arXiv:hep-ph/9401329v1 25 Jan 1994

Two-Loop Neutrino Masses and the Solar Neutrino Problem

DEBAJYOTI CHOUDHURY[(a)*], RAJ GANDHI[(b),(c)†], J.A. GRACEY[(d)‡] AND BISWARUP MUKHOPADHYAYA[(e)§]

[(a)] *Max-Planck-Inst. für Physik, Werner Heisenberg Inst.*

Föhringer Ring 6, 80805 München, Germany

[(b)] *Center for Theoretical Physics, Department of Physics, Texas A&M University*

College Station, TX 77843-4242, USA

[(c)] *Astroparticle Physics Group, Houston Advanced Research Center (HARC)*

The Woodlands, TX 77381, USA

[(d)] *DAMTP, University of Liverpool,*

P.O Box 147, Liverpool L69 3BX, U.K.

[(e)] *Mehta Research Institute, 10 Kasturba Gandhi Marg,*

Allahabad, 211002, India

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[*]debchou@iws166.mppmu.mpg.de, debchou@dmumpiwh.bitnet

[†]raj@tamphys.bitnet

[‡]jag@s-a.amtp.liv.ac.uk

[§]biswarup@mri.ernet.in

ABSTRACT

The addition of m singlet right-handed neutrinos to the Standard Model leads to radiatively generated mass corrections for the $SU(2)_L$ doublet neutrinos. For those neutrinos which are massless at the tree level after this addition, this implies a small mass generated at the two-loop level via $W^{\pm}$ exchange. We calculate these mass corrections exactly by obtaining an analytic form for the general case of n doublets and m singlets. As a phenomenological application, we consider the $m = 1$ case and examine the masses and mixings of the doublet neutrinos which arise as a result of the two-loop correction in the light of experimental data from two sources which may shed light on the question of neutrino masses. These are (a) the neutrino detectors reporting a solar neutrino deficit (and its resolution via Mikheyev-Smirnov-Wolfenstein matter oscillations), and (b) the COBE satellite data on the non-zero angular variations of the cosmic microwave background temperature (and its possible implications for hot dark matter). Within the framework of the extension considered here, which leaves the gauge group structure of the Standard Model intact, we show that it is possible for neutrinos to acquire small masses naturally, with values which are compatible with current theoretical bias and experimental data.

1 Introduction

It is fair to say that the problem of understanding the origin of fermion masses is one of the most perplexing questions facing particle physics today. The Standard model [1] can reproduce the observed fermion masses via electroweak symmetry breaking and the Higgs mechanism, but provides no explanation for their values. When such an understanding is obtained, one of the issues that it must clarify is the smallness of neutrino masses (if, indeed, neutrinos are massive) relative to those of the other fermions. An attractive explanation for this observed feature of the fermion mass spectrum is the see-saw mechanism [2]. It postulates the existence of right-handed neutrinos with masses of the order of the next energy threshold and uses this in combination with the Higgs mechanism to generate light (Majorana) neutrino masses via an effective dimension five operator.

Given our present ignorance of the origins of mass and the lack of experimental pointers towards any particular mechanism, it is important to keep an open mind on the smallness of neutrino mass. In this paper, we explore, via detailed calculation, the issue of radiatively generated neutrino masses, since this is also a natural way in which masses small compared to those of other fermions may be generated.

Any such effort needs to invoke physics beyond the Standard model. In view of the extraordinary and demonstrated robustness of the model to experimental tests over the last twenty years, we have thought it reasonable to make the simplest possible extension to the standard theory and study its effect on neutrino masses via radiative corrections, *i.e.* the addition of m $SU(2)_L \otimes U(1)_Y$ singlet right-handed neutrinos. *A priori*, there is no connection between their number and that of the doublet neutrinos, hence the simplest case corresponds to $m = 1$, *i.e.* the addition of one right-handed singlet neutrino to the standard model [3].

The gauge group structure of the weak sector remains unchanged as a consequence of this extension, but majorana mass terms incorporating the scale of new physics are now allowed. We do not speculate on their origin, but only note that it would require invoking an additional global symmetry (such as a conserved lepton number) to set these to zero. The doublet neutrinos acquire radiative (and, in some cases, tree-level) masses due to the presence of the singlets, as we discuss below. The radiative masses arise (via mixing) due to a two-loop mechanism [4, 5] involving the exchange of $W^{\pm}$ bosons. In Sections II and the Appendix, we calculate, exactly and in analytic form, the two-loop masses accquired by the initially massless doublet neutrinos. Our calculation is general and valid for any number of doublet and singlet neutrinos, but in order to obtain phenomenologically useful information, we focus, in Section III, on the $m = 1$ case. Even this simplest extension of the Standard model introduces four new parameters into the theory. On the issue of neutrino masses, it is non-accelerator experiments that provide information on the cutting edge. Hence we have chosen to examine the results for the $m = 1$ case in the light of (a) the MSW [6] solution to the solar neutrino deficit seen by the Kamiokande [7], GALLEX [8], SAGE [9] and Homestake [10] neutrino detectors and (b) the implications for hot dark matter (neutrinos) from the recent COBE observations on the anisotropy of the microwave background [11]. Invoking this experimental information restricts the parameter space and consequently, in the context considered in this paper, permits a handle on the range of the mass scale characterizing physics beyond the standard model. We show that doublet neutrino masses compatible with both (a) and (b) above can result naturally from such physics at the several hundred GeV scale.

