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PUTTING STRING/FIVEBRANE DUALITY TO THE TEST †

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ABSTRACT

According to string/fivebrane duality, the Green-Schwarz factorization of the $D = 10$ spacetime anomaly polynomial I_{12} into $X_4\, X_8$ means that just as X_4 is the anomaly polynomial of the $d = 2$ string worldsheet so X_8 should be the anomaly polynomial of the $d = 6$ fivebrane worldvolume. To test this idea we perform a fivebrane calculation of X_8 and find perfect agreement with the string one–loop result.

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The dual formulations of $D = 10$ supergravity, one with a 7–form field strength[1] and the other with a 3–form field strength[2] have long been something of an enigma from the point of view of superstrings. As field theories, each seems equally as good. In particular, provided we couple them to $E_8 \times E_8$ or $SO(32)$ super Yang-Mills, then both are anomaly–free[3, 4]. Since the 3–form version corresponds to the field theory limit of the heterotic string, it was natural to conjecture[5] that the 7–form version corresponds to the field theory limit of an extended object dual to the string: the "heterotic fivebrane". Just as the 2–form potential B_{MN} ($M = 0, 1, \ldots, 9$) couples to the $d = 2$ string worldsheet via the term

$$S_2 = \frac{1}{2\pi\alpha'} \int d^2\xi \, \frac{1}{2} \epsilon^{ij} \partial_i x^M \partial_j x^N B_{MN} = \frac{1}{2\pi\alpha'} \int B_2 \tag{1}$$

where ξ^i ($i = 1, 2$) are the worldsheet coordinates and $(2\pi\alpha')^{-1}$ is the string tension, so the 6–form potential B_{MNPQRS} ($M = 0, 1, \ldots, 9$) couples to the $d = 6$ fivebrane worldvolume via the term

$$\begin{aligned} S_6 &= \frac{1}{(2\pi)^3\beta'} \int d^6\xi \, \frac{1}{6!} \epsilon^{ijklmn} \partial_i x^M \partial_j x^N \partial_k x^P \partial_l x^Q \partial_m x^R \partial_n x^S B_{MNPQRS} \\ &= \frac{1}{(2\pi)^3\beta'} \int B_6 \end{aligned} \tag{2}$$

where ξ^i ($i = 1, \ldots, 6$) are the worldvolume coordinates and $[(2\pi)^3\beta']^{-1}$ is the fivebrane tension. Writing $H_3 = dB_2 + O(\alpha')$ and $H_7 = dB_6 + O(\beta')$, the relation

$$H_7 = e^{-\phi} * H_3 \tag{3}$$

where ϕ is the $D = 10$ dilaton and $*$ denotes the Hodge dual using the canonical metric $g_{MN}(\text{can})$, ensures that the roles of field equations and Bianchi identities of the 3–form version of supergravity are interchanged in the 7–form version. The main subject of this paper will be the $O(\alpha')$ corrections to H_3 and the $O(\beta')$ corrections to H_7, discussed below.

An existence proof for this heterotic fivebrane was provided by Strominger [6], who showed that the heterotic fivebrane emerges as a soliton solution of the 3–form version. He went on to suggest that the strong coupling regime of the string should correspond to the weak coupling regime of the fivebrane; an idea made quantitatively more precise in Ref. 7, where it was shown that the σ–model metrics $g_{MN}(\text{string})$ and $g_{MN}(\text{fivebrane})$ are related to the canonical metric by $e^{-\phi/2}\, g_{MN}(\text{string}) = g_{MN}(\text{can}) = e^{\phi/6}\, g_{MN}(\text{fivebrane})$ and hence that g(fivebrane) = g(string)$^{-1/3}$ where g(string) and g(fivebrane) are the string and fivebrane loop expansion parameters. The same paper also established the Dirac quantization rule

$$2\kappa^2 = n(2\pi)^5\alpha'\beta' \qquad , n = \text{integer} \tag{4}$$

where κ^2 is the $D = 10$ gravitational constant. Further evidence was provided by the complementary discovery that a heterotic string emerges as a soliton solution of the 7–form version [8]. Both the string and fivebrane soliton solutions break half the supersymmetries, both saturate a Bogomol'nyi bound between the mass and the topological charge, and both go over into the corresponding elementary string [9] and elementary fivebrane [10] solutions at large distances. These elementary solutions are the extreme mass = charge limit of the black string and black fivebrane which display event horizons and a singularity at the origin [11]. However, they are mutually non–singular in the sense that the string is a non–singular solution of the 7–form version and the fivebrane a non–singular solution of the 3–form version [12]. Recent work on conformal field theories [13] and other exact solutions [14] are also all consistent with this string/fivebrane duality conjecture.

