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Bubbling AdS and droplet descriptions of BPS geometries in IIB supergravity

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Abstract

This paper focuses on supergravity duals of BPS states in $\mathcal{N} = 4$ super Yang-Mills. In order to describe these duals, we begin with a sequence of breathing mode reductions of IIB supergravity: first on S^3, then $S^3 \times S^1$, and finally on $S^3 \times S^1 \times CP^1$. We then follow with a complete supersymmetry analysis, yielding 1/8, 1/4 and 1/2 BPS configurations, respectively (where in the last step we take the Hopf fibration of S^3). The 1/8 BPS geometries, which have an S^3 isometry and are time-fibered over a six-dimensional base, are determined by solving a non-linear equation for the Kähler metric on the base. Similarly, the 1/4 BPS configurations have an $S^3 \times S^1$ isometry and a four-dimensional base, whose Kähler metric obeys another non-linear, Monge-Ampère type equation.

Despite the non-linearity of the problem, we develop a universal bubbling AdS description of these geometries by focusing on the boundary conditions which ensure their regularity. In the 1/8 BPS case, we find that the S^3 cycle shrinks to zero size on a five-dimensional locus inside the six-dimensional base. Enforcing regularity of the full solution requires that the interior of a smooth, generally disconnected five-dimensional surface be removed from the base. The $\mathrm{AdS}_5 \times S^5$ ground state corresponds to excising the interior of an S^5, while the 1/8 BPS excitations correspond to deformations (including topology change) of the S^5 and/or the excision of additional droplets from the base. In the case of 1/4 BPS configurations, by enforcing regularity conditions, we identify three-dimensional surfaces inside the four-dimensional base which separate the regions where the S^3 shrinks to zero size from those where the S^1 shrinks.

We discuss a large class of examples to show the emergence of a universal bubbling AdS picture for all 1/2, 1/4 and 1/8 BPS geometries.

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1 Introduction

In its most straightforward incarnation, AdS/CFT duality is a relation between $\mathcal{N}=4$ super-Yang Mills theory and IIB string theory on $\mathrm{AdS}_5 \times S^5$. This system has been extensively studied, and recently there has been much progress in the study of various sectors of this correspondence. In general, some of the best understood aspects of this duality naturally arise through the use of supersymmetry. A particularly striking example of this was realized in a remarkable paper by Lin, Lunin and Maldacena (LLM) [1], which constructed explicit regular 1/2 BPS states in IIB supergravity and demonstrated their relation to the free fermion picture of the corresponding 1/2 BPS sector of the $\mathcal{N}=4$ super-Yang Mills theory [2, 3].

Based on the correspondence with chiral primaries satisfying $\Delta = J$, LLM examined all regular 1/2 BPS states with $SO(4)\times SO(4)$ isometry in IIB supergravity with only the metric and self-dual five-form turned on. Because of this $S^3\times S^3$ isometry, explicit construction of such 1/2 BPS 'bubbling AdS' configurations may be simplified by working in an effective four-dimensional theory of the form

$$e^{-1}\mathcal{L}_4 = e^{3H}[R + \tfrac{15}{2}\partial H^2 - \tfrac{3}{2}\partial G^2 - \tfrac{1}{4}e^{-3(H+G)}F_{\mu\nu}^2 + 12e^{-H}\cosh G]. \tag{1.1}$$

The four-dimensional metric, two scalars H and G, and the 2-form field strength $F_{\mu\nu}$ are related to their ten-dimensional counterparts according to [1, 4, 5]

$$\begin{aligned} ds_{10}^2 &= g_{\mu\nu}dx^\mu dx^\nu + e^H(e^G d\Omega_3^2 + e^{-G}d\widetilde{\Omega}_3^2), \\ F_{(5)} &= (1+*_{10})F_{(2)}\wedge\Omega_3. \end{aligned} \tag{1.2}$$

Since the supersymmetric bubbling configurations preserve a time-like Killing vector $\partial/\partial t$, the construction further simplifies into a three dimensional one. The result is that all such 1/2 BPS states are describable in terms of a single harmonic function $Z = \frac{1}{2}\tanh G$ satisfying the linear equation [1]

$$\left(\partial_1^2 + \partial_2^2 + y\partial_y \frac{1}{y}\partial_y\right) Z(x_1, x_2, y) = 0. \tag{1.3}$$

The resulting ten-dimensional metric is then of the form

$$ds_{10}^2 = -h^{-2}(dt+\omega)^2 + h^2(dx_1^2 + dx_2^2 + dy^2) + y(e^G d\Omega_3^2 + e^{-G}d\widetilde{\Omega}_3^2) \tag{1.4}$$

where $h^{-2} = 2y\cosh G$.

The bubbling picture arises through the observation that regularity of the metric (1.4) demands that only one of the three-spheres collapses (in an appropriate manner) as $y\to 0$. The necessary boundary conditions are then simply

$$Z(x_1, x_2, y=0) = \pm\tfrac{1}{2}. \tag{1.5}$$

These boundary conditions allow the $y = 0$ boundary plane to be identified with the fermion droplet phase-space plane [1], and the complete form of Z may then be obtained through an appropriate

Green's function solution to (1.3). In fact, a key feature of this 1/2 BPS bubbling $\mathrm{AdS}_5 \times S^5$ construction is precisely the linearity of the governing equation (1.3). This linearity is natural from the free fermion picture on the gauge theory side of the duality, and at first sight may be thought of as a consequence of the BPS (*i.e.* no force) condition. However, this is not necessarily the case, as for example 1/2 BPS configurations in 11-dimensional supergravity with $SO(3) \times SO(6)$ isometry are described by a Toda-type equation, which is non-linear [1]. Nevertheless, even in this case, the bubbling picture survives in terms of boundary conditions corresponding to either the S^2 or S^5 shrinking on the $y = 0$ boundary plane.

Given the elegant bubbling description for 1/2 BPS configurations in both the gauge theory and string theory side of the AdS/CFT correspondence, it is natural to extend the above LLM investigation to both 1/4 BPS [6–10] and 1/8 BPS [11–13] configurations. While there are several possibilities for obtaining reduced supersymmetry, we are primarily interested in backgrounds with multiple commuting R-charges turned on. For $\mathcal{N} = 4$ super-Yang Mills, as well as the dual description of IIB on $\mathrm{AdS}_5 \times S^5$, the relevant supergroup is $PSU(2,2|4)$, which admits the bosonic subgroup $SO(2,4) \times SO(6)$. On the gravity side, states may be labeled by (Δ, S_1, S_2) for energy and spin in AdS_5 and (J_1, J_2, J_3) for angular momentum on S^5. Focusing on the chiral primaries, we take s-wave states in AdS_5 satisfying $\Delta = J_1 + J_2 + J_3$. Given that the BPS condition takes the form

$$\Delta \geq \pm g S_1 \pm g S_2 \pm J_1 \pm J_2 \pm J_3 \tag{1.6}$$

(with an even number of minus signs, and with g the inverse radius of AdS_5), we see that the generic state with three non-vanishing R-charges preserves 1/8 of the supersymmetries. When $J_3 = 0$, the eigenvalues of the Bogomol'nyi matrix pair up, and we are left with a 1/4 BPS state. Finally, when $J_2 = J_3 = 0$, the system reduces to the familiar 1/2 BPS case.

When gravitational backreaction is taken into account, the turning on of J_1, J_2 and J_3 in succession breaks the isometries of the five-sphere from $SO(6)$ to $SO(4)$, $SO(2)$ and finally the identity. Combining this with the unbroken $SO(4)$ isometry of s-wave states in AdS_5, the natural family of backgrounds we are interested in takes the form

supersymmetries	chiral primary	isometry
1/2 BPS	$\Delta = J_1$	$S^3 \times S^3$
1/4 BPS	$\Delta = J_1 + J_2$	$S^3 \times S^1$
1/8 BPS	$\Delta = J_1 + J_2 + J_3$	S^3

(1.7)

In this paper, our main interest is the supergravity description of such backgrounds. The 1/2 BPS case was of course the subject of LLM [1] and related investigations. The other two cases have generally received less attention. However, the invariant tensor analysis of [14–17] has recently been applied towards the construction of supergravity backgrounds corresponding to these two cases. Backgrounds with $S^3 \times S^1$ isometry were initially examined in [18], and subsequent gauging of the $U(1)$ isometry was considered in [19]. In addition, solutions preserving an S^3 isometry (corresponding to the 1/8 BPS case) may be obtained by double analytic continuation of the AdS_3

solutions investigated in [20], as it was later done in [21]. (Note that 1/4 BPS and 1/8 BPS solutions of a different nature were also investigated in [4] and [22], respectively.)

In both cases of S^3 isometry [20] and $S^3 \times S^1$ isometry [18, 19], the invariant tensor analysis and resulting description of the backgrounds are essentially complete. However, unlike for LLM geometries, in these cases the supersymmetry analysis is not particularly constructive. For example, it was found in [20] that 1/8 BPS configurations with an S^3 isometry may be written using a metric of the form

$$ds_{10}^2 = -e^{2\alpha}(dt + \omega)^2 + e^{-2\alpha} h_{ij} dx^i dx^j + e^{2\alpha} d\Omega_3^2, \tag{1.8}$$

where h_{ij} is a Kähler metric of complex dimension three. In the end, the invariant tensor analysis does not provide an actual procedure for obtaining this metric short of solving a non-linear equation on its curvature [20]

$$\Box_6 R = -R_{ij} R^{ij} + \tfrac{1}{2} R^2. \tag{1.9}$$

Similarly, the 1/4 BPS analysis of [18, 19] leads to a non-linear equation of Monge-Ampère type related to the properties of the Kähler metric on a base of complex dimension two.

Although the presence of such non-linear equations complicates the analysis of 1/4 and 1/8 BPS states, it is nevertheless possible to develop a robust picture of bubbling AdS even without complete knowledge of the supergravity solution. The main point here is that the supergravity backgrounds are determined not only by the imposition of local conditions such as (1.9), but also by the boundary conditions. In particular, turning back to the LLM case, we recall that the droplet picture really originates from the LLM boundary conditions (1.5) imposed to ensure regularity of the geometry and not directly from the harmonic function equation (1.3). The LLM boundary conditions $Z(x_1, x_2, 0) = \pm 1/2$ ensure that the metric remains smooth wherever either of the S^3 factors collapses to zero size. Likewise, 1/4 BPS configurations preserving an $S^3 \times S^1$ isometry have potential singularities in the metric whenever either the S^3 or S^1 collapses. Avoiding such singularities then demands similar boundary conditions: $Z(x_i, y = 0) = \pm 1/2$, where this time $i = 1, \ldots, 4$ and

$$ds_{10}^2 = -h^{-2}(dt + \omega)^2 + y^{-1} e^{-G} h_{ij} dx^i dx^j + h^2 dy^2 + y(e^G d\Omega_3^2 + e^{-G}(d\psi + \mathcal{A})^2). \tag{1.10}$$

Note that $h^{-2} = 2y \cosh G$ is unchanged from the LLM case. What is different, however, is that now the metric h_{ij} (as well as the function G) appears rather complicated, and does not admit an easy construction.

The bubbling AdS description of 1/8 BPS configurations is particularly interesting in that it constitutes the most general case of turning on all three commuting R-charges. Since the 1/8 BPS metric, given in (1.8), does not involve a y coordinate, there is no 1/8 BPS equivalent of an LLM $y = 0$ phase-space plane. Nevertheless, the Kähler base can be given in terms of six real coordinates, x_i, $i = 1, \ldots, 6$. As highlighted in [23], it is natural to associate these coordinates with the six real adjoint scalars of the dual $\mathcal{N} = 4$ super-Yang Mills theory. In this picture, the eigenvalue distribution from the matrix description maps into configurations in $\mathbb{R}^6$ corresponding to the degeneration locus of the S^3 in AdS_5. From the gravity side, this indicates that the six-dimensional base has regions removed, with the boundary of such regions dual to the eigenvalue distribution. The $\mathrm{AdS}_5 \times S^5$ 'ground state' corresponds to removing a ball from the center of $\mathbb{R}^6$, and the addition of dual giant gravitons corresponds to removing other disconnected regions as well. Although the six-dimensional metric becomes singular as one approaches the boundary, it must behave in such a manner that, when combined with the shrinking S^3, the full ten-dimensional metric remains regular.

It is the aim of this paper to elucidate the bubbling picture of both 1/4 and 1/8 BPS configurations that we have sketched above, and to justify the connection between boundary conditions and droplets in an effective phase-space description of these geometries. Before we do so, however, we present a unified treatment of the invariant tensor analysis for 1/8, 1/4 and 1/2 BPS configurations. In particular, based on symmetry conditions, we may start with IIB supergravity with the self-dual five-form active, and perform a breathing mode reduction to seven dimensions on S^3. This seven-dimensional system is the natural place to start from when discussing 1/8 BPS configurations. A further reduction on S^1 brings the system down to six dimensions (and allows a description of 1/4 BPS geometries). Because of the abelian $U(1)$ isometry, we allow a gauge field to be turned on in this reduction [19]. Finally, we may reduce this system to four dimensions on CP^1. A generic configuration with $S^3 \times S^1 \times CP^1$ isometry will preserve 1/4 of the supersymmetries [22]. However, by making use of the Hopf fibration of S^3 as $U(1)$ bundled over CP^1, we may recover the round $S^3 \times S^3$ background of LLM, thus giving rise to the 1/2 BPS system.

Following the chain of breathing mode reductions and the supersymmetry analysis, we discuss how the bubbling AdS picture arises in the 1/4 and 1/8 BPS sectors. Essentially, this is based on an investigation of the boundary conditions needed to maintain a smooth geometry wherever any of the various spheres degenerate to zero size. Because of the difficulty in providing a constructive method for obtaining the full supergravity backgrounds, we will mainly support our arguments with a set of examples, which we treat separately for the 1/8 BPS and 1/4 BPS cases. Readers who wish to skip the details of the breathing mode reductions and invariant tensor analyses are invited to proceed directly to Section 4, where the bubbling AdS description is taken up.

The main technical results of this paper are presented in the following two sections. In Section 2, we perform a chain of breathing mode reductions, starting with S^3, then adding S^1 and finally adding CP^1. This allows us to write down effective seven, six and four-dimensional theories governing 1/8, 1/4 and 1/2 BPS configurations, respectively. The supersymmetry analysis is then taken up in Section 3; this is intended to give a unified treatment of [20], [18, 19], and [1], for the 1/8, 1/4 and 1/2 BPS cases, respectively, and show how the ansatz of these three cases are embedded into each other. The remaining parts of this paper are devoted to the development of the bubbling AdS description of 1/4 and 1/8 BPS states. In Section 4, we present a brief summary of the supergravity backgrounds, and then show how the LLM boundary conditions generalize to provide a uniform droplet picture which survives the reduction from 1/2 BPS down to 1/4 and 1/8 BPS configurations. We then turn to examples of 1/8 BPS geometries in Section 5 followed by 1/4 BPS geometries in Section 6. In Section 7 we return to the local conditions on the Kähler metric for 1/8 BPS configurations and investigate in particular the interplay between boundary conditions and regularity of the metric. Finally, we conclude in Section 8 with a summary of the 1/8 BPS droplet picture and how it also encompasses 1/4 and 1/2 BPS states as special cases. Various technical details are relegated to the appendices.

2 Breathing mode compactifications of IIB supergravity

The bosonic fields of IIB supergravity are given by the NSNS fields g_{MN}, B_{MN} and ϕ as well as the RR field strengths $F_{(1)}$, $F_{(3)}$ and $F^+_{(5)}$, while the fermionic fields are the (complex Weyl) gravitino Ψ_M and dilatino λ, both transforming with definite chirality in $D = 10$. Because we are interested in describing giant graviton configurations, which are essentially built out of D3-branes, we will only concern ourselves with the self-dual five-form $F^+_{(5)}$ in addition to the metric. In this sector, the IIB theory admits a particularly simple bosonic truncation with equations of motion

$$R_{MN} = \frac{1}{4 \cdot 4!}(F^2)_{MN}, \qquad F_{(5)} = *F_{(5)}, \qquad dF_{(5)} = 0. \tag{2.1}$$

The corresponding Lagrangian is given by

$$e^{-1}\mathcal{L}_{10} = R - \frac{1}{4 \cdot 5!}F^2_{(5)}, \tag{2.2}$$

where self-duality of $F_{(5)}$ is to be imposed only after deriving the equations of motion.

In the absence of the IIB dilaton/axion and three-form field strengths, the dilatino transformation becomes trivial. Thus the only relevant supersymmetry transformation is that of the gravitino, which becomes

$$\delta\Psi_M = \left[\nabla_M + \frac{i}{16 \cdot 5!}F_{NPQRS}\Gamma^{NPQRS}\Gamma_M\right]\epsilon. \tag{2.3}$$

Note that there is a delicate balance between self-duality of $F_{(5)}$ and the chirality of the spinor parameter ϵ. With the natural definition of self-duality $F_{M_1\cdots M_5} = \frac{1}{5!}\epsilon_{M_1\cdots M_5}{}^{N_1\cdots N_5}F_{N_1\cdots N_5}$, the spinor ϵ satisfies $\Gamma^{11}\epsilon = \epsilon$ where $\Gamma^{11} = \frac{1}{10!}\epsilon_{M_1 \cdot M_{10}}\Gamma^{M_1\cdots M_{10}}$.

The bubbling configurations that we are interested in always preserve an S^3 in AdS$_5$. However, the isometries of the S^5 are naturally broken depending on the amount of angular momentum (or R-charge) (J_1, J_2, J_3) turned on. As in [1], for 1/2 BPS configurations we take $J_2 = J_3 = 0$, and the resulting internal isometry is that of S^3. For 1/4 BPS configurations [18] we have $J_3 = 0$ and hence S^1 isometry. The generic 1/8 BPS case has all three angular-momenta non-vanishing, resulting in the loss of all manifest isometry of the original S^5.

It is then clear that, to capture this family of solutions, we ought to consider breathing mode reductions of (2.2) and (2.3) on S^3, $S^3 \times S^1$ and $S^3 \times S^3$, respectively, for 1/8, 1/4 and 1/2 BPS geometries. It is natural to proceed with this reduction in steps, at each stage adding additional symmetries to the system. Adding a $U(1)$ isometry to the S^3 reduction is straightforward, and a natural way to obtain $S^3 \times S^3$ from $S^3 \times U(1)$ is to use the Hopf fibration of the second S^3 as a $U(1)$ bundle over CP^1. This chain of reductions also provides a natural way of understanding the embedding of 1/2 BPS configurations into the 1/4 BPS system, and then finally into the 1/8 BPS case.

We note that Kaluza-Klein sphere reductions have been extensively studied in the literature. However, the main feature of the present set of reductions is the inclusion of breathing (and possibly squashing) modes [24]. Although these bosonic reductions are consistent (as any truncation to the singlet sector would be [25]), the resulting theory is however not supersymmetric, as the breathing and squashing modes are in general part of the massive Kaluza-Klein tower. Nevertheless, it is still instructive to reduce the original IIB Killing spinor equation (2.3) along with the bosonic sector fields. In this way, any solution to the reduced Killing spinor equations may then be lifted to yield a supersymmetric background of the original IIB theory. Breathing mode reductions of the supersymmetry variations were previously investigated in [26], and in the LLM context in [4, 5].

2.1 S^3 reduction to $D = 7$

The first stage of the reduction, corresponding to the generic 1/8 BPS case, is to highlight the S^3 isometry inside AdS_5, which we always retain. We thus take a natural reduction ansatz of the form

$$\begin{aligned} ds_{10}^2 &= ds_7^2 + e^{2\alpha} d\Omega_3^2, \\ {}^{10}F_{(5)} &= F_{(2)} \wedge \omega_3 + \widetilde{F}_{(5)}. \end{aligned} \tag{2.4}$$

note that self-duality of ${}^{10}F_{(5)}$ imposes the conditions

$$F_{(2)} = -e^{3\alpha} *_7 \widetilde{F}_{(5)}, \qquad \widetilde{F}_{(5)} = e^{-3\alpha} *_7 F_{(2)}. \tag{2.5}$$

The ten-dimensional Einstein equation in (2.1) reduces to yield the seven-dimensional Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 3(\partial_\mu \alpha \partial_\nu \alpha - 2 g_{\mu\nu} (\partial\alpha)^2 + \nabla_\mu \nabla_\nu \alpha - g_{\mu\nu} \Box \alpha) + \tfrac{1}{2} e^{-6\alpha} [F^2{}_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} F^2] + 3 e^{-2\alpha} \tag{2.6}$$

(in the 'string frame'), as well as the scalar equation of motion

$$\partial^\mu (3\alpha) \partial_\mu \alpha + \Box \alpha = -\tfrac{1}{8} e^{-6\alpha} F^2 + 2 e^{-2\alpha}. \tag{2.7}$$

In addition, the $F_{(5)}$ Bianchi identity and equation of motion in (2.1) reduce to their seven-dimensional counterparts

$$dF_{(2)} = 0, \qquad d(e^{-3\alpha} *_7 F_{(2)}) = 0. \tag{2.8}$$

The above equations of motion may be obtained from an effective seven-dimensional Lagrangian

$$e^{-1} \mathcal{L}_7 = e^{3\alpha} [R + 6 (\partial\alpha)^2 - \tfrac{1}{4} e^{-6\alpha} F_{(2)}^2 + 6 e^{-2\alpha}]. \tag{2.9}$$

The run-away potential term arises because of the curvature of the reduction S^3, and will remain unbalanced until the second S^3 is introduced.

2.1.1 Supersymmetry variations

In order to study supersymmetric configurations, we must also examine the reduction of the gravitino variation (2.3). In order to do so, we choose a Dirac decomposition of the form

$$\Gamma_\mu = \gamma_\mu \otimes 1 \otimes \sigma_1, \qquad \Gamma_a = 1 \otimes \sigma_a \otimes \sigma_2. \tag{2.10}$$

Defining the 10-dimensional chirality matrix as $\Gamma^{11} = \frac{1}{10!}\epsilon_{M_1\cdots M_{10}}\Gamma^{M_1\cdots M_{10}}$, we find $\Gamma^{11} = -1\otimes 1\otimes\sigma_3$ where we have taken the seven-dimensional Dirac matrices to satisfy $\frac{1}{7!}\epsilon_{\mu_1\cdots\mu_7}\gamma^{\mu_1\cdots\mu_7} = 1$. In this case, the IIB chirality condition $\Gamma^{11}\epsilon = \epsilon$ translates into the condition that ϵ has negative σ_3 eigenvalue. This allows us to decompose the complex IIB spinor as ${}^{10}\epsilon = \epsilon \otimes \chi \otimes \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ where χ is a two-component spinor on S^3 satisfying the Killing spinor equation

$$\left[\hat\nabla_a + \frac{i\eta}{2}\hat\sigma_a\right]\chi = 0, \tag{2.11}$$

with $\eta = \pm 1$.

Using the above decomposition, the 10-dimensional gravitino variation (2.3) decomposes into a seven-dimensional 'gravitino' variation

$$\delta\psi_\mu = \left[\nabla_\mu - \frac{i}{16}e^{-3\alpha}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu\right]\epsilon, \tag{2.12}$$

as well as a 'dilatino' variation

$$\delta\lambda = \left[\gamma^\mu\partial_\mu\alpha + \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \eta e^{-\alpha}\right]\epsilon \tag{2.13}$$

which arises from the components of (2.3) living on the S^3. We emphasize here that these are not necessarily the transformations of any actual seven-dimensional supersymmetric model, as we only claim the bosonic sector to form a consistent truncation of the original IIB theory. Nevertheless, based on their structure, it is useful to think of these as would-be gravitino and dilatino variations. So long as these two 'Killing spinor equations' are satisfied, we are guaranteed that the lifted solution is a supersymmetric configuration of the original IIB theory.

2.2 Additional reduction on $U(1)$ to $D = 6$

In order to describe 1/4 BPS geometries with $S^3 \times S^1$ isometry, we may further reduce the seven-dimensional system (2.9) to $D = 6$ along a $U(1)$ direction. This follows by a traditional Kaluza-Klein circle reduction, where we take

$$\begin{aligned} ds_7^2 &= ds_6^2 + e^{2\beta}(d\psi + \mathcal{A})^2, \\ {}^7F_{(2)} &= F_{(2)} + d\chi \wedge (d\psi + \mathcal{A}). \end{aligned} \tag{2.14}$$

This is the most general ansatz consistent with $U(1)$ isometry, and includes an axionic scalar χ which in the original IIB picture corresponds to five-form flux on $S^3 \times S^1$ along with a non-compact dimension. For a pure bubbling picture with S^3 inside AdS$_5$ and S^1 independently inside S^5, we would want to set $\chi = 0$. However doing so at this stage would lead to an inconsistent truncation as demonstrated below. We thus prefer to work with the most general $U(1)$ reduction including χ at this stage.

The resulting six-dimensional Einstein equation is

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R &= \tfrac{1}{4}\partial_\mu(3\alpha+\beta)\partial_\nu(3\alpha+\beta) - \tfrac{5}{8}g_{\mu\nu}(\partial(3\alpha+\beta))^2 + \nabla_\mu\nabla_\nu(3\alpha+\beta) - g_{\mu\nu}\Box(3\alpha+\beta) \\ &\quad + \tfrac{3}{4}[\partial_\mu(\alpha-\beta)\partial_\nu(\alpha-\beta) - \tfrac{1}{2}g_{\mu\nu}(\partial(\alpha-\beta))^2] + \tfrac{1}{2}e^{-6\alpha-2\beta}[\partial_\mu\chi\partial_\nu\chi - \tfrac{1}{2}g_{\mu\nu}(\partial\chi)^2] \\ &\quad + \tfrac{1}{2}e^{-6\alpha}[F^2{}_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}F^2] + \tfrac{1}{2}e^{2\beta}[\mathcal{F}^2{}_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}\mathcal{F}^2] + 3g_{\mu\nu}e^{-2\alpha} \end{aligned} \tag{2.15}$$

and the scalar equations are

$$\begin{aligned} \partial^\mu(3\alpha+\beta)\partial_\mu\alpha + \Box\alpha &= -\tfrac{1}{4}e^{-6\alpha-2\beta}(\partial\chi)^2 - \tfrac{1}{8}e^{-6\alpha}F^2 + 2e^{-2\alpha}, \\ \partial^\mu(3\alpha+\beta)\partial_\mu\beta + \Box\beta &= -\tfrac{1}{4}e^{-6\alpha-2\beta}(\partial\chi)^2 + \tfrac{1}{8}e^{-6\alpha}F^2 + \tfrac{1}{4}e^{2\beta}\mathcal{F}^2, \\ \partial^\mu(-3\alpha-\beta)\partial_\mu\chi + \Box\chi &= \tfrac{1}{2}e^{2\beta}F_{\mu\nu}\mathcal{F}^{\mu\nu}. \end{aligned} \tag{2.16}$$

In addition, the field strengths satisfy the Bianchi identities and equations of motion

$$\begin{aligned} d\mathcal{F} &= 0, \qquad & d(e^{3\alpha+3\beta} *_6 \mathcal{F}) &= -e^{-3\alpha+\beta} *_6 F\wedge d\chi, \\ dF &= d\chi\wedge\mathcal{F}, \qquad & d(e^{-3\alpha+\beta} *_6 F) &= 0. \end{aligned} \tag{2.17}$$

The above equations of motion may be derived from an effective six-dimensional Lagrangian

$$e^{-1}\mathcal{L}_6 = e^{3\alpha+\beta}[R + \tfrac{3}{4}(\partial(3\alpha+\beta))^2 - \tfrac{3}{4}(\partial(\alpha-\beta))^2 - \tfrac{1}{2}e^{-6\alpha-2\beta}(\partial\chi)^2 - \tfrac{1}{4}e^{-6\alpha}F_{(2)}^2 - \tfrac{1}{4}e^{2\beta}\mathcal{F}_{(2)}^2 + 6e^{-2\alpha}], \tag{2.18}$$

where $F_{(2)} = dA_{(1)} + \chi\mathcal{F}_{(2)}$.

Note that if we were to take $\chi = 0$, its equation of motion (2.16) would demand the constraint $F_{\mu\nu}\mathcal{F}^{\mu\nu} = 0$. This is consistent with the independence of the S^3 in AdS$_5$ and S^1 in S^5 sectors, where $F_{(2)}$ lives in AdS$_5$ while $\mathcal{F}_{(2)}$ lives in S^5.

To make a connection with the 1/4 BPS geometries investigated in [18, 19], we may let

$$\alpha = \tfrac{1}{2}(H+G), \qquad \beta = \tfrac{1}{2}(H-G). \tag{2.19}$$

This results in a metric reduction of the form

$$ds_{10}^2 = ds_6^2 + e^H[e^G d\Omega_3^2 + e^{-G}(d\psi + \mathcal{A})^2], \tag{2.20}$$

as well as an effective Lagrangian

$$e^{-1}\mathcal{L}_6 = e^{2H+G}[R + \tfrac{3}{4}(\partial(2H+G))^2 - \tfrac{3}{4}(\partial G)^2 - \tfrac{1}{4}e^{-3(H+G)}F^2 - \tfrac{1}{4}e^{H-G}\mathcal{F}^2 + 6e^{-(H+G)}]. \tag{2.21}$$

Note that we have set $\chi = 0$. So, in addition to (2.21), we must also impose the constraint $F_{\mu\nu}\mathcal{F}^{\mu\nu} = 0$ indicated above.

2.2.1 Supersymmetry variations

From the seven-dimensional point of view, the supersymmetry conditions are encoded in the gravitino and dilatino variations (2.12) and (2.13). Given the bosonic reduction (2.14), the supersymmetry variations are easily reduced along the $U(1)$ fiber to give rise to six-dimensional variations. In particular, we may use the straightforward relation between six and seven-dimensional Dirac matrices

$$\gamma_\mu \to \begin{cases} \gamma_\mu & \mu = 0, \ldots, 5, \\ \gamma^7 \equiv \frac{1}{6!}\epsilon_{\mu_1\cdots\mu_6}\gamma^{\mu_1\cdots\mu_6} & \mu = 6, \end{cases} \tag{2.22}$$

and no additional Dirac decomposition is needed.

With this convention, the two-form field strength reduces according to

$$^7F_{\mu\nu}\gamma^{\mu\nu} = F_{\mu\nu}\gamma^{\mu\nu} + 2e^{-\beta}\gamma^\mu\gamma^7\partial_\mu\chi, \tag{2.23}$$

while the spin connections reduce according to

$$^7\omega^{\alpha\gamma} = \omega^{\alpha\gamma} - \tfrac{1}{2}e^\beta\mathcal{F}^{\alpha\gamma}e^7, \qquad ^7\omega^{\alpha 7} = -e^{\mu\,\alpha}\partial_\mu\beta e^7 - \tfrac{1}{2}e^\beta\mathcal{F}^{\alpha\gamma}e^\gamma. \tag{2.24}$$

In order to properly reduce the covariant derivative $^7\nabla_\mu$ appearing in the gravitino variation (2.12), we must keep in mind that Killing spinors ϵ may in fact be charged along the $U(1)$ fiber [4]. We thus take

$$\partial_\psi \leftrightarrow -\tfrac{i}{2}n, \tag{2.25}$$

where $n \in \mathbb{Z}$, and the sign is chosen for later convenience. This integral choice of n corresponds to the period of ψ being 2π.

Putting the above together, we find the six-dimensional 'gravitino' variation

$$\delta\psi_\mu = \left[\nabla_\mu + \frac{in}{2}\mathcal{A}_\mu - \frac{i}{16}e^{-3\alpha}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu + \frac{1}{4}e^\beta\mathcal{F}_{\mu\nu}\gamma^\nu\gamma^7 + \frac{i}{8}e^{-3\alpha-\beta}\gamma^\nu\partial_\nu\chi\gamma_\mu\gamma^7\right]\epsilon, \tag{2.26}$$

as well as the two 'dilatino' variations

$$\begin{aligned} \delta\lambda_\alpha &= \left[\gamma^\mu\partial_\mu\alpha + \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{4}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^7 - \eta e^{-\alpha}\right]\epsilon, \\ \delta\lambda_\beta &= \left[\gamma^\mu\partial_\mu\beta - \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \frac{1}{4}e^\beta\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\gamma^7 + \frac{i}{4}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^7 - ine^{-\beta}\gamma^7\right]\epsilon. \end{aligned} \tag{2.27}$$

Here λ_α is identical to λ given in (2.13), while $\lambda_\beta = 2\gamma^7\psi_7$. These variations are for the general reduction, including the axionic scalar χ. If desired, we may truncate to $\chi = 0$ and furthermore make the substitution (2.19) to arrive at the transformations [18, 19]

$$\begin{aligned} \delta\psi_\mu &= \left[\nabla_\mu + \frac{in}{2}\mathcal{A}_\mu + \frac{1}{4}e^{\frac{1}{2}(H-G)}\mathcal{F}_{\mu\nu}\gamma^\nu\gamma^7 - \frac{i}{16}e^{-\frac{3}{2}(H+G)}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu\right]\epsilon, \\ \delta\lambda_H &= \left[\gamma^\mu\partial_\mu H - \frac{1}{4}e^{\frac{1}{2}(H-G)}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\gamma^7 - \eta e^{-\frac{1}{2}(H+G)} - ine^{-\frac{1}{2}(H-G)}\gamma^7\right]\epsilon, \\ \delta\lambda_G &= \left[\gamma^\mu\partial_\mu G + \frac{1}{4}e^{\frac{1}{2}(H-G)}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\gamma^7 + \frac{i}{4}e^{-\frac{3}{2}(H+G)}F_{\mu\nu}\gamma^{\mu\nu} - \eta e^{-\frac{1}{2}(H+G)} + ine^{-\frac{1}{2}(H-G)}\gamma^7\right]\epsilon, \end{aligned} \tag{2.28}$$

corresponding to the truncated Lagrangian of (2.21).

2.3 The final reduction on CP^1 to $D=4$

Noting that S^3 can be written as $U(1)$ bundled over CP^1, we may obtain an $S^3\times S^3$ solution by reducing the effective six-dimensional system to four dimensions on CP^1. This procedure will actually allow for more general geometries, where the second S^3 is squashed along the $U(1)$ fiber. The generic (squashed S^3) $\times$ (round S^3) system has $SU(2)\times U(1)\times SO(4)$ isometry, and was investigated in [22].

The CP^1 reduction proceeds by taking

$$\begin{aligned} ds_6^2 &= ds_4^2 + e^{2\gamma} ds^2(CP^1), \\ {}^6F_{(2)} &= F_{(2)} + 2m\chi J, \\ {}^6\mathcal{F}_{(2)} &= \mathcal{F}_{(2)} + 2mJ, \end{aligned} \tag{2.29}$$

where $J_{(2)}$ is the Kähler form on CP^1. We take the standard Einstein metric on CP^1 with $\hat{R}_{ab} = \lambda \hat{g}_{ab}$.

Although the reduction is straightforward, the intermediate steps are somewhat tedious. We end up with a four-dimensional Einstein equation of the form

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R &= \tfrac{1}{6}\partial_\mu(3\alpha+\beta+2\gamma)\partial_\nu(3\alpha+\beta+2\gamma) - \tfrac{7}{12} g_{\mu\nu}(\partial(3\alpha+\beta+2\gamma))^2 \\ &\quad +\nabla_\mu\nabla_\nu(3\alpha+\beta+2\gamma) - g_{\mu\nu}\Box(3\alpha+\beta+2\gamma) \\ &\quad +\tfrac{1}{6}[\partial_\mu(3\alpha-\beta-2\gamma)\partial_\nu(3\alpha-\beta-2\gamma)] - \tfrac{1}{2} g_{\mu\nu}(\partial(3\alpha-\beta-2\gamma))^2] \\ &\quad +\tfrac{2}{3}[\partial_\mu(\beta-\gamma)\partial_\nu(\beta-\gamma) - \tfrac{1}{2} g_{\mu\nu}(\partial(\beta-\gamma))^2] + \tfrac{1}{2} e^{-6\alpha-2\beta}[\partial_\mu\chi\partial_\nu\chi - \tfrac{1}{2} g_{\mu\nu}(\partial\chi)^2] \\ &\quad +\tfrac{1}{2} e^{-6\alpha}[F^2{}_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} F^2] + \tfrac{1}{2} e^{2\beta}[\mathcal{F}^2{}_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu}\mathcal{F}^2] \\ &\quad +g_{\mu\nu}[3e^{-2\alpha} + \lambda e^{-2\gamma} - m^2 e^{2\beta-4\gamma}(1+e^{-6\alpha-2\beta}\chi^2)]. \end{aligned} \tag{2.30}$$

The three scalars α, β and γ are non-canonically normalized, while the axionic scalar χ is canonical. The four scalar equations of motion are

$$\begin{aligned} \partial^\mu(3\alpha+\beta+2\gamma)\partial_\mu\alpha + \Box\alpha &= -\tfrac{1}{4} e^{-6\alpha-2\beta}(\partial\chi)^2 - \tfrac{1}{8} e^{-6\alpha} F^2 + 2e^{-2\alpha} - m^2 e^{-6\alpha-4\gamma}\chi^2, \\ \partial^\mu(3\alpha+\beta+2\gamma)\partial_\mu\beta + \Box\beta &= -\tfrac{1}{4} e^{-6\alpha-2\beta}(\partial\chi)^2 + \tfrac{1}{8} e^{-6\alpha} F^2 + \tfrac{1}{4} e^{2\beta}\mathcal{F}^2 + m^2 e^{-6\alpha-4\gamma}\chi^2 \\ &\quad +2m^2 e^{2\beta-4\gamma}, \\ \partial^\mu(3\alpha+\beta+2\gamma)\partial_\mu\gamma + \Box\gamma &= \tfrac{1}{4} e^{-6\alpha-2\beta}(\partial\chi)^2 + \tfrac{1}{8} e^{-6\alpha} F^2 + \lambda e^{-2\gamma} - m^2 e^{-6\alpha-4\gamma}\chi^2 \\ &\quad -2m^2 e^{2\beta-4\gamma}, \\ \partial^\mu(-3\alpha-\beta+2\gamma)\partial_\mu\chi + \Box\chi &= \tfrac{1}{2} e^{2\beta} F_{\mu\nu}\mathcal{F}^{\mu\nu} + 4m^2 e^{2\beta-4\gamma}\chi, \end{aligned} \tag{2.31}$$

while the field strengths satisfy the Bianchi identities and equations of motion

$$\begin{aligned} d\mathcal{F} &= 0, \qquad & d(e^{3\alpha+3\beta+2\gamma} *_4 \mathcal{F}) &= -e^{-3\alpha+\beta+2\gamma} *_4 F\wedge d\chi, \\ dF &= d\chi\wedge\mathcal{F}, \qquad & d(e^{-3\alpha+\beta+2\gamma} *_4 F) &= 0. \end{aligned} \tag{2.32}$$

The four-dimensional Lagrangian which yields the above equations of motion is then of the form

$$\begin{aligned} e^{-1}\mathcal{L}_4 &= e^{3\alpha+\beta+2\gamma}[R+\tfrac{5}{6}(\partial(3\alpha+\beta+2\gamma))^2-\tfrac{1}{6}(\partial(3\alpha-\beta-2\gamma))^2-\tfrac{2}{3}(\partial(\beta-\gamma))^2 \\ &\quad -\tfrac{1}{2}e^{-6\alpha-2\beta}(\partial\chi)^2-\tfrac{1}{4}e^{-6\alpha}F_{(2)}^2-\tfrac{1}{4}e^{2\beta}\mathcal{F}_{(2)}^2+6e^{-2\alpha}+2\lambda e^{-2\gamma} \\ &\quad -2m^2e^{2\beta-4\gamma}(1+e^{-6\alpha-2\beta}\chi^2)]. \end{aligned} \tag{2.33}$$

Although we have introduced two constant parameters, m [which is related to the fibration in (2.29)] and λ (which is the curvature of CP^1, $\hat{R}_{ab}=\lambda\hat{g}_{ab}$), they may be scaled away by adjusting the breathing and squashing mode scalars β and γ. In particular, so long as $\lambda\neq 0$ and $m\neq 0$, we may set $m=\pm 1$, $\lambda=4$ by shifting the fields according to

$$\begin{aligned} &\beta\to\beta+\log(\lambda/4|m|), \qquad \gamma\to\gamma+\tfrac{1}{2}\log(\lambda/4), \\ &\chi\to\frac{\lambda}{4|m|}\chi, \qquad \mathcal{A}_\mu\to\frac{4|m|}{\lambda}\mathcal{A}_\mu. \end{aligned} \tag{2.34}$$

Although this transformation rescales the effective Lagrangian by an overall constant, this has no effect on the classical equations of motion. Ignoring this overall factor, (2.33) takes on the parameter free form

$$\begin{aligned} e^{-1}\mathcal{L}_4 &= e^{3\alpha+\beta+2\gamma}[R+\tfrac{5}{6}(\partial(3\alpha+\beta+2\gamma))^2-\tfrac{1}{6}(\partial(3\alpha-\beta-2\gamma))^2-\tfrac{2}{3}(\partial(\beta-\gamma))^2 \\ &\quad -\tfrac{1}{2}e^{-6\alpha-2\beta}(\partial\chi)^2-\tfrac{1}{4}e^{-6\alpha}F_{(2)}^2-\tfrac{1}{4}e^{2\beta}\mathcal{F}_{(2)}^2+6e^{-2\alpha}+8e^{-2\gamma}-2e^{2\beta-4\gamma}(1+e^{-6\alpha-2\beta}\chi^2)]. \end{aligned} \tag{2.35}$$

The above system allows for a general squashed S^3 geometry, and corresponds to the case studied in [22]. To obtain a round $S^3\times S^3$ reduction, we may take

$$\gamma=\beta, \qquad \chi=0, \qquad \mathcal{F}=0, \tag{2.36}$$

where consistency of setting the scalars γ and β equal to each other is ensured by the above choice of $|m|=1$ and $\lambda=4$. The resulting truncation becomes

$$e^{-1}\mathcal{L}_4=e^{3(\alpha+\beta)}[R+\tfrac{15}{2}(\partial(\alpha+\beta))^2-\tfrac{3}{2}(\partial(\alpha-\beta))^2-\tfrac{1}{4}e^{-6\alpha}F^2+6(e^{-2\alpha}+e^{-2\beta})]. \tag{2.37}$$

Defining

$$\alpha=\tfrac{1}{2}(H+G), \qquad \beta=\tfrac{1}{2}(H-G) \tag{2.38}$$

finally gives (1.1), which was obtained in [4] by direct $S^3\times S^3$ reduction of the LLM ansatz (1.2).

