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Critical and Non-Critical Einstein-Weyl Supergravity

H. Lü [1,2], C.N. Pope [3,4], E. Sezgin[3] and L. Wulff[3]

[1] *China Economics and Management Academy*
Central University of Finance and Economics, Beijing 100081, China

[2] *Institute for Advanced Study, Shenzhen University*
Nanhai Ave 3688, Shenzhen 518060, China

[3] *George P. & Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy,*
Texas A&M University, College Station, TX 77843, USA

[4] *DAMTP, Centre for Mathematical Sciences, Cambridge University,*
Wilberforce Road, Cambridge CB3 OWA, UK

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ABSTRACT

We construct $\mathcal{N}=1$ supersymmetrisations of some recently-proposed theories of critical gravity, conformal gravity, and extensions of critical gravity in four dimensions. The total action consists of the sum of three separately off-shell supersymmetric actions containing Einstein gravity, a cosmological term and the square of the Weyl tensor. For generic choices of the coefficients for these terms, the excitations of the resulting theory around an AdS_4 background describe massive spin-2 and massless spin-2 modes coming from the metric; massive spin-1 modes coming from a vector field in the theory; and massless and massive spin-$\frac{3}{2}$ modes (with two unequal masses) coming from the gravitino. These assemble into a massless and a massive $\mathcal{N}=1$ spin-2 multiplet. In critical supergravity, the coefficients are tuned so that the spin-2 mode in the massive multiplet becomes massless. In the supersymmetrised extensions of critical gravity, the coefficients are chosen so that the massive modes lie in a "window" of lowest energies E_0 such that these ghostlike fields can be truncated by imposing appropriate boundary conditions at infinity, thus leaving just positive-norm massless supergravity modes.

Contents

1 Introduction

It was shown in [1, 2] that four-dimensional Einstein gravity with additional curvature-squared terms is perturbatively renormalisable. The renormalisability comes at a price, namely that the excitations around a Minkowski background contain states of negative norm as well as states of positive norm. Specifically, the excitations comprise massive spin-0 and massless spin-2 modes with positive norm, and massive spin-2 modes with negative norm. By choosing the curvature-squared terms to be of the form of the square of the Weyl tensor, the spin-0 modes can be eliminated. It was recently observed that if a cosmological constant is added as well, the coefficient of Weyl-squared can be adjusted so that the massive spin-2 modes become massless [3]. This theory of "critical" gravity thus describes regular massless spin-2 excitations and logarithmic spin-2 excitations around an AdS_4 background. The energies of the massless spin-2 modes are zero, whilst those of the logarithmic modes are in general nonvanishing [3]. However, as discussed in [4, 5], the energies of the general excitations can have either sign, and so one would have to truncate out the logarithmic modes in order to avoid ghostlike modes. This, unfortunately, leaves a rather empty theory with only zero-norm massless spin-2 states.

Maldacena recently considered the conformally-invariant theory with a pure Weyl-squared action, in which the massive spin-2 field in an AdS_4 background is truncated by imposing an appropriate boundary condition [6]. This is possible because the massive spin-2 mode actually has a negative mass-squared in this case, meaning that it carries a non-unitary representation of $SO(2,3)$, but it is not sufficiently negative to imply that it is tachyonic. This massive mode has a slower fall-off than the massless spin-2 mode, and so it can be eliminated, while retaining the massless mode, by imposing a suitable AdS fall-off condition at infinity.

It was subsequently observed in [7] that there exists a natural generalisation of critical gravity, in which the coefficient of Weyl-squared that is added to cosmological Einstein gravity is chosen to lie anywhere in the range where the massive spin-2 mode has negative, but not tachyonic, mass-squared. This gives a one-parameter family of theories where one can truncate out the ghostlike massive spin-2 modes by the imposition of boundary conditions, while retaining the (positive norm) massless spin-2 modes. One end of the parameter range corresponds to the pure Weyl-squared theory considered by Maldacena.

In this paper, we study an $\mathcal{N} = 1$ supersymmetric extension of cosmological gravity with the Weyl-squared term. We do this by starting from known results for an off-shell chiral superfield formulation, and then re-expressing the Lagrangian in a component field expansion. We shall work with off-shell $D = 4, \mathcal{N} = 1$ supergravity with the so-called old minimal set of auxiliary fields [8, 9][1]. Prior to adding in the Weyl-squared multiplet, the off-shell theory of Einstein supergravity plus cosmological constant contains an auxiliary vector field and an auxiliary complex scalar field. After adding in the Weyl-squared terms the auxiliary vector becomes dynamical, with propagating massive spin-1 modes. However, the complex scalar remains non-dynamical.

In section 2, we perform a component expansion of the chiral superfield expressions for the $\mathcal{N} = 1$ off-shell supersymmetric actions whose bosonic sectors correspond to Einstein gravity, the cosmological term, and the square of the Weyl tensor. These are exactly the ingredients in critical gravity and its extensions. For simplicity, we restrict attention to those terms that will contribute when computing the linearised fluctuation equations around the AdS_4 vacuum. In section 3, we derive the relevant equations of motion, and the linearised equations for the fluctuations. These give a fourth-order equation for spin-2 fluctuations, a third-order equation for spin-$\frac{3}{2}$ fluctuations, and a second-order equation for the spin-1 fluctuations. In section 4, we analyse the multiplet structure for the fluctuation fields, showing how, in general, they comprise a massless $\mathcal{N} = 1$ spin-2 multiplet, and a massive $\mathcal{N} = 1$ spin-2 multiplet. We also analyse the action of the supersymmetry transformations on the various fields.

In section 5, we examine possible ways to obtain ghost-free theories. This can be achieved by choosing the coefficient of the Weyl-squared term so that the undesirable negative-norm massive fields can be truncated from the spectrum by the imposition of appropriate boundary conditions, while still retaining the fields in the massless multiplet. We consider two cases; critical supergravity, where the massive multiplet becomes massless, giving rise to logarithmic modes that can be truncated from the spectrum; and a 1-parameter family of non-critical theories where the massive spin-2 fields are all in non-unitary representations of the AdS algebra, and which therefore have slower fall-off than the massless modes, allowing them again to be truncated by a suitable boundary condition. The paper ends with conclusions in section 6. In a set of three appendices we present some of our notation and conventions; a detailed discussion of the $\mathcal{N} = 1$ superspace constraints; and an explicit construction of the transformations, using Killing spinors, that relate the spinor and tensor harmonics in AdS_4 for all spins $s \leq 2$.

[1]For higher derivative off-shell $D = 4$, $\mathcal{N} = 1$ supergravity in the new minimal formulation, see [10, 11, 12, 13].

2 Off-Shell Supersymmetrisation of Cosmological Einstein plus Weyl-squared Gravity

There is a standard recipe for constructing a supersymmetric action from any chiral superfield r. The Lagrangian is given by [14]

$$e^{-1}\mathcal{L} = \Big(\tfrac{1}{2}D^\alpha D_\alpha + i(\bar\psi_a\sigma^a)^\alpha D_\alpha + \bar{\mathcal{M}} + \bar\psi_a\bar\sigma^{ab}\bar\psi_b\Big)\, r| + \text{h.c.}\,, \tag{2.1}$$

where the notation $X|$ denotes the lowest component (θ independent) in the θ expansion of the superfield X. The standard supergravity action is obtained by taking $r = -3\mathfrak{R}$, where $\mathfrak{R}$ is a chiral superfield whose lowest component is $\mathfrak{R}| = \frac{1}{6}\mathcal{M}$, where $\mathcal{M}$ is a complex scalar auxiliary field (see Appendix B).[2] The resulting Lagrangian is [8, 9]

$$e^{-1}\mathcal{L}_1 = \tfrac{1}{2}R + \tfrac{1}{3}(A^\mu A_\mu - \bar{\mathcal{M}}\mathcal{M}) + \tfrac{1}{2}\bar\psi_\mu\gamma^{\mu\nu\rho}\psi_{\nu\rho}\,, \tag{2.2}$$

where D_μ is the Lorentz-covariant derivative, A_μ is a real auxiliary vector field that also comes from $\mathfrak{R}$, and

$$\psi_{\mu\nu} = 2D_{[\mu}\psi_{\nu]}\ . \tag{2.3}$$

(See appendices A and B for further notation and conventions.) The Ricci scalar R in (2.2) is constructed from a spin-connection with added quadratic fermion torsion. These additional terms will not concern us here, since they will not contribute to the linearised equations in an AdS_4 background.

Taking instead $r = 1$, equation (2.1) gives

$$e^{-1}\mathcal{L}_2 \;=\; \mathcal{M} + \bar{\mathcal{M}} - \bar\psi_\mu\gamma^{\mu\nu}\psi_\nu\,. \tag{2.4}$$

In backgrounds where $\mathcal{M}$ is constant, this is the supersymmetrisation of a cosmological constant term.

Finally the Weyl-squared invariant is obtained by taking $r = -\frac{1}{4}W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$, where $W_{\alpha\beta\gamma}$ is a chiral superfield whose lowest component is proportional to the gravitino curvature (see Appendix B):

$$e^{-1}\mathcal{L}_3 = C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - \tfrac{2}{3}F^{\mu\nu}F_{\mu\nu} - \tfrac{4}{3}\bar\psi^{\mu\nu}\not{D}\psi_{\mu\nu} + \tfrac{4}{3}\bar\psi_{\mu\lambda}\gamma^{\mu\nu\rho}D_\rho\psi_\nu{}^\lambda + \cdots\,, \tag{2.5}$$

where

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\ , \tag{2.6}$$

[2] We use $\mathfrak{R}$ rather than the conventional R to denote the superfield, to avoid confusion with the Ricci scalar.

and the ellipses denote terms of the form $\psi^2 \times \nabla$(bosonic fields) and quartic fermion terms. (These terms will vanish when we study the excitations around an AdS_4 background, and so we shall not need to consider them in this paper.) Note that the square of the Weyl tensor can be written in terms of the Riemann and Ricci curvature as

$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \tfrac{1}{3}R^2\,. \tag{2.7}$$

There exists just one more independent curvature-squared invariant, modulo total derivatives, for which the superfield r is given by

$$r = \left(\bar{D}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}} - 4\mathfrak{R}\right)\mathfrak{R}\mathfrak{R}^\dagger\ . \tag{2.8}$$

In components, this gives rise to an R^2 term as well as a kinetic term for the real part of the auxiliary field $\mathcal{M}$. We shall not consider this invariant further, in this paper, so that the scalar fields remain non-dynamical.

