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New Einstein-Sasaki and Einstein Spaces from Kerr-de Sitter

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ABSTRACT

In this paper, which is an elaboration of our results in hep-th/0504225, we construct new Einstein-Sasaki spaces $L^{p,q,r_1,\cdots,r_{n-1}}$ in all odd dimensions $D = 2n+1 \geq 5$. They arise by taking certain BPS limits of the Euclideanised Kerr-de Sitter metrics. This yields local Einstein-Sasaki metrics of cohomogeneity n, with toric $U(1)^{n+1}$ principal orbits, and n real non-trivial parameters. By studying the structure of the degenerate orbits we show that for appropriate choices of the parameters, characterised by the $(n+1)$ coprime integers $(p,q,r_1,\ldots,r_{n-1})$, the local metrics extend smoothly onto complete and non-singular compact Einstein-Sasaki manifolds $L^{p,q,r_1,\cdots,r_{n-1}}$. We also construct new complete and non-singular compact Einstein spaces $\Lambda^{p,q,r_1,\cdots,r_n}$ in $D = 2n+1$ that are not Sasakian, by choosing parameters appropriately in the Euclideanised Kerr-de Sitter metrics when no BPS limit is taken.

Contents

1 Introduction **2**

2 Five-Dimensional Einstein-Sasaki Spaces **4**

2.1 The local five-dimensional metrics 4

2.2 Global structure of the five-dimensional solutions 6

2.3 Quasi-regular examples 9

2.4 Volumes and the Bishop bound 10

2.5 The $u \leftrightarrow -u$ symmetry 11

2.6 The $u = 0$ case 12

2.7 Curvature invariants 13

3 Higher-Dimensional Einstein-Sasaki Spaces **13**

4 Non-Sasakian Einstein Spaces **16**

4.1 Odd dimensions 17

4.2 Even dimensions 19

5 Conclusions **21**

A Singular Limits **22**

1 Introduction

Compact Einstein spaces have long been of interest in mathematics and physics. One of their principal applications in physics has been in higher-dimensional supergravity, string theory and M-theory, where they can provide backgrounds for reductions to lower-dimensional spacetimes. Of particular interest are Einstein spaces that admit Killing spinors, since these can provide supersymmetric backgrounds. In recent times, one of the most important applications of this type has been to supersymmetric backgrounds $\mathrm{AdS}_5 \times K_5$ of type IIB string theory, where K_5 is a compact Einstein space admitting Killing spinors. Such configurations provide examples for studying the AdS/CFT Correspondence, which relates bulk properties of the dimensionally-reduced AdS supergravity to a superconformal field theory on the four-dimensional boundary of AdS_5 [1–3].

The most studied cases have been when K_5 is the standard round 5-sphere, which is associated with an $\mathcal{N} = 4$ superconformal boundary field theory. Another case that has been extensively studied is when K_5 is the space $T^{1,1} = (SU(2) \times SU(2))/U(1)$. Until recently, these two homogeneous spaces were the only explicitly known examples of five-dimensional Einstein spaces admitting Killing spinors, although the general theory of Einstein-Sasaki spaces, which are odd-dimensional Einstein spaces admitting a Killing spinor, was well established, and some existence proofs were known (see, for example, [4, 5]).

In recent work by Gauntlett, Martelli, Sparks and Waldram, the picture changed dramatically with the construction of infinitely many explicit examples of Einstein-Sasaki spaces in five [6] and higher [7] odd dimensions. Their construction was based on some earlier results in [8, 9], in which local Einstein-Kähler metrics of cohomogeneity 1 were obtained as line bundles over Einstein-Kähler bases. Using the well-known result that an Einstein-Sasaki metric can be written as circle bundle over an Einstein-Kähler metric, this yielded the new local metrics discussed in [6, 7]. In the case of five dimensions, the resulting Einstein-Sasaki metrics are characterised by a non-trivial real parameter. They have cohomogeneity 1, with principal orbits $SU(2) \times U(1) \times U(1)$. In general these local metrics become singular where the orbits degenerate, but if the real parameter is appropriately restricted to rational values, the metric at the degeneration surfaces extends smoothly onto a complete and non-singular compact manifold. The resulting Einstein-Sasaki spaces were denoted by $Y^{p,q}$ in [6], where p and q are coprime integers with $q < p$. Further generalisations were obtained in [10, 11]

It was shown in [12] that the Einstein-Sasaki spaces $Y^{p,q}$, and the higher-dimensional generalisations obtained in [7], could be obtained by taking certain limits of the Euclideanised Kerr-de Sitter rotating black hole metrics found in five dimensions in [13], and in all higher dimensions in [14, 15]. Specifically, the limit considered in [12] involved setting the n independent rotation parameters of the general $(2n+1)$-dimensional Kerr-de Sitter metrics equal, and then sending this parameter to a limiting value that corresponds, in the Lorentzian regime, to having rotation at the speed of light at infinity. [This BPS scaling limit for the black hole metrics with two equal angular momenta was recently studied in [16].]

In a recent paper [17], we showed that vastly greater classes of complete and non-singular Einstein-Sasaki spaces could be obtained by starting from the general Euclideanised Kerr-de Sitter metrics, with unequal rotation parameters, and again taking an appropriate limit under which the metrics become locally Einstein-Sasakian. In fact, this limit can be understood as a Euclidean analogue of the BPS condition that leads to supersymmetric black hole metrics. In five dimensions, this construction leads to local Einstein-Sasaki metrics of cohomogeneity 2, with $U(1)\times U(1)\times U(1)$ principal orbits, and two non-trivial real parameters. In dimension $D=2n+1$, the local Einstein-Sasaki metrics have cohomogeneity n, with $U(1)^{n+1}$ principal orbits, and they are characterised by n non-trivial real parameters. In general the metrics are singular, but by studying the behaviour of the collapsing orbits at endpoints of the ranges of the inhomogeneous coordinates, we showed in [17] that the metrics extend smoothly onto complete and non-singular compact manifolds if the real parameters are appropriately restricted to be rational. This led to new classes of Einstein-Sasaki spaces, denoted by $L^{p,q,r_1,\cdots,r_{n-1}}$ in $2n+1$ dimensions, where $(p,q,r_1,\ldots,r_{n-1})$ are $(n+1)$ coprime integers. If the integers are specialised appropriately, the rotation parameters become equal and the spaces reduce to those obtained previously in [6] and [7]. For example, in five dimensions our general class of Einstein-Sasaki spaces $L^{p,q,r}$ reduce to those in [6] if $p+q=2r$, with $Y^{p,q}=L^{p-q,p+q,p}$ [17].

In this paper, we elaborate on some of our results that appeared in [17], and give further details about the Einstein-Sasaki spaces that result from taking BPS limits of the Euclideanised Kerr-de Sitter metrics. We also give details about new complete and non-singular compact Einstein spaces that are not Sasakian, which we also discussed briefly in [17]. These again arise by making special choices of the non-trivial parameters in the Euclideanised Kerr-de Sitter metrics, but this time without first having taken a BPS limit. The five-dimensional Einstein-Sasaki spaces are discussed in section 2, and the higher-dimensional Einstein-Sasaki spaces in section 3. In section 4 we discuss the non-Sasakian Einstein spaces, and the paper ends with conclusions in section 5. In an appendix, we discuss certain singular BPS limits, where not all the rotation parameters are taken to limiting values. This discussion also encompasses the case of even-dimensional Kerr-de Sitter metrics, which do not give rise to non-singular spaces with Killing spinors.

2 Five-Dimensional Einstein-Sasaki Spaces

2.1 The local five-dimensional metrics

Our starting point is the five-dimensional Kerr-AdS metric found in [13], which is given by

$$\begin{aligned} ds_5^2 &= -\frac{\Delta}{\rho^2}\Big[dt - \frac{a\,\sin^2\theta}{\Xi_a}\,d\phi - \frac{b\,\cos^2\theta}{\Xi_b}\,d\psi\Big]^2 + \frac{\Delta_\theta\,\sin^2\theta}{\rho^2}\Big[a\,dt - \frac{r^2+a^2}{\Xi_a}\,d\phi\Big]^2 \\ &\quad + \frac{\Delta_\theta\,\cos^2\theta}{\rho^2}\Big[b\,dt - \frac{r^2+b^2}{\Xi_b}\,d\psi\Big]^2 + \frac{\rho^2\,dr^2}{\Delta} + \frac{\rho^2\,d\theta^2}{\Delta_\theta} \\ &\quad + \frac{(1+g^2r^2)}{r^2\,\rho^2}\Big[a\,b\,dt - \frac{b\,(r^2+a^2)\,\sin^2\theta}{\Xi_a}\,d\phi - \frac{a\,(r^2+b^2)\,\cos^2\theta}{\Xi_b}\,d\psi\Big]^2\,, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \Delta &\equiv \frac{1}{r^2}\,(r^2+a^2)(r^2+b^2)(1+g^2r^2) - 2m\,, \\ \Delta_\theta &\equiv 1 - g^2a^2\,\cos^2\theta - g^2b^2\,\sin^2\theta\,, \\ \rho^2 &\equiv r^2 + a^2\,\cos^2\theta + b^2\,\sin^2\theta\,, \\ \Xi_a &\equiv 1 - g^2a^2\,, \qquad \Xi_b \equiv 1 - g^2b^2\,. \end{aligned} \tag{2.2}$$