Crucial to the solution of Ref. 8 was the observation that duality mixes up string and fivebrane loops: what is a one loop effect for the string might be a tree level effect for the fivebrane, and vice versa. At higher loop orders, this leads to an infinite number of non–renormalization theorems (including the vanishing of the cosmological term) all of which are consistent with known string calculations to higher orders both in α' (worldsheet loops) and g(string) (spacetime loops) [15]. It is this loop mixing which allows us to test string/fivebrane duality, in spite of our ignorance of how to quantize the fivebrane. If duality is correct, we should be able to reproduce string loop effects from tree–level fivebranes !

To see this, let us first consider the well–known $SO(32)$ Yang–Mills Chern–Simons corrections to H_3. ($E_8 \times E_8$ requires a separate treatment and will be discussed elsewhere). In bosonic formulation, these are obtained [16] by augmenting the action S_2 of (1) by the WZW terms

$$S_2{}' = -n_2 \frac{6\pi}{N_2} \int_{\partial M_3} \mathrm{tr} AK + n_2 \frac{2\pi}{N_2} \int_{M_3} \mathrm{tr} K^3 \qquad (5)$$

where $A = A_M\, dx^M$, $K = g^{-1}dg$ and where the gauge fields $A_M = A_M{}^a\, T^a$ and the group elements g are matrices in the fundamental representation of $SO(32)$. Here $n_2 =$ integer is the level of the Kac–Moody algebra and N_2 is a normalization constant given by the general formula [17]

$$N_{2n-2} = \Big(\frac{2\pi}{i}\Big)^n \frac{(2n-1)!}{(n-1)!} \tag{6}$$

However, in $8k+2$ dimensions for which $n = 4k+2$ we may define Majorana–Weyl fermions and the corresponding WZW terms are to be divided by 2. This is the case for the heterotic string, so $N_2 = -48\pi^2$. This agrees with Witten [18]. Let us define $F = dA + A^2$ and

$$\begin{aligned} I_{2n} &= \Big(\frac{i}{2\pi}\Big)^n \frac{1}{n!}\, \mathrm{tr} F^n \\ d\omega_{2n-1} &= I_{2n} \\ \delta\omega_{2n-1} &= d\omega^1_{2n-2} \end{aligned} \tag{7}$$

then the sum $S_2 + S_2{}'$ is gauge invariant under $\delta A = d\lambda + [A, \lambda]$ and $\delta K = d\lambda + [K, \lambda]$ provided

$$\delta B_2 = \frac{n_2}{2}\alpha'(2\pi)^2\omega^1_2 \tag{8}$$

and hence the gauge invariant field strength is given by [19]

$$\begin{aligned} H_3 &= dB_2 - \frac{n_2}{2}\alpha'(2\pi)^2\omega_3 \\ dH_3 &= -\frac{n_2}{2}\alpha'(2\pi)^2 I_4 \end{aligned} \tag{9}$$

This modification to the Bianchi identity is thus seen to be a classical string effect (i.e. tree level in the $D = 10$ string loop expansion).

Next we turn to the Green–Schwarz anomaly cancellation mechanism [3] which is a genuine quantum (string one loop) effect. The Majorana–Weyl gauginos of the $D = 10$ theory belong to the dimension 496 adjoint representation. The non–abelian anomaly polynomial is given by the abelian anomaly in $d = 12$, which is given by $\frac{1}{2} I_{12}$ of (7) with the fundamental tr replaced by the adjoint Tr. The factor of $\frac{1}{2}$ arises because the fermions are Majorana. As emphasized by Green and Schwarz [3], the miracle of $SO(32)$ is that since $\mathrm{Tr}F^6 = \mathrm{Tr}F^2\,\mathrm{Tr}F^4/48 - (\mathrm{Tr}F^2)^3/14,400$, I_{12} factorizes:

$$\frac{1}{2} I_{12} = X_4\, X_8 \tag{10}$$

In fact, since $\mathrm{Tr}F^4 = 24\,\mathrm{tr}F^4 + 3(\mathrm{tr}F^2)^2$ and $\mathrm{Tr}F^2 = 30\,\mathrm{tr}F^2$, we have the remarkable coincidence

$$X_4 = \frac{1}{2}\, I_4 \qquad\qquad X_8 = I_8 \tag{11}$$

The consistent anomaly

$$G = \frac{1}{2} \cdot 2\pi \int \Big(\frac{1}{3}\, \omega_2^1 I_8 + \frac{2}{3}\, \omega_6^1 I_4\Big) \tag{12}$$