The off-shell supersymmetry transformation rules are

$$\begin{aligned}
\delta e_\mu{}^a &= \bar{\epsilon}\gamma^a\psi_\mu\,,\\
\delta\psi_\mu &= -D_\mu\epsilon - \frac{i}{6}(2A_\mu + \gamma_{\rho\mu}A^\rho)\gamma_5\epsilon - \tfrac{1}{6}\gamma_\mu(S + i\gamma_5 P)\epsilon\,,\\
\delta S &= \bar{\epsilon}\gamma^{\mu\nu}\widehat{\psi}_{\mu\nu}\,,\\
\delta P &= i\bar{\epsilon}\gamma^{\mu\nu}\gamma_5\widehat{\psi}_{\mu\nu}\,,\\
\delta A_\mu &= \tfrac{i}{8}\bar{\epsilon}(\gamma_\mu\gamma^{\nu\rho} - 3\gamma^{\nu\rho}\gamma_\mu)\gamma_5\widehat{\psi}_{\nu\rho}\,,
\end{aligned} \tag{2.9}$$

where

$$\widehat{\psi}_{\mu\nu} = \psi_{\mu\nu} + \frac{i}{3}\gamma_5\left(2A_{[\mu} + A^\rho\gamma_{\rho[\mu}\right)\psi_{\nu]} + \tfrac{1}{3}\gamma_{[\mu}(S + i\gamma_5 P)\psi_{\nu]}\,, \tag{2.10}$$

and $\mathcal{M}$ is written in terms of real scalar and pseudoscalar fields as $\mathcal{M} = S + iP$.

Here we shall consider a linear combination of the supersymmetric Lagrangians discussed above,

$$\mathcal{L} = \mathcal{L}_1 + a\,\mathcal{L}_2 + b\,\mathcal{L}_3\,. \tag{2.11}$$

Thus the total bosonic Lagrangian is

$$\mathcal{L}_B = \tfrac{1}{2}R + \tfrac{1}{3}(A^\mu A_\mu - S^2 - P^2) + 2aS + b\,C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - \frac{2b}{3}F^{\mu\nu}F_{\mu\nu}\,, \tag{2.12}$$

and the total fermionic Lagrangian (modulo terms that will vanish in the AdS_4 background we shall consider) is

$$\mathcal{L}_F = \tfrac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\psi_{\nu\rho} - a\,\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu - \frac{4b}{3}\,\bar{\psi}^{\mu\nu}\not{D}\psi_{\mu\nu} + \frac{4b}{3}\,\bar{\psi}_{\mu\lambda}\gamma^{\mu\nu\rho}D_\rho\psi_\nu{}^\lambda\,. \tag{2.13}$$

3 Equations of Motion and Linearisation around AdS_4

3.1 Bosonic fields

The bosonic equations of motion, following from (2.12), are

$$S = 3a\,, \qquad P = 0\,, \tag{3.1}$$

$$\nabla^\mu F_{\mu\nu} + \frac{1}{4b}\, A_\nu = 0\,, \tag{3.2}$$

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \tfrac{1}{3}(S^2 + P^2 - 6aS) g_{\mu\nu} + \tfrac{2}{3}(A_\mu A_\nu - \tfrac{1}{2} A^\rho A_\rho g_{\mu\nu})$$
$$-\frac{8b}{3}(F_\mu{}^\rho\, F_{\nu\rho} - \tfrac{1}{4} F^{\rho\sigma} F_{\rho\sigma} g_{\mu\nu}) + E_{\mu\nu} = 0\,, \tag{3.3}$$

where $E_{\mu\nu}$, the contribution to the Einstein equation from the Weyl-squared term, is given by

$$E_{\mu\nu} = 8b(R_{\mu\rho}\, R_\nu{}^\rho - \tfrac{1}{4} R^{\rho\sigma} R_{\rho\sigma}\, g_{\mu\nu}) - \frac{4b}{3}\left[R\,(R_{\mu\nu} - \tfrac{1}{4} R\, g_{\mu\nu}) + g_{\mu\nu}\,\Box R - \nabla_\mu \nabla_\nu R\right]$$
$$+4b\left[\Box R_{\mu\nu} + \tfrac{1}{2}\Box R\, g_{\mu\nu} - 2\nabla_\rho \nabla_{(\mu} R_{\nu)}{}^\rho\right]\,. \tag{3.4}$$

The maximally-symmetric vacuum solution of the bosonic equations of motion is given by setting $A_\mu = 0$, and taking $g_{\mu\nu}$ to be the metric on AdS_4, satisfying

$$R_{\mu\nu\rho\sigma} = -a^2 (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})\,, \qquad R_{\mu\nu} = -3a^2 g_{\mu\nu}\,, \qquad R = -12a^2\,. \tag{3.5}$$

We may then consider the equations for linearised bosonic fluctuations around this background. For the metric, we consider $\delta g_{\mu\nu} = h_{\mu\nu}$, and define[3]

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \tfrac{1}{2} R^L\, g_{\mu\nu} + 3a^2\, h_{\mu\nu}\,, \tag{3.6}$$

$$R^L_{\mu\nu} = \nabla^\lambda \nabla_{(\mu} h_{\nu)\,\lambda} - \tfrac{1}{2}\Box h_{\mu\nu} - \tfrac{1}{2}\nabla_\mu \nabla_\nu h\,, \tag{3.7}$$

$$R^L = \nabla^\mu \nabla^\nu h_{\mu\nu} - \Box h + 3a^2 h\,, \tag{3.8}$$

where $h \equiv g^{\mu\nu} h_{\mu\nu}$. The linearised equation for $h_{\mu\nu}$ is then given by [3]

$$(4b\,\Box + 1 + 16a^2 b)\mathcal{G}^L_{\mu\nu} - \frac{4b}{3}(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box - 3a^2 g_{\mu\nu}) R^L = 0\,. \tag{3.9}$$

Noting that $g^{\mu\nu}\mathcal{G}^L_{\mu\nu} = -R^L$, we find that the trace of (3.9) gives simply

$$R^L = 0\,. \tag{3.10}$$

We may consider a 1-parameter family of possible gauge choices for $h_{\mu\nu}$, of the form

$$\nabla^\mu h_{\mu\nu} = c\nabla_\nu h\,, \tag{3.11}$$

[3] All covariant derivatives in the expressions expanded around AdS_4 are understood to be covariant with respect to the AdS_4 background connection.

where c is a constant. (de Donder gauge corresponds to $c = \frac{1}{2}$.) The trace equation (3.10) then implies

$$(c-1)\Box h + 3a^2 h = 0\,. \tag{3.12}$$

If we choose $c = 1$ in the gauge condition (3.11) then we immediately deduce that $h = 0$, as in [3]. If we instead take $c \neq 1$, then we can make residual coordinate transformations $\delta x^\mu = \xi^\mu$ with $\xi_\mu = \partial_\mu \xi$, which will therefore preserve the gauge condition (3.11) provided that ξ satisfies

$$(c-1)\Box \xi + 3a^2 \xi = 0\,. \tag{3.13}$$

Since the transformation of h is given by $h \to h + 2\Box\xi$, and since h and ξ satisfy the same equation, it follows that ξ can be used in order to set h to zero. Thus for any value of c, whether equal to 1 or not, the trace mode h can be eliminated by the gauge choice. We shall assume from now on that this has been done, and so $h_{\mu\nu}$ is in transverse traceless gauge,

$$\nabla^\mu h_{\mu\nu} = 0\,, \qquad h = 0\,. \tag{3.14}$$

The full linearised equation (3.9) for $h_{\mu\nu}$ then becomes [3]

$$(\Box + 2a^2)\left(\Box + 4a^2 + \frac{1}{4b}\right) h_{\mu\nu} = 0\,. \tag{3.15}$$

Provided that the constant terms in the two factors are unequal, the general solution to the fourth-order equation (3.15) is just a linear combination of solutions to the two second-order equations. To see this, suppose we have $(\Box+\lambda_1)(\Box+\lambda_2)h_{\mu\nu} = 0$. This can be written as

$$(\Box + \lambda_1) h^{(1)}_{\mu\nu} = 0\,, \quad \text{where} \quad (\Box + \lambda_2) h_{\mu\nu} = h^{(1)}_{\mu\nu}\,. \tag{3.16}$$

Defining

$$h_{\mu\nu} = h^{(2)}_{\mu\nu} + \frac{1}{\lambda_2 - \lambda_1}\, h^{(1)}_{\mu\nu}\,, \tag{3.17}$$

we see that provided $\lambda_2 \neq \lambda_1$, the general solution to the fourth-order equation is a linear combination of $h^{(1)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$ satisfying

$$(\Box + \lambda_1) h^{(1)}_{\mu\nu} = 0\,, \qquad (\Box + \lambda_2) h^{(2)}_{\mu\nu} = 0\,. \tag{3.18}$$

Thus, equation (3.15) implies that generically there are massless spin-2 modes satisfying

$$(\Box + 2a^2) h_{\mu\nu} = 0\,, \tag{3.19}$$

and additional massive spin-2 modes satisfying

$$(\Box + 4a^2 + \frac{1}{4b}) h_{\mu\nu} = 0\,. \tag{3.20}$$

The degenerate case where $2a^2 = 4a^2 + 1/(4b)$, i.e. $b = -1/(8a^2)$, which in fact corresponds to critical gravity, will be discussed in detail later, in section 5.