The metric satisfies $R_{\mu\nu} = -4g^2\,g_{\mu\nu}$. As shown in [18], the energy and angular momenta are given by

$$E = \frac{\pi\,m\,(2\Xi_a + 2\Xi_b - \Xi_a\,\Xi_b)}{4\Xi_a^2\,\Xi_b^2}\,, \qquad J_a = \frac{\pi\,m\,a}{2\Xi_a^2\,\Xi_b}\,, \qquad J_b = \frac{\pi\,m\,b}{2\Xi_b^2\,\Xi_a}\,. \tag{2.3}$$

As discussed in [19], the BPS limit can be found by studying the eigenvalues of the Bogomol'nyi matrix arising in the AdS superalgebra from the anticommutator of the supercharges. In $D=5$, these eigenvalues are then proportional to

$$E \pm gJ_a \pm gJ_b\,. \tag{2.4}$$

A BPS limit is achieved when one or more of the eigenvalues vanishes. For just one zero eigenvalue, the four cases in (2.4) are equivalent under reversals of the angular velocities, so we may without loss of generality consider $E - gJ_a - gJ_b = 0$. From (2.3), we see that this is achieved by taking a limit in which ga and gb tend to unity, namely, by setting $ga = 1 - \frac{1}{2}\epsilon\alpha$, $gb = 1 - \frac{1}{2}\epsilon\beta$, rescaling m according to $m = m_0\epsilon^3$, and sending ϵ to zero. As we shall see, the metric remains non-trivial in this limit. An equivalent discussion in the Euclidean regime leads to the conclusion that in the corresponding limit, one obtains five-dimensional Einstein metrics admitting a Killing spinor.

We perform a Euclideanisation of (2.1) by making the analytic continuations

$$t \to \mathrm{i}t\,, \quad g \to \frac{\mathrm{i}}{\sqrt{\lambda}}\,, \quad a \to \mathrm{i}a\,, \quad b \to \mathrm{i}b\,, \tag{2.5}$$

and then take the BPS limit by setting

$$\begin{aligned} &a = \lambda^{-\frac{1}{2}}(1 - \tfrac{1}{2}\alpha\,\epsilon)\,, \quad b = \lambda^{-\frac{1}{2}}(1 - \tfrac{1}{2}\beta\,\epsilon)\,, \\ &r^2 = \lambda^{-1}(1 - x\epsilon)\,, \quad m = \tfrac{1}{2}\lambda^{-1}\mu\epsilon^3 \end{aligned} \tag{2.6}$$

and sending $\epsilon \to 0$. The metric becomes

$$\lambda\, ds_5^2 = (d\tau + \sigma)^2 + ds_4^2\,, \tag{2.7}$$

where

$$\begin{aligned} ds_4^2 &= \frac{\rho^2\, dx^2}{4\Delta_x} + \frac{\rho^2\, d\theta^2}{\Delta_\theta} + \frac{\Delta_x}{\rho^2}(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi)^2 \\ &\quad + \frac{\Delta_\theta \sin^2\theta\cos^2\theta}{\rho^2}(\frac{\alpha - x}{\alpha}d\phi - \frac{\beta - x}{\beta}d\psi)^2\,, \\ \sigma &= \frac{(\alpha - x)\sin^2\theta}{\alpha}d\phi + \frac{(\beta - x)\cos^2\theta}{\beta}d\psi\,, \\ \Delta_x &= x(\alpha - x)(\beta - x) - \mu\,, \quad \rho^2 = \Delta_\theta - x\,, \\ \Delta_\theta &= \alpha\,\cos^2\theta + \beta\,\sin^2\theta\,. \end{aligned} \tag{2.8}$$

A straightforward calculation shows that the four-dimensional metric in (2.8) is Einstein. Note that the parameter μ is trivial, and can be set to any non-zero constant, say, $\mu = 1$, by rescaling α, β and x. The metrics depend on two non-trivial parameters, which we can take to be α and β at fixed μ. However, it is sometimes convenient to retain μ, allowing it to be determined as the product of the three roots x_i of Δ_x.

It is also straightforward to verify that the four-dimensional Einstein metric in (2.8) is Kähler, with Kähler form $J = \frac{1}{2}d\sigma$. We find that

$$J = e^1 \wedge e^2 + e^3 \wedge e^4\,, \tag{2.9}$$

when expressed in terms of the vielbein

$$\begin{aligned} e^1 &= \frac{\rho dx}{2\sqrt{\Delta_x}}\,, \qquad e^2 = \frac{\sqrt{\Delta_x}}{\rho}\,(\frac{\sin^2\theta}{\alpha}\,d\phi + \frac{\cos^2\theta}{\beta}\,d\psi)\,, \\ e^3 &= \frac{\rho d\theta}{\sqrt{\Delta_\theta}}\,, \qquad e^4 = \frac{\sqrt{\Delta_\theta}\,\sin\theta\cos\theta}{\rho}\,(\frac{\alpha - x}{\alpha}d\phi - \frac{\beta - x}{\beta}d\psi)\,. \end{aligned} \tag{2.10}$$

A straightforward calculation confirms that J is indeed covariantly constant.

2.2 Global structure of the five-dimensional solutions

Having obtained the local form of the five-dimensional Einstein-Sasaki metrics, we can now turn to an analysis of the global structure. The metrics are in general of cohomogeneity 2, with toric principal orbits $U(1) \times U(1) \times U(1)$. The orbits degenerate at $\theta = 0$ and $\theta = \frac{1}{2}\pi$, and at the roots of the cubic function Δ_x appearing in (2.8). In order to obtain metrics on complete non-singular manifolds, one must impose appropriate conditions to ensure that the collapsing orbits extend smoothly, without conical singularities, onto the degenerate surfaces. If this is achieved, one can obtain a metric on a non-singular manifold, with $0 \le \theta \le \frac{1}{2}\pi$ and $x_1 \le x \le x_2$, where x_1 and x_2 are two adjacent real roots of Δ_x. In fact, since Δ_x is negative at large negative x and positive at large positive x, and since we must also have $\Delta_x > 0$ in the interval $x_1 < x < x_2$, it follows that x_1 and x_2 must be the smallest two roots of Δ_x.

The easiest way to analyse the behaviour at each collapsing orbit is to examine the associated Killing vector ℓ whose length vanishes at the degeneration surface. By normalising the Killing vector so that its "surface gravity" κ is equal to unity, one obtains a translation generator $\partial/\partial\chi$ where χ is a local coordinate near the degeneration surface, and the metric extends smoothly onto the surface if χ has period 2π. The "surface gravity" for the Killing vector ℓ is given, in the Euclidean regime, by

$$\kappa^2 = -\frac{g^{\mu\nu}\,(\partial_\mu \ell^2)(\partial_\nu \ell^2)}{4\ell^2} \tag{2.11}$$

in the limit that the degeneration surface is reached.

The normalised Killing vectors that vanish at the degeneration surfaces $\theta = 0$ and $\theta = \frac{1}{2}\pi$ are simply given by $\partial/\partial\phi$ and $\partial/\partial\psi$ respectively. At the degeneration surfaces $x = x_1$ and $x = x_2$, we find that the associated normalised Killing vectors ℓ_1 and ℓ_2 are given by

$$\ell_i = c_i\,\frac{\partial}{\partial\tau} + a_i\,\frac{\partial}{\partial\phi} + b_i\,\frac{\partial}{\partial\psi}\,, \tag{2.12}$$

where the constants c_i, a_i and b_i are given by

$$\begin{aligned} a_i &= \frac{\alpha c_i}{x_i - \alpha}\,, \qquad b_i = \frac{\beta c_i}{x_i - \beta}\,, \\ c_i &= \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha+\beta)x_i - \alpha\beta - 3x_i^2}\,. \end{aligned} \tag{2.13}$$

Since we have a total of four Killing vectors $\partial/\partial\phi$, $\partial/\partial\psi$, ℓ_1 and ℓ_2 that span a three-dimensional space, there must exist a linear relation amongst them. Since they all generate translations with a 2π period repeat, it follows that unless the coefficients in the linear relation are rationally related, then by taking integer combinations of translations around the 2π circles, one could generate a translation implying an identification of arbitrarily nearby points in the manifold. Thus one has the requirement for obtaining a non-singular manifold that the linear relation between the four Killing vectors must be expressible as

$$p\ell_1 + q\ell_2 + r\,\frac{\partial}{\partial\phi} + s\,\frac{\partial}{\partial\psi} = 0 \tag{2.14}$$

for *integer* coefficients (p, q, r, s), which may, of course, be assumed to be coprime. We must also require that all subsets of three of the four integers be coprime too. This is because if any three had a common divisor k, then dividing (2.14) by k one could deduce that the direction associated with the Killing vector whose coefficient was not divisible by k would be identified with period $2\pi/k$, thus leading to a conical singularity.