is then cancelled by adding to the effective action

$$\Delta\Gamma_2 = -2\pi \int \Big(\frac{1}{n_2 \alpha' (2\pi)^2}\, B_2\, I_8 + \frac{1}{3}\, \omega_3\, \omega_7\Big) \tag{13}$$

and recalling the transformation rule for B_2 given in (8). Now for B_2 normalized as in (1), its kinetic term is

$$\Gamma_2 = -\frac{1}{2\kappa^2} \int \frac{1}{2} e^{-\phi}\, H_3 \wedge *H_3 \tag{14}$$

and hence the addition of (13) modifies the field equation to

$$d(e^{-\phi} * H_3) = \frac{2\kappa^2}{n_2\alpha'(2\pi)} I_8 \tag{15}$$

So far, all our considerations started with the string worldsheet. The acid test for string/fivebrane duality is to reproduce (15) starting from the fivebrane worldvolume. We begin by augmenting the action S_6 of (2) by the WZW term [20]

$$S_6{}' = \frac{70\pi n_6}{N_6}\int_{\partial M_7} C_6 - \frac{2\pi n_6}{N_6}\int_{M_7} \mathrm{tr}\, K^7 \tag{16}$$

The explicit form for C_6 is given in Ref. 20. Here we need only note that it transforms as

$$\delta C_6 = 24(2\pi)^4[\omega_6^1(A,\lambda) - \omega_6^1(K,\lambda)] \tag{17}$$

Here $n_6 =$ integer is the level of the Mickelsson–Faddeev algebra [21] and $N_6 = (2\pi)^4 7!/3!$ from (6). The sum $S_6 + S_6{}'$ is gauge invariant provided

$$\delta B_6 = -n_6\beta'(2\pi)^4\omega_6^1 \tag{18}$$

and hence the gauge invariant field strength is given by [19]

$$\begin{aligned} H_7 &= dB_6 + n_6\beta'(2\pi)^4\omega_7 && (19)\\ dH_7 &= n_6\beta'(2\pi)^4 I_8 && (20)\end{aligned}$$

Using (3), this is identical to (15) provided

$$2\kappa^2 = n_2 n_6 (2\pi)^5\alpha'\beta' \tag{21}$$

But this is just the Dirac quantization rule (4) with $n = n_2 n_6$!

Having established that the classical fivebrane correctly reproduces the quantum string result in the pure Yang–Mills sector, we now turn to the gravitational and mixed anomalies. Now one must include the $D = 10$ gravitino contribution to I_{12}, but again it factorizes as in (10) where now [22]

$$\begin{aligned} X_4 &= \frac{1}{2} \cdot \frac{1}{(2\pi)^2} \Big[-\frac{1}{2} \,\mathrm{tr} F^2 + \frac{1}{2} \,\mathrm{tr} R^2 \Big] \\ X_8 &= \Big(\frac{1}{2\pi}\Big)^4 \Big[\frac{1}{24} \,\mathrm{tr} F^4 - \frac{1}{192} \,\mathrm{tr} F^2 \,\mathrm{tr} R^2 + \frac{1}{768} (\mathrm{tr} R^2)^2 + \frac{1}{192} \,\mathrm{tr} R^4 \Big] \end{aligned} \tag{22}$$

Since we have already obtained the correct overall normalization of X_4 and X_8 it remains only to explain the relative coefficients. (We set $n_2 = n_6 = 1$ from now on). This is most easily done by changing from the bosonic WZW formalism to the fermionic one where I_4 and I_8 are the anomaly polynomials arising via the index theorem from the chiral fermions on the $d = 2$ worldsheet and $d = 6$ worldvolumes respectively. We should mention that although we have the luxury of choosing a bosonic or fermionic formulation on the $d = 2$ worldsheet, there is probably no such bose–fermi equivalence in $d = 6$ and so ultimately one will have to choose between the bosonic and fermionic formulations. This choice must await a complete covariant κ–symmetric Green–Schwarz action for the heterotic fivebrane which, to date, is still lacking [23]. Fortunately, for the present purposes of calculating anomalies, either way will do. In the Yang–Mills case, this may be seen explicitly via the Euclidean identities for the index of the Dirac operator [24]

$$(\mathrm{ind}\, i\not{D})_{2n-2} = \int I_{2n} = \Big(\frac{i}{2\pi}\Big)^n \frac{1}{n!} \int \mathrm{tr} F^n$$