For the vector A_μ, which vanishes in the AdS$_4$ background, the fluctuation equation is just given by the Proca equation (3.2). Taking the divergence, one therefore finds

$$\nabla^\mu A_\mu = 0\,, \qquad (\Box + 3a^2 + \frac{1}{4b})A_\mu = 0\,. \tag{3.21}$$

3.2 The gravitino equation

The gravitino equation of motion in the AdS$_4$ background follows from (2.13):

$$\gamma^{\mu\nu\rho}\psi_{\nu\rho} - 2a\gamma^{\mu\nu}\psi_\nu - \frac{8b}{3}\Big[2\gamma^\rho D_\nu D_\rho\psi^{\mu\nu} + \gamma_\nu{}^{\rho\sigma} D_\rho D_\sigma\psi^{\mu\nu} - \gamma^{\mu\rho}{}_\nu D^\sigma D^\nu\psi_{\rho\sigma}\Big] = 0\,. \tag{3.22}$$

Multiplying with γ_μ, and using the identity $D_{[\mu}\psi_{\nu\rho]} = -\frac{1}{2}a^2\gamma_{[\mu\nu}\psi_{\rho]}$ in the AdS$_4$ background, we obtain

$$D^\mu\psi_\mu - (\not{D} - \tfrac{3}{2}a)(\gamma^\mu\psi_\mu) = 0\,. \tag{3.23}$$

Imposing the gauge condition $\gamma^\mu\psi_\mu = 0$ implies also $D^\mu\psi_\mu = 0$, and the gravitino equation of motion (3.22) gives

$$\not{D}\Box\psi_\mu + \left(3a^2 + \frac{1}{4b}\right)\not{D}\psi_\mu + \frac{a}{4b}\psi_\mu = 0\,. \tag{3.24}$$

Using $(\not{D})^2\psi_\mu = \Box\psi_\mu + 4a^2\psi_\mu$, we can rewrite (3.24) in the factorised form

$$(\not{D} + a)\left(\not{D} - \tfrac{1}{2}a - \tfrac{1}{2}\sqrt{a^2 - b^{-1}}\right)\left(\not{D} - \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{a^2 - b^{-1}}\right)\psi_\mu = 0\,. \tag{3.25}$$

The analysis of this third-order equation is analogous to our earlier discussion for spin 2. Provided that the three constant terms in the factorised form (3.25) are unequal, the general solution will be a linear combination of the solutions to the three separate factors. In other words, there will be the massless gravitino mode satisfying

$$(\not{D} + a)\psi_\mu = 0\,, \tag{3.26}$$

and two massive gravitino modes, satisfying, respectively,

$$(\not{D} - \tfrac{1}{2}a - \tfrac{1}{2}\sqrt{a^2 - b^{-1}})\psi_\mu = 0\,, \tag{3.27}$$

$$(\not{D} - \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{a^2 - b^{-1}})\psi_\mu = 0\,, \tag{3.28}$$

The degenerate cases, where two eigenvalues coincide, will be treated later in our discussion in section 5.

3.3 The linearised supersymmetry transformations

We begin by observing that the AdS_4 background given by (3.5) is supersymmetric. This can be seen from the expression for $\delta\psi_\mu$ in (2.9), which vanishes in the AdS_4 background for any Killing spinor solution ϵ_- of

$$D_\mu \epsilon_\pm = \pm\tfrac{1}{2} a \gamma_\mu \epsilon_\pm \,. \tag{3.29}$$

In what follows, it will be understood when we use ϵ to denote a Killing spinor, that it will be of the ϵ_- type.

The linearised transformation rules, which will be useful for describing how supersymmetry acts on the fluctuation modes, are given by

$$\begin{aligned} \delta h_{\mu\nu} &= 2\bar\epsilon \gamma_{(\mu}\psi_{\nu)} \,, \\ \delta\psi_\mu &= \tfrac{1}{4}\nabla_\rho h_{\mu\sigma}\,\gamma^{\rho\sigma}\epsilon - \frac{i}{6}(2A_\mu + \gamma_{\rho\mu}A^\rho)\gamma_5\epsilon - \tfrac{1}{4} a h_{\mu\nu}\gamma^\nu \epsilon \,, \\ \delta A_\mu &= \tfrac{3}{2} i\bar\epsilon\gamma_5 (\not{D} + a)\psi_\mu \,. \end{aligned} \tag{3.30}$$

In obtaining the expression for δA_μ, we have used the gauge condition $\gamma^\mu\psi_\mu = 0$, and its consequence that $D^\mu\psi_\mu = 0$.

4 Spectrum and Multiplet Structure of the Fluctuations

In this section, we investigate the structure of the small fluctuations around the AdS_4 background, showing how the various modes assemble into $\mathcal{N} = 1$ multiplets under AdS supersymmetry.

4.1 AdS representations of the fluctuations

Subject to appropriate boundary conditions, the solutions of the linearised equations obtained in the previous section form unitary irreducible representations of the $SO(3,2)$ AdS group. These representations, denoted by $D(E_0, s)$, are labelled by their lowest energy E_0 and their spin s. The unitary irreducible representations of $\mathcal{N} = 1$ AdS supersymmetry fall into four disjoint classes [15], namely

$$\begin{aligned} &\text{Class 1}: \quad D(\tfrac{1}{2},0)\oplus D(1,\tfrac{1}{2})\,, \\ &\text{Class 2}: \quad D(E_0,0)\oplus D(E_0+\tfrac{1}{2},\tfrac{1}{2})\oplus D(E_0+1,0)\,, \qquad E_0 > \tfrac{1}{2}\,, \\ &\text{Class 3}: \quad D(s+1,s)\oplus D(s+\tfrac{3}{2},s+\tfrac{1}{2})\,, \qquad s = \tfrac{1}{2},1,\tfrac{3}{2},\ldots\,, \\ &\text{Class 4}: \quad D(E_0,s)\oplus D(E_0+\tfrac{1}{2},s+\tfrac{1}{2})\oplus D(E_0+\tfrac{1}{2},s-\tfrac{1}{2})\oplus D(E_0+1,s)\,, \quad E_0 > s+1\,. \end{aligned} \tag{4.1}$$

Class 1 is the singleton, supermultiplet; Class 2 is the Wess-Zumino supermultiplet; Class 3 comprises massless gauge supermultiplets; and Class 4 comprises massive supermultiplets.

The representations arising in our case can be determined from the eigenvalues of the D'Alembertian (for bosons) or the Dirac operator (for fermions). For the fields of spins 2, 1 and $\frac{3}{2}$ of interest to us, one has

$$\begin{aligned}
&D(E_0,2): && \Box h_{\mu\nu} = a^2[E_0(E_0-3)-2]h_{\mu\nu}\,, \\
&D(E_0,1): && \Box A_\mu = a^2[E_0(E_0-3)-1]A_\mu\,, \\
&D(E_0,\tfrac{3}{2}): && \not{D}\psi^\pm_\mu = \pm a(E_0-\tfrac{3}{2})\psi^\mp_\mu\,,
\end{aligned} \tag{4.2}$$

where $\psi^\pm_\mu = \frac{1}{2}(1\pm\gamma_5)\psi_\mu$.

Let us first consider the general situation, for generic values of the coefficient b associated with the Weyl-squared term. From (3.15) we see that there are always massless spin-2 modes satisfying (3.19), in the $D(3,2)$ representation, and from (3.25) there are always massless spin-$\frac{3}{2}$ modes satisfying (3.26), in the $(\frac{5}{2},\frac{3}{2})$ representation. These bosonic and fermionic modes form the massless supermultiplet

$$D(\tfrac{5}{2},\tfrac{3}{2})\oplus D(3,2)\,, \tag{4.3}$$

which is of Class 3 with $s=\frac{3}{2}$.

The remaining modes that we read off from (3.20) for spin-2, (3.21) for spin-1, and (3.27) and (3.28) for spin-$\frac{3}{2}$, can then be seen, respectively, to have the E_0 values

$$\begin{aligned}
&\text{Spin-2}: && E_0 = \tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{1-\frac{1}{a^2 b}}\,, \\
&\text{Spin-1}: && E_0 = \tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{1-\frac{1}{a^2 b}}\,, \\
&\text{Spin-}\tfrac{3}{2}: && E_0 = 2\pm\tfrac{1}{2}\sqrt{1-\frac{1}{a^2 b}}\,, \quad \text{and} \quad E_0 = 1\pm\tfrac{1}{2}\sqrt{1-\frac{1}{a^2 b}}\,.
\end{aligned} \tag{4.4}$$

When the plus sign is chosen in front of all the square roots, and if the parameter b is chosen so that

$$\sqrt{1-\frac{1}{a^2 b}} > 3\,, \tag{4.5}$$

i.e so that

$$-\frac{1}{8a^2} < b < 0\,, \tag{4.6}$$

then the representations in (4.4) all satisfy the bound $E_0 > s+1$, and they can be seen to form an $\mathcal{N}=1$ unitary massive supermultiplet, of the Class 4 type. If (4.5) is not satisfied, then the multiplet will be non-unitary. There is another massive multiplet, which is always non-unitary, corresponding to taking the minus sign in front of all the square roots.

If the parameter b lies in the range where $1 - 1/(a^2 b)$ is negative, then the E_0 values become complex. Since, in particular, the modes have time dependence proportional to $e^{iE_0 t}$, this would imply that they would have real exponential growth, corresponding to classical instability. Such modes are tachyonic, and are the higher-spin analogues of scalar modes that violate the Breitenlohner-Freedman bound [16]. We shall always require that b be chosen so that

$$1 - \frac{1}{a^2 b} \geq 0\,. \tag{4.7}$$

4.2 Action of supersymmetry on the fluctuation modes

In this subsection we shall study the manner in which supersymmetry maps the solutions of different spins into each other. There are two reasons why it is of interest to do this. Firstly, it provides a simple way to obtain explicit expressions for the solutions for all spins $s \leq 2$, starting from those for any particular given spin. Secondly, it will give nontrivial information about the multiplet structure, including in the critical case, which we shall discuss in section 5, when non-standard representations with logarithmic behaviour arise. In the present section, we shall consider just the non-critical case.