From (2.14), and (2.12), we have

$$\begin{aligned} &pa_1 + qa_2 + r = 0\,, \qquad pb_1 + qb_2 + s = 0\,, \\ &pc_1 + qc_2 = 0\,. \end{aligned} \tag{2.15}$$

From these relations it then follows that the ratios between each pair of the four quantities

$$a_1 c_2 - a_2 c_1\,, \quad b_1 c_2 - b_2 c_1\,, \quad c_1\,, \quad c_2 \tag{2.16}$$

must be rational. Thus in order to obtain a metric that extends smoothly onto a complete and non-singular manifold, we must choose the parameters in (2.8) so that the rationality of the ratios is achieved. In fact it follows from (2.13) that

$$1 + a_i + b_i + 3c_i = 0 \tag{2.17}$$

for all roots x_i, and using this one can show that there are only two independent rationality conditions following from the requirements of rational ratios for the four quantities in (2.16). One can also see from (2.17) that

$$p + q - r - s = 0\,, \tag{2.18}$$

and so the further requirement that all triples chosen from the coprime integers (p, q, r, s) also be coprime is automatically satisfied.

The upshot from the above discussion is that we can have complete and non-singular five-dimensional Einstein-Sasaki spaces $L^{p,q,r}$, where

$$pc_1 + qc_2 = 0\,, \quad pa_1 + qa_2 + r = 0\,. \tag{2.19}$$

These equations and (2.17) allow one to solve for α, β and the roots x_1 and x_2, for positive coprime integer triples (p,q,r). The requirements $0 < x_1 \le x_2 < x_3$, and $\alpha > x_2$, $\beta > x_2$, restrict the integers to the domain $0 < p \le q$ and $0 < r < p+q$. All such coprime triples yield complete and non-singular Einstein-Sasaki spaces $L^{p,q,r}$, and so we get infinitely many new examples.

The spaces $L^{p,q,r}$ all have the topology of $S^2 \times S^3$. We are very grateful to Krzysztof Galicki for the following argument which shows this: The total space of the Calabi-Yau cone, with metric $ds_6^2 = dy^2 + y^2\,\lambda ds_5^2$, can be viewed as a circle reduction (i.e. a symplectic quotient) of $\mathbb{C}^4$ by the diagonal action of $S^1(p,q,-r,-s)$ with $p+q-r-s=0$. The topology of the $L^{p,q,r}$ spaces has also been discussed in detail in [20].

The volume of $L^{p,q,r}$ (with $\lambda = 1$) is given by

$$V = \frac{\pi^2(x_2-x_1)(\alpha+\beta-x_1-x_2)\Delta\tau}{2k\alpha\beta}\,, \tag{2.20}$$

where $\Delta\tau$ is the period of the coordinate τ, and $k = \gcd(p,q)$. Note that the (ϕ,ψ) torus is factored by a freely-acting $\mathbb{Z}_k$, along the diagonal. $\Delta\tau$ is given by the minimum repeat distance of $2\pi c_1$ and $2\pi c_2$, i.e. the minimisation of $|2\pi c_1 M + 2\pi c_2 N|$ over the integers (M,N). We have

$$2\pi(c_1 M + c_2 N) = \frac{2\pi c_1}{q}\,(Mq - Np)\,, \tag{2.21}$$

and so if p and q have greatest common divisor k, the integers M and N can be chosen so that the minimum of $|Mq-Np|$ is k, and so

$$\Delta\tau = \frac{2\pi k|c_1|}{q}\,. \tag{2.22}$$

The volume of $L^{p,q,r}$ is therefore given by

$$V = \frac{\pi^3|c_1|(x_2-x_1)(\alpha+\beta-x_1-x_2)}{q\alpha\beta}\,, \tag{2.23}$$

There is a quartic equation expressing V purely in terms of (p,q,r). Writing

$$V = \frac{\pi^3(p+q)^3 W}{8pqrs}\,, \tag{2.24}$$

we find

$$\begin{aligned} 0 &= 27W^4 - 8(2-9h_+)W^3 - [8h_+(2-h_+)^2 - h_-^2(30+9h_+)]W^2 \\ &\quad -2h_-^2[2(2-h_+)^2 - 3h_-^2]W - (1-f^2)(1-g^2)h_-^4 \end{aligned} \tag{2.25}$$

where

$$f \equiv \frac{q-p}{p+q}\,, \qquad g \equiv \frac{r-s}{p+q}\,, \tag{2.26}$$

and $h_\pm = f^2 \pm g^2$. The central charge of the dual field theory is rational if W is rational, which, as we shall show below, is easily achieved.

If one sets $p + q = 2r$, implying that α and β become equal, our Einstein-Sasaki metrics reduce to those in [6], and the conditions we have discussed for achieving complete non-singular manifolds reduce to the conditions for the $Y^{p,q}$ obtained there, with $Y^{p,q} = L^{p-q,p+q,p}$. The quartic (2.25) then factorises over the rationals into quadrics, giving the volumes found in [6].

Further special limits also arise. For example, if we take $p = q = r = 1$, the roots x_1 and x_2 coalesce, $\alpha = \beta$, and the metric becomes the homogeneous $T^{1,1}$ space, with the four-dimensional base space being $S^2 \times S^2$. In another limit, we can set $\mu = 0$ in (2.8) and obtain the round metric on S^5, with CP^2 as the base. (In fact, we obtain S^5/Z_q if $p = 0$.) Except in these special "regular" cases, the four-dimensional base spaces themselves are singular, even though the Einstein-Sasaki spaces $L^{p,q,r}$ are non-singular. The Einstein-Sasaki space is called quasi-regular if $\partial/\partial\tau$ has closed orbits, which happens if c_1 is rational. If c_1 is irrational the orbits of $\partial/\partial\tau$ never close, and the Einstein-Sasaki space is called irregular.

2.3 Quasi-regular examples

We find that we can obtain quasi-regular Einstein-Sasaki 5-spaces, with rational values for the roots x_i, the parameters α, β, and the volume factor W if the integers (p, q, r) are chosen such that

$$\frac{q-p}{p+q} = \frac{2(v-u)(1+uv)}{4-(1+u^2)(1+v^2)}\,, \qquad \frac{r-s}{p+q} = \frac{2(v+u)(1-uv)}{4-(1+u^2)(1+v^2)}\,, \tag{2.27}$$

where u and v are any rational numbers satisfying

$$0 < v < 1\,, \qquad -v < u < v\,. \tag{2.28}$$

A convenient choice that eliminates the redundancy in the (α, β, μ) parameterisation of the local solutions is by taking $x_3 = 1$, in which case we then have rational solutions with

$$\begin{aligned} x_1 &= \tfrac{1}{4}(1+u)(1-v)\,, \qquad x_2 = \tfrac{1}{4}(1-u)(1+v)\,, \qquad x_3 = 1\,, \\ \alpha &= 1 - \tfrac{1}{4}(1+u)(1+v)\,, \qquad \beta = 1 - \tfrac{1}{4}(1-u)(1-v)\,, \qquad \mu = \tfrac{1}{16}(1-u^2)(1-v^2)\,. \end{aligned} \tag{2.29}$$

From these, we have

$$\begin{aligned} c_1 &= -\frac{2(1-u)(1+v)}{(v-u)[4-(1+u)(1-v)]}\,, \qquad c_2 = \frac{2(1+u)(1-v)}{(v-u)[4-(1-u)(1+v)]}\,, \\ a_1 &= \frac{(1+v)(3-u-v-uv)}{(v-u)[4-(1+u)(1-v)]}\,, \qquad a_2 = -\frac{(1+u)(3-u-v-uv)}{(v-u)(4-(1-u)(1+v)]}\,, \\ b_1 &= \frac{(1-u)(3+u+v-uv)}{(v-u)[4-(1+u)(1-v)]}\,, \qquad b_2 = -\frac{(1-v)(3+u+v-uv)}{(v-u)[4-(1-u)(1+v)]}\,. \end{aligned} \tag{2.30}$$

It follows that c_1 is also rational, and so these Einstein-Sasaki spaces are quasi-regular, with closed orbits for $\partial/\partial\tau$. The volume is given by (2.24), with

$$W = \frac{16(1-u^2)^2\,(1-v^2)^2}{(3-u^2-v^2-u^2v^2)^3}\,, \tag{2.31}$$

and so the ratio of V to the volume of the unit 5-sphere (which is π^3) is rational too.

Note that although we introduced the (u,v) parameterisation in order to write quasi-regular examples with rational roots and volumes, the same parameterisation is also often useful in general. One simply takes u and v to be real numbers, not in general rational, defined in terms of p, q, r and s by (2.27). They are again subject to the restrictions (2.28).

