MIFP-04-16

hep-th/0408143

August 2004

PP-waves in AdS Gauged Supergravities and Supernumerary Supersymmetry

J. Kerimo and H. Lü [1]

George P. & Cynthia W. Mitchell Institute for Fundamental Physics,

Texas A&M University, College Station, TX 77843-4242, USA

ABSTRACT

Purely gravitational pp-waves in AdS backgrounds are described by the generalised Kaigorodov metrics, and they generically preserve $\frac{1}{4}$ of the maximum supersymmetry allowed by the AdS spacetimes. We obtain $\frac{1}{2}$ supersymmetric purely gravitational pp-wave solutions, in which the Kaigorodov component is set to zero. We construct pp-waves in AdS gauged supergravities supported by a vector field. We find that the solutions in $D = 4$ and $D = 5$ can then preserve $\frac{1}{2}$ of the supersymmetry. Like those in ungauged supergravities, the supernumerary supersymmetry imposes additional constraints on the harmonic function associated with the pp-waves. These new backgrounds provide interesting novel features of the supersymmetry enhancement for the dual conformal field theory in the infinite-momentum frame.

[1] Research supported in part by DOE grant DE-FG03-95ER40917.

1 Introduction

Maximally supersymmetric pp-waves of type IIB [1] and M-theory [2] provide simple but non-trivial backgrounds for studying [3] the AdS/CFT correspondence [4, 5, 6] since string theory on such a background becomes massive and exact solvable [7]. These solutions can also be obtained from the Penrose limit of AdS×sphere backgrounds of the corresponding theories, and thus are supported by form field strengths instead of being purely gravitational. A large class of pp-waves supported by field strengths in M-theory and type-IIB supergravities were studied in [8, 9, 10]. These solutions in general have 16 "standard" Killing spinors, that is half of the maximum supersymmetry. For appropriate choices of field strengths and integration constants, supernumerary Killing spinors beyond the 16 standard ones could also arise [8, 9, 10, 11, 12, 13]. These include all of those from the Penrose limit of AdS×sphere arising from non-dilatonic p-branes and/or intersecting p-branes, and of AdS×sphere×sphere, arising from non-standard brane intersections [14].

It is natural to study the pp-waves in AdS background. The effect of introducing a pp-wave can be viewed as performing an infinite boost on the boundary conformal field theory [15, 16]. The purely gravitational pp-wave in AdS_4 was constructed in the 1960s and has been known as the Kaigorodov metric [17]. Higher dimensional generalisations, namely, the purely gravitational pp-waves in higher AdS spacetimes, were obtained in [15]. These metrics are homogeneous and preserve $\frac{1}{4}$ of the supersymmetry, consisting with the fact that in the dual conformal field theory, the original supersymmetry as well as the superconformal symmetry, are broken by the boost [15]. The Kaigorodov metrics can also be generalised to a class of inhomogeneous solutions, obtained in [18, 19, 20].

In this paper, we construct pp-waves in AdS gauged supergravities that are not only purely gravitational like Kaigorodov metrics but are also those supported in addition by a field strength of the theories as well. We show by explicit construction that supernumerary supersymmetry can arise with appropriately chosen field strength and integration constants in $D = 4$ and $D = 5$. The new solutions preserve $\frac{1}{2}$ of the supersymmetry, double the number of standard Killing spinors associated with the general pp-wave solutions including the Kaigorodov metric. In fact we show that even in the case of pure gravitational pp-waves, supernumerary supersymmetry can arise, extending the result of [16], where only $\frac{1}{4}$ supersymmetric purely gravitational pp-waves were discussed. The arising of the supernumerary supersymmetry provides novel features of supersymmetry enhancement for the dual conformal field theories in the infinite-momentum frame.

