MIFPA-10-22
TUW-10-06

Flipped $SU(5)$ GUTs from E_8 Singularities in F-theory

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Abstract

In this paper we construct supersymmetric flipped $SU(5)$ GUTs from E_8 singularities in F-theory. We start from an $SO(10)$ singularity unfolded from an E_8 singularity by using an $SU(4)$ spectral cover. To obtain realistic models, we consider $(3,1)$ and $(2,2)$ factorizations of the $SU(4)$ cover. After turning on the massless $U(1)_X$ gauge flux, we obtain the $SU(5) \times U(1)_X$ gauge group. Based on the well-studied geometric backgrounds in the literature, we demonstrate several models and discuss their phenomenology.

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1 Introduction

String theory is a ten-dimensional theory of quantum gravity and so far is the most promising candidate for a fundamental unified theory. To build connections to the physics at a low energy scale, string theorists have been using the techniques of compactification to construct models in four-dimensional spacetime. F-theory [1–3](see [4] for review) is a twelve-dimensional geometric extension of string theory where one can engineer gauge theories from a geometric approach [5, 6]. We are interested in how gauge theories realized by F-theory can accommodate Grand Unified Theory (GUT) models. Recently, extensive studies of GUT local models and their corresponding phenomenology in F-theory have been undertaken in [7–32]. In addition, supersymmetry breaking has been discussed in [33–37], and the application to cosmology has been studied in [38]. Semi-local and global model building in F-theory were particularly discussed in [39–66]. Systematic studies of how models of higher rank GUT groups, such as $SO(10)$, are embedded into the compact geometry in F-theory have not been fully investigated. To this end, we are interested in the $SO(10)$ subgroup $SU(5) \times U(1)_X$ which is realized as the flipped $SU(5)$ GUT [67–69]. Although local flipped $SU(5)$ models have been discussed in F-theory, we study the model as a semi-local construction. In this paper we shall build flipped $SU(5)$ models by unfolding an E_8 singularity via the $SO(10)$ gauge group.

To construct flipped $SU(5)$ models in the four-dimensional spacetime, we compactify F-theory on an elliptically fibered Calabi-Yau fourfold X_4 with a base threefold B_3. We adopt a bottom-up approach to construct models in the decoupling limit to avoid full F-theory on a complicated elliptically fibered Calabi-Yau fourfold. More precisely, we consider a contractible complex surface S inside B_3 such that we can reduce full F-theory on X_4 to an effective eight-dimensional supersymmetric gauge theory on $\mathbb{R}^{3,1} \times S$. In this paper the surface S is assumed to be a del Pezzo surface [70, 71]. Since we will construct flipped $SU(5)$ models from an $SO(10)$ gauge group, we have to engineer the singularities of types D_5, D_6, E_6, and E_7 in the Calabi-Yau fourfold X_4. Because these singularities can be embedded into a single singularity E_8, we start our discussion from the E_8 singularity and unfold it into a D_5 singularity.

Generally, one may turn on certain fluxes to obtain the chiral spectrum. In F-theory, there is a four-form G-flux, which consists of three-form fluxes and gauge fluxes. In type IIB theory, these three-form fluxes produce a back-reaction in the background geometry. It has been shown in [30, 72] that the three-form fluxes induce non-commutative geometric structures and also modify the texture of the Yukawa couplings. F-theory in Fuzzy space also has been studied in [63]. In this paper we shall turn off these three-form fluxes and focus only on the gauge fluxes. The gauge $U(1)_X$ flux is able to break the gauge group $SO(10)$ down to $SU(5) \times U(1)_X$. It was shown in [9, 48] that the spectral cover construction naturally encodes the unfolding information of an E_8 singularity as well as the gauge fluxes. In this paper we shall focus on the $SU(4)$ spectral cover encoding the $SO(10)$ singularity from unfolding E_8. The four-dimensional low-energy spectrum of the flipped $SU(5)$ model is then determined by the cover fluxes and the $U(1)_X$ flux.

The $SU(4)$ spectral cover has many interesting properties. From the subgroup decomposition of E_8, one can find that there is no explicit presentation of $\overline{\mathbf{10}}$. In addition, the cover associated to the $\mathbf{10}$ representation forms a double-curve and along this curve there are co-dimension two singularities. After resolving the singularities along the curve, one finds that the net chirality of the $\mathbf{10}$ curve vanishes [39]. Since the background geometry generically determines the G flux, there are not many degrees of freedom left to adjust the chirality on the $\mathbf{16}$ curve to create three-generation models. These ideas motivate us to consider factorizing the spectral cover [46, 47, 52, 54, 55] to introduce additional parameters for model building. We consider two possibilities of splitting the $SU(4)$ spectral cover: (3,1) and (2,2) factorizations. The curve of the fundamental representation is then divided into two $\mathbf{16}$ curves, while generically the $\mathbf{10}$ curve is detached into three. However, due to the monodromy structure there are only two $\mathbf{10}$ curves in the (3,1) case.

In semi-local $SO(10)$ GUTs, there exists only the $\mathbf{16}\,\mathbf{16}\,\mathbf{10}$ Yukawa coupling from the enhancement to an E_7 singularity. The GUT Higgs fields coming from the adjoints or other representations such as $\mathbf{45}$, $\mathbf{54}$, or $\mathbf{120}$ are absent in the F-theory construction. Therefore, the most convincing way to break the $SO(10)$ gauge group is turning on the $U(1)_X$ flux on the GUT surface S. This $U(1)_X$ gauge field can be massless [7, 10, 73], so we can interpret the gauge group as the flipped $SU(5)$ model after turning on such a flux. With non-trivial restrictions to the curves, this $U(1)_X$ flux generically modifies the net chirality of matter localized on these curves. We may identify the flipped $SU(5)$ superheavy Higgs fields with one of the $\mathbf{10}+\overline{\mathbf{10}}$ vector-like pairs in the spectrum for further gauge breaking to MSSM.

The organization of the rest of the paper is as follows: in section 2, we briefly review the local geometry of an elliptically fibered Calabi-Yau fourfold with ADE singularities and the $SU(4)$ spectral cover. In section 3, we study $(3,1)$ and $(2,2)$ factorizations of the $SU(4)$ cover. In section 4, we construct cover fluxes and compute the chirality of matter localized on each curve for the $(3,1)$ and $(2,2)$ cover factorizations. In section 5, we briefly review the $D3$ tadpole cancellation in F-theory. We also give explicit formulae of geometric and cover flux contributions in the tadpole cancellation. In section 6, we demonstrate several examples of flipped $SU(5)$ models and discuss their phenomenology. We summarize and conclude in section 7.

2 Preliminaries

2.1 Elliptically fibered Calabi-Yau Fourfolds and ADE Singularities

Let us consider an elliptically fibered Calabi-Yau fourfold $\pi : X_4 \to B_3$ with a section $\sigma_{B_3} : B_3 \to X_4$. Due to the presence of the section σ_{B_3}, X_4 can be described by the Weierstrass form:

$$y^2 = x^3 + fx + g, \tag{2.1}$$

where f and g are sections of suitable line bundles over B_3. More precisely, to maintain Calabi-Yau condition $c_1(X_4) = 0$, it is required that[1] $f \in \Gamma(K_{B_3}^{-4})$ and $g \in \Gamma(K_{B_3}^{-6})$, where K_{B_3} is the canonical bundle of B_3. Let $\Delta \equiv 4f^3 + 27g^2$ be the discriminant of the elliptic fibration Eq. (2.1) and S be one component of the locus $\{\Delta = 0\}$ where elliptic fibers degenerate. In the vicinity of S, one can regard X_4 as an ALE fibration over the surface S. To construct $SO(10)$ and flipped $SU(5)$ GUT models, one can start with engineering a D_5 singularity corresponding to the gauge group $SO(10)$ in the following way. Let z be a section of the normal bundle N_{S/B_3} of S in B_3 and the zero section then represents the surface S. Since f and g are sections of some line bundles over B_3, one can locally expand f and g in terms of z as follows:

$$f = 3\sum_{k=0}^{4} f_k(u,v)z^k, \qquad g = 2\sum_{l=0}^{6} g_l(u,v)z^l, \tag{2.2}$$

[1]The symbol $\Gamma(L)$ stands for a set of global sections of the bundle L.

where (u, v) are coordinates of S and the prefactors 2 and 3 are just for convenience. Then the Weierstrass form Eq. (2.1),

$$y^2 = x^3 + 3\sum_{k=0}^{4} f_k(u,v) z^k x + 2\sum_{l=0}^{6} g_l(u,v) z^l, \tag{2.3}$$

describes an ALE fibration over S, where $f_k \in \Gamma(K_{B_3}^{-4} \otimes \mathcal{O}_{B_3}(-kS))$ and $g_l \in \Gamma(K_{B_3}^{-6} \otimes \mathcal{O}_{B_3}(-lS))$.[2] According to the Kodaira classification of singular elliptic fibers, one can classify the singularity of an elliptic fibration by the vanishing order of f, g, and Δ, denoted by $\mathrm{ord}(f)$, $\mathrm{ord}(g)$, and $\mathrm{ord}(\Delta)$, respectively. We summarize the relevant ADE classification and corresponding gauge groups in Table 1. A detailed list can be found in [9]. According to Table 1, a D_5 singularity corresponds to the case of

Singularity	ord(f)	ord(g)	ord(Δ)	Gauge Group
A_n	0	0	$n+1$	$SU(n+1)$
D_{n+4}	$\geqslant 2$	3	$n+6$	$SO(2n+8)$
D_{n+4}	2	$\geqslant 3$	$n+6$	$SO(2n+8)$
E_6	$\geqslant 3$	4	8	E_6
E_7	3	$\geqslant 5$	9	E_7
E_8	$\geqslant 4$	5	10	E_8

Table 1: ADE singularities and corresponding gauge groups.

$(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta)) = (\geqslant 2, 3, 7)$ or $(2, \geqslant 3, 7)$. Recall that S is the locus $\{z = 0\}$. To obtain a D_5 singularity, the vanishing orders of f and g at $z = 0$ are required to be two and three, respectively[3]. Let us consider the sections f and g to be

$$f = 3(f_2 z^2 + f_3 z^3), \qquad g = 2(g_3 z^3 + g_4 z^4 + g_5 z^5). \tag{2.4}$$

Then the corresponding discriminant is given by

$$\begin{aligned} \Delta &= cz^6[(f_2^3 + g_3^2) + (3f_2^2 f_3 + 2g_3 g_4)z + (3f_2 f_3^2 + g_4^2 + 2g_3 g_5)z^2 \\ &+ (f_3^3 + 2g_4 g_5)z^3 + \mathcal{O}(z^4)], \end{aligned} \tag{2.5}$$

[2] By adjunction formula, $K_S = K_{B_3} \otimes N_{S/B_3}|_S$, we have $f_k \in \Gamma(K_S^{-4} \otimes N_{S/B_3}^{4-k})$ and $g_l \in \Gamma(K_S^{-6} \otimes N_{S/B_3}^{6-l})$, where K_S is the canonical bundle of S.

[3] One can show that in this case the only consistent triplet vanishing orders for a D_5 singularity is $(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta)) = (2, 3, 7)$. The higher order terms are irrelevant to the singularity. However, they may change the monodromy group [62].

where $c = 4 \cdot 27$. To obtain $\mathrm{ord}(\Delta) = 7$, let us set $f_2 = -h^2$ and $g_3 = h^3$, where $h \in \Gamma(K_{B_3}^{-2} \otimes \mathcal{O}_{B_3}(-S))$. Then the discriminant is reduced to

$$\Delta = cz^7[(3h^4 f_3 + 2h^3 g_4) + (-3h^2 f_3^2 + g_4^2 + 2h^3 g_5)z + (f_3^3 + 2g_4 g_5)z^2 + \mathcal{O}(z^3)]. \quad (2.6)$$

The singularity of ALE fibration is now characterized by the sections $\{h, f_3, g_4, g_5\}$. When $h = 0$, one can find that $(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta)) = (3, 4, 8)$ at the locus $\{z = 0\} \cap \{h = 0\}$. It follows from the Kodaira classification that the singularity is enhanced to E_6. When $3hf_3 + 2g_4 = 0$, the triplet vanishing orders becomes $(2, 3, 8)$, which implies that the singularity at the locus $\{z = 0\} \cap \{3hf_3 + 2g_4 = 0\}$ is D_6 and that the corresponding enhanced gauge group is $SO(12)$. In a similar manner, one can find the codimension two singularities corresponding to E_7 and $SO(14)$ in S. We summarize the results in Table 2.

Gauge Group	$(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta))$	Locus
$SO(10)$	$(2, 3, 7)$	$\{z = 0\}$
E_6	$(3, 4, 8)$	$\{z = 0\} \cap \{h = 0\}$
$SO(12)$	$(2, 3, 8)$	$\{z = 0\} \cap \{3hf_3 + 2g_4 = 0\}$
E_7	$(3, 5, 9)$	$\{z = 0\} \cap \{h = 0\} \cap \{g_4 = 0\}$
$SO(14)$	$(2, 3, 9)$	$\{z = 0\} \cap \{3hf_3 + 2g_4 = 0\} \cap \{3f_3^2 - 8hg_5 = 0\}$

Table 2: Gauge enhancements and corresponding loci.

For later use, it is convenient to introduce the Tate form of the fibration:

$$y^2 = x^3 + \mathbf{b}_4 x^2 z + \mathbf{b}_3 y z^2 + \mathbf{b}_2 x z^3 + \mathbf{b}_0 z^5, \quad (2.7)$$

where $\mathbf{b}_m \in \Gamma(K_S^{m-6} \otimes N_{S/B_3})$. Actually, Eq. (2.7) is nothing more than the unfolding of an E_8 singularity to a singularity of $SO(10)$. Notice that by comparing Eq. (2.7) with Eqs. (2.3) and (2.4), one can obtain the relations between $\{f_2, f_3, g_3, g_4, g_5\}$ and $\{\mathbf{b}_0, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ as follows:

$$\begin{cases} f_2 = -\frac{1}{9}\mathbf{b}_4^2 \\ f_3 = \frac{1}{3}\mathbf{b}_2 \\ g_3 = \frac{1}{27}\mathbf{b}_4^3 \\ g_4 = \frac{1}{8}\mathbf{b}_3^2 - \frac{1}{6}\mathbf{b}_2\mathbf{b}_4 \\ g_5 = \frac{1}{2}\mathbf{b}_0. \end{cases} \quad (2.8)$$

With the relations in Eq. (2.8), the discriminant Eq. (2.6) becomes

$$\begin{aligned}\Delta &= \widetilde{c}z^7\{16\mathbf{b}_3^2\mathbf{b}_4^3 + [27\mathbf{b}_3^4 - 72\mathbf{b}_2\mathbf{b}_3^2\mathbf{b}_4 - 16\mathbf{b}_4^2(\mathbf{b}_2^2 - 4\mathbf{b}_0\mathbf{b}_4)]z \\ &+ [16\mathbf{b}_2(4\mathbf{b}_2^2 - 18\mathbf{b}_0\mathbf{b}_4) + 216\mathbf{b}_0\mathbf{b}_3^2]z^2 + \mathcal{O}(z^3)\},\end{aligned} \tag{2.9}$$

where $\widetilde{c} = \frac{1}{16}$. It follows from Eq. (2.8) that the codimension one loci $\{z = 0\} \cap \{h = 0\}$ and $\{z = 0\} \cap \{3hf_3 + 2g_4\}$ in S can be equivalently expressed as $\{z = 0\} \cap \{\mathbf{b}_4 = 0\}$ and $\{z = 0\} \cap \{\mathbf{b}_3 = 0\}$, respectively. Due to the gauge enhancements, matter **16** and **10** are localized at the loci of E_6 and $SO(12)$ singularities, respectively. One can also find that the loci of codimension two singularities E_7 and $SO(14)$ in S are $\{z = 0\} \cap \{\mathbf{b}_3 = 0\} \cap \{\mathbf{b}_4 = 0\}$ and $\{z = 0\} \cap \{\mathbf{b}_3 = 0\} \cap \{\mathbf{b}_2^2 - 4\mathbf{b}_0\mathbf{b}_4 = 0\}$, respectively. At these loci, the corresponding gauge groups are enhanced to E_7 and $SO(14)$, respectively[4]. In particular, the Yukawa coupling **16 16 10** can be realized at the points with E_7 singularities. We summarize the results in Table 3.