2.4 Volumes and the Bishop bound

Note that the volume V can be expressed in terms of u, v and p as

$$V = \frac{16\pi^3(1+u)(1-v)}{p\,(3+u+v-uv)(3+u-v+uv)(3-u-v-uv)}\,, \tag{2.32}$$

where u and v are given in terms of p, q and r by 2.27. It is easy to verify that the volume is always bounded above by the volume of the unit 5-sphere,[1] as it must be by Bishop's theorem [21]. To see this, define

$$Y \equiv 1 - \frac{pV}{\pi^3}\,. \tag{2.33}$$

Since p is a positive integer, then if we can show that $Y > 0$ for all our inhomogeneous Einstein-Sasaki spaces, it follows that they must all have volumes less than π^3, the volume of the unit S^5. It is easy to see that

$$Y = \frac{(1-u)(1+v)\,F}{(3+u+v-uv)(3+u-v+uv)(3-u-v-uv)}\,, \tag{2.34}$$

where

$$F = 11 + (u+v)^2 + 2uv + 4(u-v) - u^2v^2\,. \tag{2.35}$$

With u and v restricted to the region defined by (2.28), it is clear that the sign of Y is the same as the sign of F. It also follows from (2.28) that

$$2uv > -2\,,\qquad 4(u-v) > -8\,,\qquad -u^2v^2 > -1\,, \tag{2.36}$$

and so we have $F > 0$. Thus $Y > 0$ for all the inhomogeneous Einstein-Sasaki spaces, proving that they all satisfy $V < \pi^3$.

[1]Recall that all our volume formulae are with respect to spaces normalised to $R_{ij} = 4g_{ij}$.

2.5 The $u \leftrightarrow -u$ symmetry

By making appropriate redefinitions of the coordinates, we can make manifest the discrete symmetry of the five-dimensional metrics under the transformation $u \to -u$, which, from (2.27), corresponds to the exchange of the integers (p, q) with the integers (r, s). Accordingly, we define new coordinates[2]

$$y = \Delta_\theta\,, \qquad \hat\psi = \frac{\phi - \psi}{\beta - \alpha}\,, \qquad \hat\phi = \frac{\alpha^{-1}\,\phi - \beta^{-1}\psi}{\beta - \alpha}\,, \qquad \hat\tau = \tau + \frac{\beta\psi - \alpha\phi}{\beta - \alpha}\,. \tag{2.37}$$

In terms of these, the Einstein-Sasaki metrics become

$$\lambda ds_5^2 = (d\hat\tau + \hat\sigma)^2 + ds_4^2\,, \tag{2.38}$$

with

$$\begin{aligned} ds_4^2 &= \frac{(y-x)dx^2}{4\Delta_x} + \frac{(y-x)dy^2}{4\Delta_y} + \frac{\Delta_x}{y-x}\,(d\hat\psi - yd\hat\phi)^2 + \frac{\Delta_y}{y-x}\,(d\hat\psi - xd\hat\phi)^2\,,\\ \hat\sigma &= (x+y)d\hat\psi - xyd\hat\phi\,. \end{aligned} \tag{2.39}$$

The metric functions Δ_x and Δ_y are given by

$$\Delta_x = x(\alpha - x)(\beta - x) - \mu\,, \qquad \Delta_y = -y(\alpha - y)(\beta - y)\,, \tag{2.40}$$

It is convenient now to adopt the parameterisation introduced in (2.27). Note that this can be done whether or not u and v are chosen to be rational. Then from (2.29) we have

$$\begin{aligned} \Delta_x &= \tfrac{1}{16}(x-1)[4x - (1-u)(1+v)][4x - (1+u)(1-v)]\,,\\ \Delta_y &= -\tfrac{1}{16}y[4(1-y) - (1-u)(1-v)][4(1-y) - (1+u)(1+v)]\,. \end{aligned} \tag{2.41}$$

It is now manifest that the five-dimensional metric (2.38) is invariant under sending $u \to -u$, provided that at the same time we make the coordinate transformations

$$\begin{aligned} x &\longrightarrow 1 - y\,, \qquad y \longrightarrow 1 - x\,,\\ \hat\psi &\longrightarrow -\hat\psi + \hat\phi\,, \qquad \hat\tau \longrightarrow \hat\tau + 2\hat\psi - \hat\phi\,. \end{aligned} \tag{2.42}$$

The argument above shows that to avoid double counting, we should further restrict the coprime integers (p, q, r) so that either $u \ge 0$, or else $u \le 0$. We shall make the latter choice, which implies that we should restrict (p, q, r) so that

$$0 \le p \le q \le r \le p + q\,. \tag{2.43}$$

[2]A similar redefinition of coordinates has also been given in [20].

2.6 The $u = 0$ case

Having seen that there is a symmetry under $u \leftrightarrow -u$, it is of interest to study the fixed point of this discrete symmetry, i.e. $u = 0$, which corresponds to setting $q = r$ (and hence $p = s$).

From (2.29), a convenient way of restricting to $u = 0$ is by choosing

$$\mu = \tfrac{2}{27}(2\alpha - \beta)(2\beta - \alpha)(\alpha + \beta)\,. \tag{2.44}$$

This allows us to factorise the function Δ_x, giving

$$\Delta_x = \tfrac{1}{27}(2\alpha - \beta - 3x)(\alpha - 2\beta + 3x)(2\alpha + 2\beta - 3x)\,. \tag{2.45}$$

Now we introduce new tilded coordinates, defined by

$$\begin{aligned}
\tilde{x} &= -x + \tfrac{1}{3}(\alpha + \beta)\,, \qquad \tilde{y} = \Delta_\theta - \tfrac{1}{3}(\alpha + \beta)\,,\\
\phi &= -\alpha[\tilde{\psi} + \tfrac{1}{3}(2\beta - \alpha)\tilde{\phi}]\,, \qquad \psi = -\beta[\tilde{\psi} + \tfrac{1}{3}(2\alpha - \beta)\tilde{\phi}]\,,\\
\tilde{\tau} &= \tau - \tfrac{1}{3}(\alpha + \beta)\tilde{\psi} + \tfrac{1}{9}(2\alpha^2 - 5\alpha\beta + 2\beta^2)\tilde{\phi}\,.
\end{aligned} \tag{2.46}$$

After doing this, the metric takes on the form

$$\lambda ds_5^2 = (d\tilde{\tau} + \tilde{\sigma})^2 + ds_4^2\,, \tag{2.47}$$

with

$$\begin{aligned}
ds_4^2 &= \frac{(\tilde{x} + \tilde{y})d\tilde{x}^2}{4\Delta(\tilde{x})} + \frac{(\tilde{x} + \tilde{y})d\tilde{y}^2}{4\Delta(\tilde{y})} + \frac{\Delta(\tilde{x})}{\tilde{x} + \tilde{y}}\,(d\tilde{\psi} + \tilde{y}d\tilde{\phi})^2 + \frac{\Delta(\tilde{y})}{\tilde{x} + \tilde{y}}\,(d\tilde{\psi} - \tilde{x}d\tilde{\phi})^2\,,\\
\tilde{\sigma} &= (\tilde{y} - \tilde{x})d\tilde{\psi} - \tilde{x}\tilde{y}d\tilde{\phi}\,.
\end{aligned} \tag{2.48}$$

The metric functions $\Delta(\tilde{x})$ and $\Delta(\tilde{y})$, which are now the same function with either $\tilde{x}$ or $\tilde{y}$ as argument, are given by

$$\Delta(z) = \tfrac{1}{27}(\alpha - 2\beta + 3z)(2\alpha - \beta - 3z)(\alpha + \beta + 3z)\,. \tag{2.49}$$

It is therefore natural to define new parameters γ_1 and γ_2, given by

$$\gamma_1 = \tfrac{1}{3}(2\alpha - \beta)\,, \qquad \gamma_2 = \tfrac{1}{3}(2\beta - \alpha)\,. \tag{2.50}$$

In terms of these, the function $\Delta(z)$ becomes

$$\Delta(z) = -(\gamma_1 - z)(\gamma_2 - z)(\gamma_1 + \gamma_2 + z)\,. \tag{2.51}$$

There is now a manifest discrete symmetry, under which we send

$$\tilde{x} \leftrightarrow \tilde{y}\,, \qquad \tilde{\phi} \leftrightarrow -\tilde{\phi}\,. \tag{2.52}$$

It is worth remarking that the quartic polynomial (2.25) determining the volume factorises in quadrics over the rationals in the case that $u = 0$, giving

$$V = \frac{4\pi^3}{27p^2q^2}\Big((p+q)(p-2q)(q-2p) + 2(p^2 - pq + q^2)^{3/2}\Big)\,. \tag{2.53}$$

2.7 Curvature invariants

In this section, we present some results for curvature invariants for the five-dimensional Einstein-Sasaki metrics. We find

$$\begin{aligned} I_2 &\equiv R^{ijk\ell}\,R_{ijk\ell} = \frac{192(\rho^{12} + 2\mu^2)}{\rho^{12}}\,,\\ I_3 &\equiv R^{ij}{}_{k\ell}\,R^{k\ell}{}_{mn}\,R^{mn}{}_{ij} = \frac{384(5\rho^{18} + 12\mu^2\rho^6 + 8\mu^3)}{\rho^{18}}\,,\\ J_3 &\equiv R^i{}_j{}^k{}_\ell\,R^j{}_m{}^\ell{}_n\,R^m{}_i{}^n{}_k = \frac{96(\rho^{18} - 12\mu^2\rho^6 + 16\mu^3)}{\rho^{18}}\,. \end{aligned} \tag{2.54}$$

Since these curvature invariants depend on the coordinates only via the single combination $\rho^2 = \alpha\sin^2\theta + \beta\cos^2\theta - x$, one might wonder whether the Einstein-Sasaki metrics, despite ostensibly being of cohomogeneity 2, were actually only of cohomogeneity 1, becoming manifestly so when described in an appropriate coordinate system. In fact this is not the case, as can be seen by calculating the scalar invariant

$$K = g^{\mu\nu}(\partial_\mu I_2)(\partial_\nu I_2)\,, \tag{2.55}$$

which turns out to be given by

$$\begin{aligned} K &= \frac{2^{18}3^4\mu^4}{\rho^{30}}\Big[-\rho^6 + (2\beta-\alpha)\rho^4 - \beta(\beta-\alpha)\rho^2 - \mu\\ &\qquad + (\alpha-\beta)[3\rho^2 - 2(2\beta-\alpha)]\rho^2\,\cos^2\theta - 3(\alpha-\beta)^2\,\rho^2\,\cos^4\theta\Big]\,. \end{aligned} \tag{2.56}$$

Since this invariant does not depend on the x and θ coordinates purely via the combination $\rho^2 = \alpha\sin^2\theta + \beta\cos^2\theta - x$, we see that the metrics do indeed genuinely have cohomogeneity 2. They do, of course, reduce to cohomogeneity 1 if the parameters α and β are set equal.