We can determine how supersymmetry acts on the fluctuations by making use of the linearised supersymmetry transformations given in equations (3.30). Essentially, we substitute a mode of one of the fields, satisfying (3.19), (3.20), (3.21), (3.26), (3.27) or (3.28), into the right-hand sides of the transformation rules, and thus read off the associated supersymmetry-related modes. To be precise, it is necessary also to make appropriate compensating gauge transformations (general coordinate, and/or local Lorentz), in order to ensure that the supersymmetry-related modes obey the appropriate gauge conditions we are imposing, which amount to their being divergence-free and (γ–)traceless.

To begin, we observe that if ψ_μ satisfies the massless gravitino equation (3.26), then the $\delta h_{\mu\nu}$ transformation in (3.30) generates a massless spin-2 solution, since

$$(\Box + 2a^2)[2\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + \delta_\xi h_{\mu\nu}] = 0\,, \tag{4.8}$$

where the compensating general coordinate transformation is given by

$$\delta_\xi h_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\,, \qquad \xi_\mu = \frac{1}{3a}\,\bar{\epsilon}\psi_\mu\,. \tag{4.9}$$

Note that the massless ψ_μ mode does not generate any spin-1 solution, since the $(\not{D} + a)$ operator in the δA_μ transformation in (3.30) annihilates the massless gravitino solution.

In the reverse direction, substituting the massless spin-2 solution (3.19) into the $\delta\psi_\mu$ transformation, we find that indeed

$$(\not{D}+a)[\tfrac{1}{4}\nabla_\rho h_{\mu\sigma}\,\gamma^{\rho\sigma}\epsilon-\tfrac{1}{4}ah_{\mu\nu}\,\gamma^\nu\epsilon]=0\ , \tag{4.10}$$

which shows that the $\delta\psi_\mu$ generates a massless spin-3/2 solution.

By similar reasoning, we find that the solutions of the massive fluctuation equations map into one another under the linearised supersymmetry transformations, forming the massive supermultiplet that we discussed in the previous subsection. In this case, the required compensating general coordinate transformation is given by

$$\delta_\xi h_{\mu\nu}=2\nabla_{(\mu}\xi_{\nu)}\ ,\qquad \xi_\mu=\frac{1}{2a-\lambda}\,\bar\epsilon\psi_\mu\ , \tag{4.11}$$

where $\not{D}\psi_\mu=\lambda\psi_\mu$ with λ to be read off from (3.27) and (3.28). The singular situation where $\lambda=2a$ arises in the critical case which will be discussed in section 5.2. The substitution of a massive gravitino solution into the right hand side of δA_μ generates the solution for the massive vector field obeying the Proca field equation (3.21). Finally, with the substitution of the massive graviton solution into $\delta\psi_\mu$, it solves the equation

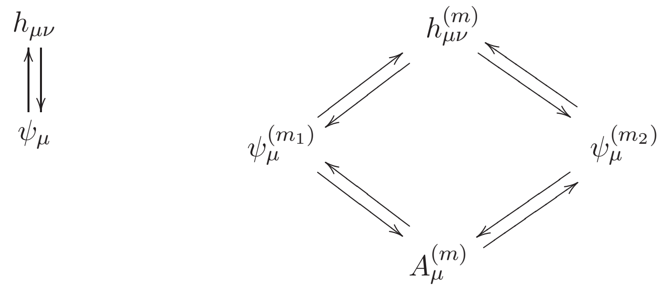
$$(\not{D}-\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b^{-1}})(\not{D}-\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b^{-1}})[\tfrac{1}{4}\nabla_\rho h_{\mu\sigma}\,\gamma^{\rho\sigma}\epsilon-\tfrac{1}{4}ah_{\mu\nu}\,\gamma^\nu\epsilon]=0\,, \tag{4.12}$$

and thus both of the massive gravitino modes arise, as a linear combination. Substituting the massive spin-1 solution in $\delta\psi_\mu$ on the other hand, again yields a linear combination of massive gravitino solutions, provided that we take into account compensating supersymmetry transformation needed to ensure that $\delta\psi_\mu$ is divergent-free and γ-traceless. This compensating transformation, whose parameter we shall denote by $\hat\epsilon$ is given by

$$\hat\epsilon=\frac{ib}{3(1+8a^2b)}\,(4aA_\mu\gamma^\mu-F_{\mu\nu}\gamma^{\mu\nu})\,\gamma^5\epsilon\ . \tag{4.13}$$

Note that only terms that are linear in fluctuation fields are to be retained in $\delta_{\hat\epsilon}\psi_\mu$. Moreover, the overall factor is divergent at the critical point that will be discussed further in section 5.2.

In summary, we have shown that away from the critical point the fluctuations form a massless and a massive supergravity multiplet, both on shell, as shown in the figure below, where the superscripts refer to massive states whose AdS energies are given in (4.4).



5 Ghost-free Supergravities

As is well known in the case of pure cosmological gravity with a Weyl-squared term, the massive spin-2 excitations around the AdS_4 background have energies that are opposite in sign to those of the massless spin-2 modes (see, for example, [3]). Thus if the overall sign of the action is chosen so that the massless graviton has positive-energy excitations, then the massive spin-2 modes will be ghostlike. In order to achieve a ghost-free theory, one may try to eliminate the massive excitations by imposing some appropriate boundary conditions at infinity. The situation for the supersymmetric extensions that we are considering in this paper is similar, and so we can again examine the circumstances under which such a truncation of the massive multiplets may be possible.

It is useful to divide the discussion into two cases. One case arises when the critical choice for the parameter b is taken, namely

$$b = b_{\rm crit} = -\frac{1}{8a^2}\,. \tag{5.1}$$

In this case, the massive spin-2 modes, satisfying (3.20), become massless, resulting in the emergence of a new type of solution to the fourth-order equation (3.15) that has a logarithmic dependence on the radial AdS_4 coordinate. An analogous phenomenon occurs also in the spin-$\frac{3}{2}$ sector. We shall discuss this case in subsection 5.2 below. The logarithmic modes have indefinite norm, and must therefore be truncated out in order to achieve a ghost-free theory. However, the massless spin-2 modes have zero norm in this case [3], and so after the truncation one is left with a rather trivial theory. A further feature, in this critical case, is that the kinetic term $-\frac{2}{3}bF^{\mu\nu}F_{\mu\nu}$ for the spin-1 fields has the "wrong sign."

The second case, which we shall consider first, corresponds to the situation where b is instead chosen so that the unitarity bound (4.5) is violated, while still respecting the condition (4.7) for avoiding tachyons. This will provide a supersymmetric generalisation of the "extended critical gravities" considered recently in [7].

5.1 Extensions of critical supergravity

In order to be able to impose boundary conditions that eliminate the ghost-like massive modes, while retaining the desired massless modes, it is necessary to choose the b parameter to lie in a range where the massive modes have a slower fall-off at infinity than the massless modes. The fall-off is governed by the lowest-energy eigenvalue E_0, with modes having larger E_0 falling off faster than those with smaller E_0. (See for example [17], where the spin-2 modes are constructed.) Thus the desired choices for the parameter b will be those for which the massive modes are all non-unitary, satisfying $E_0 < s+1$, while, by contrast, the massless modes satisfy $E_0 = s+1$. Bearing in mind that we must still require the massive modes to be non-tachyonic, in order to avoid classical instabilities, it follows from (4.5) and (4.7) that b should be chosen to satisfy

$$b \geq \frac{1}{a^2} \qquad \text{or} \qquad b \leq -\frac{1}{8a^2}\,. \tag{5.2}$$

There is a further requirement, which excludes the negative b choices in (5.2). This can be seen from the results in [3, 7], where the energies of the spin-2 modes are calculated. In order to have non-negative energies for the massless spin-2 modes, it is necessary that b should satisfy $b \geq -1/(8a^2)$. Thus we are led to consider the 1-parameter family of theories for which

$$b \geq \frac{1}{a^2}\,. \tag{5.3}$$

For all values of b within this range, the modes in the massive supermultiplet will fall off more slowly than those in the massless supermultiplet, and so they can be eliminated by imposing appropriate boundary conditions at infinity. Included in this family is the limit where b goes to infinity; after making an overall rescaling with a factor $1/b$, this corresponds to the conformally-invariant case that is the $\mathcal{N}=1$ generalisation of the pure Weyl-squared gravity that was recently considered by Maldacena [6]. In the entire range (5.3), the excitations in the massless supermultiplet will all have positive energies.

It is interesting to note that at the lower end of the range in (5.3), when $b = 1/a^2$, the two massive spin-$\frac{3}{2}$ branches in (3.27) and (3.28) become degenerate, and so there will be spin-$\frac{3}{2}$ modes with logarithmic coordinate dependence in this case, even though none of the other members of the massive supermultiplet will exhibit such behaviour. It is also worth remarking that the kinetic term $-\frac{2}{3} b F^{\mu\nu} F_{\mu\nu}$ for the spin-1 field has the correct sign throughout the range (5.3).

5.2 Critical supergravity

At the critical point we have

$$b_{\rm crit} = -\frac{1}{8a^2} \tag{5.4}$$

and the linearized equations of motion become

$$\left(\Box + 2a^2\right)^2 h_{\mu\nu} = 0 \,, \tag{5.5}$$

$$\left(\Box + a^2\right) A_\mu = 0 \,, \tag{5.6}$$

$$(\not{D} + a)^2(\not{D} - 2a)\psi_\mu = 0 \,. \tag{5.7}$$

It immediately follows that the vector field describes a massive spin-1 mode with AdS energy $E_0 = 3$. As for the graviton and gravitino field equations, to begin with they describe modes that follow from the factorization of their wave operators. These are the massless spin-2 and massless spin-3/2 modes, and a massive spin-3/2 mode satisfying $(\not{D} - 2a)\psi_\mu = 0$, thereby having AdS energy $E_0 = 7/2$. In addition to these, however, there will also be logarithmic modes that satisfy the relations

$$\begin{aligned} (\Box + 2a^2)^2 h^{\log}_{\mu\nu} &= 0 \,, \qquad & (\Box + 2a^2) h^{\log}_{\mu\nu} &\neq 0 \,, \\ (\not{D} + a)^2 \psi^{\log}_\mu &= 0 \,, \qquad & (\not{D} + a)\psi^{\log}_\mu &\neq 0 \,. \end{aligned} \tag{5.8}$$

Next we discuss how supersymmetry relates the fluctuation modes to each other, to determine the underlying multiplet structure. As in our previous discussion for the case of a generic massive multiplet, we look at the linearised supersymmetry transformations in (3.30), plug in the various modes at the critical point on the right–hand–side and then verify that the result satisfies an appropriate equation. In some cases the supersymmetry transformations have to be accompanied by an appropriate compensating gauge-transformation to preserve the gauge-conditions.