Gauge Group	Locus	Object
$SO(10)$	$\{z = 0\}$	GUT Seven-branes
E_6	$\{z = 0\} \cap \{\mathbf{b}_4 = 0\}$	Matter **16**
$SO(12)$	$\{z = 0\} \cap \{\mathbf{b}_3 = 0\}$	Matter **10**
E_7	$\{z = 0\} \cap \{\mathbf{b}_3 = 0\} \cap \{\mathbf{b}_4 = 0\}$	Yukawa Coupling **16 16 10**
$SO(14)$	$\{z = 0\} \cap \{\mathbf{b}_3 = 0\} \cap \{\mathbf{b}_2^2 - 4\mathbf{b}_0\mathbf{b}_4 = 0\}$	Extra Coupling

Table 3: Gauge enhancements in $SO(10)$ GUT geometry.

2.2 $SU(4)$ Spectral Cover

To engineer the $SO(10)$ gauge group from an E_8 singularity, let us consider the following decomposition

$$\begin{aligned} E_8 &\rightarrow SO(10) \times SU(4)_\perp \\ \mathbf{248} &\rightarrow (\mathbf{1}, \mathbf{15}) + (\mathbf{45}, \mathbf{1}) + (\mathbf{10}, \mathbf{6}) + (\mathbf{16}, \mathbf{4}) + (\overline{\mathbf{16}}, \bar{\mathbf{4}}).\end{aligned} \tag{2.10}$$

[4]One can also use Tate's algorithm to determine the singularity type of the Tate form Eq. (2.7) [6].

and the Tate form of the fibration,

$$y^2 = x^3 + \mathbf{b}_4 x^2 z + \mathbf{b}_3 y z^2 + \mathbf{b}_2 x z^3 + \mathbf{b}_0 z^5. \tag{2.11}$$

For simplicity, let us define $c_1 \equiv c_1(S)$ and $t \equiv -c_1(N_{S/B_3})$, then the homological classes of the sections x, y, z, and b_m can be expressed as

$$[x] = 3(c_1 - t),\ [y] = 2(c_1 - t),\ [z] = -t,\ [\mathbf{b}_m] = (6-m)c_1 - t \equiv \eta - mc_1. \tag{2.12}$$

Recall that locally X_4 can be described by an ALE fibration over S. Pick a point $p \in S$ and the fiber is an ALE space denoted by ALE_p. One can construct an ALE space by resolving an orbifold $\mathbb{C}^2/\Gamma_{ADE}$, where Γ_{ADE} is a discrete subgroup of $SU(2)$ [74], for more information, see [75–79]. It was shown that the intersection matrix of the exceptional 2-cycles corresponds to the Cartan matrix of ADE types. In this paper we will focus on engineering the $SO(10)$ gauge group by unfolding an E_8 singularity. To this end, let us consider $\alpha_i \in H_2(\mathrm{ALE}_p, \mathbb{Z})$, $i = 1, 2, ..., 8$ to be the roots[5] of E_8. The extended E_8 Dynkin diagram with roots and Dynkin indices are shown in Fig 1. Notice that $\alpha_{-\theta}$ is the highest root and satisfies the condition

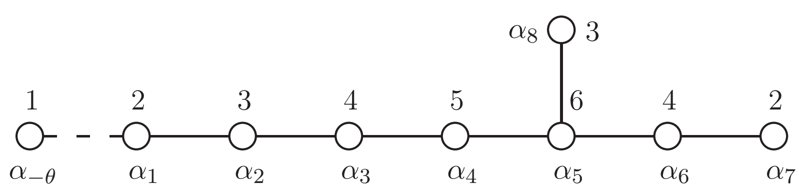

Figure 1: The extended E_8 Dynkin diagram and indices

$\alpha_{-\theta} + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0$. To obtain $SO(10)$, we keep the volume of the cycles $\{\alpha_4, \alpha_5, ..., \alpha_8\}$ vanishing and then $SU(4)_\perp$ is generated by $\{\alpha_1, \alpha_2, \alpha_3\}$. An enhancement to E_6 happens when α_3 or any of its image under the Weyl permutation shrinks to zero size. Let $\{\lambda_1, ..., \lambda_4\}$ be the periods of these 2-cycles. As described in [10, 48], the information of theses λ_i can be encoded in the

[5] By abuse of notation, the corresponding exceptional 2-cycles are also denoted by α_i

coefficients $\mathbf{b}_m$ in Eq. (2.11) via the following relations:

$$\begin{cases} \sum_i \lambda_i = \frac{b_1}{b_0} = 0 \\ \sum_{i<j} \lambda_i\lambda_j = \frac{b_2}{b_0} \\ \sum_{i<j<k} \lambda_i\lambda_j\lambda_k = \frac{b_3}{b_0} \\ \prod_l \lambda_l = \frac{b_4}{b_0}, \end{cases} \tag{2.13}$$

where $b_m \equiv \mathbf{b}_m|_{z=0}$. Equivalently, $\{\lambda_1, ..., \lambda_4\}$ can be regarded as the roots of the equation

$$b_0 \prod_k (s+\lambda_k) = b_0 s^4 + b_2 s^2 + b_3 s + b_4 = 0. \tag{2.14}$$

When $p \in S$ varies along S, Eq. (2.14) defines a fourfold cover over S, called the fundamental $SU(4)$ spectral cover. This cover is a section of the canonical bundle $K_S \to S$. When λ_i vanish, $\prod_i \lambda_i = b_4 = 0$ in which the gauge group is enhanced to E_6 and matter **16** is localized. According to the decomposition (2.10), matter **10** corresponds to the anti-symmetric representation **6** of $SU(4)_\perp$, associated to a sixfold cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ over S. This associated cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ can be constructed as follows:

$$b_0^2 \prod_{i<j} (s+\lambda_i+\lambda_j) = b_0^2 s^6 + 2b_0 b_2 s^4 + (b_2^2 - 4b_0 b_4)s^2 - b_3^2 = 0. \tag{2.15}$$

Since matter **10** corresponds to $\lambda_i + \lambda_j = 0$, $i \neq j$, it follows from Eq. (2.15) that $b_3 = 0$, which means that matter **10** is localized at the locus $\{b_3 = 0\}$ as shown in Table 3. It is not difficult to see that the spectral covers indeed encode the information of singularities and gauge group enhancements. However, the spectral cover is even more powerful. With it, we can construct a Higgs bundle to calculate the chirality of matter **16** and **10** by switching on a line bundle on the cover.

Let us define X to be the total space of the canonical bundle K_S over S. Note that X is a local Calabi-Yau threefold. However, X is non-compact. To obtain a compact space, one can compactify X to the total space $\bar{X}$ of the projective bundle over S, *i.e.*

$$\bar{X} = \mathbb{P}(\mathcal{O}_S \oplus K_S), \tag{2.16}$$

with a map $\pi : \bar{X} \to S$, where $\mathcal{O}_S$ is the trivial bundle over S. Notice that $\bar{X}$ is compact but no longer a Calabi-Yau threefold. Let $\mathcal{O}(1)$ be a hyperplane section of $\mathbb{P}^1$ fiber and denote its first Chern class by σ_∞. We define the homogeneous coordinates of the fiber by $[U : V]$. Note that $\{U = 0\}$ and $\{V = 0\}$ are sections of $\mathcal{O}(1) \otimes K_S$ and $\mathcal{O}(1)$, while the class of $\{U = 0\}$ and $\{V = 0\}$ are $\sigma \equiv \sigma_\infty - \pi^* c_1(S)$ and σ_∞, respectively. The intersection of $\{U = 0\}$ and $\{V = 0\}$ is empty. Thus, one can obtain $\sigma \cdot \sigma = -\sigma \cdot \pi^* c_1$. The affine coordinate s is defined by $s = U/V$. In $\bar{X}$, the $SU(4)$ cover Eq. (2.14) is homogenized as

$$\mathcal{C}_V^{(4)} : \quad b_0 U^4 + b_2 U^2 V^2 + b_3 U V^3 + b_4 V^4 = 0 \tag{2.17}$$

with induced map $p_4 : \mathcal{C}_V^{(4)} \to S$. It is not difficult to see that the homological class $[\mathcal{C}_V^{(4)}]$ of the cover $\mathcal{C}_V^{(4)}$ is given by $[\mathcal{C}_V^{(4)}] = 4\sigma + \pi^*\eta$. One can calculate the locus of the matter **16** curve by intersection of $[\mathcal{C}_V^{(4)}]$ with σ

$$[\mathcal{C}_V^{(4)}] \cap \sigma = (4\sigma + \pi^*\eta) \cdot \sigma = \sigma \cdot \pi^*(\eta - 4c_1), \tag{2.18}$$

which implies that $[\Sigma_{\mathbf{16}}] = \eta - 4c_1$ in S. Alternatively, one could deduce this from the fact that the locus of $\Sigma_{\mathbf{16}}$ in S is $\{b_4 = 0\}$. It follows from Eq. (2.15) that the homological class of the cover $\mathcal{C}_{\wedge^2 V}^{(6)}$ is given by

$$[\mathcal{C}_{\wedge^2 V}^{(6)}] = 6\sigma + 2\pi^*\eta \tag{2.19}$$

Notice that $\mathcal{C}_{\wedge^2 V}^{(6)}$ is generically singular. To solve this problem, one can consider intersection $\tau\mathcal{C}_V \cap \mathcal{C}_V$ and define [41, 80]

$$[D] = [\mathcal{C}_V^{(4)}] \cap [\mathcal{C}_V^{(4)}] - [\mathcal{C}_V^{(4)}] \cap \sigma - [\mathcal{C}_V^{(4)}] \cap 3\sigma_\infty \tag{2.20}$$

where τ is a $\mathbb{Z}_2$ involution $V \to -V$ acting on the spectral cover[6]. The **10** curve can then be evaluated by

$$[D]|_\sigma = 4(\eta - 3c_1), \tag{2.21}$$

which implies that $[\Sigma_{\mathbf{10}}] = 2\eta - 6c_1$ in S.

To obtain chiral spectrum, we turn on a spectral line bundle $\mathcal{L}$ on the cover $\mathcal{C}_V^{(4)}$. The corresponding Higgs bundle is given by $V = p_{4*}\mathcal{L}$. For an $SU(n)$ bundle, it is required that $c_1(V) = 0$. It follows that

$$c_1(p_{4*}\mathcal{L}) = p_{4*}c_1(\mathcal{L}) - \frac{1}{2}p_{4*}r, \tag{2.22}$$

[6] Note that there are double points on $\Sigma_{\mathbf{10}}$. One can resolve these double points by blowing-up and then obtain resolved $\tilde{\Sigma}_{\mathbf{10}}$ with a mapping $\tilde{\pi}_D : D \to \tilde{\Sigma}_{\mathbf{10}}$ of degree 4 and $[\tilde{\Sigma}_{\mathbf{10}}] = \eta - 3c_1$ [39].

where r is the ramification divisor given by $r = p_{4*}c_1 - c_1(\mathcal{C}_V^{(4)})$. It is convenient to define the cover flux γ by

$$c_1(\mathcal{L}) = \lambda\gamma + \frac{1}{2}r, \tag{2.23}$$

where λ is a parameter used to compensate the non-integral class $\frac{1}{2}r$. The traceless condition $c_1(p_{4*}\mathcal{L}) = 0$ is then equivalent to the condition $p_{4*}\gamma = 0$. One can show that

$$\gamma = (4 - p_4^* p_{4*})(\mathcal{C}_V^{(4)} \cdot \sigma) \tag{2.24}$$

satisfies the traceless condition. Since the first Chern class of a line bundle must be integral, it follows that λ and γ have to obey the following quantization condition

$$\lambda\gamma + \frac{1}{2}[p_4^* c_1 - c_1(\mathcal{C}_V^{(4)})] \in H_4(\bar{X}, \mathbb{Z}). \tag{2.25}$$

With the given cover flux γ, the net chirality of matter **16** is calculated by [39, 48]

$$N_{\mathbf{16}} = (\mathcal{C}_V^{(4)} \cdot \sigma) \cdot \lambda\gamma = -\lambda\eta \cdot (\eta - 4c_1) \tag{2.26}$$

On the other hand, the matter **10** corresponds to the anti-symmetric representation **6** in $SU(4)_\perp$, associated to the spectral cover $\mathcal{C}^{(6)}_{\wedge^2 V}$. It turns out that for the $SU(4)$ cover, the net chirality of matter **10** is given by [39]

$$N_{\mathbf{10}} = D \cdot \gamma = 0. \tag{2.27}$$

It follows from Eqs. (2.26) and (2.27) that one obtain an $SO(10)$ model with $-\lambda\eta \cdot (\eta - 4c_1)$ copies of matter on the **16** curve and nothing on the **10** curve. The flux γ does not have many degrees of freedom to tune and the candidate of **10** Higgs is absent. Therefore, in search of realistic models, we shall consider factorization of the $SU(4)$ cover $\mathcal{C}_V^{(4)}$ to enrich the configuration, along the line of the $SU(5)$ cover studied in [46, 47, 52, 54]. In the next section, we shall focus on the construction of $(3, 1)$ and $(2, 2)$ factorizations of the cover $\mathcal{C}_V^{(4)}$.

3 $SU(4)$ Cover Factorization

3.1 $(3, 1)$ Factorization

We consider the $(3, 1)$ factorization, $C_V^{(4)} \to \mathcal{C}^{(a)} \times \mathcal{C}^{(b)}$ corresponding to the factorization of Eq. (2.17) as follows:

$$\mathcal{C}^{(a)} \times \mathcal{C}^{(b)} : \quad (a_0U^3 + a_1U^2V + a_2UV^2 + a_3V^3)(d_0U + d_1V) = 0. \tag{3.1}$$

By comparing with Eq. (2.17), one can obtain the following relations:

$$b_0 = a_0 d_0, \quad b_1 = a_1 d_0 + a_0 d_1, \quad b_2 = a_2 d_0 + a_1 d_1, \quad b_3 = a_3 d_0 + a_2 d_1, \quad b_4 = a_3 d_1. \quad (3.2)$$

Let ξ_1 be the homological class $[d_1]$ of d_1 and write

$$[d_0] = c_1 + \xi_1, \qquad [a_k] = \eta - (k+1)c_1 - \xi_1, \quad k = 0, 1, 2, 3. \quad (3.3)$$

It is easy to see that the homological classes of $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(b)}$ in $\bar{X}$ are

$$[\mathcal{C}^{(a)}] = 3\sigma + \pi^*(\eta - c_1 - \xi_1), \quad [\mathcal{C}^{(b)}] = \sigma + \pi^*(c_1 + \xi_1). \quad (3.4)$$

With the classes given in Eq. (3.4), the homological classes of factorized matter curves $\Sigma_{\mathbf{16}^{(a)}}$ and $\Sigma_{\mathbf{16}^{(b)}}$ in S are given by

$$[\Sigma_{\mathbf{16}^{(a)}}] = [\mathcal{C}^{(a)}]|_\sigma = \eta - 4c_1 - \xi_1, \quad [\Sigma_{\mathbf{16}^{(b)}}] = [\mathcal{C}^{(b)}]|_\sigma = \xi_1. \quad (3.5)$$

To obtain the factorized **10** curves, we follow the method proposed in [46, 47, 52, 80] to calculate the intersection $\mathcal{C}_V^{(4)} \cap \tau\mathcal{C}_V^{(4)}$, where τ is the $\mathbb{Z}_2$ involution $\tau : V \to -V$ acting on the spectral cover. Since the calculation is straightforward, we omit the detailed calculation here and only summarize the results[7] in Table 4.