The paper is organised as follows: In section 2, we discuss the supersymmetry of the purely gravitational pp-waves in the Einstein theory with a cosmological constant in arbitrary dimensions. We show that $\frac{1}{2}$ supersymmetric solutions can arise. In section 3, we obtain pp-waves in $D=4$, $\mathcal{N}=2$ Einstein-Maxwell gauged supergravity supported by the Maxwell field. We obtain explicit supernumerary Killing spinors as well as the standard Killing spinors. In section 4, we obtain the analogous solutions in $D=5$, $\mathcal{N}=2$ Einstein-Maxwell gauged supergravity. We show that supernumerary Killing spinors also arise in this case. In section 5, we obtain $U(1)$-charged pp-wave solutions to minimal AdS gauged supergravities in $D=6$ and $D=7$. In these two cases, the solutions have only standard Killing spinors but no supernumerary ones. We conclude our paper in section 5. In appendix A, we uplift some of our solutions to M/string theories. In appendix B, we present a general class of pp-waves supported by an n-form field strength in a D-dimensional AdS gravity theory.

2 Purely gravitational pp-waves

In this section, we consider purely gravitational pp-waves in Einstein gravity with a negative cosmological constant in arbitrary dimensions. The Lagrangian is given by

$$e^{-1}\mathcal{L} = R + (D-1)(D-2)g^2, \tag{1}$$

where $e = \big(-\det(g_{MN})\big)^{1/2}$. The Killing spinor in this theory satisfies the equation

$$\nabla_M \epsilon = -\tfrac{1}{2} g\, \Gamma_M \epsilon\,. \tag{2}$$

We study AdS pp-waves using the metric ansatz

$$ds_D^2 = e^{2g\rho}(-4dx^+ dx^- + H(dx^+)^2 + dz_i\, dz_i) + d\rho^2, \tag{3}$$

where the function H depends on x^+, ρ and z_i coordinates. The Einstein equations of motion reduce to

$$\Box H \equiv \Big(\partial_\rho^2 + g(D-1)\partial_\rho + e^{2g\rho}\sum_{i=1}^{D-3}\partial_i^2\Big) H = 0\,, \tag{4}$$

where the index i stands here for z_i. To discuss the Killing spinor equations, we make a natural choice for the vielbein basis

$$e^+ = e^{g\rho} dx^+\,, \quad e^- = e^{g\rho}(-2dx^- + \tfrac{1}{2} H\, dx^+)\,, \quad e^i = e^{g\rho} dz^i\,, \quad e^\rho = d\rho\,, \tag{5}$$

such that we have $ds^2 = 2e^+e^- + e^z e^z + e^\rho e^\rho$. In this tangent basis, the spin connections are given by

$$\omega_{-\rho} = g\, e^+\,, \quad \omega_{+i} = \tfrac{1}{2} e^{-g\rho}\, \partial_i H\, e^+\,, \quad \omega_{+\rho} = g\, e^- + \tfrac{1}{2} H'\, e^+\,, \quad \omega_{i\rho} = g\, e^i\,, \tag{6}$$

where the prime denotes the derivative ∂_ρ. Note that for the metric in this basis we have $\eta_{+-} = 1$ and $\eta_{++} = \eta_{--} = 0$. In the following we use the notation that all derivatives are with respect to the curved metric and all indices on gamma matrices are vielbein indices. The Killing spinor equations are given by

$$\begin{aligned}
&[\partial_+ + \tfrac{1}{2} g e^{g\rho}\, \Gamma_+(\Gamma_\rho + 1) + \tfrac{1}{4} g e^{g\rho} H\, \Gamma_-(\Gamma_\rho + 1) + \tfrac{1}{4} e^{g\rho} H'\, \Gamma_{-\rho} + \tfrac{1}{4} \sum_i^{D-3} \partial_i H\, \Gamma_{-i}]\epsilon = 0\,,\\
&[\partial_- - g e^{g\rho}\, \Gamma_-(\Gamma_\rho + 1)]\epsilon = 0\,,\\
&[\partial_i + \tfrac{1}{2} g e^{g\rho}\, \Gamma_i(\Gamma_\rho + 1)]\epsilon = 0\,, \qquad i = 1, 2, \cdots, D-3,\\
&[\partial_\rho + \tfrac{1}{2} g\, \Gamma_\rho]\epsilon = 0\,,
\end{aligned} \tag{7}$$