2 Radiative Generation of Neutrino Masses

In this section we give a description of an exact general procedure for calculating two-loop neutrino masses applicable to any extension of the Standard model which incorporates singlet right-handed neutrinos. (We remark below on the reason why a one-loop mass does not arise in the sitiuation considered here, where *only* right-handed handed neutrinos are added to the existing particle spectrum.) After setting up the generic integral that needs to be calculated we describe the procedure for evaluating it exactly in the Appendix.

The lepton sector of the extension considered here has, in general, $n\,(\geq 3)$ doublet fields $[\nu'_{iL}\ l_{iL}]^T$ and m singlet fields $(\nu'_{AL})^c = (\nu'^c_A)_R$. (Here $i = 1....n$, $A = 1....m$ and $\nu^c \equiv C\bar{\nu}^T$ is the charge conjugate spinor.) In addition, one has the charged lepton $SU(2)_L$ singlet fields l_{iR}. The primes on the neutrino fields denote weak eigenstates as opposed to physical particle states. Without any loss of generality, we have assumed that the weak eigenstates l_i are the same as the corresponding mass eigenstates, *i.e.* the charged lepton mass matrix is diagonal.

As noted in the Introduction, in addition to the Dirac mass terms, the most general Lagrangian consistent with the gauge symmetry of the Standard model also contains possible Majorana mass terms for neutrinos of the form $m_{AB}\overline{(\nu'_{AL})^c}\nu'_{BL}$. In the minimal model under consideration here, such terms must be bare mass terms, but in a more involved model they could arise, for instance, due to the vacuum expectation value of a singlet higgs. To facilitate discussion, we combine all the left handed neutrinos into a $(n+m)$–dimensional vector in the flavour space denoting it by $\nu'_{\alpha L}$. The most general mass term is thus given by

$$\mathcal{L}_{\mathrm{m}} = \sum_{i=1}^{n} \mu_i \bar{l}_{iL} l_{iR} + \sum_{\alpha,\beta=1}^{n+m} \overline{(\nu'_{\alpha L})^c} \mathcal{M}_{\alpha\beta} \nu'_{\beta L} + h.c \tag{2.1}$$

Here $\mathcal{M}$ is a complex symmetric [1] $(n+m) \times (n+m)$ matrix of the form

$$\mathcal{M} = \begin{pmatrix} 0_{n\times n} & D_{n\times m} \\ D^T_{m\times n} & M_{m\times m} \end{pmatrix} \tag{2.2}$$

with D and M denoting the Dirac and the Majorana mass terms respectively. The first block is identically zero in the absence of a non–trivial vacuum expectation value for a $SU(2)_L$–triplet higgs field. (This restriction is imposed not only by our philosophy of minimal extension, but more importantly, by m_W/m_Z — the observed ratio of the gauge boson masses.) $\mathcal{M}$ can be diagonalized by a biunitary transformation of the form

$$V^T \mathcal{M} V = \widehat{\mathcal{M}} = \text{diagonal}(m_\alpha) \tag{2.3}$$

The mass eigenstates (ν_α) are then easily identified to be

$$\nu_L = V^\dagger \nu'_L \tag{2.4}$$

The relevant piece of the weak Lagrangian is then given by

$$\mathcal{L}_{\text{wk}} = J_\mu W^\mu \tag{2.5}$$

where

$$
\begin{aligned}
J^+_\mu &= \frac{ig}{\sqrt{2}} \sum_{i=1}^{n} \bar{l}_i \gamma_\mu P_L \nu'_i = \frac{ig}{\sqrt{2}} \sum_{i=1}^{n} \sum_{\alpha=1}^{n+m} K_{i\alpha} \bar{l}_i \gamma_\mu P_L \nu_i \\
J^3_\mu &= \frac{ig}{2c_W} \sum_{i=1}^{n} \left(\bar{l}_i \gamma_\mu P_L l_i + \bar{\nu}'_i \gamma_\mu P_L \nu'_i \right) \\
&= \frac{ig}{2c_W} \left[\sum_{i=1}^{n} \bar{l}_i \gamma_\mu P_L l_i + \sum_{\alpha,\beta=1}^{n+m} (K^\dagger K)_{\alpha\beta} \bar{\nu}_\alpha \gamma_\mu P_L \nu_\beta \right]
\end{aligned}
\tag{2.6}
$$

Here $c_W \equiv \cos\theta_W$, where θ_W is the Weinberg angle, $g = e/\sin\theta_W$, $P_L \equiv (1-\gamma_5)/2$ and

$$K = \begin{pmatrix} I_{n\times n} & 0 \\ 0 & 0 \end{pmatrix} V \tag{2.7}$$

---

[1]That $\mathcal{M}$ has to be symmetric is evident from the charge conjugation property of fermion bilinears.

is the $(n+m)$–dimensional analog of the quark sector Cabibbo–Kobayashi–Maskawa matrix. Note that though $KK^\dagger = \text{diag}(I_{n\times n}, 0)$, $(K^\dagger K)_{\alpha\beta} \neq \delta_{\alpha\beta}$. Thus we do indeed have flavour changing neutral currents (FCNC) in the neutrino sector.

Having set up the general formalism, let us now concentrate on the case where $n > m$. There exist then $n-m$ neutrinos that are strictly massless at the tree level. We now calculate the changes to such a spectrum accruing from quantum corrections.