	$[\mathcal{C}^{(b)(b)}]$	$2[\mathcal{C}^{(a)(b)}]$	$[\mathcal{C}^{(a)(a)}]$
16	$\sigma \cdot \pi^*\xi_1$	-	$\sigma \cdot \pi^*(\eta - 4c_1 - \xi_1)$
10	$\pi^*\xi_1 \cdot \pi^*(c_1 + \xi_1)$	$2[\sigma + \pi^*(c_1 + \xi_1)]$ $\cdot\, \pi^*(\eta - 3c_1 - \xi_1) + 2\sigma \cdot \pi^*\xi_1$	$[2\sigma + \pi^*(\eta - 2c_1 - \xi_1)]$ $\cdot\, \pi^*(\eta - 3c_1 - \xi_1) + 2(\sigma + \pi^*c_1) \cdot \pi^*\xi_1$
∞	$\sigma_\infty \cdot \pi^*(c_1 + \xi_1)$	$4\sigma_\infty \cdot \pi^*(c_1 + \xi_1)$	$\sigma_\infty \cdot \pi^*(\eta - c_1 - \xi_1)$ $+2\sigma_\infty \cdot \pi^*(\eta - 2c_1 - 2\xi_1)$

Table 4: The homological classes of the matter curves in the $(3, 1)$ factorization.

It follows from Table 4 that the relevant classes in $\bar{X}$ for **10** curves are

$$[\mathcal{C}^{(a)(a)}] = [2\sigma + \pi^*(\eta - 2c_1 - \xi_1)] \cdot \ \pi^*(\eta - 3c_1 - \xi_1) + 2(\sigma + \pi^*c_1) \cdot \pi^*\xi_1, \quad (3.6)$$
$$[\mathcal{C}^{(a)(b)}] = [\sigma + \pi^*(c_1 + \xi_1)] \cdot \ \pi^*(\eta - 3c_1 - \xi_1) + \sigma \cdot \pi^*\xi_1, \quad (3.7)$$

which give rise to the **10** curves

$$[\Sigma_{\mathbf{10}^{(a)(a)}}] = \eta - 3c_1, \quad [\Sigma_{\mathbf{10}^{(a)(b)}}] = \eta - 3c_1, \quad (3.8)$$

respectively.

[7]To simplify notations, we denote $\mathcal{C}^{(k)} \cap \tau\mathcal{C}^{(l)}$ by $\mathcal{C}^{(k)(l)}$. Notice that $[\mathcal{C}^{(k)(l)}] = [\mathcal{C}^{(l)(k)}]$.

3.2 $(2,2)$ Factorization

In the $(2,2)$ factorization, the cover is split as $\mathcal{C}_V^{(4)} \to \mathcal{C}^{(d_1)} \times \mathcal{C}^{(d_2)}$. More precisely, the cover defined in Eq. (2.17) is factorized into the following form:

$$\mathcal{C}^{(d_1)} \times \mathcal{C}^{(d_2)}: \quad (e_0U^2 + e_1UV + e_2V^2)(f_0U^2 + f_1UV + f_2V^2) = 0. \tag{3.9}$$

By comparing the coefficients with Eq. (2.17), one obtains

$$b_0 = e_0f_0, \quad b_1 = e_0f_1 + e_1f_0, \quad b_2 = e_0f_2 + e_1f_1 + e_2f_0, \quad b_3 = e_1f_2 + e_2f_1, \quad b_4 = e_2f_2. \tag{3.10}$$

Let ξ_2 be the homological class of f_2 and then the homological classes of other sections can be written as

$$[f_1] = c_1 + \xi_2, \quad [f_0] = 2c_1 + \xi_2, \quad [e_m] = \eta - (m+2)c_1 - \xi_2, \quad m = 0,1,2. \tag{3.11}$$

In this case, the homological classes of $\mathcal{C}^{(d_1)}$ and $\mathcal{C}^{(d_2)}$ are given by

$$[\mathcal{C}^{(d_1)}] = 2\sigma + \pi^*(\eta - 2c_1 - \xi_2), \quad [\mathcal{C}^{(d_2)}] = 2\sigma + \pi^*(2c_1 + \xi_2). \tag{3.12}$$

The homological classes of the corresponding matter curves $\Sigma_{\mathbf{16}^{(d_1)}}$ and $\Sigma_{\mathbf{16}^{(d_2)}}$ are then computed as

$$[\Sigma_{\mathbf{16}^{(d_1)}}] = [\mathcal{C}^{(d_1)}]|_\sigma = \eta - 4c_1 - \xi_2, \quad [\Sigma_{\mathbf{16}^{(d_2)}}] = [\mathcal{C}^{(d_2)}]|_\sigma = \xi_2, \tag{3.13}$$

respectively. To calculate the homological classes of the factorized **10** curves, we again follow the method proposed in [46,47,52,80] to calculate the intersection $\mathcal{C}_V^{(4)} \cap \tau\mathcal{C}_V^{(4)}$. We omit the detailed calculation here and only summarize the results in Table 5.

	$[\mathcal{C}^{(d_2)(d_2)}]$	$2[\mathcal{C}^{(d_1)(d_2)}]$	$[\mathcal{C}^{(d_1)(d_1)}]$
16	$\sigma \cdot \pi^*\xi_2$	-	$\sigma \cdot \pi^*(\eta - 4c_1 - \xi_2)$
10	$(2\sigma + \pi^*(2c_1 + \xi_2)) \cdot \pi^*(c_1 + \xi_2)$	$2(2\sigma + \pi^*(2c_1 + \xi_2)) \cdot \pi^*(\eta - 4c_1 - \xi_2)$	$\pi^*(\eta - 3c_1 - \xi_2) \cdot \pi^*(\eta - 4c_1 - \xi_2) + 2(\sigma + \pi^*c_1) \cdot \pi^*(c_1 + \xi_2)$
∞	$\sigma_\infty \cdot \pi^*(2c_1 + \xi_2)$	$4\sigma_\infty \cdot \pi^*(2c_1 + \xi_2)$	$\sigma_\infty \cdot \pi^*(\eta - 2c_1 - \xi_2) + 2\sigma_\infty \cdot \pi^*(\eta - 4c_1 - 2\xi_2)$

Table 5: The homological classes of the matter curves in the $(2,2)$ factorization.

It follows from Table 5 that the classes in $\bar{X}$ for the factorized **10** curves are as follows:

$$[\mathcal{C}^{(d_1)(d_1)}] = 2(\sigma + \pi^* c_1) \cdot \pi^*(c_1 + \xi_2) + \pi^*(\eta - 3c_1 - \xi_2) \cdot \pi^*(\eta - 4c_1 - \xi_2), \quad (3.14)$$
$$[\mathcal{C}^{(d_1)(d_2)}] = (2\sigma + \pi^*(2c_1 + \xi_2)) \cdot \ \pi^*(\eta - 4c_1 - \xi_2), \quad (3.15)$$
$$[\mathcal{C}^{(d_2)(d_2)}] = (2\sigma + \pi^*(2c_1 + \xi_2)) \cdot \ \pi^*(c_1 + \xi_2). \quad (3.16)$$

With the classes $[\mathcal{C}^{(d_1)(d_1)}]$, $[\mathcal{C}^{(d_1)(d_2)}]$, and $[\mathcal{C}^{(d_2)(d_2)}]$, one can calculate the classes of the corresponding **10** curves in S as follows:

$$[\Sigma_{\mathbf{10}^{(d_1)(d_1)}}] = c_1 + \xi_2, \quad [\Sigma_{\mathbf{10}^{(d_1)(d_2)}}] = 2\eta - 8c_1 - 2\xi_2, \quad [\Sigma_{\mathbf{10}^{(d_2)(d_2)}}] = c_1 + \xi_2. \quad (3.17)$$

4 Spectral Cover Fluxes

Let us consider the case of the cover factorization $\mathcal{C}_V^{(n)} \to \mathcal{C}^{(l)} \times \mathcal{C}^{(m)}$. To obtain well-defined cover fluxes and maintain supersymmetry, we impose the following constraints [47]:

$$c_1(p_{l*}\mathcal{L}^{(l)}) + c_1(p_{m*}\mathcal{L}^{(m)}) = 0, \quad (4.1)$$
$$c_1(\mathcal{L}^{(k)}) \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m, \quad (4.2)$$
$$[c_1(p_{l*}\mathcal{L}^{(l)}) - c_1(p_{m*}\mathcal{L}^{(m)})] \cdot_S [\omega] = 0, \quad (4.3)$$

where p_k denotes the projection map from the cover $\mathcal{C}^{(k)}$ to S, $p_k:\ \mathcal{C}^{(k)} \to S$, $\mathcal{L}^{(k)}$ is a line bundle over $\mathcal{C}^{(k)}$ and $[\omega]$ is an ample divisor dual to a Kähler form of S. The first constraint Eq. (4.1) is the traceless condition for the induced Higgs bundle[8]. The second constraint Eq. (4.2) requires that the first Chern class of a well-defined line bundle $\mathcal{L}^{(k)}$ over $\mathcal{C}^{(k)}$ must be integral. The third constraint states that the 2-cycle $c_1(p_{l*}\mathcal{L}^{(l)}) - c_1(p_{m*}\mathcal{L}^{(m)})$ in S has to be supersymmetic. Note that Eq. (4.1) can be expressed as

$$p_{l*}c_1(\mathcal{L}^{(l)}) - \frac{1}{2}p_{l*}r^{(l)} + p_{m*}c_1(\mathcal{L}^{(m)}) - \frac{1}{2}p_{m*}r^{(m)} = 0, \quad (4.4)$$

[8]One may think of Eq. (4.1) as the traceless condition of an $SU(4)$ bundle V_4 over S split into $V_3 \oplus L$ with $V_3 = p_{a*}\mathcal{L}^{(a)}$ and $L = p_{b*}\mathcal{L}^{(b)}$. Then the traceless condition of V_4 can be expressed by $c_1(V_4) = c_1(p_{a*}\mathcal{L}^{(a)}) + c_1(p_{b*}\mathcal{L}^{(b)}) = 0$.

where $r^{(l)}$ and $r^{(m)}$ are the ramification divisors for the maps p_l and p_m, respectively. Recall that the ramification divisors $r^{(k)}$ are defined by

$$r^{(k)} = p_k^* c_1 - c_1(\mathcal{C}^{(k)}), \quad k = l, m. \tag{4.5}$$

The term $c_1(\mathcal{C}^{(k)})$ in Eq. (4.5) can be calculated by the adjuction formula [82, 83],

$$c_1(\mathcal{C}^{(k)}) = (c_1(\bar{X}) - [\mathcal{C}^{(k)}]) \cdot [\mathcal{C}^{(k)}]. \tag{4.6}$$

It is convenient to define cover fluxes $\gamma^{(k)}$ as

$$c_1(\mathcal{L}^{(k)}) = \gamma^{(k)} + \frac{1}{2} r^{(k)}, \quad k = l, m. \tag{4.7}$$

With Eq. (4.7), the traceless condition Eq. (4.1) can be expressed as $p_{l*}\gamma^{(l)} + p_{m*}\gamma^{(m)} = 0$. By using Eq. (4.5) and Eq. (4.7), we can recast the quantization condition Eq. (4.2) by $\gamma^{(k)} + \frac{1}{2}[p_k^* c_1 - c_1(\mathcal{C}^{(k)})] \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}),\quad k = l, m$. Finally, the supersymmetry condition Eq. (4.3) is reduced to $p_{k*}\gamma^{(k)} \cdot_S [\omega] = 0$. We summarize the constraints as follows:

$$p_{l*}\gamma^{(l)} + p_{m*}\gamma^{(m)} = 0, \tag{4.8}$$

$$\gamma^{(k)} + \frac{1}{2}[p_k^* c_1 - c_1(\mathcal{C}^{(k)})] \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m, \tag{4.9}$$

$$p_{k*}\gamma^{(k)} \cdot_S [\omega] = 0, \quad k = l, m. \tag{4.10}$$

In the next section, we shall explicitly construct the cover fluxes $\gamma^{(k)}$ satisfying Eq. (4.8), (4.9), and (4.10) for the $(3, 1)$ and $(2, 2)$ factorizations. We also calculate the restrictions of the fluxes to each matter curve.

4.1 (3,1) Factorization

In the $(3, 1)$ factorization, the ramification divisors for the spectral covers $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(b)}$ are given by

$$r^{(a)} = [\mathcal{C}^{(a)}] \cdot [\sigma + \pi^*(\eta - 2c_1 - \xi_1)], \tag{4.11}$$

$$r^{(b)} = [\mathcal{C}^{(b)}] \cdot (-\sigma + \pi^*\xi_1), \tag{4.12}$$

respectively. We define traceless fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ by

$$\gamma_0^{(a)} = (3 - p_a^* p_{a*})\gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [3\sigma - \pi^*(\eta - 4c_1 - \xi_1)], \tag{4.13}$$

$$\gamma_0^{(b)} = (1 - p_b^* p_{b*})\gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot (\sigma - \pi^*\xi_1)\,, \tag{4.14}$$

where $\gamma^{(a)}$ and $\gamma^{(b)}$ are non-traceless fluxes and defined as

$$\gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot \sigma, \quad \gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot \sigma. \tag{4.15}$$

Then we can calculate the restriction of fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ to each matter curve. We omit the calculation here and only summarize the results in the following table.

	$\gamma_0^{(b)}$	$\gamma_0^{(a)}$
$\mathbf{16}^{(b)}$	$-\xi_1 \cdot_S (c_1 + \xi_1)$	0
$\mathbf{16}^{(a)}$	0	$-(\eta - c_1 - \xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$
$\mathbf{10}^{(a)(b)}$	$-\xi_1 \cdot_S (c_1 + \xi_1)$	$-(\eta - 3c_1 - 3\xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$
$\mathbf{10}^{(a)(a)}$	0	$(\eta - 3c_1 - 3\xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$

(4.16)

Due to the factorization, one also can define additional fluxes $\delta^{(a)}$ and $\delta^{(b)}$ by

$$\begin{aligned} \delta^{(a)} &= (1 - p_b^* p_{a*})\gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot \sigma - [\mathcal{C}^{(b)}] \cdot \pi^*(\eta - 4c_1 - \xi_1) \\ \delta^{(b)} &= (3 - p_a^* p_{b*})\gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot 3\sigma - [\mathcal{C}^{(a)}] \cdot \pi^* \xi_1. \end{aligned} \tag{4.17}$$

Another flux one can include is [47]

$$\tilde{\rho} = (3p_b^* - p_a^*)\rho, \tag{4.18}$$

for any $\rho \in H_2(S, \mathbb{R})$. We summarize the restriction of fluxes $\delta^{(a)}$, $\delta^{(b)}$ and $\tilde{\rho}$ to each matter curve in the following table.

