CTP TAMU-09/99 UPR/838-T Imperial/TP/98-99/43 **hep-th/9903057**

March 1999

Superdualities, Brane Tensions and Massive IIA/IIB Duality

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ABSTRACT

The gauge transformations of p-form fields in supergravity theories acquire a non-commuting character when one introduces potentials both for the theory's original field strengths and for their duals. This has previously been shown in the "doubled" formalism for maximal supergravities, where a generalised duality relation between original and dual field strengths replaces the equations of motion. In the doubled formalism, the gauge transformations generate a superalgebra, and the corresponding symmetries have accordingly been called "superdualities." The corresponding Noether charges form a representation of the cohomology ring on the spacetime manifold. In this paper, we show that the gauge symmetry superalgebra implies certain non-trivial relations among the various p-brane tensions, which can straightforwardly be read off from the superalgebra commutation relations. This provides an elegant derivation of the brane-tension relations purely within a given theory, without the need to make use of duality relations between different theories, such as the type IIA/IIB T-duality, although the results are consistent with such dualities. We present the complete set of brane-tension relations in M-theory, in the type IIA and type IIB theories, and in all the lower-dimensional maximal supergravities. We also construct a doubled formalism for massive type IIA supergravity, and this enables us to obtain the brane-tension relations involving the D8-brane, purely within the framework of the massive IIA theory. We also obtain explicit transformations for the nine-dimensional T-duality between the massive type IIA theory and the Scherk-Schwarz reduced type IIB theory.

[1] Research supported in part by DOE grant DE-FG02-95ER40893

[2] Research supported in part by DOE grant DE-FG03-95ER40917.

Contents

1 Introduction

A new formulation of the equations of motion in maximal supergravities was recently developed in [1], in which every field in the theory, with the exception of gravity itself, is augmented by a "double" field of the dual degree. Thus in general in D dimensions each potential of degree n is augmented by its double, of degree $D-n-2$. In this approach the doubling is performed even on the dilatons and all other scalar fields, corresponding to $n=0$. The effect of this doubling is that, with the exception of the Einstein equation, all the other bosonic equations of motion are recast into a first-order form. In fact, as was shown in [1], they can all be recast in the form of an algebraic condition on a single generalised field strength that is subject to a "twisted self-duality" condition.

One of the intriguing features of the doubled system is that when one looks at the extended set of gauge transformations for the entire set of fields, one encounters non-commutativities that were not seen in the analogous gauge transformations for the original system of fields. By associating Lie algebra generators with each field in the extended system, one can thus construct an associated symmetry algebra. Interestingly enough, since the generators associated with forms of odd degrees must themselves be odd (*i.e.* fermionic), one generally finds that the algebra encoding the gauge symmetry transformations is a Lie superalgebra [1]. (The only exception to this among the maximal supergravities is the case of the type IIB theory in $D=10$, for which all the generators are bosonic.) Formulating the system of bosonic field equations as a twisted self-duality condition is achieved by exponentiating the superalgebra generators, with the various gauge potentials as parameters, and constructing a generalised field strength $\mathcal{G} = d\mathcal{V}\,\mathcal{V}^{-1}$. The twisted self-duality condition is then expressed as $*\mathcal{G} = \mathcal{S}\,\mathcal{G}$, where $\mathcal{S}$ is a pseudo-involution operator that maps between the generators of the original fields and their doubles [1].

In fact, as we shall discuss in this paper, a careful inspection shows that the non-commutativity of certain gauge transformations can already be seen in the framework of the canonical formalism even before the introduction of the dual gauge potentials. This arises when one considers the integrated Noether charges as generators of canonical transformations. Non-vanishing Noether charges for local symmetries occur only for "large" gauge transformations, corresponding to cohomologically nontrivial p-form gauge parameters. As a consequence, one finds that the Poisson bracket algebra of the integrated charges gives a representation of the cohomology ring on the underlying spacetime manifold.

The non-commutativity of the gauge transformations allows one to establish a set of relations among the various p-brane "tensions" (which are perhaps better thought of as the units of the corresponding electric charges). By using the superalgebra in the doubled formalism, one can straightforwardly arrive at relations between the p-brane tensions that could previously be derived only using rather intricate arguments based on duality transformations and various D-brane techniques [2,3,4,5,6,7,8,9,10]. Some of the relations that we shall present in this paper have appeared previously in the literature [2,3,4,5,6,7,8,9,10], but many are new.

In section 2 we shall present a canonical discussion of the gauge transformations, and shall show how the Poisson brackets of the gauge generators can be non-vanishing even when the gauge transformations might ostensibly appear to be abelian. The explanation for this apparent discrepancy is related to the subtle distinction between gauge transformations with exact gauge parameters and transformations with closed gauge parameters. In fact, precisely the same subtlety was shown in [1] to be responsible for the non-commutativity in the doubled formalism. In section 3 we shall exploit this non-commutativity in order to derive relations between the tensions for the various p-branes supported by the fields in the eleven-dimensional and ten-dimensional maximal supergravity theories. As we shall show, there is in general a one-to-one correspondence between the set of non-vanishing (anti)commutators in the Lie superalgebra, and the set of brane-tension relations. In section 4 we shall extend this discussion to all the lower-dimensional maximal supergravities. One interesting feature is that certain sets of brane-tension relations are themselves inter-related, as a consequence of a discrete set of relations among the various non-trivial commutators in the Lie superalgebras. This application of the so-called "jade rule"[1] [1] of the Lie superalgebras leads to a significant simplification of the structures of the brane-tension relations in the theories.

The brane-tension relations in the various dimensions can be inter-related also by means of dimensional reduction, and also by exploiting the T-duality symmetry that relates the type IIA and type IIB theories. We shall discuss this is detail in sections 4 and 5. In order to obtain a complete picture, it is necessary to extend the discussion of the type IIA theory to include the massive IIA supergravity first constructed in [11]. The main topic covered in section 5 is the construction of the doubled system of equations for the massive IIA theory, yielding an extended Lie superalgebra with additional (anti)commutators related to tension relations involving the D8-brane. Finally, in an extensive appendix, we derive explicit results for the T-duality between the massive IIA and the type IIB supergravities. This involves performing a Kaluza-Klein reduction of the massive IIA theory to $D = 9$, and a generalised Scherk-Schwarz reduction of the type IIB theory to $D = 9$. We do this at

[1] The term "jade rule" was a more lapidary variant of general rules such as the golden rule, *etc.*

the level of the full doubled systems. In the last subsection of the appendix, we derive the explicit field transformations that map between the nine-dimensional massive IIA and IIB theories.

2 Local symmetry Noether charges and non-commutativity of supergravity gauge transformations

Let us begin with an elementary discussion of the gauge transformations in supergravity theories and their non-commutativity. We shall consider this issue both at the level of the gauge transformations themselves and also at the level of the corresponding charges. For this, we shall first need to consider the nature of the Noether charges that can be associated to gauge symmetries.

It is well known that if the Lagrangian of a theory is left invariant by some set of group transformations, one can always, following the Noether procedure, define a set of locally-conserved quantities, *i.e.* Noether currents. The conservation law for the Noether current follows from the equations of motion. This conservation law has the consequence that if one integrates the time component of the current over the volume of a spatial hypersurface, one obtains a globally-conserved quantity, *i.e.* a charge.

In the case of a rigid symmetry transformation, a Noether charge may be interpreted as the generator of the associated symmetry transformation. In the case of a gauge symmetry, on the other hand, the equations of motion typically imply that the Noether charge reduces to a surface integral. This surface integral can sometimes be interpreted as the generator of a non-vanishing symmetry transformation, depending on the topological character of the corresponding gauge parameter Λ. Consequently, in discussing the charges, one must take care to consider the topological character of the corresponding gauge parameter Λ. Thus, instead of just considering a charge Q for a given symmetry, one should consider the charge Q_Λ associated to a specific gauge transformation, incorporating the transformation parameter into the charge integral. For "little" local symmetry transformations, which can be continuously deformed back to the identity transformation and which fall off sufficiently fast at infinity, the total charge integral vanishes upon use of the equations of motion. For topologically nontrivial, or "large" symmetry transformations, on the other hand, this integral need not vanish. In that case, a nonvanishing integrated charge may be interpreted, *via* Poisson (or, more correctly, Dirac) brackets, as the generator of a gauge transformation. Since "little" gauge transformations have vanishing charge integrals, there is a natural equivalence relation between large symmetry transformations differing by little transformations. In view of this behaviour, the large symmetry transformations are somewhat akin to rigid symmetry transformations such as Yang-Mills colour-rotating transformations that tend to constants instead of falling off at infinity, for which nonvanishing Noether charges may be defined, and for which charge integrals corresponding to transformations differing by a "little" gauge transformation are equal.

Let us now consider the construction of Noether currents and charges more specifically. If we have a set of fields ϕ^i, where i labels the fields, and a set of transformations $\delta\phi^i = f^i(\phi^j)$ which leave the Lagrangian $\mathcal{L}$ invariant , where $f^i(\phi^j)$ are some given functions, then the conserved Noether current is given by

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\delta\phi^i \; . \tag{2.1}$$

This definition is appropriate in the case where Lagrangian itself is invariant under the symmetry transformations. If instead it is invariant only up to a total derivative, *i.e.* if it transforms as $\delta\mathcal{L} = \partial_\mu\Omega^\mu$ for some Ω^μ, then the formula (2.1) is replaced by

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\delta\phi^i - \Omega^\mu \; . \tag{2.2}$$

This last expression is the one that we shall be using, since it is indeed the case that some gauge transformations leave the supergravity Lagrangians invariant only up to total derivatives. Having established the notation, we shall now derive explicit commutation relations in the simplest of the examples, namely eleven-dimensional supergravity. Let us first note a simplifying feature of the formula (2.2). Since we are interested in commutation relations for globally-conserved charges only, we have to consider the following integral

$$Q = \int j^0 dV^{(10)}, \tag{2.3}$$

where integration is performed over the entire ten-dimensional space. Note that in the definition of j^0 (2.2), the first term is nothing but the canonical momentum multiplied by the field variation under the symmetry transformation.

The field content of eleven-dimensional supergravity includes a 3-form $A_{(3)}$. It is a gauge field, transforming as $\delta A_{(3)} = \Lambda_{(3)}$ under gauge transformations where $\Lambda_{(3)}$ is an arbitrary *closed* 3-form, $d\Lambda_{(3)} = 0$. It is a straightforward calculation to see that the eleven-dimensional Lagrangian

$$\mathcal{L}_{11} = R*1 - \tfrac{1}{2}*F_{(4)}\wedge F_{(4)} - \tfrac{1}{6}F_{(4)}\wedge F_{(4)}\wedge A_{(3)} \tag{2.4}$$

transforms as

$$\delta\mathcal{L} = d(\tfrac{1}{6}\Lambda_{(3)} \wedge A_{(3)} \wedge F_{(4)}) \ , \tag{2.5}$$

which implies, according to our previous discussion, that the following conserved charges can be defined:

$$Q_e(\Lambda_{(3)}) = \int \Lambda_{(3)} \wedge (6 * \Pi - \tfrac{1}{6}A_{(3)} \wedge F_{(4)}) \ . \tag{2.6}$$

It is understood that the integrand here is projected into a 10-dimensional spacelike hypersurface. In (2.6), we have introduced a canonical momentum 3-form $\Pi = \frac{1}{3!}\Pi_{ijk}dx^i \wedge dx^j \wedge dx^k$, with components defined by $\Pi_{ijk} = \frac{\partial\mathcal{L}}{\partial(\partial_t A_{ijk})}$. The Hodge dual is taken with respect to the ten-dimensional metric. One may verify that for a $\Lambda_{(3)}$ that is not only closed but also exact, the charge integral (2.6) vanishes upon integration by parts and the use of the equations of motion. This behaviour may be compared with the analogous charge integral in Maxwell theory, $\int \Lambda_{(1)} \wedge *\Pi$, where $\Pi^i = F^{0i}$. For an exact $\Lambda_{(1)}$ falling off sufficiently rapidly at infinity, this integral vanishes upon use of the equations of motion. But for non-exact $\Lambda_{(1)}$, the integral need not vanish.

For the charges (2.6), one may use the canonical Poisson bracket relations

$$\{A_{I_1 I_2 I_3}, \Pi'^{J_1 J_2 J_3}\} = \delta^{[J_1}_{[I_1}\,\delta^{J_2}_{I_2}\,\delta^{J_3]}_{I_3]}\,\delta^{(10)}(x - x') \ , \tag{2.7}$$

with all others vanishing, to derive the charge algebra

$$\{Q_e(\Lambda^1_{(3)}), Q_e(\Lambda^2_{(3)})\} = Q_m(\Lambda^1_{(3)} \wedge \Lambda^2_{(3)}) \ , \tag{2.8}$$

where the charge $Q_m(\Lambda_{(6)})$ is defined by

$$Q_m(\Lambda_{(6)}) = \int \Lambda_{(6)} \wedge F_{(4)} \ . \tag{2.9}$$

With this result we see that the non-commutativity is a characteristic property of the theory, rooted in the structure of the gauge symmetry, and is not just an incidental by-product of the doubled formalism that we are using. In the doubled formalism, on the other hand, we shall encounter in addition an underlying (super)algebra of gauge transformations that accords with the algebra (2.8) for the integrated charges.

Similar discussions can be given in other situations where we meet non-commutativity of form-field gauge transformations, for example in the type IIA and IIB theories. The algebras in those cases are a little more complicated, but the basic structure remains the same.

At the level of the integrated charges, the algebra (2.8) reflects the ring structure of the cohomology of p-form gauge parameters on the underlying spacetime manifold. Thus, another interpretation of the integrated charges such as (2.6) is as a representation of the cohomology ring of the spacetime manifold. For most of the examples that arise in the study of p-brane solutions in supergravity, this cohomology ring corresponds to that of a torus. It remains an interesting problem to explore more sophisticated situations with manifolds of less trivial cohomology.

3 Brane tension relations

In this section we shall derive some direct consequences of the non-commutativity of certain gauge transformations in supergravity theories. It turns out that consistency requirements impose some rather nontrivial relations among the various p-brane tensions. Some of the relations we derive here have appeared previously in the literature [2,3,4,5,6,7,8,9,10], where they were obtained by more indirect means. Typically, this involved making a sequence of mappings between different low-energy theories, for example by exploiting the T-duality that relates the type IIA and IIB theories, or even more indirectly through the requirements for certain anomaly cancellations. By contrast, the method that we shall present below represents a considerable simplification, not only technically but also conceptually, in that it allows the brane-tension relations to be derived purely within the framework of the low-energy description of a given theory.