where we have $\Gamma_+^2 = \Gamma_-^2 = 0$ and $\{\Gamma_+, \Gamma_-\} = 2$. Thus, we see that a generic pp-wave in a pure Einstein theory with a cosmological constant preserves $\frac{1}{4}$ of the maximally allowed supersymmetry. The projections are given by

$$(\Gamma_\rho + 1)\epsilon = 0 = \Gamma_-\epsilon\,. \tag{8}$$

We are interested in finding solutions that preserve more supersymmetry. One might expect that it would be helpful in this case first to analyse the integrability conditions $[\partial_M\,, \partial_N]\epsilon = 0$ among the Killing spinor equations. This calculation yields

$$\begin{aligned}
0 &= [\partial_+\,, \partial_i]\epsilon = -\tfrac{1}{4}\big[g e^{2g\rho} H'\, \Gamma_i + e^{g\rho} \partial_i H'\, \Gamma_\rho + \sum_j \partial_j \partial_i H\, \Gamma_j\big]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,, \partial_\rho]\epsilon = -\tfrac{1}{4}\big[e^{g\rho}(H'' + 2gH')\Gamma_\rho + \sum_i \partial_i H'\, \Gamma_i\big]\Gamma_-\epsilon\,.
\end{aligned} \tag{9}$$

The integrability conditions are satisfied provided that $\Gamma_-\epsilon = 0$. This is an example where integrability conditions are not enough for the existence of the Killing spinors.

To see whether the metrics can admit more supersymmetry than the $\frac{1}{4}$, let us use the less restrictive projection condition

$$g(\Gamma_\rho + 1)\epsilon = \mathrm{i} f\, \Gamma_-\epsilon\,, \tag{10}$$

where $f = f(x^+, \rho, z_i)$ is to be determined. Substituting this projection into the Killing spinor equations, we have

$$\begin{aligned}
&\Big[\partial_+ + \tfrac{\mathrm{i}}{2} e^{g\rho} f\, \Gamma_+ \Gamma_- - \tfrac{1}{4}\Big(e^{g\rho} H' + \sum_i \Gamma_i \partial_i H\Big)\Gamma_-\Big]\epsilon = 0\,,\\
&\partial_-\,\epsilon = 0\,, \qquad [\partial_i + \tfrac{\mathrm{i}}{2} e^{g\rho} f\, \Gamma_i\, \Gamma_-]\,\epsilon = 0\,, \qquad \big[\partial_\rho + \tfrac{1}{2} f\, \Gamma_- - \tfrac{1}{2} g\big]\epsilon = 0\,.
\end{aligned} \tag{11}$$

The integrability conditions $[\partial_M\,,\partial_N]\epsilon=0$ among these equations are

$$\begin{aligned}
0 &= [\partial_i\,,\partial_j]\epsilon = -\tfrac{\mathrm{i}}{2}e^{g\rho}(\Gamma_j\,\partial_i f-\Gamma_i\,\partial_j f)\Gamma_-\epsilon\,,\\
0 &= [\partial_i\,,\partial_\rho]\epsilon = \tfrac{\mathrm{i}}{2}[(e^{g\rho}f)'\,\Gamma_i-\partial_i f]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_i]\epsilon = -\tfrac12\Big[\mathrm{i}e^{g\rho}(\Gamma_i\,\partial_+ f-\Gamma_+\,\partial_i f)+e^{2g\rho}f^2\,\Gamma_i+\tfrac12 e^{g\rho}\partial_i H'\\
&\qquad +\tfrac12\sum_{j=1}^{D-3}\Gamma_j\partial_j\partial_i H\Big]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_\rho]\epsilon = -\tfrac12\Big[\mathrm{i}\partial_+ f+e^{g\rho}f^2-\mathrm{i}(e^{g\rho}f)'\,\Gamma_+ +\tfrac12\sum_i\Gamma_i\partial_i H'\\
&\qquad +\tfrac12 e^{g\rho}(H''+gH')\Big]\Gamma_-\epsilon\,.
\end{aligned}\tag{12}$$