	$\delta^{(b)}$	$\delta^{(a)}$	$\tilde{\rho}$
$\mathbf{16}^{(b)}$	$-3c_1 \cdot_S \xi_1$	$-\xi_1 \cdot_S (\eta - 4c_1 - \xi_1)$	$3\rho \cdot_S \xi_1$
$\mathbf{16}^{(a)}$	$-\xi_1 \cdot_S (\eta - 4c_1 - \xi_1)$	$-c_1 \cdot_S (\eta - 4c_1 - \xi_1)$	$-\rho \cdot_S (\eta - 4c_1 - \xi_1)$
$\mathbf{10}^{(a)(b)}$	$\xi_1 \cdot_S (2\eta - 9c_1 - 3\xi_1)$	$-(\eta - 3c_1 - \xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$	$2\rho \cdot_S (\eta - 3c_1)$
$\mathbf{10}^{(a)(a)}$	$-2\xi_1 \cdot_S (\eta - 3c_1)$	$(\eta - 3c_1 - \xi_1) \cdot_S (\eta - 4c_1 - \xi_1)$	$-2\rho \cdot_S (\eta - 3c_1)$

(4.19)

With Eqs. (4.14), (4.17), and (4.18), we define the universal cover flux Γ to be [47]

$$\Gamma = k_a \gamma_0^{(a)} + k_b \gamma_0^{(b)} + m_a \delta^{(a)} + m_b \delta^{(b)} + \tilde{\rho} \equiv \Gamma^{(a)} + \Gamma^{(b)}, \tag{4.20}$$

where $\Gamma^{(a)}$ and $\Gamma^{(b)}$ are given by

$$\Gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [(3k_a + m_a)\sigma - \pi^*(k_a(\eta - 4c_1 - \xi_1) + m_b \xi_1 + \rho)], \tag{4.21}$$

$$\Gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot [(k_b + 3m_b)\sigma - \pi^*(k_b \xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho)]. \tag{4.22}$$

Note that

$$p_{a*}\Gamma^{(a)} = -3m_b \xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho, \tag{4.23}$$

$$p_{b*}\Gamma^{(b)} = 3m_b \xi_1 - m_a(\eta - 4c_1 - \xi_1) + 3\rho. \tag{4.24}$$

Clearly, $\Gamma^{(a)}$ and $\Gamma^{(b)}$ obey the traceless condition $p_{a*}\Gamma^{(a)} + p_{b*}\Gamma^{(b)} = 0$. Besides, the quantization condition in this case becomes

$$(3k_a+m_a+\frac{1}{2})\sigma-\pi^*[k_a(\eta-4c_1-\xi_1)+m_b\xi_1+\rho-\frac{1}{2}(\eta-2c_1-\xi_1)] \in H_4(\bar{X},\mathbb{Z}), \quad (4.25)$$

$$(k_b+3m_b-\frac{1}{2})\sigma-\pi^*[k_b\xi_1+m_a(\eta-4c_1-\xi_1)-3\rho-\frac{1}{2}\xi_1] \in H_4(\bar{X},\mathbb{Z}). \quad (4.26)$$

The supersymmetry condition is given by

$$[3m_b\xi_1-m_a(\eta-4c_1-\xi_1)+3\rho]\cdot_S[\omega]=0. \quad (4.27)$$

4.2 (2,2) Factorization

We can calculate the ramification divisors $r^{(d_1)}$ and $r^{(d_2)}$ for the $(2,2)$ factorization and obtain

$$r^{(d_1)} = [\mathcal{C}^{(d_1)}]\cdot\pi^*(\eta-3c_1-\xi_2), \quad (4.28)$$

$$r^{(d_2)} = [\mathcal{C}^{(d_2)}]\cdot\pi^*(c_1+\xi_2). \quad (4.29)$$

We then define traceless cover fluxes $\gamma_0^{(d_1)}$ and $\gamma_0^{(d_2)}$ by

$$\gamma_0^{(d_1)} = (2-p^*_{d_1}p_{d_1*})\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}]\cdot[2\sigma-\pi^*(\eta-4c_1-\xi_2)], \quad (4.30)$$

$$\gamma_0^{(d_2)} = (2-p^*_{d_2}p_{d_2*})\gamma^{(d_2)} = [\mathcal{C}^{(d_2)}]\cdot(2\sigma-\pi^*\xi_2), \quad (4.31)$$

where $\gamma^{(d_1)}$ and $\gamma^{(d_21)}$ are non-traceless fluxes and given by

$$\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}]\cdot\sigma, \quad \gamma^{(d_2)} = [\mathcal{C}^{(d_2)}]\cdot\sigma. \quad (4.32)$$

We summarize the restriction of the fluxes to each factorized curve in the following table.

	$\gamma_0^{(d_2)}$	$\gamma_0^{(d_1)}$
$\mathbf{16}^{(d_2)}$	$-\xi_2\cdot_S(2c_1+\xi_2)$	0
$\mathbf{16}^{(d_1)}$	0	$-(\eta-2c_1-\xi_2)\cdot_S(\eta-4c_1-\xi_2)$
$\mathbf{10}^{(d_2)(d_2)}$	0	0
$\mathbf{10}^{(d_1)(d_2)}$	0	$-2(\eta-4c_1-2\xi_2)\cdot_S(\eta-4c_1-\xi_2)$
$\mathbf{10}^{(d_1)(d_1)}$	0	$2(\eta-4c_1-2\xi_2)\cdot_S(\eta-4c_1-\xi_2)$

(4.33)

Due to the factorization, one also can define following fluxes [47]

$$\delta^{(d_1)} = (2-p^*_{d_2}p_{d_1*})\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}]\cdot 2\sigma-[\mathcal{C}^{(d_2)}]\cdot\pi^*(\eta-4c_1-\xi_2),$$

$$\delta^{(d_2)} = (2-p^*_{d_1}p_{d_2*})\gamma^{(d_2)} = [\mathcal{C}^{(d_2)}]\cdot 2\sigma-[\mathcal{C}^{(d_1)}]\cdot\pi^*\xi_2, \quad (4.34)$$

and

$$\widehat{\rho} = (p_{d_2}^* - p_{d_1}^*)\rho, \tag{4.35}$$

for any $\rho \in H_2(S,\mathbb{R})$. We summarize the restriction of the fluxes $\delta^{(d_1)}$, $\delta^{(d_2)}$, and $\widehat{\rho}$ to each factorized curve as follows:

	$\delta^{(d_2)}$	$\delta^{(d_1)}$	$\widehat{\rho}$
$\mathbf{16}^{(d_2)}$	$-2c_1 \cdot_S \xi_2$	$-\xi_2 \cdot_S (\eta - 4c_1 - \xi_2)$	$\rho \cdot_S \xi_2$
$\mathbf{16}^{(d_1)}$	$-\xi_2 \cdot_S (\eta - 4c_1 - \xi_2)$	$-2c_1 \cdot_S (\eta - 4c_1 - \xi_2)$	$-\rho \cdot_S (\eta - 4c_1 - \xi_2)$
$\mathbf{10}^{(d_2)(d_2)}$	$2\xi_2 \cdot_S (c_1 + \xi_2)$	$-2(c_1 + \xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$	$2\rho \cdot_S (c_1 + \xi_2)$
$\mathbf{10}^{(d_1)(d_2)}$	0	$-2(\eta - 4c_1 - 2\xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$	0
$\mathbf{10}^{(d_1)(d_1)}$	$-2\xi_2 \cdot_S (c_1 + \xi_2)$	$2(\eta - 3c_1 - \xi_2) \cdot_S (\eta - 4c_1 - \xi_2)$	$-2\rho \cdot_S (c_1 + \xi_2)$

(4.36)

In this case the universal cover flux is defined by

$$\Gamma = k_{d_1}\gamma_0^{(d_1)} + k_{d_2}\gamma_0^{(d_2)} + m_{d_1}\delta^{(d_1)} + m_{d_2}\delta^{(d_2)} + \widehat{\rho} = \Gamma^{(d_1)} + \Gamma^{(d_2)}, \tag{4.37}$$

where

$$\begin{aligned}
\Gamma^{(d_1)} &= [\mathcal{C}^{(d_1)}] \cdot \{2(k_{d_1} + m_{d_1})\sigma - \pi^*[k_{d_1}(\eta - 4c_1 - \xi_2) + m_{d_2}\xi_2 + \rho]\}\,, \\
\Gamma^{(d_2)} &= [\mathcal{C}^{(d_2)}] \cdot \{2(k_{d_2} + m_{d_2})\sigma - \pi^*[k_{d_2}\xi_2 + m_{d_1}(\eta - 4c_1 - \xi_2) - \rho]\}\,.
\end{aligned} \tag{4.38}$$

Note that

$$p_{d_1*}\Gamma^{(d_1)} = -2m_{d_2}\xi_2 + 2m_{d_1}(\eta - 4c_1 - \xi_2) - 2\rho, \tag{4.39}$$

$$p_{d_2*}\Gamma^{(d_2)} = 2m_{d_2}\xi_2 - 2m_{d_1}(\eta - 4c_1 - \xi_2) + 2\rho. \tag{4.40}$$

It is easy to see that $\Gamma^{(d_1)}$ and $\Gamma^{(d_2)}$ satisfy the traceless condition $p_{d_1*}\Gamma^{(d_1)} + p_{d_2*}\Gamma^{(d_2)} = 0$. In addition, the quantization condition in this case becomes

$$2(k_{d_1} + m_{d_1})\sigma - \pi^*[k_{d_1}(\eta - 4c_1 - \xi_2) + m_{d_2}\xi_2 + \rho - \frac{1}{2}(\eta - 3c_1 - \xi_2)] \in H_4(\bar{X},\mathbb{Z}), \tag{4.41}$$

$$2(k_{d_2} + m_{d_2})\sigma - \pi^*[k_{d_2}\xi_2 + m_{d_1}(\eta - 4c_1 - \xi_2) - \rho - \frac{1}{2}(c_1 + \xi_2)] \in H_4(\bar{X},\mathbb{Z}). \tag{4.42}$$

The supersymmetry condition is then given by

$$[2m_{d_2}\xi_2 - 2m_{d_1}(\eta - 4c_1 - \xi_2) + 2\rho] \cdot_S [\omega] = 0. \tag{4.43}$$

5 $D3$-brane Tadpole Cancellation

The cancellation of tadpoles is crucial for consistent compactifications. In general, there are induced tadpoles from 7-brane, 5-brane, and 3-brane charges in F-theory. It is well known that 7-brane tadpole cancellation in F-theory is automatically satisfied since X_4 is a Calabi-Yau manifold. In spectral cover models, the cancellation of the $D5$-brane tadpole follows from the topological condition that the overall first Chern class of the Higgs bundle vanishes. Therefore, the non-trivial tadpole cancellation needed to be satisfied is the $D3$-brane tadpole. The $D3$-brane tadpole can be calculated by the Euler characteristic $\chi(X_4)$. The cancellation condition is of the form [81]

$$N_{D3} = \frac{\chi(X_4)}{24} - \frac{1}{2}\int_{X_4} G \wedge G, \tag{5.1}$$

where N_{D3} is the number of $D3$-branes and G is the four-form flux on X_4. For a non-singular elliptically fibered Calabi-Yau manifold, it was shown in [81] that the Euler characteristic $\chi(X_4)$ can be expressed as

$$\chi(X_4) = 12\int_{B_3} c_1(B_3)[c_2(B_3) + 30c_1(B_3)^2], \tag{5.2}$$

where $c_k(B_3)$ are the Chern classes of B_3. It follows from Eq. (5.2) that $\chi(X_4)/24$ is at least half-integral[9]. When X_4 admits non-abelian singularities, the Euler characteristic of X_4 is replaced by the refined Euler characteristic, the Euler characteristic of the smooth fourfold obtained from a suitable resolution of X_4. On the other hand, G-flux encodes the two-form gauge fluxes on 7-branes. It was shown in [84] that

$$\frac{1}{2}\int_{X_4} G \wedge G = -\frac{1}{2}\Gamma^2, \tag{5.3}$$

where Γ is the universal cover flux defined in section 4 and Γ^2 is the self-intersection number of Γ inside the spectral cover[10]. It is a challenge to find compactifications

[9] For a generic Calabi-Yau manifold, it was shown in [81] that $\chi(X_4)/6 \in \mathbb{Z}$, which implies that $\chi(X_4)/24$ takes value in $\mathbb{Z}_4$.

[10] Eq. (5.3) originates from the spectral cover construction in heterotic string compactifications [84]. This equation holds for F-theory compactified on elliptically fibered fourfolds possessing a heterotic dual by heterotic/F-theory duality. However, since X_4 is not a global fibration over S, we assume that Eq. (5.3) is valid for F-theory models without heterotic dual, and the fluxes can correctly described by spectral covers.

with non-vanishing G-flux and non-negative N_{D_3} to satisfy the tadpole cancellation condition Eq. (5.1). In the next two subsections, we shall derive the formulae of refined Euler characteristic $\chi(X_4)$ and the self-intersection of universal cover fluxes Γ^2 for $(3,1)$ and $(2,2)$ factorizations.

5.1 Geometric Contribution

In the presence of non-abelian singularities, X_4 becomes singular and the Euler characteristic $\chi(X_4)$ is modified by resolving the singularities. To be more concrete, let us consider X_4 with an elliptic fibration which degenerates over S to a non-abelian singularity corresponding to gauge group H and define G to be the complement of H in E_8. The Euler characteristic is modified to

$$\chi(X_4) = \chi^*(X_4) + \chi_G - \chi_{E_8}, \tag{5.4}$$

where $\chi^*(X_4)$ is the Euler characteristic for a smooth fibration over B_3 given by Eq. (5.2). The characteristic χ_{E_8} is given by [54, 84, 85]

$$\chi_{E_8} = 120 \int_S (3\eta^2 - 27\eta c_1 + 62c_1^2). \tag{5.5}$$

For the case of $G = SU(n)$, the characteristic $\chi_{SU(n)}$ is given by[11]

$$\chi_{SU(n)} = \int_S (n^3 - n)c_1^2 + 3n\eta(\eta - nc_1). \tag{5.6}$$

When G splits into a product of two groups G_1 and G_1, χ_G in Eq. (5.4) is then replaced by $\chi_{G_1}^{(k)} + \chi_{G_2}^{(l)}$ in which η is replaced by the class $\eta^{(m)}$ in the spectral cover $\mathcal{C}^{(m)}$ for $m = k, l$. For the case of $(3,1)$ factorization, the refined Euler characteristic is then calculated by

$$\begin{aligned} \chi(X_4) &= \chi^*(X_4) + \chi_{SU(3)}^{(a)} + \chi_{SU(1)}^{(b)} - \chi_{E_8} \\ &= \chi^*(X_4) + \int_S 3[c_1(38c_1 - 21t - 20\xi_1) + (3t^2 + 6t\xi_1 + 4\xi_1^2)] - \chi_{E_8}. \end{aligned} \tag{5.7}$$

[11] Eqs. (5.4)-(5.6) initially were derived in heterotic string compactifications [84, 85]. A priori, these formulae are valid only for F-theory models with a heterotic dual. It was observed in [54] that these formulae also hold for some F-theory models which do not admit a heterotic dual. However, this match fails in other examples observed in [86]. In these examples, extra gauge groups appear in regions away from S and cannot be described by spectral covers. We assume that Eqs. (5.4)-(5.6) hold for our models.

In the $(2,2)$ factorization, the refined Euler characteristic[12] is

$$\begin{aligned}\chi(X_4) &= \chi^*(X_4) + \chi^{(d_1)}_{SU(2)} + \chi^{(d_2)}_{SU(2)} - \chi_{E_8} \\ &= \chi^*(X_4) + \int_S 6[c_1(10c_1 - 6t - 4\xi_2) + (t^2 + 2t\xi_2 + 2\xi_2^2)] - \chi_{E_8}. \end{aligned} \quad (5.8)$$

5.2 Cover flux Contribution

It follows from Eqs. (5.1) and (5.3) that

$$N_{D3} = \frac{\chi(X_4)}{24} + \frac{1}{2}\Gamma^2. \quad (5.9)$$

In the previous subsection, we discussed the first term on the right hand side of Eq. (5.9). To calculate N_{D3}, it is necessary to compute the self-intersection Γ^2 of the universal cover flux Γ. Recall that in section 4, the universal cover flux was defined by

$$\Gamma = \sum_k \Gamma^{(k)}, \quad (5.10)$$

where $\Gamma^{(k)}$ are cover fluxes satisfying the traceless condition,

$$\sum_k p_{k*}\Gamma^{(k)} = 0. \quad (5.11)$$

In what follows, we will compute Γ^2 for both the $(3,1)$ and $(2,2)$ factorizations.

5.2.1 $(3,1)$ Factorization

Recall that for the case of $(3,1)$ factorization, the universal cover flux is given by

$$\Gamma = k_a\gamma_0^{(a)} + k_b\gamma_0^{(b)} + m_a\delta^{(a)} + m_b\delta^{(b)} + \tilde{\rho} = \Gamma^{(a)} + \Gamma^{(b)}, \quad (5.12)$$

where $\Gamma^{(a)}$ and $\Gamma^{(b)}$ are

$$\Gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [(3k_a + m_a)\sigma - \pi^*(k_a[a_3] + m_b[d_1] + \rho)] \equiv [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}], \quad (5.13)$$

$$\Gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot [(k_b + 3m_b)\sigma - \pi^*(k_b[d_1] + m_a[a_3] - 3\rho)] \equiv [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}]. \quad (5.14)$$

[12]For the $(3,1)$ factorization, $\eta^{(a)} = (\eta - c_1 - \xi_1)$ and $\eta^{(b)} = (c_1 + \xi_1)$. For the $(2,2)$ factorization, $\eta^{(d_1)} = (\eta - 2c_1 - \xi_2)$ and $\eta^{(d_2)} = (2c_1 + \xi_2)$.