2.3.1 Supersymmetry variations

Turning to the supersymmetry variations, our aim is to reduce the six-dimensional 'gravitino' and 'dilatino' variations (2.28) on CP^1 to four-dimensions. To do so, we start by introducing a Dirac decomposition

$${}^6\gamma_\mu=\gamma_\mu\otimes 1, \qquad {}^6\gamma_a=\gamma^5\otimes\sigma_a, \tag{2.39}$$

where $a = 1, 2$ correspond to the two directions on CP^1. Note that we define $\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}$, so that $\gamma^7 = \frac{1}{6!}\epsilon_{\mu_1\cdots\mu_6}{}^6\gamma^{\mu_1\cdots\mu_6} = \gamma^5 \otimes \sigma_3$.

From (2.29), and the definition of the Kähler form, we see that the two-form field strengths reduce according to

$$
\begin{aligned}
{}^6F_{\mu\nu}\gamma^{\mu\nu} &= F_{\mu\nu}\gamma^{\mu\nu} + 4ime^{-2\gamma}\chi\sigma_3, \\
{}^6\mathcal{F}_{\mu\nu}\gamma^{\mu\nu} &= \mathcal{F}_{\mu\nu}\gamma^{\mu\nu} + 4ime^{-2\gamma}\sigma_3.
\end{aligned} \tag{2.40}
$$

Inserting this into (2.28) gives rise to a straightforward reduction of the 'dilatino' variations

$$
\begin{aligned}
\delta\lambda_\alpha &= \left[\gamma^\mu\partial_\mu\alpha + \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{4}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5\sigma_3 - \frac{1}{2}me^{-3\alpha-2\gamma}\chi\sigma_3 - \eta e^{-\alpha}\right]\epsilon, \\
\delta\lambda_\beta &= \left[\gamma^\mu\partial_\mu\beta - \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \frac{1}{4}e^{\beta}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\gamma^5\sigma_3 + \frac{i}{4}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5\sigma_3\right. \\
&\qquad \left. + \frac{1}{2}me^{-3\alpha-2\gamma}\chi\sigma_3 - i(me^{\beta-2\gamma} + ne^{-\beta}\sigma_3)\gamma^5\right]\epsilon.
\end{aligned} \tag{2.41}
$$

In order to reduce the 'gravitino' variation, we use the spin connections

$$
{}^6\omega^{\alpha\beta} = \omega^{\alpha\beta}, \qquad {}^6\omega^{\alpha b} = -e^{\mu\,\alpha}\partial_\mu\gamma e^b, \qquad {}^6\omega^{ab} = e^{-\gamma}\hat{\omega}_c^{ab}e^c, \tag{2.42}
$$

where $\hat{\omega}_c^{ab}$ is the spin connection on CP^1. This results in the four-dimensional 'gravitino' variation

$$
\begin{aligned}
\delta\psi_\mu &= \left[\nabla_\mu + \frac{in}{2}\mathcal{A}_\mu - \frac{i}{16}e^{-3\alpha}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu + \frac{1}{4}e^{\beta}\mathcal{F}_{\mu\nu}\gamma^\nu\gamma^5\sigma_3 + \frac{i}{8}e^{-3\alpha-\beta}\gamma^\nu\partial_\nu\chi\gamma_\mu\gamma^5\sigma_3\right. \\
&\qquad \left. + \frac{1}{4}me^{-3\alpha-2\gamma}\chi\gamma_\mu\sigma_3\right]\epsilon,
\end{aligned} \tag{2.43}
$$

as well as the variation on CP^1

$$
\begin{aligned}
\delta\psi_i &= \left[\hat{\nabla}_i + \frac{in}{2}\mathcal{A}_i\right]\epsilon + \frac{1}{2}e^{\gamma}\gamma^5\hat{\sigma}_i\left[\gamma^\mu\partial_\mu\gamma - \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \frac{i}{4}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5\sigma_3\right. \\
&\qquad \left. - \frac{1}{2}me^{-3\alpha-2\gamma}\chi\sigma_3 + ime^{\beta-2\gamma}\gamma^5\right]\epsilon.
\end{aligned} \tag{2.44}
$$

At this stage, there are several ways to proceed. Since we are interested in writing the squashed S^3 as $U(1)$ bundled over CP^1, we assume from now on that both λ and m are non-vanishing. In this case, the scaling of (2.34) allows us to set $\lambda = 4$ and $m = \hat{\eta}$, where $\hat{\eta} = \pm 1$ is a choice of sign. Two-component Killing spinors $\hat{\epsilon}$ on the squashed sphere can then be taken to either satisfy

$$
\left[\hat{\nabla}_i + \frac{in}{2}\mathcal{A}_i\right]\hat{\epsilon} = 0, \qquad n \neq 0, \tag{2.45}
$$

or

$$
\left[\hat{\nabla}_i + \frac{i\hat{\eta}}{2}\hat{\sigma}_i\right]\hat{\epsilon} = 0, \qquad n = 0. \tag{2.46}
$$

This second possibility corresponds to ordinary Killing spinors on CP^1. However, the sign in the Killing spinor equation (2.46) is not arbitrary, but rather is fixed to ensure that these Killing spinors descend properly from those on the squashed S^3. At this point a note is in order concerning the $\hat{\eta}$, which is the sign of m. From (2.29), we my infer that changing the sign of $\hat{\eta}$ corresponds to changing the sign of the gauge bundle on the $U(1)$ fiber, which in term corresponds to orientation reversal on the squashed S^3. In general, orientation issues may be rather subtle in squashed sphere compactifications, with only one choice of sign yielding a supersymmetric configuration [27, 28]. It is for this reason that we have kept $\hat{\eta}$ as a parameter. Nevertheless, it is important to keep in mind that $\hat{\eta}$ is a parameter specifying the bosonic field configuration, and that changing the sign of $\hat{\eta}$ (flipping the orientation) in principle changes the solution. For this reason, $\hat{\eta}$ ought to be thought of as a fixed constant, unlike the Killing spinor sign parameters η and $\widetilde{\eta}$ (defined below), which may be chosen freely.

For the first case ($n \neq 0$), the Killing spinors are charged along the $U(1)$ fiber, but are (gauge) covariantly constant on CP^1. Integrability of (2.45) shows that the $U(1)$ charge is given by $n = \pm 2$, with corresponding projection condition

$$\sigma_3 \hat{\epsilon} = \widetilde{\eta} \hat{\epsilon}, \tag{2.47}$$

where $\widetilde{\eta} = \pm 1$. The sign in the projection is correlated with the $U(1)$ charge according to $n = -2\hat{\eta}\widetilde{\eta}$. Taking these various signs into account, we end up with the 'gravitino' and 'dilatino' variations

$$\begin{aligned}
\delta\psi_\mu &= \Big[\nabla_\mu - i\hat{\eta}\widetilde{\eta}\mathcal{A}_\mu - \frac{i}{16}e^{-3\alpha}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu + \frac{1}{4}\widetilde{\eta}e^{\beta}\mathcal{F}_{\mu\nu}\gamma^\nu\gamma^5 + \frac{i}{8}\widetilde{\eta}e^{-3\alpha-\beta}\gamma^\nu\partial_\nu\chi\gamma_\mu\gamma^5 \\
&\qquad + \frac{1}{4}\hat{\eta}\widetilde{\eta}e^{-3\alpha-2\gamma}\chi\gamma_\mu\Big]\epsilon, \\
\delta\lambda_\alpha &= \Big[\gamma^\mu\partial_\mu\alpha + \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{4}\widetilde{\eta}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5 - \frac{1}{2}\hat{\eta}\widetilde{\eta}e^{-3\alpha-2\gamma}\chi - \eta e^{-\alpha}\Big]\epsilon, \\
\delta\lambda_\beta &= \Big[\gamma^\mu\partial_\mu\beta - \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \frac{1}{4}\widetilde{\eta}e^{\beta}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\gamma^5 + \frac{i}{4}\widetilde{\eta}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5 + \frac{1}{2}\hat{\eta}\widetilde{\eta}e^{-3\alpha-2\gamma}\chi \\
&\qquad + i\hat{\eta}(2e^{-\beta} - e^{\beta-2\gamma})\gamma^5\Big]\epsilon, \\
\delta\lambda_\gamma &= \Big[\gamma^\mu\partial_\mu\gamma - \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \frac{i}{4}\widetilde{\eta}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5 - \frac{1}{2}\hat{\eta}\widetilde{\eta}e^{-3\alpha-2\gamma}\chi + i\hat{\eta}e^{\beta-2\gamma}\gamma^5\Big]\epsilon.
\end{aligned} \tag{2.48}$$

Note that $\delta\lambda_\gamma$ is obtained from the gravitino variation $\delta\psi_i$ on CP^1. Because of the projection (2.47), a complete set of Killing spinors is obtained only after taking into account both signs of $\widetilde{\eta}$.

For the second case ($n = 0$), the Killing spinors are uncharged along the $U(1)$ fiber. In this case, we end up with the variations

$$\begin{aligned}
\delta\psi_\mu &= \left[\nabla_\mu - \frac{i}{16}e^{-3\alpha}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu + \frac{1}{4}\widetilde{\eta}e^{\beta}\mathcal{F}_{\mu\nu}\gamma^\nu\gamma^5 + \frac{i}{8}\widetilde{\eta}e^{-3\alpha-\beta}\gamma^\nu\partial_\nu\chi\gamma_\mu\gamma^5 + \frac{1}{4}\hat{\eta}\widetilde{\eta}e^{-3\alpha-2\gamma}\chi\gamma_\mu\right]\epsilon, \\
\delta\lambda_\alpha &= \left[\gamma^\mu\partial_\mu\alpha + \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{4}\widetilde{\eta}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5 - \frac{1}{2}\hat{\eta}\widetilde{\eta}e^{-3\alpha-2\gamma}\chi - \eta e^{-\alpha}\right]\epsilon, \\
\delta\lambda_\beta &= \left[\gamma^\mu\partial_\mu\beta - \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \frac{1}{4}\widetilde{\eta}e^{\beta}\mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\gamma^5 + \frac{i}{4}\widetilde{\eta}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5 + \frac{1}{2}\hat{\eta}\widetilde{\eta}e^{-3\alpha-2\gamma}\chi \right. \\
&\qquad \left. -i\hat{\eta}e^{\beta-2\gamma}\gamma^5\right]\epsilon, \\
\delta\lambda_\gamma &= \left[\gamma^\mu\partial_\mu\gamma - \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \frac{i}{4}\widetilde{\eta}e^{-3\alpha-\beta}\gamma^\mu\partial_\mu\chi\gamma^5 - \frac{1}{2}\hat{\eta}\widetilde{\eta}e^{-3\alpha-2\gamma}\chi - i\hat{\eta}(2e^{-\gamma} - e^{\beta-2\gamma})\gamma^5\right]\epsilon,
\end{aligned} \tag{2.49}$$

where $\delta\lambda_\gamma$ was obtained by substituting (2.46) into (2.44). Although no σ_3 projection is involved in this case, it is nevertheless still convenient to break up the Killing spinor expressions into definite σ_3 eigenvalues corresponding to (2.47). In addition to the lack of gauge connection $\mathcal{A}_\mu$ in the 'gravitino' variation, these expressions differ from those in the first case, (2.48), in the 'superpotential' gradient terms in the λ_β and λ_γ variations. Note that, in both cases, the orientation sign $\hat{\eta}$ may be removed by taking $\chi \to \hat{\eta}\chi$, $\mathcal{A}_\mu \to \hat{\eta}\mathcal{A}_\mu$ and $\gamma^5 \to \hat{\eta}\gamma^5$. It is the latter transformation on γ^5 that highlights the orientation reversal nature of this map.

The above supersymmetry variations simplify considerably in the round $S^3 \times S^3$ limit, given by (2.36). Here, we obtain

$$\begin{aligned}
\delta\psi_\mu &= \left[\nabla_\mu - \frac{i}{16}e^{-3\alpha}F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu\right]\epsilon, \\
\delta\lambda_\alpha &= \left[\gamma^\mu\partial_\mu\alpha + \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} - \eta e^{-\alpha}\right]\epsilon, \\
\delta\lambda_\beta &= \left[\gamma^\mu\partial_\mu\beta - \frac{i}{8}e^{-3\alpha}F_{\mu\nu}\gamma^{\mu\nu} \pm i\hat{\eta}e^{-\beta}\gamma^5\right]\epsilon,
\end{aligned} \tag{2.50}$$

where the $+$ sign corresponds to the $U(1)$ charged Killing spinor case, and the $-$ sign to the uncharged case. These expressions reproduce the supersymmetry variations of the LLM construction, [1,4], as they must. Here we see that the sign choice in the last term of the $\delta\lambda_\beta$ variation comes from the two types of Killing spinors on the (un)squashed sphere, and not from the orientation sign $\hat{\eta}$ (which can be absorbed by a redefinition of γ^5).

3 Supersymmetry analysis

3.1 1/8 BPS configurations

We begin with the general 1/8 BPS bubbling case, which only an S^3 inside AdS$_5$ is preserved. In this case, the relevant supersymmetry variations are (2.12) and (2.13). A double Wick rotated version of this system (*i.e.* one with AdS_3 instead of S^3 isometry) was recently investigated in [20], and the results are directly applicable to the present case.

The analysis of [20] demonstrated that the seven-dimensional metric may be written as time fibered over a six (real) dimensional Kähler base which satisfies an appropriate geometric condition. Here we briefly review this construction.

For a Dirac spinor ϵ in seven dimensions, we start by forming a set of Dirac bilinears

$$f = i\bar{\epsilon}\epsilon, \qquad K^\mu = \bar{\epsilon}\gamma^\mu\epsilon, \qquad V^{\mu\nu} = \bar{\epsilon}\gamma^{\mu\nu}\epsilon, \qquad Z^{\mu\nu\lambda} = i\bar{\epsilon}\gamma^{\mu\nu\lambda}\epsilon. \tag{3.1}$$

The factors of i are chosen to make these quantities real. In addition to the above, we may also form a set of (complex) Majorana bilinears

$$f^m = \epsilon^c\epsilon, \qquad Z^m_{\mu\nu\lambda} = \epsilon^c\gamma_{\mu\nu\lambda}\epsilon. \tag{3.2}$$

Counting the individual tensor components of the above, we find 64 real Dirac bilinear components and 36 complex Majorana bilinear components, giving rise to $136 = \frac{1}{2}(16\cdot 17)$ total real components. Since this matches the number of bilinears formed out of a spinor ϵ with 16 real components, we see that this set of bilinears is complete.

Of course, these tensor quantities are highly constrained by the algebraic identities (corresponding to Fierz rearrangement). Here we do not aim to be comprehensive, but simply list some relevant identities. First we have the normalization conditions

$$K^2 = -f^2 - |f^m|^2, \qquad V^2 = 6f^2 + 6|f^m|^2, \qquad Z^2 = -18f^2 + 24|f^m|^2, \qquad |Z^m|^2 = 48f^2 + 6|f^m|^2. \tag{3.3}$$

Then there are the orthogonality conditions

$$K^\mu V_{\mu\nu} = 0, \qquad K^\mu Z^m_{\mu\nu\lambda} = f^m V_{\nu\lambda}. \tag{3.4}$$

Finally, there are the identities which are directly useful for determining the structure

$$fZ + K\wedge V + \Re(f^{m*}Z^m) = 0, \tag{3.5}$$
$$V\wedge Z^m = -2f^m * V, \tag{3.6}$$
$$V\wedge V = -2*(K\wedge V), \tag{3.7}$$
$$K\wedge Z^m = -i*(fZ^m - f^m Z), \tag{3.8}$$
$$Z^m\wedge Z^{m*} = 8if*K. \tag{3.9}$$

Here f^{m*} and Z^{m*} denote the complex conjugates of f^m and Z^m, respectively.

As shown in [20], backgrounds preserving (at least) 1/8 of the supersymmetries necessarily have SU(3) structure. To see this, we first note that (3.3) constrains the norm of K^μ to be non-positive. Furthermore, from (A.11), we see that K^μ satisfies the Killing equation. We may thus choose K^μ as a preferred time like Killing vector $K^\mu\partial_\mu = \partial/\partial t$. (Although the null possibility may be of interest, we do not pursue it here, as we are mainly interested in bubbling AdS configurations.) In fact, we may deduce a fair bit more about the structure by noting from (A.14) that the Majorana scalar invariant f^m necessarily vanishes. This gives us the norms of the tensors

$$K^2 = -f^2, \qquad V^2 = 6f^2, \qquad Z^2 = -18f^2, \qquad |Z^m|^2 = 48f^2, \tag{3.10}$$

as well as the conditions that V and Z^m are orthogonal to K^μ

$$i_K V = i_K Z^m = 0. \tag{3.11}$$

Using (3.5), we may also solve for Z

$$Z = -f^{-1} K \wedge V, \tag{3.12}$$

demonstrating that Z is not an independent tensor quantity. As a result, the structure is implicitly defined by the time-like Killing vector K^μ along with a real 2-form V and complex 3-form Z^m. Using (3.6), (3.7) and (3.9), it as easy to see that

$$V \wedge Z^m = 0, \qquad V \wedge V \wedge V = \tfrac{3i}{4} f Z^m \wedge Z^{m\,*} = -6f^2 * K. \tag{3.13}$$

But this is simply the requirement for SU(3) structure in 6 + 1 dimensions. Thus the seven-dimensional space splits naturally into time and a six (real) dimensional base with SU(3) structure.

To proceed with an explicit construction, we may now solve (A.2) to obtain $f = e^\alpha$. We then make a choice of metric of the form

$$ds_7^2 = -e^{2\alpha}(dt + \omega)^2 + e^{-2\alpha} h_{ij} dx^i dx^j. \tag{3.14}$$

The one-form associated with the Killing vector $K^\mu \partial_\mu = \partial_t$ is then $K_\mu dx^\mu = -e^{2\alpha}(dt+\omega)$. Following [20], we define the canonical two-form J and the holomorphic three-form

$$J = e^\alpha V, \qquad \Omega = e^{2\alpha} e^{-2i\eta t} Z^m. \tag{3.15}$$

Note that Ω is independent of time. The restriction (3.13) onto the six-dimensional base gives the usual SU(3) structure conditions

$$J \wedge \Omega = 0, \qquad J \wedge J \wedge J = \tfrac{3i}{4} \Omega \wedge \Omega^* = -6 *_6 1, \tag{3.16}$$

while the differential identities (A.6) and (A.17) give the integrability equations

$$dJ = 0, \qquad d\Omega = 2i\eta\omega \wedge \Omega. \tag{3.17}$$

This ensures that the six-dimensional base has U(3) holonomy. In other words, it is Kähler, with the Kähler form

$$J = i h_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} = \frac{1}{2} J_{ij} dx^i \wedge dx^j, \tag{3.18}$$

and the Ricci form

$$\mathcal{R} = i R_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = 2\eta d\omega.$$

In addition, the differential identities constrain the two-form F and scalar α to satisfy

$$F = d[e^{4\alpha}(dt+\omega)] - 2\eta J, \qquad e^{-4\alpha} = -\tfrac{1}{8}R, \tag{3.19}$$

where R is the scalar curvature of h_{ij} [20].

Finally, to guarantee that the above is a true solution to the equations of motion, we may apply the Bianchi identity and equation of motion for $F_{(2)}$. From (3.19) along with $dJ = 0$ the Bianchi identity turns out to be trivial, while the $F_{(2)}$ equation of motion gives

$$\Box_6 e^{-4\alpha} = \tfrac{1}{8}(R_{ij}R^{ij} - \tfrac{1}{2}R^2), \tag{3.20}$$

where $\Box_6$ as well as the tensor contraction is with respect to the base metric h_{ij}. Substituting in the expression for $e^{-4\alpha}$ in (3.19) then gives a condition on the curvature

$$\Box_6 R = -R_{ij}R^{ij} + \tfrac{1}{2}R^2. \tag{3.21}$$

In summary, 1/8 BPS configurations preserving an S^3 isometry may be described by a seven-dimensional metric (3.14) with form field and scalar given by (3.19). The one-form ω is defined according to $\mathcal{R} = 2\eta d\omega$, where the sign η is related to the orientation of the Killing spinor on S^3. The full solution is determined in terms of a six-real dimensional Kähler metric h_{ij} satisfying the curvature condition (3.21).

From a ten-dimensional point of view, the solution is essentially given by time and S^3 fibered over the six-dimensional base. In order to ensure regularity, we may focus on regions on the base where the S^3 fiber shrinks to zero size. This corresponds to regions where $e^\alpha \to 0$, which by (3.19) corresponds to $R \to \infty$. Thus the six-dimensional base generally will be bounded by surfaces of infinite curvature where the S^3 degenerates. At the same time, the $e^{-2\alpha}$ factor in front of the six-dimensional metric ought to be such that the physical ten-dimensional metric remains regular. Furthermore, the collapsing S^3 along with the transverse direction to the degeneration surface must locally yield $\mathbb{R}^4$ to ensure the absence of conical singularities. Examination of these boundary conditions will be taken up in Sections 4 and 7 below.

3.2 1/4 BPS configurations

Following the above analysis, we now turn to the 1/4 BPS case preserving $S^3 \times S^1$ isometry. Here there are at least two possible approaches that may be taken. The first is to realize that, since 1/4 BPS configurations form a subset of all 1/8 BPS solutions, we may simply take the above 1/8 BPS analysis and demand that the resulting geometry admits a further $U(1)$ isometry. The second is to directly analyze the effective six-dimensional supersymmetry variations (2.26) and (2.27). The advantage of this method, which was recently employed in [18, 19], is that it leads to a natural choice of coordinates with which to parameterize the solution.

Before turning to the full supersymmetry analysis of [18, 19], we first examine the possibility of imposing an additional $U(1)$ isometry on the 1/8 BPS solutions described above. Noting that the generic solution is given in terms of a complex three-dimensional Kähler base identified by

(3.17) and with curvature satisfying (3.21), we may locally choose an appropriate set of complex coordinates

$$z_1, \quad z_2, \quad z_3 \equiv re^{i\psi}, \tag{3.22}$$

and impose symmetry under ψ translation (*i.e.* by demanding that $\partial/\partial\psi$ is a Killing vector). This indicates that the Kähler potential ought to be of the form

$$K(z_i, \overline{z}_i, r^2) \qquad i = 1, 2. \tag{3.23}$$

This Kähler potential leads to a metric on the base of the form

$$\begin{aligned} h_{ij} dx^i dx^j &= h_{i\bar{j}} dz^i dz^{\bar{j}} + c.c. = 2h_{i\bar{j}} dz^i dz^{\bar{j}} = 2\partial_i \partial_{\bar{j}} K dz^i dz^{\bar{j}} \\ &= 2\partial_i \partial_{\bar{j}} K dz_i d\overline{z}_j + 2(r^2K')'(dr^2 + r^2 d\psi^2) + 4r dr \Re(\partial_i K' dz_i) + 4r^2 d\psi \Im(\partial_i K' dz_i), \end{aligned} \tag{3.24}$$

where a prime denote partial differentiation with respect to r^2, and $\Re$ and $\Im$ denote real and imaginary parts, respectively. After completing the square, this may be rewritten as

$$\begin{aligned} h_{ij} dx^i dx^j &= 2\left(\partial_i \partial_{\bar{j}} K - \frac{r^2}{(r^2K')'} \partial_i K' \partial_{\bar{j}} K'\right) dz_i d\overline{z}_j + \frac{1}{2r^2(r^2K')'} d(r^2K')^2 \\ &\quad + 2r^2(r^2K')' \left(d\psi + \frac{1}{(r^2K')'} \Im(\partial_i K' dz_i)\right)^2. \end{aligned} \tag{3.25}$$

A change of variables $y^2 = 2r^2K'$ brings this to the form

$$\begin{aligned} & h_{ij} dx^i dx^j = 2\left(\partial_i \partial_{\bar{j}} K - \frac{2r^2}{(y^2)'} \partial_i K' \partial_{\bar{j}} K'\right) dz_i d\overline{z}_j + \frac{y^2}{r^2(y^2)'} dy^2 + r^2(y^2)'(d\psi + \mathcal{A})^2, \\ & \mathcal{A} = \frac{2}{(y^2)'} \Im(\partial_i K' dz_i)\,, \end{aligned} \tag{3.26}$$

where $(y^2)' = (2r^2K')'$, and r is to be eliminated by inverting the above transformation.

Although this form of the metric is suggestive that the complex three-dimensional base splits into a two-dimensional piece along with a 'radial' coordinate y and fiber direction ψ, the physical understanding of this solution is somewhat obscure. For this reason, it is instructive to perform the supersymmetry analysis directly with the actual variations (2.26) and (2.27). This analysis, which was initiated in [18, 19], starts with the definition of the (Dirac and Majorana) spinor bilinears

$$\begin{aligned} & f_1 = \overline{\epsilon}\gamma^7\epsilon, \qquad f_2 = i\overline{\epsilon}\epsilon, \qquad K^\mu = \overline{\epsilon}\gamma^\mu\epsilon, \qquad L^\mu = \overline{\epsilon}\gamma^\mu\gamma^7\epsilon, \\ & V^{\mu\nu} = \overline{\epsilon}\gamma^{\mu\nu}\epsilon, \qquad Y^{\mu\nu} = i\overline{\epsilon}\gamma^{\mu\nu}\gamma^7\epsilon, \qquad Z^{\mu\nu\lambda} = i\overline{\epsilon}\gamma^{\mu\nu\lambda}\epsilon, \\ & f^m = \epsilon^c\epsilon, \qquad Y^m_{\mu\nu} = \epsilon^c\gamma_{\mu\nu}\gamma^7\epsilon, \qquad Z^m_{\mu\nu\lambda} = \epsilon^c\gamma_{\mu\nu\lambda}\epsilon. \end{aligned} \tag{3.27}$$

We have highlighted the close relation between six and seven-dimensional Dirac spinors by using an identical notation with the bilinears defined above in (3.1) and (3.2), except for the cases where γ^7 is involved (and with a rewriting $f \to f_2$ consistent with the LLM notation). The 'new' bilinears with γ^7 are of course the components of the seven-dimensional bilinears (3.1) and (3.2) along the circle direction.

Although the six-dimensional Fierz identities may in principle be derived from the seven-dimensional ones, some of the expressions we are interested in cannot be written in a seven-dimensional covariant manner. Thus we work directly with the above bilinears in six dimensions. In this case, we have the normalization conditions

$$\begin{aligned}
&K^2 = -L^2 = -f_1^2 - f_2^2 - |f^m|^2, \qquad V^2 = -2f_1^2 + 4f_2^2 + 4|f^m|^2,\\
&Y^2 = 4f_1^2 - 2f_2^2 + 4|f^m|^2, \qquad Z^2 = -12f_1^2 - 12f_2^2 + 12|f^m|^2,\\
&|Y^m|^2 = 8f_1^2 + 8f_2^2 + 2|f^m|^2, \qquad |Z^m|^2 = -24f_1^2 + 24f_2^2.
\end{aligned} \tag{3.28}$$

We also have identities related to the projection of the various tensors onto K^μ and L^μ

$$\begin{aligned}
&K \cdot L = 0,\\
&K^\mu V_{\mu\nu} = f_1 L_\nu, \qquad L^\mu V_{\mu\nu} = f_1 K_\nu,\\
&K^\mu Y_{\mu\nu} = f_2 L_\nu, \qquad L^\mu Y_{\mu\nu} = f_2 K_\nu,\\
&K^\mu Y^m_{\mu\nu} = f^m L_\nu, \qquad L^\mu Y^m_{\mu\nu} = f^m K_\nu,\\
&K^\mu Z_{\mu\nu\lambda} = -f_1 Y_{\nu\lambda} + f_2 V_{\nu\lambda}, \qquad L^\mu Z_{\mu\nu\lambda} = \Im(f^m Y^{m\,*}_{\nu\lambda}),\\
&K^\mu Z^m_{\mu\nu\lambda} = -f_1 Y^m_{\nu\lambda} + f^m V_{\nu\lambda}, \qquad L^\mu Z^m_{\mu\nu\lambda} = -if_2 Y^m_{\nu\lambda} + if^m Y_{\nu\lambda}.
\end{aligned} \tag{3.29}$$

Finally, the following Fierz identities are useful for determining the structure

$$\begin{aligned}
&f_1 V + f_2 Y + \Re(f^m Y^{m\,*}) = -K \wedge L,\\
&K \wedge Z = *\Im(f^m Y^{m\,*}), \qquad L \wedge Z = *(f_2 V - f_1 Y),\\
&K \wedge Z^m = -i * (f_2 Y^m - f^m Y), \qquad L \wedge Z^m = - * (f_1 Y^m - f^m Y).
\end{aligned} \tag{3.30}$$

Since the six-dimensional bilinears parallel those of the seven-dimensional case, it is not surprising to see from (B.6) that the Majorana scalar invariant f^m vanishes in this case as well. Setting $f^m = 0$, we now obtain

$$K^2 = -L^2 = -f_1^2 - f_2^2, \qquad K \cdot L = 0, \tag{3.31}$$

which we note is identical to the LLM case, even though we are working in six dimensions instead of four. This ensures that K^μ is time-like while L_μ is space-like and orthogonal to K^μ. This gives rise to a natural decomposition of the six-dimensional space into a four-dimensional base along with a preferred time-like and a preferred space-like direction.

Furthermore, the above identities allow us to decompose the bilinears into components along K^μ and L_μ and those orthogonal to them. The result is

$$\begin{aligned}
&V = -\frac{f_1}{f_1^2 + f_2^2} K \wedge L - f_2 I^3, \qquad Y = -\frac{f_2}{f_1^2 + f_2^2} K \wedge L + f_1 I^3, \qquad Z = K \wedge I^3,\\
&Y^m = -\sqrt{f_1^2 + f_2^2}(I^1 - iI^2), \qquad Z^m = -\frac{1}{\sqrt{f_1^2 + f_2^2}}(f_1 K - if_2 L) \wedge (I^1 - iI^2),
\end{aligned} \tag{3.32}$$

where the triplet of two-forms I^i are orthogonal to both K^μ and L_μ and satisfy the SU(2) structure equation

$$I^i_{ab} I^j_{bc} = -\delta_{ac}\delta^{ij} - \epsilon^{ijk} I^k_{ac}, \tag{3.33}$$

as well as the self-duality condition

$$I^i_{ab} = \tfrac{1}{2}\epsilon_{abcd} I^i_{cd}, \tag{3.34}$$

on the four-dimensional base. It should be noted, however that since the Majorana bilinears are charged under the $U(1)$ gauge symmetry carried by $\mathcal{A}_\mu$, the two-forms $I^1 \pm iI^2$ carry $U(1)$ charge, while only I^3 is neutral. The implication of this is that only I^3 is gauge invariant, and as a result we conclude that the system has $U(2)$ structure in $5+1$ dimensions, except for backgrounds with vanishing $\mathcal{A}_\mu$, which instead carry $SU(2)$ structure. In either case, the structure group is a subgroup of $SU(3)$, which showed up as the structure group pertaining to the 1/8 BPS solutions found above.

In contrast to the 1/8 BPS analysis given above, an explicit construction of 1/4 BPS configurations is complicated by the fact that many more field components now need to be specified. In addition to the six-dimensional metric $g_{\mu\nu}$, we have the three scalars α, β and χ as well as the field strengths $F_{(2)}$ and $\mathcal{F}_{(2)}$. We note, however, that the axionic scalar χ is related to the IIB five-form flux threading both S^3 and S^1 in the reduction in the sense that

$${}^{10}F_{(5)} = d\chi \wedge (d\psi + \mathcal{A}) \wedge \omega_3 + \cdots . \tag{3.35}$$

While this is certainly allowed by the isometries, any excitation of χ necessarily falls outside of the 'bubbling AdS' interpretation, as non-zero χ corresponds to mixed components of five-form flux (where S^3 is inside AdS$_5$ and S^1 is inside S^5). We thus specialize the analysis by taking $\chi = 0$. At the same time, we recall that such a truncation leads to the requirement $F_{\mu\nu}\mathcal{F}^{\mu\nu} = 0$, which will be expected to show up as additional constraints on the solution.

Following [18, 19], the supersymmetry analysis begins by using the one-form identities given in (B.8) through (B.13) to obtain the scalar bilinears f_1 and f_2 in terms of the fields α and β and then to solve for the components of the field strengths $F_{(2)}$ and $\mathcal{F}_{(2)}$. Noting from (B.8) that $d(e^{-\alpha} f_2) = 0$, we may immediately write $f_2 = ae^\alpha$ for some constant a. However, obtaining an expression for f_1 is somewhat more involved. To proceed, we make the simplifying assumption that $i_K\mathcal{F} = 0$, which was also imposed in [19]. This assumption that the electric component of $\mathcal{F}_{(2)}$ vanishes ensures that the $U(1)$ bundle is only fibered over the spatial components of the metric. This is consistent with taking the gauged $U(1)$ to be contained inside the original S^5 as opposed to AdS$_5$, so we do not believe this assumption to be overly restrictive, at least as far as bubbling geometries are concerned. In any case, we keep in mind that the following supersymmetry analysis only pertains to the specialization of the most general $S^3 \times S^1$ system to the case when

$$\chi = 0, \qquad i_K\mathcal{F} = 0. \tag{3.36}$$

Having imposed $i_K\mathcal{F} = 0$, (B.11) may then be solved to yield $f_1 = be^\beta$ for constant b. As a result, all scalar bilinears are now fully determined

$$f^m = 0, \qquad f_1 = be^\beta, \qquad f_2 = ae^\alpha. \tag{3.37}$$

At this point, it is useful to specialize the form of the six-dimensional metric. Noting from (B.23) that K^μ is a Killing vector, we take $K^\mu\partial_\mu = \partial_t$. Furthermore, (B.9) then gives $L = -\eta b\, de^{\alpha+\beta}$, so that L is a closed one-form. In particular, using (3.37), we may express $y = -\eta a^{-1} f_1 f_2$ if desired. From (3.31), we may now specialize the choice of coordinates to take $L = dy$. As a result, we now make a choice of metric of the form

$$ds_6^2 = -h^{-2}(dt+\omega)^2 + f_2^{-2} h_{ij} dx^i dx^j + h^2 dy^2, \tag{3.38}$$

where

$$h^{-2} = f_1^2 + f_2^2, \qquad K = -h^{-2}(dt+\omega), \qquad L = dy, \tag{3.39}$$

and we have included a factor of f_2^{-2} in front of the four-dimensional metric h_{ij} for latter convenience.

Given the above, the remaining one-form differential identities (B.10) through (B.13) allow us to determine most components of $F_{(2)}$ and $\mathcal{F}_{(2)}$. We find

$$\begin{aligned} a^3 F_{(2)} &= d(f_2^4) \wedge (dt+\omega) + 4h^2 f_2^5 I_i^{3j} \partial_j f_1\, dx^i \wedge dy + \tfrac{1}{2} a^3 F_{ij} dx^i \wedge dx^j, \\ \mathcal{F}_{(2)} &= \tfrac{1}{2} \mathcal{F}_{ij} dx^i \wedge dx^j, \end{aligned} \tag{3.40}$$

where

$$a^3 I_{ij}^3 F^{ij} = -8 f_2^{-1} \partial_y f_1, \qquad I_{ij}^3 \mathcal{F}^{ij} = -4b^2 f_1^{-2} f_2^{-4} \left(\frac{a}{b}\eta - n\right). \tag{3.41}$$

Note that indices on the four-dimensional base are raised and lowered with the metric h_{ij}.

Before completing the determination of the two-form field strengths, we examine the content of the three-form identity (B.25), which states $d(f_2 V) = b^{-1} f_1 f_2 \mathcal{F} \wedge dy$. Using the structure identities (3.32), we may write $V = f_1(dt+\omega)\wedge dy - f_2 I^3$. As a result, (B.25) leads to the identities

$$\tilde{d}(f_2^2 I^3) = 0, \qquad \tilde{d}\omega = b^{-1}\mathcal{F} + (f_1 f_2)^{-1} \partial_y (f_2^2 I^3), \tag{3.42}$$

where $\tilde{d} = dx^i \partial_i$ acts only on the four-dimensional base. At this stage, it ought to be clear why we have chosen a prefactor f_2^{-2} in front of the base metric h_{ij} in (3.38). This is because, by defining

$$I^3 = f_2^{-2} J, \tag{3.43}$$

we obtain the canonical two-form J which is closed ($\tilde{d}J = 0$), and which satisfies $J \wedge J = 2 *_4 1$, where the volume form is given in terms of h_{ij}. This in particular indicates that the four-dimensional base is Kähler.

Additional information on the form of the solution remains to be extracted from the $\nabla_\mu V_{\nu\lambda}$ identity, (B.34). Examining $\nabla_y V_{ij}$ and $\nabla_i V_{jk}$ yield the identities

$$\nabla_i^4 J_{jk} = 0, \qquad \partial_y J_i{}^j = 0, \tag{3.44}$$

confirming that J is covariantly constant with respect to the metric h_{ij}. Note, however, that while $J_i{}^j$ is independent of y, in general both J_{ij} and h_{ij} are highly non-trivial functions of y. The remaining components of (B.34) serve to complete the determination of the two-forms

$$\begin{aligned} F &= \frac{1}{a^3}d[f_2^4(dt+\omega)] + \frac{y^2}{a}(d\omega - b^{-1}\mathcal{F}) + \frac{2\eta}{a^2}J, \\ \mathcal{F} &= \tfrac{1}{2}\mathcal{F}_{ij}dx^i\wedge dx^j, \qquad \mathcal{F}_{ij}^{(+)} = -\frac{b^2}{a^2y^2}\left(\frac{a}{b}\eta - n\right)J_{ij}, \\ d\omega &= \frac{1}{b}\mathcal{F} - \frac{\eta}{ay}\left(\partial_y J - J_i{}^j\partial_j Z dx^i\wedge dy\right). \end{aligned} \tag{3.45}$$

Here, as in [18, 19], we have defined the LLM function

$$Z = \frac{1}{2}\frac{f_2^2 - f_1^2}{f_2^2 + f_1^2}. \tag{3.46}$$

Note that the anti-self-dual part of $\mathcal{F}$ is unconstrained by the differential identities.