3.1 M-brane tensions

Let us start with the simplest example, namely eleven-dimensional supergravity. We shall be rather brief, since the doubled formalism has been developed in detail in an earlier paper [1]; we refer the interested reader there for additional information. The bosonic Lagrangian for eleven-dimensional supergravity is given by (2.4). Varying with respect to $A_{(3)}$, we obtain the equation of motion

$$d{*F_{(4)}} + \tfrac{1}{2} F_{(4)} \wedge F_{(4)} = 0 \ . \tag{3.1}$$

Equation (3.1) can be written as $d({*F_{(4)}} + \frac{1}{2} A_{(3)} \wedge F_{(4)}) = 0$, and so we can write the field equation in the first-order form

$$* F_{(4)} = F_{(7)} \equiv dA_{(6)} - \tfrac{1}{2} A_{(3)} \wedge F_{(4)} \ , \tag{3.2}$$

where we have introduced the dual potential $A_{(6)}$. It is easy to check that the first-order equation (3.2) is invariant under the following gauge transformations [1]

$$\delta A_{(3)} = \Lambda_{(3)}, \qquad \delta A_{(6)} = \Lambda_{(6)} - \tfrac{1}{2} \Lambda_{(3)} \wedge A_{(3)} \ , \tag{3.3}$$

where $\Lambda_{(3)}$ and $\Lambda_{(6)}$ are closed 3-form and 6-form gauge parameters, satisfying $d\Lambda_{(3)} = 0$ and $d\Lambda_{(6)} = 0$. The commutators of infinitesimal gauge transformations are given by

$$\begin{aligned}
[\delta_{\Lambda_{(3)}}, \delta_{\Lambda'_{(3)}}] &= \delta_{\Lambda''_{(6)}}, \qquad \Lambda''_{(6)} = \Lambda_{(3)} \wedge \Lambda'_{(3)}, \\
[\delta_{\Lambda_{(3)}}, \delta_{\Lambda_{(6)}}] &= 0, \qquad [\delta_{\Lambda_{(6)}}, \delta_{\Lambda'_{(6)}}] = 0 \ .
\end{aligned} \tag{3.4}$$

Since these transformations are to be thought of as gauge symmetries, it follows that not only the eleven-dimensional equations of motion for the massless fields, but also the low-energy actions for all extended objects, including massive p-branes, must be invariant under these symmetries. In order to investigate the restrictions imposed by their non-commutativity, we need to incorporate the couplings of the $(p+1)$-forms to the world-volume fields. Fortunately, the nature of these couplings is well known. For example, the term in the world-volume action describing the coupling of the 3-form $A_{(3)}$ has the minimal form

$$T_{(3)} \int A_{(3)}, \tag{3.5}$$

where $T_{(3)}$ is the membrane "tension." In this paper, we shall use the letter T exclusively for brane tensions. In particular $T_{(d)}$, with $d = p+1$ denotes the tension for the p-brane.

Let us suppose now that the space-time contains a compact six-dimensional sub-manifold with non-trivial third and sixth homology groups, $H^3(M)$ and $H^6(M)$. To simplify the discussion, we shall take this compact sub-manifold to be the six-torus. Now, if we wrap a membrane around one of the homology three-cycles $M_{(3)}$, and consider making gauge transformations of the form $\delta A_{(3)} = \omega_{(3)}$, where $\omega_{(3)}$ is a closed 3-form such that $\int_{M_{(3)}} \omega_{(3)} \neq 0$, then invariance of the term (3.5) imposes the following restriction on the value of the integral

$$T_{(3)} \int_{M_{(3)}} \omega_{(3)} = 2\pi k, \tag{3.6}$$

where k is an arbitrary integer, in order that the quantum effective action be invariant. This condition implies that if we take any closed 3-form gauge parameter, and integrate it over an arbitrary homology 3-cycle, then result must be quantised in terms of the inverse membrane tension.

From our above discussion we know that the gauge transformations do not commute. However, since the commutator of two symmetry transformations must itself also be a symmetry, we conclude that it too should leave everything invariant. We have seen above that the commutator of two gauge transformations of the potential $A_{(3)}$ gives rise to a gauge transformation of the potential $A_{(6)}$ (3.4). It is natural to think of $A_{(6)}$ as the gauge potential for the magnetic field-strength, and as such it must couple minimally to the world-volume of the five-brane through the term

$$T_{(6)} \int A_{(6)} \ . \tag{3.7}$$

The $A_{(6)}$ potential has its own independent symmetry; namely, we can shift it by an arbitrary closed 6-form, $\omega_{(6)}$. By the same argument as for the membrane, provided that five-brane is also wrapped over a certain homology 6-cycle the invariance of the world-volume action implies the quantisation condition

$$T_{(6)} \int_{M_{(6)}} \omega_{(6)} = 2\pi\ell \ , \tag{3.8}$$

where ℓ is again an arbitrary integer. Now, if we commute two gauge transformations for the $A_{(3)}$ potential we obtain the following shift in the five-brane world-volume action:

$$\delta\mathcal{S} = T_{(6)} \int_{M_{(6)}} \omega^1_{(3)} \wedge \omega^2_{(3)}, \tag{3.9}$$

where $\omega^1_{(3)}$ and $\omega^2_{(3)}$ are the parameters of the first and second gauge transformations respectively. For the torus this integral can be decomposed into the sum of products of integrals over 3-cycles. But we already know that these integrals are quantised in terms of the membrane tension (3.6). This is consistent with the equation (3.8) if and only if

$$T_{(6)} = \frac{1}{2\pi} T^2_{(3)} \ . \tag{3.10}$$

At this stage it is worthwhile to make an observation that significantly simplifies the calculations in more complicated cases, such as the type IIA or IIB theories in ten or lower dimensions. One can recast the commutation relations (3.4) as commutators in an ordinary super Lie algebra, by introducing generators V and $\tilde{V}$ for the $\Lambda_{(3)}$ and $\Lambda_{(6)}$ transformations respectively. We see that commutation relations (3.4) translate into the super Lie algebra [1]

$$\{V,V\} = -\tilde{V}, \qquad [V,\tilde{V}] = 0, \qquad [\tilde{V},\tilde{V}] = 0. \tag{3.11}$$

Note that commutators are even or odd according to whether the degrees of the associated field strengths are odd or even. (We shall in general, unless severe ambiguity might arise, avoid clumsy language by referring to commutators and anti-commutators generally as commutators in what follows.) Again we refer the reader to [1] for all details about these algebras. Here, we wish only to point out that the complete structure of all relations among the p-brane tensions is encoded in these algebras, and can be directly read off from the commutators of the generators. For each non-vanishing commutator, one simply needs to replace the bracket on the left by the product of corresponding inverse p-brane tensions, each multiplied by 2π, and likewise with each term on the right. For example, if one takes the first commutator in (3.11), and replaces $\{V,V\}$ by $(2\pi/T_{(3)})^2$ and $\tilde{V}$ by $(2\pi/T_{(6)})$ (minus signs must be dropped), then one arrives at the relation (3.10).

3.2 Type IIA brane tensions

The gauge potentials in type IIA massless supergravity in $D=10$ are $A_{(3)}$, $A_{(2)1}$, arising from the dimensional reduction of the three-form potential in $D=11$, together with the Kaluza-Klein vector $\mathcal{A}^1_{(1)}$. (Note that the index 1 implies that it is the first step in the reduction, from $D=11$ to $D=10$.) Their dual potentials are $\widetilde{A}_{(5)}$, $\widetilde{A}^1_{(6)}$ and $\widetilde{A}_{(7)1}$ respectively. These fields, together with the dilaton ϕ and its 8-form dual ψ can be used to construct a "coset representative" as follows:

$$\mathcal{V} = e^{\frac{1}{2}\phi\, H}\, e^{\mathcal{A}^1_{(1)}\, W_1}\, e^{A_{(2)1}\, V^1}\, e^{A_{(3)}\, V}\, e^{\tilde{A}_{(5)}\, \widetilde{V}}\, e^{\tilde{A}^1_{(6)}\, \widetilde{V}_1}\, e^{\widetilde{A}_{(7)1}\, \widetilde{W}^1}\, e^{\frac{1}{2}\psi\, \tilde{H}}\ . \tag{3.12}$$

Here, the generators H, W_1, V^1, V, $\widetilde{V}$, $\widetilde{V}_1$, $\widetilde{W}^1$ and $\widetilde{H}$ satisfy the following super Lie algebra [1]

$$\begin{aligned}
[H,W_1] &= \tfrac{3}{2}W_1\ , \qquad [H,V^1]=-V^1\ , \qquad [H,V]=\tfrac{1}{2}V\ ,\\
[H,\widetilde{W}_1] &= -\tfrac{3}{2}\widetilde{W}_1\ , \qquad [H,\widetilde{V}^1]=\widetilde{V}^1\ , \qquad [H,\widetilde{V}]=-\tfrac{1}{2}\widetilde{V}\ ,\\
[W_1,V^1] &= -V\ , \qquad \{W_1,\widetilde{V}\}=-\widetilde{V}_1\ , \qquad [V^1,V]=-\widetilde{V}\ ,\\
[V^1,\widetilde{V}] &= -\widetilde{W}^1\ , \qquad \{V,V\}=-\widetilde{V}_1\ , \qquad \{W_1,\widetilde{W}^1\}=-\tfrac{3}{8}\widetilde{H}\ ,\\
[V^1,\widetilde{V}_1] &= -\tfrac{1}{4}\widetilde{H}\ , \qquad \{V,\widetilde{V}\}=-\tfrac{1}{8}\widetilde{H}\ ,
\end{aligned} \tag{3.13}$$

with all other commutators vanishing. The equations of motion are then given by $*\mathcal{G}=\mathcal{S}\,\mathcal{G}$, where $*$ is the Hodge dual and $\mathcal{G}=d\mathcal{V}\,\mathcal{V}^{-1}$. The operator $\mathcal{S}$ is an involution (or, according to circumstance, a pseudo-involution) that exchanges each generators for a field with that of its partners under the doubling [1].

It is now a rather straightforward procedure to read off a variety of relations among all the p-brane tensions in the type IIA theory. For instance, the brackets involving W_1 or $\widetilde{W}^1$, associated with the Kaluza-Klein vector and its dual, give rise to the following identities

$$T_{(7)} = \frac{1}{2\pi}\,T_{(2)}\,T_{(5)}\ ,\quad T_{(6)} = \frac{1}{2\pi}\,T_{(1)}\,T_{(5)}\ ,\quad T_{(3)} = \frac{1}{2\pi}\,T_{(1)}\,T_{(2)}\ . \tag{3.14}$$

The brackets involving only V's and $\widetilde{V}$'s, associated with the fields coming from the dimensional reduction of $A_{(3)}$ and its dual in $D=11$, give rise to

$$T_{(6)} = \frac{1}{2\pi}\,T_{(3)}^2\ ,\qquad T_{(5)} = \frac{1}{2\pi}\,T_{(2)}\,T_{(3)}\ . \tag{3.15}$$

Note that the first relation in (3.15) is the same as the one found already in $D = 11$, and hence it can be viewed as a vertical dimensional reduction of the result in $D = 11$. The second relation in (3.15) can be viewed as a double-dimensional reduction from (3.10) in $D = 11$. The relations in (3.14) involve the tensions of the D0-brane and D6-brane, which are associated with the Kaluza-Klein vector, and hence they are not related to (3.10) by dimensional reduction. Note that there is a conservation rule for the subscripts that denote the word-volume dimensions of the various branes appearing in the tension relations.

It is worth mentioning that all the tensions in this paper are measured using the p-branes' own metrics, *e.g.* string tension is measured in the string metric; membrane tension is measured in the membrane metric, *etc.*. In such metrics, the tensions are independent of the moduli. One can of course also discuss the tensions in a given fixed metric. In that case, the tensions would in general depend on the moduli, since the metrics are related by modulus-dependent Weyl transformations. It is straightforward to generalise to these cases, following from the fact that if we have an algebra $[X, Y\} = Z$, then we have a dilaton summation rule that the dilaton vector coupled to the field associated with the generator Z is the sum of the dilaton vectors of the fields associated with X and Y. This dilaton summation rule guarantees the proper dilaton dependence in the brane tension relations in any given metric.

We should draw attention to a subtlety in the use of such algebras for extracting relations among p-brane tensions. In the super Lie algebra, there are generators H and $\widetilde{H}$ associated with the dilaton and its dual ψ. However, there seem to be no BPS objects in the supergravity theories that naturally couple to a dilaton. It follows that commutators involving H and $\widetilde{H}$ do not imply any tension relations. Furthermore, only the non-vanishing commutators of generators associated with gauge potentials (which can include axions) are associated with non-trivial tension relations among the corresponding p-branes.