Then the self-intersection of the cover flux Γ is calculated by [47]

$$\Gamma^2 = [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] + [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}]. \tag{5.15}$$

In the $(3,1)$ factorization, $[\mathcal{C}^{(a)}] = 3\sigma + \pi^*(\eta - c_1 - \xi_1)$ and $[\mathcal{C}^{(b)}] = \sigma + \pi^*(c_1 + \xi_1)$. By Eqs. (5.13) and (5.14), one can obtain

$$\begin{aligned}
[\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] &= -(3k_a + m_a)^2([a_3] \cdot_S c_1) - k_a(3k_a + 2m_a)[a_3]^2 + 3m_b^2[d_1]^2 \\
&- 2m_b m_a([a_3] \cdot_S [d_1]) - 2(m_a[a_3] - 3m_b[d_1]) \cdot_S \rho \\
&+ 3(\rho \cdot_S \rho),
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
[\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}] &= -(k_b + 3m_b)^2([d_1] \cdot_S c_1) - k_b(k_b + 6m_b)[d_1]^2 + m_a^2[a_3]^2 \\
&- 6m_b m_a([a_3] \cdot_S [d_1]) - 6(m_a[a_3] - 3m_b[d_1]) \cdot_S \rho \\
&+ 9(\rho \cdot_S \rho).
\end{aligned} \tag{5.17}$$

Putting everything together, one obtains

$$\Gamma^2 = -\frac{1}{3}(3k_a + m_a)^2([a_0] \cdot_S [a_3]) - (k_b + 3m_b)^2([d_0] \cdot_S [d_1]) + \frac{4}{3}(m_a[a_3] - 3m_b[d_1] - 3\rho)^2. \tag{5.18}$$

5.2.2 $(2,2)$ Factorization

Recall that in the $(2,2)$ factorization, the universal flux is given by

$$\Gamma = k_{d_1}\gamma_0^{(d_1)} + k_{d_2}\gamma_0^{(d_2)} + m_{d_1}\delta^{(d_1)} + m_{d_2}\delta^{(d_2)} + \widehat{\rho} \equiv \Gamma^{(d_1)} + \Gamma^{(d_2)}, \tag{5.19}$$

where $\Gamma^{(d_1)}$ and $\Gamma^{(d_2)}$ are

$$\Gamma^{(d_1)} = [\mathcal{C}^{(d_1)}] \cdot [2(k_{d_1} + m_{d_1})\sigma - \pi^*(k_{d_1}[e_2] + m_{d_2}[f_2] + \rho)] \equiv [\mathcal{C}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}], \tag{5.20}$$

$$\Gamma^{(d_2)} = [\mathcal{C}^{(d_2)}] \cdot [2(k_{d_2} + m_{d_2})\sigma - \pi^*(k_{d_2}[f_2] + m_{d_1}[e_2] - \rho)] \equiv [\mathcal{C}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}]. \tag{5.21}$$

Then the self-intersection Γ^2 can be computed as

$$\Gamma^2 = [\mathcal{C}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}] + [\mathcal{C}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}]. \tag{5.22}$$

Notice that $[\mathcal{C}^{(d_1)}] = 2\sigma + \pi^*(\eta - 2c_1 - \xi_2)$ and $[\mathcal{C}^{(d_2)}] = 2\sigma + \pi^*(2c_1 + \xi_2)$ in the $(2,1)$ factorization. It follows from Eqs. (5.20) and (5.21) that

$$\begin{aligned}
[\mathcal{C}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}] \cdot [\widetilde{\mathcal{C}}^{(d_1)}] &= -4(k_{d_1} + m_{d_1})^2([e_2] \cdot_S c_1) - 2k_{d_1}(k_{d_1} + 2m_{d_1})[e_2]^2 + 2m_{d_2}^2[f_2]^2 \\
&- 4m_{d_1}m_{d_2}([e_2] \cdot_S [f_2]) - 4(m_{d_1}[e_2] - m_{d_2}[f_2]) \cdot_S \rho \\
&+ 2(\rho \cdot_S \rho),
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
[\mathcal{C}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}] \cdot [\widetilde{\mathcal{C}}^{(d_2)}] &= -4(k_{d_2} + m_{d_2})^2([f_2] \cdot_S c_1) - 2k_{d_2}(k_{d_2} + 2m_{d_2})[f_2]^2 + 2m_{d_1}^2[e_2]^2 \\
&- 4m_{d_1}m_{d_2}([f_2] \cdot_S [e_2]) - 4(m_{d_1}[e_2] - m_{d_2}[f_2]) \cdot_S \rho \\
&+ 2(\rho \cdot_S \rho).
\end{aligned} \tag{5.24}$$

Therefore, Γ^2 is given by

$$\Gamma^2 = -2(k_{d_1} + m_{d_1})^2([e_0] \cdot_S [e_2]) - 2(k_{d_2} + m_{d_2})^2([f_0] \cdot_S [f_2]) + 4(m_{d_1}[e_2] - m_{d_2}[f_2] - \rho)^2. \tag{5.25}$$

6 Models

6.1 $U(1)_X$ Flux and Spectrum

Let us start with the $(3,1)$ factorization. Consider the breaking pattern as follows:

$$\begin{array}{ccl}
SU(4)_\perp & \rightarrow & SU(3) \times U(1) \\
\mathbf{15} & \rightarrow & \mathbf{8}_0 + \mathbf{3}_{-4} + \overline{\mathbf{3}}_4 + \mathbf{1}_0 \\
\mathbf{6} & \rightarrow & \mathbf{3}_2 + \overline{\mathbf{3}}_{-2} \\
\mathbf{4} & \rightarrow & \mathbf{3}_{-1} + \mathbf{1}_3
\end{array} \tag{6.1}$$

Then the representations $(\mathbf{16}, \mathbf{4})$ and $(\mathbf{10}, \mathbf{6})$ in Eq. (2.10) are decomposed as

$$(\mathbf{16}, \mathbf{4}) \rightarrow (\mathbf{16}_{-1}, \mathbf{3}) + (\mathbf{16}_3, \mathbf{1}), \quad (\mathbf{10}, \mathbf{6}) \rightarrow (\mathbf{10}_2, \mathbf{3}) + (\mathbf{10}_{-2}, \overline{\mathbf{3}}) \tag{6.2}$$

On the other hand, we can further break $SO(10)$ in Eq. (2.10) by $U(1)_X$ flux as follows:

$$\begin{array}{ccl}
SO(10) & \rightarrow & SU(5) \times U(1)_X \\
\mathbf{16} & \rightarrow & \mathbf{10}_{-1} + \overline{\mathbf{5}}_3 + \mathbf{1}_{-5} \\
\mathbf{10} & \rightarrow & \mathbf{5}_2 + \overline{\mathbf{5}}_{-2}
\end{array} \tag{6.3}$$

Curve	Matter	Bundle	Chirality
$\mathbf{16}^{(a)}_{-1}$	$\mathbf{10}_{-1,-1}$	$V_{\mathbf{16}} \otimes L_X^{-1}\|_{\Sigma^{(a)}_{\mathbf{16}}}$	M_a
	$\bar{\mathbf{5}}_{-1,3}$	$V_{\mathbf{16}} \otimes L_X^{3}\|_{\Sigma^{(a)}_{\mathbf{16}}}$	$M_a + N_a$
	$\mathbf{1}_{-1,-5}$	$V_{\mathbf{16}} \otimes L_X^{-5}\|_{\Sigma^{(a)}_{\mathbf{16}}}$	$M_a - N_a$
$\mathbf{16}^{(b)}_{3}$	$\mathbf{10}_{3,-1}$	$V_{\mathbf{16}} \otimes L_X^{-1}\|_{\Sigma^{(b)}_{\mathbf{16}}}$	M_b
	$\bar{\mathbf{5}}_{3,3}$	$V_{\mathbf{16}} \otimes L_X^{3}\|_{\Sigma^{(b)}_{\mathbf{16}}}$	$M_b + N_b$
	$\mathbf{1}_{3,-5}$	$V_{\mathbf{16}} \otimes L_X^{-5}\|_{\Sigma^{(b)}_{\mathbf{16}}}$	$M_b - N_b$
$\mathbf{10}^{(a)(a)}_{-2}$	$\mathbf{5}_{-2,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(a)(a)}_{\mathbf{10}}}$	$M_{aa} + N_{aa}$
	$\bar{\mathbf{5}}_{-2,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(a)(a)}_{\mathbf{10}}}$	M_{aa}
$\mathbf{10}^{(a)(b)}_{2}$	$\mathbf{5}_{2,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(a)(b)}_{\mathbf{10}}}$	$M_{ab} + N_{ab}$
	$\bar{\mathbf{5}}_{2,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(a)(b)}_{\mathbf{10}}}$	M_{ab}

Table 6: Chirality of matter localized on matter curves **16** and **10** in the (3,1) factorization.

We suppose that $V_{\mathbf{16}} \otimes L_X^{-1}$ has restriction of degree M_k to $\Sigma_{\mathbf{16}^{(k)}}$ while L_X^4 has restriction of degree N_k. Similarly, we define $V_{\mathbf{10}} \otimes L_X^{-2}$ has restriction of degree M_{kl} to $\Sigma_{\mathbf{10}^{(k)(l)}}$ while L_X^4 has restriction of degree N_{kl}. We summarize the chirality on each matter curve in Table 6. For the $(2,2)$ factorization, the analysis is similar to the case of the $(3,1)$ factorization. We summarize the chirality induced from the cover and $U(1)_X$ fluxes in Table 7.

6.2 (3,1) Factorization and CY_4 with a dP_2 Surface

In this section, we shall explicitly realize models in specific geometries. We first consider the Calabi-Yau fourfold constructed in [45] to be our X_4. This Calabi-Yau fourfold contains a dP_2 surface embedded into the base B_3. For the detailed geometry of this Calabi-Yau fourfold, we refer readers to [45]. Here we only collect the relevant

Curve	Matter	Bundle	Chirality
$\mathbf{16}^{(d_2)}_{-1}$	$\mathbf{10}_{-1,-1}$	$V_{\mathbf{16}} \otimes L_X^{-1}\|_{\Sigma^{(d_2)}_{\mathbf{16}}}$	M_{d_2}
	$\bar{\mathbf{5}}_{-1,3}$	$V_{\mathbf{16}} \otimes L_X^{3}\|_{\Sigma^{(d_2)}_{\mathbf{16}}}$	$M_{d_2} + N_{d_2}$
	$\mathbf{1}_{-1,-5}$	$V_{\mathbf{16}} \otimes L_X^{-5}\|_{\Sigma^{(d_2)}_{\mathbf{16}}}$	$M_{d_2} - N_{d_2}$
$\mathbf{16}^{(d_1)}_{1}$	$\mathbf{10}_{1,-1}$	$V_{\mathbf{16}} \otimes L_X^{-1}\|_{\Sigma^{(d_1)}_{\mathbf{16}}}$	M_{d_1}
	$\bar{\mathbf{5}}_{1,3}$	$V_{\mathbf{16}} \otimes L_X^{3}\|_{\Sigma^{(d_1)}_{\mathbf{16}}}$	$M_{d_1} + N_{d_1}$
	$\mathbf{1}_{1,-5}$	$V_{\mathbf{16}} \otimes L_X^{-5}\|_{\Sigma^{(d_1)}_{\mathbf{16}}}$	$M_{d_1} - N_{d_1}$
$\mathbf{10}^{(d_2)(d_2)}_{-2}$	$\mathbf{5}_{-2,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(d_2)(d_2)}_{\mathbf{10}}}$	$M_{d_2 d_2} + N_{d_2 d_2}$
	$\bar{\mathbf{5}}_{-2,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(d_2)(d_2)}_{\mathbf{10}}}$	$M_{d_2 d_2}$
$\mathbf{10}^{(d_1)(d_2)}_{0}$	$\mathbf{5}_{0,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(d_1)(d_2)}_{\mathbf{10}}}$	$M_{d_1 d_2} + N_{d_1 d_2}$
	$\bar{\mathbf{5}}_{0,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(d_1)(d_2)}_{\mathbf{10}}}$	$M_{d_1 d_2}$
$\mathbf{10}^{(d_1)(d_1)}_{2}$	$\mathbf{5}_{2,2}$	$V_{\mathbf{10}} \otimes L_X^{2}\|_{\Sigma^{(d_1)(d_1)}_{\mathbf{10}}}$	$M_{d_1 d_1} + N_{d_1 d_1}$
	$\bar{\mathbf{5}}_{2,-2}$	$V_{\mathbf{10}} \otimes L_X^{-2}\|_{\Sigma^{(d_1)(d_1)}_{\mathbf{10}}}$	$M_{d_1 d_1}$

Table 7: Chirality of matter localized on matter curves **16** and **10** in the (2,2) factorization.

geometric data[13] for calculation. The basic geometric data of X_4 is

$$c_1 = 3H - E_1 - E_2, \quad t = -c_1(N_{S/B_3}) = H, \quad \chi^*(X_4) = 13968. \tag{6.4}$$

From Eq. (6.4), we can conclude $\eta = 17H - 6E_1 - 6E_2$, $\eta^2 = 217$, $c_1 \cdot \eta = 39$, and $c_1^2 = 7$. For the (3,1) factorization, it follows from Eq. (5.7) that the refined Euler characteristic is

$$\chi(X_4) = 10746 + (12\xi_1^2 - 18\xi_1\eta + 48\xi_1 c_1). \tag{6.5}$$

The self-intersection of the cover flux Γ is then given by

$$\begin{aligned}\Gamma^2 \;=\; & -(3k_a^2 + 2k_a m_a)(50 + \xi_1^2 - 2\xi_1\eta + 5\xi_1 c_1) + m_a^2(6 + \xi_1^2 - 2\xi_1\eta + 9\xi_1 c_1) \\ & -(k_b + 3m_b)^2(\xi_1^2 + \xi_1 c_1) + 12m_b^2\xi_1^2 + 8m_a m_b(\xi_1^2 - \xi_1\eta + 4\xi_1 c_1) \\ & +12\rho^2 - 8m_a(\rho\eta - \rho\xi_1 - 4\rho c_1) + 24m_b\rho\xi_1,\end{aligned} \tag{6.6}$$

[13] In section 6, H and E_m, $m = 1, 2, .., k$ are defined to be the hyperplane divisor and exceptional divisors of dP_k, respectively.

and the number of generations for matter **16** and **10** on the curves are

$$
\begin{aligned}
N_{\mathbf{16}^{(b)}} &= (m_a - k_b)\xi_1^2 - m_a\xi_1\eta + (4m_a - k_b - 3m_b)\xi_1 c_1 + 3\rho\xi_1, & (6.7)\\
N_{\mathbf{16}^{(a)}} &= -(50k_a + 11m_a) + (m_b - k_a)\xi_1^2 + (2k_a - m_b)\xi_1\eta \\
&\quad +(4m_b - 5k_a + m_a)\xi_1 c_1 - \rho\eta + 4\rho c_1 + \rho\xi_1, & (6.8)\\
N_{\mathbf{10}^{(a)(b)}} &= -28(k_a + m_a) - (k_b + 3k_a + m_a + 3m_b)\xi_1^2 + (4k_a + 2m_a + 2m_b)\xi_1\eta \\
&\quad -(k_b + 15k_a + 7m_a + 9m_b)\xi_1 c_1 + 2\rho\eta - 6\rho c_1, & (6.9)\\
N_{\mathbf{10}^{(a)(a)}} &= 28(k_a + m_a) + (3k_a + m_a)\xi_1^2 - (4k_a + 2m_a + 2m_b)\xi_1\eta \\
&\quad +(15k_a + 7m_a + 6m_b)\xi_1 c_1 - 2\rho\eta + 6\rho c_1. & (6.10)
\end{aligned}
$$

In this case, the supersymmetric condition Eq. (4.10) reduces to

$$[(3m_b + m_a)\xi_1 - m_a(\eta - 4c_1) + 3\rho] \cdot_S [\omega], \tag{6.11}$$

where we choose $[\omega] = \alpha(E_1 + E_2) + \beta(H - E_1 - E_2),\ 2\alpha > \beta > \alpha > 0$ to be an ample divisor in dP_2. In the (3,1) factorization, one more constraint that we may impose is that the ramification of the degree-one cover should be trivial. In other words, we impose the following constraint:

$$(c_1 + \xi_1) \cdot \xi_1 = 0. \tag{6.12}$$

In what follows, we show three examples based on this geometry. We find that there are only finite number of solutions for parameters.