Before proceeding, in view of the fact there exist massive neutrinos and also FCNC's in the neutrino sector, it is appropriate at this point to remark on the possibility of one–loop graphs with a Z or a Higgs exchange introducing a non–trivial correction to the neutrino mass matrix. However, it can be easily seen that it is possible to rotate the neutrino states such that only m of them have Yukawa couplings to the Higgs. Thus, only those doublet states that are massive at tree level obtain a Higgs induced mass at the one-loop level. In addition, since the flavor-changing Z couplings have the same mixing parameters as the flavor-changing Yukawa couplings, the one-loop Z exchange diagrams do not contribute to the masses of the $n - m$ neutrinos which are massless at the tree level. This reasoning applies at all orders to any diagram where all virtual particles are neutral. Hence the relevant diagram to compute is that given in Fig.(1).

We shall work in the weak interaction basis for the external neutrinos and the mass basis for all the virtual particles. Furthermore, we shall concentrate only on the first $n \times n$ block of $\mathcal{M}$, *i.e.* on the generation of Majorana mass terms for the doublet neutrinos. In the unitary gauge, the correction to the neutrino propagator is then given by

$$
\begin{aligned}
i\Sigma_{ij}^{(2)}(p) \; = \; & \left(\frac{ig}{\sqrt{2}}\right)^4 \sum_{\alpha=1}^{n+m} \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \gamma_\mu P_R \frac{i}{\slashed{p} + \slashed{q} - \mu_i} K_{\alpha i}^\dagger \gamma_\nu P_R \\
& \frac{i}{\slashed{p} + \slashed{q} + \slashed{k} - m_\alpha} K_{\alpha j}^\dagger \gamma_\sigma P_L \frac{i}{\slashed{p} + \slashed{k} - \mu_j} \gamma_\lambda P_L \\
& \frac{-i\left(g^{\mu\sigma} - q^\mu q^\sigma/m_W^2\right)}{q^2 - m_W^2} \frac{-i\left(g^{\nu\lambda} - k^\nu k^\lambda/m_W^2\right)}{k^2 - m_W^2}
\end{aligned}
\tag{2.8}
$$

5

The mass correction is of course given by $\mathcal{M}_{ij}^{(2)} = \Sigma_{ij}^{(2)}(p = 0)$, and after some algebra this leads to

$$\mathcal{M}_{ij}^{(2)} = \frac{g^4}{4} \sum_{\alpha=1}^{n+m} m_\alpha K_{\alpha j}^\dagger K_{\alpha i}^\dagger \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{(k+q)^2\, k \cdot q}{\mathcal{D}_{ij;\alpha}}$$
$$\left[\left(4 + \frac{k^2 q^2}{m_W^4} \right) - 4 \frac{q^2 + k^2}{m_W^2} \right] \tag{2.9}$$

where

$$\mathcal{D}_{ij;\alpha} = (k+q)^2 \left\{ (k+q)^2 - m_\alpha^2 \right\} (q^2 - \mu_i^2)(q^2 - m_W^2)(k^2 - m_W^2)(k^2 - \mu_j^2) \tag{2.10}$$

We see that the mass corrections would be identically zero if $m_\alpha = 0$, $\forall \alpha$. This ought to be so as any mass renormalization must be proportional to the bare mass terms. The integral above has a naive degree of divergence of 4. However, note that

$$\sum_{\alpha=1}^{n+m} m_\alpha K_{\alpha j}^\dagger K_{\alpha i}^\dagger = \mathcal{M}_{ij} = 0 \tag{2.11}$$

and hence

$$\sum_{\alpha=1}^{n+m} m_\alpha K_{\alpha i}^\dagger K_{\alpha j}^\dagger \frac{(k+q)^2}{(k+q)^2 - m_\alpha^2} = \sum_{\alpha=1}^{n+m} \frac{K_{\alpha i}^\dagger K_{\alpha j}^\dagger m_\alpha^3}{(k+q)^2 - m_\alpha^2} \tag{2.12}$$

This clearly is analogous to the GIM mechanism in the quark sector. Even on substitution of eqn(2.12) in eqn(2.9), the integral in the latter is still formally divergent. Notice, however, that this is but an artifact of the unitary gauge and is not a real divergence [12]. In fact, by invoking identities similar to eqn(2.12) or equivalently, by working in the Feynman gauge, one obtains [2]

$$\mathcal{M}_{ij}^{(2)} = g^4 \sum_{\alpha=1}^{n+m} m_\alpha^3 K_{\alpha j}^\dagger K_{\alpha i}^\dagger \left[4 + 4 \frac{\mu_i^2 + \mu_j^2}{m_W^2} + \frac{\mu_i^2 \mu_j^2}{m_W^4} \right] \Lambda(\mu_i^2, m_W^2, m_\alpha^2, 0, \mu_l^2, m_W^2) \tag{2.13}$$

where

$$\Lambda(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) \equiv$$

[2]This has often been cited in the literature as a GIM-like cancellation, but in our view the two are quite different.

$$\int \frac{d^4k \; d^4q \; k \cdot q}{(q^2 + m_1^2)(q^2 + m_2^2)\left\{(k+q)^2 + m_3^2\right\}\left\{(k+q)^2 + m_4^2\right\}(k^2 + m_5^2)(k^2 + m_6^2)}$$

$$(2.14)$$

is an Euclidean integral evaluated in the Appendix.

The expression in eqn(2.13) thus represents the Majorana mass generated for the doublet neutrino at the two–loop level. In operator language, it arises from terms of the form

$$\overline{(L_{iL})^c}L_{jL}\phi\phi S \qquad (2.15)$$

where L_{iL} represent the doublet lepton fields, ϕ is the usual higgs field and S represents the lepton number violating operator (whether a singlet higgs or a bare mass term). We note that this five dimensional effective operator for the radiative masses is the same as that for the conventional see-saw mechanism. The difference between the two resides in the scale of mass generation. Two-loop radiative masses compatible with the solar and COBE data can arise from right-handed neutrinos at the several hundred GeV scale, as we show below, whereas the see-saw mechanism generates similar valued masses via heavy neutrinos at the grand unified scale.