From these integrability conditions we see that if we insist on more supersymmetry than the usual $\frac14$ we must set

$$\partial_i f=0=\partial_i H'\qquad\text{and}\qquad\partial_i\partial_j H=0\,,\qquad i\neq j\,.\tag{13}$$

We then have

$$(e^{g\rho}f)' = 0\,,\tag{14}$$

$$\mathrm{i}\partial_+ f+e^{g\rho}f^2+\tfrac12 e^{-g\rho}\partial_i\partial_i H = 0\,,\qquad i=1,2,\cdots,D-3\,,\tag{15}$$

$$\mathrm{i}\partial_+ f+e^{g\rho}f^2+\tfrac12 e^{g\rho}(H''+gH') = 0\,.\tag{16}$$

The conditions in (13), together with (4), imply that H is given by

$$H=\tfrac12\sum_{i=1}^{D-3}c_i z_i^2+\frac{e^{-2g\rho}}{2g^2(D-3)}\sum_{i=1}^{D-3}c_i+b\,e^{-(D-1)g\rho}\,,\tag{17}$$

where c_i and b are functions depending on x^+ only. Equation (15) implies that all c_i's are equal, and hence we let $c_i=c(x^+)$. From eqs.(14) and (16) it follows that we must set $b=0$. It is straightforward to solve for f, given by

$$f=e^{-g\rho}U(x^+)\,,\tag{18}$$

where U satisfies the following first-order non-linear equation

$$\mathrm{i}\frac{dU}{dx^+}+U^2+\tfrac12 c=0\,.\tag{19}$$

Making use of eq.(19) together with the solutions for f and H we can now solve the Killing spinor equations given in (11). The Killing spinor solution is

$$\begin{aligned}
\epsilon &= e^{\frac12 g\rho}\Big(1-\tfrac{\mathrm{i}}{2}U\sum_{i=1}^{D-3}z_i\,\Gamma_i\,\Gamma_-\Big)\Big(1+\tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-\Big)\\
&\qquad\times\Big[1-\tfrac12\Big(1-e^{-\mathrm{i}\int U dx^+}\Big)\Gamma_+\,\Gamma_-\Big]\epsilon_0\,,
\end{aligned}\tag{20}$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$. Thus, the metric preserves $\frac{1}{2}$ of the supersymmetry. It is important that the final result of our Killing spinors (20) satisfy the projection condition (10), which can be easily verified to be true.

Note that the special case of $c = 0$, $b \neq 0$ is the Kaigorodov metric. The above analysis implies that it preserves $\frac{1}{4}$ of the supersymmetry. In order to have $\frac{1}{2}$ BPS solutions, it is necessary to set the Kaigorodov component to zero.

Note that in general c is any function depending on x^+. The simplest case is that c is a constant. The x^+ dependence of c has no effect on the existence of the Killing spinors, but only modifies the explicit Killing spinor solutions.

3 PP-waves in $D = 4$ gauged supergravity

3.1 The solution

In this section we continue our investigations of supernumerary supersymmetry by including a U(1) charge. We start with gauged $\mathcal{N} = 2$ Einstein-Maxwell AdS supergravity, whose Lagrangian for the bosonic sector is given by

$$e^{-1}\mathcal{L}_4 = R - \tfrac{1}{4}F_{(2)}^2 + 6g^2, \tag{21}$$

where $F_{(2)} = dA_{(1)}$. The supersymmetry transformation rule for the complex gravitino $\Psi_M = \Psi_M^1 + \mathrm{i}\Psi_M^2$ is [21, 22]

$$\delta\Psi_M = \Big[\nabla_M - \tfrac{\mathrm{i}}{2}gA_M + \tfrac{\mathrm{i}}{8}F_{AB}\,\Gamma^{AB}\,\Gamma_M + \tfrac{1}{2}g\,\Gamma_M\Big]\epsilon\,. \tag{22}$$