6.2.1 Model 1

In this model we represent a three-generation example. The numerical parameters are listed in Table 8.

k_b	k_a	m_b	m_a	ρ	ξ_1	α	β
-1.5	-0.5	-2	1	$H + 3E_1 + E_2$	E_2	9	11

Table 8: Parameters of Model 1 of the (3,1) factorization in dP_2.

The matter content and the corresponding classes are listed in Table 9. By using Eqs. (6.5) and (6.6), we obtain $\chi(X_4) = 10674$ and $\Gamma^2 = -159.5$. It follows from Eq. (5.9) that $N_{D3} = 365$.

Matter	Class in S	Class with fixed ξ_1	Generation	Restr. of $[F_X]$
$\mathbf{16}^{(b)}$	ξ_1	E_2	0	1
$\mathbf{16}^{(a)}$	$\eta - 4c_1 - \xi_1$	$5H - 2E_1 - 3E_2$	3	-1
$\mathbf{10}^{(a)(b)}$	$\eta - 3c_1$	$8H - 3E_1 - 3E_2$	14	0
$\mathbf{10}^{(a)(a)}$	$\eta - 3c_1$	$8H - 3E_1 - 3E_2$	-14	0

Table 9: Model 1 matter content with $[F_X] = E_1 - E_2$. It is a three-generation model with non-trivial flux restrictions.

6.2.2 Model 2

Model 2 is another example of a three-generation model with $\chi(X_4) = 10674$, $\Gamma^2 = -159.5$, and $N_{D3} = 365$. The construction is similar to the model 1. We list the numerical parameters in Table 10.

k_b	k_a	m_b	m_a	ρ	ξ_1	α	β
-1.5	0.5	-2	-2	$-4H + 4E_1 + 5E_2$	E_1	9	11

Table 10: Parameters of Model 2 of the (3,1) factorization in dP_2.

The matter content and the corresponding classes are shown in Table 11.

Matter	Class in S	Class with fixed ξ_1	Generation	Restr. of $[F_X]$
$\mathbf{16}^{(b)}$	ξ_1	E_1	0	1
$\mathbf{16}^{(a)}$	$\eta - 4c_1 - \xi_1$	$5H - 3E_1 - 2E_2$	3	-1
$\mathbf{10}^{(a)(b)}$	$\eta - 3c_1$	$8H - 3E_1 - 3E_2$	14	0
$\mathbf{10}^{(a)(a)}$	$\eta - 3c_1$	$8H - 3E_1 - 3E_2$	-14	0

Table 11: Model 2 matter content with $[F_X] = E_1 - E_2$.

6.2.3 Model 3

Next we build a four-generation model in $SO(10)$. The reason why we would like to discuss such a case is that the only choice for the $U(1)_X$ flux on dP_2 is $[F_X] =$

$\pm(E_1 - E_2)$, and then the restrictions of $[F_X]$ to the **16** curves are always non-zero, which results in the variation of the chirality numbers of the $SU(5)$ matter descended from the **16** curves. The two examples shown above only make sense for an three-generation $SO(10)$ model, and they are no longer three-generation models after gauge breaking. Since we expect to build a three-generation model at $SU(5)$ level, we slightly increase the generation number at the $SO(10)$ level to prevent the chirality being too small. The numerical parameters are listed in Table 12. In this model, it is not difficult to obtain $\chi(X_4) = 10674$ and $\Gamma^2 = -355.5$. It turns out that $N_{D3} = 267$ is a positive integer.

k_b	k_a	m_b	m_a	ρ	ξ_1	α	β
-1.5	-0.5	-2	1	$5E_1 + E_2$	E_2	12	17

Table 12: Parameters of Model 3 of the (3,1) factorization in dP_2.

The matter content and the corresponding classes are listed in Table 13.

Matter	Class in S	Class with fixed ξ_1	Generation	Restr. of $[F_X]$
$\mathbf{16}^{(b)}$	ξ_1	E_2	0	1
$\mathbf{16}^{(a)}$	$\eta - 4c_1 - \xi_1$	$5H - 2E_1 - 3E_2$	4	-1
$\mathbf{10}^{(a)(b)}$	$\eta - 3c_1$	$8H - 3E_1 - 3E_2$	10	0
$\mathbf{10}^{(a)(a)}$	$\eta - 3c_1$	$8H - 3E_1 - 3E_2$	-10	0

Table 13: Model 3 matter content with $[F_X] = E_1 - E_2$. There are four generations on the $\mathbf{16}^{(a)}$ curve.

6.2.4 Discussion

Model 1 and Model 2 of (3,1) factorization have the following $SO(10)$ structure:

$$\begin{array}{c|c|c} \text{Maatter} & \text{Copy} & U(1)_C \\ \hline \mathbf{16}^{(b)} & 0 & -3 \\ \mathbf{16}^{(a)} & 3 & 1 \\ \mathbf{10}^{(a)(b)} & 14 & -2 \\ \mathbf{10}^{(a)(a)} & -14 & 2 \end{array} \tag{6.13}$$

where $U(1)_C$ is from the cover, the $U(1)^3$ Cartan subalgebra of $SU(4)_\perp$ that is not removed from the monodromy. The Yukawa coupling is filtered by the conservation of this $U(1)_C$. Before turning on the $U(1)_X$ flux, this spectrum can fit the minimum requirement by forming the Yukawa coupling $\mathbf{16}^{(a)}_{-1}\mathbf{16}^{(a)}_{-1}\mathbf{10}^{(a)(b)}_{2}$ of the $SO(10)$ GUT with some exotic $\mathbf{10}$s. However, when $U(1)_X$ flux is turned on, the non-vanishing restriction of the flux to each $\mathbf{16}$ curve changes the chirality, while the chirality on the $\mathbf{10}$ curves remain untouched. The analysis in Table 6 suggests that a three-generation model may descend from a four-generation $SO(10)$ model after the gauge group is broken to $SU(5) \times U(1)_X$ by $[F_X] = E_1 - E_2$. Here we try to explain Model 3 as a flipped $SU(5)$ model with its spectrum presented in Table 14.

Matter	Rep.	Generation
$\mathbf{10}_M$	$\mathbf{10}_{-1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{-1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{-1,-5}$	3
$\mathbf{10}_H + \overline{\mathbf{10}}_H$	$\mathbf{10}_{-1,-1} + \overline{\mathbf{10}}_{-1,1}$	1
$\mathbf{5}_h$	$\mathbf{5}_{2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{2,-2}$	1
$\mathbf{10}$	$\mathbf{10}_{-1,-1}$	1
$\bar{\mathbf{5}}$	$\bar{\mathbf{5}}_{3,3}$	1
$\mathbf{1}$	$\mathbf{1}_{-1,-5}$	2
$\mathbf{1}$	$\mathbf{1}_{3,5}$	1
$\mathbf{5}+\bar{\mathbf{5}}$ exotics	$\mathbf{5}_{-2,2} + \bar{\mathbf{5}}_{-2,-2}$	9
	$\mathbf{5}_{2,2} + \bar{\mathbf{5}}_{2,-2}$	-10

Table 14: Flipped $SU(5)$ spectrum of Model 3.

In this case, the Yukawa couplings are

$$\begin{aligned}\mathcal{W} \supset\ & \mathbf{10}_{-1,-1M}\mathbf{10}_{-1,-1M}\mathbf{5}_{2,2h} + \mathbf{10}_{-1,-1M}\bar{\mathbf{5}}_{-1,3M}\bar{\mathbf{5}}_{2,-2h} + \bar{\mathbf{5}}_{-1,3M}\mathbf{1}_{-1,-5M}\mathbf{5}_{2,2h} \\ + \ & \mathbf{10}_{-1,-1H}\mathbf{10}_{-1,-1H}\mathbf{5}_{2,2h} + \overline{\mathbf{10}}_{-1,1H}\overline{\mathbf{10}}_{-1,1H}\bar{\mathbf{5}}_{2,-2h} + \ldots . \end{aligned} \tag{6.14}$$

We may identify the flipped $SU(5)$ superheavy Higgs fields with one of the $\mathbf{10}+\overline{\mathbf{10}}$ vector-like pairs on the $\mathbf{16}^{(a)}$ curve, which is not obvious from this configuration. Since the restrictions of the flux to the curves change the chirality, there are unavoidable exotic fermions, like the examples studied in [47]. In the following subsection, we will study models from a different geometric backgrounds to see if it is possible to retain the chirality unchanged while the flux F_X is turned on.

6.3 (3,1) Factorization and CY_4 with a dP_7 Surface

Although dP_2 surface is elegant, it does not possess enough degrees of freedom in the number of exceptional divisors for model building. Therefore, we turn to the geometry of the compact Calabi-Yau fourfold realized as complete intersections of two hypersurfaces with an embedded dP_7 surface[14]. The detailed construction can be found in [54]. Again here we only collect relevant geometric data for calculation. The basic geometric data is as follows:

$$\begin{aligned} c_1 &= 3H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7, \\ t &= 2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6, \\ \eta &= 16H - 5E_1 - 5E_2 - 5E_3 - 5E_4 - 5E_5 - 5E_6 - 6E_7. \end{aligned} \tag{6.15}$$

with $\chi^*(X_4) = 1728$. From Eq. (6.15), we have $\eta^2 = 70$, $\eta \cdot c_1 = 12$, and $c_1^2 = 2$. The refined Euler characteristic is given by

$$\chi(X_4) = 738 + (12\xi_1^2 - 18\xi_1\eta + 48\xi_1 c_1), \tag{6.16}$$

and the self-intersection of the cover flux Γ is

$$\begin{aligned} \Gamma^2 = \; & -(3k_a^2 + 2k_a m_a)(18 + \xi_1^2 - 2\xi_1\eta + 5\xi_1 c_1) + m_a^2(2 + \xi_1^2 - 2\xi_1\eta + 9\xi_1 c_1) \\ & -(k_b + 3m_b)^2(\xi_1^2 + \xi_1 c_1) + 12m_b^2\xi_1^2 + 8m_a m_b(\xi_1^2 - \xi_1\eta + 4\xi_1 c_1) \\ & +12\rho^2 - 8m_a(\rho\eta - \rho\xi_1 - 4\rho c_1) + 24m_b\rho\xi_1. \end{aligned} \tag{6.17}$$

[14]By abuse of notation, we also denote this Calabi-Yau fourfold by X_4.

Again we summarize the generation number on each curve as follows:

$$\begin{aligned} N_{\mathbf{16}^{(b)}} &= (m_a - k_b)\xi_1^2 - m_a\xi_1\eta + (4m_a - k_b - 3m_b)\xi_1 c_1 + 3\rho\xi_1, && (6.18)\\ N_{\mathbf{16}^{(a)}} &= -(18k_a + 4m_a) + (m_b - k_a)\xi_1^2 + (2k_a - m_b)\xi_1\eta \\ &\quad +(4m_b - 5k_a + m_a)\xi_1 c_1 - \rho\eta + 4\rho c_1 + \rho\xi_1, && (6.19)\\ N_{\mathbf{10}^{(a)(b)}} &= -10(k_a + m_a) - (k_b + 3k_a + m_a + 3m_b)\xi_1^2 + (4k_a + 2m_a + 2m_b)\xi_1\eta \\ &\quad -(k_b + 15k_a + 7m_a + 9m_b)\xi_1 c_1 + 2\rho\eta - 6\rho c_1, && (6.20)\\ N_{\mathbf{10}^{(a)(a)}} &= 10(k_a + m_a) + (3k_a + m_a)\xi_1^2 - (4k_a + 2m_a + 2m_b)\xi_1\eta \\ &\quad +(15k_a + 7m_a + 6m_b)\xi_1 c_1 - 2\rho\eta + 6\rho c_1. && (6.21) \end{aligned}$$

The supersymmetry condition is then

$$[(3m_b + m_a)\xi_1 - m_a(\eta - 4c_1) + 3\rho] \cdot_S [\omega] = 0, \tag{6.22}$$

where $[\omega]$ is an ample divisor dual to a Kähler form of dP_7. For simplicity, we choose $[\omega]$ to be

$$[\omega] = 14\beta H - (5\beta - \alpha)\sum_{i=1}^{7} E_i, \tag{6.23}$$

with constraints $5\beta > \alpha > 0$.

In what follows, we present one example based on this geometry. This model is three-generation with vanishing restrictions of the $U(1)_X$ flux to the **16** curves.

6.3.1 Model

We present a three-generation model in this example. The numerical result of the parameters is listed in Table 15. With data in Table15 and Table16, one can obtain $\chi(X_4) = 648$ and $\Gamma^2 = -42$ by using Eqs. (6.16) and (6.17). It follows from Eq. (5.9) that $N_{D3} = 6$.

k_b	k_a	m_b	m_a	ρ	ξ_1	α	β
-1.5	-1	0	1.5	$\frac{1}{2}(2E_1 + 2E_2 + E_4)$	$2H - E_1 - E_2 - E_3 - E_5 - E_6$	3	2

Table 15: Parameters of the (3,1) factorization model in dP_7.

The matter content and the corresponding classes are listed in Table 16.

Matter	Class in S	Class with fixed ξ_1	Generation
$\mathbf{16}^{(b)}$	ξ_1	$2H - E_1 - E_2 - E_3 - E_5 - E_6$	0
$\mathbf{16}^{(a)}$	$\eta - 4c_1 - \xi_1$	$2H - E_4 - 2E_7$	3
$\mathbf{10}^{(a)(b)}$	$\eta - 3c_1$	$7H - 2\sum_{i=1}^{6} E_i - 3E_7$	1
$\mathbf{10}^{(a)(a)}$	$\eta - 3c_1$	$7H - 2\sum_{i=1}^{6} E_i - 3E_7$	-1

Table 16: The dP_7 model matter content. Since it is a three-generation model, the flux is chosen to have trivial restriction. For example, $[F_X] = E_5 - E_6$.

6.3.2 Discussion

In this example we tune $[F_X] = E_4 - E_5$ to obtain trivial restrictions on all the curves, so the chirality on each curve remains unchanged. By the analysis of Table 6, we can create a flipped $SU(5)$ spectrum as shown in Table 17. The Yukawa couplings turn out to be

$$\begin{aligned} \mathcal{W} \supset\ & \mathbf{10}_{-1,-1M}\mathbf{10}_{-1,-1M}\mathbf{5}_{2,2h} + \mathbf{10}_{-1,-1M}\bar{\mathbf{5}}_{-1,3M}\bar{\mathbf{5}}_{2,-2h} + \bar{\mathbf{5}}_{-1,3M}\mathbf{1}_{-1,-5M}\mathbf{5}_{2,2h} \\ +\ & \mathbf{10}_{-1,-1H}\mathbf{10}_{-1,-1H}\mathbf{5}_{2,2h} + \overline{\mathbf{10}}_{-1,1H}\overline{\mathbf{10}}_{-1,1H}\bar{\mathbf{5}}_{2,-2h} + \cdots . \end{aligned} \tag{6.24}$$

Matter	Rep.	Generation
$\mathbf{10}_M$	$\mathbf{10}_{-1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{-1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{-1,-5}$	3
$\mathbf{5}_h$	$\mathbf{5}_{2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{2,-2}$	1
$\mathbf{10}_H + \overline{\mathbf{10}}_H$	$\mathbf{10}_{-1,-1} + \overline{\mathbf{10}}_{-1,1}$	1
$\mathbf{5} + \bar{\mathbf{5}}$ exotics*		

Table 17: Flipped $SU(5)$ spectrum with vanishing restrictions of $[F_X]$ on the curves in (3,1) factorization in dP_7.