Given these field strengths, the second expression in (3.41) is identically satisfied. On the other hand, compatibility of $J_{ij}F^{ij}$ between the first expression in (3.41) and the form of F given in (3.45) gives rise to an important condition on the volume of the Kähler base

$$J^{ij}\partial_y J_{ij} \equiv \partial_y \log\det h_{ij} = 4h^2\left[\frac{2f_1^2}{f_2}\partial_y f_2 + \frac{bf_2}{f_1}\left(\frac{a}{b}\eta - n\right)\right]. \tag{3.47}$$

By substituting in

$$f_1 f_2 = -a\eta y, \qquad \frac{f_1}{f_2} = e^{-G}, \tag{3.48}$$

the above expression may be brought into the form

$$\tfrac{1}{2}\partial_y \log\det h_{ij} = \frac{2e^{-G}}{e^G + e^{-G}}\partial_y G + \frac{2}{y(1+e^{2G})}\left(2 - \frac{b}{a}n\eta\right) - \frac{2}{y}\left(1 - \frac{b}{a}n\eta\right), \tag{3.49}$$

originally given in [19]. The factor of 1/2 on the left hand side arises because here we still take h_{ij} as a real metric.

To ensure a complete solution to the equations of motion, we now apply the Bianchi identities and equations of motions (2.17), which for $\chi = 0$ take on the simple form

$$0 = dF = d\mathcal{F} = d(f_1 f_2^{-3} *_6 F) = d(f_1^3 f_2^3 *_6 \mathcal{F}). \tag{3.50}$$

We begin with the Bianchi identities. Since $\mathcal{F}$ is incompletely specified, we are left with the requirement $d\mathcal{F} = 0$, which admits no particular simplification. For $dF = 0$, however, we see from (3.45) that it is automatically satisfied, provided $\mathcal{F}$ and $d\omega$ are both closed. Actually $d^2\omega = 0$ is not guaranteed in the above expression. Instead, just as in the LLM case [1], it gives rise to the second-order condition

$$iy\partial_y \frac{1}{y}\partial_y J_{ij} + 2J_{[j}{}^k\nabla_{i]}\nabla_k Z = 0. \tag{3.51}$$

Introducing a Kähler potential K with

$$h_{ij} = \tfrac{1}{2}(\nabla_i\nabla_j + J_i{}^k J_j{}^l \nabla_k\nabla_l)K, \tag{3.52}$$

we see that the condition (3.51) may be solved by taking

$$Z(x^i, y) = -\frac{1}{2} y \partial_y \frac{1}{y} \partial_y K(x^i, y). \tag{3.53}$$

Note that, while an arbitrary harmonic function may be added to Z, this may be absorbed by making an appropriate Kähler transformation on K.

Turning to the equations of motion, we see that the $\mathcal{F}$ equation of motion given in (3.50) is equivalent to $d(y^3 *_6 \mathcal{F}) = 0$. Through appropriate manipulations, and using the fact that $*_4\mathcal{F} = \mathcal{F}^{(+)} - \mathcal{F}^{(-)} = 2\mathcal{F}^{(+)} - \mathcal{F}$, we may show that this is equivalent to

$$\mathcal{F}_{ij}\mathcal{F}^{ij} = \frac{\eta b}{ay}\mathcal{F}^{ij}\partial_y J_{ij}. \tag{3.54}$$

Using the Bianchi identity $d\mathcal{F} = 0$, and in particular $\partial_y \mathcal{F}_{ij} = 0$, we obtain

$$\mathcal{F}^{ij}\partial_y J_{ij} = -\partial_y(\mathcal{F}^{ij} J_{ij}) = -\frac{8b^2}{a^2y^3}\left(\frac{a}{b}\eta - n\right). \tag{3.55}$$

As a result, the $\mathcal{F}$ equation of motion reduces to

$$\mathcal{F}_{ij}\mathcal{F}^{ij} = -\frac{4b^4}{a^4y^4}\left(2\frac{a}{b}\eta\right)\left(\frac{a}{b}\eta - n\right). \tag{3.56}$$

Since the self-dual component of $\mathcal{F}$ is known from (3.45), the above may be rewritten in the equivalent form

$$\mathcal{F}_{ij} *_4 \mathcal{F}^{ij} = \frac{8b^4}{a^4y^4}\left(\frac{a}{b}\eta - n\right)\left(2\frac{a}{b}\eta - n\right), \tag{3.57}$$

which is identical to the $\mathcal{F}\wedge\mathcal{F}$ constraint given in [19]. Incidentally, we note that the self-dual and anti-self-dual components of $\partial_y J$ may be expressed as

$$\begin{aligned}
(\partial_y J)^{(+)} &= \tfrac{1}{4} J \partial_y \log\det h_{ij}, \\
(\partial_y J)^{(-)} &= \partial_y J - \tfrac{1}{4} J \partial_y \log\det h_{ij}.
\end{aligned} \tag{3.58}$$

In addition, as a consequence of (3.56), we may verify that both the F equation of motion and the $F_{\mu\nu}\mathcal{F}^{\mu\nu} = 0$ constraint are automatically satisfied.

Finally, to complete the solution, we note that the $U(2)$ structure of the base is highlighted by both the canonical two-form J identified in (3.43) and a holomorphic two-form Ω, which may be defined by

$$\Omega = -i f_2^2 (I^1 - iI^2). \tag{3.59}$$

The structure equation (3.33) along with self-duality is then equivalent to the statement

$$J \wedge \Omega = 0, \qquad J \wedge J = \tfrac{1}{2}\Omega\wedge\Omega^* = 2 *_4 1. \tag{3.60}$$

Along with $\tilde{d}J = 0$ shown above, we are also interested in the integrability of Ω. This may be investigated by considering (B.31), where $Y^m = i f_2^{-2} h\Omega$ according to (3.32). We find

$$D\Omega = \left[-ib\left(2\frac{a}{b}\eta - n\right)(dt + \omega) + \tfrac{1}{2}\tilde{d}\log(Z + \tfrac{1}{2}) + \tfrac{1}{4}\partial_y \log h\, dy\right] \wedge \Omega. \tag{3.61}$$

To interpret this result, we examine each component separately. Along the time direction, we have

$$\partial_t\Omega = -ib\left(2\frac{a}{b}\eta - n\right)\Omega, \tag{3.62}$$

indicating that we may take

$$\Omega = e^{-ib\left(2\frac{a}{b}\eta-n\right)t}\Omega_0, \tag{3.63}$$

where Ω_0 is independent of time. Note that this time dependence is analogous to that found in (3.15) for the 1/8 BPS solutions given above. Along the y direction, (3.61) gives

$$\partial_y\Omega = \tfrac{1}{4}\partial_y \log\det h_{ij}\,\Omega, \tag{3.64}$$

which is compatible with $\Omega\wedge\Omega^*$ being proportional to the volume form on the base.

What we are mainly interested in, of course, is $\tilde{d}\Omega$ on the base. Taking into account that $D = d + in\mathcal{A}$, we see that

$$\tilde{d}\Omega = \left[-in\mathcal{A} - ib\left(2\frac{a}{b}\eta - n\right)\omega + \tfrac{1}{2}\tilde{d}\log(Z+\tfrac{1}{2})\right]\wedge\Omega. \tag{3.65}$$

From this, we may extract the Ricci form on the base

$$\begin{aligned}\mathcal{R} &= \left(-n\mathcal{F} - b\left(2\frac{a}{b}\eta - n\right)\tilde{d}\omega - \tfrac{1}{2}\tilde{d}\left(J_i{}^j\partial_j\log(Z+\tfrac{1}{2})dx^i\right)\right)\\ &= \left(-2\frac{a}{b}\eta\mathcal{F} - i\frac{b}{a}\eta\left(2\frac{a}{b}\eta - n\right)\frac{1}{y}\partial_y J - \tfrac{1}{2}\tilde{d}\left(J_i{}^j\partial_j\log(Z+\tfrac{1}{2})dx^i\right)\right),\end{aligned} \tag{3.66}$$

where in the second line we have used the expression (3.45) for $\tilde{d}\omega$. For a Kähler metric h_{ij}, the Ricci form may be given as

$$\mathcal{R}_{ij} = -\tfrac{1}{2}J_{[j}{}^k\nabla_{i]}\nabla_k\log\det h_{lm}. \tag{3.67}$$

In this case, we may take a y derivative of (3.66) and substitute in the expression (3.47) to obtain

$$\begin{aligned}-2J_{[j}{}^k\nabla_{i]}\nabla_k\left[h^2\left(\frac{2f_1^2}{f_2}\partial_y f_2 + \frac{bf_2}{f_1}\left(\frac{a}{b}\eta - n\right)\right)\right] &= -2\frac{a}{b}\eta\partial_y\mathcal{F}_{ij} - i\frac{b}{a}\eta\left(2\frac{a}{b}\eta - n\right)\partial_y\frac{1}{y}\partial_y J_{ij}\\ &\quad -J_{[j}{}^k\nabla_{i]}\nabla_k\partial_y\log(Z+\tfrac{1}{2}).\end{aligned} \tag{3.68}$$

Noting that $\partial_y\mathcal{F}_{ij}=0$, and using (3.51) to rewrite $\partial_y y^{-1}\partial_y J_{ij}$ in terms of derivatives of Z, we may see that the above expression is automatically satisfied. Thus compatibility of (3.66) with (3.47) is ensured.

As may be evidenced by the above discussion, the supersymmetry analysis leading to the complete 1/4 BPS system is rather involved. In order to summarize the results, and to make a comparison with [18, 19], we may reexpress the scalars α and β in terms of the coordinate y and the function G through (3.48). In this case, the full ten-dimensional metric takes the form

$$ds_{10}^2 = -h^{-2}(dt+\omega)^2 + h^2[2(Z+\tfrac{1}{2})^{-1}\partial_i\partial_{\bar{j}}K dz^i d\bar{z}^{\bar{j}} + dy^2] + y[e^G d\Omega_3^2 + e^{-G}(d\psi+\mathcal{A})^2], \tag{3.69}$$

where

$$h^{-2} = 2y\cosh G, \qquad Z = \tfrac{1}{2}\tanh G. \tag{3.70}$$

In the equation above we have switched to a complex notation for the Kähler base, so that in particular the metric is given by

$$ds_4^2 = h_{ij}dx^i dx^j = 2h_{i\bar{j}}dz^i dz^{\bar{j}} = 2\partial_i\partial_{\bar{j}}K(z_i, \bar{z}_{\bar{i}}; y)dz^i d\bar{z}^{\bar{j}}, \tag{3.71}$$

This is the complex form of the expression given previously in real notation in (3.52).

The LLM function Z is constrained according to (3.53)

$$Z = -\frac{1}{2}y\partial_y \frac{1}{y}\partial_y K(z_i, \bar{z}_{\bar{i}}; y), \tag{3.72}$$

and furthermore the Kähler metric must satisfy a Monge-Ampère type equation (3.49)

$$\partial_y \log\det h_{i\bar{j}} = \frac{2e^{-G}}{e^G + e^{-G}}\partial_y G + \frac{2}{y(e^{2G}+1)}(2 - n\eta) - \frac{2}{y}(1 - n\eta). \tag{3.73}$$

Note that, for simplicity, we have set the constants $a = b = -\eta$. This equation can be integrated to yield

$$\log\det h_{i\bar{j}} = \log(Z + \frac{1}{2}) + n\eta\log y + \frac{1}{y}(2 - n\eta)\partial_y K + D(z_i, \bar{z}_{\bar{j}}), \tag{3.74}$$

where $D(z_i, \bar{z}_{\bar{j}})$ arises as an integration constant as we peel off a ∂_y derivative from (3.73). Furthermore, the Ricci form on the base must satisfy the constraint (3.66). When expressed in complex coordinates, this reduces to

$$\mathcal{R} = i\partial\bar{\partial}\log\det h_{i\bar{j}} = i\left(2i\eta\mathcal{F} + (2 - n\eta)\frac{1}{y}\partial\bar{\partial}\partial_y K + \partial\bar{\partial}\log(Z + \frac{1}{2})\right), \tag{3.75}$$

where the holomorphic and anti-holomorphic differential operators ∂ and $\bar{\partial}$ are defined by

$$\partial = dz^i\partial_i, \qquad \bar{\partial} = d\bar{z}^{\bar{j}}\partial_{\bar{j}}, \tag{3.76}$$

and where we recall that the Kähler form is $J = ih_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} = i\partial\bar{\partial}K$. Substituting the solution to the Monge-Ampère equation (3.74) into (3.75), we find that

$$\partial\bar{\partial}D = 2i\eta\mathcal{F}\,, \tag{3.77}$$

where $\mathcal{F} = d\mathcal{A}$ is the field strength corresponding to the gauging of the S^1 isometry.

Of course, the complete solution also involves the two-forms given in (3.45). In particular, with $a = b = -\eta$, we have

$$\begin{aligned}
\eta F &= -d[y^2 e^{2G}(dt + \omega)] - y^2(d\omega + \eta\mathcal{F}) + 2i\partial\bar{\partial}K, \\
\mathcal{F}^{(+)} &= -\frac{i}{y^2}(\eta - n)\partial\bar{\partial}K, \\
d\omega &= -\eta\mathcal{F} + \frac{i}{y}(\partial\bar{\partial}\partial_y K - (\partial - \bar{\partial})Z \wedge dy).
\end{aligned} \tag{3.78}$$

Note that only the self-dual part of $\mathcal{F}$ is determined. Comparing $\mathcal{F}^{(+)}$ with (3.77) then implies

$$(1 + *_4)\partial\bar{\partial}D = \frac{4}{y^2}(1 - n\eta)\partial\bar{\partial}K. \tag{3.79}$$

Finally, one last condition on the solution arises from the $\mathcal{F}$ equation of motion, namely the $\mathcal{F} \wedge \mathcal{F}$ constraint (3.57)

$$\mathcal{F} \wedge \mathcal{F} = \frac{4}{y^4}(1 - n\eta)(2 - n\eta) *_4 1. \tag{3.80}$$

As demonstrated in [19], the BPS solutions with $S^3 \times S^1$ isometry fall into several families, depending on the $U(1)$ charge n of the Killing spinor. A particularly simple case, first considered in [18], is the ungauged ansatz, where $\mathcal{A} = 0$, corresponding to ψ being trivially fibered over the base. In this case, $\mathcal{F}$ vanishes, and (3.77) reduces to

$$\partial\bar{\partial} D = 0\,. \tag{3.81}$$

This indicates that D can be an arbitrary harmonic function of z_1, z_2. Furthermore, from (3.79) we see that this condition corresponds to having

$$n\eta = 1, \tag{3.82}$$

which is also consistent with the vanishing of the $\mathcal{F} \wedge \mathcal{F}$ constraint in (3.80). Curiously, this constraint also takes on a simple form when $n\eta = 2$. As shown in [19], this allows the embedding of the 1/2 BPS LLM ansatz into the gauged 1/4 BPS ansatz. The case $n\eta = 3$ is also interesting, as it allows for solutions of the form AdS_5 times a Sasaki-Einstein space.

3.3 1/2 BPS configurations

Continuing along the chain of reductions, the final case to consider corresponds to taking S^3 times squashed S^3 isometry, as described in Section 2.3, where the squashed S^3 is written as $U(1)$ bundled over CP^1. In general, squashing the S^3 inside S^5 (while keeping the round S^3 inside AdS_5) further reduces the supersymmetries of the original LLM system from 1/2 down to 1/8 BPS. The complete analysis of the supersymmetry variations (2.48) and (2.49) is quite involved, and will not be pursued below. The first system, (2.48), corresponding to Killing spinors charged along the $U(1)$ fiber was thoroughly analyzed in [22].

We are of course more directly interested in the sequence of 1/2, 1/4 and 1/8 BPS states corresponding to the successive turning on of R-charges J_1, J_2 and J_3. In this case, we limit our consideration to the round $S^3 \times S^3$ reduction, which is nothing but the original LLM system of [1]. Although the supersymmetry analysis of this system has been thoroughly investigated in [1] and subsequent work, for completeness, and to highlight the complete 1/2, 1/4 and 1/8 BPS family of solutions, we review the analysis here.

For the round $S^3 \times S^3$ reduction, the relevant supersymmetry variations are given by (2.50). Replacing $\pm\hat{\eta}$ in (2.50) by $-\tilde{\eta}$ to simplify notation, the supersymmetry variations read

$$\begin{aligned}
\delta\psi_\mu &= \left[\nabla_\mu - \frac{i}{16} e^{-3\alpha} F_{\nu\lambda}\gamma^{\nu\lambda}\gamma_\mu\right]\epsilon, \\
\delta\lambda_\alpha &= \left[\gamma^\mu\partial_\mu\alpha + \frac{i}{8} e^{-3\alpha} F_{\mu\nu}\gamma^{\mu\nu} - \eta e^{-\alpha}\right]\epsilon, \\
\delta\lambda_\beta &= \left[\gamma^\mu\partial_\mu\beta - \frac{i}{8} e^{-3\alpha} F_{\mu\nu}\gamma^{\mu\nu} - i\tilde{\eta} e^{-\beta}\gamma^5\right]\epsilon.
\end{aligned} \tag{3.83}$$

Since ϵ may be viewed as a Dirac spinor in four dimensions, we may form the following bilinears [1]

$$\begin{aligned}
&f_1 = \bar{\epsilon}\gamma^5\epsilon, \qquad f_2 = i\bar{\epsilon}\epsilon, \qquad K^\mu = \bar{\epsilon}\gamma^\mu\epsilon, \qquad L^\mu = \bar{\epsilon}\gamma^\mu\gamma^5\epsilon, \qquad Y^{\mu\nu} = i\bar{\epsilon}\gamma^{\mu\nu}\gamma^5\epsilon, \\
&K^m_\mu = \epsilon^c\gamma_\mu\epsilon, \qquad Y^m_{\mu\nu} = \epsilon^c\gamma_{\mu\nu}\gamma^5\epsilon.
\end{aligned} \tag{3.84}$$

Note that K^m, viewed as a complex one-form, was denoted ω in [1].

The above bilinears are normalized according to the Fierz relations

$$K^2 = -L^2 = -f_1^2 - f_2^2, \qquad Y^2 = 2f_1^2 - 2f_2^2, \qquad |K^m|^2 = 2f_1^2 + 2f_2^2, \qquad |Y^m|^2 = -4f_1^2 + 4f_2^2. \tag{3.85}$$

In addition, they satisfy the identities

$$K \cdot L = K \cdot K^m = L \cdot K^m = 0, \qquad K^\mu Y_{\mu\nu} = f_2 L_\nu, \qquad L^\mu Y_{\mu\nu} = f_2 K_\nu. \tag{3.86}$$

Following [1], we note that K^μ defines a time-like (Killing) direction, while L_μ is space-like and orthogonal to K^μ. The four-dimensional space then splits into a two-dimensional base (the LLM x_1–x_2 plane) along with a preferred time-like and a preferred space-like (the LLM y coordinate) direction.

The structure defined by the above bilinears is highlighted by noting that they may be decomposed according to

$$Y = -\frac{f_2}{f_1^2 + f_2^2} K \wedge L + f_1 I, \qquad K^m = \sqrt{f_1^2 + f_2^2}\,\widetilde{\Omega}, \qquad Y^m = \frac{1}{\sqrt{f_1^2 + f_2^2}}(f_1 K - i f_2 L) \wedge \widetilde{\Omega}, \tag{3.87}$$

where

$$I_{ab} I_{bc} = -\delta_{ac}, \qquad |\widetilde{\Omega}|^2 = 2. \tag{3.88}$$

These expressions are the analog of (3.32) for the present case. In particular, here the real two-form I along with the complex one-form $\widetilde{\Omega}$ together define a preferred $U(1)$ structure.

The familiar analysis of [1] proceeds by solving the one-form identities (C.5) through (C.10) for the bilinears f_1 and f_2 as well as for the field strength $F_{(2)}$. For simplicity with signs, we choose

$$f_1 = -\eta e^\beta, \qquad f_2 = -\tilde{\eta} e^\alpha, \tag{3.89}$$

so that

$$e^{\alpha+\beta} = y, \tag{3.90}$$

where we have chosen to write $L = dy$, which is compatible with L being a closed one-form, as indicated by (C.14). In this case, $F_{(2)}$ is given by

$$F_{(2)} = \tilde{\eta}(dt + \omega) \wedge de^{4\alpha} - \eta h^2 e^{3\alpha - 3\beta} *_3 de^{4\beta}, \tag{3.91}$$

where we have chosen to write the four-dimensional metric as

$$ds_4^2 = -h^{-2}(dt + \omega)^2 + h^2[h_{ij} dx^i dx^j + dy^2], \tag{3.92}$$

with

$$h^{-2} = f_1^2 + f_2^2 = e^{2\alpha} + e^{2\beta}, \qquad K = -h^{-2}(dt+\omega), \qquad L = dy. \tag{3.93}$$

Note that $*_3$ is the Hodge dual with respect to the three-dimensional metric given inside the square brackets above.

We now note that (C.11) gives rise to the condition [1]

$$d\omega = -\eta\tilde{\eta}\frac{1}{y} *_3 dZ, \tag{3.94}$$

where

$$Z = \frac{1}{2}\frac{f_2^2 - f_1^2}{f_2^2 + f_1^2} = \frac{1}{2}\frac{e^{2\alpha} - e^{2\beta}}{e^{2\alpha} + e^{2\beta}}. \tag{3.95}$$

In terms of ω and Z, the expression (3.91) can be rewritten as

$$F_{(2)} = -\tilde{\eta} d[e^{4\alpha}(dt+\omega)] - \tilde{\eta} y^2 d\omega - 2\eta(Z + \tfrac{1}{2}) *_3 dy. \tag{3.96}$$

It is now easy to see that $F_{(2)}$ is automatically closed, so long as $d\omega$ is [1, 4]. Of course, the requirement $d\omega = 0$ for $d\omega$ given in (3.94) yields the LLM condition that Z be a harmonic function

$$d\left(\frac{1}{y} *_3 dZ\right) = 0, \tag{3.97}$$

which is the basis for the bubbling AdS picture. The Hodge dual is evaluated with respect to the three-dimensional metric $h_{ij}dx^i dx^j + dy^2$.

To complete the 1/2 BPS picture, it is worth noting that the metric h_{ij} on the two-dimensional base can be specified by defining the canonical-two form J and holomorphic one-form Ω

$$I = h^2 J, \qquad \widetilde{\Omega} = h\Omega. \tag{3.98}$$

where I and $\widetilde{\Omega}$ are given in (3.87). Using the decomposition of Y in (3.87) and the differential identities (C.17) and (C.20)

$$d(f_1 Y) = 0, \qquad d(f_2 * Y) = 0, \tag{3.99}$$

we see that

$$d(f_1^2 I) = *_3 dZ \wedge dy, \qquad d(f_2^2 I) = - *_3 dZ \wedge dy, \tag{3.100}$$

so that

$$dJ = d(h^{-2} I) = 0. \tag{3.101}$$

Furthermore, comparing (3.98) with (3.87) demonstrates that $\omega = K^m$, in which case (C.31) immediately shows that

$$d\Omega = 0. \tag{3.102}$$

This combination of $dJ = 0$ and $d\Omega = 0$ now demonstrates that the two-dimensional base is flat, in which case we can rewrite (3.92) using the trivial base metric

$$\begin{aligned} ds_4^2 &= -h^{-2}(dt+\omega)^2 + h^2[dx_1^2 + dx_2^2 + dy^2] \\ &= -h^{-2}(dt+\omega)^2 + h^2[dzd\overline{z} + dy^2]. \end{aligned} \tag{3.103}$$

This essentially completes the summary of the LLM analysis [1]. In the remaining sections of this paper, we will make use of the results of the above supersymmetry analyses to develop a universal picture of bubbling AdS geometries.

4 Bubbling AdS

The above reductions on S^3, $S^3 \times S^1$ and $S^3 \times S^3$ and the supersymmetry analyses provide a uniform framework for describing the corresponding 1/8, 1/4 and 1/2 BPS configurations in IIB supergravity. However, we are interested in much more than simply a useful means of characterizing the supergravity solutions. What we desire is a complete understanding of the geometries and how they are mapped into states in the dual $\mathcal{N} = 4$ Yang Mills theory.

The best developed picture for these bubbling AdS states is of course in the 1/2 BPS sector, where the x_1–x_2 plane of [1] has a direct counterpart in the phase plane of the dual free fermion picture of the 1/2 BPS sector of the $\mathcal{N} = 4$ Yang Mills theory [2,3]. Furthermore, 'droplets' in the LLM plane are related to non-trivial topology of the gravity solution, and are directly equivalent to giant gravitons expanding either in AdS_5 or S^5.

What we would like to obtain is a similar understanding of the 1/4 and 1/8 BPS sectors of the theory. However, this task is made rather more complicated for several reasons. For one thing, on general grounds, we expect that the 1/2 BPS states (which preserve 16 real supersymmetries) are described by wave-functions of a non-interacting free fermion system. (Note, however, that the system appears to be interacting when the fermionic degrees of freedom are changed to bosons.) The reduced supersymmetry cases do of course admit descriptions as *e.g.* multi-matrix models on the gauge theory side. However, we expect the resulting system to be a system of interacting bosons without a dual free fermion description, and hence more complicated to describe on the gravity side of the duality. This is in fact borne out by the explicit 1/8 and 1/4 BPS analysis of [20, 18, 19], as reviewed above in Section 3. In particular, both the 1/8 and 1/4 cases involve non-linear equations, in contrast with the linear LLM equation (3.97), which is the basis for harmonic superposition of 1/2 BPS states.

Nevertheless, there is an elegant structure underlying the sequence of 1/2, 1/4 and 1/8 BPS states. As discussed in Section 3, these configurations are characterized by $U(1)$, $U(2)$ and $SU(3)$ structure, respectively, and are described by specifying appropriate field configurations on the corresponding one-, two- and three-complex dimensional base manifolds. Since these manifolds are Kähler, they can also be considered symplectic, which is perhaps more natural for a phase-space description. In the 1/2 and 1/4 BPS cases, there is an additional y direction where y is directly related to the volume of $S^3 \times S^3$ for the 1/2 BPS case, or somewhat indirectly related to the volume of $S^3 \times S^1$ in the 1/4 BPS case. Although the 1/8 BPS metric, (3.14), has no room for an extra y coordinate, we may nevertheless define $y \equiv e^{\alpha}$, and thereby obtain an effective y variable related to the volume of S^3.

At this point, it is perhaps worthwhile to summarize the main features of the 1/2, 1/4 and 1/8

BPS geometries. From (3.92), (3.38) and (3.14), along with the liftings of Section 2, we have

$$
\begin{aligned}
&\text{1/2 BPS:} && ds^2 = -h^{-2}(dt+\omega)^2 + h^2[h_{ij}dx^i dx^j + dy^2] + e^{2\alpha}d\Omega_3^2 + e^{2\beta}d\widetilde{\Omega}_3^2,\\
&\text{1/4 BPS:} && ds^2 = -h^{-2}(dt+\omega)^2 + e^{-2\alpha}h_{ij}dx^i dx^j + h^2 dy^2 + e^{2\alpha}d\Omega_3^2 + e^{2\beta}(d\psi+\mathcal{A})^2,\\
&\text{1/8 BPS:} && ds^2 = -e^{2\alpha}(dt+\omega)^2 + e^{-2\alpha}h_{ij}dx^i dx^j + e^{2\alpha}d\Omega_3^2,
\end{aligned}
\tag{4.1}
$$

where in all cases $h^{-2} = e^{2\alpha} + e^{2\beta}$. In addition

$$
y = e^{\alpha+\beta} \quad \text{(for 1/2 and 1/4 BPS)} \quad \text{or} \quad y = e^{\alpha} \quad \text{(for 1/8 BPS)}. \tag{4.2}
$$

Although the metric and form fields must satisfy various local conditions (some of which may be rather complicated, especially in the 1/4 BPS case) in order to ensure a valid solution, the global features that we are mainly interested in are encoded by the boundary conditions imposed to ensure regularity of the above metrics. As in the LLM analysis [1], we are concerned with regularity as any one of the spheres (or circle) in (4.1) shrinks to zero size. Since this occurs at $y = 0$, we obtain a natural generalization of the LLM condition (1.5)

$$
Z(x_i, y=0) = \pm\tfrac{1}{2} \quad \text{(for 1/2 and 1/4 BPS)}, \tag{4.3}
$$

where in both cases $Z = \frac{1}{2}(e^{2\alpha} - e^{2\beta})/(e^{2\alpha} + e^{2\beta})$. The analogous $y = 0$ boundary condition for the 1/8 BPS system is more difficult to characterize, but is similar in spirit to the above.

In the 1/2 BPS (LLM) case, for geometries asymptotic to $\mathrm{AdS}_5 \times S^5$, the $y = 0$ plane consists of regions of $Z = -1/2$ (shrinking S^3 inside AdS_5) in a background of $Z = 1/2$ (shrinking S^3 inside S^5). The $\mathrm{AdS}_5 \times S^5$ 'ground state' corresponds to a circular disk of $Z = -1/2$; at $y = 0$, the interior of this disk is mapped to the 'center' of AdS, while the exterior is mapped to the point where S^3 shrinks inside S^5. In general, the boundary between $Z = 1/2$ and $Z = -1/2$ is the locus where both of the three-spheres simultaneously shrink to zero size. As a result, the LLM solution essentially maps the non-trivial topology of the 1/2 BPS background entirely onto a plane (the $y = 0$ plane). The configuration is then fully determined by specifying one-dimensional curves in the plane, corresponding to the boundary between the $Z = 1/2$ and $Z = -1/2$ regions. This is of course the dual picture of the 'droplet' description where regions, or droplets, are specified.

The extension of this picture to the 1/4 BPS case is then straightforward. In this case, the topology of the background is again determined by the structure of the solution on the $y = 0$ hyperplane. This time, the hyperplane is four-dimensional, and may be divided into regions of $Z = 1/2$ and $Z = -1/2$ by three-dimensional surfaces. This time, however, $Z = 1/2$ corresponds to a shrinking one-cycle in S^5, while $Z = -1/2$ corresponds as usual to shrinking S^3 inside AdS_5. As we show below, the $\mathrm{AdS}_5 \times S^5$ ground state in this case consists of a ball of $Z = -1/2$ in a background of $Z = 1/2$. We do note, however, that in contrast with the LLM picture, this $y = 0$ hyperplane has a non-trivial (Kähler) metric, and hence is not flat. Nevertheless, so long as the bubbling picture relies only on the topology of the droplets, it will remain valid. This distortion of the geometry is of course to be expected for reduced supersymmetry configurations, which can no longer be treated as non-interacting collective modes.

The 1/8 BPS case is particularly interesting, both because it no longer incorporates a $y = 0$ hyperplane, and because it is the most general case encompassing the other two in appropriate limits. Defining the variable $y = e^{\alpha}$, as in (4.2), the locus of shrinking S^3 inside AdS_5 then corresponds to five-dimensional surfaces of $y = 0$ within the six-dimensional base. In order to obtain a regular geometry, the 1/8 BPS metric in (4.1) must then approach a solution of the form

$$ds^2 = \cdots + (dy^2 + y^2 d\Omega_3^2), \qquad \text{as} \quad y \to 0. \tag{4.4}$$

In other words, the shrinking S^3 combines with the y direction to locally form $\mathbb{R}^4$. In this case, y is non-negative, and may be considered as a local coordinate normal to the five-dimensional boundary surfaces. Viewed in this manner, since y terminates at zero and does not become negative, the six-dimensional base space ends at these five-dimensional surfaces. In particular, the interiors are unphysical; they simply do not exist. Another way to understand this is to note from (3.19) that y is related to the scalar curvature of the base according to $R = -8/y^4$. Thus these five-dimensional surfaces of vanishing y are singular (from the six-dimensional point of view), and space simply ends there, as there is no natural extension for going past such singularities. Of course, the full ten-dimensional solution remains regular, so long as the $y = 0$ surfaces are locally of the form (4.4).

The general picture of 1/8 BPS states is thus one of S^3 and time fibered over a six-dimensional base, where various regions (*i.e.* droplets) have been excised. Since the S^3 inside AdS_5 shrinks on the (in general disconnected) five-dimensional boundary surface, this surface may be related to the locus of D3-branes wrapped on the S^3, which are simply dual giant gravitons expanding in AdS_5 [23]. In cases with additional supersymmetries (1/4 or 1/2 BPS), this six-dimensional base admits an additional S^1 or S^3 isometry. In such cases, the S^1 or S^3 can be pulled out explicitly, along with the y variable, which can be promoted to an actual coordinate normal to the shrinking S^3 inside AdS_5. This transformation, which maps the five-dimensional boundary surfaces to the $y = 0$ hyperplane, is highly non-trivial, but has the feature of placing much of the interesting topological data onto a single hyperplane within the full ten-dimensional space-time.

Abstracting the details for a moment, we see a uniform picture emerging, where 1/2, 1/4 and 1/8 BPS configurations are described by one, three and five-dimensional surfaces embedded within two, four and six-dimensional hyperplanes. Equivalently, we may use a dual description of two, four and six-dimensional droplets. Only in the 1/2 BPS case is the $y = 0$ hyperplane actually flat. In the other cases, we expect them to be diffeomorphic to $\mathbb{R}^4$ and $\mathbb{R}^6$ [23], although such global properties cannot be seen directly from the local supersymmetry analysis of Section 3. In particular, bubbling orientifold models [29] can be constructed by making appropriate discrete identifications on the base spaces.

From the $\mathcal{N} = 4$ Yang-Mills side of the duality, the 1/2, 1/4 and 1/8 BPS configurations may be described by one, two and three (complex) matrix models corresponding to the three complexified adjoint scalars $X = \phi_1 + i\phi_2$, $Y = \phi_3 + i\phi_4$ and $Z = \phi_5 + i\phi_6$ of the $\mathcal{N} = 4$ theory [2]. As a result, there is a natural map between the space of matrix eigenvalues (*i.e.* the free fermion phase space in the 1/2 BPS case) and the corresponding one, two and three complex dimensional base spaces $ds^2 = h_{ij} dx^i dx^j$ in (4.1). In all such cases, the $\mathrm{AdS}_5 \times S^5$ ground state corresponds to taking a round ball in the base space (at $y = 0$ when appropriate). Turning on giant graviton excitations on top of the ground state then corresponds to introducing disconnected droplets, either inside the ball (giant gravitons expanding in S^5) or outside (dual giant gravitons expanding in AdS_5). Of course, for 1/8 BPS configurations, only the giant gravitons expanding in AdS_5 are manifest, as the interior of the ball is completely removed.

Until now, we have said very little about the non-linear equations characterizing the 1/8 and 1/4 BPS solutions. For the former, the main condition on the solution is given by (3.21), while for the latter, one has (3.74), along with the subsidiary conditions (3.77), (3.79) and (3.80). In general, these conditions are difficult to work with, and hence we are unable to present an explicit construction of these reduced supersymmetry bubbling AdS geometries. We do note, however, that in the case of LLM, the 1/2 BPS geometries are fully characterized by the LLM boundary condition (1.5) $Z = \pm\frac{1}{2}$ at $y = 0$. In particular, the LLM Laplacian (3.97) is only of secondary importance in developing the bubbling AdS interpretation of the solutions. This linear equation does of course facilitate the writing of explicit solutions, and furthermore is presumably intimately tied to the non-interacting nature of 1/2 BPS states. Nevertheless, the topology of the system, and hence much of the information on giant gravitons, is contained in the LLM boundary condition itself, and not necessarily the harmonic superposition rule derived from (3.97). Of course, this was already noted in [1] in the case of 1/2 BPS configurations of M-theory, where a droplet picture emerged from consideration of the boundary conditions, despite the fact that the full solution involves the Toda equation.

Likewise for the 1/8 and 1/4 BPS systems, we expect that each choice of boundary conditions (specified either as $y = 0$ surfaces in a six-dimensional base, or as droplets in the $y = 0$ hyperplane) gives rise to a unique bubbling AdS geometry. Because of the non-linear nature of the expressions involved, however, we do not envision a simple proof of either the existence or uniqueness of the solutions. We certainly expect large classes of solutions to exist, although it would also be interesting to see if the conditions on the solutions preclude any particular classes of droplets from existing as regular bubbling AdS geometries.

5 Examples fitting into the 1/8 BPS case

Although we have not been able to solve the 1/8 and 1/4 BPS conditions (3.21) and (3.47) completely, we may nevertheless use the existing (known) solutions, as well as a specific class of new 1/4 BPS solutions, to present evidence for the general droplet picture. We start with several 1/8 BPS (actually S^3 isometry) examples before turning, in Section 6, to 1/4 BPS geometries. We should also note that in Section 7 we will analyze the regularity conditions for a rather generic class of 1/8 BPS solutions, and see that a picture of six-dimensional droplets will emerge by requiring their ten-dimensional metric to be regular.

The general 1/8 BPS system falls into the S^3 isometry analysis of Section 3.1. This solution is presented in terms of a seven-dimensional metric $g_{\mu\nu}$, two-form $F_{(2)}$ and scalar α, which are given by (3.14) and (3.19). Our main concern here is with the metric, which when lifted to ten dimensions takes the form (4.1)

$$ds^2 = -y^2(dt+\omega)^2 + \frac{1}{y^2}h_{ij}dx^i dx^j + y^2 d\Omega_3^2, \tag{5.1}$$

where we have made explicit the identification of $y(x_i)$ with $e^{\alpha(x_i)}$, as in (4.2). The complete solution is determined (at least up to diffeomorphisms) in terms of a Kähler metric h_{ij} with curvature satisfying (3.21)

$$\Box_6 R = -R_{ij}R^{ij} + \tfrac{1}{2}R^2, \tag{5.2}$$

and with $y = (-8/R)^{1/4}$. Note that this identification of y demands that the Kähler base has non-vanishing negative scalar curvature, with $R \to -\infty$ on the five-dimensional degeneration surfaces where $y \to 0$. Given these preliminaries, we now turn to some examples.

5.1 $\mathbf{AdS}_3 \times S^3 \times T^4$

While we are mainly interested in geometries which are asymptotically connected to $\mathrm{AdS}_5 \times S^5$, we note that (5.2) admits a simple solution where the base is taken to be the direct product of a hyperbolic space with a torus, $\mathbb{H}^2 \times T^4$, with curvature given by

$$R_{ij} = \begin{cases} -4h_{ij} & i,j = 1,2, \\ 0 & i,j = 3,\ldots,6 \end{cases} \tag{5.3}$$

(using real coordinates). This base can be obtained from a Kähler potential

$$K(z_1,z_2,z_3) = -\tfrac{1}{2}\log(1-|z_1|^2) + \tfrac{1}{2}(|z_2|^2+|z_3|^2). \tag{5.4}$$

Because y is a constant (which in our normalization is simply $y=1$), this solution has constant scalar curvature, and hence no shrinking three-cycles. Of course, we recall that, since here y is a function and not a coordinate, there is no problem with setting it to a constant.

When this $\mathbb{H}^2 \times T^4$ base is incorporated into the full metric (5.1), it is easy to see that the resulting geometry is that of $\mathrm{AdS}_3 \times S^3 \times T^4$. In particular, by writing the metric on $\mathbb{H}^2$ as

$$ds_2^2 = d\rho^2 + \tfrac{1}{4}\sinh^2(2\rho)d\psi^2, \tag{5.5}$$

and by taking

$$\omega = \sinh^2\rho\, d\psi, \tag{5.6}$$

(which is compatible with the condition $\mathcal{R} = 2d\omega$), we end up with $\mathrm{AdS}_3 \times S^3 \times T^4$ written as

$$ds_{10}^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\,(d\psi - dt)^2 + d\vec{x}_4^2 + d\Omega_3^2. \tag{5.7}$$

Note that the natural coordinates implicit in the fibration of time over the Kähler base involve motion at the speed of light along the angular direction in AdS_3.

This example is of course the double analytic continuation of the similar example given in [20], which realized $\mathrm{AdS}_3 \times S^3 \times T^4$ using an $S^2 \times T^4$ base.

5.2 $\mathrm{AdS}_5 \times S^5$

Our primary interest is of course with developing a droplet picture for excitations on top of $\mathrm{AdS}_5 \times S^5$. To proceed in this direction, we first consider the realization of the $\mathrm{AdS}_5 \times S^5$ ground state itself. In this case, we take the ten-dimensional metric[1]

$$ds_{10}^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + d\Omega_5^2, \tag{5.8}$$

and identify the S^3 in AdS_5 with the S^3 of (5.1). This determines

$$y = \sinh\rho \tag{5.9}$$

along with the remaining seven-dimensional metric

$$ds_7^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + d\Omega_5^2. \tag{5.10}$$

Here there are multiple ways of proceeding. What we would like, of course, is to rewrite this metric using giant graviton speed of light angular coordinates of the form

$$\phi = \psi - t, \tag{5.11}$$

where ϕ is a rotation angle in S^5, and ψ its natural giant graviton counterpart. Because of the symmetry of the five-sphere, it is natural to parameterize it in terms of three rotation planes (with three angular coordinates ϕ_i and corresponding angular momenta J_i). However, it is also possible, and perhaps more convenient, to write S^5 as $U(1)$ bundled over CP^2. While CP^2 does not admit a spin-structure, it nevertheless admits a spin^c-structure, and that is the main reason why we must allow for charged Killing spinors along the fiber when reducing to six dimensions.