We consider the following pp-wave ansatz

$$\begin{aligned} ds^2 &= e^{2g\rho}(-4dx^+\,dx^- + H(dx^+)^2 + dz^2) + d\rho^2,\\ A_{(1)} &= g^{-1}S(1 - e^{-g\rho})\,dx^+, \end{aligned} \tag{23}$$

where $H = H(x^+, \rho, z)$ and S is here a function of x^+. The equations of motion imply that H satisfies

$$\Box H \equiv H'' + 3g\,H' + e^{-2g\rho}\partial_z^2 H = -S^2 e^{-4g\rho}\,. \tag{24}$$

The solution can be expressed as

$$H = S^2\Big(\tfrac{1}{2}c\,z^2 + g^{-2}\,(\tfrac{1}{2}c\,e^{-2g\rho} - \tfrac{1}{4}e^{-4g\rho} + b\,e^{-3g\rho})\Big) + H_0\,, \tag{25}$$

where b and c are functions of x^+ and H_0 satisfies $\Box H_0 = 0$. (Note that the terms associated with b and c actually belong to H_0. We extract them since they are necessary for the solution to reduce under $g \to 0$ to the pp-wave that is the Penrose limit of $\mathrm{AdS}_2 \times S^2$ of the corresponding ungauged theory.)

If we turn off the field strength by setting $S = 0$, and let H_0 depend only on ρ, namely $H_0 = c_0 + b\, e^{-3g\rho}$, then we recover the Kaigorodov metric.

3.2 Standard supersymmetry

Here we investigate the supersymmetry of the "charged" pp-wave we derived. The Killing spinor equations in this background are given by

$$
\begin{aligned}
&[\partial_+ + \tfrac{1}{2} g e^{g\rho}\, \Gamma_+(\Gamma_\rho + 1) + \tfrac{1}{4} g e^{g\rho} H\, \Gamma_-(\Gamma_\rho + 1) + \tfrac{1}{4}\Gamma_{-z}\partial_z H + \tfrac{1}{4} e^{g\rho} H'\, \Gamma_{-\rho} \\
&\qquad + \tfrac{\mathrm{i}}{2} S(e^{-g\rho} - 1) + \tfrac{\mathrm{i}}{4} e^{-g\rho} S\, \Gamma_\rho \Gamma_- \Gamma_+]\epsilon = 0\,, \\
&[\partial_- - g e^{g\rho}\, \Gamma_-(\Gamma_\rho + 1)]\epsilon = 0\,, \\
&[\partial_z + \tfrac{1}{2} g e^{g\rho}\, \Gamma_z(\Gamma_\rho + 1) + \tfrac{\mathrm{i}}{4} e^{-g\rho} S\, \Gamma_{z\rho}\Gamma_-]\epsilon = 0\,, \\
&[\partial_\rho - \tfrac{\mathrm{i}}{4} e^{-2g\rho} S\, \Gamma_- + \tfrac{1}{2} g\, \Gamma_\rho]\epsilon = 0\,.
\end{aligned} \tag{26}
$$

Imposing the following projections

$$
(\Gamma_\rho + 1)\epsilon = 0\,, \qquad \Gamma_-\epsilon = 0\,, \tag{27}
$$

the Killing spinor equations become

$$
[\partial_+ - \tfrac{\mathrm{i}}{2} S]\epsilon = 0\,, \qquad \partial_-\, \epsilon = 0\,, \qquad \partial_z\, \epsilon = 0\,, \qquad [\partial_\rho - \tfrac{1}{2} g]\epsilon = 0\,. \tag{28}
$$

Thus the Killing spinor is given by

$$
\epsilon = e^{\frac{1}{2} g\rho + \frac{\mathrm{i}}{2}\int S\, dx^+}\, \epsilon_0\,, \tag{29}
$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$ and $\Gamma_-\epsilon_0 = 0$. The solution therefore preserves $\frac{1}{4}$ of the supersymmetry. We follow the literature [8, 9] and call these spinors the standard Killing spinors, since there is no further requirement on the function H for the existence of the ϵ, as long as H satisfies the equation of motion (24).