This spectrum looks standard, and the advantage is that there are no exotic fermions and the quantum numbers(charges) of the matter are typical. We again

*There is one $(\mathbf{5}, \bar{\mathbf{5}})$ on the $\mathbf{10}^{(a)(a)}$ curve.

assume that the superheavy Higgses $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ come from one of the vector-like $\mathbf{10}+\overline{\mathbf{10}}$ pairs on the $\mathbf{16}^{(a)}$ curve. It is not obvious to calculate the number of such pairs. For simplicity, we just extract one pair for phenomenology purposes.

6.4 (2,2) Factorization and CY_4 with a dP_2 Surface

Let us consider the $(2,2)$ factorization with the geometric background in Eq. (6.4) [45]. In this case, the refined Euler characteristic turns out to be

$$\chi(X_4) = 10446 + (12\xi_2^2 - 12\xi_2\eta + 48\xi_2 c_1). \tag{6.25}$$

The self-intersection of the cover flux Γ is

$$\begin{aligned}\Gamma^2 \;=\;& -2(k_{d_1}+m_{d_1})^2(39+\xi_2^2-2\xi_2\eta+6\xi_2c_1)+4m_{d_1}^2(17+\xi_2^2-2\xi_2\eta+8\xi_2c_1)\\ &-2(k_{d_2}+m_{d_2})^2(\xi_2^2+2\xi_2c_1)+4m_{d_2}^2\xi_2^2+8m_{d_1}m_{d_2}(\xi_2^2-\xi_2\eta+4\xi_2c_1)\\ &+4\rho^2-8m_{d_1}(\rho\eta-\rho\xi_2-4\rho c_1)+8m_{d_2}\rho\xi_2.\end{aligned} \tag{6.26}$$

In this case, we can find models with integral N_{D3}. However, to have more degrees of freedom for model building, we shall focus on the geometry of the CY_4 with an embedded dP_7 surface [54] in the next subsection.

6.5 (2,2) Factorization and CY_4 with a dP_7 Surface

We again consider the geometric background in Eq. (6.15)and the (2,2) factorization. In this case, the refined Euler characteristic is given by

$$\chi(X_4) = 636 + (12\xi_2^2 - 12\xi_2\eta + 48\xi_2 c_1). \tag{6.27}$$

The self-intersection of the cover flux Γ is

$$\begin{aligned}\Gamma^2 \;=\;& -2(k_{d_1}+m_{d_1})^2(14+\xi_2^2-2\xi_2\eta+6\xi_2c_1)+4m_{d_1}^2(6+\xi_2^2-2\xi_2\eta+8\xi_2c_1)\\ &-2(k_{d_2}+m_{d_2})^2(\xi_2^2+2\xi_2c_1)+4m_{d_2}^2\xi_2^2+8m_{d_1}m_{d_2}(\xi_2^2-\xi_2\eta+4\xi_2c_1)\\ &+4\rho^2-8m_{d_1}(\rho\eta-\rho\xi_2-4\rho c_1)+8m_{d_2}\rho\xi_2.\end{aligned} \tag{6.28}$$

The generations of matter on the curves are

$$N_{\mathbf{16}^{(d_2)}} = (m_{d_1}-k_{d_2})\xi_2^2 - m_{d_1}\xi_2\eta + (4m_{d_1}-2k_{d_2}-2m_{d_2})\xi_2 c_1 + \rho\xi_2, \quad (6.29)$$

$$\begin{aligned} N_{\mathbf{16}^{(d_1)}} &= -(14k_{d_1}+8m_{d_1}) + (m_{d_2}-k_{d_1})\xi_2^2 + (2k_{d_1}-m_{d_2})\xi_2\eta \\ &\quad +(4m_{d_2}-6k_{d_1}+2m_{d_1})\xi_2 c_1 - \rho\eta + 4\rho c_1 + \rho\xi_2, \end{aligned} \quad (6.30)$$

$$\begin{aligned} N_{\mathbf{10}^{(d_2)(d_2)}} &= -8m_{d_1} + 2(m_{d_1}+m_{d_2})\xi_2^2 + 2(m_{d_2}+5m_{d_1})\xi_2 c_1 - 2m_{d_1}\xi_2\eta \\ &\quad +2\rho c_1 + 2\rho\xi_2, \end{aligned} \quad (6.31)$$

$$N_{\mathbf{10}^{(d_1)(d_2)}} = -2(k_{d_1}+m_{d_1})(6+2\xi_2^2-3\xi_2\eta+12\xi_2 c_1), \quad (6.32)$$

$$\begin{aligned} N_{\mathbf{10}^{(d_1)(d_1)}} &= (12k_{d_1}+20m_{d_1}) + (4k_{d_1}+2m_{d_1}-2m_{d_2})\xi_2^2 - 2(3k_{d_1}+2m_{d_1})\xi_2\eta \\ &\quad +(24k_{d_1}-2m_{d_2}+14m_{d_1})\xi_2 c_1 - 2\rho c_1 - 2\rho\xi_2. \end{aligned} \quad (6.33)$$

The supersymmetry condition is then

$$[2m_{d_2}\xi_2 - 2m_{d_1}(\eta - 4c_1 - \xi_2) + 2\rho] \cdot_S [\omega] = 0, \quad (6.34)$$

where $[\omega]$ is an ample divisor dual to a Kähler form of dP_7. For simplicity, we choose $[\omega]$ to be

$$[\omega] = 14\beta H - (5\beta-\alpha)\sum_{i=1}^{7} E_i, \quad (6.35)$$

with constraints $5\beta > \alpha > 0$.

In the (2,2) factorization of the $SU(4)$ cover, we expect the matter spectrum for an $SO(10)$ model as

Maatter	Copy	$U(1)_C$
$\mathbf{16}^{(d_2)}$	0/3	-1
$\mathbf{16}^{(d_1)}$	3/0	1
$\mathbf{10}^{(d_2)(d_2)}$	n_1	-2
$\mathbf{10}^{(d_1)(d_2)}$	n_2	0
$\mathbf{10}^{(d_1)(d_1)}$	n_3	2

(6.36)

The $U(1)_C$ is of the $U(1)^3$ Cartan subalgebra of $SU(4)_\perp$ that is not removed from the monodromy. The Yukawa coupling is filtered by the conservation of this $U(1)_C$. The possible Yukawa couplings for constructing a minimum $SO(10)$ GUT are then $\mathbf{16}^{(d_1)}\mathbf{16}^{(d_1)}\mathbf{10}^{(d_2)(d_2)}$ and $\mathbf{16}^{(d_2)}\mathbf{16}^{(d_2)}\mathbf{10}^{(d_1)(d_1)}$. We will demonstrate examples of the flipped $SU(5)$ GUT model from the following models.

6.5.1 Model 1

In this example we demonstrate a three-generation model. The numerical parameters are shown in Table 18, and the matter content and the corresponding classes with the flux $[F_X] = E_2 - E_3$ are listed in Table 19. By using Eqs. (6.25) and (6.26), we obtain $\chi(X_4) = 600$ and $\Gamma^2 = -18$ which gives rise to $N_{D3} = 16$.

k_{d_2}	k_{d_1}	m_{d_2}	m_{d_1}	ρ	ξ_2	α	β
-1	0	1.5	-0.5	$-\frac{1}{2}(H - 2E_1 + 2E_2 + 2E_3 + 2E_4 + E_7)$	$H - E_1$	1	3

Table 18: Parameters of Model 1 of the (2,2) Factorization in dP_7.

Matter	Class in S	Class with fixed ξ_2	Generation	Restr. of F_X
$\mathbf{16}^{(d_2)}$	ξ_2	$H - E_1$	0	0
$\mathbf{16}^{(d_1)}$	$\eta - 4c_1 - \xi_2$	$3H - \sum_{i=2}^{6} E_i - 2E_7$	3	0
$\mathbf{10}^{(d_2)(d_2)}$	$c_1 + \xi_2$	$4H - 2E_1 - \sum_{i=2}^{6} E_i - 2E_7$	4	0
$\mathbf{10}^{(d_1)(d_2)}$	$2\eta - 8c_1 - 2\xi_2$	$6H - 2\sum_{i=2}^{6} E_i - 4E_7$	-3	0
$\mathbf{10}^{(d_1)(d_1)}$	$c_1 + \xi_2$	$4H - 2E_1 - \sum_{i=2}^{6} E_i - 2E_7$	-1	0

Table 19: The Matter content of Model 1. The flux is tuned that the restriction is zero on each curve.

6.5.2 Model 2

k_{d_2}	k_{d_1}	m_{d_2}	m_{d_1}	ρ	ξ_2	α	β
1	0	-0.5	-0.5	$-\frac{1}{2}(H - 2E_1 + 2E_2 - 2E_3 - E_7)$	$2H - E_1 - E_2 - E_3 - E_7$	1	3

Table 20: Parameters of Model 2 of the (2,2) Factorization in dP_7.

In this model, we show a four-generation example with non-zero restrictions of F_X on the matter curves. The spectrum can maintain a three-generation model after the gauge is broken to $SU(5) \times U(1)_X$ by F_X. The parameters are presented in Table 20, while the matter content and the corresponding classes with the flux $[F_X] = E_3 - E_4$

are listed in Table 21. In this model, we have $\chi(X_4) = 600$ and $\Gamma^2 = -26$ which gives rise to $N_{D3} = 12$.

Matter	Class in S	Class with fixed ξ_2	Gen.	Restr. of F_X
$\mathbf{16}^{(d_2)}$	ξ_2	$2H - E_1 - E_2 - E_3 - E_7$	0	1
$\mathbf{16}^{(d_1)}$	$\eta - 4c_1 - \xi_2$	$2H - E_4 - E_5 - E_6 - E_7$	4	-1
$\mathbf{10}^{(d_2)(d_2)}$	$c_1 + \xi_2$	$5H - 2E_1 - 2E_2 - 2E_3 - \sum_{i=4}^{6} E_i - 2E_7$	4	1
$\mathbf{10}^{(d_1)(d_2)}$	$2\eta - 8c_1 - 2\xi_2$	$4H - 2E_4 - 2E_5 - 2E_6 - 2E_7$	-3	-2
$\mathbf{10}^{(d_1)(d_1)}$	$c_1 + \xi_2$	$5H - 2E_1 - 2E_2 - 2E_3 - \sum_{i=4}^{6} E_i - 2E_7$	-1	1

Table 21: Matter content of Model 2. The flux $[F_X] = E_3 - E_4$ has restrictions on the curves.

6.5.3 Discussion

Matter	Rep.	Generation
$\mathbf{10}_M$	$\mathbf{10}_{1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{1,-5}$	3
$\mathbf{5}_h$	$\mathbf{5}_{-2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{-2,-2}$	1
$\mathbf{10}_H + \overline{\mathbf{10}}_H$	$\mathbf{10}_{1,-1} + \overline{\mathbf{10}}_{1,1}$	1
$\mathbf{5} + \bar{\mathbf{5}}$ exotics	$\mathbf{5}_{-2,2} + \bar{\mathbf{5}}_{-2,-2}$	3
	$\mathbf{5}_{0,2} + \bar{\mathbf{5}}_{0,-2}$	3
	$\mathbf{5}_{2,2} + \bar{\mathbf{5}}_{2,-2}$	-1

Table 22: Flipped $SU(5)$ spectrum of Model 1 of the (2,2) factorization in dP_7.

The number of (-2) 2-cycles in dP_7 is large enough that it is possible to remain the chirality unchanged by tuning F_X with vanishing restrictions on all the curves. An example is presented in Model 1, and the corresponding flipped $SU(5)$ spectrum can be found in Table 22.

The Yukawa couplings of the flipped $SU(5)$ model from Model 1 then are

$$\begin{aligned}\mathcal{W} \supset\ & \mathbf{10}_{1,-1M}\mathbf{10}_{1,-1M}\mathbf{5}_{-2,2h} + \mathbf{10}_{1,-1M}\bar{\mathbf{5}}_{1,3M}\bar{\mathbf{5}}_{-2,-2h} + \bar{\mathbf{5}}_{1,3M}\mathbf{1}_{1,-5M}\mathbf{5}_{-2,2h} \\ + \ & \mathbf{10}_{1,-1H}\mathbf{10}_{1,-1H}\mathbf{5}_{-2,2h} + \overline{\mathbf{10}}_{1,1H}\overline{\mathbf{10}}_{1,1H}\bar{\mathbf{5}}_{-2,-2h} + \ldots . \end{aligned} \tag{6.37}$$

Similar to the examples with trivial restriction of F_X in the previous models, the spectrum in this model is standard in the sense that there are no exotic chiral fermions, and the quantum numbers of the matter are typical. We claim that the superheavy Higgses $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ come from a vector-like pair on the $\mathbf{16}^{(d_1)}$ curve, however again it is not obvious and we are not able to fix the number of such pairs. In addition, there exist a few exotic $\mathbf{5}$ fields from the $\mathbf{10}$ curves.

On the other hand, the restrictions of the flux F_X on the curves in Model 2 are non-vanishing, thus they contribute to the chirality on the curves. From the information in Table 7 we can interpret the matter content to fit the flipped $SU(5)$ GUT spectrum in Table 23.

Matter	Rep.	Generation
$\mathbf{10}_M$	$\mathbf{10}_{1,-1}$	3
$\bar{\mathbf{5}}_M$	$\bar{\mathbf{5}}_{1,3}$	3
$\mathbf{1}_M$	$\mathbf{1}_{1,-5}$	3
$\mathbf{10}_H + \overline{\mathbf{10}}_H$	$\mathbf{10}_{1,-1} + \overline{\mathbf{10}}_{1,1}$	1
$\mathbf{5}_h$	$\mathbf{5}_{-2,2}$	1
$\bar{\mathbf{5}}_h$	$\bar{\mathbf{5}}_{-2,-2}$	1
$\bar{\mathbf{5}}$	$\bar{\mathbf{5}}_{-1,3}$	1
$\mathbf{1}$	$\mathbf{1}_{-1,5}$	1
$\mathbf{1}$	$\mathbf{1}_{1,-5}$	2
$\mathbf{5}+\bar{\mathbf{5}}$ exotics from the $\mathbf{10}$ curves†		

Table 23: Flipped $SU(5)$ spectrum of Model 2 of the (2,2) factorization in dP_7.

In this case, the Yukawa couplings for flipped $SU(5)$ are the same:

$$\begin{aligned}\mathcal{W} \supset\ & \mathbf{10}_{-1,-1M}\mathbf{10}_{-1,-1M}\mathbf{5}_{2,2h} + \mathbf{10}_{-1,-1M}\bar{\mathbf{5}}_{1,3M}\bar{\mathbf{5}}_{0,-2h'} + \bar{\mathbf{5}}_{1,3M}\mathbf{1}_{-1,-5M}\mathbf{5}_{0,2h'} \\ + \ & \mathbf{10}_{-1,-1H}\mathbf{10}_{-1,-1H}\mathbf{5}_{2,2h} + \overline{\mathbf{10}}_{1,1H}\overline{\mathbf{10}}_{1,1H}\bar{\mathbf{5}}_{-2,-2h} + \ldots . \end{aligned} \tag{6.38}$$

†The $(\mathbf{5}, \bar{\mathbf{5}})$ exotics from the $\mathbf{10}$ curves of $SO(10)$ can be obtained from Table 7.

The $\mathbf{10}+\overline{\mathbf{10}}$ superheavey Higgses are identified as a vector-like pair from the **16** curve. In this model there are a few unavoidable exotic fields descended from both **16** and **10** curves.

6.5.4 The Singlet Higgs

In the flipped $SU(5)$ model, the matter singlet is the right-handed electron, while it is the right-handed neutrino in the Georgi-Glashow $SU(5)$ GUT. Different from the $SU(5)$ spectral cover construction, the flipped $SU(5)$ matter singlet is naturally embedded into the **16** representation of $SO(10)$ in the $SU(4)$ spectral cover configuration. Thus there is no need of additional effort to identify it in the spectrum.