Writing the S^5 metric as

$$d\Omega_5^2 = ds^2(CP^2) + (d\phi + \mathcal{A})^2, \qquad d\mathcal{A} = 2J, \tag{5.12}$$

and performing the angular shift (5.11) yields the seven-dimensional metric

$$ds_7^2 = -\sinh^2\rho\left(dt + \sinh^{-2}\rho(d\psi + \mathcal{A})\right)^2 + \sinh^{-2}\rho\left(\sinh^2\rho(d\rho^2 + ds^2(CP^2)) + \cosh^2\rho(d\psi + \mathcal{A})^2\right). \tag{5.13}$$

As a result, the six-dimensional metric on the base is

$$ds_6^2 = (r^2 - 1)ds^2(CP^2) + dr^2 + r^2(d\psi + \mathcal{A})^2, \tag{5.14}$$

where we have defined $r = \cosh\rho$. The Ricci tensor is

$$R_{ij} = \begin{cases} -4(r^2-1)^{-2}h_{ij} & i,j = 3,\ldots,6\ (CP^2), \\ 4(r^2-1)^{-2}h_{ij} & i,j = 1,2\ (r \text{ and } \psi). \end{cases} \tag{5.15}$$

[1] Note that here we have taken the AdS_5 radius L to be unity.

The alternate more symmetrical decomposition of S^5 follows by introducing the complex coordinates

$$\begin{aligned} z_1 &= r\cos\theta_1 e^{i\phi_1}, \\ z_2 &= r\sin\theta_1\cos\theta_2 e^{i\phi_2}, \\ z_3 &= r\sin\theta_1\sin\theta_2 e^{i\phi_3}. \end{aligned} \tag{5.16}$$

In this case, we have

$$\begin{aligned} |dz_i|^2 &= dr^2 + r^2 d\Omega_5^2, \\ |\overline{z}_i dz_i|^2 &= r^2 dr^2 + r^4(\cos^2\theta_1 d\phi_1 + \sin^2\theta_1\cos^2\theta_2 d\phi_2 + \sin^2\theta_1\sin^2\theta_2 d\phi_3)^2. \end{aligned} \tag{5.17}$$

Taking the seven-dimensional metric (5.10) and shifting

$$\phi_i = \psi_i - t \tag{5.18}$$

we obtain

$$ds_7^2 = -\sinh^2\rho(dt+\omega)^2 + \sinh^{-2}\rho\left(\sinh^2\rho\, d\rho^2 - \cosh^2\rho\frac{dr^2}{r^2} + \sinh^2\rho\frac{|dz_i|^2}{r^2} + \frac{|\overline{z}_i dz_i|^2}{r^4}\right), \tag{5.19}$$

where

$$\omega = \sinh^{-2}\rho\left(\cos^2\theta_1 d\psi_1 + \sin^2\theta_1\cos^2\theta_2 d\psi_2 + \sin^2\theta_1\sin^2\theta_2 d\psi_3\right), \tag{5.20}$$

and now the z_i's are defined with the angles ψ_i.

In order to eliminate the original ρ coordinate, we may define

$$r = \cosh\rho. \tag{5.21}$$

The resulting six-dimensional base metric has the simple form

$$ds_6^2 = (|z_i|^2 - 1)\frac{|dz_i|^2}{|z_i|^2} + \frac{|\overline{z}_i dz_i|^2}{(|z_i|^2)^2}, \tag{5.22}$$

and may be obtained from a Kähler potential

$$K = \tfrac{1}{2}|z_i|^2 - \tfrac{1}{2}\log(|z_i|^2). \tag{5.23}$$

For completeness, we note that

$$\omega = \frac{1}{|z_i|^2 - 1}\frac{\Im(\overline{z}_i dz_i)}{|z_i|^2}. \tag{5.24}$$

5.2.1 Boundary conditions

It is now instructive to examine the form of the six-dimensional base given by (5.22). The complex coordinates z_i cover the space completely, and are furthermore restricted to the region $|z_i|^2 \geq 1$, as it is evident from (5.21). Moreover, since

$$y^2 = |z_i|^2 - 1, \tag{5.25}$$

we see that y naturally parameterizes the radial direction in $\mathbb{C}^3$ starting from the unit five-sphere on outward. This confirms the picture developed above in Section 4 that the $\mathrm{AdS}_5 \times S^5$ vacuum corresponds to removing a round ball from the Kähler base which, while not flat, is nevertheless diffeomorphic to $\mathbb{C}^3$. Note also that this description matches perfectly with the matrix wave-function picture explored recently in [23].

5.3 Three-charge smooth solutions

Given the picture of the $\mathrm{AdS}_5 \times S^5$ ground state as a round ball removed from $\mathbb{C}^3$, we may in general consider two types of excitations. As in [1], the first consist of deformations of the surface of the ball, corresponding to Kaluza-Klein excitations (gravitational ripples), and the second consists of introducing topology changing droplets, corresponding to giant gravitons.

In principle, excitations corresponding to ripples on the Fermi surface can be fully explored in the linearized regime. By consistency, the result must reproduce the subsector of Kaluza-Klein modes of IIB theory on $\mathrm{AdS}_5 \times S^5$ [30] that is consistent with the 1/8 BPS condition. In the 1/2 BPS case, this connection was explicitly demonstrated in [31].

Here we choose not to carry out the complete linearized analysis at this time. Instead, we consider a class of smooth, three-charge 'AdS bubble' solutions which were studied in [32]. These solutions are smoothed out (no horizon) versions of the R-charged black holes (*i.e.* superstars), and are described by a five-dimensional field configuration

$$\begin{aligned}
&ds_5^2 = -(H_1H_2H_3)^{-2/3} f\,dt^2 + (H_1H_2H_3)^{1/3}(f^{-1}dr^2 + r^2 d\Omega_3^2),\\
&A^i_{(1)} = -H_i^{-1}dt, \qquad X_i = (H_1H_2H_3)^{1/3}\, H_i^{-1}, \qquad \cosh\varphi_i = (RH_i)',\\
&f = 1 + r^2 H_1H_2H_3,
\end{aligned} \tag{5.26}$$

where $R \equiv r^2$, and where a prime denotes a derivative with respect to R. Furthermore, the functions H_i obey the equation

$$f(RH_i)'' = [1 - (RH_i)'^2]\,(H_1H_2H_3)H_i^{-1}. \tag{5.27}$$

This non-linear coupled set of equations admits the trivial solution $(RH_i)' = 1$ (for all $i = 1,2,3$), in which case $\varphi_i = 0$ and $H_i = 1 + Q_i/R$. This simply reproduces the three-charge superstar solutions of [33, 34]. On the other hand, while the general exact solution to this system of equations is not known (except in the one-charge, *i.e.* LLM, case), numerical investigations indicate that it admits a six-parameter family of solutions, corresponding to three charges Q_i and three corresponding scalar deformations related to turning on $\varphi_i \neq 0$. For fixed charges, the three scalar parameters may then be adjusted to ensure regularity of the solution as $R \to 0$. In particular, regularity here means that both H_i and its derivatives H_i' remain bounded as $R \to 0$.

These three-charge solutions preserve 1/8 of the supersymmetries, and are furthermore regular without horizons. As such, they must fall under the classification of Section 3.1. To see how they may be expressed in the bubbling metric form of (5.1), we first lift (5.26) to ten dimensions following the procedure outlined in [35]:

$$ds_{10}^2 = \sqrt{\Delta}\, ds_5^2 + \frac{1}{\sqrt{\Delta}}\, T_{IJ}^{-1}\, D\mu^I D\mu^J, \tag{5.28}$$

where

$$\Delta \equiv T_{IJ}\,\mu^I\mu^J, \qquad \sum_{I=1}^{6}\mu^I\mu^I = 1, \qquad D\mu^I \equiv d\mu^I + A_{(1)}^{IJ}\mu^J. \tag{5.29}$$

The constrained scalars X_i, along with the fields φ_i are given by the decomposition

$$T_{IL} = \mathrm{diag}(X_1\, e^{-\varphi_1}, X_1\, e^{\varphi_1}, X_2\, e^{-\varphi_2}, X_2\, e^{\varphi_2}, X_3\, e^{-\varphi_3}, X_3\, e^{\varphi_3}), \tag{5.30}$$

and the $U(1)^3$ gauge fields are

$$A_{(1)}^{12} = A_{(1)}^1, \qquad A_{(1)}^{34} = A_{(1)}^2, \qquad A_{(1)}^{56} = A_{(1)}^3. \tag{5.31}$$

More explicitly, using (5.31) and $A_{(1)}^i = -H_i^{-1}\, dt$ we have the three pairs of expressions

$$\begin{aligned}
D\mu_1 &= d\mu_1 - \mu_2\, H_1^{-1}\, dt, \qquad D\mu_2 = d\mu_2 + \mu_1\, H_1^{-1}\, dt,\\
D\mu_3 &= d\mu_3 - \mu_4\, H_2^{-1}\, dt, \qquad D\mu_4 = d\mu_4 + \mu_3\, H_2^{-1}\, dt,\\
D\mu_5 &= d\mu_5 - \mu_6\, H_3^{-1}\, dt, \qquad D\mu_6 = d\mu_6 + \mu_5\, H_3^{-1}\, dt.
\end{aligned} \tag{5.32}$$

Also, we find that

$$\Delta = X_1(e^{-\varphi_1}\mu_1^2 + e^{\varphi_1}\mu_2^2) + X_2(e^{-\varphi_2}\mu_3^2 + e^{\varphi_2}\mu_4^2) + X_3(e^{-\varphi_3}\mu_5^2 + e^{\varphi_3}\mu_6^2). \tag{5.33}$$

The uplifted metric can then be written as

$$\begin{aligned}
ds_{10}^2 &= \sqrt{\Delta}\Big[-\frac{f}{(H_1H_2H_3)^{2/3}}\, dt^2 + (H_1H_2H_3)^{1/3}(f^{-1}dr^2 + r^2 d\Omega_3^2)\Big]\\
&\quad + \frac{1}{\sqrt{\Delta}}\Bigg[H_1\frac{e^{\varphi_1}(D\mu_1)^2 + e^{-\varphi_1}(D\mu_2)^2}{(H_1H_2H_3)^{1/3}} + H_2\frac{e^{\varphi_2}(D\mu_3)^2 + e^{-\varphi_2}(D\mu_4)^2}{(H_1H_2H_3)^{1/3}}\\
&\qquad + H_3\frac{e^{\varphi_3}(D\mu_5)^2 + e^{-\varphi_3}(D\mu_6)^2}{(H_1H_2H_3)^{1/3}}\Bigg].
\end{aligned} \tag{5.34}$$

In addition, we make the following explicit choice of coordinates on the five-sphere

$$\mu = (\tilde\mu_1 \sin\phi_1, \tilde\mu_1\cos\phi_1, \tilde\mu_2\sin\phi_2, \tilde\mu_2\cos\phi_2, \tilde\mu_3\sin\phi_3, \tilde\mu_3\cos\phi_3), \tag{5.35}$$

where

$$\tilde\mu_1 = \sin\theta, \qquad \tilde\mu_2 = \cos\theta\sin\alpha, \qquad \tilde\mu_3 = \cos\theta\cos\alpha. \tag{5.36}$$

These ‘direction cosines’ obey

$$\sum_{I=1}^{6}\mu_I^2 = \sum_{i=1}^{3}\tilde\mu_i^2 = 1. \tag{5.37}$$

The first step in transforming this solution into the 1/8 BPS form (5.1) is to identify the three-sphere inside AdS_5. In this case, examination of (5.34) directly yields

$$y^2 = \sqrt{\Delta}\, r^2\, (H_1 H_2 H_3)^{1/3}. \tag{5.38}$$

Next, by properly collecting the time components, we may write the remaining seven-dimensional part of the metric in the standard form

$$ds_7^2 = -y^2(dt+\omega)^2 + y^{-2} h_{mn}\, dx^m\, dx^n, \tag{5.39}$$

where

$$\begin{aligned} \omega_{\phi_i} &= -\frac{\tilde{\mu}_i^2}{r^2\,\Delta\,(H_1H_2H_3)^{2/3}}\left[(\cos\phi_i)^2\, e^{-\varphi_i} + (\sin\phi_i)^2\, e^{\varphi_i}\right], \\ \omega_{\tilde{\mu}_i} &= \frac{2\tilde{\mu}_i \sin\phi_i \cos\phi_i \sinh\varphi_i}{(H_1H_2H_3)^{2/3}}, \end{aligned} \tag{5.40}$$

and the metric on the six-dimensional base is given by

$$\begin{aligned} h_{rr} &= \frac{r^2\,(H_1H_2H_3)^{2/3}\,\Delta}{f}, \\ h_{\phi_i\,\phi_j} &= \delta_{ij}\, r^2\, H_i\, \tilde{\mu}_i^2\, [\cos^2\phi_i e^{-\varphi_i} + \sin^2\phi_i e^{\varphi_i}] \\ &\quad + \frac{\tilde{\mu}_i^2\,\tilde{\mu}_j^2}{\Delta\,(H_1H_2H_3)^{2/3}}\, [\cos^2\phi_i e^{-\varphi_i} + \sin^2\phi_i e^{\varphi_i}]\, [\cos^2\phi_j e^{-\varphi_j} + \sin^2\phi_j e^{\varphi_j}], \\ h_{\tilde{\mu}_i\,\tilde{\mu}_j} &= \delta_{ij} r^2\, H_i\, [\cos^2\phi_i e^{\varphi_i} + \sin^2\phi_i e^{-\varphi_i}] \\ &\quad + \frac{4\tilde{\mu}_i\,\tilde{\mu}_j\, \cos\phi_i\, \sin\phi_i\, \cos\phi_j\, \sin\phi_j}{\Delta\,(H_1H_2H_3)^{2/3}}\, \sinh\varphi_i\, \sinh\varphi_j, \\ h_{\tilde{\mu}_i\,\phi_j} &= -\delta_{ij}\, 2r^2 \tilde{\mu}_i \cos\phi_i \sin\phi_i H_i \sinh\varphi_i \\ &\quad - \frac{2\cos\phi_i\, \sin\phi_i}{\Delta\,(H_1H_2H_3)^{2/3}}\, [\cos^2\phi_j e^{-\varphi_j} + \sin^2\phi_j e^{\varphi_j}]\, \sinh\varphi_i. \end{aligned} \tag{5.41}$$

To show that h_{mn} is Kähler, we introduce complex coordinates

$$z_i = \rho_i(r^2)\, \tilde{\mu}_i \left[\cos\phi_i e^{\varphi_i/2} + i\, \sin\phi_i e^{-\varphi_i/2}\right], \qquad i = 1,2,3. \tag{5.42}$$

The functions ρ_i are implicitly defined through the equation

$$\partial_R \log \rho_i^2 = \frac{H_1H_2H_3}{H_i f}\, \cosh\varphi_i, \qquad R \equiv r^2. \tag{5.43}$$

For the Kähler potential, we postulate the following dependence on the complex coordinates

$$K = K(\tfrac{1}{2}(z_i^2 + \bar{z}_i^2), |z_i|^2), \tag{5.44}$$

and for convenience we define the quantities

$$\begin{aligned} x_i &= \tfrac{1}{2}(z_i^2 + \bar{z}_i^2)\,, \\ y_i &= |z_i|^2. \end{aligned} \tag{5.45}$$

One can then read off from the $\tilde{\mu}_i$ and ϕ_i metric components in (5.41) the following differential conditions for the Kähler potential:

$$
\begin{aligned}
\partial_{y_i} K(x_i, y_i) &= \frac{RH_i}{2\rho_i^2}, \\
\partial_{x_i}\partial_{x_j} K(x_i, y_i) &= \frac{1}{2\Lambda\, H_1 H_2 H_3} \frac{\sinh\varphi_i\, \sinh\varphi_j}{\rho_i^2\, \rho_j^2}, \\
\partial_{x_i}\partial_{y_j} K(x_i, y_i) &= -\frac{1}{2\Lambda\, H_1 H_2 H_3} \frac{\sinh\varphi_i\, \cosh\varphi_j}{\rho_i^2\, \rho_j^2}, \\
\partial_{y_i}\partial_{y_j} K(x_i, y_i) &= \frac{1}{2\Lambda\, H_1 H_2 H_3} \frac{\cosh\varphi_i\, \cosh\varphi_j}{\rho_i^2\, \rho_j^2},
\end{aligned} \tag{5.46}
$$

where

$$
\Lambda = \frac{\Delta}{(H_1 H_2 H_3)^{1/3}}. \tag{5.47}
$$

Furthermore, consistency of the above equations implies the following differential conditions for the function $R(x_i, y_i)$:

$$
\partial_{x_i} R = -\frac{f\, \sinh\varphi_i}{\rho_i^2\, \Lambda\, H_1 H_2 H_3}, \qquad \partial_{y_i} R = \frac{f\, \cosh\varphi_i}{\rho_i^2\, \Lambda\, H_1 H_2 H_3}. \tag{5.48}
$$

In the end, we have only been able to obtain the Kähler potential implicitly in terms of its derivatives (5.46). To check that we have obtained the correct metric, we compute

$$
\begin{aligned}
ds_6^2 &= 2\partial_{z_i}\partial_{\bar{z}_j} K\, dz_i\, d\bar{z}_j \\
&= \sum_i \frac{R\, H_i}{\rho_i^2}\, dz_i d\bar{z}_i + \sum_{i,j} \frac{1}{\Lambda\, H_1 H_2 H_3\, \rho_i^2\, \rho_j^2} \\
&\quad \times \big[(\bar{z}_i\, \cosh\varphi_i - z_i\, \sinh\varphi_i)\, dz_i\big] \big[(z_j\, \cosh\varphi_j - \bar{z}_j\, \sinh\varphi_j)\, d\bar{z}_j\big].
\end{aligned} \tag{5.49}
$$

After some algebra, and using

$$
dz_i = z_i \frac{d\tilde{\mu}_i}{\tilde{\mu}_i} + i(z_i \cosh\varphi_i - \bar{z}_i \sinh\varphi_i)\, d\phi_i + \frac{r\, H_1 H_2 H_3}{H_i\, f}(z_i \cosh\varphi_i - \bar{z}_i \sinh\varphi_i)\, dr, \tag{5.50}
$$

one can recover the metric components listed in (5.41).

As a special limit of the regular three-charge solution discussed above, we may consider the three-charge extremal black hole (superstar) obtained by setting all the scalar fields φ_i to zero. The resulting singular solution has

$$
H_i = 1 + \frac{Q_i}{r^2}, \tag{5.51}
$$

with Q_i representing the black hole charges. Thus, the three-charge black hole can be embedded into the 1/8 BPS ansatz simply by taking the $\varphi_i = 0$ limit of the Kähler metric found above. Complex coordinates will now take the form

$$
z_i = \rho_i(r^2)\, \tilde{\mu}_i\, e^{i\phi_i}\,, \qquad i = 1, 2, 3, \tag{5.52}
$$

with the functions ρ_i defined through

$$
\partial_R \log \rho_i^2 = \frac{H_1 H_2 H_3}{f\, H_i}. \tag{5.53}
$$

Defining again $y_i = |z_i|^2$, we find that the Kähler potential is now only a function of the magnitudes

$$K = K(|z_i|^2) = K(y_i). \tag{5.54}$$

The differential equations for the Kähler potential reduce to

$$\begin{aligned} \partial_{y_i} K &= \frac{R H_i}{2\rho_i^2}, \\ \partial_{y_i}\partial_{y_j} K &= \frac{1}{2\Lambda\, H_1 H_2 H_3\, \rho_i^2\, \rho_j^2}, \end{aligned} \tag{5.55}$$

where

$$\Lambda = \sum_{i=1}^{3} \frac{y_i}{\rho_i^2\, H_i} = \frac{\Delta}{(H_1 H_2 H_3)^{1/3}}. \tag{5.56}$$

Consistency of the equations above yields the equation for the function $r^2(y_i)$:

$$\partial_{y_i} r^2 = \frac{f}{\Lambda\, H_1 H_2 H_3\, \rho_i^2}. \tag{5.57}$$

5.3.1 Boundary conditions

Our main interest in examining the three-charge smooth solutions is of course to explore the boundary surface where the S^3 inside AdS$_5$ collapses. As indicated by (5.38), the y function is given by

$$y^2 = \sqrt{\Delta}\, r^2 (H_1 H_2 H_3)^{1/3}, \tag{5.58}$$

and we are interested in the locus where this vanishes. Although this is a product of several functions, we first note that regularity and smoothness of the solution demands that the functions H_i never vanish. In particular, they must approach a non-zero constant as $r \to 0$. This in turn keeps Δ finite and non-zero. As a result, we conclude that $y = 0$ only when $r = 0$.

Since y is an implicit function of the three complex coordinates

$$z_i = \rho_i(r^2)\, \tilde{\mu}_i \left[\cos\phi_i e^{\varphi_i/2} + i \sin\phi_i e^{-\varphi_i/2}\right] \tag{5.59}$$

defined in (5.42), the algebraic condition $y = 0$ (or equivalently $r = 0$) imposes a single real constraint on the z_i coordinates, yielding a five real dimensional surface embedded in $\mathbb{C}^3$. To examine the shape of this surface, we first use

$$\begin{aligned} \tilde{\mu}_i \cos\phi_i &= \Re\Big(\frac{z_i}{\rho_i} e^{-\varphi_i/2}\Big), \\ \tilde{\mu}_i \sin\phi_i &= \Im\Big(\frac{z_i}{\rho_i} e^{\varphi_i/2}\Big), \end{aligned} \tag{5.60}$$

to find

$$\tilde{\mu}_i^2 = \frac{e^{-\varphi_i}}{\rho_i^2}\Big(\frac{z_i + \bar{z}_i}{2}\Big)^2 - \frac{e^{\varphi_i}}{\rho_i^2}\Big(\frac{z_i - \bar{z}_i}{2}\Big)^2. \tag{5.61}$$

Finally, using the constraint

$$\sum_{i=1}^{3} \tilde{\mu}_i^2 = 1, \tag{5.62}$$

we obtain

$$\sum_{i=1}^{3}\Big[\frac{\cosh\varphi_i}{\rho_i^2}|z_i|^2-\frac{\sinh\varphi_i}{\rho_i^2}\Big(\frac{z_i^2+\bar z_i^2}{2}\Big)\Big]=1. \tag{5.63}$$

The degeneration surface that we are interested in lies at $r=0$. Since the functions ρ_i and φ_i given above are functions of r, we define

$$\bar\rho_i\equiv\rho_i(r=0),\qquad \bar\varphi_i\equiv\varphi_i(r=0), \tag{5.64}$$

to be their boundary values. Regularity of the three-charge solution ensures that these values are all non-vanishing. In this case, the five-dimensional surface is given simply by

$$\sum_{i=1}^{3}\Big[\frac{\cosh\bar\varphi_i}{\bar\rho_i^2}|z_i|^2-\frac{\sinh\bar\varphi_i}{\bar\rho_i^2}\Big(\frac{z_i^2+\bar z_i^2}{2}\Big)\Big]=1. \tag{5.65}$$

This is an ellipsoid, as can be seen more clearly by writing it in terms of real and imaginary parts $z_i=x_i+iy_i$:

$$\sum_{i=1}^{3}\frac{1}{\bar\rho_i^2}\Big[e^{-\bar\varphi_i}\,x_i^2+e^{\bar\varphi_i}\,y_i^2\Big]=1. \tag{5.66}$$

This ellipsoid may be considered to be a deformation of the round sphere corresponding to the AdS ground state discussed above in Section 5.2.1. One way to see this is to note that turning off the deformation scalars, $\bar\varphi_i\to0$, forces $H_i\to1$ (to avoid the potential singularity at $r=0$). In this case, the three-charge solution reduces to the $\mathrm{AdS}_5\times S^5$ vacuum, and (5.43) is trivially integrated to give $\rho_i^2=1+r^2$. This in turn gives $\bar\rho_i=1$, in which case (5.66) reduces to the equation for a sphere of unit radius

$$\sum_{i=1}^{3}(x_i^2+y_i^2)=1, \tag{5.67}$$

corresponding to the ground state 'Fermi surface' which yields the $\mathrm{AdS}_5\times S^5$ vacuum.

As we noted for the $\mathrm{AdS}_5\times S^5$ vacuum, only the outside of the ellipsoid (5.66) is allowed. To see this, it is enough to show that both $\rho_i^2e^{\varphi_i}$ and $\rho_i^2e^{-\varphi_i}$ are monotonically increasing functions of r^2. Using $\cosh\varphi_i=(RH_i)'$ and the equation of motion (5.27), we find

$$\partial_R\,\varphi_i=\frac{H_1H_2H_3}{H_if}\,(-\sinh\varphi_i). \tag{5.68}$$

This may be combined with the expression for $\partial_R\log\rho_i^2$ from (5.43) to obtain

$$\begin{aligned}
\partial_R\Big(\frac{\rho_i^2}{e^{\varphi_i}}\Big)&=\frac{\partial_R\rho_i^2-\rho_i^2\partial_R\,\varphi_i}{e^{\varphi_i}}=\frac{\rho_i^2\,H_1H_2H_3}{e^{\varphi_i}H_i\,f}(\cosh\varphi_i+\sinh\varphi_i)=\frac{\rho_i^2\,H_1H_2H_3}{H_i\,f}\geq0,\\
\partial_R\Big(\frac{\rho_i^2}{e^{-\varphi_i}}\Big)&=\frac{\partial_R\rho_i^2+\rho_i^2\partial_R\,\varphi_i}{e^{-\varphi_i}}=\frac{\rho_i^2\,H_1H_2H_3}{e^{-\varphi_i}H_i\,f}(\cosh\varphi_i-\sinh\varphi_i)=\frac{\rho_i^2\,H_1H_2H_3}{H_i\,f}\geq0.
\end{aligned} \tag{5.69}$$

Thus the six axes of the ellipsoid $\rho_ie^{\varphi_i/2}$ and $\rho_ie^{-\varphi_i/2}$ all increase with r, which shows that only the region outside the smallest ellipsoid (given by $r=0$) is occupied.

Deforming the round ball into an ellipsoid corresponds to turning on angular momentum two harmonics on S^5. These modes are part of the standard Kaluza-Klein spectrum [30]. Likewise, the three-charge smooth gravity solution of [32], given by the fields (5.26), is dual to $\mathcal{N}=4$ Yang-Mills in a 1/8 BPS sector built on top of a combination of $\mathrm{Tr}(X^2)$, $\mathrm{Tr}(Y^2)$ and $\mathrm{Tr}(Z^2)$.

It is also instructive to consider the superstar (singular R-charged black hole) limit of the above three-charge solution, which is obtained by taking $\varphi_i = 0$ while keeping at least one of the three R charges turned on. In this case, from (5.65), we can read off the corresponding five-dimensional degeneration surface

$$\sum_{i=1}^{3} \frac{|z_i|^2}{\bar{\rho}_i^2} = 1. \tag{5.70}$$

However, for a complete picture, we also need information on the values of $\bar{\rho}_i$ for the superstar. For three non-vanishing charges, we may integrate (5.53) using (5.51) to arrive at

$$\rho_i^2 = \prod_{a=1}^{3} (R - R_a)^{\lambda_a^i}, \tag{5.71}$$

where R_a are the three roots of the cubic expression

$$0 = R^2 f = R^2 + \prod_{a=1}^{3} (R + Q_a). \tag{5.72}$$

Note that, so long as the charges Q_a are non-negative (which we always assume as a physical condition), then none of the roots R_a can lie on the positive real axis. The exponents in (5.71) are given by

$$\lambda_a^i = \frac{(R_a + Q_{i+1})(R_a + Q_{i+2})}{(R_a - R_{a+1})(R_a - R_{a+2})}, \tag{5.73}$$

where the subscripts are to be taken modulo three (*i.e.* to lie in the range $1,2,3$). For a fixed i, these exponents satisfy

$$\sum_{a=1}^{3} \lambda_a^i = 1, \qquad \sum_{a=1}^{3} R_a \lambda_a^i = -Q_i - 1. \tag{5.74}$$

As a result, the large R behavior of (5.71) is simply

$$\rho_i^2(R) \sim R + 1 + Q_i + \mathcal{O}\Big(\frac{1}{R}\Big). \tag{5.75}$$

We are of course more interested in the fate of the ellipsoid (5.70), which is obtained from the minimum values $\bar{\rho}_i$. The three non-vanishing charge case is somewhat unusual, in that the naked singularity is generally reached for $R < 0$ [33]. This occurs at the first zero of the function $R^3 H_1 H_2 H_3 = \prod(R + Q_i)$, which we may take to be at $R = -Q_3$ by appropriate ordering of the charges (*i.e.* $Q_1 \ge Q_2 \ge Q_3 > 0$). By expanding (5.71) near this singularity, we obtain

$$\begin{aligned} \rho_i^2 \;&=\; \bar{\rho}_i^2 \Big[1 + (R + Q_3)\delta_{i3} \frac{(Q_1 - Q_3)(Q_2 - Q_3)}{Q_3^2} \\ &\qquad + \tfrac{1}{2}(R + Q_3)^2 \left(|\epsilon_{ij3}| \frac{Q_j - Q_3}{Q_3^2} + \delta_{i3} \frac{2Q_1Q_2 - (Q_1 + Q_2)Q_3}{Q_3^3} \right) + \cdots \Big], \end{aligned} \tag{5.76}$$

where

$$\bar{\rho}_i^2(R) = \prod_{a=1}^{3}(-R_a - Q_3)^{\lambda_a^i}. \tag{5.77}$$

This shows that, despite the presence of the naked singularity, the ellipsoid defined by (5.70), and with interior removed, is still present for the generic three charge superstar. Here, the singularity of the solution is rather subtle, and arises not because of degeneration of the boundary surface, but rather because vanishing of the linear term for ρ_1^2 and ρ_2^2 in (5.76) results in unwanted singular behavior of the Kähler base near the ellipsoid. While the curvature of the Kähler base for a regular solution is supposed to blow up as $R \sim -8/y^4$ where y is the normal to the boundary, here the singularity is apparently of a different nature.

The above expressions are slightly modified in the case of one or more vanishing charges. For $Q_3 = 0$, the ρ_i are given by

$$\begin{aligned}
\rho_1^2 &= \sqrt{(R-R_+)(R-R_-)}\left(\frac{R-R_-}{R-R_+}\right)^{\frac{1+Q_1-Q_2}{2\sqrt{(Q_1+Q_2+1)^2-4Q_1Q_2}}}, \\
\rho_2^2 &= \sqrt{(R-R_+)(R-R_-)}\left(\frac{R-R_-}{R-R_+}\right)^{\frac{1-Q_1+Q_2}{2\sqrt{(Q_1+Q_2+1)^2-4Q_1Q_2}}}, \\
\rho_3^2 &= R\left(\frac{R-R_-}{R-R_+}\right)^{\frac{1}{\sqrt{(Q_1+Q_2+1)^2-4Q_1Q_2}}},
\end{aligned} \tag{5.78}$$

where

$$R_\pm = -\tfrac{1}{2}[(Q_1+Q_2+1) \mp \sqrt{(Q_1+Q_2+1)^2 - 4Q_1Q_2}] \tag{5.79}$$

are the two non-zero roots of (5.72). Note that $R_- < R_+ < 0$. As a result, the naked singularity is reached at $R = 0$, where ρ_3^2 vanishes. This demonstrates that $\bar{\rho}_3^2 = 0$ in the two charge case. Hence in this case the ellipsoid (5.70) collapses, and the singularity of the solution is manifest.

The one-charge superstar is even more straightforward. If Q_1 is the only non-vanishing charge, we have

$$\begin{aligned}
\rho_1^2 &= R + Q_1 + 1, \\
\rho_2^2 &= R^{\frac{Q_1}{Q_1+1}}(R+Q_1+1)^{\frac{1}{Q_1+1}}, \\
\rho_3^2 &= R^{\frac{Q_1}{Q_1+1}}(R+Q_1+1)^{\frac{1}{Q_1+1}}.
\end{aligned} \tag{5.80}$$

Taking $R \to 0$, we read off $\bar{\rho}_2^2 = \bar{\rho}_3^2 = 0$, and thus the ellipsoid collapses in two of the three complex directions. The remaining direction defines a circle in the z_1 plane, corresponding to the LLM disk with intermediate value of the LLM $Z(z_1, \bar{z}_1, y)$ function at $y = 0$, as originally demonstrated in [1].

5.4 LLM

The exploration of the three charge smooth solutions in the previous subsection has allowed us to gain some intuition on the nature of turning on Kaluza-Klein excitations, corresponding to smooth deformations of the Fermi surface. However, we are also interested in the case of topology change and the emergent picture of droplets (particle and hole excitations). While we do not have a particularly constructive way of obtaining complete 1/8 BPS solutions with non-trivial topology, there is in fact a large class of topologically interesting solutions which we may investigate, and these are nothing but the LLM ones. The LLM geometries of course preserve 1/2 of the supersymmetries, so comprise a very special subclass of the configurations described by the 1/8 BPS system of (5.1) and (5.2).

The 1/2 BPS LLM solution (4.1) has the form [1]

$$ds_{10}^2 = -h^{-2}(dt+V)^2 + h^2(|dz_1|^2 + dy^2) + ye^G d\Omega_3^2 + ye^{-G} d\widetilde{\Omega}_3^2, \tag{5.81}$$

where $V = V_z dz_1 + V_{\overline{z}} d\overline{z}_1$ satisfies the relations

$$y\partial_y V_z = i\partial_{z_1} Z, \qquad y\partial_y V_{\overline{z}} = -i\partial_{\overline{z}_1} Z, \qquad 2iy(\partial_{\overline{z}_1} V_z - \partial_{z_1} V_{\overline{z}}) = \partial_y Z, \tag{5.82}$$

and

$$Z = \tfrac{1}{2}\tanh G, \qquad h^{-2} = 2y\cosh G. \tag{5.83}$$

Here we have deliberately chosen to follow the LLM notation [1] so as to avoid confusion with the corresponding quantities in the 1/8 BPS system. Furthermore, here we reserve y to only refer to the y coordinate of LLM, and not to the y variable used in (5.1). In particular, the 1/8 BPS metric will be taken in the form

$$ds_{10}^2 = -e^{2\alpha}(dt+\omega)^2 + e^{-2\alpha} h_{ij} dx^i dx^j + e^{2\alpha} d\Omega_3^2. \tag{5.84}$$

By identifying the two three-spheres defined by $d\Omega_3$ in (5.81) and (5.84), we see that

$$e^{2\alpha} = ye^G. \tag{5.85}$$

The remaining seven-dimensional metric then has the form

$$ds_7^2 = -(e^{2\alpha} + y^2 e^{-2\alpha})(dt+V)^2 + e^{-2\alpha}(Z+\tfrac{1}{2})(dy^2 + |dz_1|^2) + y^2 e^{-2\alpha} d\widetilde{\Omega}_3^2. \tag{5.86}$$

We again wish to shift the angular coordinates on $d\widetilde{\Omega}_3$. This may be done by writing

$$d\widetilde{\Omega}_3^2 = d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2, \tag{5.87}$$

and then shifting

$$\phi_1 = \psi_1 - t, \qquad \phi_2 = \psi_2 - t. \tag{5.88}$$

Performing this shift and completing the square in dt now yields

$$\begin{aligned} ds_7^2 \;=\; & -e^{2\alpha}(dt+\omega)^2 + e^{-2\alpha}\bigg[\frac{y^2}{Z+\frac{1}{2}}(V^2 + 2V(\cos^2\theta d\psi_1 + \sin^2\theta d\psi_2)) \\ & + y^2\frac{1-2Z}{1+2Z}(\cos^2\theta d\psi_1 + \sin^2\theta d\psi_2)^2 + y^2 d\widetilde{\Omega}_3^2 + (Z+\tfrac{1}{2})(dy^2 + |dz_1|^2)\bigg], \end{aligned} \tag{5.89}$$

where

$$\omega = \frac{1}{Z+\frac{1}{2}}V + \frac{1-2Z}{1+2Z}(\cos^2\theta d\psi_1 + \sin^2\theta d\psi_2). \tag{5.90}$$

As a result, the metric on the six-dimensional base can be read off from the terms inside the square brackets above.

To show that this metric is Kähler, and to read off the Kähler potential, we introduce complex coordinates

$$z_2 = r\cos\theta e^{i\psi_1}, \qquad z_3 = r\sin\theta e^{i\psi_2}, \tag{5.91}$$

so that

$$\begin{aligned} |dz_i|^2 &= dr^2 + r^2 d\widetilde{\Omega}_3^2, \\ |\overline{z}_i dz_i|^2 &= r^2 dr^2 + r^4(\cos^2\theta d\psi_1 + \sin^2\theta d\psi_2)^2, \\ \Im(\overline{z}_i dz_i) &= r^2(\cos^2\theta d\psi_1 + \sin^2\theta d\psi_2), \end{aligned} \tag{5.92}$$

where here $i = 2, 3$ only. In this case, the metric on the six-dimensional base becomes

$$\begin{aligned} ds_6^2 &= (Z+\tfrac{1}{2})(dy^2 + |dz_1|^2) - \frac{y^2}{Z+\frac{1}{2}}\frac{dr^2}{r^2} + \frac{y^2}{r^2}|dz_i|^2 + \frac{y^2}{r^4}\frac{1-2Z}{1+2Z}|\overline{z}_i dz_i|^2 \\ &\quad + \frac{y^2}{Z+\frac{1}{2}}(V_z dz_1 + V_{\overline{z}} d\overline{z}_1)^2 + \frac{2y^2}{r^2(Z+\frac{1}{2})}(V_z dz_1 + V_{\overline{z}} d\overline{z}_1)\Im(\overline{z}_i dz_i). \end{aligned} \tag{5.93}$$

Note that $r^2 = |z_i|^2$. Since the original LLM coordinate y is somehow out of place, we need to find a transformation relating y with the complex coordinates z_1, z_2, z_3. To obtain this transformation, we take a hint from the dr and dy sector of the metric

$$ds_6^2 = \frac{y^2}{Z+\frac{1}{2}}\left((Z+\frac{1}{2})^2\frac{dy^2}{y^2} - \frac{dr^2}{r^2}\right) + \cdots . \tag{5.94}$$

This suggests that we take

$$r^2(z_1, \overline{z}_1, y) = \exp\int^{y^2} \left(Z(z_1, \overline{z}_1, y') + \tfrac{1}{2}\right)\frac{d(y'^2)}{y'^2}, \tag{5.95}$$

where we are somewhat sloppy about the limits of the indefinite integral. Because of the $z_1, \overline{z}_1$ dependence on the right hand side, this relation is somewhat subtle to manipulate. For example

$$\begin{aligned} \frac{dr}{r} &= \left(\int^y \partial_{z_1} Z\frac{dy'}{y'}\right)dz_1 + \left(\int^y \partial_{\overline{z}_1} Z\frac{dy'}{y'}\right)d\overline{z}_1 + \frac{Z+\frac{1}{2}}{y}dy \\ &= -i\left[\left(\int^y \partial_{y'} V_z dy'\right)dz_1 - \left(\int^y \partial_{y'} V_{\overline{z}} dy'\right)d\overline{z}_1\right] + \frac{Z+\frac{1}{2}}{y}dy \\ &= -i(V_z dz_1 - V_{\overline{z}} d\overline{z}_1) + \frac{Z+\frac{1}{2}}{y}dy, \end{aligned} \tag{5.96}$$

where we have used (5.82). Here we assume that the integration in (5.95) may be defined so that this differential relation holds. Inserting this relation into (5.93) finally gives the complex Hermitian metric

$$\begin{aligned} ds_6^2 &= \left((Z+\tfrac{1}{2}) + \frac{4y^2}{Z+\frac{1}{2}}V_z V_{\overline{z}}\right)|dz_1|^2 + \frac{y^2}{r^2}\left(|dz_2|^2 + |dz_3|^2\right) + \frac{y^2}{r^4}\frac{1-2Z}{1+2Z}|\overline{z}_2 dz_2 + \overline{z}_3 dz_3|^2 \\ &\quad - \frac{4y^2}{r^2(Z+\frac{1}{2})}\Re\left(iV_{\overline{z}}(\overline{z}_2 dz_2 + \overline{z}_3 dz_3)d\overline{z}_1\right), \end{aligned} \tag{5.97}$$

where $r^2 = |z_2|^2 + |z_3|^2$, and y is implicitly defined from (5.95).