As we have seen previously, when we substitute the critical massive gravitino mode satisfying $(\not{D} - 2a)\psi_\mu = 0$ into $\delta h_{\mu\nu}$, the compensating gauge-transformation (4.11) that is needed in order to preserve the gauge condition diverges. This means that supersymmetry does not map the critical massive gravitino mode to a transverse traceless spin-2 mode. Similarly we have seen that when the critical massive vector mode is substituted in $\delta\psi_\mu$, the required compensating gauge transformation (4.13) again diverges, which means that the critical massive vector mode is not mapped to a gravitino mode in the $\gamma^\mu\psi_\mu = 0$, $D^\mu\psi_\mu = 0$ gauge by supersymmetry. When substituted into δA_μ, the critical massive gravitino will however give rise to a critical massive spin-1 mode, as follows immediately from the analysis we gave previously.

It remains only to analyse what happens when the logarithmic modes satisfying (5.8) are substituted into the supersymmetry transformations. Let us start with the supersymmetry variation of the vector. It is not hard to verify that

$$(\Box + a^2)\left(\bar{\epsilon}\gamma_5(\not{D} + a)\psi_\mu^{\log}\right) = 0\,. \tag{5.9}$$

This means that the gravitino log mode is mapped by supersymmetry into the critical massive spin-1 mode. Next we consider what happens when the graviton log mode is substituted into $\delta\psi_\mu$. Since the log modes satisfy the same gauge conditions as the regular modes, no compensating gauge transformation is needed and one finds

$$(\not{D} + a)[\tfrac{1}{4}\nabla_\rho h_{\mu\sigma}^{\log}\,\gamma^{\rho\sigma}\epsilon - \tfrac{1}{4}a h_{\mu\nu}^{\log}\,\gamma^\nu\epsilon] = \tfrac{1}{4}(\Box + 2a^2)h_{\mu\nu}^{\log}\,\gamma^\nu\epsilon \neq 0\ , \tag{5.10}$$

and

$$(\not{D} + a)^2(\not{D} - 2a)[\tfrac{1}{4}\nabla_\rho h_{\mu\sigma}^{\log}\,\gamma^{\rho\sigma}\epsilon - \tfrac{1}{4}a h_{\mu\nu}^{\log}\,\gamma^\nu\epsilon] = 0\,. \tag{5.11}$$

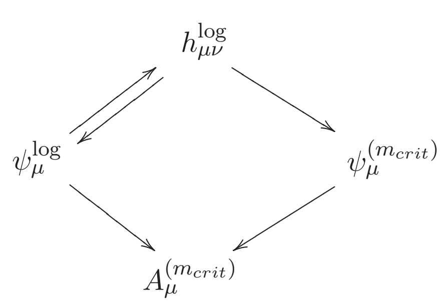
This shows that the graviton log mode is mapped by supersymmetry to a linear combination of the gravitino log mode and the critical massive gravitino mode. Finally we need to analyse what happens when the gravitino log mode is substituted into $\delta h_{\mu\nu}$. In this case a compensating general coordinate transformation will be needed to preserve the gauge condition. With some work, one can show that

$$(\Box + 2a^2)^2[2\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}^{\log} + \delta_\xi h_{\mu\nu}] = 0\,, \tag{5.12}$$

where the compensating general coordinate transformation takes the form

$$\delta_\xi h_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\,, \qquad \xi_\mu = \frac{1}{9a^2}\bar{\epsilon}(\not{D} + 4a)\psi_\mu^{\log}\,. \tag{5.13}$$

This shows that the gravitino log mode is mapped by supersymmetry into the graviton log mode. This completes the analysis of the supermultiplet structure at the critical point. In addition to the massless supergravity multiplet we have the non-standard multiplet



where supersymmetry transformations are indicated by the arrows.

Note that the logarithmic modes are not eigenstates of the AdS energy generator. Indeed, all of these modes are given as a product of a universal logarithmic dependent factor and the solution for the massless mode as [17]

$$\phi^{\log} = (2it + \log\sinh 2\rho - \log\tanh\rho)\phi^{\rm massless} , \tag{5.14}$$

where ϕ generically denotes any field that has logarithmic mode, in a coordinate system in which the AdS_4 metric is given by

$$a^2 ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right] . \tag{5.15}$$

We are not aware of a group theoretical analysis of the representations of AdS superalgebra which accommodates such states. The analysis of supersymmetry transformations, nonetheless, seems to suggest that that if boundary conditions that exclude logarithmic modes are to be imposed, then the full multiplet containing these modes are to be excluded. In any event, in view of the recent developments in the study of the critical bosonic gravity dynamical content [6, 7], we shall not pursue further the supersymmetric version of the story here.

6 Conclusions

In this paper, we have constructed an $\mathcal{N} = 1$ supersymmetrisation of a class of four-dimensional gravities with a quadratic curvature modification proportional to the square of the Weyl tensor. The resulting supergravities encompass supersymmetrisations of critical gravity [3], where the coefficient of Weyl-squared is adjusted so that the generically massive spin-2 excitations become massless; pure conformally-invariant Weyl-squared gravity, which was recently proposed in [6] as providing an equivalent description of ordinary gravity in the long-wavelength regime; and a class of generalisations of critical gravity considered recently in [7].

We showed that the excitations of the $\mathcal{N} = 1$ theory around its AdS_4 vacuum generically describe a massless spin-2 multiplet and a massive spin-2 multiplet. In the critical gravity limit, the massive spin-2 field becomes massless, leading to the emergence of spin-2 and spin-$\frac{3}{2}$ modes with logarithmic coordinate dependence. The formerly massive multiplet becomes a non-standard one in this limit, which lies outside the usual classification of unitary $\mathcal{N} = 1$ representations described in [15].

The extensions beyond the critical limit, which are the supersymmetric generalisation of the theories considered in [7], arise when the coefficient of the Weyl-squared term is chosen to lie in the range where the massive fields carry non-unitary representations of $SO(2,3)$. For the bosons (spin-2 and spin-1), this means that they have mass-squared values, defined as $(\Box + 2a^2 - M_2^2)h_{\mu\nu} = 0$ and $(\Box + 3a^2 - M_1^2)A_\mu = 0$, that are negative. They are, however, not sufficiently negative to be tachyonic, meaning that their lowest energies E_0, given by

$$E_0^{(2)} = \frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{M_2^2}{a^2}}\,, \qquad E_0^{(1)} = \frac{3}{2} \pm \sqrt{\frac{1}{4} + \frac{M_1^2}{a^2}}\,, \tag{6.1}$$

are still real. Because the lowest energies of the massive fields all violate the unitarity bounds $E_0^{(s)} \geq s+1$, they have a slower fall-off at large distance than the massless fields, and thus they can be eliminated, while retaining the massless fields, by imposing appropriate boundary conditions. The same is true also for the logarithmic modes in the case of critical gravity. Eliminating the massive or logarithmic modes is desirable from a physical point of view, since they can have negative norms, and hence are ghost-like.

Although for physical reasons one would probably wish to truncate out the ghost-like massive modes, there may be circumstances where it could be of interest to retain them. It has, for example, been suggested that the retention of the logarithmic modes in critical gravity could give rise to an interesting relation to a dual three-dimensional logarithmic CFT on the AdS_4 boundary [17]. A preliminary investigation of this idea has been initiated in [20], where a toy model with a scalar field satisfying a fourth-order field equation has been considered.

The extensions beyond critical supergravity, i.e. the theories where the parameter b characterising the Weyl-squared action satisfies $b \geq 1/a^2$, may provide a family of toy models for renormalisable supergravities without ghosts, provided that one truncates out the negative mass-squared spin-2 modes. It would be interesting to investigate further the properties of these theories at the quantum level.