3 Higher-Dimensional Einstein-Sasaki Spaces

The construction of five-dimensional Einstein-Sasaki spaces that we have given in section 2 can be extended straightforwardly to all higher odd dimensions. We take the rotating Kerr-de Sitter metrics obtained in [14,15], and impose the Bogomol'nyi conditions $E - g\sum_i J_i = 0$, where E and J_i are the energy and angular momenta that were calculated in [18], and given in (2.3). We find that a non-trivial BPS limit exists where $g a_i = 1 - \frac{1}{2}\alpha_i \epsilon$, $m = m_0 \epsilon^{n+1}$. After Euclideanisation of the $D = 2n+1$ dimensional rotating black hole metrics obtained in [14], which is achieved by sending $\tau \to \mathrm{i}\,\tau$, and $a_i \to \mathrm{i}\, a_i$ in equation (3.1) of that paper (and using y rather than r as the radial variable, to avoid a clash of notations later), one has

$$\begin{aligned} ds^2 &= W\,(1-\lambda\, y^2)\, d\tau^2 + \frac{U\, dy^2}{V - 2m} + \frac{2m}{U}\Big(d\tau - \sum_{i=1}^n \frac{a_i\, \mu_i^2\, d\varphi_i}{1 - \lambda\, a_i^2}\Big)^2 \\ &\quad + \sum_{i=1}^n \frac{y^2 - a_i^2}{1 - \lambda\, a_i^2}\, [d\mu_i^2 + \mu_i^2\, (d\varphi_i + \lambda\, a_i\, d\tau)^2] + \frac{\lambda}{W\,(1-\lambda y^2)}\Big(\sum_{i=1}^n \frac{(y^2 - a_i^2)\mu_i\, d\mu_i}{1 - \lambda\, a_i^2}\Big)^2\,, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} V &\equiv \frac{1}{y^2}\,(1 - \lambda\, y^2) \prod_{i=1}^n (y^2 - a_i^2)\,, \qquad W \equiv \sum_{i=1}^n \frac{\mu_i^2}{1 - \lambda\, a_i^2}\,, \\ U &= \sum_{i=1}^n \frac{\mu_i^2}{y^2 - a_i^2} \prod_{j=1}^n (y^2 - a_j^2)\,. \end{aligned} \tag{3.2}$$

The BPS limit is now achieved by setting

$$\begin{aligned} &a_i = \lambda^{-\frac{1}{2}} (1 - \tfrac{1}{2}\alpha_i\, \epsilon)\,, \\ &y^2 = \lambda^{-1}(1 - x\epsilon)\,, \quad m = \tfrac{1}{2}\lambda^{-1}\mu\epsilon^{n+1}\,, \end{aligned} \tag{3.3}$$

and then sending $\epsilon \to 0$. We then obtain $D = 2n+1$ dimensional Einstein-Sasaki metrics ds^2, given by

$$\lambda ds^2 = (d\tau + \sigma)^2 + d\bar{s}^2\,, \tag{3.4}$$

with $R_{\mu\nu} = 2n\lambda g_{\mu\nu}$, where the $2n$-dimensional metric $d\bar{s}^2$ is Einstein-Kähler, with Kähler form $J = \frac{1}{2} d\sigma$, and

$$\begin{aligned} d\bar{s}^2 &= \frac{Y dx^2}{4xF} - \frac{x(1-F)}{Y}\Big(\sum_i \alpha_i^{-1}\, \mu_i^2 d\varphi_i\Big)^2 + \sum_i (1 - \alpha_i^{-1}\, x)(d\mu_i^2 + \mu_i^2 d\varphi_i^2) \\ &\quad + \frac{x}{\sum_i \alpha_i^{-1} \mu_i^2}\,\Big(\sum_j \alpha_j^{-1}\, \mu_j d\mu_j\Big)^2 - \sigma^2\,, \\ \sigma &= \sum_i (1 - \alpha_i^{-1} x)\mu_i^2\, d\varphi_i\,, \\ Y &= \sum_i \frac{\mu_i^2}{\alpha_i - x}\,, \qquad F = 1 - \frac{\mu}{x}\prod_i (\alpha_i - x)^{-1}\,, \end{aligned} \tag{3.5}$$

where $\sum_i \mu_i^2 = 1$. The $D = 2n+1$ dimensional Einstein-Sasaki metrics have cohomogeneity n, with $U(1)^{n+1}$ principal orbits.

The discussion of the global properties is completely analogous to the one we gave previously for the five-dimensional case. The n Killing vectors $\partial/\partial\varphi_i$ vanish at the degenerations of the $U(1)^{n+1}$ principal orbits where each μ_i vanishes, and conical singularities are avoided if each coordinate φ_i has period 2π. The Killing vectors

$$\ell_i = c(i)\,\frac{\partial}{\partial\tau} + \sum_j b_j(i)\,\frac{\partial}{\partial\varphi_j} \tag{3.6}$$

vanish at the roots $x = x_i$ of $F(x)$, and have unit surface gravities there, where

$$b_j(i) = -\frac{c(i)\alpha_j}{\alpha_j - x_i}\,, \quad c(i)^{-1} = \sum_j \frac{x_i}{\alpha_j - x_i} - 1\,. \tag{3.7}$$

The metrics extend smoothly onto complete and non-singular manifolds if

$$p\ell_1 + q\ell_2 + \sum_j r_j \frac{\partial}{\partial\varphi_j} = 0 \tag{3.8}$$

for coprime integers (p, q, r_j), where in addition all possible subsets of $(n+1)$ of the integers are also coprime (which is again automatic–see below). This implies the algebraic equations

$$p\,c(1) + q\,c(2) = 0\,, \quad p\,b_j(1) + q\,b_j(2) + r_j = 0\,, \tag{3.9}$$

determining the roots x_1 and x_2, and the parameters α_j. The two roots of $F(x)$ must be chosen so that $F > 0$ when $x_1 < x < x_2$. With these conditions satisfied, we obtain infinitely many new complete and non-singular compact Einstein-Sasaki spaces in all odd dimensions $D = 2n + 1$.

It follows from (3.7) that

$$\sum_j b_j(i) + (n+1)c(i) + 1 = 0\,, \tag{3.10}$$

and hence using (3.9) we have

$$p + q = \sum_j r_j\,. \tag{3.11}$$

This can be used to eliminate r_n in favour of the other $(n+1)$ integers. The Einstein-Sasaki spaces, which we denote by $L^{p,q,r_1,\cdots,r_{n-1}}$, are therefore characterised by specifying $(n+1)$ coprime integers, which must lie in an appropriate domain. Without loss of generality, we may choose $p < q$, and order the two roots x_1 and x_2 so that $x_1 < x_2$. It follows that we shall have

$$c_1 < 0\,, \qquad c_2 > 0\,, \qquad |c_1| > c_2\,. \tag{3.12}$$

The parameters α_j must all satisfy $\alpha_j > x_2$, to ensure that Y is always positive. From (3.9) we have therefore have

$$r_j = \frac{qc(2)\,\alpha_j(x_2-x_1)}{(\alpha_j-x_1)(\alpha_j-x_2)} > 0\,. \tag{3.13}$$

To avoid overcounting, we can therefore specify the domain by

$$0 < p < q\,, \qquad 0 < r_1 \le r_2 \le \cdots \le r_{n-1} \le r_n\,. \tag{3.14}$$

The n-torus of the φ_j coordinates is in general factored by a freely-acting $\mathbb{Z}_k$, where $k = \gcd(p,q)$. The volume (with $\lambda = 1$) is given by

$$V = \frac{|c(1)|}{q}\,\mathcal{A}_{2n+1}\Big[\prod_i\Big(1-\frac{x_1}{\alpha_i}\Big) - \prod_i\Big(1-\frac{x_2}{\alpha_i}\Big)\Big]\,, \tag{3.15}$$

since $\Delta\tau$ is given by $2\pi k|c(1)|/q$, where $\mathcal{A}_{2n+1}$ is the volume of the unit $(2n+1)$-sphere. In the special case that the rotations α_i are set equal, the metrics reduce to those obtained in [7].