We also note that though the corrections ostensibly are proportional to m_α^3 (eqn.(2.13)), the actual dependence is linear (apart from logarithmic corrections) due to suppressions hidden in Λ. As m_α becomes larger and terms of the order of $(\mu_i/m_\alpha)^2$ become negligible, the correction goes as $\Sigma_\alpha K_{\alpha i}^\dagger K_{\alpha j}^\dagger m_\alpha$, which is simply the $(ij)^{\text{th}}$ element of the tree-level mass matrix, and hence zero for the cases of interest here.

Finally, we remark that a complex $\mathcal{M}$ in eqn.(2.1) obviously leads to a complex digonalizing matrix V and hence possibly to CP–violating processes. However, since there is no evidence as yet of any such non-conservation in the leptonic sector, we have, in the interests of simplicity, chosen to perform all numerical calculations assuming a real neutrino mass matrix.

3 Application: The Solar Neutrino Deficit and COBE Data

In order to make a connection to experiment and phenomenology, we now specialize to the $n = 3$ and $m = 1$ case and examine the two loop mass corrections in the context of (a) the MSW solution [6] to the solar neutrino deficit reported by various detectors [7, 8, 9, 10] and (b) recent COBE [11] data and its implications for neutrinos as dark matter.

The solar deficit is the only long-standing possible evidence for physics beyond the Standard model, and the MSW mechanism is its most popular resolution. In its essence, the mechanism requires neutrinos to be massive (and non-degenerate), allowing the interaction eigenstate ν_e (assumed to comprise predominantly of the lightest mass eigenstate) to oscillate to ν_μ or ν_τ due to the difference in the forward scattering potential seen by the two states in their passage through solar matter. It thus identifies a range of vacuum mixing angle and mass squared difference values which are compatible with the deficit observed by the various detectors. Figure 2, excluding curves labelled (a), (b) and (c), is taken from Ref. [13] and shows the familiar two-flavor mixing MSW solution space, where θ is the Cabibbo mixing angle and Δm^2 is the difference of the squares of the two neutrino masses, which, in the present context, are acquired at the two-loop level.

COBE data on the anisotropy of the microwave background, while not making a definitive statement on the nature of dark matter, seem to suggest that it may have both hot and cold components, with the former being a neutrino (since it is the only known hot dark matter candidate) with a mass of ≈ 10 eV.

We use both of the above considerations to restrict the rather large parameter space available to us.

In the scenario with one additional singlet, we have two massive and two massless neu-

trinos at tree-level. The two massive ones acquire both one-loop and two-loop corrections, which we neglect, and the massless states acquire small masses at the two-loop level. The two tree-level masses and all the radiative corrections are expressible in terms of four input mass parameters for the matrix $\mathcal{M}$. For various plausible (fixed) values of m_α, (the singlet mass, signifying the scale of new physics) and the added constraint that the other neutrino with a tree level mass lie in the 10 eV range, we obtain a one parameter set of curves (see Fig. 2) which denotes the intersection of the "two-loop space" with the MSW solution space. Note that restricting ourslves to the two dimensional MSW space imposes a third constraint, *i.e.* that the ν_e mixes predominantly with only one other state. Curve (a) in Figure 2 corresponds to a singlet mass of 100 GeV and a ν_τ mass of ≈ 8.6 eV. The two-loop masses and mixings of ν_e and ν_μ are then such that they span the MSW space as shown. Curve (b) corresponds to a singlet mass of 400 GeV and a ν_μ mass of ≈ 7 eV. ν_τ and ν_e then acquire radiative masses and mixings that span the solution space as shown. For $\sin^2 2\theta$ greater than $\approx 3 \times 10^{-1}$, ν_τ becomes lighter than ν_e, and MSW oscillations occur between anti-neutrino rather than neutrino states, and are thus not relevant. We note that (b) passes through the (small-angle, non-adiabatic) MSW region that is compatible with all detectors and also represents a value of m_{ν_μ} (7 eV) that provides a very good fit to COBE data in the context of a hot plus cold dark matter scenario. Finally, curve (c) represents a singlet mass of 1 TeV and a ν_μ mass of ≈ 9.8 eV, and terminates where it does because for larger mixing angles the ν_e becomes heavier than the ν_τ. Note that the determination of which flavor the ν_e oscillates to is made by examining the mixing (diagonalizing) matrix of the full (*i.e.* tree + loop) mass matrix. A (reasonable) assumption built into the results is that ν_e is the lightest state.

We stress that these curves represent a phenomenological exercise more than anything else to demonstrate that our calculations can make connection with experiment when the full

parameter space, which is quite large, is constrained by imposing physically and empirically well-motivated restrictions.

We note that the singlet mass values chosen by us (100 GeV, 400 GeV and 1 TeV) are not in conflict with accelerator [14] or cosmological [15] bounds on these particles.

Finally, we remark that a disparity between the mass scales of the $\nu_e, \nu_\mu(\nu_\tau)$ and that of the $\nu_\tau(\nu_\mu)$ seems to be required if we take both the solar and COBE implications for neutrino masses seriously. In the simple model under consideration here, such a disparity arises naturally since the neutrino which contributes to dark matter has a tree level mass while the other two have loop masses.