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A Notation and Conventions

The σ-matrices satisfy

$$\sigma^a \bar{\sigma}^b + \sigma^b \bar{\sigma}^a = -2\eta^{ab} \,. \tag{A.1}$$

Other useful relations are

$$\varepsilon^{abcd}\sigma_{cd} = -2i\sigma^{ab} \,, \quad \varepsilon^{abcd}\bar{\sigma}_{cd} = 2i\bar{\sigma}^{ab} \,, \quad \sigma^{abc} = i\varepsilon^{abcd}\sigma_d \,, \quad \bar{\sigma}^{abc} = -i\varepsilon^{abcd}\bar{\sigma}_d \,, \tag{A.2}$$

and

$$\mathrm{tr}(\sigma^{ab}\sigma_{cd}) = -4\delta^a_{[c}\delta^b_{d]} - 2i\varepsilon^{ab}{}_{cd} \,. \tag{A.3}$$

Dirac gamma-matrices satisfying

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab} \tag{A.4}$$

are constructed as

$$\gamma^a = \begin{pmatrix} 0 & i\sigma^a \\ i\bar{\sigma}^a & 0 \end{pmatrix} \,. \tag{A.5}$$

We also have

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \,. \tag{A.6}$$

B $D = 4$, $\mathcal{N} = 1$ supergravity

In this Appendix we will describe the superspace constraints of $D = 4$, $\mathcal{N} = 1$ supergravity. To facilitate the comparison to other superspace literature we will use the notation of Wess and Bagger [18], which differs slightly from that used in the rest of the paper. In particular, letters from the beginning of the alphabet denote tangent space indices, lower case Latin indices are vector indices, while lower case Greek indices are spinor indices and capital Latin indices run over both (in the rest of the paper coordinate vector indices are denoted $\mu, \nu, \ldots$). The Bianchi identities for the torsion and curvature read

$$D_{[A}T_{BC]}{}^D + T_{[AB}{}^E T_{|E|C]}{}^D + R_{[ABC]}{}^D = 0 \tag{B.1}$$

$$D_{[A}R_{BC]}{}^{DE} + T_{[AB}{}^F R_{|F|C]}{}^{DE} = 0 \,. \tag{B.2}$$

In the next section we will describe their solution up to mass dimension 3/2, with some results that we will need also at dimension 2 and 5/2. The superspace covariant derivative satisfies

$$[D_A, D_B] = -T_{AB}{}^C D_C + \frac{1}{2}R_{AB}{}^{cd}\ell_{cd} \,, \tag{B.3}$$

where

$$\ell_{cd}V_a = 2\eta_{a[c}V_{d]} \qquad \ell_{cd}\psi_\alpha = -\frac{1}{2}(\sigma_{cd})_\alpha{}^\beta\psi_\beta\,, \tag{B.4}$$

on a vector and spinor respectively. This means that the spin-connection satisfies

$$\Omega_\alpha{}^\beta = -\frac{1}{4}\Omega^{cd}(\sigma_{cd})_\alpha{}^\beta \tag{B.5}$$

and similarly for dotted spinor indices.

B.1 Supergravity constraints

The non-vanishing components of the torsion and curvature, organized according to mass-dimension, are

Dimension 0

$$T_{\alpha\dot\beta}{}^a = -i\sigma^a_{\alpha\dot\beta}\,. \tag{B.6}$$

Dimension 1

$$T_{a\dot\alpha}{}^\beta = i(\sigma_a)^\beta{}_{\dot\alpha}\mathfrak{R}\,, \qquad T_{a\alpha}{}^{\dot\beta} = i(\bar\sigma_a)^{\dot\beta}{}_\alpha\mathfrak{R}^\dagger \tag{B.7}$$

and

$$T_{a\alpha}{}^\beta = 2i\delta^\beta_\alpha G_a - i(\sigma_{ab})_\alpha{}^\beta G^b\,, \qquad T_{a\dot\alpha}{}^{\dot\beta} = 2i\delta^{\dot\beta}_{\dot\alpha}G_a + i(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha}G^b\,, \tag{B.8}$$

where $\mathfrak{R}$ is a scalar superfield and G_a is a real vector superfield whose lowest components are the auxiliary fields of the so-called old minimal formulation. Note in particular the constraint

$$T_{ab}{}^c = 0 \tag{B.9}$$

which determines the spin-connection.

The curvature components are

$$R_{\alpha\beta cd} = -2(\sigma_{cd})_{\alpha\beta}\mathfrak{R}^\dagger\,, \quad R_{\dot\alpha\dot\beta cd} = -2(\bar\sigma_{cd})_{\dot\alpha\dot\beta}\mathfrak{R}\,, \quad R_{\alpha\dot\beta cd} = -2(\sigma_{bcd})_{\alpha\dot\beta}G^b\,. \tag{B.10}$$

Dimension 3/2

At this dimension one finds that $\mathfrak{R}$ is chiral,

$$D_{\dot\alpha}\mathfrak{R} = 0\,, \qquad D_\alpha\mathfrak{R}^\dagger = 0\,, \tag{B.11}$$

as well as

$$\begin{aligned}
D_\alpha \mathfrak{R} &= -\frac{1}{6}(\sigma^{cd})_{\alpha\gamma} T_{cd}{}^{\gamma} \\
D_{\dot\alpha} \mathfrak{R}^\dagger &= \frac{1}{6}(\bar\sigma^{cd})_{\dot\alpha\dot\gamma} T_{cd}{}^{\dot\gamma} \\
D_\alpha G_a &= \frac{1}{48}(3(\sigma^{cd}\sigma^a)_{\alpha\dot\gamma} T_{cd}{}^{\dot\gamma} - (\sigma^a\bar\sigma^{cd})_{\alpha\dot\gamma} T_{cd}{}^{\dot\gamma}) \\
D_{\dot\alpha} G_a &= \frac{1}{48}(3(\bar\sigma^{cd}\bar\sigma^a)_{\dot\alpha\gamma} T_{cd}{}^{\gamma} - (\bar\sigma^a\sigma^{cd})_{\dot\alpha\gamma} T_{cd}{}^{\gamma}) \,. \qquad \text{(B.12)}
\end{aligned}$$

The curvature components of this dimension are

$$\begin{aligned}
R_{\alpha bcd} &= \frac{i}{2}(\sigma_b)_{\alpha\dot\gamma} T_{cd}{}^{\dot\gamma} - \frac{i}{2}(\sigma_d)_{\alpha\dot\gamma} T_{bc}{}^{\dot\gamma} - \frac{i}{2}(\sigma_c)_{\alpha\dot\gamma} T_{db}{}^{\dot\gamma} \\
R_{\dot\alpha bcd} &= -\frac{i}{2}(\bar\sigma_b)_{\dot\alpha\gamma} T_{cd}{}^{\gamma} + \frac{i}{2}(\bar\sigma_d)_{\dot\alpha\gamma} T_{bc}{}^{\gamma} + \frac{i}{2}(\bar\sigma_c)_{\dot\alpha\gamma} T_{db}{}^{\gamma} \,. \qquad \text{(B.13)}
\end{aligned}$$

Dimension 2

One finds

$$\begin{aligned}
D_\alpha T_{bc}{}^{\gamma} &= -2i\delta_\alpha^\gamma G_{bc} + 2i(\sigma_{d[b})_\alpha{}^\gamma D_{c]} G^d + 2(\sigma_{bc})_\alpha{}^\gamma G^2 + 4(\sigma_{d[b})_\alpha{}^\gamma G_{c]} G^d + 2(\sigma_{bc})_\alpha{}^\gamma \mathfrak{R}\mathfrak{R}^\dagger \\
&\quad + \frac{1}{4} R_{bc}{}^{de}(\sigma_{de})_\alpha{}^\gamma \qquad \text{(B.14)} \\
D_{\dot\alpha} T_{bc}{}^{\gamma} &= 2i(\sigma_{[b})^\gamma{}_{\dot\alpha} D_{c]}\mathfrak{R} + 16(\sigma_{[b})^\gamma{}_{\dot\alpha} G_{c]}\mathfrak{R} + 4(\sigma_{bc}\sigma^e)^\gamma{}_{\dot\alpha} G_e \mathfrak{R} \,, \qquad \text{(B.15)}
\end{aligned}$$

where $G_{ab} = 2D_{[a}G_{b]}$ is the field strength of G_a and similar expressions for $T_{bc}{}^{\dot\gamma}$.

In terms of the superfield

$$W^{\alpha\beta\gamma} = (\sigma^{bc})^{(\alpha\beta} T_{bc}{}^{\gamma)} \qquad \text{(B.16)}$$

this implies

$$\begin{aligned}
D_\alpha W^{\beta\gamma\delta} &= -3i\delta_\alpha^{(\delta}(\sigma^{bc})^{\beta\gamma)} G_{bc} + \frac{1}{4} R_{bc}{}^{de}(\sigma_{de})_\alpha{}^{(\delta}(\sigma^{bc})^{\beta\gamma)} \qquad \text{(B.17)} \\
D_{\dot\alpha} W^{\alpha\beta\gamma} &= 0 \,, \qquad \text{(B.18)}
\end{aligned}$$

in particular $W^{\alpha\beta\gamma}$ is a chiral superfield. Similar relations hold for $\bar W^{\dot\alpha\dot\beta\dot\gamma}$.

Using the equation for $D_\alpha\mathfrak{R}$ as well as that for $D_\alpha T_{bc}{}^\gamma$ one finds

$$D^\alpha D_\alpha \mathfrak{R} = -\frac{1}{6}(R_{ab}{}^{ab} - 12iD_a G^a + 24G^2 + 48\mathfrak{R}\mathfrak{R}^\dagger) \,. \qquad \text{(B.19)}$$

Similarly one can compute two spinor derivatives on $\mathfrak{R}^\dagger$ and G_a but we will not do this here as we will not need them.

Dimension 5/2

From the Bianchi identities one finds that

$$\begin{aligned} D_\alpha R_{bc}{}^{de} &= -2D_{[b}R_{c]\alpha}{}^{de} - 2iG_{[b}R_{c]\alpha}{}^{de} + 2i(\sigma_{[b}\bar{\sigma}^f)_\alpha{}^\beta R_{c]\beta}{}^{de}G_f + 2i(\sigma_{[b})_\alpha{}^{\dot\beta} R_{c]\dot\beta}{}^{de}\mathfrak{R}^\dagger \\ &\quad + 2(\sigma^{de})_{\alpha\beta}T_{bc}{}^\beta\mathfrak{R}^\dagger - 2(\sigma_{def})_{\alpha\dot\beta}T_{bc}{}^{\dot\beta}G^f \,. \end{aligned} \tag{B.20}$$

Using the expression for the dimension 3/2 curvatures this implies that

$$(\sigma^{bc})^{(\beta\gamma}(\sigma_{de})^{\delta)\alpha}D_\alpha R_{bc}{}^{de} = 40W^{\beta\gamma\delta}\mathfrak{R}^\dagger + 10(\sigma^{bc})^{(\beta\gamma}(\sigma^d)^{\delta)}{}_{\dot\beta}(iD_bT_{cd}{}^{\dot\beta} - T_{cd}{}^{\dot\beta}G_b + 2T_{bc}{}^{\dot\beta}G_d) \tag{B.21}$$

Using this expression one computes

$$D^\alpha D_\alpha W^{\beta\gamma\delta} = -10W^{\beta\gamma\delta}\mathfrak{R}^\dagger + 2(\sigma^{bc})^{(\beta\gamma}(\sigma^d)^{\delta)}{}_{\dot\beta}(iD_dT_{bc}{}^{\dot\beta} - 4T_{cd}{}^{\dot\beta}G_b - 7T_{bc}{}^{\dot\beta}G_d) \,. \tag{B.22}$$

One could also derive other relations from the Bianchi identities but we will not need more than these here.