3.3 Type IIB brane tensions

The doubled formalism for type IIB theory can be constructed by introducing the dual potentials ψ, $\widetilde{\chi}$, $A^i_{(6)}$ for the original fields ϕ, χ, and $A^i_{(2)}$. Note that of the index values $i = 1, 2$, the value $i = 1$ corresponds to NS-NS fields, while $i = 2$ corresponds to R-R fields. Introducing a generator for each potential as before, one can construct a coset representative [1]

$$\mathcal{V} = e^{\frac{1}{2}\phi\, H}\, e^{\chi\, E_+}\, e^{(A^1_{(2)}\, V_+ + A^2_{(2)}\, V_-)}\, e^{B_{(4)}\, U}\, e^{(A^1_{(6)}\, \widetilde{V}_+ + A^2_{(6)}\, \widetilde{V}_-)}\, e^{\widetilde{\chi}\, \widetilde{E}_+}\, e^{\frac{1}{2}\psi\, \widetilde{H}}\ . \tag{3.16}$$

The equations of motion can then be written as $*\mathcal{G} = \mathcal{S}\,\mathcal{G}$, with $\mathcal{G} = d\mathcal{V}\,\mathcal{V}^{-1}$, provided that the generators satisfy the super Lie algebra [1]

$$\begin{aligned}
&[H, E_+] = 2E_+\ , \qquad [H, V_+] = V_+\ , \qquad [H, V_-] = -V_-\ ,\\
&[H, \widetilde{E}_+] = -2\widetilde{E}_+\ , \qquad [H, \widetilde{V}_+] = -\widetilde{V}_+\ , \qquad [H, \widetilde{V}_-] = \widetilde{V}_-\ ,\\
&[E_+, V_-] = V_+\ , \qquad [E_+, \widetilde{V}_+] = -\widetilde{V}_-\ , \qquad [V_+, V_-] = -U\ ,\\
&[V_+, U] = \widetilde{V}_-\ , \qquad [V_-, U] = -\widetilde{V}_+\ , \qquad [V_-, \widetilde{V}_+] = \widetilde{E}_+\ ,\\
&[E_+, \widetilde{E}_+] = \tfrac{1}{2}\widetilde{H}\ , \qquad [V_+, \widetilde{V}_+] = \tfrac{1}{4}\widetilde{H}\ , \qquad [V_-, \widetilde{V}_-] = -\tfrac{1}{4}\widetilde{H}\ .
\end{aligned} \tag{3.17}$$

Thus, this set of algebraic relations again enables us simply to read off the relations among the type IIB p-brane tensions, namely

$$T^{\rm NS}_{(6)} = \frac{1}{2\pi}\, T^{\rm RR}_{(2)}\, T_{(4)}\ , \qquad T^{\rm RR}_{(6)} = \frac{1}{2\pi}\, T^{\rm NS}_{(2)}\, T_{(4)}\ , \qquad T_{(4)} = \frac{1}{2\pi}\, T^{\rm NS}_{(2)}\, T^{\rm RR}_{(2)}\ , \tag{3.18}$$

$$T_{(8)} = \frac{1}{2\pi}\, T^{\rm NS}_{(2)}\, T^{\rm RR}_{(6)}\ , \qquad T^{\rm RR}_{(2)} = \frac{1}{2\pi}\, T_{(0)}\, T^{\rm NS}_{(2)}\ , \qquad T^{\rm NS}_{(6)} = \frac{1}{2\pi}\, T_{(0)}\, T^{\rm RR}_{(6)}\ . \tag{3.19}$$

Note that the tension relation (3.18) is $SL(2,{\rm I\!R})$ covariant, whilst (3.19) is not. This is understandable, since the higher-degree gauge potentials form linear representations under $SL(2,{\rm I\!R})$, and hence so do their associated tensions. The tensions $T_{(0)}$ and $T_{(8)}$ are associated with the axion and its dual, which do not transform linearly under $SL(2,{\rm I\!R})$, and hence (3.19) is not $SL(2,{\rm I\!R})$ covariant.

4 Lower-dimensional brane tensions

In the previous sections, we have showed that the brane tension relations in M-theory or in the type II theories can be derived from the non-commutativity of the gauge transformations in the corresponding supergravities. In particular, they can be read off directly from the super Lie algebras of the associated doubled formalisms constructed in [1]. The super Lie algebras for all lower-dimensions maximal massless supergravities were also obtained in [1], and from these it is straightforward to read off the complete set of brane tensions in all the toroidally-reduced theories.

4.1 The reduction rule and the brane-tension "jade rule"

We begin with a brief review of the super Lie algebra of the lower dimensional maximal massless supergravities. These can be obtained by dimensional reduction from $D = 11$ supergravity or type IIB supergravity. In the bosonic sector, in additional to the metric, the theory contains the dilatons $\vec{\phi}$ and the gauge potentials $\mathcal{A}^i_{(0)j}$, $\mathcal{A}^i_{(1)}$, $A_{(0)ijk}$, $A_{(1)ij}$, $A_{(2)i}$ and $A_{(3)}$. In the doubled formalism, a dual field is introduced for each field (except for the metric), giving $\vec{\psi}$, $\widetilde{\mathcal{A}}^i_{(D-2)j}$, $\widetilde{\mathcal{A}}_{(D-3)i}$, $\widetilde{A}^{ijk}_{(D-2)}$, $\widetilde{A}^{ij}_{(D-3)}$, $\widetilde{A}^i_{(D-4)}$ and $\widetilde{A}_{(D-5)}$. The associated generators for all these fields are given by $\vec{H}$, $E_i{}^j$, W_i, V^{ijk}, V^{ij}, V^i and V for the original fields, and $\widetilde{E}^i{}_j$, $\widetilde{W}^i$, $\widetilde{V}_{ijk}$, $\widetilde{V}_{ij}$, $\widetilde{V}_i$ and $\widetilde{V}$ for the doubled fields.

The generators form a deformed cotangent super Lie algebra. To be precise, let use U^a to denote the set of generators of $\{\vec{H}, E_i{}^j, W_i\}$, that is associated with the fields coming from the dimensional reduction of the metric, and $U^{\bar{a}}$ to denote the generators of $\{V^{ijk}.V^{ij}, V^i, V\}$ that are associated with the fields coming from the dimensional reduction of the three-form potential in $D = 11$. Then the superalgebra has the following form [1]

$$[U^a, U^b\} = f^{ab}{}_c\, U^c\ , \qquad [U^a, U^{\bar{b}}\} = f^{a\bar{b}}{}_{\bar{c}}\, U^{\bar{c}}\ , \qquad [U^{\bar{a}}, U^{\bar{b}}\} = g^{\bar{a}\bar{b}\bar{c}}\, \widetilde{U}_{\bar{c}}\ ,$$
$$[U^a, \widetilde{U}_b\} = f^{ca}{}_b\, \widetilde{U}_c\ , \qquad [U^a, \widetilde{U}_{\bar{b}}\} = f^{\bar{c}a}{}_{\bar{b}}\, \widetilde{U}_{\bar{c}}\ , \qquad [U^{\bar{a}}, \widetilde{U}_{\bar{b}}\} = f^{c\bar{a}}{}_{\bar{b}}\, \widetilde{U}_c\ . \qquad (4.1)$$

This algebra satisfies the so-called "jade rule", which states that if we have untilded generators X, Y and Z where $[X, Y\} = Z$, then it follows that we will necessarily also have $[X, \widetilde{Z}\} = (-1)^{XY+1}\, \widetilde{Y}$ [1]. This implies that once the structure constants in the first line in (4.1) are given, the structure constants for the second line can be deduced from the jade rule. Thus it is only necessary for us to present the commutation relations for U^a and $U^{\bar{a}}$, which are given by

$$[E_i{}^j, E_k{}^\ell] = \delta^j_k\, E_i{}^\ell - \delta^\ell_i\, E_k{}^j\ , \qquad [E_i{}^j, E^{k\ell m}] = -3\delta_i^{[k}\, E^{\ell m]j}\ ,$$
$$[E_i{}^j, V^k] = -\delta_i^k\, V^j\ , \qquad [E_i{}^j, V^{k\ell}] = 2\delta_i^{[k}\, V^{\ell]j}\ , \qquad [E_i{}^j, W_k] = \delta^j_k\, W_i\ ,$$
$$[W_i, E^{jk\ell}] = -3\delta_i^{[j}\, V^{k\ell]}\ , \qquad \{W_i, V^{jk}\} = -2\delta_i^{[j}\, V^{k]}\ , \qquad [W_i, V^j] = -\delta^j_i\, V\ ,$$
$$[V^{\bar{a}}, V^{\bar{b}}\} = -(-1)^{[\bar{b}]}\, \epsilon^{\bar{c}\bar{a}\bar{b}}\, \widetilde{V}_{\bar{c}}\ , \qquad (4.2)$$

together with $[\vec{H}, X] = \vec{\mu}\, X$ where $\vec{\mu}$ is the dilaton vector for any generator X. Note that here we use generic indices $\bar{a}, \bar{b}, \ldots$ to represent antisymmetrised sets of $i, j, \ldots$ indices. The symbol $[\bar{a}]$ denotes the number of such $i, j, \ldots$ indices. Appropriate $1/[\bar{a}]!$ combinatoric factors are understood in summations over repeated generic indices. It is easy to see from (4.2) that the algebra for the generators $\{\vec{H}, E_i{}^j, W_i\}$ is $G = SL_+(11 - D|1)$, and the generators $\{V, V^i, V^{ij}, V^{ijk}\}$ form representations under G.

The jade rule for the algebra (4.1) has the consequence that if we have a tension relation

$$T_{(n+m)} = \frac{1}{2\pi}\, T_{(n)}\, T_{(m)} \qquad (4.3)$$

then we must also have two further tension relations

$$T_{(D-2-n)} = \frac{1}{2\pi}\, T_{(D-2-n-m)}\, T_{(m)} \qquad \text{and} \qquad T_{(D-2-m)} = \frac{1}{2\pi}\, T_{(D-2-n-m)}\, T_{(n)} \ . \tag{4.4}$$

For example, the M-brane tension relation (3.10) is invariant under this jade rule. The full set of brane tension relations of the type IIA theory given in (3.14) and (3.15) can be obtained from applying the jade rule on the first equations in (3.14) and (3.15) respectively. The same story goes for the type IIB case, with the complete set of tension relations given in (3.18) and (3.19).

It follows from the above discussion that the complete set of brane-tension relations in lower dimensions is given by

$$\begin{aligned}
&T_{(0)i}{}^{j} = \frac{1}{2\pi}\, T_{(0)i}{}^{k}\, T_{(0)k}{}^{j}\ , \qquad T_{(0)}^{ijk} = \frac{1}{2\pi}\, T_{(0)\ell}{}^{i}\, T_{(0)}^{\ell jk}\ ,\\
&T_{(2)}^{i} = \frac{1}{2\pi}\, T_{(0)j}{}^{i}\, T_{(2)}^{j}\ , \qquad T_{(1)}^{ij} = \frac{1}{2\pi}\, T_{(0)k}{}^{i}\, T_{(1)}^{jk}\ , \qquad T_{(1)i} = \frac{1}{2\pi}\, T_{(0)i}{}^{j}\, T_{(1)j}\ ,\\
&T_{(1)}^{ij} = \frac{1}{2\pi}\, T_{(1)k}\, T_{(0)}^{ijk}\ , \qquad T_{(2)}^{i} = \frac{1}{2\pi}\, T_{(1)j}\, T_{(1)}^{ij}\ , \quad T_{(3)} = \frac{1}{2\pi}\, T_{(1)i}\, T_{(2)}^{i}\ ,\\
&T_{(D-2-[\bar a])}^{\bar a} = \frac{1}{2\pi}\, T_{([\bar b])}^{\bar b}\, T_{([\bar c])}^{\bar c}\ ,
\end{aligned} \tag{4.5}$$

together with those which can be derived from the jade rule. Here, we are using a self-explanatory notation for labelling the brane tensions that parallels the index labelling on the corresponding gauge potentials listed previously. In the last equation in (4.5), it is only when $\{\bar a, \bar b, \bar c\}$ collectively saturate the range of the internal indices without repetition that there is a non-trivial relation between the associated tensions. There is no sum over the repeated indices in (4.5); rather it meant that the relation holds for different values of the repeated indices.

Having obtained the complete set of the brane-tension relations in lower dimensions, it is of interest to see how they are related by dimensional reduction. If in $D+1$ dimensions there is a tension relation given (4.3), then in D dimensions, there exist relations

$$T_{(n+m)} = \frac{1}{2\pi}\, T_{(n)}\, T_{(m)}\ , \qquad T_{(n+m-1)} = \frac{1}{2\pi}\, T_{(n-1)}\, T_{(m)}\ , \qquad T_{(n+m-1)} = \frac{1}{2\pi}\, T_{(n)}\, T_{(m-1)}\ . \tag{4.6}$$

The first relation can be viewed as coming from vertical dimensional reduction, whilst the second and third come from diagonal reduction. Of course, additional brane tensions emerge from the introduction of a new Kaluza-Klein vector, associated with the generator W_i, whose algebra is given in (4.2).

4.2 IIA/IIB T-duality

The standard dimensional reduction of the type IIA and type IIB supergravities on a circle gives rise to two $D = 9$ supergravities which are identical, modulo field redefinitions. The identification of the type IIA/IIB gauge potentials leads to an identification of their associated electric and magnetic brane tensions. It is straightforward then to see that the $D = 9$ brane tensions relations are the same in the two theories obtained from standard dimensional reduction of the IIA or IIB theories. In this scheme, the vertical dimensional reduction of the brane-tension relation between the 7-brane, the NS-NS string and R-R 5-brane would lead to the $D = 9$ relation

$$T_{(8)} = \frac{1}{2\pi} T^{\rm NS}_{(2)} T^{\rm RR}_{(6)} \,. \tag{4.7}$$

However, this could not actually arise within the framework of a standard Kaluza-Klein reduction, since there is no seven-brane in $D = 9$ *massless* supergravity. It is, however, nevertheless consistent to perform instead a generalised Scherk-Schwarz dimensional reduction, which gives rise to a massive supergravity in $D = 9$ [12], within which the above brane-tension relation does hold. Applying T-duality and oxidising back to $D = 10$, one is led to expect that there should be a brane-tension relation

$$T_{(9)} = \frac{1}{2\pi} T_{(2)} T_{(7)} \tag{4.8}$$

in ten dimensions. There is no eight-brane in massless type IIA supergravity, but there is such a solution in massive type IIA supergravity. In the next sections, we shall show that the brane tension relation (4.8) does indeed hold within the framework of the massive type II theory.

5 Massive IIA supergravity

5.1 Doubled formalism for massive IIA supergravity

As originally formulated, the massive $N = 2$ supergravity in ten dimensions involved a fixed mass parameter m. After a transformation of variables, given in [12], its bosonic sector can be described by the Lagrangian[2]

$$\mathcal{L} \;=\; R\,{*\mathbb{1}} - \tfrac{1}{2}{*d\phi}\wedge d\phi - \tfrac{1}{2}e^{\frac{3}{2}\phi}\,{*F_{(2)}}\wedge F_{(2)} - \tfrac{1}{2}e^{-\phi}\,{*F_{(3)}}\wedge F_{(3)} - \tfrac{1}{2}e^{\frac{1}{2}\phi}\,{*F_{(4)}}\wedge F_{(4)}$$

[2]Our notation and conventions are different from those used in [11]; here we use a convenient notation, using differential forms. The Lagrangian is written as a 10-form. When there is no ambiguity, we often omit the wedge-product symbol between differential forms in a product, for example writing $A \wedge B$ as $A\,B$, and $A \wedge A$ as $(A)^2$, *etc.*

$$-\tfrac{1}{2}dA_{(3)}\wedge dA_{(3)}\wedge A_{(2)}-\tfrac{1}{6}m\,dA_{(3)}\wedge(A_{(2)})^3-\tfrac{1}{40}m^2\,(A_{(2)})^5-\tfrac{1}{2}m^2\,e^{\frac{5}{2}\phi}{*\rlap{1}\mkern2mu 1}\,,\quad(5.1)$$

where the field strengths are given in terms of potentials by

$$\begin{aligned}F_{(2)}&=dA_{(1)}+m\,A_{(2)}\,,\qquad F_{(3)}=dA_{(2)}\,,\\F_{(4)}&=dA_{(3)}+A_{(1)}\wedge dA_{(2)}+\tfrac{1}{2}m\,A_{(2)}\wedge A_{(2)}\,.\end{aligned}\qquad(5.2)$$

The formulation where m is a constant is an inappropriate one in the context of string theory, where one wishes to describe sets of D8-branes that can carry different values of the "charge" m. One can easily reformulate the Lagrangian (5.1) so that m is treated as a spacetime-independent field, subject to the Bianchi identity $dm=0$. This Bianchi identity can be enforced by adding a Lagrange multiplier term $\mathcal{L}_{LM}=m\,dA_{(9)}$ to (5.1).