4 Conclusions

We have explicitly obtained an analytic form for the radiative two-loop masses acquired by doublet neutrinos in models where right-handed singlets are present. We have made an effort to keep our calculation general and the expression for the mass correction that we obtain may have applications in other models with right-handed neutrinos. We have calculated these masses (for the one singlet case) in the light of experimental data from solar neutrino detectors and from COBE, within the confines of the MSW solution to the solar deficit. By doing so we have made an effort to demonstrate that intermediate scale physics (*i.e.* physics at ≤ 1 TeV) can lead, in a simple way, to naturally small masses for neutrinos which have physically meaningful values, without requiring drastic changes in the presently known particle spectrum or gauge group structure.

A   Appendix: Evaluation of Λ_{123456}

In this section we discuss the exact evaluation of the fundamental finite two loop four dimensional integral underlying the mechanism. As a first step, though, we consider the more general two loop Euclidean space integral, Λ_{123456}, defined by

$$\Lambda_{123456} = \int_{pq} \frac{p \cdot q}{(p^2 + m_1^2)(p^2 + m_2^2)((p+q)^2 + m_3^2)((p+q)^2 + m_4^2)(q^2 + m_5^2)(q^2 + m_6^2)} \quad (A.1)$$

which we will evaluate analytically and then specialize to the case we are concerned with. For reasons which we explain below we choose to calculate eqn (A.1) in d-dimensions where

$$\int_k = \frac{\mu^{4-d}}{(2\pi)^d} \int d^d k \quad (A.2)$$

and μ is an arbitrary mass parameter introduced to ensure the coupling constant remains dimensionless in our d-dimensional manipulations. The subscripts on Λ_{123456} correspond to the masses m_i^2 of the integral and we note that the function has certain obvious symmetries, $\Lambda_{123456} = \Lambda_{213456} = \Lambda_{563412}$, which ought to be preserved in the final expression. The strategy to evaluate eqn (A.1) is to use partial fractions to obtain a sum of 2-loop integrals with three propagators and then to substitute for the value of each of these sub-integrals, which have been considered by other authors in different contexts before, [16, 17, 18, 19]. For instance, if we define

$$J_{ijk} = \int_p \int_q \frac{p \cdot q}{(p^2 + m_i^2)(q^2 + m_j^2)((p+q)^2 + m_k^2)} \quad (A.3)$$

then eqn (A.1) is built out of a sum of eight such integrals where its only symmetry is $J_{ijk} = J_{jik}$. Rewriting the numerator of eqn (A.3) one finds

$$J_{ijk} = \tfrac{1}{2}[I_i I_j - I_j I_k - I_k I_i - (m_k^2 - m_i^2 - m_j^2)I_{ijk}] \quad (A.4)$$

where

$$I_i = \int_p \frac{1}{(p^2 + m_i^2)} \quad (A.5)$$

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$$I_{ijk} \;=\; \int_p \int_q \frac{1}{(p^2 + m_i^2)((p+q)^2 + m_j^2)(q^2 + m_k^2)} \qquad (\text{A.6})$$

and the latter function is totally symmetric, corresponding to a two loop vacuum graph (ie zero external momentum). The integral I_{ijk} has been considered in [16, 17] and a single integral representation of it exists, [18, 19, 20]. For our purposes, however, we have chosen to use the elegant formula given in [20] since it is explictly symmetric in the masses. Although Λ_{123456} is itself ultraviolet finite the sub-integrals, eqns (A.3) and (A.4), are divergent and therefore require regularization. In [19, 20] dimensional regularization was introduced to control these infinities, which is why we choose to calculate eqn (A.1) in d-dimensions, so that I_{ijk} involves double and simple poles in ϵ where $d = 4 - 2\epsilon$. Therefore in the final result these must cancel for all m_i^2. As a first step, it is trivial to observe that in the partial fraction decomposition of eqn (A.1) the $I_i I_j$ type terms, which are also divergent, formally cancel to leave only the I_{ijk} terms. To proceed we recall the important properties of I_{ijk} which have been discussed in more detail in [20]. In d-dimensions the exact value, for arbitrary (mass)^2, x, y and z, is

$$I(x,y,z) \;=\; I(2a,0,0) + \Gamma'[F(\tfrac{1}{2}c - y) + F(\tfrac{1}{2}c - z) - F(x - \tfrac{1}{2}c)] \qquad (\text{A.7})$$

where

$$
\begin{aligned}
I_{ijk} &= I(m_i^2, m_j^2, m_k^2) \\
\Gamma' &= \frac{(\mu^2)^{4-d}}{(4\pi)^d} \Gamma(2 - \tfrac{1}{2}d)\Gamma(1 - \tfrac{1}{2}d) \\
a &= \tfrac{1}{2}[x^2 + y^2 + z^2 - 2xy - 2yz - 2zx]^{1/2} \\
c &= x + y + z \qquad\qquad\qquad\qquad\qquad\qquad (\text{A.8})
\end{aligned}
$$

and

$$F(w) \;=\; \int_a^w ds\, \frac{1}{(s^2 - a^2)^{(4-d)/2}} \qquad (\text{A.9})$$

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The result (A.7) is valid in the region of (x, y, z) space where $a^2 \geq 0$. For the case when $a^2 < 0$, then the solution is, with $b^2 = -a^2$,

$$
\begin{aligned}
I(x, y, z) \;=\; & -I(2b, 0, 0) \sin(\tfrac{1}{2}\pi d) \\
& + \; \Gamma'[G(\tfrac{1}{2}c - x) + G(\tfrac{1}{2}c - y) + G(\tfrac{1}{2}c - z)]
\end{aligned}
\tag{A.10}
$$

