STRESS AND STRAIN

The stress/strain constitutive equations for the generalized, anisotropic, heterogeneous, linear, Kelvin–Voigt and Maxwell materials are obtained when the constitutive tensor is a function only of the stress and strain and their time derivatives. While these conditions can be postulated, whether or not they hold must be determined either empirically or on the basis of the microstructure of the medium.

These constitutive equations affect the 'virtual-strain-energy' term (see eq. 24):

$$\begin{aligned}\psi(\mathbf{r},\,t)&=\boldsymbol{\sigma}(\mathbf{r},\,t)\mathbf{:}\nabla\mathbf{U}(\mathbf{r},\,\tau)-\boldsymbol{\Sigma}(\mathbf{r},\,\tau)\mathbf{:}\nabla\mathbf{u}(\mathbf{r},\,t)\\&=\boldsymbol{\sigma}(\mathbf{r},\,t)\mathbf{:E}(\mathbf{r},\,\tau)-\boldsymbol{\Sigma}(\mathbf{r},\,\tau)\mathbf{:e}(\mathbf{r},\,t)\,.\end{aligned} \tag{B1}$$

The gradients of the displacements, which are second-order tensors composed of skew-symmetric as well as symmetric tensors, can be replaced by the symmetric strain tensors because the stress tensors are symmetric—at least for all continua that have zero net moment. We assume that this is the case in the following.

Kelvin-Voigt medium

The constitutive equations for the generalized Kelvin–Voigt medium are (see eq. 26)

$$\boldsymbol{\sigma}(\mathbf{r}, t) = \sum_{n=0}^{N} \mathbf{C}_n(\mathbf{r}) : \frac{d^n \mathbf{e}}{dt^n}, \qquad \boldsymbol{\Sigma}(\mathbf{r}, \tau) = \sum_{n=0}^{N} \tilde{\mathbf{C}}_n(\mathbf{r}) : \frac{d^n \mathbf{E}}{d\tau^n}, \tag{B2}$$

where we distinguish between the fourth-order moduli tensors, $\mathbf{C}_n$ and $\tilde{\mathbf{C}}_n$, for the original and adjoint problems, respectively. Then the 'virtual-strain-energy' term becomes

$$\psi(\mathbf{r}, t) = \sum_{n=0}^{N} \left[\mathbf{C}_n(\mathbf{r}) \vdots \frac{d^n \mathbf{e}}{dt^n} \mathbf{E} - \tilde{\mathbf{C}}_n(\mathbf{r}) \vdots \frac{d^n \mathbf{E}}{d\tau^n} \mathbf{e} \right] = \sum_{n=0}^{N} \psi_n(\mathbf{r}, t), \tag{B3}$$

and, using the fact that $d^n\mathbf{E}/d\tau^n = (-1)^n d^n\mathbf{E}/dt^n$, we obtain

$$\psi(\mathbf{r}, t) = \sum_{n=0}^{N} \left[\mathbf{C}_n(\mathbf{r}) \vdots \frac{d^n \mathbf{e}}{dt^n} \mathbf{E} - (-1)^n \tilde{\mathbf{C}}_n(\mathbf{r}) \vdots \frac{d^n \mathbf{E}}{dt^n} \mathbf{e} \right]. \tag{B4}$$

For $n=0$, we have, with $\mathbf{C}_0 = \tilde{\mathbf{C}}_0^{\mathrm{T3412}}$,

$$\psi_0(\mathbf{r}, t) = [\mathbf{C}_0 - \tilde{\mathbf{C}}_0^{\mathrm{T3412}}] \vdots \mathbf{e}\mathbf{E} = 0. \tag{B5}$$

For $n=1$, we have

$$\psi_1(\mathbf{r}, t) = \left[\mathbf{C}_1(\mathbf{r}) \vdots \frac{d\mathbf{e}}{dt} \mathbf{E} + \tilde{\mathbf{C}}_1(\mathbf{r}) \vdots \frac{d\mathbf{E}}{dt} \mathbf{e} \right], \tag{B6}$$

or, with $\mathbf{C}_1 = \tilde{\mathbf{C}}_1^{\mathrm{T3412}}$ and $d\mathbf{C}_1/dt = 0$,

$$\psi_1(\mathbf{r}, t) = \frac{d}{dt} [\mathbf{C}_1(\mathbf{r}) \vdots \mathbf{e}\mathbf{E}]. \tag{B7}$$

For $n=2$, we have

$$\psi_2(\mathbf{r}, t) = \left[\mathbf{C}_2(\mathbf{r}) \vdots \frac{d^2 \mathbf{e}}{dt^2} \mathbf{E} - \tilde{\mathbf{C}}_2(\mathbf{r}) \vdots \frac{d^2 \mathbf{E}}{dt^2} \mathbf{e} \right], \tag{B8}$$

or, with $\mathbf{C}_2 = \tilde{\mathbf{C}}_2^{\mathrm{T3412}}$ and $d\mathbf{C}_2/dt = 0$,

$$\psi_2(\mathbf{r}, t) = \frac{d}{dt} \left[\mathbf{C}_2(\mathbf{r}) \vdots \left(\frac{d\mathbf{e}}{dt} \mathbf{E} - \mathbf{e} \frac{d\mathbf{E}}{dt} \right) \right]. \tag{B9}$$