In this section, we shall reformulate the massive IIA theory in a "doubled formalism," following the same ideas and procedures as those developed in [1], where they were applied to the usual massless theories of $D=11$ supergravity, type IIB supergravity, and their toroidal dimensional reductions. The philosophy of the doubled formalism is essentially to recast the system of second-order differential equations of motion for the original potentials of the theory into a first-order form, by introducing a dual potential for every original one. The ostensible doubling of the physical degrees of freedom that would result from this is removed by the imposition of algebraic constraints that equate the new "doubled" set of field strengths to the duals of the original field strengths. In fact, these constraint equations actually encode the original system of field equations.

The strategy used in [1] for constructing the doubled systems was first to obtain the system of field equations from the original Lagrangian describing the theory, and then to show by a systematic procedure that each equation could be reformulated in a first-order form, by introducing an appropriate dual potential. In our present massive IIA example, we begin by considering the equation of motion for the 3-form potential $A_{(3)}$ that follows from (5.1), namely

$$d(e^{\frac{1}{2}\phi}\,{*F_{(4)}})+dA_{(2)}\,dA_{(3)}+\tfrac{1}{2}m\,A_{(2)}\,A_{(2)}\,dA_{(2)}=0\,.\qquad(5.3)$$

We note that an overall exterior derivative can be extracted from this equation, so that we may write it as $d[e^{\frac{1}{2}\phi}\,{*F_{(4)}}+A_{(2)}\,dA_{(3)}+\frac{1}{6}m\,(A_{(2)})^3]=0$. This allows us to re-express the equation of motion as

$$e^{\frac{1}{2}\phi}\,{*F_{(4)}}\equiv F_{(6)}=dA_{(5)}-A_{(2)}\,dA_{(3)}-\tfrac{1}{6}m\,(A_{(2)})^3\,,\qquad(5.4)$$

where the 5-form potential $A_{(5)}$ dual to $A_{(3)}$ has now been introduced. Next, we consider the equation of motion for $A_{(1)}$, which is $d(e^{\frac{3}{2}\phi}\,{*F_{(2)}})+e^{\frac{1}{2}\phi}\,{*F_{(4)}}\,dA_{(2)}=0$. Substituting the previously-derived result (5.4) into this, we can then remove the derivative from this equation, introducing a new doubled potential $A_{(7)}$. Continuing this process, we can rewrite the entire theory in a first-order form by introducing an additional double potential for each of the original fields (including the dilaton, but excluding the metric itself). Summarising the results, we obtain the following first-order system:

$$
\begin{aligned}
e^{\frac{1}{2}\phi} {*F_{(4)}} &\equiv F_{(6)} = dA_{(5)} - A_{(2)}\, dA_{(3)} - \tfrac{1}{6} m\, (A_{(2)})^3 \,, \\
e^{\frac{3}{2}\phi} {*F_{(2)}} &\equiv F_{(8)} = dA_{(7)} - A_{(2)}\, dA_{(5)} + \tfrac{1}{2} A_{(2)}\, A_{(2)} dA_{(3)} + \tfrac{1}{24} m\, (A_{(2)})^4 \,, \qquad (5.5) \\
e^{-\phi} {*F_{(3)}} &\equiv F_{(7)} = dA_{(6)} - (dA_{(5)} - A_{(2)}\, dA_{(3)} - \tfrac{1}{6} m\, (A_{(2)})^3\,)\, \mathcal{A}_{(1)} \\
&\quad -\tfrac{1}{2} A_{(3)}\, dA_{(3)} - m\, A_{(7)} \,, \\
{*d\phi} &\equiv F^{\phi}_{(9)} = dA^{\phi}_{(8)} - \tfrac{5}{4} m\, A_{(9)} + \tfrac{1}{2} m\, A_{(2)}\, A_{(7)} - \tfrac{3}{4} A_{(1)}\, dA_{(7)} - \tfrac{1}{2} A_{(2)}\, dA_{(6)} \\
&\quad -\tfrac{1}{4} A_{(3)}\, dA_{(5)} + \tfrac{3}{4} A_{(1)}\, A_{(2)}\, dA_{(5)} + \tfrac{1}{4} A_{(2)}\, A_{(3)}\, dA_{(3)} \\
&\quad -\tfrac{3}{8} A_{(1)}\, A_{(2)}\, A_{(2)}\, dA_{(3)} - \tfrac{1}{32} m\, A_{(1)}\, (A_{(2)})^4 \,, \\
m\, e^{\frac{5}{2}\phi} {*\mathbb{1}} &\equiv F_{(10)} = dA_{(9)} - A_{(2)}\, dA_{(7)} + \tfrac{1}{2} A_{(2)}\, A_{(2)}\, dA_{(5)} \\
&\quad -\tfrac{1}{6} (A_{(2)})^3\, dA_{(3)} - \tfrac{1}{120} m\, (A_{(2)})^5 \,.
\end{aligned}
$$

In deriving the last equation, we have treated m as a spacetime-dependent field, and derived its "equation of motion" by varying the Lagrangian with respect to m.

For future reference, we note that among the gauge symmetries of the double theory is one with a 1-form gauge parameter $\lambda_{(1)}$, under which the various potentials transform as follows:

$$
\begin{aligned}
&\delta A_{(1)} = -m\, \lambda_{(1)} \,, \qquad \delta A_{(2)} = d\lambda_{(1)} \,, \qquad \delta A_{(3)} = -m\, \lambda_{(1)}\, A_{(2)} \,, \\
&\delta A_{(5)} = \lambda_{(1)}\, dA_{(3)} - \tfrac{1}{2} m\, A_{(2)}\, A_{(2)}\, \lambda_{(1)} \,, \qquad \delta A_{(6)} = \tfrac{1}{2} m\, A_{(2)}\, A_{(3)}\, \lambda_{(1)} \,, \qquad (5.6) \\
&\delta A_{(7)} = \lambda_{(1)}\, dA_{(5)} - \tfrac{1}{6} m\, (A_{(2)})^3\, \lambda_{(1)} \,, \qquad \delta A_{(9)} = \lambda_{(1)}\, dA_{(7)} - \tfrac{1}{24} m\, (A_{(2)})^4\, \lambda_{(1)} \,.
\end{aligned}
$$

If we had been treating m as a constant parameter in the Lagrangian, this transformation would have had the interpretation of describing a Stückelberg symmetry, which would allow the field $A_{(1)}$ to be set to zero, reflecting the fact that this field is eaten by $A_{(2)}$ when it becomes massive. However since we have changed to a viewpoint in which m is a field in the theory, we can no longer interpret $\delta A_{(1)} = -m\, \lambda_{(1)}$ as the inhomogeneous term in a Stückelberg symmetry; rather, it is just one of many terms in the generally non-linear full set of transformations (5.6).

In fact, it is worth remarking that the treatment of the field m can be put on a more equal footing with the other fields if we adopt the formal device of regarding the 0-form field strength m as arising from the exterior derivative of a (-1)-form:

$$m = dA_{(-1)}\ . \tag{5.7}$$

Having done this, a sequence of transformations under the other gauge symmetries of the theory allow us to move the exterior derivatives off the $A_{(-1)}$ potentials in (5.6), and instead onto the gauge parameters $\lambda_{(1)}$. Having done so, we can then replace the exact 2-form $d\lambda_{(1)}$ by the closed 2-form $\Lambda_{(2)}$, putting the gauge transformation of the 2-form potential $A_{(2)}$ on a par with the way we have described the gauge transformations for all the other potentials in the doubled formalism. We find that the full set of gauge transformations then takes the form

$$\begin{aligned}
&\delta A_{(-1)} = \Lambda_{(-1)}\ , \qquad \delta A_{(1)} = \Lambda_{(1)} - \Lambda_{(2)}\, A_{(-1)}\ , \qquad \delta A_{(2)} = \Lambda_{(2)}\ ,\\
&\delta A_{(3)} = \Lambda_{(3)} - \Lambda_{(2)}\, A_{(-1)}\, A_{(2)} + \Lambda_{(1)}\, A_{(2)}\ ,\\
&\delta A_{(5)} = \Lambda_{(5)} + \tfrac{1}{2}\Lambda_{(1)}\, (A_{(2)})^2 + \Lambda_{(2)}\, A_{(3)} - \tfrac{1}{2}\Lambda_{(2)}\, A_{(-1)}\, (A_{(2)})^2\ ,\\
&\delta A_{(6)} = \Lambda_{(6)} - \Lambda_{(1)}\, A_{(5)} + \tfrac{1}{2}\Lambda_{(1)}\, A_{(2)}\, A_{(3)} + \Lambda_{(2)}\, A_{(-1)}\, A_{(5)}\\
&\qquad\quad -\tfrac{1}{2}\Lambda_{(2)}\, A_{(-1)}\, A_{(2)}\, A_{(3)} - \tfrac{1}{2}\Lambda_{(3)}\, A_{(3)} - \Lambda_{(7)}\, A_{(-1)}\ ,\\
&\delta A_{(7)} = \Lambda_{(7)} + \tfrac{1}{6}\Lambda_{(1)}\, (A_{(2)})^3 + \Lambda_{(2)}\, A_{(5)} - \tfrac{1}{6}\Lambda_{(2)}\, A_{(-1)}\, (A_{(2)})^2\ ,\\
&\delta A_{(8)} = \Lambda_{(8)} + \tfrac{3}{4}\Lambda_{(1)}\, A_{(7)} - \tfrac{1}{8}\Lambda_{(1)}\, (A_{(2)})^2\, A_{(3)} - \tfrac{1}{2}\Lambda_{(2)}\, A_{(6)} - \tfrac{3}{4}\Lambda_{(2)}\, A_{(-1)}\, A_{(7)}\\
&\qquad\quad +\tfrac{1}{8}\Lambda_{(2)}\, A_{(-1)}\, A_{(3)}\, (A_{(2)})^2 + \tfrac{1}{4}\Lambda_{(3)}\, A_{(5)} + \tfrac{5}{4}\Lambda_{(9)}\, A_{(-1)}\ ,\\
&\delta A_{(9)} = \Lambda_{(9)} + \tfrac{1}{24}\Lambda_{(1)}\, (A_{(2)})^4 + \Lambda_{(2)}\, A_{(7)} - \tfrac{1}{24}\Lambda_{(2)}\, A_{(-1)}\, (A_{(2)})^4\ .
\end{aligned} \tag{5.8}$$

From the definitions of the original field strengths in (5.2), and the doubled fields in (5.5), it is easy to calculate the Bianchi identities for the full set of field strengths. We find

$$\begin{aligned}
&dF_{(2)} = m\, F_{(3)}\ , \qquad dF_{(3)} = 0\ , \qquad dF_{(4)} = F_{(2)}\wedge F_{(3)}\ ,\\
&dF_{(6)} = -F_{(3)}\wedge F_{(4)}\ , \qquad dF_{(8)} = -F_{(3)}\wedge F_{(6)}\ ,\\
&dF_{(7)} = -\tfrac{1}{2}F_{(4)}\wedge F_{(4)} - m\, F_{(8)} - F_{(2)}\wedge F_{(6)}\ ,\\
&dF^{\phi}_{(9)} = \tfrac{5}{4}\, m\, F_{(10)} - \tfrac{3}{4}F_{(2)}\wedge F_{(8)} - \tfrac{1}{2}F_{(3)}\wedge F_{(7)} - \tfrac{1}{4}F_{(4)}\wedge F_{(6)}\ ,\\
&dF_{(10)} = 0\ , \qquad d\, m = 0\ .
\end{aligned} \tag{5.9}$$

Note that these are all bilinear relations (m is viewed a 0-form field strength here). It is interesting to note that although $F_{(10)}$ is by definition a closed 10-form, since we are in ten dimensions, we could nevertheless choose to consider the system of field strengths in (5.2) and (5.5) as being defined in some arbitrary dimension $D > 10$. In this case, we can simply calculate $dF_{(10)}$ from the definition of $F_{(10)}$ in (5.5), finding

$$dF_{(10)} = -F_{(3)} \wedge F_{(8)} \ . \tag{5.10}$$

Thus even though there was no *a priori* reason for it to do so, the field $F_{(10)}$ satisfies a bilinear Bianchi identity in $D > 10$.

From the Bianchi identities, it is a simple matter to read off the commutation relations for the generators associated with the various fields. To do this, we first, as in [1], define the generalised field strength obtained by summing over products of all field strengths multiplied by their associated generators:

$$\begin{aligned} \mathcal{G} &= \tfrac{1}{2} d\phi\, H + m\, e^{\frac{5}{4}\phi}\, Y + e^{\frac{3}{4}\phi}\, F_{(2)}\, W_1 + e^{-\frac{1}{2}\phi}\, F_{(3)}\, V^1 + e^{\frac{1}{4}\phi}\, F_{(4)}\, V \\ &\quad + e^{-\frac{1}{4}\phi}\, F_{(6)}\, \widetilde V + e^{\frac{1}{2}\phi}\, F_{(7)}\, \widetilde V_1 + e^{-\frac{3}{4}\phi}\, F_{(8)}\, \widetilde W^1 + e^{-\frac{5}{4}\phi}\, F_{(10)}\, \widetilde Y + \tfrac{1}{2} F^\phi_{(9)}\, \widetilde H \ . \end{aligned} \tag{5.11}$$

Here, in addition to the generators already introduced for the usual type IIA theory in [1], we have the generators Y and $\widetilde Y$ associated with the 0-form and 10-form fields m and $F_{(10)}$ respectively. Note that these are both fermionic in nature, since the associated potentials are odd-degree forms.