In order to show that the above metric is Kähler, we may directly obtain the Kähler potential $K(z_1, \overline{z}_1, r^2)$ by integrating the differential relations

$$\partial_{r^2} K = \frac{y^2}{2r^2}, \qquad \partial_{r^2}\partial_{r^2} K = \frac{y^2}{2r^4}\frac{1-2Z}{1+2Z}, \qquad \partial_{z_1}\partial_{\overline{z}_1} K = \tfrac{1}{2}(Z+\tfrac{1}{2}) + \frac{2y^2}{Z+\frac{1}{2}} V_z V_{\overline{z}},$$
$$\partial_{z_1}\partial_{r^2} K = -\frac{y^2}{2r^2(Z+\frac{1}{2})} i V_{\overline{z}}, \qquad \partial_{\overline{z}_1}\partial_{r^2} K = \frac{y^2}{2r^2(Z+\frac{1}{2})} i V_z. \tag{5.98}$$

The result is particularly simple

$$K(z_1, \overline{z}_1, y^2) = \tfrac{1}{2} \int^{y^2} (Z(z_1, \overline{z}_1, y') + \tfrac{1}{2}) d(y'^2). \tag{5.99}$$

Of course, y^2 has to be rewritten in terms of z_1, $\overline{z}_1$ and r^2 using (5.95). In order to verify that this is correct, we need the chain rule expressions

$$\begin{aligned} \partial_r f(z_1, \overline{z}_1, r) &= \frac{1}{\partial r/\partial y} \partial_y f(z_1, \overline{z}_1, y) = \frac{y}{r(Z+\frac{1}{2})} \partial_y f(z_1, \overline{z}_1, y), \\ \partial_{z_1} f(z_1, \overline{z}_1, r) &= \left(\partial_{z_1} - \frac{\partial r/\partial z_1}{\partial r/\partial y} \partial_y \right) f(z_1, \overline{z}_1, y) = \left(\partial_{z_1} + \frac{iyV_z}{Z+\frac{1}{2}} \partial_y \right) f(z_1, \overline{z}_1, y), \end{aligned} \tag{5.100}$$

where $r = r(z_1, \overline{z}_1, y)$.

Linearity of the LLM Laplacian (3.97) allows a Green's function solution for Z of the form [1]

$$Z(z_1, \overline{z}_1, y) = \frac{1}{2} - \frac{y^2}{\pi} \int_D \frac{dx_1' dx_2'}{[|z_1 - z_1'|^2 + y^2]^2}, \tag{5.101}$$

where the integral is only over the areas of the two-dimensional droplets ($Z = -1/2$) sitting in the $Z = 1/2$ background. This allows us to rewrite (5.95) as

$$\log(r^2) = \log(y^2) + \frac{1}{\pi} \int_D \frac{dx_1' dx_2'}{|z_1 - z_1'|^2 + y^2}, \tag{5.102}$$

at least up to an unimportant y-independent function arising from the indefinite y integral in (5.95). As y approaches 0, there are two cases to consider: i) $z_1 \in D$ and ii) $z_1 \notin D$. In the first case $r^2\big|_{y=0}$ is finite and (5.102) defines a five-dimensional surface, whereas in the latter $r^2 = y^2 + \mathcal{O}(y^4)$.

In addition, substituting (5.101) into (5.99) while ensuring proper asymptotic behavior gives an expression for the Kähler potential

$$K = \tfrac{1}{2} y^2 + \tfrac{1}{2} |z_1|^2 + \frac{1}{2\pi} \int_D \left(\frac{y^2}{|z_1 - z_1'|^2 + y^2} - \log[|z_1 - z_1'|^2 + y^2] \right) dx_1' dx_2'. \tag{5.103}$$

5.4.1 The LLM vacuum

As a simple example, we may consider the $\mathrm{AdS}_5 \times S^5$ vacuum, which is specified by a circular disk in the LLM plane. Taking this disk to have radius L, the Green's function integral (5.101) gives [1]

$$Z = \frac{|z_1|^2 + y^2 - L^2}{2\sqrt{(|z_1|^2 + y^2 - L^2)^2 + 4y^2 L^2}}. \tag{5.104}$$

Before working out the Kähler potential, we may use (5.102) to determine

$$r^2 = \frac{1}{2}\left(L^2 + y^2 - |z_1|^2 + \sqrt{(|z_1|^2 + y^2 - L^2)^2 + 4y^2L^2}\right), \tag{5.105}$$

which in turn may be inverted to yield

$$y^2 = r^2\left(1 - \frac{L^2}{r^2 + |z_1|^2}\right). \tag{5.106}$$

The $y = 0$ surface reduces to $r^2 + |z_1|^2 = L^2$ for $|z_1| < L$ and to $r = 0$ for $|z_1| > L$, corresponding to the cases i) and ii) mentioned in the previous section, after (5.102).

The Kähler potential itself is obtained from (5.103):

$$\begin{aligned} K &= \frac{1}{4}\Big[|z_1|^2 + y^2 + L^2 + \sqrt{(|z_1|^2 + y^2 - L^2)^2 + 4y^2L^2} \\ &\quad -2L^2 \log\left(\frac{1}{2}\left(|z_1|^2 + y^2 + L^2 + \sqrt{(|z_1|^2 + y^2 - L^2)^2 + 4y^2L^2}\right)\right)\Big]. \end{aligned} \tag{5.107}$$

Using (5.105), this may be rewritten as

$$K = \tfrac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2) - \tfrac{1}{2}L^2 \log(|z_1|^2 + |z_2|^2 + |z_3|^2), \tag{5.108}$$

where we have used $r^2 = |z_2|^2 + |z_3|^2$. This of course recovers the symmetrical $\mathrm{AdS}_5 \times S^5$ Kähler potential (5.23), but this time with the AdS radius L restored.

5.4.2 Multi-disk configurations

Given the vacuum solution corresponding to a single LLM disk, there is in fact a natural procedure for building up topologically non-trivial configurations through linear superposition. Suppose we have n disks, each with radius b_i, centered at the complex position a_i in the z_1 plane. So long as the disks are non-overlapping, the function Z obtained by (5.101) has a superposition solution of the form

$$\begin{aligned} Z &= \frac{1}{2} + \sum_{i=1}^{n}\left(\frac{|z_1 - a_i|^2 + y^2 - b_i^2}{2\sqrt{(|z_1 - a_i|^2 + y^2 - b_i^2)^2 + 4y^2b_i^2}} - \frac{1}{2}\right) \\ &= \frac{1-n}{2} + \sum_{i=1}^{n}\frac{|z_1 - a_i|^2 + y^2 - b_i^2}{2\sqrt{(|z_1 - a_i|^2 + y^2 - b_i^2)^2 + 4y^2b_i^2}}. \end{aligned} \tag{5.109}$$

In addition, the form of the integral (5.102) relating r^2 with y^2 indicates that r^2 may be obtained by superposing n individual terms, each of the form given by (5.105)

$$\begin{aligned} r^2 &= y^2 \prod_{i=1}^{n} \frac{1}{2y^2}\left[b_i^2 + y^2 - |z_1 - a_i|^2 + \sqrt{(|z_1 - a_i|^2 + y^2 - b_i^2)^2 + 4y^2b_i^2}\right] \\ &= \frac{y^{2(1-n)}}{2^n}\prod_{i=1}^{n}\left[b_i^2 + y^2 - |z_1 - a_i|^2 + \sqrt{(|z_1 - a_i|^2 + y^2 - b_i^2)^2 + 4y^2b_i^2}\right]. \end{aligned} \tag{5.110}$$

Similarly, the Kähler potential may be obtained by superposing individual terms of the form (5.107)

$$\begin{aligned} K &= \tfrac{1}{2}y^2 + \tfrac{1}{2}|z_1|^2 + \sum_{i=1}^{n} \frac{1}{4}\Bigg[b_i^2 - y^2 - |z_1 - a_i|^2 + \sqrt{(|z_1-a_i|^2 + y^2 - b_i^2)^2 + 4y^2 b_i^2} \\ &\qquad -2b_i^2 \log\left(\frac{1}{2}\left(b_i^2 + y^2 + |z_1-a_i|^2 + \sqrt{(|z_1-a_i|^2 + y^2 - b_i^2)^2 + 4y^2 b_i^2}\right)\right)\Bigg]. \end{aligned} \tag{5.111}$$

In principle, (5.110) ought to be inverted to give y^2 as a function of z_1, $\overline{z}_1$ and r^2. In turn, this could then be inserted into (5.111) to obtain the final expression for the Kähler potential. Unfortunately, however, (5.110) is a rather unwieldy function to invert. Nevertheless, we can learn a fair bit about the boundary conditions even without an explicit form of the Kähler potential.

Our main interest is to examine the degeneration surface when $e^{2\alpha} \to 0$ (*i.e.* when the S^3 inside AdS_5 shrinks). From (5.85), this requires that $y \to 0$ (along with some possible requirement on e^G, which we are not so concerned about). Recalling that $r^2 = |z_2|^2 + |z_3|^2$ in our notation, setting $y = 0$ in (5.110) then defines a five-dimensional degeneration surface through a real algebraic equation in $\mathbb{C}^3$. Actually, because of the $y^{2(1-n)}$ prefactor, some care must be taken before we can let $y = 0$ in (5.110). To proceed, we may start with the small y expansion of (5.110), and then subsequently take $y \to 0$.

Because of the square root expressions, this small y expansion is dependent on our location in the z_1 plane. In particular, as $y \to 0$, we have

$$\begin{aligned} &\left[b_i^2 + y^2 - |z_1-a_i|^2 + \sqrt{(|z_1-a_i|^2 + y^2 - b_i^2)^2 + 4y^2 b_i^2}\right] \\ &\qquad = \begin{cases} \dfrac{|z_1-a_i|^2}{|z_1-a_i|^2 - b_i^2}(2y^2) + \mathcal{O}(y^4), & |z_1 - a_i| > |b_i|; \\ 2(b_i^2 - |z_1-a_i|^2) + \mathcal{O}(y^2), & |z_1-a_i| < |b_i|. \end{cases} \end{aligned} \tag{5.112}$$

The first case corresponds to z_1 outside the i-th disk, and the second to z_1 inside. Because of the non-overlapping condition, z_1 can fall inside a single disk, at most. Suppose we look at the region inside the j-th disk. In this case, the expression for r^2 in (5.110) receives $n-1$ contributions of the first type (when $i \neq j$), and a single contribution of the second type. This combination of expansions introduces a $y^{2(n-1)}$ factor in the product, canceling the $y^{2(1-n)}$ factor in (5.110). So the result for this j-th region is

$$r^2 \equiv |z_2|^2 + |z_3|^2 = \left(b_j^2 - |z_1 - a_j|^2\right) \prod_{i\neq j} \frac{|z_1-a_i|^2}{|z_1-a_i|^2 - b_i^2}. \tag{5.113}$$

Note that this equation is exact, even though we had to expand in $y = 0$ in order to obtain it.

We recall that this equation defines a five-dimensional surface inside $\mathbb{C}^3$ where the S^3 inside AdS_5 shrinks to a point. To understand the implication of this equation better, we may consider the single-disk limit, when the other $n-1$ finite disks are very far away from the jth disk. In this case, $|z_1 - a_i| \gg b_i$ for $i \neq j$, and we get the simplified expression

$$|z_2|^2 + |z_3|^2 = b_j^2 - |z_1 - a_j|^2. \tag{5.114}$$

This describes a round five-sphere centered at $z_1 = a_j$, with radius b_j. When the disks are not so well separated, the additional factors in (5.113) lead to a distortion of the five-sphere. Nevertheless, the picture that emerges is clear. The interior of each LLM disk gets mapped into a (possibly distorted) five-sphere degeneration surface inside $\mathbb{C}^3$. Equation (5.113) simply describes the j-th disconnected component of the complete five-dimensional degeneration surface.

We have now shown that non-trivial LLM topology has a natural generalization in the 1/8 BPS system. In particular, individual LLM droplets (with disk topology) map directly into degeneration surfaces which are topologically five-spheres, and which may be considered as canonical 1/8 BPS droplets. Since the interior of each droplet is not present, the 1/8 BPS system can be described using a set of coordinates spanning $\mathbb{C}^3$, but with various regions removed. In the LLM picture, a large disk surrounded by small droplets corresponds to a collection of dual giant gravitons, all expanding in AdS_5 [1]. Each droplet modifies the topology, and may be considered as a backreacted version of a giant graviton. In the general 1/8 BPS description, this has a corresponding picture as a large spherical void at the center of $\mathbb{C}^3$ surrounded by a set of five-sphere 'bubbles', each bubble being one of the dual giant gravitons.

Given this understanding of dual giants in the 1/8 BPS context, there is still one remaining question, and that is how giant gravitons expanding on S^5 fit in the above framework. In terms of the LLM picture, turning on these giant gravitons corresponds to introducing holes in the AdS disk itself. Before we consider the effect of holes, however, we first consider the $y \to 0$ behavior of (5.110) in the case that z_1 lies outside all of the disks. In this case, all n expressions in (5.110) are of the form of the top line in (5.112), and we thus end up with

$$r^2 \approx y^2 \prod_{i=1}^{n} \frac{|z_1 - a_i|^2}{|z_1 - a_i|^2 - b_i^2}, \tag{5.115}$$

as $y \to 0$. The extra y^2 factor then ensures that $r \to 0$ as $y \to 0$, so long as z_1 lies outside the disks. Recalling that $r^2 = |z_2|^2 + |z_3|^2$, this limit corresponding to shrinking S^3 inside S^5, which of course agrees with the 1/2 BPS bubbling picture of [1].

We are now in a position to consider adding holes (giant gravitons expanding in S^5) to the above multi-disk configuration. Again, because of linear superposition, we may consider holes as simply 'negative' regions inside a disk (provided, of course, that they are entirely contained within the corresponding disk). In this case, for n disks as above along with m circular holes (each with radius $\tilde{b}_i$ and centered at $\tilde{a}_i$), the generalization of (5.110) is simply

$$\begin{aligned} r^2 \;=\; & y^2 \prod_{i=1}^{n} \frac{1}{2y^2}\left[b_i^2 + y^2 - |z_1 - a_i|^2 + \sqrt{(|z_1 - a_i|^2 + y^2 - b_i^2)^2 + 4y^2 b_i^2}\right] \\ & \times \prod_{i=1}^{m} 2y^2 \left[\tilde{b}_i^2 + y^2 - |z_1 - \tilde{a}_i|^2 + \sqrt{(|z_1 - \tilde{a}_i|^2 + y^2 - \tilde{b}_i^2)^2 + 4y^2 \tilde{b}_i^2}\right]^{-1}. \end{aligned} \tag{5.116}$$

For a single hole inside the AdS disk, the degeneration surface can be obtained by taking the $y \to 0$ limit of this expression for the case where z_1 lies in the disk, but not the hole. The resulting surface

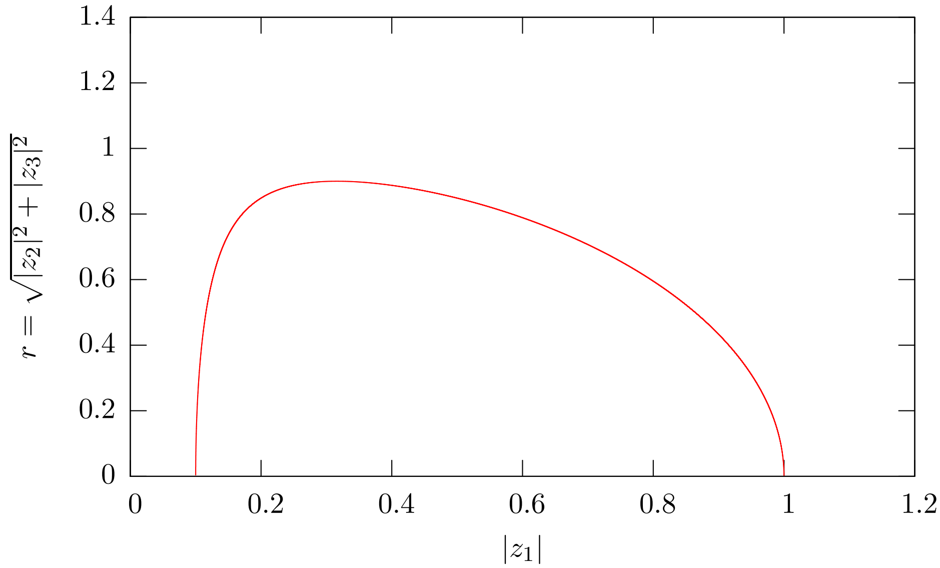

Figure 1: Profile of r versus $|z_1|$ for the configuration corresponding to a single hole of radius 0.1 centered at the origin of the AdS disk (of unit radius). This picture corresponds to a maximal giant graviton expanding on S^5.

is described by

$$r^2 \equiv |z_2|^2 + |z_3|^2 = \frac{(L^2 - |z_1|^2)(|z_1 - \tilde{a}|^2 - \tilde{b}^2)}{|z_1 - \tilde{a}|^2}, \tag{5.117}$$

where we have taken the AdS disk to be centered at the origin and to have radius L. The hole is centered at $\tilde{a}$, and has radius $\tilde{b}$. This describes a five-dimensional surface of topology $S^4 \times S^1$, which was in fact already noticed in [1] when fibering $\tilde{S}^3$ over an annulus in the LLM plane. As an example, we plot the profile of the surface given by (5.117) in Fig. 1.

On the gauge theory side of the duality, the picture shown in Fig. 1 presumably corresponds to the numerical eigenvalue distribution studied recently in [23] for the one hole state. We note that, at least in this coordinate system, the change of r is very steep near the central hole of the giant graviton. This may account for the failure of the numerical eigenvalue distribution to close on this hole observed in [23]. However, it remains to be seen whether or not the present coordinate system is in fact the one which is preferred when matching to the eigenvalue distribution.

As more holes are introduced into the AdS disk, more and more non-trivial topology is generated; adding m holes gives rise to a corresponding five-dimensional surface which may be described as S^3 fibered over the disk with m holes. Thus the five-dimensional boundary surface has a very physical interpretation as the distortion of the original five-sphere of the $\mathrm{AdS}_5 \times S^5$ background. A complete 1/2 BPS bubbling geometry, with both giant gravitons and dual giants, thus involves an AdS disk along with both particle and hole excitations. The holes in the AdS disk change the topology of the original five-sphere, while the particles outside the disk give rise to additional degeneration surfaces. Consider, for example, the LLM geometry specified in Fig. 2, corresponding to the excitation of

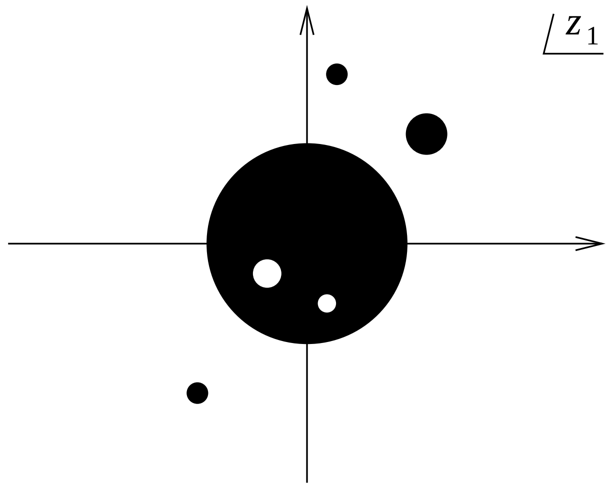

Figure 2: An LLM configuration with three droplets and two holes.

two giant gravitons and three dual giants. When written in the 1/8 BPS framework, the resulting degeneration surfaces, as given by (5.116), take on the form shown in Fig. 3. More complicated geometries, corresponding to non-circular droplets, are of course possible. However, for 1/2 BPS states, the boundary surfaces always contain an additional unbroken S^3 isometry related to the angular directions not indicated in Fig 3. This isometry would not be present for more generic 1/4 and 1/8 BPS bubbles. Nevertheless, even in such cases, the overall picture of droplets as removed volumes of $\mathbb{R}^6$ remains valid.

6 Examples fitting into the 1/4 BPS case

After having studied the general 1/8 BPS case, we now turn to explicit solutions for the case of 1/4 BPS configurations. These backgrounds have an additional S^1 isometry compared with the generic 1/8 BPS backgrounds, and have a ten-dimensional metric of the form

$$\begin{aligned} ds^2_{10} &= -h^{-2}(dt+\omega)^2 + h^2((Z+\tfrac{1}{2})^{-1}2h_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} + dy^2) + y(e^G d\Omega_3^2 + e^{-G}(d\psi + \mathcal{A})^2), \\ h^{-2} &= 2y\cosh G, \qquad h_{i\bar{j}} = \partial_i\partial_{\bar{j}}K. \end{aligned} \tag{6.1}$$

We have also defined

$$Z \equiv \frac{1}{2}\mathrm{tanh}G = -\frac{1}{2}y\partial_y\frac{1}{y}\partial_y K\,, \tag{6.2}$$

which is the 1/4 BPS version of the LLM function. The four-dimensional base metric $h_{i\bar{j}}$ is Kähler, and is further constrained by a Monge-Ampère type equation (3.74), along with auxiliary condition (3.79)

$$\begin{aligned} &\log\det h_{i\bar{j}} = \log(Z+\frac{1}{2}) + n\eta\log y + \frac{1}{y}(2-n\eta)\partial_y K + D(z_i,\bar{z}_{\bar{j}}), \\ &(1+*_4)\partial\bar{\partial}D = \frac{4}{y^2}(1-n\eta)\partial\bar{\partial}K. \end{aligned} \tag{6.3}$$

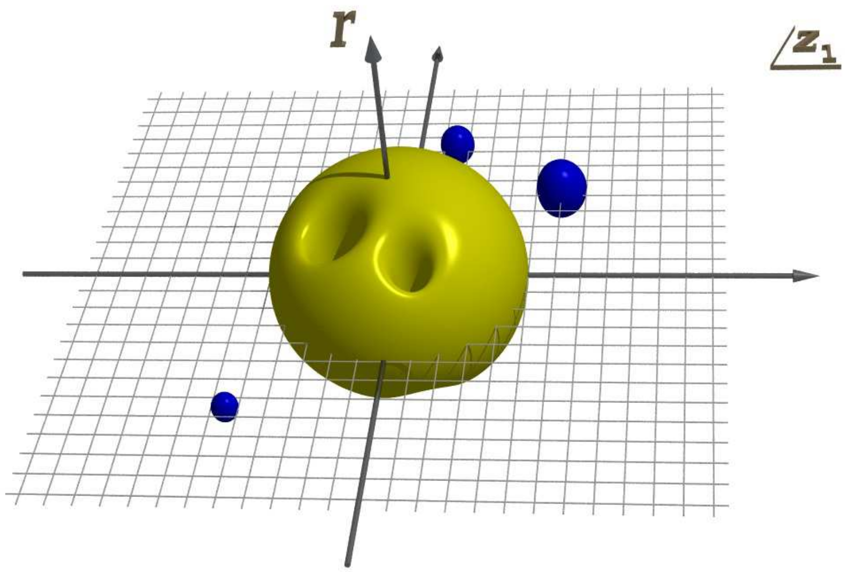

Figure 3: The LLM configuration of Fig. 2 shown as droplets in the six-dimensional base given by (5.97). Here $r^2 = |z_2|^2 + |z_3|^2$, and the additional S^3 directions are suppressed. Note that the physical space is comprised of the region outside of the droplets only.

Since we are mainly interested in the form of the Kähler metric on the base, we do not repeat here the expressions for the two-forms $\mathcal{F} = d\mathcal{A}$ and $d\omega$, nor for the IIB self-dual five-form. These expressions, along with details of the analysis, may be found above in Sections 2.2 and 3.2. We do note, however, that the two-form $\mathcal{F}$ must satisfy the additional constraint given in (3.80).

Since the construction of arbitrary new backgrounds by solving the Monge-Ampère equation (6.3) is a rather challenging task, we instead look at several classes of existing solutions and see how they may be transformed into the 1/4 BPS form (6.1). In this way, we are able to deduce the generic 1/4 BPS bubbling picture without having to turn directly to the construction of explicit solutions.

Note, however, that (6.3) becomes much simpler to analyze in certain special cases, such as when the complex two-dimensional base decomposes into a direct product of two Riemann surfaces. We will study this case at the end of this section and show its connection to the embedding of the LLM solution into the gauged ansatz.

6.1 $\mathbf{AdS}_5 \times S^5$

Before expanding on the 1/4 BPS droplet picture, we start with the embedding of the $\mathrm{AdS}_5 \times S^5$ ground state into the framework given by (6.1). We will then move on to more complicated geometries.

As in Section 5.2, we take global $\mathrm{AdS}_5 \times S^5$ written as:

$$\begin{aligned} ds^2 &= -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + d\Omega_5^2 \\ &= -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 \\ &\qquad + \sin^2\theta d\psi^2 + d\theta^2 + \cos^2\theta[\cos^2\alpha d\phi_1^2 + d\alpha^2 + \sin^2\alpha d\phi_2^2], \end{aligned} \tag{6.4}$$

where in the second line we have chosen an explicit parameterization of the five-sphere metric. In order to embed this into the 1/4 BPS system of (6.1), we must identify the appropriate $S^3 \times S^1$ isometry for the embedding. While the S^3 factor inside AdS_5 is obvious, there are several possible choices for the circle factor inside S^5. By writing the five-sphere metric as above, we have chosen to follow the ungauged 1/4 BPS ansatz, where we set $\mathcal{A} = 0$ from the start. Then, after comparing with (6.1), we choose to identify the time coordinate t, as well as the $S^3 \times S^1$ factors $d\Omega_3$ and $d\psi$. (Another possibility, which we do not pursue, would be to write S^5 as $U(1)$ bundled over CP^2 as in (5.12), and then to follow the gauged 1/4 BPS ansatz.)

The above identification allows us to deduce

$$\begin{aligned} ye^G &= \sinh^2\rho, & ye^{-G} &= \sin^2\theta, \\ h^{-2} &= \sinh^2\rho + \sin^2\theta, & y &= \sinh\rho\sin\theta. \end{aligned} \tag{6.5}$$

Thus the y coordinate is easily given in terms of the original global $\mathrm{AdS}_5 \times S^5$ variables. In fact, these expressions are identical to their 1/2 BPS LLM counterparts. This suggests that we simply use the LLM coordinate transformation

$$y = \sinh\rho\sin\theta, \qquad r = \cosh\rho\cos\theta, \tag{6.6}$$

to map between (ρ, θ) and (r, y) coordinates. In particular, this yields

$$dy^2 + dr^2 = h^{-2}(d\rho^2 + d\theta^2). \tag{6.7}$$

For the remaining coordinates, we note, just as in the 1/8 BPS case of (5.18), that the azimuthal angles ψ_1 and ψ_2 need to be shifted

$$\phi_1 = \psi_1 - t, \qquad \phi_2 = \psi_2 - t. \tag{6.8}$$

After completing the square in dt, and comparing with (6.1), we now obtain the one-form

$$\omega = h^2\cos^2\theta(\cos^2\alpha d\psi_1 + \sin^2\alpha d\psi_2), \tag{6.9}$$

as well as the four-dimensional Kähler metric

$$\begin{aligned} ds_4^2 &= h^2\sinh^2\rho\big[dr^2 + h^{-2}\cos^2\theta(\cos^2\alpha d\psi_1^2 + d\alpha^2 + \sin^2\alpha d\psi_2^2) \\ &\qquad + \cos^4\theta(\cos^2\alpha d\psi_1 + \sin^2\alpha d\psi_2)^2\big] \\ &\equiv A\, dr^2 + B\, d\Omega_3^2 + C(\cos^2\alpha d\psi_1 + \sin^2\alpha d\psi_2)^2\,. \end{aligned} \tag{6.10}$$

In order to demonstrate that (6.10) is indeed Kähler, we identify the Kähler potential K. To do so, we first write the metric entirely in terms of the coordinates $(r, \alpha, \psi_1, \psi_2)$. This may be done by inverting (6.6) to obtain

$$\begin{aligned}
\sinh^2 \rho &= \tfrac{1}{2}(r^2 + y^2 - 1) + \sqrt{\tfrac{1}{4}(r^2 + y^2 - 1)^2 + y^2}, \\
\sin^2 \theta &= -\tfrac{1}{2}(r^2 + y^2 - 1) + \sqrt{\tfrac{1}{4}(r^2 + y^2 - 1)^2 + y^2},
\end{aligned} \tag{6.11}$$

which gives us expressions for A, B and C in terms of (r, y) only. We now introduce complex coordinates z_1, z_2 and, based on symmetry, assume that the Kähler potential is only a function of $r^2 = |z_1|^2 + |z_2|^2$ and y, namely $K = K(r^2, y)$. We then find that the metric takes the form

$$ds^2 = 2K'(|dz_1|^2 + |dz_2|^2) + 2K''|\bar{z}_1 dz_1 + \bar{z}_2 dz_2|^2, \tag{6.12}$$

where primes are derivatives with respect to r^2. To make contact with (6.10), we choose a parameterization of z_1 and z_2 as

$$z_1 = r \cos \alpha e^{i\psi_1}, \qquad z_2 = r \sin \alpha e^{i\psi_2}. \tag{6.13}$$

Using

$$\begin{aligned}
|dz_1|^2 + |dz_2|^2 &= dr^2 + r^2 d\Omega_3^2, \\
|\bar{z}_1 dz_1 + \bar{z}_2 dz_2|^2 &= r^2 dr^2 + r^4(\cos^2 \alpha d\psi_1 + \sin^2 \alpha d\psi_2)^2,
\end{aligned} \tag{6.14}$$

the Kähler metric (6.12) becomes

$$ds^2 = 2(K' + r^2 K'')dr^2 + 2r^2 K' d\Omega_3^2 + 2r^4 K''(\cos^2 \alpha d\psi_1 + \sin^2 \alpha d\psi_2)^2. \tag{6.15}$$

Comparing (6.15) with (6.10) gives the identifications

$$K' + r^2 K'' = \tfrac{1}{2}A, \qquad r^2 K' = \tfrac{1}{2}B, \qquad r^4 K'' = \tfrac{1}{2}C. \tag{6.16}$$

Notice that this system is overdetermined, since the function $K(r^2)$ is determined by three equations. However, we may verify that $B + C = r^2 A$ and $A = B'$. As a result, the three equations are redundant, and we are left with only $K' = B/2r^2$, which may be integrated to give the Kähler potential

$$K(r^2, y) = \frac{1}{2} \int^{r^2} \frac{B(r^2, y)}{r^2} d(r^2). \tag{6.17}$$

Although it is not particularly illuminating, we can perform the integral explicitly. Using the expression for B,

$$B = \tfrac{1}{2}(r^2 - y^2 - 1) + \sqrt{\tfrac{1}{4}(r^2 + y^2 - 1)^2 + y^2}, \tag{6.18}$$

we find that the Kähler potential is

$$\begin{aligned}
K &= \tfrac{1}{2}\left(\tfrac{1}{2}(r^2 + y^2 + 1) + \sqrt{\tfrac{1}{4}(r^2 + y^2 - 1)^2 + y^2}\right) \\
&\quad - \tfrac{1}{2}\log\left(\tfrac{1}{2}(r^2 + y^2 + 1) + \sqrt{\tfrac{1}{4}(r^2 + y^2 - 1)^2 + y^2}\right) \\
&\quad - \tfrac{1}{2}y^2 \log\left(\tfrac{1}{2}(-r^2 + y^2 + 1) + \sqrt{\tfrac{1}{4}(r^2 + y^2 - 1)^2 + y^2}\right) + \tfrac{1}{2}y^2 \log(y).
\end{aligned} \tag{6.19}$$

The final function of y ensures that K satisfies the relation (6.2).

6.1.1 Boundary conditions

In analogy with the 1/8 BPS embedding of $AdS_5 \times S^5$ as well as the 1/2 BPS LLM embedding, we expect to find that boundary conditions at $y = 0$ will give us a spherical surface. To make this apparent, we start by pointing out that our complex coordinates are such that

$$|z_1|^2 + |z_2|^2 = r^2. \tag{6.20}$$

The coordinate $y = \sinh\rho \sin\theta$ vanishes in two cases, either when $\rho = 0$ or $\theta = 0$. The $\rho = 0$ case, corresponding to the S^3 shrinking to zero size, tells us from (6.6) that $r \leq 1$. In turn, this translates into the interior of a spherical (unit radius) droplet:

$$|z_1|^2 + |z_2|^2 \leq 1. \tag{6.21}$$

On the other hand, the $\theta = 0$ limit, which describes collapse of the S^1, corresponds to the outside of the spherical droplet,

$$|z_1|^2 + |z_2|^2 \geq 1. \tag{6.22}$$

Thus the two regions are separated by a three-dimensional sphere of unit radius[2]. This may be viewed as a higher-dimensional realization of the unit LLM circle, which describes the 1/2 BPS embedding of $\mathrm{AdS}_5 \times S^5$, as well as a lower-dimensional realization of the five-sphere which describes the 1/8 BPS embedding.

6.2 Two-charge smooth solutions

Starting from the round three-sphere, which describes the $\mathrm{AdS}_5 \times S^5$ ground state, we now move on to less trivial backgrounds. In particular, we now turn to the case of the smooth, two-charge (1/4 BPS) solutions which can be obtained from the more general three-charge case (5.34) by setting one of the charges to zero. To be specific, we choose to set $H_1 = 1$ and the corresponding scalar field $\varphi_1 = 0$ in (5.34).

Using the explicit expressions (5.35) for μ_1 and μ_2, the metric takes the form

$$\begin{aligned} ds_{10}^2 \;=\; & -\frac{f\sqrt{\Delta}}{(H_2H_3)^{2/3}}\,dt^2 + \sqrt{\Delta}(H_2H_3)^{1/3}\,f^{-1}\,dr^2 + r^2\,\sqrt{\Delta}(H_2H_3)^{1/3}\,d\Omega_3^2 \\ & + \frac{1}{\sqrt{\Delta}(H_2H_3)^{1/3}}\left[\cos^2\theta d\theta^2 + \sin^2\theta(d\phi_1 - dt)^2\right] \\ & + \frac{1}{\sqrt{\Delta}}\left\{\frac{H_2^{2/3}}{H_3^{1/3}}\left[e^{\varphi_2}\Big(d\mu_3 - H_2^{-1}\,\mu_4\,dt\Big)^2 + e^{-\varphi_2}\Big(d\mu_4 + H_2^{-1}\,\mu_3\,dt\Big)^2\right]\right. \\ & \qquad \left. + \frac{H_3^{2/3}}{H_2^{1/3}}\left[e^{\varphi_3}\Big(d\mu_5 - H_3^{-1}\,\mu_6\,dt\Big)^2 + e^{-\varphi_3}\Big(d\mu_6 + H_3^{-1}\,\mu_5\,dt\Big)^2\right]\right\}. \end{aligned} \tag{6.23}$$

If we let $d\psi = d\phi_1 - dt$, we can think of ψ as parameterizing the S^1 direction of the 1/4 BPS ansatz. In particular, this suggests the ungauged ansatz, as $d\psi$ is trivially fibered over the remaining

[2] The reason for the unit radius is that we have taken the AdS_5 radius to be one.

directions of the metric (6.23). However, for convenience in subsequent manipulations, we will formally allow $\mathcal{A} \neq 0$ for the moment. Along with S^1, the S^3 is also clearly visible, which brings us to the following identifications:

$$y\, e^G = r^2 (H_2 H_3)^{1/3} \sqrt{\Delta}, \qquad y\, e^{-G} = \frac{\sin^2\theta}{\sqrt{\Delta}(H_2 H_3)^{1/3}}, \tag{6.24}$$

with

$$\begin{aligned} \Delta \;=\; & (H_2 H_3)^{1/3} \sin^2\theta + \frac{H_3^{1/3}}{H_2^{2/3}} \cos^2\theta \sin^2\alpha (e^{-\varphi_2} \sin^2\phi_2 + e^{\varphi_2} \cos^2\phi_2) \\ & + \frac{H_2^{1/3}}{H_3^{2/3}} \cos^2\theta \cos^2\alpha (e^{-\varphi_3} \sin^2\phi_3 + e^{\varphi_3} \cos^2\phi_3). \end{aligned} \tag{6.25}$$

Thus, we find

$$\begin{aligned} & y = r\, \sin\theta, \qquad e^G = \frac{\sqrt{\Delta}\, (H_2 H_3)^{1/3}\, r}{\sin\theta}, \\ & h^{-2} = y\, e^G + y\, e^{-G} = \sqrt{\Delta} (H_2 H_3)^{1/3} \left(r^2 + \frac{\sin^2\theta}{\Delta (H_2 H_3)^{2/3}} \right). \end{aligned} \tag{6.26}$$

We will come back to these relations when we discuss boundary conditions.

To show that this solution fits into the 1/4 BPS ansatz (6.1), we could of course try to embed it directly, by first identifying the four-dimensional base, and expressing it in terms of complex coordinates. However, for the case of non-vanishing scalar fields φ_i, this calculation turns out to be particularly cumbersome. We will instead make use of the 1/8 BPS embedding of the three-charge solution given in Section 5.3, and require that the solution has an additional $U(1)$ isometry. Note that this is the same strategy that was employed in the general 1/4 BPS discussion of Section 3.2.

To impose an additional $U(1)$, we take the Kähler potential of the 1/8 BPS solution to be of the form

$$K = K(|z_1|^2, z_i, \bar{z}_i), \qquad i = 2, 3, \tag{6.27}$$

where $z_1 = \tilde{r}\, e^{i\psi}$. This clearly corresponds to setting one of the scalar fields to zero, $\varphi_1 = 0$ (and also $H_1 = 1$). The six-dimensional base of the 1/8 BPS ansatz (5.39) then becomes

$$\begin{aligned} ds_6^2 = h_{mn} dx^m\, dx^n \;=\; & 2\Big[\partial_i \bar{\partial}_j K - \frac{\tilde{r}^2}{(\tilde{r}^2 K')'} \partial_i K'\, \bar{\partial}_j K'\Big] dz_i d\bar{z}_j + \frac{d(\tilde{r}^2 K')^2}{2\tilde{r}^2 (\tilde{r}^2 K')'} \\ & + 2\tilde{r}^2 (\tilde{r}^2 K')' \Big[d\psi + \frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'}\Big]^2, \end{aligned} \tag{6.28}$$

where now a prime denotes derivatives with respect to $|z_1|^2 = \tilde{r}^2$. Next, we would like to make the somewhat natural identification

$$y^2 = 2\tilde{r}^2 K', \tag{6.29}$$

which allows us to rewrite the base as

$$ds_6^2 = 2\Big[\partial_i \bar{\partial}_j K - \frac{\tilde{r}^2}{(\tilde{r}^2 K')'} \partial_i K'\, \bar{\partial}_j K'\Big] dz_i d\bar{z}_j + \frac{K'}{(\tilde{r}^2 K')'} dy^2 + 2\tilde{r}^2 (\tilde{r}^2 K')' \Big[d\psi + \frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'}\Big]^2. \tag{6.30}$$

The ten-dimensional metric then becomes

$$
\begin{aligned}
ds_{10}^2 &= e^{2\alpha} d\Omega_3^2 + 2e^{-2\alpha}\Big[\partial_i \bar{\partial}_j K - \frac{\tilde{r}^2}{(\tilde{r}^2 K')'}\partial_i K'\,\bar{\partial}_j K'\Big] dz_i d\bar{z}_j + e^{-2\alpha}\frac{K'}{(\tilde{r}^2 K')'}dy^2 \\
&\quad + 2e^{-2\alpha}\tilde{r}^2(\tilde{r}^2 K')'\Big[d\psi + \frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'}\Big]^2 - e^{2\alpha}(dt + \omega_{1/8})^2, \qquad (6.31)
\end{aligned}
$$

where we are adopting the notation $\omega_{1/8}$ for the three-charge (1/8 BPS) solution, so as to avoid confusion with the 1/4 BPS ω. Clearly, the condition (6.29), if general, might shed some light on the meaning of the y coordinate inside of the 1/8 BPS ansatz. Specifically, it is natural to ask whether $K' = 0$ plays a crucial role in determining boundary conditions on the $y = 0$ plane.