In order to compute the Weyl-squared invariant we need two spinor derivatives of $W^2 = W^{\beta\gamma\delta}W_{\beta\gamma\delta}$. With a bit of work one finds

$$\begin{aligned} D^\alpha D_\alpha(W^2) &= -\frac{2}{3}(R_{ab}{}^{cd}R^{ab}{}_{cd} + 2R_{ab}{}^{ab}R_{cd}{}^{cd} + 5R_{ab}{}^{cd}R_{cd}{}^{ab} - 12R_{ab}{}^{ac}R_{cd}{}^{bd}) + 96G^{ab}G_{ab} \\ &\quad - 20W^2\mathfrak{R}^\dagger + 4(\sigma^{ab})^{(\alpha\beta}(\sigma^c)^{\gamma)}{}_{\dot\beta}(iD_cT_{ab}{}^{\dot\beta} - 4T_{bc}{}^{\dot\beta}G_a - 7T_{ab}{}^{\dot\beta}G_c)W_{\alpha\beta\gamma} \\ &\quad - \frac{i}{3}\varepsilon^{abcd}(R^{ef}{}_{ab}R_{efcd} + R_{ab}{}^{ef}R_{cdef} + 4R_{ab}{}^{ef}R_{efcd} - 8R_{ea}{}^{ef}R_{fbcd} - 144G_{ab}G_{cd}) \,. \end{aligned} \tag{B.23}$$

Note that the terms in the last line are imaginary an will therefore not contribute to the action.

B.2 Components

Here we collect some component results which we need. The lowest component of the superfields $\mathfrak{R}$ and G_a are the auxiliary fields of the old minimal formulation of $D = 4$ supergravity,

$$\mathfrak{R}| = \frac{1}{6}\mathcal{M} \,, \qquad G_a| = \frac{1}{6}A_a \,. \tag{B.24}$$

(The vector field A_a is customarily called b_a in the superspace literature.) The gravitino is defined as the lowest component of the spinorial supervielbein

$$E_m{}^\alpha| = \psi_m{}^\alpha \,. \tag{B.25}$$

Using this fact, the gravitino field-strength $\psi_{ab} = 2D_{[a}\psi_{b]}$ can be written

$$\psi_{ab}{}^{\gamma} \equiv e_b{}^n e_a{}^m T_{mn}{}^{\gamma}| = T_{ab}{}^{\gamma}| - i\psi^{\gamma}_{[a}\, A_{b]} + \frac{i}{3}(\sigma_{[a}\sigma^c\psi_{b]})^{\gamma}\, A_c - \frac{i}{3}(\sigma_{[a}\bar{\psi}_{b]})^{\gamma}\, \mathcal{M} \quad \text{(B.26)}$$

$$\bar{\psi}_{ab}{}^{\dot{\gamma}} \equiv e_b{}^n e_a{}^m T_{mn}{}^{\dot{\gamma}}| = T_{ab}{}^{\dot{\gamma}}| + i\bar{\psi}^{\dot{\gamma}}_{[a}\, A_{b]} - \frac{i}{3}(\bar{\sigma}_{[a}\sigma^c\bar{\psi}_{b]})^{\dot{\gamma}}\, A_c - \frac{i}{3}(\bar{\sigma}_{[a}\psi_{b]})^{\dot{\gamma}}\, \bar{\mathcal{M}}\,, \quad \text{(B.27)}$$

which defines the 'covariantized' gravitino field strength

$$\begin{aligned}
\psi_{ab}{}^{(cov)\gamma} &\equiv T_{ab}{}^{\gamma}| = \psi_{ab}{}^{\gamma} + i\psi^{\gamma}_{[a}\, A_{b]} - \frac{i}{3}(\sigma_{[a}\bar{\sigma}^c\psi_{b]})^{\gamma}\, A_c + \frac{i}{3}(\sigma_{[a}\bar{\psi}_{b]})^{\gamma}\, \mathcal{M} \\
\bar{\psi}_{ab}{}^{(cov)\dot{\gamma}} &\equiv T_{ab}{}^{\dot{\gamma}}| = \bar{\psi}_{ab}{}^{\dot{\gamma}} - i\bar{\psi}^{\dot{\gamma}}_{[a}\, A_{b]} + \frac{i}{3}(\bar{\sigma}_{[a}\sigma^c\bar{\psi}_{b]})^{\dot{\gamma}}\, A_c + \frac{i}{3}(\bar{\sigma}_{[a}\psi_{b]})^{\dot{\gamma}}\, \bar{\mathcal{M}}\,. \qquad \text{(B.28)}
\end{aligned}$$

For the Riemann tensor, $\mathcal{R}_{ab}{}^{cd}$ which is computed in the standard way from the spin–connection ω^{cd}, we find[4]

$$\begin{aligned}
\mathcal{R}_{ab}{}^{cd} \equiv e_b{}^n e_a{}^m R_{mn}{}^{cd}| = R_{ab}{}^{cd}| &+ i\psi_{[a}\sigma_{b]}\bar{\psi}^{(cov)}_{cd} - i\psi_{[a}\sigma^d\bar{\psi}^{(cov)}_{b]c} + i\psi_{[a}\sigma^c\bar{\psi}^{(cov)}_{b]d} \\
&+ i\bar{\psi}_{[a}\bar{\sigma}_{b]}\psi^{(cov)}_{cd} - i\bar{\psi}_{[a}\bar{\sigma}^d\psi^{(cov)}_{b]c} + i\bar{\psi}_{[a}\bar{\sigma}^c\psi^{(cov)}_{b]d} - \frac{1}{3}\psi_a\sigma_{cd}\psi_b\,\bar{\mathcal{M}} - \frac{1}{3}\bar{\psi}_a\bar{\sigma}_{cd}\bar{\psi}_b\,\mathcal{M} \\
&- \frac{2}{3}\psi_{[a}\sigma^{cde}\bar{\psi}_{b]}\, A_e\,, \qquad \text{(B.29)}
\end{aligned}$$

which gives

$$R_{ab}{}^{ab}| = \mathcal{R} - 2i\psi^a\sigma^b\bar{\psi}^{(cov)}_{ab} - 2i\bar{\psi}^a\bar{\sigma}^b\psi^{(cov)}_{ab} + \frac{1}{3}\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b\,\mathcal{M} + \frac{1}{3}\psi_a\sigma^{ab}\psi_b\,\bar{\mathcal{M}} + \frac{2}{3}\psi_a\sigma^{abc}\bar{\psi}_b\, A_c\,. \quad \text{(B.30)}$$

B.3 Supersymmetry transformations

For completeness we give also the supersymmetry transformations of the component fields. They are given by

$$\begin{aligned}
\delta e_m{}^a &= -\epsilon^{\beta}T_{\beta m}{}^a| + \bar{\epsilon}^{\dot{\beta}}T_{\dot{\beta}m}{}^a| \\
\delta\psi_m{}^{\alpha} &= -D_m\epsilon^{\alpha} - \epsilon^{\beta}T_{\beta m}{}^{\alpha}| + \bar{\epsilon}^{\dot{\beta}}T_{\dot{\beta}m}{}^{\alpha}| \\
\delta\bar{\psi}_m{}^{\dot{\alpha}} &= -D_m\bar{\epsilon}^{\dot{\alpha}} - \epsilon^{\beta}T_{\beta m}{}^{\dot{\alpha}}| + \bar{\epsilon}^{\dot{\beta}}T_{\dot{\beta}m}{}^{\dot{\alpha}}| \\
\delta\mathcal{M} &= -6\epsilon^{\alpha}D_{\alpha}\mathfrak{R}| \\
\delta\bar{\mathcal{M}} &= 6\bar{\epsilon}^{\dot{\alpha}}D_{\dot{\alpha}}\mathfrak{R}^{\dagger}| \\
\delta A_a &= -6(\epsilon^{\alpha}D_{\alpha} - \bar{\epsilon}^{\dot{\alpha}}D_{\dot{\alpha}})G_a|\,. \qquad \text{(B.31)}
\end{aligned}$$

[4]The form of ω^{cd} can be found from the constraint $T_{ab}{}^c = 0$ but we will not need its explicit form. Note that it will contain ψ^2-terms but these will not contribute to the equations of motion in our case.