It was shown in [1] that the equations of motion can be derived by requiring that the generalised field strength $\mathcal{G}$ satisfy the Cartan-Maurer equation

$$d\mathcal{G} = \mathcal{G} \wedge \mathcal{G} \ . \tag{5.12}$$

This requirement then gives a determination of the commutation relations for the various generators. Thus we find by comparing with (5.9) that the non-vanishing commutators are precisely those found in [1] for the usual massless type IIA theory, and presented here in (3.13), together with some additional ones resulting from the inclusion of the additional fields m and $F_{(10)}$. We find that these extra commutators are

$$\begin{aligned} [V^1, Y] &= W_1 \ , \qquad \{\widetilde W^1, Y\} = -\widetilde V_1 \ , \\ [H, Y] &= -\tfrac{5}{2}\, Y \ , \qquad \{Y, \widetilde Y\} = -\tfrac{5}{8}\, \widetilde H \ . \end{aligned} \tag{5.13}$$

Note that if we consider the theory in a dimension $D > 10$, so that there is the additional non-trivial Bianchi identity for $dF_{(10)}$, in (5.10), we obtain one further non-vanishing commutator, namely

$$[V^1, \widetilde W^1] = -\widetilde Y \ . \tag{5.14}$$

Actually, there are also direct ways of deriving this commutator that do not require the use of the "dimensionally-extended" Bianchi identity (5.10). For example, one can read it off from the gauge transformations given in (5.8). Alternatively, one can read it off from the fact that the exponentiation $\mathcal{V}$ of the superalgebra generators in this theory, analogous to (3.16), must give rise to the generalised field strength $\mathcal{G} = d\mathcal{V}\,\mathcal{V}^{-1}$ given in (5.11). In general, all three procedures give identical conclusions about the commutation relations in the superalgebra [1], but the method where one reads them off from the Bianchi identities degenerates in the case of a field strength of degree D in D dimensions unless one "dimensionally extends" the spacetime to $D+1$ dimensions.

5.2 Massive type IIA brane tensions

The additional non-trivial commutators in the massive type IIA supergravity imply additional brane-tension relations, over and above those arising in the massless IIA theory. In particular, the commutator (5.14) implies that

$$T_{(9)} = \frac{1}{2\pi}\,T_{(2)}\,T_{(7)}\ . \tag{5.15}$$

The existence of this relation is essential for type IIA/IIB T-duality, as explained in section 4.2. It should be emphasised again that the above relation is derived within the framework of the massive type IIA supergravity itself, without needing to invoke type IIA/IIB T-duality. The first line of (5.13) can be understood by applying the jade rule to (5.14). Acting with the jade rule on the brane tension (5.15), we would obtain two more relations that involve the brane tension of a (-2)-brane. It is not clear whether such BPS states exist.

Acknowledgments

We are grateful to R. Dijkgraaf and G. Papadopoulos for useful discussions. H.L. and K.S.S. would like to thank Texas A&M University for hospitality.

Appendices

A Massive IIA/IIB T-duality

A.1 Reduction of massive IIA to $D=9$

In order to find the T-duality transformation between the type IIB and the massive type IIA theories, it is necessary to reduce each of them to $D=9$. For the type IIB theory the reduction will be of of the generalised Scherk-Schwarz type, whereas for the massive type IIA, it will be a standard Kaluza-Klein reduction on a circle. In this appendix, we now perform the reduction of the massive IIA theory. We shall be interested in obtaining the full doubled system in $D=9$. For the most part, this can be done by dimensionally reducing the already-doubled system in $D=10$. However, since we are unable to double gravity itself, it follows that a reduction of the doubled $D=10$ system will not of itself generate the doubled fields for the Kaluza-Klein vector and the new Kaluza-Klein dilaton, which come from the ten-dimensional metric upon dimensional reduction. Thus for these fields, it is necessary to perform a doubling after having reduced the ten-dimensional theory to $D=9$. For all other fields, however, one can easily check that the nine-dimensional theory obtained by doubling in $D=10$ and then reducing to $D=9$ is the same as the one obtained by reducing the original theory from $D=10$ to $D=9$ and then doubling in the lower dimension. Here, since the algebra is somewhat lengthy, we shall just present our results for the full set of nine-dimensional fields. Since the doubling of the Kaluza-Klein fields must be performed in $D=9$, it is useful first to present the nine-dimensional Lagrangian:

$$\begin{aligned}
\mathcal{L}_9 \;\; = \;\; & R\,{*\mathbb{1}} - \tfrac12 {*d\phi}\wedge d\phi - \tfrac12 {*d\varphi}\wedge d\varphi - \tfrac12 e^{\frac32\phi+2\alpha\varphi}\,{*F_{(2)}}\wedge F_{(2)} \\
& -\tfrac12 e^{\frac32\phi-14\alpha\varphi}\,{*F_{(1)1}}\wedge F_{(1)1} - \tfrac12 e^{-\phi+4\alpha\varphi}\,{*F_{(3)}}\wedge F_{(3)} - \tfrac12 e^{-\phi-12\alpha\varphi}\,{*F_{(2)1}}\wedge F_{(2)1} \\
& -\tfrac12 e^{\frac12\phi+6\alpha\varphi}\,{*F_{(4)}}\wedge F_{(4)} - \tfrac12 e^{\frac12\phi-10\alpha\varphi}\,{*F_{(3)1}}\wedge F_{(3)1} - \tfrac12 e^{16\alpha\varphi}\,{*\mathcal{F}_{(2)}}\wedge \mathcal{F}_{(2)} \\
& -\tfrac12 m^2\, e^{\frac52\phi-2\alpha\varphi}\,{*\mathbb{1}} - \tfrac12 A_{(1)1}\, dA_{(3)}\, dA_{(3)} - A_{(2)}\, dA_{(2)1}\, dA_{(3)} \\
& -\tfrac16 m\,(A_{(2)})^3\, dA_{(2)1} - \tfrac12 m\, A_{(1)1}\,(A_{(2)})^2\, dA_{(3)} - \tfrac18 m^2\,(A_{(2)})^4\, A_{(1)1}\;, \qquad \text{(A.1)}
\end{aligned}$$

where we have defined $\alpha = 1/(4\sqrt{7})$.

Now, we present the results for the doubled system. Firstly, we list the "original" field strengths in $D=9$:

$$\begin{aligned}
& F_{(1)1} = dA_{(0)1} + m\,A_{(1)1}\;, \qquad F_{(2)} = dA_{(1)} + m\,A_{(2)} + \mathcal{A}_{(1)}\,F_{(1)1}\;, \\
& F_{(2)1} = dA_{(1)1}\;, \qquad F_{(3)} = dA_{(2)} - \mathcal{A}_{(1)}\,dA_{(1)1}\;, \\
& F_{(3)1} = dA_{(2)1} - A_{(0)1}\,dA_{(2)} + A_{(1)}\,dA_{(1)1} + m\,A_{(1)1}\,A_{(2)}\;, \\
& F_{(4)} = dA_{(3)} + + A_{(1)}\,dA_{(2)} + \tfrac12 m\,(A_{(2)})^2 + \mathcal{A}_{(1)}\,F_{(3)1}\;, \\
& \mathcal{F}_{(2)} = d\mathcal{A}_{(1)}\;, \qquad \text{(A.2)}
\end{aligned}$$

Next, we list the doubled field strengths:

$$\begin{aligned}
F_{(5)1} &= dA_{(4)1} - A_{(1)1}\,dA_{(3)} - A_{(2)}\,dA_{(2)1} - \tfrac12 m\,(A_{(2)})^2\,A_{(1)1}\;, \\
F_{(6)} &= dA_{(5)} - A_{(2)}\,dA_{(3)} - \tfrac16 m\,(A_{(2)})^3 + \mathcal{A}_{(1)}\,F_{(5)1}\;,
\end{aligned}$$

$$
\begin{aligned}
F_{(6)1} &= dA_{(5)1} - A_{(0)1}\, dA_{(5)} - A_{(1)}\, dA_{(4)1} - A_{(2)1}\, dA_{(3)} + A_{(0)1}\, A_{(2)}\, dA_{(3)} + A_{(1)}\, A_{(1)1}\, dA_{(3)} \\
&\quad + A_{(1)}\, A_{(2)}\, dA_{(2)1} - m\, A_{(6)1} + \tfrac{1}{2} m\, A_{(1)}\, A_{(1)1}\, (A_{(2)})^2 + \tfrac{1}{6} m\, A_{(0)1}\, (A_{(2)})^3 \ , \\
F_{(7)} &= dA_{(6)} - \mathcal{A}_{(1)}\, F_{(6)1} - A_{(1)}\, dA_{(5)} - \tfrac{1}{2} A_{(3)}\, dA_{(3)} + A_{(1)}\, A_{(2)}\, dA_{(3)} - m\, A_{(7)} \\
&\quad + \tfrac{1}{6} m\, A_{(1)}\, (A_{(2)})^3 \ , \\
F_{(7)1} &= dA_{(6)1} - A_{(1)1}\, dA_{(5)} - A_{(2)}\, dA_{(4)1} + A_{(1)1}\, A_{(2)}\, dA_{(3)} + \tfrac{1}{2} (A_{(2)})^2\, dA_{(2)1} \\
&\quad + \tfrac{1}{6} m\, A_{(1)1}\, (A_{(2)})^3 \ , \\
F_{(8)} &= dA_{(7)} + \mathcal{A}_{(1)}\, F_{(7)1} - A_{(2)}\, dA_{(5)} + \tfrac{1}{2} (A_{(2)})^2\, dA_{(3)} + \tfrac{1}{24} m\, (A_{(2)})^4 \ , \\
F_{(9)1} &= dA_{(8)} - A_{(1)1}\, dA_{(7)} - A_{(2)}\, dA_{61} + A_{(1)1}\, A_{(2)}\, dA_{(5)} + \tfrac{1}{2} (A_{(2)})^2\, dA_{(4)1} \\
&\quad - \tfrac{1}{2} A_{(1)1}\, (A_{(2)})^2\, dA_{(3)} - \tfrac{1}{6} (A_{(2)})^3\, dA_{(2)1} - \tfrac{1}{24} m\, A_{(1)1}\, (A_{(2)})^4 \ , \\
\mathcal{F}_{(7)} &= d\mathcal{A}_{(6)} - A_{(0)1}\, dA_{(6)1} - A_{(1)1}\, dA_{(5)1} - A_{(2)1}\, dA_{(4)1} + A_{(0)1}\, A_{(1)1}\, dA_{(5)} \\
&\quad + A_{(0)1}\, A_{(2)}\, dA_{(4)1} + A_{(1)1}\, A_{(2)1}\, dA_{(3)} - \tfrac{1}{2} A_{(2)1}\, A_{(3)}\, dA_{(1)1} \\
&\quad - A_{(0)1}\, A_{(1)1}\, A_{(2)}\, dA_{(3)} - \tfrac{1}{2} A_{(0)1}\, (A_{(2)})^2\, dA_{(2)1} + m\, A_{(1)1}\, A_{(6)1} \\
&\quad - \tfrac{1}{6} m\, A_{(0)1}\, A_{(1)1}\, (A_{(2)})^3 \ .
\end{aligned} \tag{A.3}
$$

Note that $\mathcal{F}_{(7)}$ is the doubled field strength corresponding to the dual of Kaluza-Klein field strength $\mathcal{F}_{(2)}$. We are using a notation where a subscript "1" that is not enclosed in parentheses indicates the internal index associated with the $D = 10$ to $D = 9$ reduction step. Finally, the doubled fields associated with the two dilatons ϕ and φ are:

$$
\begin{aligned}
F^{\phi}_{(8)1} &= dA^{\phi}_{(7)1} - \tfrac{3}{4} A_{(0)1}\, dA_{(7)} - \tfrac{3}{4} A_{(1)}\, dA_{(6)1} + \tfrac{1}{2} A_{(1)1}\, dA_{(6)} - \tfrac{1}{2} A_{(2)}\, dA_{(5)1} \\
&\quad - \tfrac{1}{4} A_{(2)1}\, dA_{(5)} - \tfrac{1}{4} A_{(3)}\, dA_{(4)1} + \tfrac{3}{4} A_{(0)1}\, A_{(2)}\, dA_{(5)} + \tfrac{3}{4} A_{(1)}\, A_{(1)1}\, dA_{(5)} \\
&\quad + \tfrac{3}{4} A_{(1)}\, A_{(2)}\, dA_{(4)1} - \tfrac{1}{4} A_{(1)1}\, A_{(3)}\, dA_{(3)} + \tfrac{1}{4} A_{(2)}\, A_{(2)1}\, dA_{(3)} \\
&\quad + \tfrac{1}{4} A_{(2)}\, A_{(3)}\, dA_{(2)1} - \tfrac{3}{8} A_{(0)1}\, (A_{(2)})^2\, dA_{(3)} - \tfrac{3}{4} A_{(1)}\, A_{(1)1}\, A_{(2)}\, dA_{(3)} \\
&\quad - \tfrac{3}{8} A_{(1)}\, (A_{(2)})^2\, dA_{(2)1} - \tfrac{5}{4} m\, A_{(8)} - \tfrac{1}{2} m\, A_{(1)1}\, A_{(7)} + \tfrac{1}{2} m A_{(2)}\, A_{(6)1} \\
&\quad - \tfrac{1}{32} m\, A_{(0)1}\, (A_{(2)})^4 - \tfrac{1}{8} m\, A_{(1)}\, A_{(1)1}\, (A_{(2)})^3 \ ,
\end{aligned}
$$