where

$$
G(w) \;=\; \int_0^w ds \, \frac{1}{(s^2 + b^2)^{(4-d)/2}}
\tag{A.11}
$$

and, for example,

$$
I(x, 0, 0) \;=\; \frac{\Gamma(2 - \tfrac{1}{2}d)\Gamma(3 - d)\Gamma^2(\tfrac{1}{2}d - 1)x^{d-3}}{(4\pi)^d \Gamma(\tfrac{1}{2}d)(\mu^2)^{d-4}}
\tag{A.12}
$$

which is clearly singular in four dimensions. To obtain the finite part of Λ_{123456} each part of $I(x, y, z)$ needs to be expanded in powers of ϵ to the $O(1)$ term and the poles in ϵ cancelled. The non-trivial part of this exercise is the ϵ-expansion of the $F(w)$ and $G(w)$ integrals. These have been given in [20] and we record that to the ϵ-finite term,

$$
\begin{aligned}
(4\pi)^4 I(x, y, z) \;=\; & -\frac{c}{2\epsilon^2} - \frac{1}{\epsilon}\left[\frac{3c}{2} - L_1\right] - \tfrac{1}{2}[L_2 - 6L_1 + \xi(x, y, z) \\
& + c(7 + \zeta(2)) + (y + z - x)\overline{\ln}y\overline{\ln}z \\
& + (z + x - y)\overline{\ln}z\overline{\ln}x + (y + x - z)\overline{\ln}y\overline{\ln}x]
\end{aligned}
\tag{A.13}
$$

where $\zeta(n)$ is the Riemann zeta function, $L_i = x\overline{\ln}^i x + y\overline{\ln}^i y + z\overline{\ln}^i z$, $\overline{\ln}x = \ln(x/\hat{\mu}^2)$, $\hat{\mu}^2 = 4\pi e^{-\gamma}\mu^2$ and γ is Euler's constant, and for $a^2 > 0$,

$$
\xi(x, y, z) \;=\; 8a[M(\phi_z) + M(\phi_y) - M(-\phi_x)]
\tag{A.14}
$$

where

$$
M(t) \;=\; -\int_0^t d\phi \, \ln \sinh \phi
\tag{A.15}
$$

and the angles ϕ_x are defined by

$$
\phi_x \;=\; \coth^{-1}\left[\frac{\tfrac{1}{2}c - x}{a}\right]
\tag{A.16}
$$

For $a^2 < 0$, then

$$\xi(x, y, z) = 8b[L(\theta_x) + L(\theta_y) + L(\theta_z) - \tfrac{1}{2}\pi \ln 2] \qquad (A.17)$$

where the θ_x angles are given by

$$\theta_x = \tan^{-1}\left[\frac{\tfrac{1}{2}c - x}{b}\right] \qquad (A.18)$$

and $L(t)$ is the Lobachevskij function,

$$L(t) = -\int_0^t d\theta \ln \cos \theta \qquad (A.19)$$

Equation (A.17) can also be rewritten as

$$\xi(x, y, z) = 8b[\tilde{L}(\theta_z) + \tilde{L}(\theta_y) - \tilde{L}(-\theta_x)] \qquad (A.20)$$

where $\tilde{L}(t) = \int_t^{\pi/2} d\theta \ln \cos \theta$ in order to make the obvious analytic continuation across $a^2 = 0$ more apparent. It is worth noting that essentially eqn (A.1) has been reduced to a single simple function, eqn (A.19), whose properties are well known. We have used the following identities in order to write an efficient programme to calculate Λ_{123456} for a range of physical mass values. For instance, [21],

$$\begin{aligned} L(t) &= -L(-t) & \text{for} \quad -\tfrac{1}{2}\pi \le t \le \tfrac{1}{2}\pi \\ L(t) &= L(\tfrac{1}{2}\pi - t) + (t - \tfrac{1}{4}\pi)\ln 2 - \tfrac{1}{2}L(\tfrac{1}{2}\pi - 2t) & \text{for} \quad 0 \le t \le \tfrac{1}{4}\pi \\ L(t) &= \pm L(\pi \pm t) \mp \pi \ln 2 & (A.21) \end{aligned}$$

Therefore, when the argument of the Lobachevskij function is known, the identities of eqn (A.21) mean that one need only write a routine to evaluate $L(t)$ numerically in the range $[0, \tfrac{1}{2}\pi)$. For example, if $0 \le \lambda < 2\pi$ then for any integer n

$$L(2\pi n + \lambda) = 2\pi n \ln 2 + L(\lambda) \qquad (A.22)$$

and so on.

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Returning to the partial fraction form of Λ_{123456} with the result for I_{ijk}, the c and L_i terms of the ϵ expansion cancel in the final expression and we can therefore take the limit back to four dimensions, $\epsilon \to 0$. Consequently, we end up with the following analytic expression:

$$
\begin{aligned}
\Lambda_{123456} \;=\; & -\frac{1}{4(4\pi)^4(m_1^2 - m_2^2)(m_3^2 - m_4^2)(m_5^2 - m_6^2)} \\
\times \; & \Bigg[(m_3^2 - m_1^2 - m_5^2)\Bigg[\xi_{135} - m_1^2 \ln\left(\frac{m_1^2}{m_3^2}\right)\ln\left(\frac{m_1^2}{m_5^2}\right) \\
& - m_3^2 \ln\left(\frac{m_3^2}{m_1^2}\right)\ln\left(\frac{m_3^2}{m_5^2}\right) - m_5^2 \ln\left(\frac{m_5^2}{m_1^2}\right)\ln\left(\frac{m_5^2}{m_3^2}\right)\Bigg] \\
& - (m_3^2 - m_1^2 - m_6^2)\Bigg[\xi_{136} - m_1^2 \ln\left(\frac{m_1^2}{m_3^2}\right)\ln\left(\frac{m_1^2}{m_6^2}\right) \\
& - m_3^2 \ln\left(\frac{m_3^2}{m_1^2}\right)\ln\left(\frac{m_3^2}{m_6^2}\right) - m_6^2 \ln\left(\frac{m_6^2}{m_1^2}\right)\ln\left(\frac{m_6^2}{m_3^2}\right)\Bigg] \\
& - (m_4^2 - m_1^2 - m_5^2)\Bigg[\xi_{145} - m_1^2 \ln\left(\frac{m_1^2}{m_4^2}\right)\ln\left(\frac{m_1^2}{m_5^2}\right) \\
& - m_4^2 \ln\left(\frac{m_4^2}{m_1^2}\right)\ln\left(\frac{m_4^2}{m_5^2}\right) - m_5^2 \ln\left(\frac{m_5^2}{m_1^2}\right)\ln\left(\frac{m_5^2}{m_4^2}\right)\Bigg] \\
& + (m_4^2 - m_1^2 - m_6^2)\Bigg[\xi_{146} - m_1^2 \ln\left(\frac{m_1^2}{m_4^2}\right)\ln\left(\frac{m_1^2}{m_6^2}\right) \\
& - m_4^2 \ln\left(\frac{m_4^2}{m_1^2}\right)\ln\left(\frac{m_4^2}{m_6^2}\right) - m_6^2 \ln\left(\frac{m_6^2}{m_1^2}\right)\ln\left(\frac{m_6^2}{m_4^2}\right)\Bigg] \\
& - (m_3^2 - m_2^2 - m_5^2)\Bigg[\xi_{235} - m_2^2 \ln\left(\frac{m_2^2}{m_3^2}\right)\ln\left(\frac{m_2^2}{m_5^2}\right) \\
& - m_3^2 \ln\left(\frac{m_3^2}{m_2^2}\right)\ln\left(\frac{m_3^2}{m_5^2}\right) + m_5^2 \ln\left(\frac{m_5^2}{m_2^2}\right)\ln\left(\frac{m_5^2}{m_3^2}\right)\Bigg] \\
& + (m_3^2 - m_2^2 - m_6^2)\Bigg[\xi_{236} - m_2^2 \ln\left(\frac{m_2^2}{m_3^2}\right)\ln\left(\frac{m_2^2}{m_6^2}\right) \\
& - m_3^2 \ln\left(\frac{m_3^2}{m_2^2}\right)\ln\left(\frac{m_3^2}{m_6^2}\right) - m_6^2 \ln\left(\frac{m_6^2}{m_2^2}\right)\ln\left(\frac{m_6^2}{m_3^2}\right)\Bigg] \\
& + (m_4^2 - m_2^2 - m_5^2)\Bigg[\xi_{245} - m_2^2 \ln\left(\frac{m_2^2}{m_4^2}\right)\ln\left(\frac{m_2^2}{m_5^2}\right) \\
& - m_4^2 \ln\left(\frac{m_4^2}{m_2^2}\right)\ln\left(\frac{m_4^2}{m_5^2}\right) - m_5^2 \ln\left(\frac{m_5^2}{m_2^2}\right)\ln\left(\frac{m_5^2}{m_4^2}\right)\Bigg] \\
& - (m_4^2 - m_2^2 - m_6^2)\Bigg[\xi_{246} - m_2^2 \ln\left(\frac{m_2^2}{m_4^2}\right)\ln\left(\frac{m_2^2}{m_6^2}\right)
\end{aligned}
$$

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$$- m_4^2 \ln \left(\frac{m_4^2}{m_2^2} \right) \ln \left(\frac{m_4^2}{m_6^2} \right) - m_6^2 \ln \left(\frac{m_6^2}{m_2^2} \right) \ln \left(\frac{m_6^2}{m_4^2} \right) \Big] \Big] \tag{A.23}$$

where $\xi_{ijk} = \xi(m_i^2, m_j^2, m_k^2)$ and it is evaluated according to eqns (A.14) or (A.17) depending on whether the particular a^2 is positive or negative. A further check on our manipulations to obtain eqn (A.23) is the absence of the arbitrary mass μ which was required at intermediate steps to have logarithms whose arguments were dimensionless quantities.

Although it may appear that the final result is singular in certain cases through denominator factors like $(m_1^2 - m_2^2)$ when $m_1^2 = m_2^2$, the expression within the square brackets also vanishes. Moreover, if one sets $m_2^2 = m_1^2 + \delta$, where δ is small, and expands in powers of δ then in the limit as $\delta \to 0$ a non-zero non-singular function of the independent mass remains. Further, there is no difficulty with singularities when one or more masses is zero. To illustrate this point explicitly we consider the integral Λ_{123056} where the zero subscript means the corresponding mass of eqn (A.1) is zero. Its form can readily be deduced from eqn (A.23) by taking the $m_4^2 \to 0$ limit. However, to do this the behaviour of $\xi(x, y, z)$ in the $z \to 0$ limit is required since eqn (A.23) has terms like $\ln m_4^2$ which are potentially infinite in the limit we require. It is is easy to deduce from the explicit representation, eqn (A.14), that