$$= \quad (-1)^{n-1}\Big(\frac{i}{2\pi}\Big)^n \frac{(n-1)!}{(2n-1)!} \int \mathrm{tr}K^{2n-1} \qquad (23)$$

which provide an independent check on the equivalence of the WZW and chiral fermion calculations. For the gravitational anomalies, the WZW method is unknown to us and we shall follow the fermionic approach where [24]

$$\begin{aligned} I_4 &= \frac{1}{(2\pi)^2}\Big[\frac{i^2}{2}\,\mathrm{tr}F^2 + \frac{r}{48}\,\mathrm{tr}R^2\Big] \\ I_8 &= \frac{1}{(2\pi)^4}\Big[\frac{i^4}{24}\,\mathrm{tr}F^4 + \frac{i^2}{96}\,\mathrm{tr}F^2\,\mathrm{tr}R^2 + \frac{r}{4608}\,(\mathrm{tr}R^2)^2 + \frac{r}{5760}\,\mathrm{tr}R^4\Big] \end{aligned} \qquad (24)$$

Here the Yang–Mills trace is in whatever representation the fermions are in and r is its dimensionality and the Lorentz trace is in the vector representation. In the Green–Schwarz formalism the heterotic string is described by superspace coordinates (X^M, θ^α) where the θ's are in the 16 of $SO(1,9)$ and the $SO(32)$ quantum numbers are carried by Majorana–Weyl fermions in the 32–dimensional fundamental representation. Although the θ's are worldsheet scalars, they are anticommuting and obey a first order Lagrangian. However, because of κ symmetry, only 8 of the θ's are physical (equal to $10-2$, the number of transverse X's). The net effect of integrating out the θ's is thus equivalent to 8 Majorana–Weyl fermions but with the opposite chirality to the 32 gauge fermions [25]. In summary we calculate X_4 by dividing I_4 by two (because the fermions are Majorana), taking the tr in the fundamental representation and setting $r = 32 - 8 = 24$. This agrees with (22). In the case of the fivebrane we again have the same (X^M, θ^α) and we again take the gauge fermions in the 32 of $SO(32)$. Because we are in $d = 6$, however, the fermions are no longer Majorana–Weyl (which exist only in 2 mod 8 dimensions) and so we do not divide I_8 by 2. Moreover in $d=6$, the number of physical θ's is only 4 (equal to $10-6$, the number of transverse X's) since κ–symmetry halves the number and (unlike $d=2$) going on–shell halves it again. The net effect of integrating out the θ's is thus equivalent to only two $d=6$ fermions [25] and hence $r=32-2=30$. Thus

$$I_8=\frac{1}{(2\pi)^4}\Big[\frac{1}{24}\,\mathrm{tr}F^4-\frac{1}{96}\,\mathrm{tr}F^2\,\mathrm{tr}R^2+\frac{15}{2304}\,(\mathrm{tr}R^2)^2+\frac{1}{192}\,\mathrm{tr}R^4\Big] \tag{25}$$

This is not quite in agreement with X_8 of (22) but the difference is proportional to $\mathrm{tr}\,R^2\,X_4$ and hence the discrepancy is easily remedied. Up until now we have been assuming that the fivebrane B_6 which appears in (2) was identical to the $D=10$ supergravity B_6 which satisfies (3). However, this may need to be modified when we include the gravitational Chern–Simons corrections which are of higher order in the low–energy expansion than those of Yang–Mills. The Green–Schwarz result is $dH_7(\text{supergravity})=\beta'(2\pi)^4\,X_8$, whereas our fivebrane calculations have lead us to $dH_7(\text{fivebrane})=\beta'(2\pi)^4\,I_8$. So if we define

$$B_6(\text{fivebrane})=B_6(\text{supergravity})-c\,\frac{\beta'}{\alpha'}\,\mathrm{tr}R^2\,B_2 \tag{26}$$

then the gauge–invariant field strength $H_7(\text{supergravity})$ satisfies

$$dH_7(\text{supergravity})=\beta'(2\pi)^4\,(I_8-\frac{c}{(2\pi)^2}\,\mathrm{tr}\,R^2\,X_4) \tag{27}$$

on using (9). Comparing X_8 of (22) with I_8 of (25) and using X_4 of (22), we find perfect agreement with the Green–Schwarz result with the choice

$c = 1/48$.

In their discussion of anomalies, Green, Schwarz and Witten [22] say "The case where something really new happens is that of $4k+2$ dimensions and this is the case of interest to superstring theory since the worldsheet has dimension two and the spacetime has dimension ten !". The results of the present paper might be summarized by adding "... and the fivebrane worldvolume has dimension six !".

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