$$
\begin{aligned}
F^{\varphi}_{(8)} &= dA^{\varphi}_{(7)} - \alpha \Big\{ -8 \mathcal{A}_{(1)}\, d\mathcal{A}_{(6)} + 7 A_{(0)1}\, dA_{(7)} - A_{(1)}\, dA_{(6)1} + 6 A_{(1)1}\, dA_{(6)} \\
&\quad + 2 A_{(2)}\, dA_{(5)1} + 5 dA_{(2)1}\, dA_{(5)} - 3 dA_{(3)}\, dA_{(4)1} - 7 A_{(0)1}\, A_{(2)}\, dA_{(5)} \\
&\quad + 8 A_{(0)1}\, \mathcal{A}_{(1)}\, dA_{(6)1} + A_{(1)}\, A_{(1)1}\, dA_{(5)} + A_{(1)}\, A_{(2)}\, dA_{(4)1} - 3 A_{(1)1}\, A_{(3)}\, dA_{(3)} \\
&\quad - 8 A_{(1)1}\, \mathcal{A}_{(1)}\, dA_{(5)1} - 2 A_{(2)}\, A_{(2)1}\, dA_{(3)} + A_{(2)1}\, A_{(3)}\, dA_{(2)} + 8 A_{(2)1}\, \mathcal{A}_{(1)}\, dA_{(4)1} \\
&\quad + 8 A_{(0)1}\, A_{(1)1}\, \mathcal{A}_{(1)}\, dA_{(5)} + \tfrac{7}{2} A_{(0)1}\, (A_{(2)})^2\, dA_{(3)} - 8 A_{(0)1}\, A_{(2)}\, \mathcal{A}_{(1)}\, dA_{(4)1} \\
&\quad - A_{(1)}\, A_{(1)1}\, A_{(2)}\, dA_{(3)} - \tfrac{1}{2} A_{(1)}\, (A_{(2)})^2\, dA_{(2)1} + 8 A_{(1)1}\, A_{(2)1}\, \mathcal{A}_{(1)}\, dA_{(3)}
\end{aligned}
$$

$$-4A_{(2)1}\, A_{(3)}\, \mathcal{A}_{(1)}\, dA_{(1)1} - 8A_{(0)1}\, A_{(1)1}\, A_{(2)}\, \mathcal{A}_{(1)}\, dA_{(3)} + 4A_{(0)1}\, (A_{(2)})^2\, \mathcal{A}_{(1)}\, dA_{(2)1}$$
$$+m\, A_{(8)} - 6m\, A_{(1)1}\, A_{(7)} - 2mA_{(2)}\, A_{(6)1} + 8m\, A_{(1)1}\, \mathcal{A}_{(1)}\, A_{(6)1}$$
$$+\tfrac{7}{24} m\, A_{(0)1}\, (A_{(2)})^4 - \tfrac{1}{6} m\, A_{(1)}\, A_{(1)1}\, (A_{(2)})^3 + \tfrac{4}{3} m\, A_{(0)1}\, A_{(1)1}\, \mathcal{A}_{(1)}\, (A_{(2)})^3 \Big\} \ , \qquad (A.4)$$

where again $\alpha = 1/(4\sqrt{7})$.

Before presenting the full set of nine-dimensional field equations in the doubled formalism, it is useful to present a general lemma for the dimensional reduction of field strengths and their duals. If we start with a metric in $(D+1)$ dimensions, and perform a reduction on a circle to D dimensions, the metric ansatz will be

$$d\hat{s}^2 = e^{-2\alpha\varphi}\, ds^2 + e^{2(D-2)\alpha\varphi}\, (dz + \mathcal{A}_{(1)})^2 \ , \qquad (A.5)$$

where $\alpha = [2(D-1)(D-2)]^{-1/2}$. The dimensional reduction of an n-form field strength $\hat{F}_{(n)}$ will give

$$\hat{F}_{(n)} = F_{(n)} + F_{(n-1)} \wedge (dz + \mathcal{A}_{(1)}) \ . \qquad (A.6)$$

Denoting the Hodge dual in $(D+1)$ dimensions by $\hat{*}$, and in D dimensions by $*$, it is easy to show that the dimensional reduction of the dual of $\hat{F}_{(n)}$ is given by

$$\hat{*}\hat{F}_{(n)} = (-1)^n\, e^{2(n-1)\alpha\varphi}\, {*F_{(n)}} \wedge (dz + \mathcal{A}_1) + e^{-2(D-n)\alpha\varphi}\, {*F_{(n-1)}} \ . \qquad (A.7)$$

Consequently, we find that the dimensional reduction of the single $(D+1)$-dimensional equation $e^{a\phi}\, \hat{*}\hat{F}_{(n)} = \hat{\widetilde{F}}_{(D+1-n)}$ will in general give rise to the two D-dimensional equations

$$\begin{aligned} e^{a\phi + 2(n-1)\alpha\varphi}\, {*F_{(n)}} &= (-1)^n\, \widetilde{F}_{(D-n)1} \ , \\ e^{a\phi - 2(D-n)\alpha\varphi}\, {*F_{(n-1)1}} &= \widetilde{F}_{(D+1-n)} \ . \end{aligned} \qquad (A.8)$$

Note therefore that although we *defined* the signs of our doubled field strengths in the massive ten-dimensional theory to be such that $e^{a\phi}\, \hat{*}\hat{F}_{(n)} = +\hat{\widetilde{F}}_{(10-n)}$, we are taking the doubled fields in $D=9$ to be precisely those obtained by dimensional reduction of the ten-dimensional doubled fields. Consequently, we will have certain minus signs in the duality equations in the nine-dimensional theory, whenever n is odd, as indicated in the first line in (A.8).

With these preliminaries, we now present the nine-dimensional equations of motion for the reduced massive IIA theory:

$$e^{\frac{1}{2}\phi + 6\alpha\varphi}\, {*F_{(4)}} = F_{(5)1} \ , \qquad e^{\frac{1}{2}\phi - 10\alpha\varphi}\, {*F_{(3)1}} = F_{(6)} \ ,$$
$$e^{-\phi + 4\alpha\varphi}\, {*F_{(3)}} = -F_{(6)1} \ , \qquad e^{-\phi - 12\alpha\varphi}\, {*F_{(2)1}} = F_{(7)} \ ,$$

$$
\begin{aligned}
&e^{\frac{3}{2}\phi+2\alpha\varphi} *F_{(2)} = F_{(7)1} \ , \qquad e^{\frac{3}{2}\phi-14\alpha\varphi} *F_{(1)1} = F_{(8)} \ , \\
&*d\phi = -F^{\phi}_{(8)} \ , \qquad m\, e^{\frac{5}{1}\phi-2\alpha\varphi} *\mathbb{1} = F_{(9)1} \ , \\
&e^{16\alpha\varphi} *\mathcal{F}_{(2)} = \mathcal{F}_{(7)} \ , \qquad *d\varphi = F^{\varphi}_{(8)} \ .
\end{aligned} \tag{A.9}
$$

Note that the two equations in the final line are associated with the doubling of the new fields $\mathcal{A}_{(1)}$ and φ that have emerged from the metric under dimensional reduction. Since these have not descended from any doubled equations in the higher dimension we have simply chosen our definitions of the associated doubled field strengths $\mathcal{F}_{(7)}$ and $F^{\varphi}_{(8)}$ so that there are plus signs in these equations of motion.

A.2 Doubled formalism for type IIB supergravity

The doubled formalism for type IIB supergravity was worked out in detail in [1]. Here, we shall just summarise the results. We do, however, make one change to the formalism, in anticipation of the fact that we shall subsequently be using it for describing Scherk-Schwarz generalised reductions. It is therefore convenient to make appropriate field redefinitions prior to constructing the doubled formalism, such that the axion χ is covered by a derivative everywhere. The Lagrangian describing the bosonic sector of type IIB supergravity may thus be written as

$$
\begin{aligned}
\mathcal{L} \;=\; & R *\mathbb{1} - \tfrac{1}{2} *d\phi\wedge d\phi - \tfrac{1}{2} e^{2\phi} *d\chi\wedge d\chi - \tfrac{1}{2} e^{-\phi} *G^{\rm NS}_{(3)}\wedge G^{\rm NS}_{(3)} - \tfrac{1}{2} e^{\phi} *G^{\rm RR}_{(3)}\wedge G^{\rm RR}_{(3)} \\
& - \tfrac{1}{4} *G_{(5)}\wedge G_{(5)} + \tfrac{1}{2} B_{(4)}\, dB^{\rm NS}_{(2)}\, dB^{\rm RR}_{(2)} + \tfrac{1}{2} B_{(4)}\, dB^{\rm NS}_{(2)}\, dB^{\rm NS}_{(2)}\, d\chi \ ,
\end{aligned} \tag{A.10}
$$

where the various field strengths are defined by

$$
\begin{aligned}
G^{\rm NS}_{(3)} &= dB^{\rm NS}_{(2)} \ , \qquad G^{\rm RR}_{(3)} = dB^{\rm RR}_{(2)} + B^{\rm NS}_{(2)}\, d\chi \ , \\
G_{(5)} &= dB_{(4)} + \tfrac{1}{2} B^{\rm NS}_{(2)}\, dB^{\rm RR}_{(2)} - \tfrac{1}{2} B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(2)} + \tfrac{1}{2} B^{\rm NS}_{(2)}\, B^{\rm NS}_{(2)}\, d\chi \ .
\end{aligned} \tag{A.11}
$$

As described in [13], the self-duality of $G_{(5)}$ is to be imposed here after varying the Lagrangian (A.10) to obtain the equations of motion. This can be done consistently, since the equation of motion for $G_{(5)}$ turns out to be $d{*G_{(5)}} = dB^{\rm NS}_{(2)}\, dB^{\rm RR}_{(2)} + dB^{\rm NS}_{(2)}\, dB^{\rm NS}_{(2)}\, d\chi$, and the right-hand side is identical to the expression for the Bianchi identity for $G_{(5)}$, following from (A.11).

Following steps analogous to those used for the massive IIA theory in the previous section, and described in detail in [1], we can now construct the doubled formalism for the type IIB theory, effectively re-expressing the second-order equations of motion following from (A.10) in first-order form. We find the following:

$$\begin{aligned}
*G_{(5)} &\equiv G_{(5)} = dB_{(4)} + \tfrac{1}{2} B^{\rm NS}_{(2)} \, dB^{\rm RR}_{(2)} - \tfrac{1}{2} B^{\rm RR}_{(2)} \, dB^{\rm NS}_{(2)} + \tfrac{1}{2} B^{\rm NS}_{(2)} \, B^{\rm NS}_{(2)} \, d\chi \, , \\
e^{\phi} *G^{\rm RR}_{(3)} &\equiv G^{\rm RR}_{(7)} = dB^{\rm RR}_{(6)} - B^{\rm NS}_{(2)} \, dB_{(4)} - \tfrac{1}{4} B^{\rm NS}_{(2)} \, B^{\rm NS}_{(2)} \, dB^{\rm RR}_{(2)} - \tfrac{1}{6} B^{\rm NS}_{(2)} \, B^{\rm NS}_{(2)} \, B^{\rm NS}_{(2)} \, d\chi \, , \\
e^{-\phi} *G^{\rm NS}_{(3)} &\equiv G^{\rm NS}_{(7)} = dB^{\rm NS}_{(6)} - B^{\rm RR}_{(6)} \, d\chi + B^{\rm RR}_{(2)} \, dB_{(4)} - \tfrac{1}{4} B^{\rm RR}_{(2)} \, B^{\rm RR}_{(2)} \, dB^{\rm NS}_{(2)} + \tfrac{1}{4} B^{\rm NS}_{(2)} \, B^{\rm NS}_{(2)} \, B^{\rm RR}_{(2)} \, d\chi \, , \\
e^{2\phi} *d\chi &\equiv G_{(9)} = dB_{(8)} - B^{\rm NS}_{(2)} \, dB^{\rm RR}_{(6)} + \tfrac{1}{2} B^{\rm NS}_{(2)} \, B^{\rm NS}_{(2)} \, dB_{(4)} + \tfrac{1}{8} (B^{\rm NS}_{(2)})^3 \, dB^{\rm RR}_{(2)} \\
& \qquad + \tfrac{1}{8} B^{\rm NS}_{(2)} \, B^{\rm NS}_{(2)} \, B^{\rm RR}_{(2)} \, dB^{\rm NS}_{(2)} + \tfrac{1}{24} (B^{\rm NS}_{(2)})^4 \, d\chi \, , \\
*d\phi &\equiv G^{\phi}_{(9)} = dB^{\phi}_{(8)} - B_{(8)} \, d\chi - \tfrac{1}{2} B^{\rm NS}_{(2)} \, dB^{\rm NS}_{(6)} + \tfrac{1}{2} B^{\rm RR}_{(2)} \, dB^{\rm RR}_{(6)} \\
& \qquad + \tfrac{1}{2} B^{\rm NS}_{(2)} \, B^{\rm RR}_{(6)} \, d\chi - \tfrac{1}{2} B^{\rm NS}_{(2)} \, B^{\rm RR}_{(2)} \, dB_{(4)} - \tfrac{1}{12} (B^{\rm NS}_{(2)})^3 \, B^{\rm RR}_{(2)} \, d\chi \, . \qquad \text{(A.12)}
\end{aligned}$$

A.3 Scherk-Schwarz reduction of IIB to $D = 9$

Just as for the massive IIA theory discussed in Appendix A.1, here too we may perform a dimensional reduction to obtain the doubled formalism for type IIB in $D = 9$. This time, in order to make contact with the nine-dimensional massive IIA theory, we must make a generalised Scherk-Schwarz type reduction, where the axion χ in $D = 10$ is reduced according to $\chi(x, z) = \chi(x) + m \, z$. All other fields will be reduced according to the usual z-independent Kaluza-Klein scheme. Again, for all except the new Kaluza-Klein vector, which we denote by $\mathcal{B}_{(1)}$ here, and the new Kaluza-Klein dilaton φ, the doubled fields in $D = 9$ can be obtained simply by dimensionally reducing the doubled fields in $D = 10$. However, to obtain the doubles of the two new Kaluza-Klein fields we need to perform a doubling in $D = 9$. It is therefore useful to begin by presenting the type IIB Lagrangian in $D = 9$:

$$\begin{aligned}
\mathcal{L}_9 = \; & R * 1\!\!1 - \tfrac{1}{2} *d\phi \wedge d\phi - \tfrac{1}{2} *d\varphi \wedge d\varphi - \tfrac{1}{2} e^{2\phi} *G_{(1)} \wedge G_{(1)} - \tfrac{1}{4} e^{-8\alpha\varphi} *G_{(4)} \wedge G_{(4)} \\
& - \tfrac{1}{4} e^{8\alpha\varphi} *G_{(5)} \wedge G_{(5)} - \tfrac{1}{2} e^{-\phi + 4\alpha\varphi} *G^{\rm NS}_{(3)} \wedge G^{\rm NS}_{(3)} - \tfrac{1}{2} e^{-\phi - 12\alpha\varphi} *G^{\rm NS}_{(2)} \wedge G^{\rm NS}_{(2)} \\
& - \tfrac{1}{2} e^{\phi + 4\alpha\varphi} *G^{\rm RR}_{(3)} \wedge G^{\rm RR}_{(3)} - \tfrac{1}{2} e^{\phi - 12\alpha\varphi} *G^{\rm RR}_{(2)} \wedge G^{\rm RR}_{(2)} - \tfrac{1}{2} e^{16\alpha\varphi} *\mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} \qquad \text{(A.13)} \\
& - \tfrac{1}{2} m^2 \, e^{2\phi - 16\alpha\varphi} * 1\!\!1 + \tfrac{1}{2} B_{(3)} \, dB^{\rm NS}_{(2)} \, dB^{\rm RR}_{(2)} - \tfrac{1}{2} B_{(4)} \, dB^{\rm NS}_{(1)} \, dB^{\rm RR}_{(2)} + \tfrac{1}{2} B_{(4)} \, dB^{\rm NS}_{(2)} \, dB^{\rm RR}_{(1)} \\
& + \tfrac{1}{2} (B_{(4)} \, B^{\rm NS}_{(1)} \, dB^{\rm NS}_{(2)} - B_{(4)} \, B^{\rm NS}_{(2)} \, dB^{\rm NS}_{(1)} + B_{(3)} \, B^{\rm NS}_{(2)} \, dB^{\rm NS}_{(2)}) \, d\chi + \tfrac{1}{2} m \, B_{(4)} \, B^{\rm NS}_{(2)} \, dB^{\rm NS}_{(2)} \, .
\end{aligned}$$

This is obtained by performing the Scherk-Schwarz reduction on the ten-dimensional Lagrangian (A.10).