4 Non-Sasakian Einstein Spaces

So far in this paper, we have concentrated on the situations, in odd dimensions, where a limit of the Euclideanised Kerr-de Sitter metrics can be taken in which one has a Killing spinor. In this section, we shall describe the more general situation in which no limit is taken, and so one has Einstein metrics that do not have Killing spinors. They are therefore Einstein spaces that are not Sasakian. Again, the question arises as to whether these metrics can, for suitable choices of the parameters, extend smoothly onto complete and non-singular compact manifolds. As in the previous discussion in the Einstein-Sasaki limit, this question reduces to whether smooth extensions onto the surfaces where the principal orbits degenerate are possible.

This question was partially addressed in [14, 15], where the problem was studied in the case that the two roots defining the endpoints of the range of the radial variable were taken to be coincident. This ensured that the surface gravities at the endpoints of the (rescaled) radial variable were equal in magnitude. However, as we have seen in the discussion for the Einstein-Sasaki limits, the requirement of equal surface gravities is more restrictive than is necessary for obtaining non-singular spaces. In this section, we shall study the problem of obtaining non-singular spaces within this more general framework.

4.1 Odd dimensions

The Euclideanised Kerr-de Sitter metrics in odd dimensions $D = 2n+1$ are given in equation (3.1). From the results in [14, 15], the Killing vector

$$\tilde{\ell} \equiv \frac{\partial}{\partial\tau} - \sum_{j=1}^{n} \frac{a_j\,(1-\lambda y_0^2)}{y_0^2 - a_j^2}\,\frac{\partial}{\partial\varphi_j} \tag{4.1}$$

has vanishing norm at a root $y = y_0$ of $V(y) - 2m = 0$, and it has a "surface gravity" given by

$$\kappa = y_0\,(1-\lambda y_0^2) \sum_{j=1}^{n} \frac{1}{y_0^2 - a_j^2} - \frac{1}{y_0}\,. \tag{4.2}$$

Following the strategy we used for studying the degenerate orbits of the metrics in the Einstein-Sasaki limit, we now introduce a rescaled Killing vector $\ell = c\,\tilde{\ell}$ with c chosen so that ℓ has unit surface gravity. Thus we define Killing vectors

$$\begin{aligned} \ell_1 &= c(1)\,\frac{\partial}{\partial\tau} + \sum_{j=1}^{n} b_j(1)\,\frac{\partial}{\partial\varphi_j}\,,\\ \ell_2 &= c(2)\,\frac{\partial}{\partial\tau} + \sum_{j=1}^{n} b_j(2)\,\frac{\partial}{\partial\varphi_j}\,, \end{aligned} \tag{4.3}$$

which vanish at two adjacent roots $y = y_1$ and $y = y_2$ respectively, each of whose surface gravities is of unit magnitude. The constants are therefore given by

$$\begin{aligned} c(i)^{-1} &= y_i\,(1-\lambda y_i^2) \sum_{j=1}^{n} \frac{1}{y_i^2 - a_j^2} - \frac{1}{y_i}\,, \qquad i = 1,2\,,\\ b_j(i) &= -\frac{a_j\,(1-\lambda y_i^2)\,c(i)}{y_i^2 - a_j^2}\,, \qquad i = 1,2\,. \end{aligned} \tag{4.4}$$

We shall assume, without loss of generality, that $y_1 < y_2$, and we require that $V(y) - 2m > 0$ for $y_1 < y < y_2$, to ensure that the metric has positive definite signature. With these assumptions, we shall have

$$c(1) > 0\,, \qquad c(2) < 0\,. \tag{4.5}$$

The Killing vectors ℓ_1, ℓ_2 and $\partial/\partial\varphi_j$ have zero norm at the degeneration surfaces $r = y_1$ and $r = y_2$ respectively. The Killing vector $\partial/\partial\varphi_j$ has zero norm at the degeneration surface where $\mu_j = 0$. Since the $(n+2)$ Killing vectors ℓ_1, ℓ_2 and $\partial/\partial\varphi_j$ span a vector space of dimension $(n+1)$, it follows that they must be linearly dependent. Since each of the Killing vectors generates a translation that closes with period 2π at its own degeneration surface, it follows that the coefficients in the linear relation must be rationally related in order not to have arbitrarily nearby points being identified, and so we may write the linear relation as

$$p\,\ell_1 + q\,\ell_2 + \sum_{j=1}^{n} r_j \frac{\partial}{\partial \varphi_j} = 0 \tag{4.6}$$

for coprime integers (p, q, r_j). For the same reason we discussed for the Einstein-Sasaki cases, here too no subset of $(n+1)$ of these integers must have any common factor either. Thus we have the equations

$$p\,c(1) + q\,c(2) = 0\,, \qquad p\,b_j(1) + q\,b_j(2) + r_j = 0\,. \tag{4.7}$$

Unlike the Einstein-Sasaki limits, however, in this general case we do not have any relation analogous to (3.11) that imposes a linear relation on the $(n+2)$ integers. This is because in the general case the local metrics (3.1) have $(n+1)$ non-trivial continuous parameters, namely m and the rotations a_j, which can then be viewed as being determined, via the $(n+1)$ equations (4.7), in terms of the $(n+1)$ rational ratios p/q and r_j/q. By contrast, in the Einstein-Sasaki limits the local metrics in $D = 2n+1$ dimensions have only n non-trivial parameters, and so there must exist an equation (namely (3.11)) that relates the $(n+1)$ ratios p/q and r_j/q.

Since, without loss of generality, we are taking $r_1 < r_2$, the integers p and q must be such that $p < q$.[3] From (4.7) we have

$$r_j = \frac{p\,a_j\,c(1)\,(y_2^2 - y_1^2)(1 - \lambda a_j^2)}{(y_1^2 - a_j^2)(y_2^2 - a_j^2)}\,. \tag{4.8}$$

We must have $y > a_j$ in the entire interval $y_1 \le y \le y_2$, in order to ensure that the metric remains positive definite, and hence from (4.8) it follows that we must have $r_j > 0$. We shall denote the associated $D = 2n+1$ dimensional Einstein spaces by $\Lambda^{p,q,r_1,\ldots,r_n}$.

From an expression for the determinant of the Kerr-de Sitter metrics given in [18], it is easily seen that the volume of the Einstein space $\Lambda^{p,q,r_1,\ldots,r_n}$ is given by

$$V = \frac{\mathcal{A}_{2n+1}}{(\prod_j \Xi_j)}\,\frac{\Delta\tau}{2\pi}\,\Big[\prod_{j=1}^{n}(y_2^2 - a_j^2) - \prod_{j=1}^{n}(y_1^2 - a_j^2)\Big]\,\Big(\prod_{k=1}^{n}\int \frac{d\varphi_k}{2\pi}\Big)\,, \tag{4.9}$$

where $\mathcal{A}_{2n+1}$ is the volume of the unit $(2n+1)$-sphere. If p and q have a greatest common divisor $k = \gcd(p,q)$ then the n-torus of the φ_j coordinates will be factored by a freely-acting $\mathbb{Z}_k$, and hence it will have volume $(2\pi)^n/k$. The period of τ will be $\Delta\tau = 2\pi\,k\,c(1)/q$,

[3]The case $p = q$, corresponding to a limit with $r_1 = r_2$, was discussed extensively in [14, 15], and so for convenience we shall exclude this from the analysis here.

and hence the volume of $\Lambda^{p,q,r_1,\ldots,r_n}$ is

$$V = \frac{\mathcal{A}_{2n+1}}{(\prod_j \Xi_j)} \frac{c(1)}{q} \Big[\prod_{j=1}^n (y_2^2 - a_j^2) - \prod_{j=1}^n (y_1^2 - a_j^2) \Big] . \tag{4.10}$$

4.2 Even dimensions

A similar discussion applies to the Euclideanised Kerr-de Sitter metrics in even dimensions $D = 2n$, which are also given in [14, 15]. There is, however, a crucial difference, stemming from the fact that while there are n latitude coordinates μ_i with $1 \le i \le n$, there are only $n-1$ azimuthal coordinates φ_j, with $1 \le j \le n-1$. Since the μ_i coordinates are subject to the condition $\sum_{i=1}^n \mu_i^2 = 1$, this means that now, unlike the odd-dimensional case, there exist surfaces where *all* the azimuthal Killing vectors $\partial/\partial\varphi_j$ simultaneously have vanishing norm. This is achieved by taking

$$\mu_j = 0 , \qquad 1 \le j \le n-1 ; \qquad \mu_n = \pm 1 . \tag{4.11}$$

Thus if we consider the Killing vectors $\partial/\partial\varphi_j$ together with ℓ_1 whose norm vanishes at $y = y_1$ and ℓ_2 whose norm vanishes at $y = y_2$, then from the relation

$$p\,\ell_1 + q\,\ell_2 + \sum_j r_j \frac{\partial}{\partial\varphi_j} = 0 , \tag{4.12}$$

which can be written as

$$\ell_2 = -\frac{p}{q}\,\ell_1 - \sum_j \frac{r_j}{q} \frac{\partial}{\partial\varphi_j} , \tag{4.13}$$

we see that at $(y = y_1, \mu_n = \pm 1)$ it will also be the case that ℓ_2 has vanishing norm.[4] Since ℓ_1, ℓ_2 and $\partial/\partial\varphi_j$ all, by construction generate translations at their respective degeneration surfaces that close with period 2π, it follows from (4.13) that there will in general be a conical singularity at $y = y_2$ associated with a factoring by $\mathbb{Z}_q$. A similar argument shows there will in general be a conical singularity at $y = y_1$ associated with a factoring by $\mathbb{Z}_p$.