A first check of whether we have identified the y coordinate correctly is to show that the g_{yy} component of the metric takes the expected form, h^2. To do so, we will use the explicit relations for the Kähler potential of the three-charge solution. Recall that, in the notation of Section 5.3, we had $z_1 = \rho_1(r^2)\sin\theta\, e^{i\phi_1}$. Setting $\phi_1 = \psi$, and using (5.46), we then find that

$$
\begin{aligned}
K' &= \partial_{z_1}\partial_{\bar{z}_1} K = \frac{r^2}{2\rho_1^2} = \frac{r^2\sin^2\theta}{2\tilde{r}^2}, \\
(\tilde{r}^2 K')' &= \frac{\sin^2\theta}{2\tilde{r}^2}\frac{h^2}{\sqrt{\Delta}(H_2H_3)^{1/3}}, \qquad (6.32)
\end{aligned}
$$

which allows us to show that

$$
g_{yy} = e^{-2\alpha}\frac{K'}{(\tilde{r}^2 K')'} = h^2. \qquad (6.33)
$$

Using $e^{2\alpha} = ye^G$ and $e^{-2\alpha}\,\tilde{r}^2(2\tilde{r}^2 K')' = h^{-2}e^{-2G}$, we find that the ten-dimensional metric becomes

$$
ds_{10}^2 = ye^G d\Omega_3^2 + \frac{1}{ye^G}ds_4^2 + h^2\,dy^2 + h^{-2}e^{-2G}\Big[d\psi + \frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'}\Big]^2 - e^{2\alpha}(dt + \omega_{1/8})^2, \qquad (6.34)
$$

where we have defined

$$
ds_4^2 = 2\Big[\partial_i\bar{\partial}_j K - \frac{\tilde{r}^2}{(\tilde{r}^2 K')'}\partial_i K'\,\bar{\partial}_j K'\Big]dz_i d\bar{z}_j. \qquad (6.35)
$$

Notice that the g_{tt} component is still not in the appropriate 1/4 BPS form. To obtain the correct form of g_{tt}, it is enough to let $\psi = \tilde{\psi} - t$. After using the following decomposition,

$$
\omega_{1/8} = \omega_\psi\, d\psi + \tilde{\omega}, \qquad (6.36)
$$

we shift the angle and find

$$
\begin{aligned}
ds_{10}^2 &= ye^G d\Omega_3^2 + \frac{1}{ye^G}ds_4^2 + h^2\,dy^2 \\
&\quad + h^{-2}e^{-2G}\Big[d\tilde{\psi} - dt + \frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'}\Big]^2 - e^{2\alpha}(dt(1-\omega_\psi) + \omega_\psi d\tilde{\psi} + \tilde{\omega})^2. \qquad (6.37)
\end{aligned}
$$

Furthermore, using (5.40), we find that $\omega_\psi = -e^{-2G}$. It is then easy to show that the g_{tt} and $g_{\tilde{\psi}\tilde{\psi}}$ terms take the expected form:

$$
\begin{aligned}
g_{tt} &= h^{-2}e^{-2G} - ye^G(1-\omega_\psi)^2 = -h^{-2}\,, \\
g_{\tilde{\psi}\tilde{\psi}} &= h^{-2}e^{-2G} - e^{2\alpha}\omega_\psi^2 = ye^{-G}, \qquad (6.38)
\end{aligned}
$$

so that the metric becomes

$$\begin{aligned} ds_{10}^2 &= ye^G d\Omega_3^2 + \frac{1}{ye^G} ds_4^2 + h^2\, dy^2 + ye^{-G} d\tilde{\psi}^2 - h^{-2}(dt+\omega)^2 + h^{-2}\omega^2 \\ &+ h^{-2} e^{-2G} \Big[2d\tilde{\psi}\, \frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'} + \Big(\frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'} \Big)^2 \Big] - e^{2\alpha}(\tilde{\omega}^2 + 2\omega_\psi\, d\tilde{\psi}\,\tilde{\omega})\,, \end{aligned} \tag{6.39}$$

where

$$\omega = \tilde{\omega} + e^{-2G}\, \frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'}\,. \tag{6.40}$$

We now deal with a possible $U(1)$ gauging by completing the square in $d\tilde{\psi}$. In particular, by defining

$$\mathcal{A} = \tilde{\omega} + \frac{h^{-2}}{y\, e^G}\, \frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'}\,, \tag{6.41}$$

the metric can then be put into precisely the gauged form of the 1/4 BPS ansatz:

$$ds_{10}^2 = ye^G d\Omega_3^2 + \frac{1}{ye^G}\, ds_4^2 + h^2\, dy^2 - h^{-2}(dt+\omega)^2 + ye^{-G}(d\tilde{\psi} + \mathcal{A})^2. \tag{6.42}$$

where we have used the fact that

$$-\, ye^{-G}\mathcal{A}^2 + h^{-2}\Big[e^{-2G}\frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'} + \tilde{\omega}\Big]^2 + h^{-2} e^{-2G}\Big(\frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'}\Big)^2 - e^{2\alpha}\tilde{\omega}^2 = 0. \tag{6.43}$$

As indicated by the form of the initial metric (6.23), where the circle defined by $d\psi = d\phi_1 - dt$ is trivially fibered over the base, it is surprising to see that the gauged form of the 1/4 BPS ansatz has now turned up. However, we still have not used the explicit relations (5.46) for the Kähler potential to simplify $\mathcal{A}$. Once we do this, we indeed find $\mathcal{A} = 0$, which brings the solution to the ungauged form:

$$ds_{10}^2 = ye^G d\Omega_3^2 + \frac{1}{ye^G}\, ds_4^2 + h^2\, dy^2 - h^{-2}(dt+\omega)^2 + ye^{-G} d\tilde{\psi}^2, \tag{6.44}$$

in agreement with initial expectations. Furthermore, we can use the condition $\mathcal{A} = 0$ to express ω in terms of the 1/8 BPS one-form $\omega_{1/8}$:

$$\omega = -\frac{\Im(\partial_i K' dz_i)}{(\tilde{r}^2 K')'} = y\, h^2 \Big[e^G\, \omega_{1/8} + e^{-G}\, d\psi \Big]. \tag{6.45}$$

The final step is to find the explicit expression for the four-dimensional Kähler metric (6.35). Using the following expressions for derivatives of the Kähler potential

$$\begin{aligned} \partial_i K' &= \frac{\bar{z}_i \cosh\varphi_i - z_i \sinh\varphi_i}{2\Lambda H_2 H_3 \rho_i^2}, \\ \bar{\partial}_j K' &= \frac{z_j \cosh\varphi_j - \bar{z}_j \sinh\varphi_j}{2\Lambda H_2 H_3 \rho_j^2}, \\ (\tilde{r}^2 K')' &= \frac{r^2\, \Delta\, (H_2 H_3)^{2/3} + \tilde{r}^2}{2\Delta\, (H_2 H_3)^{2/3}}, \\ \partial_i \bar{\partial}_j K dz_i d\bar{z}_j &= \sum_i \frac{r^2 H_i}{2\rho_i^2} dz_i d\bar{z}_i \\ &\quad + \sum_{i,j} \frac{(\bar{z}_i \cosh\varphi_i - z_i \sinh\varphi_i)(z_j \cosh\varphi_j - \bar{z}_j \sinh\varphi_j)}{2\Lambda H_2 H_3\, \rho_i^2\, \rho_j^2} dz_i d\bar{z}_j, \end{aligned} \tag{6.46}$$

where

$$\Lambda = \frac{\Delta}{(H_2\, H_3)^{1/3}}, \tag{6.47}$$

we finally obtain

$$ds_4^2 = \frac{r^2 H_i}{\rho_i^2}|dz_i|^2 + \frac{(\bar{z}_i \cosh\varphi_i - z_i \sinh\varphi_i)(z_j \cosh\varphi_j - \bar{z}_j \sinh\varphi_j)}{\rho_i^2\rho_j^2\left[\Delta(H_2H_3)^{2/3} + y^2/r^4\right]} dz_i d\bar{z}_j. \tag{6.48}$$

Note that one can obtain the special case of the singular two-charge black hole (superstar) from the expressions above by setting $\varphi_i = 0$. This is very similar to the three-charge extremal black hole solution (superstar) which we described with the 1/8 BPS examples. In this case, the harmonic functions are given by

$$H_i = 1 + \frac{Q_i}{r^2}, \tag{6.49}$$

where Q_i label the black hole charges. The expression for Δ now simplifies

$$\Delta = (H_2H_3)^{1/3}\sin^2\theta + \frac{H_3^{1/3}}{H_2^{2/3}}\cos^2\theta\sin^2\alpha + \frac{H_2^{1/3}}{H_3^{2/3}}\cos^2\theta\cos^2\alpha, \tag{6.50}$$

and so do the derivatives of K:

$$\begin{aligned} \partial_i K' &= \frac{\bar{z}_i}{2\Lambda H_2 H_3 \rho_i^2}, \\ (\tilde{r}^2 K')' &= \frac{r^2\,\Delta\,(H_2H_3)^{2/3} + \tilde{r}^2}{2\Delta\,(H_2H_3)^{2/3}}, \\ \partial_i\bar{\partial}_j K dz_i d\bar{z}_j &= \sum_i \frac{r^2 H_i}{2\rho_i^2} dz_i d\bar{z}_i + \sum_{i,j} \frac{\bar{z}_i\, z_j}{2\Lambda H_2 H_3\, \rho_i^2\, \rho_j^2} dz_i d\bar{z}_j. \end{aligned} \tag{6.51}$$

The final metric then becomes

$$ds_4^2 = \frac{r^2 H_i}{\rho_i^2}|dz_i|^2 + \frac{\bar{z}_i\, z_j}{\rho_i^2\rho_j^2\left[\Delta(H_2H_3)^{2/3} + y^2/r^4\right]} dz_i d\bar{z}_j. \tag{6.52}$$

We note that, for the specific case of two equal charges, we were also able to embed this solution directly, without resorting to the 1/8 reduction, and found agreement.

6.2.1 Boundary conditions

We would like to emphasize again that for the LLM 1/2 BPS picture of [1], the boundary value (on the $y = 0$ plane) of the function Z gave the black and white coloring of all the solutions, and was a crucial element in the development of the droplet picture. In this respect, the 1/4 BPS system is more similar to the LLM case than to the 1/8 BPS case, as it also involves a $y = 0$ boundary plane and a binary choice of either the S^3 or the S^1 collapsing. As in the LLM case, this boundary condition is encoded in the behavior of Z [defined in the usual manner according to (6.2)] as the y coordinate vanishes. We now investigate this for the two charge bubble solutions.

We first combine the expressions (6.24) and (6.25) above to find Z for the smooth two-charge solutions:

$$Z = \frac{1}{2}\tanh G \;=\; \frac{1}{2}\frac{r^2\Delta(H_2H_3)^{2/3} - \sin^2\theta}{r^2\Delta(H_2H_3)^{2/3} + \sin^2\theta}$$
$$= \frac{1}{2} - \frac{\sin^2\theta}{r^2\Delta(H_2H_3)^{2/3} + \sin^2\theta}. \tag{6.53}$$

Since $y = r\sin\theta$ from (6.26), the $y \to 0$ boundary is reached when either $r \to 0$ or $\theta \to 0$. Looking at the non-trivial denominator of the expression above,

$$\begin{aligned} r^2\Delta(H_2H_3)^{2/3} + \sin^2\theta \;&=\; \sin^2\theta(1 + r^2 H_2H_3) \\ &\quad + r^2\, H_3\, \cos^2\theta \sin^2\alpha(\sin^2\phi_2 e^{-\varphi_2} + \cos^2\phi_2 e^{\varphi_2}) \\ &\quad + r^2\; H_2\, \cos^2\theta\cos^2\alpha(\sin^2\phi_3 e^{-\varphi_3} + \cos^2\phi_3 e^{\varphi_3}), \end{aligned} \tag{6.54}$$

one finds that

$$Z(\theta \to 0) = +\frac{1}{2}. \tag{6.55}$$

The case of $r \to 0$ with deformations turned on is more delicate, especially since explicit solutions for $H_{2,3}$ are not known. However, we note that as long as H_2 and H_3 approach a constant (and even in the case $H_{2,3} \sim 1/r$) as $r \to 0$, we find

$$Z(r \to 0) = -\frac{1}{2} \qquad (\text{for} \quad \varphi_{2,3} \neq 0). \tag{6.56}$$

Next, we would like to ask whether the boundary conditions found above translate into the presence of a three-dimensional surface embedded in four dimensions. From (6.26) we know that the y coordinate of the two-charge solution was identified to be

$$y = r\sin\theta = r\tilde{\mu}_1\,. \tag{6.57}$$

Clearly y vanishes when either $r = 0$ or when $\theta = 0$. Using $\sum_i \tilde{\mu}_i^2 = 1$ and the definition of our complex coordinates, we find

$$\begin{aligned} y^2 \;&=\; r^2(1 - \tilde{\mu}_2^2 - \tilde{\mu}_3^2) \\ &=\; r^2\Big[1 - \sum_{i=2,3}\frac{1}{4\rho_i^2}\Big(e^{-\varphi_i}(z_i + \bar{z}_i)^2 - e^{\varphi_i}(z_i - \bar{z}_i)^2\Big)\Big]\,. \end{aligned} \tag{6.58}$$

Thus, we see that the $\theta = 0$ condition guaranteeing $y = 0$ corresponds to the surface

$$\Big[1 - \sum_{i=2,3}\frac{1}{4\bar{\rho}_i^2}\Big(e^{-\bar{\varphi}_i}(z_i + \bar{z}_i)^2 - e^{\bar{\varphi}_i}(z_i - \bar{z}_i)^2\Big)\Big] = 0, \tag{6.59}$$

where $\bar{\rho}_i \equiv \rho_i(r = 0)$ and $\bar{\varphi}_i \equiv \varphi_i(r = 0)$. The surface denotes the boundary between regions where the S^3 shrinks ($r \to 0$) and regions where the S^1 shrinks ($\theta \to 0$). To see more explicitly that this surface is in fact an ellipsoid embedded in four dimensions, we can rewrite it using $z_i = x_i + iy_i$ in the following way:

$$\sum_{i=2,3}\Big[x_i^2\,\frac{e^{-\bar{\varphi}_i}}{\bar{\rho}_i^2} + y_i^2\,\frac{e^{\bar{\varphi}_i}}{\bar{\rho}_i^2}\Big] = 1. \tag{6.60}$$

We would like to make a few simple comments about the relation between the ellipsoid above and the five-dimensional one (5.66) obtained in the 1/8 BPS case. The 1/4 BPS ellipsoid (6.60) can be thought of as the $\varphi_1 = 0$, $\bar{\rho}_1 = 1$ limit of the 1/8 BPS ellipsoid (5.66), with the S^1 which rotates x_1 and x_2 shrinking to zero. Furthermore, we can consider the 1/2 BPS limit of (6.60) by setting another charge to zero (say $Q_2 = 0$, or $\bar{\varphi}_2 = 0$), and looking at the subspace where $x_2^2 + y_2^2 = 0$. By doing so, we find a simpler one-dimensional surface described by

$$x_3^2 \frac{e^{-\bar{\varphi}_3}}{\bar{\rho}_3^2} + y_3^2 \frac{e^{\bar{\varphi}_3}}{\bar{\rho}_3^2} = 1, \tag{6.61}$$

which is an ellipse in the two-dimensional (LLM) droplet plane. This corresponds to a horizon-free, smoothed-out solution for the 1/2 BPS singular black hole. One can alternatively arrive at this one-dimensional ellipse by considering another limit of the 1/8 BPS ellipsoid (5.66), in which $\varphi_1 = \varphi_2 = 0$, $\bar{\rho}_1 = \bar{\rho}_2 = 1$, and the S^3 rotating the x_1, x_2, x_3 and x_4 coordinates is shrinking to zero.

Let us now turn to the two-charge singular black hole (superstar) case, which is obtained by turning off the deformations, *i.e.* by setting $\varphi_i = 0$. Recalling that $H_i = 1 + Q_i/r^2$, we find that the function Z becomes

$$Z = \frac{1}{2} - \frac{1}{1 + Q_2 + Q_3 + r^2 + Q_2\,Q_3/r^2 + \cot^2\theta[(r^2+Q_3)\sin^2\alpha + (r^2+Q_2)\cos^2\alpha]}. \tag{6.62}$$

We can now see that Z approaches the same constant value independently of how y is going to zero,

$$Z(\theta \to 0) = +\frac{1}{2}, \tag{6.63}$$
$$Z(r \to 0) = +\frac{1}{2}, \tag{6.64}$$

provided neither charge vanishes. In particular, it is the $Q_2\,Q_3/r^2$ factor in the denominator of Z which causes $Z \to +\frac{1}{2}$ even when $r \to 0$. This is consistent with what we find if we look at what happens to the radii of S^3 and S^1 as $r \to 0$:

$$\begin{aligned} &r(S^3) \to \sqrt{Q_2 Q_3}\sin\theta, \\ &r(S^1) \to 0. \end{aligned} \tag{6.65}$$

On the other hand, when $\theta \to 0$, one recovers the usual result, with the S^3 staying finite and the S^1 shrinking to zero. It is precisely the fact that $S^1 \to 0$ in both limits which makes $Z = 1/2$ all the time.

Clearly, if we take one of the two charges in (6.62) to vanish, our result should be comparable to the one-charge superstar configuration. In that case it was found that, as $r \to 0$, the function Z approached a Q-dependent factor [1][3]

$$Z \to \frac{1}{2}\frac{Q-1}{Q+1}. \tag{6.66}$$

[3] Studies have shown that this distribution corresponds to "typical states" given by approximately triangular Young diagrams [36].

Indeed, if we take, for example, $H_1 = H_2 = 1$ and $H_3 = 1 + Q/r^2$, we find that Z becomes

$$Z = \frac{1}{2} - \frac{1}{1 + r^2 + Q + \cot^2\theta(r^2 + Q\sin^2\alpha)} \longrightarrow \frac{1}{2}\frac{Q(1+\cot^2\theta\sin^2\alpha) - 1}{Q(1+\cot^2\theta\sin^2\alpha)+1} \quad \text{as} \quad r \to 0\,,$$

a result that is similar to (6.66), except for some additional angular dependence. To conclude, we would like to note that the $r \to 0$ behavior (6.64) of Z for the two-charge black hole is due to the additional presence of flux, forcing the second term in (6.62) to approach zero.

Finally, we would like to identify, for the superstar, the regions in the four-dimensional subspace where $y = 0$. We can take the smooth two-charge solution result (6.58) and set $\varphi_i = 0$. We then see that y vanishes either when $r = 0$ or on the ellipsoidal surface

$$\sum_{i=2,3} \frac{|z_i|^2}{\bar{\rho}_i^2} = 1. \tag{6.67}$$

Similarly to the three-charge black hole case, if the charges are the same the surface degenerates into a sphere.

6.3 LLM

We now turn to the embedding of configurations which preserve 1/2 of the available supersymmetries, namely the LLM solutions. These are clearly a subset of the 1/4 BPS states. Recall the general form of the LLM metric, which is given in (5.81), and which we repeat here for convenience

$$ds_{10}^2 = -\hat{h}^{-2}(d\hat{t} + V)^2 + \hat{h}^2(|dz_1|^2 + d\hat{y}^2) + \hat{y}(e^{\hat{G}}d\Omega_3^2 + e^{-\hat{G}}d\widetilde{\Omega}_3^2). \tag{6.68}$$

Note that we have added a hat over LLM quantities to distinguish them from their 1/4 BPS counterparts. The most straightforward way to embed this into the 1/4 BPS ansatz (6.1) is to write the second three-sphere $\tilde{S}^3$ of (6.68) as the Hopf fibration of U(1) bundled over CP^1, and then to proceed with the gauged form of the 1/4 BPS ansatz. This is done by grouping z_1 with the complex coordinate on CP^1 to form a four-dimensional Kähler base

$$\begin{aligned} ds_{10}^2 &= -\hat{h}^{-2}(d\hat{t} + V_{z_1}dz_1 + V_{\bar{z}_1}d\bar{z}_1)^2 + \hat{h}^2 d\hat{y}^2 + \left[\hat{h}^2 dz_1 d\bar{z}_1 + \hat{y}e^{-\hat{G}}ds^2(CP^1)\right] \\ &\quad + \hat{y}e^{\hat{G}}d\Omega_3^2 + \hat{y}e^{-\hat{G}}(d\hat{\psi} + \hat{\mathcal{A}})^2, \end{aligned} \tag{6.69}$$

where $d\hat{\mathcal{A}} = 2\hat{J}$ and $\hat{J}$ is the Kähler form on CP^1.

A direct comparison of the above with the 1/4 BPS form of the metric (6.1) allows us to make the identifications:

$$\begin{aligned} &h = \hat{h} = (2\hat{y}\cosh\hat{G})^{-\frac{1}{2}}, \qquad t = \hat{t}, \qquad y = \hat{y}, \qquad e^G = e^{\hat{G}}, \qquad \hat{\psi} = \psi, \\ &\omega = V_{z_1}dz_1 + V_{\bar{z}_1}d\bar{z}_1, \qquad \mathcal{A} = \hat{\mathcal{A}}, \qquad \mathcal{F} = d\mathcal{A} = 2\hat{J}. \end{aligned} \tag{6.70}$$

The field strength $\mathcal{F}$ has flux through CP^1 and is quantized. We also infer that the four-dimensional subspace is given by:

$$\begin{aligned} h_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} &= ye^G\left[h^2 dz_1 d\bar{z}_1 + ye^{-G}\frac{dz_2 d\bar{z}_2}{(1+|z_2|^2)^2}\right] \\ &= (Z + \frac{1}{2})dz_1 d\bar{z}_1 + y^2\frac{dz_2 d\bar{z}_2}{(1+|z_2|^2)^2}, \end{aligned} \tag{6.71}$$

where we have written out the explicit metric on CP^1. Here $Z = Z(z_1, \overline{z}_1, y) = \frac{1}{2}\tanh G$ is just the LLM harmonic function introduced in [1] and satisfying (3.97)

$$4\partial_1\partial_{\bar{1}}Z + y\partial_y(\frac{1}{y}\partial_y Z) = 0, \tag{6.72}$$

where we have used that $z_1 = x_1 + ix_2$ and have rewritten the two-dimensional Laplacian in terms of complex derivatives.

It is now clear that the four-dimensional base with metric (6.71) decomposes into a direct product of two complex subspaces, the first being related to the two-dimensional LLM base and the second being simply CP^1 warped by y^2. To be explicit, we may write out the Kähler potential yielding (6.71) as a sum of two terms

$$K = \tfrac{1}{2}y^2\log(1 + |z_2|^2) + \tfrac{1}{2}\iint^{z_1,\bar{z}_1}(Z(z_1', \bar{z}_1', y) + \tfrac{1}{2})dz_1'd\bar{z}_1', \tag{6.73}$$

where the above integral is an indefinite integral (which allows for Kähler transformations). Lastly, we observe that the harmonic function Z obeys the 1/4 BPS constraint (6.2), $Z = -(y/2)\partial_y y^{-1}\partial_y K$. (Note that this condition removes the freedom to perform Kähler transformations on K.) To see this, it is useful to act on both sides with $\partial_1\bar{\partial}_1$, substitute (6.73), and notice that the ensuing equation is nothing but the harmonic equation (6.72).

To ensure that we really have a valid embedding, we would like to verify that the non-linear Monge-Ampère equation (6.3) is satisfied as well:

$$\det h_{i\bar{j}} = \frac{y^2(Z + \frac{1}{2})}{(1 + |z_2|^2)^2} = e^D(Z + \tfrac{1}{2})y^{n\eta}\exp\left(\frac{1}{y}(2 - n\eta)\partial_y K\right), \tag{6.74}$$

where we have used the explicit form of the four-dimensional metric (6.71). We now see that the y-dependence matches, provided that we identify the $U(1)$ charge of the Killing spinor with

$$n\eta = 2\,. \tag{6.75}$$

In this case, the final term in (6.74) becomes trivial, and we are left with the identification

$$e^D = \frac{1}{(1 + |z_2|^2)^2}, \tag{6.76}$$

which must be compatible with (6.3), which constrains D. Since $D = -2\log(1 + |z_2|^2)$, we see that $\partial\bar{\partial}D = 4iJ_2$ where J_2 is the Kähler form on CP^1. In this case, it is easy to verify that

$$(1 + *_4)\partial\bar{\partial}D = \frac{4i}{y^2}J_4, \tag{6.77}$$

where $J_4 = i\partial\bar{\partial}K$ is the Kähler form on the full base metric (6.71). This verifies that the constraint (6.3) is indeed satisfied.

6.3.1 Boundary conditions

Finally, we are interested in the lifting of the LLM boundary conditions into the gauged 1/4 BPS ansatz. Here, we notice from (6.70) that, since both $y = \hat{y}$ and $G = \hat{G}$, the 1/4 BPS function Z is identified with the corresponding LLM one

$$Z(z_1, z_2, \bar{z}_1, \bar{z}_2, y) = Z_{\text{LLM}}(z_1, \bar{z}_1, y). \tag{6.78}$$

As usual, the boundary conditions are imposed on the $y = 0$ subspace where either S^3 or S^1 (inside $\tilde{S}^3$) shrinks to zero size. The LLM solutions are regular if either $Z = -1/2$, which corresponds to shrinking S^3, or if $Z = 1/2$, which corresponds to shrinking $\tilde{S}^3$.

When lifted to the gauged 1/4 BPS ansatz, the boundary surfaces implied by (6.78) are z_2 independent. This indicates that the 1/2 BPS LLM droplets lift into four-dimensional droplets which are simply the direct product of of the two-dimensional droplet in the $(z_1, \bar{z}_1)$ plane with the CP^1 formed by $(z_2, \bar{z}_2)$. The boundaries of these droplets are then three-real dimensional surfaces formed from the direct product of the boundary lines of the LLM droplets with CP^1.

Unlike the above two examples of the $\text{AdS}_5 \times S^5$ sphere and the ellipsoidal deformations of the two-charge BPS bubble solution, here the shapes of the droplets are different. The reason for this is because we have used a different choice of embedding for the LLM system, corresponding to the gauged ansatz, instead of the ungauged ansatz which was used above. In fact, because the LLM configurations preserve the full $\tilde{S}^3$ isometry, and since the gauged ansatz has an explicit S^1 fiber, the three dimensional boundary surfaces necessarily have a CP^1 invariance (so that S^1 fibered over CP^1 forms the round $\tilde{S}^3$). Therefore these surfaces must be of the form of a direct product of a real curve in the LLM plane with CP^1. (The CP^1 is determined from the solution for the function D in (6.76).)

Note that, unlike in the case of the $(z_1, \bar{z}_1)$ LLM plane, which has a regular flat metric, here the four-dimensional $y = 0$ subspace given in (6.71) has non-trivial geometry; it is in fact singular since the CP^1 metric vanishes as $y \to 0$. (In general, the behavior of the base may be different for the two separate cases $Z \to \frac{1}{2}$ and $Z \to -\frac{1}{2}$.) This singularity as $y \to 0$ is reminiscent of the 1/8 BPS case, where the six-dimensional base also develops a curvature singularity as the 1/8 BPS y variable approaches zero. Although the full ten-dimensional metric is non-singular, this nevertheless complicates the issue of making any direct comparison of the four-dimensional boundary subspace with any corresponding phase space in the dual gauge theory.

To make a closer comparison with the 1/8 BPS lifting of Section 5.4, it may be advantageous to turn instead to an ungauged embedding of LLM into the 1/4 BPS ansatz. This is perhaps most straightforwardly accomplished by reducing the 1/8 BPS lift of Section 5.4 on a circle according to either (3.22) or some variation thereof. However, since the result of doing so would only yield a modified interpretation of the 1/8 BPS picture considered in Section 5.4, we will not pursue this here.

6.4 General analysis with a decomposable four-dimensional base

The above LLM embedding in the 1/4 BPS ansatz was facilitated by taking the four-dimensional base to be a warped product of the LLM plane with CP^1. In this subsection, we address the question of whether new classes of 1/4 BPS solutions may be obtained where the four-dimensional base, parameterized by the complex coordinates z_1, z_2, is a direct product of two Riemann surfaces. In particular, if the base is factorizable, then the Kähler potential would be given by the sum

$$K = K_1(z_1, \bar{z}_1, y) + K_2(z_2, \bar{z}_2, y). \tag{6.79}$$

Following the general outline of the LLM embedding, we shall also assume that

$$Z = Z(z_1, \bar{z}_1, y), \qquad D = D(z_2, \bar{z}_2, y). \tag{6.80}$$

Since Z is related to K by (6.2), the requirement that Z is independent of $z_2, \bar{z}_2$ translates into

$$\partial_2 \partial_y (\frac{1}{y} \partial_y K) = 0, \qquad \partial_{\bar{2}} \partial_y (\frac{1}{y} \partial_y K) = 0. \tag{6.81}$$

Therefore, we find that the y-dependence of K_2 is fixed:

$$K_2 = y^2 k_2(z_2, \bar{z}_2) + \tilde{k}_2(z_2, \bar{z}_2\,). \tag{6.82}$$

Also, from the second equation in (6.3), we find that

$$D = \frac{4}{y^2}(1 - n\eta) K_2 + d(z_2, y) + \bar{d}(\bar{z}_2, y)\,. \tag{6.83}$$

The immediate advantage of the assumptions we have made is that the non-linear Monge-Ampère equation factorizes. Under these conditions, the first equation in (6.3) is replaced by the following two equations:

$$\begin{aligned} \partial_1 \partial_{\bar{1}} K_1 &= \frac{k(y)}{2}(Z + \tfrac{1}{2}) \exp\left(\frac{1}{y}(2 - n\eta)\partial_y K_1\right), \\ \partial_2 \partial_{\bar{2}} K_2 &= \frac{1}{2k(y)} y^{n\eta} \exp\left(\frac{1}{y}(2 - n\eta)\partial_y K_2\right) e^D, \end{aligned} \tag{6.84}$$

where $k(y)$ is an arbitrary function. Substituting (6.82) and (6.83) into (6.84), we find

$$y^2 \partial_2 \partial_{\bar{2}} k_2 + \partial_2 \partial_{\bar{2}} \tilde{k}_2 = \frac{1}{2k(y)} y^{n\eta} \exp\bigl(2(2 - n\eta)k_2\bigr) \exp\left(\frac{4}{y^2}(1 - n\eta)(y^2 k_2 + \tilde{k}_2) + d + \bar{d}\right). \tag{6.85}$$

Since k_2, and $\tilde{k}_2$ are y-independent, matching the y-dependence on both sides of the previous equation requires that

$$4(1 - n\eta)\frac{1}{y^2}\tilde{k}_2 + d + \bar{d} = 0, \tag{6.86}$$

and

$$y^2 = \frac{1}{k(y)} y^{n\eta}. \tag{6.87}$$

Here we used the fact that the left-hand side of (6.85) is a polynomial of degree two in y to infer that the infinite series in y on the right-hand side must truncate. After the y-dependence has been factored out, we are left with

$$\partial_2\partial_{\bar{2}}k_2 = \tfrac{1}{2}\exp\bigl(2(4-3n\eta)k_2\bigr). \tag{6.88}$$

Alternatively, we can rewrite this as a Liouville equation for D:

$$\frac{1}{1-n\eta}\partial_2\partial_{\bar{2}}D = 2\exp\left(\frac{4-3n\eta}{2(1-n\eta)}D\right). \tag{6.89}$$

The z_1 dependence of the four-dimensional Kähler base is dictated by the remaining equation:

$$\partial_1\partial_{\bar{1}}K_1 = \tfrac{1}{2}y^{n\eta-2}(Z+\tfrac{1}{2})\exp\left(\frac{1}{y}(2-n\eta)\partial_y K_1\right). \tag{6.90}$$

A further restriction, namely

$$n\eta = 2, \tag{6.91}$$

which is identical to the LLM embedding case (6.75), then allows us to find explicit solutions.

Using (6.91), the Liouville equation for D becomes

$$\partial_2\partial_{\bar{2}}D + 2e^D = 0, \tag{6.92}$$

whose solutions are expressed in terms of an arbitrary holomorphic function $\mathcal{D}(z_2)$:

$$e^D = \frac{|\partial_2\mathcal{D}(z_2)|^2}{(1+|\mathcal{D}(z_2)|^2)^2}. \tag{6.93}$$

The choice of Killing spinor $U(1)$ charge according to (6.91) leads to a drastic simplification of (6.90)

$$\partial_1\partial_{\bar{1}}K_1 = \tfrac{1}{2}(Z+\tfrac{1}{2}), \tag{6.94}$$

which can be easily integrated. Of course, Z is constrained by (6.2). The compatibility of these two equations yields

$$4\partial_1\partial_{\bar{1}}Z + y\partial_y\frac{1}{y}\partial_y Z = 0\,, \tag{6.95}$$

which is the harmonic equation that we encountered before in the context of LLM solutions.

Given the above, it is now easy to see that the base has a metric of the form

$$\begin{aligned} ds_4^2 &= (Z+\tfrac{1}{2})dz_1 d\bar{z}_1 + y^2 e^D dz_2 d\bar{z}_2 \\ &= (Z+\tfrac{1}{2})dz_1 d\bar{z}_1 + y^2\frac{|\partial_2\mathcal{D}(z_2)|^2}{(1+|\mathcal{D}(z_2)|^2)^2}dz_2 d\bar{z}_2. \end{aligned} \tag{6.96}$$

A change of variables $z_2 \to w \equiv \mathcal{D}(z_2)$ then results in

$$ds_4^2 = (Z+\tfrac{1}{2})dz_1 d\bar{z}_2 + y^2\frac{dw\,d\bar{w}}{(1+|w|^2)^2}, \tag{6.97}$$

which is identical in form to that of (6.71). This demonstrates that the LLM lift examined in Section 6.3 is essentially the unique configuration corresponding to a decomposable base. Additional possibilities may exist, however, where the Killing spinors carry a different $U(1)$ charge, $n\eta \neq 2$.

6.5 Flux quantization

Until now, we have focused on developing a droplet picture by examining the loci of shrinking surfaces (S^3 or S^1) while ignoring flux issues. However, we conclude this section by considering the IIB five-form flux integral near $y=0$, with the goal of obtaining a flux quantization condition. To obtain explicit results, we limit the following analysis to the LLM embedding, where the four-dimensional base is decomposable. In this case, the ten-dimensional metric and flux take the form

$$\begin{aligned} ds^2_{10} &= -h^{-2}(dt+\omega)^2 + h^2 dy^2 + ye^G d\Omega_3^2 + ye^{-G}(d\psi+\mathcal{A})^2 \\ &\quad + \frac{1}{ye^G}\left[(\tfrac{1}{2}+Z)|dz_1|^2 + y^2 e^D |dz_2|^2\right], \end{aligned} \tag{6.98}$$

$$\begin{aligned} F_{(5)} &= (1+*_{10})\Big(d[y^2 e^{2G}(dt+\omega)] + y^2(d\omega - d\mathcal{A}) \\ &\quad -i\left[(\tfrac{1}{2}+Z)dz_1\wedge d\bar{z}_1 + y^2 e^D dz_2 \wedge d\bar{z}_2\right]\Big)\wedge\Omega_3, \end{aligned} \tag{6.99}$$

where we used (3.45) to obtain the components of the five-form. Note that here we have explicitly set $\eta=-1$[4].

We first want to consider the flux that is orthogonal to the $(dt+\omega)\wedge dy\wedge\Omega_3$ directions. This flux component is easy to identify using (6.99) and (3.45). The integral of its Hodge dual is given by:

$$\begin{aligned} \int_{Z=-\frac{1}{2}} *_{10}F_{(5)} &= \int_{y=0} *_{10}[\partial_y(y^2e^{2G})dy\wedge(dt+\omega)\wedge\Omega_3] \\ &= \int_{y=0}\left[2(\tfrac{1}{2}-Z)+\frac{y\partial_y Z}{\frac{1}{2}+Z}\right]_{y=0} e^D\frac{i}{2}dz_1\wedge d\bar{z}_1\wedge\frac{i}{2}dz_2\wedge d\bar{z}_2\wedge(d\psi+\mathcal{A}). \end{aligned} \tag{6.100}$$

It can be seen from (3.45) that $\mathcal{A}$ has components along the coordinates on the four-dimensional Kähler base only, so $dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\wedge\mathcal{A}=0$. We can perform the flux integral by first integrating out $d\psi$, and then reducing it to an integral over the four-dimensional Kähler base (at $y=0$). Notice that (always assuming we are at $y=0$) when $Z=-\frac{1}{2}$, which corresponds to the S^3 collapsing to zero size, we have

$$\left[2(\tfrac{1}{2}-Z)+\frac{y\partial_y Z}{\frac{1}{2}+Z}\right]_{y=0} = 4. \tag{6.101}$$

Thus, the flux integral reduces to

$$\begin{aligned} \int_{Z=-\frac{1}{2}} *_{10}F_{(5)} &= \int_{y=0}(2\pi)4e^D\big|_{y=0}\frac{i}{2}dz_1\wedge d\bar{z}_1\frac{i}{2}\wedge dz_2\wedge d\bar{z}_2 \\ &= 4\ \mathrm{Vol}\big(\Sigma_3\big|_{y=0}\big)\int_{Z=-\frac{1}{2}}\frac{i}{2}dz_1\wedge d\bar{z}_1 \sim N_{Z=-\frac{1}{2}}, \end{aligned} \tag{6.102}$$

[4] In general, taking the period of ψ to be 2π, choosing $\eta=1$ or -1 corresponds to choosing chirality $(1,2)$ or $(2,1)$ under $SU(2)_L\times SU(2)_R$ for the Killing spinors on S^3 in (2.11).

where $\mathrm{Vol}(\Sigma_3\big|_{y=0}) = \int_{y=0} e^D\big|_{y=0} \frac{i}{2} dz_2 \wedge d\bar{z}_2 \wedge (d\psi + \mathcal{A})$ is the volume of a three dimensional surface at $y = 0$. This corresponds to the case of D3-branes originally wrapping the S^3 in AdS$_5$ being replaced by five-form fluxes through dual five-cycles (*i.e.* $\Sigma_3\big|_{y=0}$ fibered over the $Z = -\frac{1}{2}$ region of the z_1 plane).

Next, we consider the self-dual five-form with component along $dz_1 \wedge d\bar{z}_1 \wedge \Omega_3$, and evaluate its flux integral:

$$\begin{aligned} \int_{Z=\frac{1}{2}} *_{10} F_{(5)} &= \int_{y=0} [-i(\tfrac{1}{2} + Z)_{y=0} dz_1 \wedge d\bar{z}_1 \wedge \Omega_3 + \left(y^2 e^{2G} \frac{1}{y} \partial_y J \right)_{y=0} \wedge \Omega_3] \\ &= \int_{y=0} \left[2(\tfrac{1}{2} + Z) - \frac{y \partial_y Z}{\frac{1}{2} - Z} \right]_{y=0} \frac{-i}{2} dz_1 \wedge d\bar{z}_1 \wedge \Omega_3. \end{aligned} \tag{6.103}$$

The second term in the first line comes from the $y^2 e^{2G} d\omega \wedge \Omega_3$ term in the flux near $y = 0$ in expression (3.45). Notice that, similarly to what happened in (6.100), when $Z = +\frac{1}{2}$ (corresponding to the three-cycle $\Sigma_3\big|_{y=0}$ collapsing), we have

$$\left[2(\tfrac{1}{2} + Z) - \frac{y \partial_y Z}{\frac{1}{2} - Z} \right]_{y=0} = 4. \tag{6.104}$$

Thus, the flux integral reduces to

$$- \int_{Z=\frac{1}{2}} *_{10} F_{(5)} = 4 \ \mathrm{Vol}(S^3) \int_{Z=\frac{1}{2}} \frac{i}{2} dz_1 \wedge d\bar{z}_1 \sim N_{Z=\frac{1}{2}}. \tag{6.105}$$

Once again, this corresponds to the case of D3-branes, originally wrapping the $\Sigma_3\big|_{y=0}$ in S^5, being replaced by five-form fluxes through dual five-cycles (*i.e.* S^3 fibered over the $Z = \frac{1}{2}$ region of the z_1 plane).

7 Regularity conditions for 1/8 BPS configurations

In the previous few sections, we have been concerned with developing a droplet description of generic 1/8 and 1/4 BPS smooth solutions of type IIB supergravity, corresponding to bubbling AdS configurations. These configurations have either an S^3 isometry, or an $S^3 \times S^1$ isometry. The only non-trivial ten-dimensional fields are the self-dual five-form field strength and the metric. We have also studied in detail several classes of explicit solutions, and investigated their corresponding boundary conditions at $y = 0$. It should be noted, however, that by starting with known regular solutions (such as the three-charge smooth solutions of [32] or the original 1/2 BPS LLM solutions [1]), we are necessarily guaranteed to obtain regular examples of 1/4 and 1/8 BPS embeddings.