Our results for the doubled system of fields in the massive nine-dimensional type IIB theory are as follows. Firstly, the "original" fields in $D = 9$ are:

$$G_{(1)} = d\chi - m \, \mathcal{B}_{(1)} \, , \qquad G^{\rm RR}_{(2)} = dB^{\rm RR}_{(1)} + m \, B^{\rm NS}_{(2)} - \mathcal{B}_{(1)} \, d\chi \, ,$$

$$
\begin{aligned}
&G^{\rm NS}_{(2)} = dB^{\rm NS}_{(1)}\ ,\qquad G^{\rm NS}_{(3)} = dB^{\rm NS}_{(2)} - \mathcal{B}_{(1)}\, dB^{\rm NS}_{(1)}\ ,\\
&G^{\rm RR}_{(3)} = dB^{\rm RR}_{(2)} + B^{\rm NS}_{(2)}\, d\chi - \mathcal{B}_{(1)}\, dB^{\rm RR}_{(1)} - B^{\rm NS}_{(1)}\, \mathcal{B}_{(1)}\, d\chi - m\, B^{\rm NS}_{(2)}\, \mathcal{B}_{(1)}\ ,\\
&G_{(4)} = dB_{(3)} - \tfrac12 B^{\rm NS}_{(1)}\, dB^{\rm RR}_{(2)} + \tfrac12 B^{\rm NS}_{(2)}\, dB^{\rm RR}_{(1)} + \tfrac12 B^{\rm RR}_{(1)}\, dB^{\rm NS}_{(2)} - \tfrac12 B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(1)}\\
&\qquad\quad - B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, d\chi + \tfrac12 m\, (B^{\rm NS}_{(2)})^2\ ,\\
&G_{(5)} = dB_{(4)} + \tfrac12 B^{\rm NS}_{(2)}\, dB^{\rm RR}_{(2)} - \tfrac12 B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(2)} - \mathcal{B}_{(1)}\, dB_{(3)} - \tfrac12 B^{\rm NS}_{(1)}\, \mathcal{B}_{(1)}\, dB^{\rm RR}_{(2)}\\
&\qquad\quad + \tfrac12 (B^{\rm NS}_{(2)})^2\, d\chi + \tfrac12 B^{\rm NS}_{(2)}\, \mathcal{B}_{(1)}\, dB^{\rm RR}_{(1)} + \tfrac12 B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, dB^{\rm NS}_{(2)} + \tfrac12 B^{\rm RR}_{(2)}\, \mathcal{B}_{(1)}\, dB^{\rm NS}_{(1)}\\
&\qquad\quad - B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, \mathcal{B}_{(1)}\, d\chi - \tfrac12 m\, (B^{\rm NS}_{(2)})^2\, \mathcal{B}_{(1)}\ ,\\
&\mathcal{F}_{(2)} = d\mathcal{B}_{(1)}\ .
\end{aligned}
\tag{A.14}
$$

We have presented these fields in the same order as the corresponding fields of the nine-dimensional massive IIA theory in (A.2). Of course in this case we also have $G_{(5)}$ classified as an "original" field, since we had already effectively doubled the $G_{(5)}$ field in $D = 10$. In the type IIA picture, the corresponding field $F_{(5)1}$ appears among the list of doubled fields in (A.3).

We find that the doubled fields in $D = 9$ are as follows:

$$
\begin{aligned}
&G^{\rm RR}_{(6)} = dB^{\rm RR}_{(5)} + B^{\rm NS}_{(1)}\, dB_{(4)} - B^{\rm NS}_{(2)}\, dB_{(3)} + \tfrac12 N^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, dB^{\rm RR}_{(2)} - \tfrac14 (B^{\rm NS}_{(2)})^2\, dB^{\rm RR}_{(1)}\\
&\qquad\quad + \tfrac12 B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^2\, d\chi - \tfrac16 m\, (B^{\rm NS}_{(2)})^3\ ,\\
&G^{\rm NS}_{(6)} = dB^{\rm NS}_{(5)} - B^{\rm RR}_{(1)}\, dB_{(4)} + B^{\rm RR}_{(2)}\, dB_{(3)} + B^{\rm RR}_{(5)}\, d\chi + \tfrac12 B^{\rm RR}_{(1)}\, B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(2)}\\
&\qquad\quad - \tfrac14 (B^{\rm RR}_{(2)})^2\, dB^{\rm NS}_{(1)} - \tfrac12 B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, B^{\rm RR}_{(2)}\, d\chi - \tfrac14 (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(1)}\, d\chi + \tfrac14 m\, (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)}\ ,\\
&G^{\rm NS}_{(7)} = dB^{\rm NS}_{(6)} + B^{\rm RR}_{(2)}\, dB_{(4)} - B^{\rm RR}_{(6)}\, d\chi - \mathcal{B}_{(1)}\, dB^{\rm NS}_{(5)} - B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, dB_{(4)} - \tfrac14 (B^{\rm RR}_{(2)})^2\, dB^{\rm NS}_{(2)}\\
&\qquad\quad - B^{\rm RR}_{(2)}\, \mathcal{B}_{(1)}\, dB_{(3)} + B^{\rm RR}_{(5)}\, \mathcal{B}_{(1)}\, d\chi + \tfrac14 (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)}\, d\chi + \tfrac12 B^{\rm RR}_{(1)}\, B^{\rm RR}_{(2)}\, \mathcal{B}_1\, dB^{\rm NS}_{(2)}\\
&\qquad\quad + \tfrac14 (B^{\rm RR}_{(2)})^2\, \mathcal{B}_{(1)}\, dN^{\rm NS}_{(1)} - \tfrac12 B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, B^{\rm RR}_{(2)}\, \mathcal{B}_{(1)}\, d\chi\\
&\qquad\quad - \tfrac14 (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, d\chi + m\, B^{\rm RR}_{(6)}\, \mathcal{B}_{(1)}\ ,\\
&G^{\rm RR}_{(7)} = dB^{\rm RR}_{(6)} - B^{\rm NS}_{(2)}\, dB_{(4)} - \mathcal{B}_{(1)}\, dB^{\rm RR}_{(5)} + B^{\rm NS}_{(1)}\, \mathcal{B}_{(1)}\, dB_{(4)} - \tfrac14 (B^{\rm NS}_{(2)})^2\, dB^{\rm RR}_{(2)} + B^{\rm NS}_{(2)}\, \mathcal{B}_{(1)}\, dB_{(3)}\\
&\qquad\quad + \tfrac12 B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, \mathcal{B}_{(1)}\, dB^{\rm RR}_{(2)} - \tfrac16 (B^{\rm NS}_{(2)})^3\, d\chi + \tfrac14 (B^{\rm NS}_{(2)})^2\, \mathcal{B}_{(1)}\, dB^{\rm RR}_{(1)}\\
&\qquad\quad + \tfrac12 B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^2\, \mathcal{B}_{(1)}\, d\chi + \tfrac16 m\, (B^{\rm NS}_{(2)})^3\, \mathcal{B}_{(1)}\ ,\\
&G_{(8)} = dB_{(7)} + B_{(1)}\, dB^{\rm RR}_{(6)} - B^{\rm NS}_{(2)}\, dB^{\rm RR}_{(5)} - B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, dB_{(4)} + \tfrac12 (B^{\rm NS}_{(2)})^2\, dB_{(3)}\\
&\qquad\quad - \tfrac38 B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^2\, dB^{\rm RR}_{(2)} - \tfrac14 B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(2)} - \tfrac18 (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(1)}\, dB^{\rm NS}_{(2)}\\
&\qquad\quad + \tfrac18 (B^{\rm NS}_{(2)})^3\, dB^{\rm RR}_{(1)} + \tfrac18 (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(1)} - \tfrac16 B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^3\, d\chi + \tfrac1{24} m\, (B^{\rm NS}_{(2)})^4\ ,\\
&G_{(9)} = dB_{(8)} - B^{\rm NS}_{(2)}\, dB^{\rm RR}_{(6)} + \tfrac12 (B^{\rm NS}_{(2)})^2\, dB_{(4)} + \tfrac18 (B^{\rm NS}_{(2)})^3\, dB^{\rm RR}_{(2)} + \tfrac18 (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(2)}\\
&\qquad\quad + \tfrac1{24} (B^{\rm NS}_{(2)})^4\, d\chi - \mathcal{B}_{(1)}\, G_{(8)}\ ,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{F}_{(7)} = d\mathcal{B}_{(6)} &- \tfrac{1}{2}B_{(3)}\, dB_{(3)} - B^{\rm NS}_{(1)}\, dB^{\rm NS}_{(5)} - B^{\rm RR}_{(1)}\, dB^{\rm RR}_{(5)} + B^{\rm NS}_{(1)}\, B^{\rm RR}_{(1)}\, dB_{(4)} - \tfrac{1}{2}B^{\rm NS}_{(1)}\, B^{\rm RR}_{(2)}\, dB_{(3)} \\
&- B^{\rm NS}_{(1)}\, B^{\rm RR}_{(5)}\, d\chi + \tfrac{1}{2}B^{\rm NS}_{(2)}\, B^{\rm RR}_{(1)}\, dB_{(3)} + \tfrac{1}{4}B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, B^{\rm RR}_{(1)}\, dB^{\rm RR}_{(2)} + \tfrac{1}{8}B^{\rm NS}_{(1)}\, (B^{\rm RR}_{(2)})^2\, dB^{\rm NS}_{(1)} \\
&+ \tfrac{1}{8}(B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(1)}\, dB^{\rm RR}_{(1)} + \tfrac{1}{4}B^{\rm NS}_{(2)}\, B^{\rm RR}_{(1)}\, B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(1)} + \tfrac{1}{4}B^{(n)}_{(1)} 2\, (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(1)}\, d\chi \\
&- m\, B_{(7)} + m\, B^{\rm NS}_{(1)}\, B^{\rm RR}_{(6)} - \tfrac{1}{8}m\, B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)} + \tfrac{1}{24}m\, (B^{\rm NS}_{(2)})^3\, B^{\rm RR}_{(1)}\ . \qquad \text{(A.15)}
\end{aligned}$$

Finally, the doubled fields associated with the two dilatons ϕ and φ turn out to be:

$$\begin{aligned}
G^{\phi}_{(8)} = dB^{\phi}_{(7)} &+ B_{(7)}\, d\chi + \tfrac{1}{2}B^{\rm NS}_{(1)}\, dB^{\rm NS}_{(6)} - \tfrac{1}{2}B^{\rm NS}_{(2)}\, dB^{\rm NS}_{(5)} - \tfrac{1}{2}B^{\rm RR}_{(1)}\, dB^{\rm RR}_{(6)} + \tfrac{1}{2}B^{\rm RR}_{(2)}\, dB^{\rm RR}_{(5)} \\
&+ \tfrac{1}{2}B^{\rm NS}_{(1)}\, B^{\rm RR}_{(2)}\, dB_{(4)} - \tfrac{1}{2}B^{\rm NS}_{(1)}\, B^{\rm RR}_{(6)}\, d\chi + \tfrac{1}{2}B^{\rm NS}_{(2)}\, B^{\rm RR}_{(1)}\, dB_{(4)} - \tfrac{1}{2}B^{\rm NS}_{(2)}\, B^{\rm RR}_{(2)}\, dB_{(3)} \\
&- \tfrac{1}{2}B^{\rm NS}_{(2)}\, B^{\rm RR}_{(5)}\, d\chi + \tfrac{1}{4}B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)}\, d\chi + \tfrac{1}{12}(B^{\rm NS}_{(2)})^3\, B^{\rm RR}_{(1)}\, d\chi \\
&- m\, B_{(8)} + \tfrac{1}{2}m\, B^{\rm NS}_{(2)}\, B^{\rm RR}_{(6)} - \tfrac{1}{12}(B^{\rm NS}_{(2)})^3\, B^{\rm RR}_{(2)}\ , \\
G^{\varphi}_{(8)} = dB^{\varphi}_{(7)} &- 2\alpha \Big\{ 2B_{(3)}\, dB_{(4)} + 3B^{\rm NS}_{(1)}\, dB^{\rm NS}_{(6)} + B^{\rm NS}_{(2)}\, dB^{\rm NS}_{(5)} + 3B^{\rm RR}_{(1)}\, dB^{\rm RR}_{(6)} + B^{\rm RR}_{(2)}\, dB^{\rm RR}_{(5)} \\
&- 4\mathcal{B}_{(1)}\, d\mathcal{B}_{(6)} - 2B_{(3)}\, \mathcal{B}_{(1)}\, dB_{(3)} + 2B^{\rm NS}_{(1)}\, B^{\rm RR}_{(2)}\, dB_{(4)} - 3B^{\rm NS}_{(1)}\, B^{\rm RR}_{(6)}\, d\chi - 4B^{\rm NS}_{(1)}\, \mathcal{B}_{(1)}\, dB^{\rm NS}_{(5)} \\
&- 2B^{\rm NS}_{(2)}\, B^{\rm RR}_{(1)}\, dB_{(4)} + B^{\rm NS}_{(2)}\, B^{\rm RR}_{(5)}\, d\chi - 4B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, dB^{\rm RR}_{(5)} - 4B^{\rm NS}_{(1)}\, B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, dB_{(4)} \\
&- \tfrac{1}{4}B^{\rm NS}_{(1)}\, (B^{\rm RR}_{(2)})^2\, dB^{\rm NS}_{(2)} - 2B^{\rm NS}_{(1)}\, B^{\rm RR}_{(2)}\, \mathcal{B}_{(1)}\, dB_{(3)} + 4B^{\rm NS}_{(1)}\, B^{\rm RR}_{(5)}\, \mathcal{B}_{(1)}\, d\chi \\
&- \tfrac{1}{4}(B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(1)}\, dB^{\rm RR}_{(2)} - \tfrac{1}{4}(B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{2}\, dB^{\rm RR}_{(1)} + 2B^{\rm NS}_{(2)}\, B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, dB_{(3)} \\
&- \tfrac{1}{4}(B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)}\, dB^{\rm NS}_{(1)} + \tfrac{1}{4}B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)}\, d\chi - B^{\rm NS}_{(1)}\, B^{\rm NS}_{(2)}\, B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, dB^{\rm RR}_{(2)} \\
&+ \tfrac{1}{2}B^{\rm NS}_{(1)}\, (B^{\rm RR}_{(2)})^2\, \mathcal{B}_{(1)}\, dB^{\rm NS}_{(1)} - \tfrac{1}{4}(B^{\rm NS}_{(2)})^3\, B^{\rm RR}_{(1)}\, d\chi + \tfrac{1}{2}(B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, dB^{\rm RR}_{(1)} \\
&+ B^{\rm NS}_{(2)}\, B^{\rm RR}_{(1)}\, B^{\rm RR}_{(2)}\, \mathcal{B}_{(1)}\, dB^{\rm NS}_{(1)} - B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)}\, d\chi + 4m\, B_{(8)} \\
&- 4m\, B_{(7)}\, \mathcal{B}_{(1)} - m\, B^{\rm NS}_{(2)}\, B^{\rm RR}_{(6)} + 4m\, B^{\rm NS}_{(1)}\, B^{\rm RR}_{(6)}\, \mathcal{B}_{(1)} + \tfrac{1}{2}m\, (B^{\rm NS}_{(2)})^3\, B^{\rm RR}_{(2)} \\
&- \tfrac{1}{2}m\, B^{\rm NS}_{(1)}\, (B^{\rm NS}_{(2)})^2\, B^{\rm RR}_{(2)}\, \mathcal{B}_{(1)} + \tfrac{1}{6}m\, (B^{\rm NS}_{(2)})^3\, B^{\rm RR}_{(1)}\, \mathcal{B}_{(1)} \Big\}\ . \qquad \text{(A.16)}
\end{aligned}$$

It follows that the nine-dimensional equations of motion may again be read off from the ten-dimensional ones, by using (A.8). Thus we find

$$\begin{aligned}
&e^{8\alpha\varphi} *G_{(5)} = -G_{(4)}\ , \qquad e^{-8\alpha\varphi} *G_{(4)} = G_{(5)}\ , \\
&e^{\phi 4\alpha\varphi} *G^{\rm RR}_{(3)} = -G^{\rm RR}_{(6)}\ , \qquad e^{\phi - 12\alpha\varphi} *G^{\rm RR}_{(2)} = G^{\rm RR}_{(7)}\ , \\
&e^{-\phi + 4\alpha\varphi} *G^{\rm NS}_{(3)} = -G^{\rm NS}_{(6)}\ , \qquad e^{-\phi - 12\alpha\varphi} *G^{\rm NS}_{(2)} = G^{\rm NS}_{(7)}\ , \\
&e^{2\phi} *G_{(1)} = -G_{(8)}\ , \qquad m\, e^{2\phi - 16\alpha\varphi} *\mathbb{1} = G_{(9)}\ , \qquad \text{(A.17)} \\
&*d\phi = -G^{\phi}_{(8)}\ , \\
&e^{16\alpha\varphi} *\mathcal{F}_{(2)} = \mathcal{F}_{(7)}\ , \qquad *d\varphi = G^{\varphi}_{(8)}\ .
\end{aligned}$$

Note that the two equations on the top line are actually equivalent. As in the type IIA reduction, the equations in the final line correspond to the new fields $\mathcal{B}_{(1)}$ and φ coming from the dimensional reduction of the metric, and we have chosen the conventions for their doubled field strengths so that there are plus signs in these two equations of motion.

A.4 Massive IIA/IIB T-duality in $D = 9$

Having obtained the doubled formalism for both the massive type IIA and type IIB theories in $D = 9$, it is straightforward, albeit tedious, to verify that the two sets of equations of motion are the same, after appropriate field redefinitions. The T-duality between massive type IIA and type IIB was proven in [12], making use of the Stückelberg symmetry. In this section, we shall present the explicit T-duality transformation rules for the doubled formalism. We shall present these field transformation rules in two sets, namely the R-R sector and NS-NS sector. For the R-R sector, we find that the expressions for the type IIB fields in terms of the type IIA fields are:

$$
\begin{aligned}
\chi &= -A_{(0)1}\ , \qquad B^{\rm RR}_{(1)} = A_{(1)}\ , \qquad B^{\rm RR}_{(2)} = -A_{(2)1} + A_{(0)1}\, A_{(2)} + A_{(1)}\, A_{(1)1}\ , \\
B_{(3)} &= A_{(3)} - \tfrac12 A_{(1)}\, A_{(2)} - \tfrac12 \mathcal{A}_{(1)}\, A_{(2)1} + \tfrac12 A_{(0)1}\, \mathcal{A}_{(1)}\, A_{(2)}\ , \\
B_{(4)} &= A_{(4)1} - \tfrac12 A_{(2)}\, A_{(2)1} - \tfrac12 A_{(1)}\, A_{(1)1}\, A_{(2)} \\
&\quad + \tfrac12 A_{(1)1}\, \mathcal{A}_{(1)}\, A_{(2)1} - \tfrac12 A_{(0)}\, A_{(1)1}\, \mathcal{A}_{(1)}\, A_{(2)}\ , \\
B^{\rm RR}_{(5)} &= A_{(5)} - \tfrac14 A_{(1)}\, (A_{(2)})^2 - \tfrac12 \mathcal{A}_{(1)}\, A_{(2)}\, A_{(2)1} + \tfrac12 A_{(0)1}\, \mathcal{A}_{(1)}\, (A_{(2)})^2\ , \\
B^{\rm RR}_{(6)} &= A_{(6)1} - \tfrac14 (A_{(2)})^2\, A_{(2)1} + \tfrac1{12} A_{(0)1}\, (A_{(2)})^3 - \tfrac14 A_{(1)}\, A_{(1)1}\, (A_{(2)})^2 \\
&\quad + \tfrac12 A_{(1)1}\, \mathcal{A}_{(1)}\, A_{(2)}\, A_{(2)1} - \tfrac12 A_{(0)}\, A_{(1)1}\, \mathcal{A}_{(1)}\, (A_{(2)})^2\ , \\
B_{(7)} &= A_{(7)} - \tfrac18 A_{(1)}\, (A_{(2)})^3 - \tfrac18 \mathcal{A}_{(1)}\, A_{(2)1}\, (A_{(2)})^2 + \tfrac18 A_{(0)1}\, \mathcal{A}_{(1)}\, (A_{(2)})^3\ , \\
B_{(8)} &= A_{(8)} - \tfrac1{24} (A_{(2)})^3\, A_{(2)1} - \tfrac18 A_{(1)}\, A_{(1)1} (A_{(2)})^3 + \tfrac18 A_{(1)1}\, \mathcal{A}_{(1)}\, (A_{(2)})^2\, A_{(2)1} \\
&\quad - \tfrac18 A_{(0)}\, A_{(1)1}\, \mathcal{A}_{(1)}\, (A_{(2)})^3\ ,
\end{aligned} \tag{A.18}
$$

For the NS-NS sector, we find that the T-duality transformations are given by

$$
\begin{aligned}
\mathcal{B}_{(1)} &= A_{(1)1}\ , \qquad B^{\rm NS}_{(1)} = \mathcal{A}_{(1)}\ , \qquad B^{\rm NS}_{(2)} = A_{(2)} - A_{(1)1}\, \mathcal{A}_{(1)}\ , \\
B^{\rm NS}_{(5)} &= A_{(5)1} - A_{(0)1}\, A_{(5)} + \tfrac12 A_{(2)1}\, A_{(3)} - \tfrac12 A_{(1)}\, A_{(2)}\, A_{(2)1} \\
&\quad - \tfrac14 \mathcal{A}_{(1)}\, (A_{(2)1})^2 + \tfrac12 A_{(0)1}\, A_{(1)}\, (A_{(2)})^2 + \tfrac12 A_{(0)1}\, \mathcal{A}_{(1)}\, A_{(2)}\, A_{(2)1} \\
&\quad - \tfrac14 (A_{(0)1})^2\, \mathcal{A}_{(1)}\, (A_{(2)})^2\ , \\
B^{\rm NS}_{(6)} &= \mathcal{A}_{(6)} - A_{(0)1}\, A_{(6)1} - \tfrac12 A_{(1)1}\, A_{(2)1}\, A_{(3)} - \tfrac14 A_{(2)}\, (A_{(2)1})^2 \\
&\quad - \tfrac12 A_{(1)}\, A_{(1)1}\, A_{(2)}\, A_{(2)1} + \tfrac14 A_{(1)1}\, \mathcal{A}_{(1)}\, (A_{(2)1})^2 + \tfrac1{12} (A_{(0)1})^2\, (A_{(2)})^3 \\
&\quad + \tfrac12 A_{(0)1}\, A_{(1)}\, A_{(1)1}\, (A_{(2)})^2 - \tfrac12 A_{(0)1}\, A_{(1)1}\, \mathcal{A}_{(1)}\, A_{(2)}\, A_{(2)1}
\end{aligned}
$$

$$
\begin{aligned}
& +\tfrac{1}{4}(A_{(0)1})^2\, A_{(1)1}\, \mathcal{A}_{(1)}\, (A_{(2)})^2 \ , \\
\mathcal{B}_{(6)} &= A_{(6)} - \tfrac{1}{4} A_1\, A_{(2)}\, A_{(3)} - \tfrac{1}{4}\mathcal{A}_{(1)}\, A_{(2)1}\, A_{(3)} + \tfrac{1}{4} A_{(0)1}\, \mathcal{A}_{(1)}\, A_{(2)}\, A_{(3)} \\
& -\tfrac{1}{8} A_{(1)}\, \mathcal{A}_{(1)}\, A_{(2)}\, A_{(2)1} + \tfrac{1}{8} A_{(0)1}\, A_{(1)}\, \mathcal{A}_{(1)}\, (A_{(2)})^2 \ , \\
B^{\phi}_{(7)} &= \tfrac{3}{4} A^{\phi}_{(7)} + \tfrac{\sqrt{7}}{4} A^{\varphi}_{(7)} - A_{(0)1}\, A_{(7)} + \tfrac{1}{16} A_{(2)}\, A_{(2)1}\, A_{(3)} \\
& -\tfrac{1}{8} A_{(1)}\, (A_{(2)})^2\, A_{(2)1} - \tfrac{1}{8} A_{(2)}\, (A_{(2)1})^2\, \mathcal{A}_{(1)} + \tfrac{1}{8} A_{(0)1}\, A_{(1)}\, (A_{(2)})^3 \\
& +\tfrac{1}{4} A_{(0)1}\, \mathcal{A}_{(1)}\, (A_{(2)})^2\, A_{(2)1} - \tfrac{1}{8}(A_{(0)1})^2\, \mathcal{A}_{(1)}\, (A_{(2)})^3 \ , \\
B^{\varphi}_{(7)} &= \tfrac{\sqrt{7}}{4}\, A^{\phi}_{(7)} - \tfrac{3}{4} A^{\varphi}_{(7)} + \tfrac{5}{8} A_{(2)}\, A_{(2)1}\, A_{(3)} + A_1\, A_{(1)1}\, A_{(2)}\, A_{(3)} \\
& - A_{(1)1}\, \mathcal{A}_{(1)}\, A_{(2)1}\, A_{(3)} + A_{(0)1}\, A_{(1)1}\, \mathcal{A}_{(1)}\, A_{(2)}\, A_{(3)} \ ,
\end{aligned} \tag{A.19}
$$

The relation between the dilatonic scalars in the two nine-dimensional theories are given by

$$
\begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{\rm IIA} = \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{7}}{4} \\ -\frac{\sqrt{7}}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{\rm IIB} \equiv M \begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{\rm IIB} \ . \tag{A.20}
$$

Note that we have $M^{-1} = M$. The dimensional reduction of the ten-dimensional string metric to $D = 9$ is given by

$$
\begin{aligned}
ds^2_{\rm str} &= e^{\frac{1}{2}\phi}\, ds^2_{10} \\
&= e^{\frac{1}{2}\phi}\, (e^{-\varphi/(2\sqrt{7})}\, ds^2_9 + e^{\sqrt{7}\varphi/2}\, (dz_2 + \mathcal{A})^2) \ ,
\end{aligned} \tag{A.21}
$$

where ds^2_{10} and ds^2_9 are the Einstein-frame metrics in $D = 10$ and $D = 9$. The radius of the compactifying circle, measured using the ten-dimensional string metric, is therefore given by $R = e^{\frac{1}{4}\phi + \frac{\sqrt{7}}{4}\varphi}$. Note that the dilaton vector $\{\frac{1}{4}, \frac{1}{4}\sqrt{7}\}$ defining the radius is an eigenvector of M, with eigenvalue -1. It follows that the radii $R_{\rm IIA}$ and $R_{\rm IIB}$ of the compactifying circles, measured using their respective ten-dimensional string metrics, are related by $R_{\rm IIA} = 1/R_{\rm IIB}$.

Note that all the T-duality transformations between the massive type IIA and type IIB theories are independent of m. In particular, this means that the relations between the type IIA and type IIB fields in nine-dimensions are the same whether one is looking at the massive theories or the massless ones. Note also that the relations between the original "undoubled" sets of fields do not involve any of the extended "doubled" system, and so by restricting attention just to the original undoubled fields in (A.18) and (A.19), one obtains the explicit field relations for the standard undoubled systems.

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