The upshot from the above discussion is that one can only get non-singular Einstein spaces in the $D = 2n$ dimensional case if $p = q = 1$. Since $p = q$, this implies that the two roots y_1 and y_2 coincide, and hence the analysis reduces to that which was given in [14, 15].

Since the calculations in four dimensions are very simple, it is instructive to examine this example in greater detail. The Euclideanised four-dimensional Kerr-de Sitter metric is

[4] Note that in a positive-definite metric signature, if two two vectors A and B have vanishing norm at any point, then so does $A + \lambda\, B$ for any λ. This can be seen from $(A \pm B)^2 \ge 0$, which shows that if A^2 and B^2 vanish at a point, then so does $A \cdot B$.

given by

$$ds^2 = \rho^2\Big(\frac{dy^2}{\Delta_y} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\Big(a d\tau + (y^2 - a^2)\frac{d\phi}{\Xi}\Big)^2 + \frac{\Delta_y}{\rho^2}\Big(d\tau - a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2, \quad (4.14)$$

where

$$\rho^2 = y^2 - a^2\cos^2\theta\,, \qquad \Delta_y = (y^2 - a^2)(1 - \lambda\, y^2) - 2my\,,$$
$$\Delta_\theta = 1 - \lambda\, a^2\cos^2\theta\,, \qquad \Xi = 1 - \lambda a^2\,. \qquad (4.15)$$

The function Δ_y is a quartic polynomial in y, which goes to $-\infty$ for $y \to \pm\infty$. Thus one necessary condition for obtaining a non-singular space is that there exist at least two real roots. If there are exactly two real roots, y_1 and y_2, with $y_1 \le y_2$, then radial variable y must lie in the interval $y_1 \le y \le y_2$. If there four real roots, then we should choose y_1 and y_2 to be adjacent roots, which are either the smallest pair or the largest pair.

The Killing vectors that vanish at $y = y_1$ and $y = y_2$ are given by

$$\ell_i = c_i \frac{\partial}{\partial\tau} + b_i \frac{\partial}{\partial\phi}\,, \qquad (4.16)$$

where

$$b_i = \frac{a(1-a^2)c_i}{a^2 - y_i^2}\,, \qquad c_i = \frac{2(a^2 - y_i^2)y_i}{a^2 + y_i^2 + a^2 y_i^2 - 3y_i^4} \qquad (4.17)$$

The Killing vector that vanishes at $\sin\theta = 0$ is given by

$$\ell_3 = \frac{\partial}{\partial\phi}\,. \qquad (4.18)$$

Thus we have the conditions

$$p\,\ell_1 + q\ell_2 + r\,\ell_3 = 0 \qquad (4.19)$$

for (p, q, r) which are pairwise coprime integers.

For a four-dimensional compact Einstein space, the Euler number is given by

$$\chi = \frac{1}{32\pi^2}\int |\text{Riem}|^2\, \sqrt{g}\, d^4x\,. \qquad (4.20)$$

This is easily evaluated for the four-dimensional metrics given above. With the angular coordinates having periods

$$\Delta\phi = 2\pi\,, \qquad \Delta\tau = \frac{2\pi c_1}{q}\,, \qquad (4.21)$$

we find that

$$\chi = \frac{2}{p} + \frac{2}{q}\,. \qquad (4.22)$$

If $p = q = 1$ we have $\chi = 4$. This is indeed the correct Euler number for the S^2 bundle over S^2, which, as shown in [22] and [14, 15], is the only non-singular case that arises when the roots y_1 and y_2 coincide. If one were to consider cases where $p \neq 1$ or $q \neq 1$, then in general χ would not be an integer, in accordance with our observations above that in such cases there are $\mathbb{Z}_p$ and $\mathbb{Z}_q$ orbifold type singularities in the space. It is possible that one might be able to blow up these singularities, and thereby obtain a non-singular space with a more complicated topology. The fact that the "Euler number" χ given in (4.22) has a simple rational form can perhaps be taken as supporting evidence.

5 Conclusions

In this paper, we have elaborated on the results which we obtained in [17], constructing new Einstein-Sasaki spaces $L^{p,q,r_1,\cdots,r_{n-1}}$ and non-Sasakian Einstein spaces $\Lambda^{p,q,r_1,\cdots,r_n}$, in all odd dimensions $D = 2n + 1 \geq 5$. These spaces are all complete and non-singular compact manifolds. The metrics have cohomogeneity n, with isometry group $U(1)^{n+1}$, which acts transitively on the $(n + 1)$-dimensional principal orbits.

The Einstein-Sasaki metrics arise after Euclideanisation, by taking certain BPS limits of the Kerr-de Sitter spacetimes constructed in $D = 5$ in [13], and in all higher dimensions in [14, 15]. The BPS limit effectively implies that there is a relation between the mass and the n rotation parameters of the $(2n + 1)$-dimensional Kerr-de Sitter metric, and thus the local Einstein-Sasaki metrics have n non-trivial free parameters. These metrics are in general singular, but by imposing appropriate restrictions on the parameters, we find that the metrics extend smoothly onto complete and non-singular compact manifolds, which we denote by $L^{p,q,r_1,\cdots,r_{n-1}}$, where the integers $(p, q, r_1, \ldots, r_{n-1})$ are coprime.

In the case of the five-dimensional Einstein-Sasaki spaces $L^{p,q,r}$, we have been able to obtain an explicit formula expressing the volume in terms of the coprime integers (p, q, r), via a quartic polynomial. In the AdS/CFT correspondence, it is expected that the boundary field theory dual to the type IIB string on $\mathrm{AdS}_5 \times L^{p,q,r}$ will be a quiver gauge theory. In particular, the central charge of the quiver theory should be related to the inverse of the volume of $L^{p,q,r}$. The central charges have recently been calculated using the technique of a-maximisation, and it has been shown that they are indeed precisely in correspondence with the volumes given by our polynomial (2.25) [23]. (See [24–27] for the analysis of the dual quiver theories, and a-maximisation, for the previous $Y^{p,q}$ examples.)

We have also shown, for the five-dimensional $L^{p,q,r}$ spaces, that a convenient characterisation can be given in terms of the two parameters u and v, introduced in equation (2.27), where $0 < u < v < 1$. If u and v are taken to be arbitrary rational numbers in this range, then we obtain a corresponding Einstein-Sasaki space that is "quasi-regular," meaning that the orbits of $\partial/\partial\tau$ are closed. In general, when u and v are irrational numbers, again related to the coprime integers (p, q, r) by (2.27), the orbits of $\partial/\partial\tau$ will never close, and the corresponding Einstein-Sasaki space is called "irregular."[5]

Several other papers have appeared making use of our results in [17]. These include [28], where branes in backgrounds involving the $L^{p,q,r}$ spaces are constructed, and [29], where marginal deformations of eleven-dimensional backgrounds involving the L^{p,q,r_1,r_2} spaces are constructed.

In addition to discussing the Einstein-Sasaki spaces $L^{p,q,r_1,\cdots,r_{n-1}}$, we have also elaborated in this paper on the non-Sasakian Einstein spaces $\Lambda^{p,q,r_1,\cdots,r_n}$ that were constructed in $D = 2n+1$ dimensions in [17]. These arise by Euclideanising the Kerr-de Sitter metrics without taking any BPS limit. As local metrics they are characterised by $(n+1)$ parameters, corresponding to the mass and the n independent rotations of the $(2n+1)$-dimensional rotating black holes. Again, by choosing these parameters appropriately, so that they are characterised by the $(n+2)$ coprime integers $(p, q, r_1, \ldots, r_n)$, we find that the local metrics extend smoothly onto complete and non-singular compact manifolds.

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Appendix

A Singular Limits

In this paper we have focussed on limits of the Euclideanised Kerr de Sitter metrics in odd dimensions, in which all the rotation parameters a_i are subjected to the limiting procedure given in equation (3.3). As we have seen, this yields a rich source of non-singular Einstein-Sasaki spaces, which we denote by $L^{p,q,r_1,\cdots,r_{n-1}}$ in $D = 2n+1$ dimensions.

[5]One should not be misled by the terminology "quasi-regular" and "irregular" that is applied to Einstein-Sasaki spaces; all the spaces $L^{p,q,r}$ are complete and non-singular.

In fact, a BPS limit also arises if only a subset of the rotation parameters are subjected to the limiting procedure given in (3.3). Here, we study the resulting metrics, and we show that they are singular in all such cases.