Using the superspace constraints in section B.1 and the component results in section B.2 we find

$$\begin{aligned}
\delta e_m{}^a &= -i\bar{\psi}_m\bar{\sigma}^a\epsilon - i\psi_m\sigma^a\bar{\epsilon} \\
\delta\psi_m{}^\alpha &= -D_m\epsilon^\alpha + \frac{i}{3}\epsilon^\alpha A_m + \frac{i}{6}(\sigma_{mb}\,\epsilon)_\alpha A^b - \frac{i}{6}(\sigma_m\,\bar{\epsilon})^\alpha \mathcal{M} \\
\delta\bar{\psi}_m{}^{\dot\alpha} &= -D_m\bar{\epsilon}^{\dot\alpha} - \frac{i}{3}\bar{\epsilon}^{\dot\alpha} A_m - \frac{i}{6}(\bar{\sigma}_{mb}\,\bar{\epsilon})^{\dot\alpha} A^b - \frac{i}{6}(\bar{\sigma}_m\,\epsilon)^{\dot\alpha} \bar{\mathcal{M}} \\
\delta\mathcal{M} &= -\epsilon\,\sigma^{cd}\psi_{cd}^{(cov)} \\
\delta\bar{\mathcal{M}} &= -\bar{\epsilon}\,\bar{\sigma}^{cd}\bar{\psi}_{cd}^{(cov)} \\
\delta A_a &= \frac{1}{8}(3\bar{\epsilon}\,\bar{\sigma}^{cd}\bar{\sigma}^a\psi_{cd}^{(cov)} - \bar{\epsilon}\,\bar{\sigma}^a\sigma^{cd}\psi_{cd}^{(cov)} - 3\epsilon\,\sigma^{cd}\sigma^a\bar{\psi}_{cd}^{(cov)} + \epsilon\,\sigma^a\bar{\sigma}^{cd}\bar{\psi}_{cd}^{(cov)})\,. \qquad \text{(B.32)}
\end{aligned}$$

B.4 Quadratic gravitino terms from Weyl^2 invariant in AdS$_4$

Supersymmetric Lagrangians can be constructed as

$$e^{-1}\mathcal{L} = \left(\tfrac{1}{2}D^\alpha D_\alpha + i(\bar{\psi}_a\sigma^a)^\alpha D_\alpha + \bar{\mathcal{M}} + \bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b\right) r| + \text{h.c.}\,, \qquad \text{(B.33)}$$

where r is a chiral superfield. Taking $r = -\frac{1}{4}W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$ gives the Weyl-squared invariant. In this section we shall compute the terms quadratic in the gravitino in the AdS$_4$ background given by $\mathcal{M} = 3a$ and $\mathcal{R}_{ab}{}^{cd}$ given in (3.5).

Using these expressions it is not hard to see that the curvature terms in (B.23) do not give any contribution to the quadratic gravitino terms in the action in this background. Similarly the term $G_{ab}G^{ab}|$ can not give any quadratic gravitino contribution. Using (B.23) and (B.17) we find that the quadratic gravitino terms in the AdS$_4$ background are

$$e^{-1}\mathcal{L}_\psi = \frac{a}{2}W^2| - \frac{i}{2}(\sigma^{ab})^{(\alpha\beta}(\sigma^c)^{\gamma)}{}_{\dot\beta}D_cT_{ab}{}^{\dot\beta}|\,W_{\alpha\beta\gamma}| + \text{h.c.}\,. \qquad \text{(B.34)}$$

Using the fact that

$$D_aT_{bc}{}^{\dot\gamma}| = D_a\bar{\psi}_{bc}{}^{\dot\gamma} + \frac{i}{3}(\bar{\sigma}_{[b}D_{|a|}\psi_{c]})^{\dot\gamma}\bar{\mathcal{M}} + \dots\,, \qquad \text{(B.35)}$$

together with

$$W^{\alpha\beta\gamma}| = (\sigma^{de})^{(\alpha\beta}\psi_{de}^{\gamma)} + \dots\,, \qquad \text{(B.36)}$$

where ... denotes terms that vanish in the AdS$_4$ background when expanded to linear order the two W^2-terms cancel and we find

$$e^{-1}\mathcal{L}_\psi = \frac{i}{2}(\sigma^{ab})^{\alpha\beta}(D_c\bar{\psi}_{ab}\bar{\sigma}^c)^\gamma\,(\sigma^{de})_{(\alpha\beta}\psi_{de\gamma)} + \text{h.c.}\,. \qquad \text{(B.37)}$$

Simplifying and dropping total derivatives we finally arrive at the Lagrangian

$$e^{-1}\mathcal{L}_\psi = \frac{4}{3}\left(iD_d\bar{\psi}^{ab}\bar{\sigma}^d\psi_{ab} - i\bar{\psi}^{ab}\bar{\sigma}^dD_d\psi_{ab} + iD_d\bar{\psi}_a{}^c\bar{\sigma}^{abd}\psi_{bc} + i\bar{\psi}_b{}^c\bar{\sigma}^{abd}D_d\psi_{ac}\right). \qquad \text{(B.38)}$$

C Relations between regular modes in AdS_4

AdS_4 admits four Killing spinors ϵ_+ and four Killing spinors ϵ_-, satisfying

$$\nabla_\mu \epsilon_+ = \tfrac{1}{2} a\, \gamma_\mu\, \epsilon_+ \,, \qquad \nabla_\mu \epsilon_- = -\tfrac{1}{2} a\, \gamma_\mu\, \epsilon_- \,. \tag{C.1}$$

These can be used in order to map between modes of different spins. We begin by defining the second-order operators, and eigenvalues, for each spin:

$$\begin{aligned}
\text{Spin } 0: &\quad \Delta_0\, \phi \equiv -\Box \phi = \lambda_0\, \phi \,, \\
\text{Spin } \tfrac{1}{2}: &\quad \not{D}\psi \equiv \gamma^\mu \nabla_\mu \psi = \lambda_{1/2}\, \psi \,, \\
\text{Spin } 1: &\quad \Delta_1 V_\mu \equiv -\Box V_\mu + R_{\mu\nu}\, V^\nu = \lambda_1\, V_\mu \,, \\
\text{Spin } \tfrac{3}{2}: &\quad \not{D}\psi_\mu \equiv \gamma^\nu \nabla_\nu \psi_\mu = \lambda_{3/2}\, \psi_\mu \,, \\
\text{Spin } 2: &\quad \Delta_L h_{\mu\nu} \equiv -\Box h_{\mu\nu} - 2R_{\mu\rho\nu\sigma}\, h^{\rho\sigma} + R_{\mu\rho}\, h^\rho{}_\nu + R_{\nu\rho}\, h_\mu{}^\rho = \lambda_L\, h_{\mu\nu} \,.
\end{aligned} \tag{C.2}$$

Note that we assume transverse and traceless conditions for the modes of spins 1, $\frac{3}{2}$ and 2, and so

$$\nabla^\mu V_\mu = 0, \qquad \nabla^\mu \psi_\mu = 0\,, \qquad \gamma^\mu \psi_\mu = 0\,, \qquad \nabla^\mu h_{\mu\nu} = 0\,, \qquad h^\mu{}_\mu = 0\,. \tag{C.3}$$

In the AdS_4 background, and setting $a = 1$ for convenience, we have

$$R_{\mu\nu\rho\sigma} = -g_{\mu\rho}\, g_{\nu\sigma} + g_{\mu\sigma}\, g_{\nu\rho}\,, \qquad R_{\mu\nu} = -3 g_{\mu\nu}\,, \tag{C.4}$$

and so the spin 1 and spin 2 operators become

$$\Delta_1 = -\Box - 3\,, \qquad \Delta_L = -\Box - 8\,. \tag{C.5}$$

By default, we shall consider the case where the Killing spinors ϵ_+ are used for relating the various modes, and for brevity we shall just denote these by ϵ. We find that the relations between the modes are implemented as follows:

$$\begin{aligned}
\psi &= \phi\, \epsilon + \frac{1}{\lambda_{1/2} + 1}\, \nabla_\mu \phi\, \gamma^\mu \epsilon\,, \\
V_\mu &= \bar{\epsilon} \gamma_\mu \psi - \frac{1}{\lambda_{1/2} + \frac{3}{2}}\, \bar{\epsilon}\, \nabla_\mu\, \psi\,, \\
\psi_\mu &= V_\mu\, \epsilon + \tfrac{1}{4} c (1 - 2\lambda_{3/2} - 2\lambda_{3/2}^2)\, \gamma_{\mu\nu}\, V^\nu\, \epsilon + c(1 + \lambda_{3/2})\, \nabla_\nu V_\mu\, \gamma^\nu\, \epsilon - c\, \nabla_\mu V_\nu\, \gamma^\nu\, \epsilon \\
&\quad - \frac{1}{2\lambda_{3/2}}\, \gamma_{\mu\nu\rho}\, \nabla^\nu V^\rho\, \epsilon + \tfrac{1}{2} c\, \nabla_{(\mu} \nabla_{\nu)} V_\rho\, \gamma^{\nu\rho}\, \epsilon\,, \\
h_{\mu\nu} &= \bar{\epsilon} \gamma_{(\mu}\, \psi_{\nu)} - \frac{2}{2\lambda_{3/2} + 5}\, \bar{\epsilon} \nabla_{(\mu}\, \psi_{\nu)}\,,
\end{aligned} \tag{C.6}$$

where $c^{-1} = \lambda_{3/2}\,(2+\lambda_{3/2})$. (The relative coefficients between the terms in each expression are uniquely determined by requiring that the irreducibility conditions in (C.3) hold, and that the constructions should map eigenfunctions into eigenfunctions.) These formulae furnish a systematic way of constructing the spin 1/2, 1, 3/2 and 2 solutions, starting from the spin 0 solution, and the knowledge of the Killing spinor. The spin 0 solution has been studied in great detail in [19]. Alternatively, starting from the spin 2 solution, we can obtain from it the spin 3/2, 1, 1/2 and 0 solutions by employing the formulae

$$\begin{aligned}
\psi_\mu &= h_{\mu\nu}\,\gamma^\nu\,\epsilon - \frac{1}{\lambda_{3/2}}\,\nabla_\rho h_{\mu\nu}\,\gamma^{\nu\rho}\,\epsilon\,,\\
V_\mu &= \bar\epsilon\psi_\mu\,,\\
\psi &= V_\mu\,\gamma^\mu\,\epsilon + \frac{1}{\lambda_{1/2}}\,\nabla_\mu V_\nu\,\gamma^{\mu\nu}\,\epsilon\,,\\
\phi &= \bar\epsilon\psi\ .
\end{aligned} \tag{C.7}$$

The corresponding relations between the eigenvalues are

$$\begin{aligned}
\lambda_0 &= -\lambda_{1/2}^2 + \lambda_{1/2} + 2\,,\\
\lambda_1 &= -\lambda_{1/2}^2 - \lambda_{1/2}\,,\\
\lambda_1 &= -\lambda_{3/2}^2 + \lambda_{3/2}\,,\\
\lambda_L &= -\lambda_{3/2}^2 - \lambda_{3/2} - 4\,.
\end{aligned} \tag{C.8}$$

(If ϵ_- is used instead of ϵ_+, the effect is to reverse the signs of the fermion eigenvalues in these expressions.)

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