It would be desirable, of course, to explore both regularity conditions as well as boundary conditions on the BPS geometries directly, without prior knowledge of explicit solutions. What we mean here by boundary conditions are the conditions specifying the droplets, *i.e.* the one or three-dimensional droplet boundaries on the $y = 0$ subspaces for the cases of 1/2 and 1/4 BPS solutions, or the five-dimensional droplet boundaries for the 1/8 BPS case. For 1/2 BPS LLM solutions, the uniqueness of the Green's function solution to (3.97) ensures that each droplet picture corresponds to a unique geometry[5]. Furthermore, in the absence of cusps or other pathologies in the droplets, all such 1/2 BPS solutions are regular. Hence no additional regularity conditions need to be imposed, at least for generic smooth droplets.

Because of the nonlinear equations underlying the supersymmetry analysis, however, the regularity situation for 1/4 and 1/8 BPS configurations is less clear. In principle, just as in the LLM case, it appears that droplets can have any arbitrary shape or configuration; we simply choose any desired three or five-dimensional boundary surface inside $\mathbb{R}^4$ or $\mathbb{R}^6$, respectively, for the 1/4 and 1/8 BPS cases. However, it is not obvious that an arbitrary choice would always lead to a regular smooth geometry in the full ten-dimensional sense. After all, it is the nature of non-linear equations that they do not always admit well behaved solutions throughout their entire parameter range. Furthermore, even if a regular geometry exists, its uniqueness could be questioned.

For the droplet picture that we have presented to be useful, each droplet configuration ought to give rise to a unique geometry. Based on the LLM experience, it certainly seems to be the case that droplet collections would be unique, so long as we demand the geometry to be asymptotically $\mathrm{AdS}_5 \times S^5$. We are, however, unable to prove such uniqueness. Nevertheless, we will motivate this statement by examining the approach to $\mathrm{AdS}_5 \times S^5$ in the asymptotic regime. Before doing so, however, we first examine conditions on the regularity of the geometry near the $y = 0$ boundary.

For concreteness, we focus our attention on the 1/8 BPS configurations. (This also encompasses 1/4 and 1/2 BPS configurations as special cases.[6]) These solutions can be viewed as $\mathbb{R} \times S^3$ fibrations over a six-dimensional Kähler base which ends, as $y \to 0$, on (generally disconnected) five-dimensional surfaces, where the S^3 fiber shrinks to zero size. We are interested in understanding the necessary conditions which ensure the regularity of such solutions as $y \to 0$. These conditions then allow us to understand the behavior of the Kähler potential near the five-dimensional droplet boundaries, and will provide additional insight into the moduli space of droplets in reduced supersymmetry configurations.

7.1 Regular boundary conditions near $y = 0$

Focusing on 1/8 BPS configurations, we recall from (4.1) that the full ten-dimensional metric is of the form

$$ds_{10}^2 = -y^2(dt + \omega)^2 + \frac{2}{y^2}\partial_i \partial_{\bar{j}} K dz^i d\bar{z}^{\bar{j}} + y^2 d\Omega_3^2, \tag{7.1}$$

where the radial direction y corresponds to the size of the S^3

$$y^2 = e^{2\alpha}(z^i, \bar{z}^{\bar{j}}). \tag{7.2}$$

If the scalar field α is constant (as in the case of the $\mathrm{AdS}_3 \times S^3 \times T^4$ solution), then the only regularity condition which must be enforced is on the six-dimensional Kähler metric $h_{i\bar{j}}$. Otherwise, $y = 0$

[5] Note also that boundary conditions at $y \to \infty$ are encoded in the Green's function. These are necessary to ensure a proper asymptotic AdS_5 geometry.

[6] In Appendix D we perform a regularity analysis directly on the 1/4 BPS solutions with an ungauged $S^3 \times S^1$ isometry.

corresponds to a potentially singular locus, with the three-sphere $d\Omega_3^2$ shrinking to zero size. To avoid this singularity, the ten-dimensional metric must take the form

$$ds_{10}^2 = -y^2(dt+\omega)^2 + \frac{1}{y^2}\Big(y^2dy^2 + y^2d\Sigma_4^2 + \mathcal{N}_\psi^2(d\psi + A)^2\Big) + y^2 d\Omega_3^2, \qquad y \ll 1. \tag{7.3}$$

As long as the four-dimensional subspace $d\Sigma_4^2$ is y-independent, then the $y^2d\Omega_3^2 + dy^2$ line element yields a regular (locally flat) four-dimensional component of the ten-dimensional geometry. The four-dimensional component $d\Sigma_4^2$ is similarly regular (at least in terms of taking the $y \to 0$ limit). Here $\mathcal{N}_\psi$ is a function of the coordinates on $d\Sigma_4^2$, and is finite at $y = 0$. However, the remaining two-dimensional component involving t and ψ is still potentially singular, as $g_{tt} \to 0$ and $g_{\psi\psi} \to \infty$.

To completely elucidate the $y \to 0$ behavior of the 1/8 BPS solution, we first turn to the requirement that the six-dimensional base is Kähler. In this case, we may take the three complex coordinates to be given by

$$z_j = r_j e^{i\phi_j}, \qquad j = 1, 2, 3. \tag{7.4}$$

Furthermore, the metric is determined by the Kähler potential $K(z^i, \bar{z}^{\bar{j}})$

$$ds_6^2 = h_{mn}dx^m dx^n = 2h_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} = 2\partial_i\partial_{\bar{j}}K dz^i d\bar{z}^{\bar{j}}, \tag{7.5}$$

where m and n are real indices and i and $\bar{j}$ are complex indices. Assuming toric geometry, we now introduce a new real function F, defined in the following way:

$$y^2 \equiv F(r_1^2, r_2^2, r_3^2). \tag{7.6}$$

We are looking for a Kähler potential which will give us, in the region near $y = 0$, a metric of the form

$$ds_6^2 = y^2dy^2 + y^2d\Sigma_4^2 + \mathcal{N}_\psi^2(d\psi + A)^2. \tag{7.7}$$

Henceforth our analysis will refer strictly to the $y \ll 1$ region. A Kähler potential satisfying our requirement is

$$K(z^i, z^{\bar{j}}) = \frac{1}{4}y^4 + \mathcal{O}(y^6), \tag{7.8}$$

up to an irrelevant constant. Given the definition of y in (7.6), it follows that for $y \ll 1$, the six-dimensional base is toric, with a $U(1)^3$ isometry. This may be too strong a requirement, but it allows us to consider a rather large class of solutions (F is only required to be a smooth non-singular function of r_1^2, r_2^2, r_3^2), and at the same time to be very specific. Then, using the chain rule

$$ydy = F_1r_1dr_1 + F_2r_2dr_2 + F_3r_3dr_3, \qquad F_i = \frac{\partial F}{\partial r_i^2}, \tag{7.9}$$

we find

$$ds_6^2 = \sum_{a=1}^{3}\left[F_a^2r_a^2 + y^2(F_{aa}r_a^2 + F_a)\right]\left(dr_a^2 + r_a^2d\phi_a^2\right) + 2\sum_{a<b}^{3}\left[F_aF_b + y^2F_{ab}\right]r_a^2r_b^2\left(\frac{dr_adr_b}{r_ar_b} + d\phi_a d\phi_b\right), \tag{7.10}$$

where

$$F_{ij} = \frac{\partial^2 F}{\partial r_i^2 \partial r_j^2}. \tag{7.11}$$

From $y^2 = F(r_1^2, r_2^2, r_3^2)$, by eliminating, say, r_1 in favor of y,

$$r_1^2 = f(y^2, r_2^2, r_3^2), \tag{7.12}$$

we can express the metric in terms of the $\{y, \phi_1, r_2, \phi_2, r_3, \phi_3\}$ coordinates.

The leading order terms of the six-dimensional metric are

$$\begin{aligned} ds_6^2 \;=\;& dy^2\, y^2 + \frac{1}{f_y^2}(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3)^2 \\ &+ \Bigg[dr_2^2\, \frac{y^2}{f_y}\left(-f_2 - f_{22} r_2^2 + \frac{f_2^2 r_2^2}{f}\right) + dr_3^2\, \frac{y^2}{f_y}\left(-f_3 - f_{33} r_3^2 + \frac{f_3^2 r_3^2}{f}\right) \\ &+ 2 dr_2 dr_3\, \frac{y^2}{f_y} r_2 r_3 \left(-f_{23} + \frac{f_2 f_3}{f}\right) \\ &+ d\phi_2^2\, \frac{y^2 r_2^2}{f_y^2}\left(2 r_2^2 f_{2y} f_2 - f_2 f_y - r_2^2 f_y f_{22} - \frac{r_2^2 f_2^2 f_{yy}}{f_y}\right) \\ &+ d\phi_3^2\, \frac{y^2 r_3^2}{f_y^2}\left(2 r_3^2 f_{3y} f_3 - f_3 f_y - r_3^2 f_y f_{33} - \frac{r_3^2 f_3^2 f_{yy}}{f_y}\right) \\ &+ 2 d\phi_2 d\phi_3\, \frac{y^2 r_2^2 r_3^2}{f_y}(-f_{23} + f_{2y} f_3 + f_{3y} f_2 - f_2 f_3 f_{yy}) \Bigg], \end{aligned} \tag{7.13}$$

where

$$f_y = \frac{\partial f}{\partial y^2}, \qquad f_2 = \frac{\partial f}{\partial r_2^2}, \qquad f_{2y} = \frac{\partial^2 f}{\partial r_2^2 \partial y^2}, \qquad \text{etc} \ldots. \tag{7.14}$$

The subleading terms in this metric are given in Appendix E. A direct comparison of (7.13) and (7.7) shows that in the $y \ll 1$ region they are identical, provided that we identify $\psi \equiv \phi_1$. Therefore, as anticipated, the Kähler potential $K = y^4/4$ yields a six-dimensional metric which is of the desired form, as in (7.7).

We now have all the necessary ingredients to study the regularity of the ten-dimensional metric. As discussed above, any potentially singular behavior as $y \to 0$ would come from the following two-dimensional part of the ten-dimensional metric

$$ds_2^2 = -y^2(dt + \omega)^2 + \frac{1}{y^2 f_y^2}(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3)^2. \tag{7.15}$$

We now recall that the one-form ω is determined by the Kähler potential of the six-dimensional base

$$2\eta d\omega = \mathcal{R}, \tag{7.16}$$

where $\mathcal{R}$ is the Ricci form of the base. Noting that

$$\mathcal{R} = i R_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} = \frac{i}{2} \partial_i \partial_{\bar{j}} \log(\det h_{mn}) dz^i \wedge d\bar{z}^{\bar{j}} \tag{7.17}$$

is a $(1,1)$ form and that $d = \partial + \bar{\partial}$, we have

$$\omega = \omega_i dz^i + \bar{\omega}_{\bar{j}} d\bar{z}^{\bar{j}}, \qquad \omega_i = -\frac{i\eta}{8}\partial_i \log(\det h_{mn}), \qquad \bar{\omega}_{\bar{j}} = (\omega_j)^*. \tag{7.18}$$

Since $\sqrt{\det h_{mn}}$ is a scalar density, this means that ω in (7.18) is locally defined. From (7.10) we find that $\det h_{mn} = \mathcal{O}(y^8)$ in the coordinate system of $\{r_i, \phi_i\}$. Thus, the leading order term in ω is

$$\begin{aligned} \omega &= \frac{\eta}{8}\sum_{a=1}^{3} \partial_{r_a} \log(\det h_{mn})\, r_a d\phi_a \\ &= \frac{\eta}{8}\frac{8}{y^2}(F_1 r_1^2 d\phi_1 + F_2 r_2^2 d\phi_2 + F_3 r_3^2 d\phi_3) + \mathcal{O}(y^0), \end{aligned} \tag{7.19}$$

where, on the second line, we have used the chain rule to evaluate $(dz_1\partial_1 - d\bar{z}_1\partial_{\bar{1}})\log(y^8)$ *etc.* To leading order in y, we find that

$$\omega = \frac{1}{y^2 f_y}(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3) + \mathcal{O}(y^0). \tag{7.20}$$

Plugging this expression back into (7.15), the potentially singular terms cancel, and we arrive at

$$ds_2^2 = -\frac{2}{f_y} dt (f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3) + \mathcal{O}(y^2), \tag{7.21}$$

which is regular.

To summarize, we have investigated the region of the 1/8 BPS solutions near $y=0$. Assuming a toric base, we have seen that the $y=0$ locus is a five-dimensional surface Σ_5 specified by

$$F(r_1^2, r_2^2, r_3^2) = 0. \tag{7.22}$$

Furthermore, the y coordinate is orthogonal to Σ_5. The complete ten-dimensional solution is generated by choosing an arbitrary smooth (generally disconnected) five-dimensional surface embedded in the six-dimensional Kähler base. Then the ten-dimensional solution will be non-singular provided that, in the vicinity of the Σ_5 surface,

$$\begin{aligned} ds_{10}^2 &= -g_{tt}\big|_{y=0}\, dt^2 + \left(\frac{f}{f_y} + 9a\frac{f^2}{f_y^2} - \frac{3f^2 f_{yy}}{f_y^3}\right)\left(d\phi_1 - \frac{f_2 r_2^2}{f}d\phi_2 - \frac{f_3 r_3^2}{f}d\phi_3 - w_t dt\right)^2\Bigg|_{y=0} \\ &\quad + \frac{2f}{f_y}\left(d\phi_1 - \frac{f_{2y} r_2^2}{f_y}d\phi_2 - \frac{f_{3y} r_3^2}{f_y}d\phi_3\right)^2\Bigg|_{y=0} + d\widetilde{\Sigma}_4{}^2(r_2,\phi_2,r_3,\phi_3) + d\mathbb{R}_4{}^2, \end{aligned} \tag{7.23}$$

where $g_{tt}\big|_{y=0}$ is finite and $d\widetilde{\Sigma}_4{}^2(r_2,\phi_2,r_3,\phi_3)$ is the metric of a four dimensional surface, and $d\mathbb{R}_4{}^2 = dy^2 + y^2 d\Omega_3^2$. More details of the intermediate steps are presented in Appendix E.

The cancellation of the leading order $\mathcal{O}(y^{-2})$ terms in $d\phi_1^2$, which was necessary to ensure the regularity of the solution at $y=0$, forces us to keep the subleading $\mathcal{O}(y^2)$ terms from (7.13). As we show in Appendix E, we also take into account the leading order terms generated from the *correction* to the Kähler potential, $\delta K = ay^6$. For the ten-dimensional metric, all the terms collected in (7.23) are of the same order, namely $\mathcal{O}(y^0)$. Note that, for regularity, one must also require that $(f/f_y)\big|_{y=0}$ is finite as a function of r_2^2, r_3^2. We remind the reader that the function f is defined through $r_1^2 = f(y^2, r_2^2, r_3^2)$, so the five-dimensional surface at $y = 0$ is given by the constraint $r_1^2 = f(0, r_2^2, r_3^2)$.

The full Kähler potential is obtained by evolving the approximate $K = y^4/4 + \mathcal{O}(y^6)$ according to (3.21)

$$\Box_6 R = -R_{mn}R^{mn} + \frac{1}{2}R^2, \tag{7.24}$$

where R is the Ricci scalar of the six-dimensional Kähler base, and $m, n = 1, \ldots, 6$ are real indices.

For completeness we shall also verify two consistency conditions. Since we have identified the three-sphere warp factor $e^{2\alpha}$ with y^2, and since $y = (-8/R)^{1/4}$, we must check that indeed $R = -8/y^4$ to leading order for $y \ll 1$. From the expression of the Ricci tensor on a Kähler space

$$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \sqrt{\det h_{mn}}, \tag{7.25}$$

we find that, to leading order in y,

$$R_{i\bar{j}} = -2\frac{F_i F_j r_i r_j}{y^4} + \mathcal{O}(y^{-2}), \qquad F_i = \partial_{r_i^2} F, \qquad \text{etc}\ldots. \tag{7.26}$$

Hence

$$R_{r_a r_b} = -4\frac{F_a F_b r_a r_b}{y^4} + \mathcal{O}(y^{-2}), \qquad R_{\phi_a \phi_b} = -4\frac{F_a F_b r_a^2 r_b^2}{y^4} + \mathcal{O}(y^{-2}). \tag{7.27}$$

By inverting the Kähler metric (7.10) we can evaluate the Ricci scalar

$$R = R_{mn}h^{mn} = -\frac{8}{y^4}, \tag{7.28}$$

as anticipated. The corrections to the Kähler potential are expected to cancel any potential contributions to order y^{-2} from the Kähler metric (7.10). The second check we perform on the Kähler potential is that, to leading order in y, the equation (7.24) is satisfied. Indeed this is so, since

$$R_{mn}R^{mn} = \frac{96}{y^8} + \mathcal{O}(y^{-6}), \qquad R^2 = \frac{64}{y^8} + \mathcal{O}(y^{-6}), \qquad \Box_6 R = -\frac{64}{y^8} + \mathcal{O}(y^{-6}). \tag{7.29}$$

We now turn to a discussion of the fluxes. Near each disconnected component of the five dimensional surface, we may perform an integral of F_5 over the five-surfaces and measure the number of flux-quanta threading it. We use the $y \ll 1$ metric (7.3) and the flux

$$F_5 = *_{10} F_5 = (1 + *_{10})(d[y^4(dt + \omega)] - 2\eta J^{(6)}) \wedge \Omega_3. \tag{7.30}$$

The component of F_5 which is needed contains $(dt + \omega) \wedge dy \wedge \Omega_3$. We consider its Hodge dual,

$$*_{10}\, F_5 = 4y^3 \frac{1}{y^3} \mathcal{N}_\psi (d\psi + A) \wedge \mathrm{Vol}_{\Sigma_4} + \cdots, \tag{7.31}$$

and see that the y dependence cancels nicely, which is a consequence of the regularity of the expression (7.3) for the metric. Hence the integral of the five-form flux through the i-th disconnected piece of the 5d surface $\Sigma_5^{(i)}$ is

$$\int\limits_{\Sigma_5^{(i)}} *_{10} F_5 = \int 4\mathcal{N}_\psi (d\psi + A) \wedge \mathrm{Vol}_{\Sigma_4} = N_i, \tag{7.32}$$

which is expected to be quantized. The total D3 brane flux quanta N of the solution is the sum of the flux quanta threading each disconnected component of the surfaces, *i.e.* $N = \sum_i N_i$.

We have thus seen that, in order for the ten-dimensional 1/8 BPS configurations to be regular, we need to specify the following boundary conditions. We begin with defining a five-dimensional surface via the algebraic constraint $y^2 \equiv F(r_1^2, r_2^2, r_3^2) = 0$ (for generic non-toric geometries, we should allow for a dependence on the three angular coordinates as well, even though we have not done so here). Then we require that the Kähler potential behaves (up to an irrelevant constant) as $y^4/4$, to leading order in y for $y \ll 1$. This guarantees that the $dy^2 + y^2 d\Omega_3^2$ part of the ten-dimensional metric will be regular, and it ensures that, at leading order, there will be no mixing between y and the remaining coordinates. Further requiring that the remaining part of the metric be regular imposes additional constraints on the function $F(r_1^2, r_2^2, r_3^2)$. In particular, a necessary condition for regularity is that $(f/f_y)\big|_{y=0}$ is finite, where f was defined in (7.12). So, in the end, the six-dimensional Kähler base is allowed to end only on *smooth* five-dimensional surfaces.

Other than for this smoothness condition, we have shown (at least locally near $y = 0$) that arbitrary droplet configurations are allowed by regularity. Of course, it remains to be seen whether this conclusion holds globally as well. Proving this appears to be highly non-trivial, although there are no obvious obstructions to the existence of global solutions starting from arbitrary droplet data.

7.2 Asymptotic conditions at large y

Finally, while we do not address the uniqueness of solutions directly, we now turn to an examination of the asymptotic boundary conditions. In addition to addressing regularity and uniqueness issues, these asymptotic conditions are also useful for identifying the 1/8 BPS $\mathcal{N} = 4$ SYM states that are dual to this class of regular supergravity solutions. (Other asymptotic boundary conditions could correspond to 1/4 BPS or 1/2 BPS states of $\mathcal{N} = 2$ or $\mathcal{N} = 1$ gauge theories arising from D3 branes.)

As we have seen earlier, demanding that the asymptotic geometry approaches $\mathrm{AdS}_5 \times S^5$ gives rise to a leading Kähler potential of the form (5.23)

$$K = \tfrac{1}{2}|z_i|^2 - \tfrac{1}{2}\log(|z_i|^2) + \cdots . \tag{7.33}$$

Since the small y Kähler potential behaves as (7.8)

$$K = \frac{1}{4} y^4(z_i, \bar{z}_i) + \cdots , \tag{7.34}$$

a complete solution would interpolate between (7.33) in the asymptotic region and various expressions behaving as (7.34), one for each disconnected component of the $y = 0$ boundary. The question of uniqueness is then whether the 1/8 BPS condition (7.24) admits a unique solution with these boundary conditions.

As a preliminary step, we may consider the asymptotic expansion of K, and in particular the form of the correction terms in (7.33). Recall that a general 1/8 BPS droplet configuration can be described by excising regions from $\mathbb{C}^3$, coordinatized by z_1, z_2 and z_3. Near asymptotic infinity, the geometry of these excised regions may then be encoded by generalized multipole moments. This then allows a multipole expansion of the Kähler potential at infinity. Instead of developing the general multipole expansion, we give as an example the next-to-leading expression of K for 1/8 BPS solutions with three $U(1)$ R-charges $(J_1, J_2, J_3) \propto (Q_1, Q_2, Q_3)$ turned on. It suffices to obtain this term from the asymptotic expression of the 1/8 BPS smooth configuration given by the elliptic surface in (5.66).

Since we need the next-to-leading terms, we can start from the expression in (5.75) and keep leading terms in $1/(R+1)$ or $1/R$ and linear in Q_i:

$$\rho_i^2 \simeq (R+1)(1+\frac{Q_i}{R+1}). \tag{7.35}$$

Note that if $Q_i = 0$, we find $\rho_i^2 = (R+1)$, corresponding to the $\mathrm{AdS}_5 \times S^5$ vacuum. We want to solve for R in terms of $|z_i|^2$. We have the constraint equation

$$\sum_i \frac{|z_i|^2}{\rho_i^2} \simeq \sum_i \frac{|z_i|^2}{R+1}(1-\frac{Q_i}{R+1}) \simeq 1, \tag{7.36}$$

which then gives

$$R+1 \simeq \sum_i |z_i|^2 - \frac{\sum_i Q_i |z_i|^2}{\sum_i |z_i|^2}. \tag{7.37}$$

We also checked that the above expression satisfies (5.57) by plugging in (5.56):

$$\Lambda = \sum_i \frac{|z_i|^2}{\rho_i^2 H_i} \simeq 1 - \frac{\sum_i Q_i |z_i|^2}{(\sum_i |z_i|^2)^2}. \tag{7.38}$$

In the asymptotic region, the leading and next-to-leading terms in the Kähler potential are expected to be a function of $|z_i|^2$, $i = 1, 2, 3$,

$$K = K(|z_i|^2). \tag{7.39}$$

Note that the derivatives of K are known, since they were evaluated in (5.55)

$$\partial_{|z_j|^2}\partial_{|z_i|^2} K = \frac{1}{2\Lambda H_1 H_2 H_3 \rho_j^2 \rho_i^2} \simeq \frac{1}{2(\sum_i |z_i|^2)^2} + \frac{3\sum_i Q_i |z_i|^2}{2(\sum_i |z_i|^2)^4} - \frac{Q_j + Q_i}{2(\sum_i |z_i|^2)^3} - \frac{\sum_i Q_i}{2(\sum_i |z_i|^2)^3}. \tag{7.40}$$

After integrating $\int d|z_j|^2 \int d|z_i|^2$ we get

$$K \simeq \tfrac{1}{2}\sum_i |z_i|^2 - \tfrac{1}{2}\log\Big(\sum_i |z_i|^2\Big) + \frac{1}{4}\frac{\sum_i Q_i |z_i|^2}{(\sum_i |z_i|^2)^2} - \frac{1}{8}\frac{\sum_i Q_i}{(\sum_i |z_i|^2)}. \tag{7.41}$$

The first two terms provide the leading $\mathrm{AdS}_5 \times S^5$ behavior of (7.33), while the latter two terms give the first order deviations from the $\mathrm{AdS}_5 \times S^5$ vacuum that are linear in the R-charges, which characterize the solutions.

In principle, this expansion can be carried out to higher orders, and with more general multipole distributions. In this case, individual complex components z_i and $\bar{z}_i$ would also begin to enter into the expansion of K. Nevertheless, since any arbitrary distribution of droplets in $\mathbb{C}^3$ may be fully characterized by their (infinite set of) multipole moments, and since the multipole expansion of K appears to be unique (although we have not proven this), this provides evidence that the droplet description of bubbling AdS is well defined in the sense that there is a one-to-one mapping between droplets and geometries.

8 Conclusions

In this paper we investigated the supergravity duals of BPS states in $\mathcal{N} = 4$ super Yang-Mills. We found evidence for a universal bubbling AdS picture for all 1/2, 1/4 and 1/8 BPS geometries in IIB supergravity for these states. This picture emerges from a careful consideration of the necessary conditions which ensure the regularity of these supergravity solutions.

In the case of generic 1/8 BPS solutions, which have an S^3 isometry and are time-fibered over a six-real dimensional Kähler base, regularity is enforced when the radius of S^3 (denoted by y) vanishes: $y = 0$. Since y is a function of all the base coordinates, $y = y(x^i)$, $i = 1, \ldots, 6$, the geometric locus where the S^3 shrinks to zero size is a generally disconnected five-dimensional boundary surface. We have found that regular 1/8 BPS geometries are determined by the following boundary data: the general smooth five-dimensional surfaces located at $y = 0$ and the six-dimensional Kähler potential $K = \frac{1}{4}y^4 + \mathcal{O}(y^6)$ near $y = 0$. The interior of these five-dimensional surfaces is excised from the six-dimensional base, since the base ends at $y = 0$. Each regular solution is thus associated with a smooth five-dimensional surface. For example, the boundary data for the $\mathrm{AdS}_5 \times S^5$ ground state is a five-dimensional round sphere, whose interior, *i.e.* a round ball, is removed from the six-dimensional base. A generic 1/8 BPS state is then characterized by a combination of topologically trivial deformations of the S^5 (gravitons), topologically non-trivial ones (giant gravitons), and/or excisions of other six-dimensional droplets from the base (dual giant gravitons). One may view these surfaces as the locus where the matrix eigenvalues of the three complex scalars in the dual theory are distributed. In order for these configurations to be dual to $\mathcal{N} = 4$ super Yang-Mills states, we must impose additional conditions such that asymptotically one recovers an $\mathrm{AdS}_5 \times S^5$ geometry.

In the case of 1/4 BPS solutions, which have an $S^3 \times S^1$ isometry, we have identified a four-dimensional Kähler base where the regularity conditions must be imposed. The droplets are four-dimensional regions of shrinking S^3 inside a background where the S^1 shrinks to zero size. This is a natural extension of the LLM droplet picture of 1/2 BPS states, which was obtained by specifying the two-dimensional regions inside a two-dimensional phase-space where the S^3 inside AdS_5 collapses. Therefore the 1/4 BPS regular solutions are characterized by three dimensional surfaces separating the regions where either the S^3 or the S^1 collapses. For example, in the ungauged 1/4 BPS case, the $\mathrm{AdS}_5 \times S^5$ ground state corresponds to a round three-sphere in the four-dimensional base space, and a generic 1/4 BPS state is given by a deformation and/or topologically non-trivial distortion of the round three-sphere.

We discussed several examples to better illustrate the universality of the 'bubbling AdS' picture

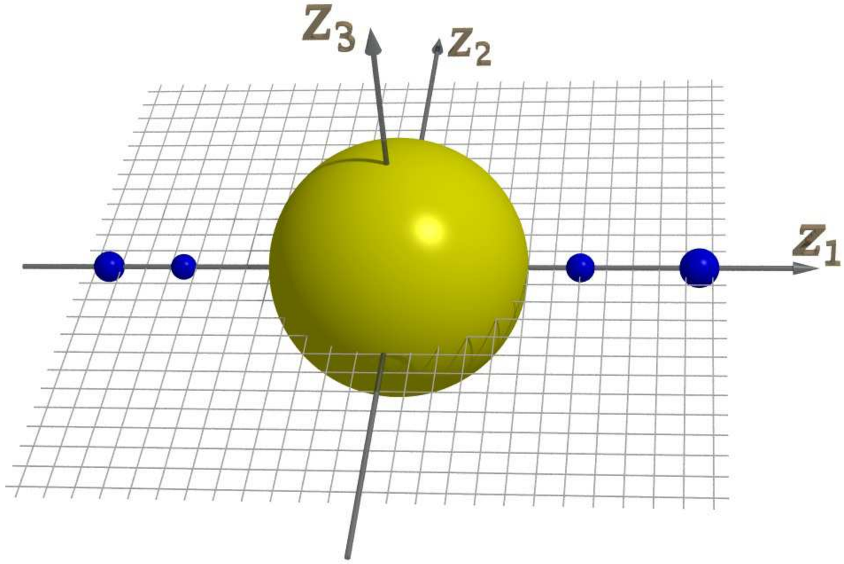

Figure 4: Schematic picture of a 1/2 BPS configuration corresponding to four dual giant gravitons excited on top of the AdS vacuum (central sphere). Giant gravitons expanding on S^5 are not pictured, but would correspond to giving the AdS sphere a non-trivial topology. These 1/2 BPS configurations always preserve an $\tilde{S}^3$ invariance corresponding to rotations in the z_2-z_3 planes.

in the 1/2, 1/4 and 1/8 BPS sectors. Given the non-linearity of the equations which determine the explicit form of the 1/4 and 1/8 BPS solutions, our regularity analysis focused on the small y region of the ten dimensional geometry and our analysis of the boundary behavior of the Kähler potential is perturbative in small y; the boundary conditions ensure the regularity of the ten-dimensional solution in a neighborhood patch near $y = 0$. Although we have given plausibility arguments, we have not rigorously shown that the solutions which are generated after specifying the boundary data are unique, nor can we say whether the perturbative analysis near the $y = 0$ region is sufficient to guarantee the regularity in the whole space at arbitrary non-zero y. Clearly such questions deserve a more thorough investigation. Although the differential equations determining the whole geometry are non-linear, the mapping between the topology of the boundary surfaces in the Kähler base and the topology of the eigenvalue distributions of the complex scalars in the dual $\mathcal{N} = 4$ gauge theory should be quite straightforward and robust.

The family of 1/2, 1/4 and 1/8 BPS geometries may be summarized using the generic 1/8 BPS picture, where the droplets live on $\mathbb{C}^3$, the coordinate space of the six-real dimensional Kähler base. As shown in Section 5.4, 1/2 BPS (*i.e.* lifted LLM) configurations are described by $\tilde{S}^3$ invariant droplets in the z_1 plane. Such configurations are shown schematically in Figure 4. Moving to 1/4 BPS geometries entails generalizing the droplets to lie anywhere in the z_1-z_2 planes, but to maintain

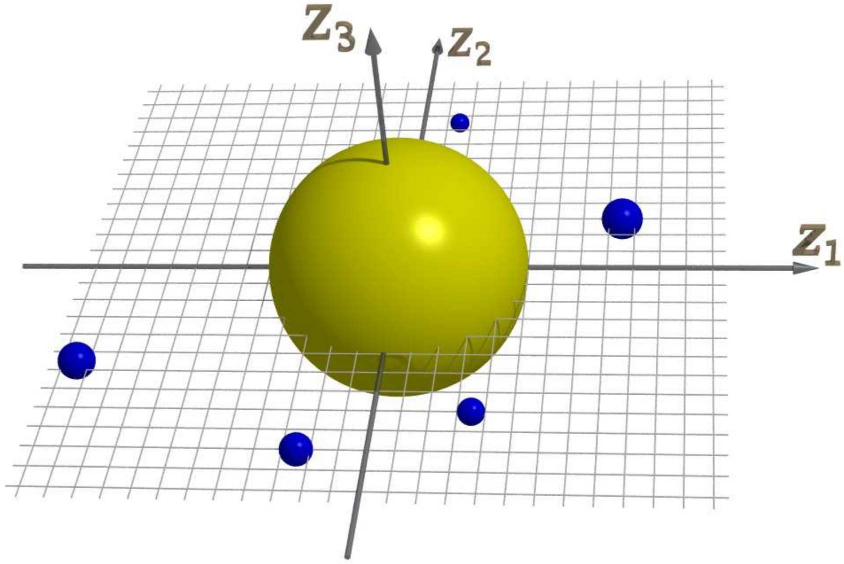

Figure 5: Picture of a 1/4 BPS configuration with five dual giant gravitons. The configuration is symmetric under S^1 rotations in the z_3 plane (which, however, cannot be directly visualized since the imaginary components of the axes are suppressed).

an S^1 invariance corresponding to rotations in the z_3 plane. This is shown in Figure 5. Finally, generic 1/8 BPS droplets may lie anywhere in $\mathbb{C}^3$, as indicated in Figure 6.

It is interesting to note that the droplets which comprise the boundary data for $1/2^n$ BPS geometries belong to a $2n$ ($n = 1, 2, 3$) real-dimensional Kähler space, which is naturally endowed with a symplectic form, and therefore admits a phase-space interpretation. It is also endowed with a complex structure, which is naturally related to the existence of the n complex scalars in the dual theory. This observation should be sharpened after quantizing the 1/4 and 1/8 BPS classical solutions discussed here. The five-dimensional surfaces that we observe are expected to become non-commutative after the quantization.

It is expected that the 1/2 BPS droplets of Figure 4 are non-interacting (as they admit a dual free-fermion description). This is supported by the linearity of the LLM harmonic function equation (3.97). Furthermore, the complex z_1 plane is unaffected by the presence of the droplets, and hence remains flat regardless of the details of the 1/2 BPS configuration. This is no longer true in the reduced supersymmetry cases. In particular, note that Figures 5 and 6 visualize the 1/4 and 1/8 BPS droplet data in coordinate space, given by Euclidean $\mathbb{C}^3$. The Kähler metric itself is highly non-trivial, so the geometry of the Kähler base is curved by the droplets themselves; in fact, the curvature on the 1/8 BPS base blows up ($R \to -\infty$) as one approaches the boundaries of the droplets. This suggests that the 1/4 and 1/8 BPS droplets will have non-trivial interactions, as

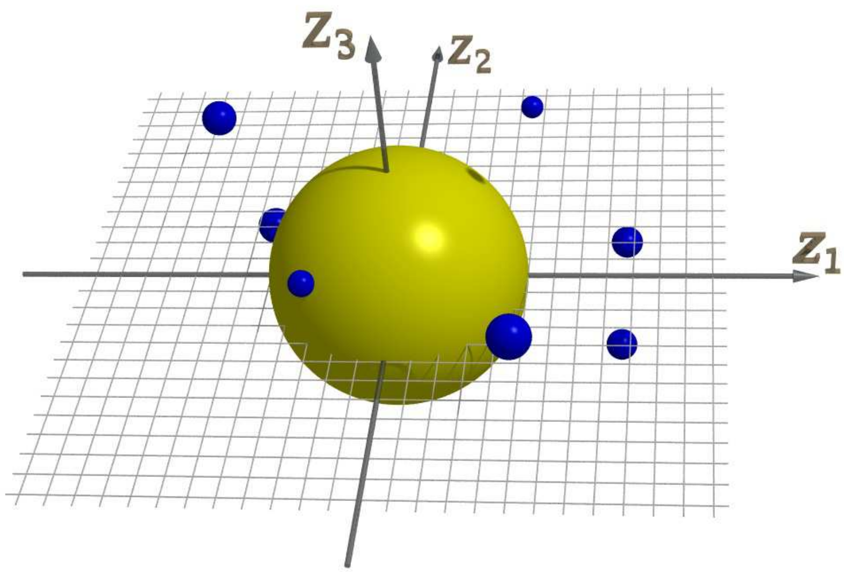

Figure 6: Schematic picture of a 1/8 BPS configuration with seven dual giant gravitons. In general, 1/8 BPS droplets may have any topology and geometry allowed by regularity.

would also be expected based on reduced supersymmetry.

Understanding this non-trivial geometry on the Kähler base and its implications for droplet dynamics seems to be essential in constructing the moduli space of these BPS configurations. Among other things, this geometry should shed light on the scattering of BPS droplets in a non-trivial background.

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A Differential identities for the S^3 reduction

The seven-dimensional system given in Section 2.1 comprises a metric, scalar and two-form field strength, $(g_{\mu\nu}, \alpha, F_{(2)})$. The differential identities are obtained by taking the supersymmetry variations (2.12) and (2.13) and contracting on the left with either $\overline{\epsilon}$ or ϵ^c along with a complete set of Dirac matrices $\{1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\lambda}\}$.

After appropriate rearrangement, most of the differential identities for the Dirac bilinears $\{f, K, V, Z\}$ may be written in form notation

$$
\begin{aligned}
i_K d\alpha &= 0, && \text{(A.1)}\\
d(e^{-\alpha} f) &= 0, && \text{(A.2)}\\
d(e^{3\alpha} f) &= -i_K F, && \text{(A.3)}\\
d(e^{2\alpha} K) &= -e^{-\alpha} f F - 2\eta e^{\alpha} V, && \text{(A.4)}\\
d(e^{-2\alpha} K) &= -e^{-5\alpha} * (F \wedge Z) + 2\eta e^{-3\alpha} V, && \text{(A.5)}\\
d(e^{\alpha} V) &= 0, && \text{(A.6)}\\
d(e^{2\alpha} * V) &= 2\eta e^{\alpha} * K, && \text{(A.7)}\\
d(e^{4\alpha} Z) &= e^{\alpha} F \wedge V - 4\eta e^{3\alpha} * Z, && \text{(A.8)}\\
d(e^{3\alpha} * Z) &= 0. && \text{(A.9)}
\end{aligned}
$$

The remaining identities are of the form

$$
\begin{aligned}
0 &= F_{\mu\nu} V^{\mu\nu} + 8\eta e^{2\alpha} f, && \text{(A.10)}\\
\nabla_{(\mu} K_{\nu)} &= 0, && \text{(A.11)}\\
\nabla_\mu V_{\nu\lambda} &= \tfrac{1}{4} e^{-3\alpha} (2 Z_{\mu[\nu}{}^{\rho} F_{\lambda]\rho} - Z_{\nu\lambda}{}^{\rho} F_{\mu\rho} - g_{\mu[\nu} Z_{\lambda]\rho\sigma} F^{\rho\sigma}), && \text{(A.12)}\\
\nabla_\mu Z_{\nu\lambda\rho} &= \tfrac{1}{4} e^{-3\alpha} (-\tfrac{1}{2} \epsilon_{\mu\nu\lambda\rho}{}^{\alpha\beta\gamma} F_{\alpha\beta} K_\gamma + 3 F_{\mu[\nu} V_{\lambda\rho]} + 3 F_{[\nu\lambda} V_{\rho]\mu} + 6 g_{\mu[\nu} F_{\lambda}{}^{\sigma} V_{\rho]\sigma}). && \text{(A.13)}
\end{aligned}
$$

Note, in particular, that (A.11) demonstrate that K^μ is a Killing vector. Although the 'dilatino' variation (2.13) leads to algebraic expressions on the spinor bilinears, they naturally combine with the gravitino variation expressions, and this is what we have done above in writing down a complete set of differential identities on the Dirac bilinears.