$$\xi(x, y, z) \sim (x - y) \left[2\mathrm{Li}_2 \left(1 - \frac{y}{x} \right) + \ln \left(\frac{x}{y} \right) \ln \left(\frac{x}{z} \right) \right] \tag{A.24}$$

as $z \to 0$. Thus a little algebra leads to the compact expression,

$$\begin{aligned}
\Lambda_{123056} &= -\frac{1}{4(4\pi)^4 (m_1^2 - m_2^2) m_3^2 (m_5^2 - m_6^2)} \\
&\times \Big[(m_3^2 - m_1^2 - m_5^2)\xi_{135} - (m_3^2 - m_1^2 - m_6^2)\xi_{136} \\
&\quad - (m_3^2 - m_2^2 - m_5^2)\xi_{235} + (m_3^2 - m_2^2 - m_6^2)\xi_{236} \\
&\quad - \rho(m_3^2, m_1^2, m_5^2) + \rho(m_3^2, m_1^2, m_6^2) \\
&\quad + \rho(m_3^2, m_2^2, m_5^2) - \rho(m_3^2, m_2^2, m_6^2)
\end{aligned}$$

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$$+ \quad \lambda(m_1^2, m_5^2) - \lambda(m_1^2, m_6^2) - \lambda(m_1^2, m_5^2) + \lambda(m_2^2, m_6^2) \Big] \tag{A.25}$$

with

$$\rho(x, y, z) \;=\; (x - y - z) \left[x \ln\left(\frac{x}{y}\right) \ln\left(\frac{x}{z}\right) + y \ln\left(\frac{y}{x}\right) \ln\left(\frac{y}{z}\right) \right.$$
$$\left. + \; z \ln\left(\frac{z}{x}\right) \ln\left(\frac{z}{y}\right) \right] \tag{A.26}$$

and

$$\lambda(x, y) \;=\; (x + y) \left[2(x - y) \mathrm{Li}_2\left(1 - \frac{y}{x}\right) - y \ln\left(\frac{x}{y}\right) \right] \tag{A.27}$$

where $\mathrm{Li}_2(t)$ is the dilogarithm function. Its properties have been discussed extensively in [22] but we make use of the following ones here

$$\mathrm{Li}_2(-t) + \mathrm{Li}_2(-1/t) \;=\; -\zeta(2) - \tfrac{1}{2} \ln^2 t \qquad \text{for } t > 0$$

$$\mathrm{Li}_2(t) + \mathrm{Li}_2(1 - t) \;=\; \zeta(2) - \ln t \ln(1 - t) \tag{A.28}$$

and its integral representation is, [22],

$$\mathrm{Li}_2(t) \;=\; - \int_0^t \frac{ds}{s} \ln(1 - s) \tag{A.29}$$

where $\mathrm{Li}_2(1) = \zeta(2) = \pi^2/6$.

Finally, another check on our overall expression eqn (A.23) is the comparison with the earlier result of [5] where only m_3^2 and m_4^2 are non-zero, ie Λ_{003400}, which was evaluated by an independent method. We can easily deduce an expression for Λ_{003400} from eqn (A.25) by using the relation (A.24) or by returning to the I_{ijk} representation of eqn (A.1) and taking the appropriate limits in that case. Useful for the former approach are the properties of the dilogarithm function, [22]. Whilst in the latter instance we made use of the Tyalor expansion of the I_{ijk} about zero mass and in particular,

$$\left. \frac{\partial^2 I(x, y, z)}{\partial y \partial z} \right|_{y = z = 0} = \frac{\Gamma^2(\tfrac{1}{2}d - 2)\Gamma(4 - \tfrac{1}{2}d)\Gamma(5 - d)x^{d-5}}{(4\pi)^d(\mu^2)^{d-4}\Gamma(\tfrac{1}{2}d)} \tag{A.30}$$

whose ϵ expansion is easy to determine. Consequently, we find

$$\Lambda_{003400} = -\frac{1}{(4\pi)^4(m_3^2 - m_4^2)}\ln\left(\frac{m_3^2}{m_4^2}\right) \tag{A.31}$$

This is in total agreement with the explicit calculation of [5] and is a necessary non-trivial check that we have the overall normalization of our integral correct, in terms of signs and factors of 2π.

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[12] A consistent use of any regularization scheme would, of course, demonstrate that all the infinities in eqn.(2.12) cancel, thereby leaving a finite result. However there is a simpler way to see this. Note that the mass correction is independent of the gauge choice and instead of the unitary gauge we could have chosen to work in the general 'tHooft–Veltman R_ξ gauge instead. Of these, the Feynman gauge has the added advantage that the gauge boson propagators are free of the longitudinal mode. This directly reduces the naive degree of divergence by 4. Of course, now we have to include, in addition to Fig 1, three further diagrams with the charged Goldstones replacing one or both of the W's. But these obviously lead to terms proportional to the fermion masses.

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Figure 1: The two-loop diagram which gives rise to the mass corrections considered in this paper.

Figure 2: The MSW solution space for the solar neutrino deficit, from Ref. [13]. Superposed on it are the 3 curves (a), (b) and (c) which represent sample calculations using our results. Each curve shows the mass squared differences and mixings for the two light neutrinos which acquire masses radiatively, for fixed values of the masses of the other two neutrinos which are massive at tree level. Curve (a) in Figure 2 corresponds to a singlet mass of 100 GeV and a ν_τ mass of ≈ 8.6 eV. Curve (b) corresponds to a singlet mass of 400 GeV and a ν_μ mass of 7 eV. Finally, curve (c) represents a singlet mass of 1TeV and a ν_μ mass of ≈ 9.8 eV.

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