Moreover, in flipped $SU(5)$ models, a Yukawa coupling needed to explain neutrino masses with the seesaw mechanism is [87, 88]

$$\mathbf{10}_{1M}\overline{\mathbf{10}}_{-1H}\mathbf{1}_{0\phi}. \tag{6.39}$$

This singlet $\mathbf{1}_0$ is an $SO(10)$ object and descends neither from the **16** nor from the **10** curves. Naively, one might think that it can be captured by the spectral cover associated to the adjoint representation in $SU(4)$ and the matter curve corresponds to $\pm(\lambda_i-\lambda_j)=0$ with $i\neq j$. The locus would then be given by [47]

$$b_0^5\prod_{i<j}^{4}(\lambda_i-\lambda_j)^2=-4b_2^3b_3^2-27b_0b_3^4+16b_2^4b_4+144b_0b_2b_3^2b_4-128b_0b_2^2b_4^2+256b_0^2b_4^3=0.$$

However, this is not the case. In fact, this singlet matter curve lives in the base B_3 instead of the surface S and can not be described by the spectral cover. To calculate the matter chirality on this singlet matter curve, we need the information of global geometry transverse to the surface S. In other words, we need to go beyond the spectral cover construction[15]. In the future, we hope there will be a global understanding of this singlet curve [47]. Therefore, we just assume this singlet exists and can provide the above Yukawa coupling.

[15]Recently this singlet has been discussed in [90] for the $SU(5)$ GUT, and it is possible to apply the same idea in this case. We leave this topic for our future work.

7 Conclusions

In this paper we built flipped $SU(5)$ models from the $SO(10)$ singularity by the $SU(4)$ spectral cover construction in F-theory. The **10** curve in the $SU(4)$ spectral cover configuration forms a double curve, and there are codimension two singularities on this curve [39]. It has been also shown that the net chirality on the **10** curve vanishes [39]. In order to obtain more degrees of freedom and non-zero generation number on the **10** curve, we split the $SU(4)$ cover into two factorizations. In the (3,1) factorization there are two **16** curves and two **10** curves on S, while in the (2,2) factorization there are two **16** curves and three **10** curves. The fluxes are also spread over the curves, providing additional parameters for model building.

We start model building from setting up appropriate $SO(10)$ spectrum on the **16** and **10** curves. Some Higgs fields, such as **210**, **120**, and $\mathbf{126}+\overline{\mathbf{126}}$ breaking the $SO(10)$ gauge group are absent in this construction. Therefore, we introduce a $U(1)_X$ flux to break $SO(10)$ to $SU(5)\times U(1)_X$. We interpret the resulting spectrum as a flipped $SU(5)$ model. The flux may have non-vanishing restrictions on the curves such that the corresponding chiralities may be modified. The superheavy Higgs fields $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ needed for breaking the gauge group to the MSSM are not obvious from the spectrum. We assume that they are a vector-like pair from the **16** curve including the fermion representations, but we are not able to fix the number of such pairs.

In the (3,1) factorization, we discuss first the construction on the geometry of the Calabi-Yau fourfold with an embedded dP_2 surface constructed in [45]. We demonstrated three examples. Two of them have three-generation, minimal $SO(10)$ GUT matter spectra. The $U(1)_X$ flux has always non-vanishing restrictions on the **16** curves, while it generically has vanishing restrictions on the **10** curves. Therefore, on a **16** curve, the chiralities of the **10**, **5**, and **1** representations are modified in the factor of the $U(1)_X$ charges, and the model no longer has three generations after the $SO(10)$ gauge symmetry is broken. To solve this problem, we constructed a four-generation model such that its corresponding flipped $SU(5)$ spectrum can possess at least three generations after the $U(1)_X$ flux is turned on. On the other hand, the $U(1)_X$ flux in the case of dP_7 geometry background [54] can be tuned to have trivial restrictions on the **16** curves so the chiralities remain untouched. We presented one three-generation example of the (3,1) factorization based on this geometry.

In the (2,2) factorization, to have more degrees of freedom for model building, we focused only on the geometry of the Calabi-Yau fourfold with an embedded dP_7 surface [54] and presented two examples. The first was a three-generation flipped $SU(5)$ model from the $SO(10)$ gauge group broken by the flux with trivial restrictions on all the matter curves. The second example, however, starts from a four-generation $SO(10)$ model whose gauge group is broken to $SU(5) \times U(1)_X$ by the flux with non-trivial restrictions on the matter curves. The resulting chiralities are modified by the flux restrictions to achieve the spectrum of a three-generation flipped $SU(5)$ model. Generically, the flipped $SU(5)$ models from a four-generation $SO(10)$ setup with non-vanishing flux restrictions to the **16** curves results in exotic fields from the **16** curves.

There remain some interesting directions for future research. First, we could construct $SO(10)$ singularities directly on Calabi-Yau fourfolds. Some examples in toric geometry are discussed in [86], and it would be interesting to consider more general fourfolds. Second, the $SO(10)$ singlet is important for the neutrino mass problem in the flipped $SU(5)$ phenomenology, however the mechanism of defining this singlet remains unclear. Third, we could investigate flipped $SU(5)$ models that do not descend from a D_5 singularity. The flipped $SU(5)$ models can be built from the anomaly-cancellation of the $U(1)$s of the monodromy group [89] in the well-studied $SU(5)$ spectral cover configuration in F-theory. A recent study on the abelian gauge factor from a certain global restriction of the Tate model [90] may be useful to study the $U(1)$ gauge groups. In addition, it is also exciting if we can turn on a non-abelian flux to break the $SO(10)$ gauge symmetry down to a standard-like model, such as the Pati-Salam model. We leave these questions for our future study.

Acknowledgments

We would like to thank A. Braun, J. Knapp, M. Kreuzer, C. Mayrhofer and J. Marsano for valuable communications and discussions. We also gratefully acknowledges hospitality and support from the Simons Center for Geometry and Physics, Stony Brook. YCC would like to thank the Physics Department at University of Cincinnati for hospitality during the preparation of this work. The work of CMC is supported in part by the Austrian Research Funds FWF under grant I192. The work of YCC is supported in part by the NSF under grant PHY-0555575 and by Texas A&M University.

References

[1] C. Vafa, Nucl. Phys. B **469**, 403 (1996) [arXiv:hep-th/9602022].

[2] D. R. Morrison and C. Vafa, Nucl. Phys. B **473**, 74 (1996) [arXiv:hep-th/9602114].

[3] D. R. Morrison and C. Vafa, Nucl. Phys. B **476**, 437 (1996) [arXiv:hep-th/9603161].

[4] F. Denef, arXiv:0803.1194 [hep-th].

[5] S. H. Katz and C. Vafa, Nucl. Phys. B **497**, 146 (1997) [arXiv:hep-th/9606086].

[6] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, Nucl. Phys. B **481**, 215 (1996) [arXiv:hep-th/9605200].

[7] C. Beasley, J. J. Heckman and C. Vafa, JHEP **0901**, 058 (2009) [arXiv:0802.3391 [hep-th]].

[8] C. Beasley, J. J. Heckman and C. Vafa, Predictions," arXiv:0806.0102 [hep-th].

[9] R. Donagi and M. Wijnholt, arXiv:0802.2969 [hep-th].

[10] R. Donagi and M. Wijnholt, arXiv:0808.2223 [hep-th].

[11] J. J. Heckman and C. Vafa, arXiv:0809.1098 [hep-th].

[12] J. J. Heckman and C. Vafa, arXiv:0809.3452 [hep-ph].

[13] J. J. Heckman and C. Vafa, arXiv:0811.2417 [hep-th].

[14] J. J. Heckman, G. L. Kane, J. Shao and C. Vafa, arXiv:0903.3609 [hep-ph].

[15] V. Bouchard, J. J. Heckman, J. Seo and C. Vafa, arXiv:0904.1419 [hep-ph].

[16] J. J. Heckman and C. Vafa, arXiv:0904.3101 [hep-th].

[17] R. Blumenhagen, arXiv:0812.0248 [hep-th].

[18] J. L. Bourjaily, arXiv:0901.3785 [hep-th].

[19] J. L. Bourjaily, arXiv:0905.0142 [hep-th].

[20] A. Font and L. E. Ibanez, arXiv:0811.2157 [hep-th].

[21] A. Font and L. E. Ibanez, Unification," JHEP **0909**, 036 (2009) [arXiv:0907.4895 [hep-th]].

[22] L. Randall and D. Simmons-Duffin, arXiv:0904.1584 [hep-ph].

[23] J. Jiang, T. Li, D. V. Nanopoulos and D. Xie, arXiv:0811.2807 [hep-th].

[24] J. Jiang, T. Li, D. V. Nanopoulos and D. Xie, arXiv:0905.3394 [hep-th].

[25] T. Li, arXiv:0905.4563 [hep-th].

[26] C.-M. Chen and Y.-C. Chung, Nucl. Phys. B **824**, 273 (2010) [arXiv:0903.3009 [hep-th]].

[27] J. J. Heckman, A. Tavanfar and C. Vafa, arXiv:0906.0581 [hep-th].

[28] J. P. Conlon and E. Palti, arXiv:0907.1362 [hep-th].

[29] J. P. Conlon and E. Palti, arXiv:0910.2413 [hep-th].

[30] S. Cecotti, M. C. N. Cheng, J. J. Heckman and C. Vafa, arXiv:0910.0477 [hep-th].

[31] Y. C. Chung, JHEP **1003**, 006 (2010) [arXiv:0911.0427 [hep-th]].

[32] S. F. King, G. K. Leontaris and G. G. Ross, arXiv:1005.1025 [hep-ph].

[33] E. I. Buchbinder, JHEP **0809**, 134 (2008) [arXiv:0805.3157 [hep-th]].

[34] J. J. Heckman, J. Marsano, N. Saulina, S. Schafer-Nameki and C. Vafa, arXiv:0808.1286 [hep-th].

[35] J. Marsano, N. Saulina and S. Schafer-Nameki, arXiv:0808.1571 [hep-th].

[36] J. Marsano, N. Saulina and S. Schafer-Nameki, arXiv:0808.2450 [hep-th].

[37] R. Blumenhagen, J. P. Conlon, S. Krippendorf, S. Moster and F. Quevedo, JHEP **0909**, 007 (2009) [arXiv:0906.3297 [hep-th]].

[38] J. J. Heckman, A. Tavanfar and C. Vafa, arXiv:0812.3155 [hep-th].

[39] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, Nucl. Phys. B **806**, 224 (2009) [arXiv:0805.1057 [hep-th]].

[40] A. P. Braun, A. Hebecker, C. Ludeling and R. Valandro, arXiv:0811.2416 [hep-th].

[41] G. Aldazabal, P. G. Camara and J. A. Rosabal, arXiv:0811.2900 [hep-th].

[42] A. Collinucci, arXiv:0812.0175 [hep-th].

[43] H. Hayashi, T. Kawano, R. Tatar and T. Watari, arXiv:0901.4941 [hep-th].

[44] B. Andreas and G. Curio, arXiv:0902.4143 [hep-th].

[45] J. Marsano, N. Saulina and S. Schafer-Nameki, JHEP **0908**, 030 (2009) [arXiv:0904.3932 [hep-th]].

[46] J. Marsano, N. Saulina and S. Schafer-Nameki, JHEP **0908**, 046 (2009) [arXiv:0906.4672 [hep-th]].

[47] J. Marsano, N. Saulina and S. Schafer-Nameki, JHEP **1004**, 095 (2010) [arXiv:0912.0272 [hep-th]].

[48] R. Donagi and M. Wijnholt, arXiv:0904.1218 [hep-th].

[49] C. Cordova, arXiv:0910.2955 [hep-th].

[50] R. Tatar, Y. Tsuchiya and T. Watari, Nucl. Phys. B **823**, 1 (2009) [arXiv:0905.2289 [hep-th]].

[51] A. Collinucci, arXiv:0906.0003 [hep-th].

[52] R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, JHEP **0909**, 053 (2009) [arXiv:0906.0013 [hep-th]].

[53] P. Aluffi and M. Esole, arXiv:0908.1572 [hep-th].

[54] R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, arXiv:0908.1784 [hep-th].

[55] T. W. Grimm, T. W. Ha, A. Klemm and D. Klevers, Compactifications," arXiv:0909.2025 [hep-th].

[56] H. Hayashi, T. Kawano, Y. Tsuchiya and T. Watari, arXiv:0910.2762 [hep-th].

[57] M. Cvetic, I. Garcia-Etxebarria and R. Richter, instantons," AIP Conf. Proc. **1200**, 246 (2010) [arXiv:0911.0012 [hep-th]].

[58] E. Dudas and E. Palti, JHEP **1001**, 127 (2010) [arXiv:0912.0853 [hep-th]].

[59] T. W. Grimm, T. W. Ha, A. Klemm and D. Klevers, arXiv:0912.3250 [hep-th].

[60] H. Jockers, P. Mayr and J. Walcher, arXiv:0912.3265 [hep-th].

[61] T. W. Grimm, S. Krause and T. Weigand, arXiv:0912.3524 [hep-th].

[62] H. Hayashi, T. Kawano, Y. Tsuchiya and T. Watari, Surface, arXiv:1004.3870 [hep-th].

[63] J. J. Heckman and H. Verlinde, arXiv:1005.3033 [hep-th].

[64] R. Blumenhagen, A. Collinucci and B. Jurke, arXiv:1002.1894 [hep-th].

[65] K. S. Choi and T. Kobayashi, arXiv:1003.2126 [hep-th].

[66] M. Cvetic, I. Garcia-Etxebarria and J. Halverson, arXiv:1003.5337 [hep-th].

[67] S. M. Barr, Phys. Lett. B **112**, 219 (1982).

[68] J. P. Derendinger, J. E. Kim and D. V. Nanopoulos, Phys. Lett. B **139**, 170 (1984).

[69] I. Antoniadis, J. R. Ellis, J. S. Hagelin and D. V. Nanopoulos, Phys. Lett. B **194**, 231 (1987).

[70] Demazure, Pinkham et Teissier, "Seminaire sur les Singularities des Surfaces," Ecole Polytechnique, 1976-1977.

[71] Y. I. Manin, Cubic forms: Algebra, geometry, arithmetic. North-Holland Publishing Co., Amsterdam, second ed., 1986. Translated from the Russian by M. Hazewinkel.

[72] F. Marchesano and L. Martucci, Phys. Rev. Lett. **104**, 231601 (2010) [arXiv:0910.5496 [hep-th]].

[73] H. Verlinde and M. Wijnholt, JHEP **0701**, 106 (2007) [arXiv:hep-th/0508089].

[74] M. R. Douglas and G. W. Moore, arXiv:hep-th/9603167.

[75] Eguchi, Gilkey, and Hanson, Phys. Rep. **66**, 214 (1980).

[76] P. Kronheimer, J. Diff. Geom. **28**, 665 (1989).

[77] P. Kronheimer, J. Diff. Geom. **29**, 685 (1989).

[78] N. Hitchin, Math. Proc. Camb. Phil. Soc **85**, 465 (1979).

[79] N. Hitchin, Sem. Bourbaki, Asterisque **206**, 137 (1992).

[80] R. Donagi, Y. H. He, B. A. Ovrut and R. Reinbacher, JHEP **0412**, 054 (2004) [arXiv:hep-th/0405014].

[81] S. Sethi, C. Vafa and E. Witten, Nucl. Phys. B **480**, 213 (1996) [arXiv:hep-th/9606122].

[82] R. Hartshorne, "Algebraic geometry," New York : Springer-Verlag, 1977.

[83] P. Griffith and J. Harris, "Principles of Algebraic Geometry," Wiley NY 1994.

[84] G. Curio and R. Y. Donagi, Nucl. Phys. B **518**, 603 (1998) [arXiv:hep-th/9801057].

[85] B. Andreas and G. Curio, Adv. Theor. Math. Phys. **3**, 1325 (1999) [arXiv:hep-th/9908193].

[86] C. M. Chen, J. Knapp, M. Kreuzer and C. Mayrhofer, arXiv:1005.5735 [hep-th].

[87] D. V. Nanopoulos, arXiv:hep-ph/0211128.

[88] H. Georgi and D. V. Nanopoulos, Nucl. Phys. B **155**, 52 (1979).

[89] E. Dudas and E. Palti, arXiv:1007.1297 [hep-ph].

[90] T. W. Grimm and T. Weigand, arXiv:1006.0226 [hep-th].