For the Majorana bilinears $\{f^m, Z^m\}$, we find instead

$$\begin{aligned}
\eta f^m &= 0, && \text{(A.14)}\\
d(e^{\alpha} f^m) &= 0, && \text{(A.15)}\\
d(e^{-3\alpha} f^m) &= \tfrac{i}{2} e^{-\alpha}(Z^m_{\mu\nu\lambda} F^{\nu\lambda}) dx^\mu, && \text{(A.16)}\\
d(e^{2\alpha} Z^m) &= -2\eta e^{\alpha} * Z^m, && \text{(A.17)}\\
d(e^{\alpha} * Z^m) &= \tfrac{i}{4} e^{-2\alpha} f^m * F, && \text{(A.18)}\\
\nabla_\mu Z^m_{\nu\lambda\rho} &= \tfrac{i}{16} e^{-3\alpha}(-\tfrac{2}{3}\delta^\alpha_\mu \epsilon_{\nu\lambda\rho}{}^{\beta\gamma\delta\epsilon} + g_{\mu[\nu}\epsilon_{\lambda\rho]}{}^{\alpha\beta\gamma\delta\epsilon} - 2\delta^\alpha_{[\nu}\epsilon_{\lambda\rho]\mu}{}^{\beta\gamma\delta\epsilon}) F_{\alpha\beta} Z^m_{\gamma\delta\epsilon}. && \text{(A.19)}
\end{aligned}$$

Since $\eta = \pm 1$ is non-vanishing (for Killing spinors on S^3), the first expression, (A.14), immediately demonstrates that the Majorana scalar invariant vanishes, $f^m = 0$. This leads to the identification of SU(3) structure and a resulting simplification of the above expressions, as discussed in Section 3.1.

B Differential identities for the $S^3 \times S^1$ reduction

In Section 2.2, we presented the reduction of the bosonic fields of IIB supergravity on $S^3 \times S^1$ along with the relevant set of supersymmetry variations (2.26) and (2.27). Here we present a partial list of differential identities related to these variations. However, before doing so, we recall that the bosonic fields in six dimensions are the metric $g_{\mu\nu}$, two abelian gauge fields A_μ and $\mathcal{A}_\mu$, as well as two 'dilatonic' scalars α and β and one 'axionic' scalar χ. The differential identities serve to related these fields with each other, as well as the Dirac $\{f_1, f_2, K, L, V, Y, Z\}$ and Majorana $\{f^m, Y^m, Z^m\}$ bilinears given in (3.27).

Because of the large number of fields and bilinears, the complete list of differential identities is rather long. Here we only list the more relevant ones to the supersymmetry analysis. We begin with the scalar identities

$$\begin{aligned}
0 &= i_K d\alpha = i_K d\beta = i_K d\chi, && \text{(B.1)}\\
0 &= F_{\mu\nu} V^{\mu\nu} + 2e^{-\beta} L^\mu \partial_\mu \chi + 8\eta e^{2\alpha} f_2, && \text{(B.2)}\\
0 &= F_{\mu\nu} Y^{\mu\nu} - 8e^{3\alpha} L^\mu \partial_\mu \alpha - 8\eta e^{2\alpha} f_1, && \text{(B.3)}\\
0 &= \mathcal{F}_{\mu\nu} V^{\mu\nu} + 4e^{-\beta} L^\mu \partial_\mu(\alpha+\beta) + 4\eta e^{-\alpha-\beta} f_1 + 4ne^{-2\beta} f_2, && \text{(B.4)}\\
0 &= \mathcal{F}_{\mu\nu} Y^{\mu\nu} + 2e^{-3\alpha-2\beta} L^\mu \partial_\mu \chi + 4\eta e^{-\alpha-\beta} f_2 - 4ne^{-2\beta} f_1, && \text{(B.5)}\\
0 &= \eta f^m = n f^m, && \text{(B.6)}\\
0 &= F_{\mu\nu} Y^{m\,\mu\nu} = \mathcal{F}_{\mu\nu} Y^{m\,\mu\nu}. && \text{(B.7)}
\end{aligned}$$

Although the $U(1)$ charge n of the Killing spinor may vanish, the S^3 Killing spinor parameter $\eta = \pm 1$ cannot vanish. As a result, (B.6) indicates that $f^m = 0$. This vanishing of the Majorana scalar invariant simplifies the structure analysis of Section 3.2, and is needed for the demonstration of U(2) structure.

After some rearrangement, the one-form identities may be written as

$$
\begin{aligned}
d(e^{-\alpha}f_2) &= 0, && \text{(B.8)}\\
d(e^{2\alpha+\beta}f_1 + e^{-\alpha}f_2\chi) &= -2\eta e^{\alpha+\beta}L, && \text{(B.9)}\\
d(e^{3\alpha}f_2) &= -i_K F + e^{-\beta}f_1 d\chi, && \text{(B.10)}\\
d(e^{-\beta}f_1) &= -i_K\mathcal{F}, && \text{(B.11)}\\
d(e^{\alpha+2\beta}f_2) &= -\tfrac{1}{2}e^{\alpha+3\beta} * Z_\mu{}^{\nu\lambda}\mathcal{F}_{\nu\lambda}dx^\mu + e^{-2\alpha+\beta}f_1 d\chi - 2ne^{\alpha+\beta}L, && \text{(B.12)}\\
d(e^{-2\alpha+\beta}f_1) &= \tfrac{1}{2}e^{-5\alpha+\beta} * Z_\mu{}^{\nu\lambda}F_{\nu\lambda}dx^\mu + 2\eta e^{-3\alpha+\beta}L, && \text{(B.13)}\\
D(e^{\alpha}f^m) &= 0, && \text{(B.14)}\\
D(e^{-3\alpha}f^m) &= \tfrac{i}{2}e^{-6\alpha}Z_\mu^{m\nu\lambda}F_{\nu\lambda}dx^\mu + ie^{-6\alpha-\beta}Y_\mu^{m\nu}\partial_\nu\chi dx^\mu, && \text{(B.15)}\\
D(e^{-\alpha-2\beta}f^m) &= \tfrac{1}{2}e^{-\alpha-\beta} * Z_\mu^{m\nu\lambda}\mathcal{F}_{\nu\lambda}dx^\mu + ie^{-4\alpha-3\beta}Y_\mu^{m\nu}\partial_\nu\chi dx^\mu, && \text{(B.16)}
\end{aligned}
$$

where $D = d + in\mathcal{A}$ is the $U(1)$ gauge covariant derivative.

Turning to the two-form identities, we have

$$
\begin{aligned}
d(e^{2\alpha}K) &= -e^{-\alpha}f_2F - e^{2\alpha+\beta}f_1\mathcal{F} - 2\eta e^{\alpha}V, && \text{(B.17)}\\
d(e^{-2\alpha}K) &= [-\tfrac{1}{4}e^{-5\alpha} * Y_{\mu\nu}{}^{\lambda\sigma}F_{\lambda\sigma} + \tfrac{1}{2}e^{-5\alpha-\beta} * Z_{\mu\nu}{}^{\lambda}\partial_\lambda\chi]dx^\mu\wedge dx^\nu - e^{-2\alpha+\beta}f_1\mathcal{F} + 2\eta e^{-3\alpha}V, && \text{(B.18)}\\
d(e^{2\beta}K) &= [-\tfrac{1}{4}e^{-3\alpha+2\beta} * Y_{\mu\nu}{}^{\lambda\sigma}F_{\lambda\sigma} - \tfrac{1}{4}e^{3\beta} * V_{\mu\nu}{}^{\lambda\sigma}\mathcal{F}_{\lambda\sigma}]dx^\mu\wedge dx^\nu - 2ne^{\beta}Y, && \text{(B.19)}\\
d(e^{\alpha+\beta}L) &= 0, && \text{(B.20)}\\
dL &= \tfrac{1}{2}e^{\beta}\mathcal{F}_\mu{}^\lambda V_{\nu\lambda}dx^\mu\wedge dx^\nu - \tfrac{1}{4}e^{-3\alpha-\beta}Z_{\mu\nu}{}^\lambda\partial_\lambda\chi dx^\mu\wedge dx^\nu, && \text{(B.21)}\\
d(e^{2\alpha}L) &= \tfrac{1}{2}e^{-\alpha}F_\mu{}^\lambda Y_{\nu\lambda}dx^\mu\wedge dx^\nu + \tfrac{1}{2}e^{2\alpha+\beta}\mathcal{F}_\mu{}^\lambda V_{\nu\lambda}dx^\mu\wedge dx^\nu. && \text{(B.22)}
\end{aligned}
$$

In addition

$$
\begin{aligned}
\nabla_{(\mu}K_{\nu)} &= 0, && \text{(B.23)}\\
\nabla_{(\mu}L_{\nu)} &= \tfrac{1}{8}e^{-3\alpha}(4F_{(\mu}{}^\lambda Y_{\nu)\lambda} - g_{\mu\nu}F_{\lambda\sigma}Y^{\lambda\sigma}) + \tfrac{1}{2}e^{\beta}\mathcal{F}_{(\nu}{}^\lambda V_{\nu)\lambda}. && \text{(B.24)}
\end{aligned}
$$

In particular, this shows that K^μ is a Killing vector. Also, while L_μ is not a closed one-form, the combination $e^{\alpha+\beta}L$ is.

For the three-form identities, we have

$$
\begin{aligned}
d(e^{\alpha}V) &= e^{\alpha+\beta}\mathcal{F}\wedge L, && \text{(B.25)}\\
d(e^{-\alpha+2\beta}V) &= \tfrac{1}{2}e^{-4\alpha+2\beta}Z_{\mu\nu}{}^{\sigma}F_{\lambda\sigma}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda} - e^{-\alpha+3\beta}i_k*\mathcal{F} - 2ne^{-\alpha+\beta}*Z, && \text{(B.26)}\\
d(e^{3\alpha+2\beta}V) &= -e^{3\alpha+3\beta}i_K*\mathcal{F} - e^{\beta}Y\wedge d\chi - 2ne^{3\alpha+\beta}*Z, && \text{(B.27)}\\
d(e^{\beta}Y) &= -e^{-3\alpha+\beta}i_K*F, && \text{(B.28)}\\
d(e^{\alpha}Y) &= -\tfrac{1}{2}e^{-2\alpha}(i_K*F - F\wedge L) + \tfrac{1}{4}e^{\alpha+\beta}Z_{\mu\nu}{}^{\sigma}\mathcal{F}_{\lambda\sigma}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda} + \eta*Z, && \text{(B.29)}\\
d(e^{3\alpha}Y) &= F\wedge L + \tfrac{1}{4}e^{3\alpha+\beta}Z_{\mu\nu}{}^{\sigma}\mathcal{F}_{\lambda\sigma}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda} + \tfrac{1}{2}e^{-\beta}V\wedge d\chi + 3\eta e^{2\alpha}*Z, && \text{(B.30)}\\
D(e^{2\alpha+\beta}Y^m) &= 2\eta e^{\alpha+\beta}*Z^m - ine^{2\alpha}Z^m, && \text{(B.31)}\\
D(e^{\beta}Y^m) &= [\tfrac{i}{4}e^{-3\alpha+\beta}*Z^{m}_{\mu\nu}{}^{\sigma}F_{\lambda\sigma} + \tfrac{i}{12}e^{-3\alpha}*Y^{m}_{\mu\nu\lambda}{}^{\sigma}\partial_{\sigma}\chi]dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda} - inZ^m, && \text{(B.32)}\\
D(e^{\alpha}Y^m) &= [\tfrac{1}{4}e^{\alpha+\beta}Z^{m}_{\mu\nu}{}^{\sigma}\mathcal{F}_{\lambda\sigma} + \tfrac{i}{12}e^{-2\alpha-\beta}*Y^{m}_{\mu\nu\lambda}{}^{\sigma}\partial_{\sigma}\chi]dx^{\mu}\wedge dx^{\nu}\wedge dx^{\lambda} + \eta*Z^m. && \text{(B.33)}
\end{aligned}
$$

When the indices are not taken to be fully antisymmetric, we must also include the identities

$$
\begin{aligned}
\nabla_{\mu}V_{\nu\lambda} &= -\tfrac{1}{4}e^{-3\alpha}(Z_{\nu\lambda}{}^{\sigma}F_{\mu\sigma} - 2Z_{\mu[\nu}{}^{\sigma}F_{\lambda]\sigma} + g_{\mu[\nu}Z_{\lambda]}{}^{\alpha\beta}F_{\alpha\beta}) + e^{\beta}\mathcal{F}_{\mu[\nu}L_{\lambda]}\\
&\quad -\tfrac{1}{4}e^{-3\alpha-\beta}(Y_{\nu\lambda}\partial_{\mu}\chi - 2Y_{\mu[\nu}\partial_{\lambda]}\chi + 2g_{\mu[\nu}Y_{\lambda]}{}^{\sigma}\partial_{\sigma}\chi), && \text{(B.34)}\\
\nabla_{\mu}Y_{\nu\lambda} &= \tfrac{1}{4}e^{-3\alpha}(*F_{\mu\nu\lambda}{}^{\sigma}K_{\sigma} - F_{\nu\lambda}L_{\mu} + 2F_{\mu[\nu}L_{\lambda]} - 2g_{\mu[\nu}F_{\lambda]}{}^{\sigma}L_{\sigma}) + \tfrac{1}{2}e^{\beta}Z_{\nu\lambda}{}^{\sigma}\mathcal{F}_{\mu\sigma}\\
&\quad +\tfrac{1}{4}e^{-3\alpha-\beta}(V_{\nu\lambda}\partial_{\mu}\chi - 2V_{\mu[\nu}\partial_{\lambda]}\chi + 2g_{\mu[\nu}V_{\lambda]}{}^{\sigma}\partial_{\sigma}\chi), && \text{(B.35)}\\
D_{\mu}Y^{m}_{\nu\lambda} &= -\tfrac{i}{4}e^{-3\alpha}(*Z^{m}_{\nu\lambda}{}^{\sigma}F_{\mu\sigma} - 2*Z^{m}_{\mu[\nu}{}^{\sigma}F_{\lambda]\sigma} + g_{\mu[\nu}*Z^{m}_{\lambda]\alpha\beta}F^{\alpha\beta}) + \tfrac{1}{2}e^{\beta}Z^{m}_{\nu\lambda}{}^{\sigma}\mathcal{F}_{\mu\sigma}\\
&\quad +\tfrac{i}{4}e^{-3\alpha-\beta}(*Y^{m}_{\mu\nu\lambda}{}^{\sigma}\partial_{\sigma}\chi - 2f^{m}g_{\mu[\nu}\partial_{\lambda]}\chi). && \text{(B.36)}
\end{aligned}
$$

C Differential identities for the $S^3\times S^3$ reduction

For the round $S^3\times S^3$ reduction, corresponding to the original LLM system of [1], the relevant supersymmetry variations are given by (3.83). Many of the differential identities for this system have been tabulated in Appendix C of [4]. We nevertheless give them here again, using our present notation.

Most of the differential identities can be presented in form notation. For the Dirac bilinears $\{f_1, f_2, K, L, Y\}$, the scalar (or zero-form) identities are

$$
\begin{aligned}
&0 = i_K d\alpha = i_K d\beta, && \text{(C.1)}\\
&0 = i_L d(\alpha+\beta) + \eta e^{-\alpha}f_1 + \tilde{\eta}e^{-\beta}f_2, && \text{(C.2)}\\
&0 = i_L d(\alpha-\beta) - \tfrac{1}{4}e^{-3\alpha}F_{\mu\nu}Y^{\mu\nu} + \eta e^{-\alpha}f_1 - \tilde{\eta}e^{-\beta}f_2, && \text{(C.3)}\\
&\tfrac{1}{8}e^{-3\alpha}F_{\mu\nu}*Y^{\mu\nu} = \tilde{\eta}e^{-\beta}f_1 = \eta e^{-\alpha}f_2. && \text{(C.4)}
\end{aligned}
$$

The one-form identities are

$$\begin{aligned}
d(e^{-\beta} f_1) &= 0, & \text{(C.5)}\\
d(e^{\alpha} f_1) &= -\eta L, & \text{(C.6)}\\
d(e^{3\beta} f_1) &= e^{-3\alpha+3\beta} i_K * F, & \text{(C.7)}\\
d(e^{-\alpha} f_2) &= 0, & \text{(C.8)}\\
d(e^{\beta} f_2) &= -\tilde{\eta} L, & \text{(C.9)}\\
d(e^{3\alpha} f_2) &= -i_K F. & \text{(C.10)}
\end{aligned}$$

The identities given here are derived by taking linear combinations of those in [4]. Of course, the particular choice we have made for which linear combinations to take is not unique. However, we find the above choice particularly useful when completing the solution in Section 3.3. Continuing with the two-form identities, we have

$$\begin{aligned}
dK &= -\tfrac{1}{2} e^{-3\alpha}(f_2 F - f_1 * F), & \text{(C.11)}\\
d(e^{2\alpha} K) &= -e^{-\alpha} f_2 F + 2\eta e^{\alpha} * Y, & \text{(C.12)}\\
d(e^{2\beta} K) &= -e^{-3\alpha+2\beta} f_1 * F - 2\tilde{\eta} e^{\beta} Y, & \text{(C.13)}\\
dL &= 0, & \text{(C.14)}\\
d(e^{\alpha+\beta} L) &= 0, & \text{(C.15)}\\
d(e^{2\alpha} L) &= \tfrac{1}{2} e^{-\alpha} F_\mu{}^\lambda Y_{\nu\lambda} dx^\mu \wedge dx^\nu. & \text{(C.16)}
\end{aligned}$$

Finally, we give the three-form identities

$$\begin{aligned}
d(e^{\beta} Y) &= 0, & \text{(C.17)}\\
d(e^{-\alpha} Y) &= -\eta e^{-2\alpha} * K, & \text{(C.18)}\\
d(e^{-3\beta} Y) &= -e^{-3\alpha-3\beta} L \wedge F, & \text{(C.19)}\\
d(e^{\alpha} * Y) &= 0, & \text{(C.20)}\\
d(e^{-\beta} * Y) &= -\tilde{\eta} e^{-2\beta} * L, & \text{(C.21)}\\
d(e^{-3\alpha} * Y) &= e^{-6\alpha} L \wedge *F. & \text{(C.22)}
\end{aligned}$$

Additional information is contained in the original (non-form notation) differential identities obtained from the gravitino variation

$$\begin{aligned}
\nabla_\mu K_\nu &= -\tfrac{1}{4} e^{-3\alpha}(f_2 F_{\mu\nu} - f_1 * F_{\mu\nu}),\\
\nabla_\mu L_\nu &= \tfrac{1}{4} e^{-3\alpha}(2F_{(\mu}{}^\lambda Y_{\nu)\lambda} - \tfrac{1}{2} g_{\mu\nu} F_{\rho\lambda} Y^{\rho\lambda}),\\
\nabla_\mu Y_{\nu\lambda} &= -\tfrac{1}{4} e^{-3\alpha}(F_{\nu\lambda} L_\mu + 2g_{\mu[\nu} F_{\lambda]}{}^\sigma L_\sigma - 2F_{\mu[\nu} L_{\lambda]}). & \text{(C.23)}
\end{aligned}$$

Note that the vector identities may be decomposed into antisymmetric and symmetric parts. The former are contained in (C.11) and (C.14), while the latter are

$$\begin{aligned}
2\nabla_{(\mu} K_{\nu)} &= 0, & \text{(C.24)}\\
2\nabla_{(\mu} L_{\nu)} &= e^{-3\alpha}(F_{(\mu}{}^\lambda Y_{\nu)\lambda} - \tfrac{1}{4} g_{\mu\nu} F_{\rho\lambda} Y^{\rho\lambda}). & \text{(C.25)}
\end{aligned}$$

For the Majorana bilinears $\{K^m, Y^m\}$, we have the gravitino differential identities

$$\begin{aligned}
\nabla_\mu K^m_\nu &= \tfrac{1}{8}e^{-3\alpha}(\tfrac{1}{2}g_{\mu\nu}F_{\rho\sigma} * Y^{m\,\rho\sigma} - 2F_{(\mu}{}^{\lambda} * Y^m_{\nu)\lambda}), \\
\nabla_\mu Y^m_{\nu\lambda} &= \tfrac{1}{2}e^{-3\alpha}(*F_{\mu[\nu}K^m_{\lambda]} - g_{\mu[\nu} * F_{\lambda]\rho}K^{m\,\rho} - \tfrac{1}{2} * F_{\nu\lambda}K^m_\mu),
\end{aligned} \tag{C.26}$$

as well as the zero-form identities

$$F^{\mu\nu}Y^m_{\mu\nu} = 0, \tag{C.27}$$

$$d(*K^m) = 0, \tag{C.28}$$

$$d(e^{\alpha+\beta} * K^m) = 0, \tag{C.29}$$

$$d(e^{4\alpha} * K^m) = -e^{\alpha}F \wedge Y^m, \tag{C.30}$$

two-form identities

$$dK^m = 0, \tag{C.31}$$

$$d(e^{\alpha}K^m) = \tfrac{1}{4}e^{-2\alpha}F_\mu{}^{\lambda} * Y^m_{\nu\lambda}dx^\mu \wedge dx^\nu - i\eta * Y^m, \tag{C.32}$$

$$d(e^{\beta}K^m) = -\tfrac{1}{4}e^{-3\alpha+\beta}F_\mu{}^{\lambda} * Y^m_{\nu\lambda}dx^\mu \wedge dx^\nu + i\tilde{\eta}Y^m, \tag{C.33}$$

and three-form identities

$$d(e^{-\beta}Y^m) = 0, \tag{C.34}$$

$$d(e^{\alpha}Y^m) = i\eta * K^m, \tag{C.35}$$

$$d(e^{3\beta}Y^m) = e^{-3\alpha+3\beta}(*F) \wedge K, \tag{C.36}$$

$$d(e^{-\alpha} * Y^m) = 0, \tag{C.37}$$

$$d(e^{\beta} * Y^m) = i\tilde{\eta} * K^m, \tag{C.38}$$

$$d(e^{3\alpha} * Y^m) = -F \wedge K^m. \tag{C.39}$$

D Regularity analysis for 1/4 BPS solutions

As an example of how we uncover the droplet picture for the 1/4 BPS geometries from a regularity analysis, we consider the case when the U(1) charge of the Killing spinor is

$$n\eta = 1, \tag{D.1}$$

which corresponds to the ungauged $S^3 \times S^1$ reduction, as discussed in Section 3.2. Under this assumption, since $D(z_i, \bar{z}_{\bar{j}})$ is constrained by $(1 + *_4)\partial\bar{\partial}D = 0$, it follows that D is a harmonic function of the four-dimensional Kähler base parametrized by z_1, z_2.

The metric given in (6.1) is potentially singular when $y = 0$, i.e. when the radius of either the S^3 or S^1 shrinks to zero. To avoid conical singularities at $y = 0$, G ought to behave such that $e^{\pm G} = yf_\pm(z_1, \bar{z}_1, z_2, \bar{z}_2) + \mathcal{O}(y^2)$ where the $\pm$ sign corresponds to having either the S^3 or S^1

collapse to zero size. Since $Z = \frac{1}{2}\tanh G$ is also tied to the four-dimensional base Kähler potential $Z = -\frac{1}{2} y \partial_y \frac{1}{y} \partial_y K$, this yields

$$
\begin{aligned}
K &= \frac{1}{2} y^2 \ln y + f_0 + \frac{y^2}{2} f_2 - \frac{y^4}{4} f_4 + \dots \\
&\text{or} \\
K &= -\frac{1}{2} y^2 \ln y + g_0 + \frac{y^2}{2} g_2 + \frac{y^4}{4} g_4 + \dots
\end{aligned}
\tag{D.2}
$$

where $f_{0,2,4}$ and $g_{0,2,4}$ are functions of z_1, z_2 and their complex conjugates. In the first case $Z \to -\frac{1}{2}$ as $y \to 0$ and in the second, $Z \to \frac{1}{2}$ as $y \to 0$. The $y = 0$ four-dimensional base is then decomposed into regions ("droplets") with $Z \to \pm\frac{1}{2}$, similar to the LLM decomposition of the two-dimensional base. The requirement that the asymptotics of the 1/4 BPS solutions be $AdS_5 \times S^5$ introduce the additional constraint that the droplet distribution must be such that, at large $|z_1|^2 + |z_2|^2$, one sees a large spherical droplet plus small distortions which can appear as deformations of the large droplet and/or as additional disconnected small droplets.

To confirm that the complete ten-dimensional geometry is non-singular we first notice that $h^{-2} = 2y \cosh G$ is finite at $y = 0$. Second, from $\ln \det h_{i\bar{j}} = \ln(Z + \frac{1}{2}) + \ln y + \frac{1}{y}\partial_y K + D$ we find that

$$
\det h_{i\bar{j}} = y^4 e^{D+\frac{1}{2}+f_2} f_4 + \dots, \qquad \text{or} \qquad \det h_{i\bar{j}} = y^0 e^{D-\frac{1}{2}+g_2} + \dots \tag{D.3}
$$

The regularity of the full ten-dimensional metric is assured since the Kähler subspace, together with its warp factor $(Z + \frac{1}{2})^{-1}$, is non-singular. This follows from the evaluation of the volume of this subspace at $y = 0$:

$$
\det(h^2 (Z + \frac{1}{2})^{-1} h_{i\bar{j}}) = \text{finite at } y = 0. \tag{D.4}
$$

E Detailed analysis of regularity conditions for 1/8 BPS configurations

In this appendix we present the details of the regularity analysis of 1/8 BPS configurations as discussed in Section 7.

In order to bring the ten-dimensional metric near $y = 0$ to the form (7.3), the Kähler potential has a Taylor expansion of the form

$$
K(z^i, z^{\bar{j}}) = \frac{1}{4} y^4 + a y^6 + \cdots , \tag{E.1}
$$

up to a unimportant shift via a Kähler transformation, and where $a = a(r_2^2, r_3^2)$ at $y = 0$.

First we calculate the metric of the six-dimensional base to leading order in y. The leading order (except in the 1st, 2nd, and 4th lines, which also contain $\mathcal{O}(y^2)$ terms) six-dimensional base metric is given by

$$
\begin{aligned}
ds_6^2 \;=\;& dy^2\, y^2\left[1+\frac{y^2}{f_y}\left(f_y-\frac{f f_{yy}}{f_y}\right)\right] \\
&+2dydr_2\; y^3 r_2\left(f_2-\frac{f f_{2y}}{f_y}\right)+2dydr_3\; y^3 r_3\left(f_3-\frac{f f_{3y}}{f_y}\right) \\
&+\frac{1}{f_y^2}(f d\phi_1-f_2 r_2^2 d\phi_2-f_3 r_3^2 d\phi_3)^2 \\
&+d\phi_1^2\;\frac{f y^2}{f_y^2}\left(f_y-\frac{f f_{yy}}{f_y}\right)+2d\phi_1 d\phi_2\;\frac{y^2 f r_2^2}{f_y^2}\left(\frac{f_2 f_{yy}}{f_y}-f_{2y}\right)+2d\phi_1 d\phi_3\;\frac{y^2 f r_3^2}{f_y^2}\left(\frac{f_3 f_{yy}}{f_y}-f_{3y}\right) \\
&+dr_2^2\;\frac{y^2}{f_y}\left(-f_2-f_{22}r_2^2+\frac{f_2^2 r_2^2}{f}\right)+dr_3^2\;\frac{y^2}{f_y}\left(-f_3-f_{33}r_3^2+\frac{f_3^2 r_3^2}{f}\right) \\
&+2dr_2 dr_3\;\frac{y^2}{f_y}r_2 r_3\left(-f_{23}+\frac{f_2 f_3}{f}\right) \\
&+d\phi_2^2\;\frac{y^2 r_2^2}{f_y^2}\left(2r_2^2 f_{2y} f_2-f_2 f_y-r_2^2 f_y f_{22}-\frac{r_2^2 f_2^2 f_{yy}}{f_y}\right) \\
&+d\phi_3^2\;\frac{y^2 r_3^2}{f_y^2}\left(2r_3^2 f_{3y} f_3-f_3 f_y-r_3^2 f_y f_{33}-\frac{r_3^2 f_3^2 f_{yy}}{f_y}\right) \\
&+2d\phi_2 d\phi_3\;\frac{y^2 r_2^2 r_3^2}{f_y}(-f_{23}+f_{2y} f_3+f_{3y} f_2-f_2 f_3 f_{yy}).
\end{aligned}
\tag{E.2}
$$

The leading metric pertaining to the four-dimensional surface is

$$
\begin{aligned}
d\Sigma_4^2 \;=\;& dr_2^2\;\frac{1}{f_y}\left(-f_2-f_{22}r_2^2+\frac{f_2^2 r_2^2}{f}\right)+dr_3^2\;\frac{1}{f_y}\left(-f_3-f_{33}r_3^2+\frac{f_3^2 r_3^2}{f}\right) \\
&+2dr_2 dr_3\;\frac{1}{f_y}r_2 r_3\left(-f_{23}+\frac{f_2 f_3}{f}\right) \\
&+d\phi_2^2\;\frac{r_2^2}{f_y^2}\left(2r_2^2 f_{2y} f_2-f_2 f_y-r_2^2 f_y f_{22}-\frac{r_2^2 f_2^2 f_{yy}}{f_y}\right) \\
&+d\phi_3^2\;\frac{r_3^2}{f_y^2}\left(2r_3^2 f_{3y} f_3-f_3 f_y-r_3^2 f_y f_{33}-\frac{r_3^2 f_3^2 f_{yy}}{f_y}\right) \\
&+2d\phi_2 d\phi_3\;\frac{r_2^2 r_3^2}{f_y}(-f_{23}+f_{2y} f_3+f_{3y} f_2-f_2 f_3 f_{yy}).
\end{aligned}
\tag{E.3}
$$

We also notice that the leading piece of the $(1/y^2 f_y^2)(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3)^2$ term cancels with the leading $y^2\omega^2$ term coming from g_{tt} in the ten-dimensional metric, as discussed in Section 7. We therefore need to consider the subleading contributions of these terms to the ten-dimensional metric.

We focus on the $\mathcal{O}(y^2)$ piece of the metric components for $d\phi_1^2$, $2d\phi_1 d\phi_2$ and $2d\phi_1 d\phi_3$ from the six-dimensional base, and refer to these as a subspace of the ten-dimensional metric. These components come from the subleading terms of $K = \frac{1}{4}y^4$ and the leading terms of $\delta K = ay^6$.

The terms originating from $K = \frac{1}{4}y^4$ are

$$
\begin{aligned}
ds_{10}^2\big|_{\text{subspace}} &= \frac{-2f_{yy}}{f_y^3}(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3)^2 \\
&+\frac{2}{f_y^2}(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3)(f_y d\phi_1 - f_{2y} r_2^2 d\phi_2 - f_{3y} r_3^2 d\phi_3) \\
&+\frac{f}{f_y^2}\left(f_y - \frac{f f_{yy}}{f_y}\right) d\phi_1^2 \\
&+2\frac{f r_2^2}{f_y^2}\left(\frac{f_2 f_{yy}}{f_y} - f_{2y}\right) d\phi_1 d\phi_2 + 2\frac{f r_3^2}{f_y^2}\left(\frac{f_3 f_{yy}}{f_y} - f_{3y}\right) d\phi_1 d\phi_3.
\end{aligned}
\tag{E.4}
$$

Next we calculate the terms from $\delta K = ay^6$. We define

$$
\widetilde{F} = \sqrt{2a}y^3 = \sqrt{2a}F^{3/2}, \qquad F \equiv y^2,
\tag{E.5}
$$

and we have

$$
\widetilde{F}_a = \sqrt{2a}\frac{3}{2}F^{1/2}F_a = \sqrt{2a}\frac{3}{2}yF_a.
\tag{E.6}
$$

Notice that we can use the general formula (7.10):

$$
ds_6^2 = 2\sum_{a=1}^{3}\left[\widetilde{F}_a^2 r_a^2 + \widetilde{F}(\widetilde{F}_{aa} r_a^2 + \widetilde{F}_a)\right]\left(dr_a^2 + r_a^2 d\phi_a^2\right) + 4\sum_{a<b}^{3}\left[\widetilde{F}_a\widetilde{F}_b + \widetilde{F}\widetilde{F}_{ab}\right] r_a^2 r_b^2\left(\frac{dr_a dr_b}{r_a r_b} + d\phi_a d\phi_b\right).
\tag{E.7}
$$

It is easy to see that the first terms in each of the two sums are of order $\mathcal{O}(y^2)$, while the second terms are of order $\mathcal{O}(y^4)$ or higher. So we only keep

$$
\begin{aligned}
ds_6^2 &= 2\sum_{a=1}^{3}\left[\widetilde{F}_a^2 r_a^2\right]\left(dr_a^2 + r_a^2 d\phi_a^2\right) + 4\sum_{a<b}^{3}\left[\widetilde{F}_a\widetilde{F}_b\right] r_a^2 r_b^2\left(\frac{dr_a dr_b}{r_a r_b} + d\phi_a d\phi_b\right) \\
&= 9ay^2\left\{\sum_{a=1}^{3}\left[F_a^2 r_a^2\right]\left(dr_a^2 + r_a^2 d\phi_a^2\right) + 2\sum_{a<b}^{3}\left[F_a F_b\right] r_a^2 r_b^2\left(\frac{dr_a dr_b}{r_a r_b} + d\phi_a d\phi_b\right)\right\}.
\end{aligned}
\tag{E.8}
$$

In other words, the subleading contribution from $\delta K = ay^6$ is actually $9ay^2$ times the leading order metric coming from $\frac{1}{4}y^4$.

Focusing on the subspace mentioned above, we find that the contribution from $\delta K = ay^6$ is:

$$
ds_{10}^2\big|_{\text{subspace}} = \frac{9a}{f_y^2}(f d\phi_1 - f_2 r_2^2 d\phi_2 - f_3 r_3^2 d\phi_3)^2.
\tag{E.9}
$$

The total contribution to the ten dimensional metric in this subspace is then given by:

$$
\begin{aligned}
ds^2_{10}\big|_{\text{subspace}} &= \left(\frac{-2f_{yy}}{f_y^3}+\frac{9a}{f_y^2}\right)(f\,d\phi_1 - f_2r_2^2d\phi_2 - f_3r_3^2d\phi_3)^2 \\
&\quad +\frac{2}{f_y^2}(f\,d\phi_1 - f_2r_2^2d\phi_2 - f_3r_3^2d\phi_3)(f_yd\phi_1 - f_{2y}r_2^2d\phi_2 - f_{3y}r_3^2d\phi_3) \\
&\quad +\frac{f}{f_y^2}\left(f_y - \frac{f f_{yy}}{f_y}\right)d\phi_1^2 \\
&\quad +2\frac{fr_2^2}{f_y^2}\left(\frac{f_2f_{yy}}{f_y}-f_{2y}\right)d\phi_1d\phi_2 + 2\frac{fr_3^2}{f_y^2}\left(\frac{f_3f_{yy}}{f_y}-f_{3y}\right)d\phi_1d\phi_3 \\
&= \left(\frac{-3f_{yy}}{f_y^3}+\frac{9a}{f_y^2}+\frac{f_y}{ff_y^2}\right)(f\,d\phi_1 - f_2r_2^2d\phi_2 - f_3r_3^2d\phi_3)^2 \\
&\quad +2\frac{f}{f_y^3}(f_yd\phi_1 - f_{2y}r_2^2d\phi_2 - f_{3y}r_3^2d\phi_3)^2 \\
&\quad +2\frac{1}{f_y^2}f_2f_{2y}r_2^4d\phi_2^2 + 2\frac{1}{f_y^2}f_3f_{3y}r_3^4d\phi_3^2 + 2\frac{1}{f_y^2}(f_2f_{3y}+f_3f_{2y})r_2^2r_3^2d\phi_2d\phi_3 \\
&\quad -\frac{1}{f_y^2}\left(\frac{f_y}{f}-\frac{f_{yy}}{f_y}\right)(f_2^2r_2^4d\phi_2^2 + f_3^2r_3^4d\phi_3{}^2 + 2f_2f_3r_2^2r_3^2d\phi_2d\phi_3) \\
&\quad -2\frac{f}{f_y^3}(f_{2y}^2r_2^4d\phi_2^2 + f_{3y}^2r_3^4d\phi_3^2 + 2f_{2y}f_{3y}r_2^2r_3^2d\phi_2d\phi_3),
\end{aligned}
\tag{E.10}
$$

where the $d\phi_2^2$, $d\phi_3^2$ and $d\phi_2d\phi_3$ terms in the last three lines will be combined into $d\widetilde{\Sigma}_4^2$.

Thus the ten dimensional metric near $y=0$ goes like

$$
\begin{aligned}
ds^2_{10} &= -\frac{2}{f_y}dt(f\,d\phi_1 - f_2r_2^2d\phi_2 - f_3r_3^2d\phi_3)\Big|_{y=0} \\
&\quad +\left(\frac{-3f_{yy}}{f_y^3}+\frac{9a}{f_y^2}+\frac{f_y}{ff_y^2}\right)(f\,d\phi_1 - f_2r_2^2d\phi_2 - f_3r_3^2d\phi_3)^2\Big|_{y=0} \\
&\quad +2\frac{f}{f_y^3}(f_yd\phi_1 - f_{2y}r_2^2d\phi_2 - f_{3y}r_3^2d\phi_3)^2\Big|_{y=0} + d\widetilde{\Sigma}_4{}^2(r_2,\phi_2,r_3,\phi_3) \\
&\quad +d\mathbb{R}_4{}^2.
\end{aligned}
\tag{E.11}
$$

We can also rewrite it in the form

$$
\begin{aligned}
ds^2_{10} &= -g_{tt}\Big|_{y=0}dt^2 + \left(\frac{f}{f_y}+\frac{9af^2}{f_y^2}-\frac{3f^2f_{yy}}{f_y^3}\right)\left(d\phi_1 - \frac{f_2r_2^2}{f}d\phi_2 - \frac{f_3r_3^2}{f}d\phi_3 - w_tdt\right)^2\Big|_{y=0} \\
&\quad +\frac{2f}{f_y}\left(d\phi_1 - \frac{f_{2y}r_2^2}{f_y}d\phi_2 - \frac{f_{3y}r_3^2}{f_y}d\phi_3\right)^2\Big|_{y=0} + d\widetilde{\Sigma}_4{}^2(r_2,\phi_2,r_3,\phi_3) \\
&\quad +d\mathbb{R}_4{}^2,
\end{aligned}
\tag{E.12}
$$

where

$$
\begin{aligned}
d\widetilde{\Sigma}_4{}^2(r_2,\phi_2,r_3,\phi_3) \;\; = \;\; & d\Sigma_4^2 + 2\frac{1}{f_y^2}f_2 f_{2y} r_2^4 d\phi_2^2 + 2\frac{1}{f_y^2}f_3 f_{3y} r_3^4 d\phi_3^2 + 2\frac{1}{f_y^2}(f_2 f_{3y} + f_3 f_{2y}) r_2^2 r_3^2 d\phi_2 d\phi_3 \\
& -\frac{1}{f_y^2}\left(\frac{f_y}{f} - \frac{f_{yy}}{f_y}\right)(f_2^2 r_2^4 d\phi_2^2 + f_3^2 r_3^4 d\phi_3^2 + 2 f_2 f_3 r_2^2 r_3^2 d\phi_2 d\phi_3) \\
& -2\frac{f}{f_y^3}(f_{2y}^2 r_2^4 d\phi_2^2 + f_{3y}^2 r_3^4 d\phi_3^2 + 2 f_{2y} f_{3y} r_2^2 r_3^2 d\phi_2 d\phi_3) \\
= \;\; & dr_2^2\,\frac{1}{f_y}\left(-f_2 - f_{22} r_2^2 + \frac{f_2^2 r_2^2}{f}\right) + dr_3^2\,\frac{1}{f_y}\left(-f_3 - f_{33} r_3^2 + \frac{f_3^2 r_3^2}{f}\right) \\
& +2 dr_2 dr_3\,\frac{1}{f_y} r_2 r_3\left(-f_{23} + \frac{f_2 f_3}{f}\right) \\
& +d\phi_2^2\,\frac{r_2^4}{f_y^2}\left(4 f_{2y} f_2 - \frac{f_2 f_y}{r_2^2} - f_y f_{22} - \frac{f_y f_2^2}{f} - \frac{2 f f_{2y}^2}{f_y}\right) \\
& +d\phi_3^2\,\frac{r_3^4}{f_y^2}\left(4 f_{3y} f_3 - \frac{f_3 f_y}{r_3^2} - f_y f_{33} - \frac{f_y f_3^2}{f} - \frac{2 f f_{3y}^2}{f_y}\right) \\
& +2 d\phi_2 d\phi_3\,\frac{r_2^2 r_3^2}{f_y}\Bigg(-f_{23} + f_{2y} f_3 + f_{3y} f_2 - f_2 f_3 f_{yy} + \frac{f_2 f_{3y}}{f_y} + \frac{f_3 f_{2y}}{f_y} \\
& \qquad\qquad - \frac{f_2 f_3}{f} + \frac{f_2 f_3 f_{yy}}{f_y^2} - \frac{2 f f_{2y} f_{3y}}{f_y^2}\Bigg). \qquad \text{(E.13)}
\end{aligned}
$$

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