After performing the Euclideanisation as in section 3, we now take limits as follows:

$$a_i = \lambda^{-\frac{1}{2}}(1 - \tfrac{1}{2}\alpha_i\,\epsilon)\,, \quad i = 1, \ldots, p \le n$$
$$r^2 = \lambda^{-1}(1 - x\epsilon)\,, \quad m = \tfrac{1}{2}\lambda^{-1}\mu\epsilon^{p+1} \tag{A.1}$$

and then send $\epsilon \to 0$. Note that for $p < n$, the angular momentum parameters $a_j < \lambda^{-\frac{1}{2}}$ with $j = p+1, \ldots, n$ are not scaled. This limit corresponds to Lorentzian BPS solutions with

$$E = g \sum_{i=1}^{p} J_i\,, \tag{A.2}$$

and has more supersymmetry than in the $p = n$ case that we discussed in section 3. The metrics (3.1) become

$$\lambda\, ds^2_{2n+1} = (d\tau + \sigma)^2 + ds^2_{2n}\,, \tag{A.3}$$

where

$$\begin{aligned}
d\bar{s}^2_{2n} &= \frac{Y dx^2}{4xF} - \frac{x(1-F)}{Y}\Big(\sum_{i=1}^{p} \alpha_i^{-1}\,\mu_i^2 d\varphi_i\Big)^2 + \sum_{i=1}^{p}(1-\alpha_i^{-1}\,x)(d\mu_i^2 + \mu_i^2 d\varphi_i^2) \\
&\quad + \frac{x}{\sum_{i=1}^{p}\alpha_i^{-1}\mu_i^2}\Big(\sum_{j=1}^{p}\alpha_j^{-1}\,\mu_j d\mu_j\Big)^2 - \sigma^2 + \sum_{k=p+1}^{n}(d\mu_k^2 + \mu_k^2 d\varphi_k^2) \\
&\quad + \sum_{i=1}^{p}(1-\alpha_i^{-1}\,x)(d\mu_i^2 + \mu_i^2 d\varphi_i^2) \\
\sigma &= \sum_{i=1}^{p}(1-\alpha_i^{-1}x)\mu_i^2\, d\varphi_i\,,
\end{aligned} \tag{A.4}$$

where $\sum_{i=1}^{n}\mu_i^2 = 1$, and

$$Y = \sum_{i=1}^{p}\frac{\mu_i^2}{\alpha_i - x}\,, \qquad F = 1 - \frac{\mu}{x}\prod_{i=1}^{p}(\alpha_i - x)^{-1}\,. \tag{A.5}$$

Note that in this scaling limit all the dependence on a_j for $j = p+1, \ldots, n$ has been absorbed into a redefinition of the mass parameter μ. For $p < n$, one can introduce a new set of coordinates:

$$\begin{aligned}
\mu_i &= \nu_i\,\sin\gamma\,, \quad i = 1, \ldots, p\,, \\
\mu_{j+p} &= \tilde{\mu}_j\,\cos\gamma\,, \quad j = 1, \ldots, (n-p)\,,
\end{aligned} \tag{A.6}$$

where $\sum_{i=1}^{p} \nu_i^2 = 1$ and $\sum_{j=1}^{n-p} \tilde{\mu}_j^2 = 1$. The metric can now be cast in the form:

$$ds_{2n+1}^2 = d\gamma^2 + \cos^2\gamma\, d\Omega_{2n-2p-1}^2 + \sin^2\gamma\, dS_{2p+1}^2\,. \tag{A.7}$$

Here $d\Omega_{2n-2p-1}^2$ is the metric on the unit $(2n-2p-1)$-sphere, and dS_{2p+1}^2 is the previously-obtained Einstein-Sasaki metric (3.4) in $2p+1$ dimensions with all $\alpha_i \neq 0$ $(i = 1 \cdots p)$. Note that since dS_{2p+1}^2 is an Einstein-Sasaki space (and not a round sphere), the metric (A.7) is singular at $\gamma = 0$. A cone over (A.7) produces a direct product of spaces in $D = 2n+2$, namely $\mathbf{R}^{2(p-n)} \times C_{2p+2}$, where C_{2p+2} is a $2p+2$ dimensional Calabi-Yau cone.

The above discussion was for the case of BPS limits for the Kerr-de Sitter metric in odd dimension $D = 2n+1$, in which only p of the n rotation parameters were subjected to the limiting procedure in (3.3). A very similar discussion can be given for BPS limits of the Kerr-de Sitter metrics in even dimension $D = 2n$, whose metrics are also given in [14, 15]. In this case there are again n coordinates μ_i, but only $(n-1)$ azimuthal angles φ_i, and so the situation is similar to the odd-dimensional case when one of the rotation parameters is set to zero. For this reason, we find that all BPS limits give rise to singular metrics, similar in form to (A.7). In fact, applying the limiting procedure in (3.3) to all $(n-1)$ rotations, we obtain the metric

$$ds_{2n}^2 = d\gamma^2 + \sin^2\gamma\, dS_{2n-1}^2\,, \tag{A.8}$$

where dS_{2n-1}^2 is an Einstein-Sasaki metric of the kind obtained in section 3. Clearly(A.8) is singular at $\gamma = 0$, and the metric does not extend onto a non-singular manifold.

References

[1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2**, 231 (1998), hep-th/9711200.

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428**, 105 (1998), hep-th/9802109.

[3] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.

[4] C.P. Boyer and K. Galicki, *New Einstein metrics in dimension five*, J. Diff. Geom. **57**, 443 (2001), math.FG/0003174.

[5] C.P. Boyer and K. Galicki, *Sasakian geometry, hypersurface singularities, and Einstein metrics*, math.DG/0405256.

[6] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on* $S^2 \times S^3$, hep-th/0403002.

[7] J.P. Gauntlett, D. Martelli, J.F. Sparks and D. Waldram, *A new infinite class of Sasaki-Einstein manifolds*, hep-th/0403038.

[8] L. Berard-Bergery, *Quelques examples de varietes riemanniennes completes non compactes a courbure de Ricci positive*, C.R. Acad. Sci, Paris, Ser. 1302, 159 (1986).

[9] D.N. Page and C.N. Pope, *Inhomogeneous Einstein metrics on complex line bundles*, Class. Quant. Grav. **4**, 213 (1987).

[10] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS backgrounds in string and M-theory*, hep-th/0411194.

[11] W. Chen, H. Lü, C.N. Pope and J.F. Vazquez-Poritz, *A note on Einstein-Sasaki metrics in* $D \geq 7$, hep-th/0411218.

[12] Y. Hashimoto, M. Sakaguchi and Y. Yasui, *Sasaki-Einstein twist of Kerr-AdS black holes*, Phys. Lett. **B600**, 270 (2004), hep-th/0407114.

[13] S.W. Hawking, C.J. Hunter and M.M. Taylor-Robinson, *Rotation and the AdS/CFT correspondence*, Phys. Rev. **D59**, 064005 (1999), hep-th/9811056.

[14] G.W. Gibbons, H. Lü, D.N. Page and C.N. Pope, *The general Kerr-de Sitter metrics in all dimensions,* J. Geom. Phys. **53**, 49 (2005), hep-th/0404008.

[15] G.W. Gibbons, H. Lü, D.N. Page and C.N. Pope, *Rotating black holes in higher dimensions with a cosmological constant,* Phys. Rev. Lett. **93**, 171102 (2004), hep-th/0409155.

[16] M. Cvetič, P. Gao and J. Simon, *Supersymmetric Kerr-anti-de Sitter solutions*, hep-th/0504136.

[17] M. Cvetič, H. Lü, D.N. Page and C.N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, hep-th/0504225.

[18] G.W. Gibbons, M.J. Perry and C.N. Pope, *The first law of thermodynamics for Kerr-anti-de Sitter black holes*, hep-th/0408217, to appear in Class. Quantum Grav.

[19] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Rotating black holes in gauged supergravities: Thermodynamics, supersymmetric limits, topological solitons and time machines*, hep-th/0504080.

[20] D. Martelli and J. Sparks, *Toric Sasaki-Einstein metrics on* $S^2 \times S^3$ hep-th/0505027.

[21] R.L Bishop, Notices Amer. Math. Soc. **10**, 363 (1963).

[22] D.N. Page, *A compact rotating gravitational instanton*, Phys. Lett. **B79**, 235 (1978).

[23] S. Benvenuti and M. Kruczenski, *From Sasaki-Einstein spaces to quivers via BPS geodesics:* L^{pqr}, hep-th/0505206.

[24] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, hep-th/0411238.

[25] M. Bertolini, F. Bigazzi and A.L. Cotrone, *New checks and subtleties for AdS/CFT and a-maximization*, JHEP **0412**, 024 (2004), hep-th/0411249.

[26] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, hep-th/0411264.

[27] D. Martelli, J. Sparks and S.T. Yau, *The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds*, hep-th/0503183.

[28] D. Gepner and S.S. Pal, *Branes in* $L(p,q,r)$, hep-th/0505039.

[29] C. Ahn and J.F. Vazquez-Poritz, *Marginal deformations with* $U(1)^3$ *global symmetry*, hep-th/0505168.