For general $n = m+1$, we have, with $\mathbf{C}_{m+1} = \tilde{\mathbf{C}}_{m+1}^{\mathrm{T3412}}$ and $d\mathbf{C}_{m+1}/dt = 0$,

$$\psi_{m+1}(\mathbf{r}, t) = \frac{d}{dt} \left[\mathbf{C}_{m+1} \vdots \left(\frac{d^m \mathbf{e}}{dt^m} \mathbf{E} + \frac{d^{m-1} \mathbf{e}}{dt^{m-1}} \frac{d\mathbf{E}}{d\tau} + \cdots + \frac{d\mathbf{e}}{dt} \frac{d^{m-1} \mathbf{E}}{d\tau^{m-1}} + \mathbf{e} \frac{d^m \mathbf{E}}{d\tau^m} \right) \right]. \tag{B10}$$

Because the 'virtual strain energy' is the time derivative of some scalar function, it can be included with the time derivative of the $\mathbf{q}(\mathbf{r}, t)$ term in eq. (3 or 4). Therefore, the Kelvin–Voigt medium gives a self-adjoint problem provided that the moduli tensors have the appropriate transpose symmetries. If, throughout the region V, the strain, $\mathbf{e}$, and all its time derivatives are zero for $t \le t_0$, and $\mathbf{E}$ and its time derivatives are all zero for $t \ge t_1$, then the virtual strain energy term goes to zero in eq. (9) and the Kelvin–Voigt medium manifests reciprocity.

Maxwell medium

The constitutive equations for the generalized Maxwell medium are (see eq. 30)

$$\mathbf{e}(\mathbf{r}, t) = \sum_{n=0}^{N} \mathbf{S}_n(\mathbf{r}) : \frac{d^n \boldsymbol{\sigma}}{dt^n}, \qquad \mathbf{E}(\mathbf{r}, \tau) = \sum_{n=0}^{N} \tilde{\mathbf{S}}_n(\mathbf{r}) : \frac{d^n \boldsymbol{\Sigma}}{d\tau^n}, \tag{B11}$$

where we distinguish between the fourth-order compliance tensors, $\mathbf{S}_n$ and $\tilde{\mathbf{S}}_n$, for the original and adjoint problems, respectively. Then the 'virtual-strain-energy' term becomes

$$\psi(\mathbf{r}, t) = \sum_{n=0}^{N} \left[\tilde{\mathbf{S}}_n(\mathbf{r}) \vdots \frac{d^n \boldsymbol{\Sigma}}{d\tau^n} \boldsymbol{\sigma} - \mathbf{S}_n(\mathbf{r}) \vdots \frac{d^n \boldsymbol{\sigma}}{dt^n} \boldsymbol{\Sigma} \right] = \sum_{n=0}^{N} \psi_n(\mathbf{r}, t), \tag{B12}$$

and, using the fact that $d^n\boldsymbol{\Sigma}/d\tau^n = (-1)^n d^n\boldsymbol{\Sigma}/dt^n$, we obtain

$$\psi(\mathbf{r}, t) = -\sum_{n=0}^{N} \left[\mathbf{S}_n(\mathbf{r}) \vdots \frac{d^n \boldsymbol{\sigma}}{dt^n} \boldsymbol{\Sigma} - (-1)^n \tilde{\mathbf{S}}_n(\mathbf{r}) \vdots \frac{d^n \boldsymbol{\Sigma}}{dt^n} \boldsymbol{\sigma} \right]. \tag{B13}$$

Except for the minus sign, this is the same as eq. (B3) with **e, E**, $\mathbf{C}_n$ and $\tilde{\mathbf{C}}_n$ replaced by $\boldsymbol{\sigma}$, $\boldsymbol{\Sigma}$, $\mathbf{S}_n$ and $\tilde{\mathbf{S}}_n$.

Consequently, for general $n = m+1$, we have, with $\mathbf{S}_{m+1} = \tilde{\mathbf{S}}_{m+1}^{\mathrm{T3412}}$ and $d\mathbf{S}_{m+1}/dt = 0$,

$$\psi_{m+1}(\mathbf{r}, t) = \frac{d}{dt} \left[\mathbf{S}_{m+1} \vdots \left(\frac{d^m \boldsymbol{\sigma}}{dt^m} \boldsymbol{\Sigma} + \frac{d^{m-1} \boldsymbol{\sigma}}{dt^{m-1}} \frac{d\boldsymbol{\Sigma}}{d\tau} + \cdots + \frac{d\boldsymbol{\sigma}}{dt} \frac{d^{m-1} \boldsymbol{\Sigma}}{d\tau^{m-1}} + \boldsymbol{\sigma} \frac{d^m \boldsymbol{\Sigma}}{d\tau^m} \right) \right]. \tag{B14}$$

Because the 'virtual strain energy' is the time derivative of some scalar function, it can be included with the time derivative of the $\mathbf{q}(\mathbf{r}, t)$ term in eq. (3 or 4). Therefore, the Maxwell medium gives a self-adjoint problem provided that the moduli tensors have the appropriate transpose symmetries. If, throughout the region V, the stress, $\boldsymbol{\sigma}$, and all its time derivatives are zero for $t \le t_0$, and $\boldsymbol{\Sigma}$ and its time derivatives are all zero for $t \ge t_1$, then the Maxwell medium manifests reciprocity.