3.3 Supernumerary supersymmetry

When the integration constants of H satisfy further conditions, there can arise additional Killing spinors, which are called supernumerary Killing spinors in [8, 9]. In order to obtain these Killing spinors, we consider the integrability conditions $[\partial_{M}\,,\partial_{N}]\epsilon=0$. We find that

$$
\begin{aligned}
0 &= [\partial_z\,,\ \partial_\rho]\epsilon=\tfrac{\mathrm{i}}{4}ge^{-g\rho}S\,\Gamma_z\,\Gamma_-(\Gamma_\rho+1)\epsilon\,,\\
0 &= [\partial_+\,,\partial_-]\epsilon=-\tfrac{\mathrm{i}}{2}gS\,\Gamma_-(\Gamma_\rho+1)\epsilon\,,\\
0 &= [\partial_+\,,\partial_z]\epsilon=\tfrac{\mathrm{i}}{4}gS(3-2\Gamma_+\Gamma_-)\Gamma_z(\Gamma_\rho+1)\epsilon-\tfrac{\mathrm{i}}{4}e^{-g\rho}\partial_+S\,\Gamma_{z\rho}\,\Gamma_-\epsilon\\
&\quad -\tfrac{1}{4}e^{g\rho}\partial_zH'\,\Gamma_\rho\,\Gamma_-\epsilon-\tfrac{1}{4}\big[ge^{2g\rho}H'+\partial_z^2H+\tfrac{1}{2}e^{-2g\rho}S^2\big]\Gamma_z\,\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_\rho]\epsilon=-\tfrac{\mathrm{i}}{4}ge^{-g\rho}S(3-\Gamma_+\Gamma_-)(\Gamma_\rho+1)\epsilon+\tfrac{\mathrm{i}}{4}e^{-2g\rho}\partial_+S\,\Gamma_-\epsilon\\
&\qquad -\tfrac{1}{4}\partial_zH'\,\Gamma_z\,\Gamma_-\epsilon+\tfrac{1}{4}e^{g\rho}\big[gH'+e^{-2g\rho}\partial_z^2H+\tfrac{1}{2}e^{-4g\rho}S^2\big]\Gamma_\rho\,\Gamma_-\epsilon\,. \qquad (30)
\end{aligned}
$$

To arrive at the last integrability condition we made use of equation (24) for H. It is clear that the integrability conditions are satisfied with the projections given in (27). However, we now show that it is possible to relax these projections. We find that the integrability conditions can also be satisfied, with the following less restrictive projection

$$
g(\Gamma_\rho+1)\epsilon=\mathrm{i}f\,\Gamma_-\epsilon\,, \qquad (31)
$$

where $f=f(x^+,\rho,z)$. This gives the projected Killing spinor equations

$$
\begin{aligned}
&[\partial_+-\tfrac{\mathrm{i}}{2}S-\tfrac{1}{2}g^{-1}e^{-g\rho}fS\,\Gamma_-+\tfrac{\mathrm{i}}{2}(e^{g\rho}f+\tfrac{1}{2}e^{-g\rho}S)\Gamma_+\Gamma_-\\
&\quad -\tfrac{1}{4}(e^{g\rho}H'+\Gamma_z\partial_zH)\Gamma_-]\epsilon=0\,,\qquad \partial_-\,\epsilon=0\,,\\
&[\partial_z+\tfrac{\mathrm{i}}{2}(e^{g\rho}f+\tfrac{1}{2}e^{-g\rho}S)\Gamma_z\,\Gamma_-]\epsilon=0\,,\\
&[\partial_\rho+\tfrac{\mathrm{i}}{2}(f-\tfrac{1}{2}e^{-2g\rho}S)\Gamma_--\tfrac{1}{2}g]\epsilon=0\,. \qquad (32)
\end{aligned}
$$

The integrability conditions among these equations are

$$
\begin{aligned}
0 &= [\partial_z\,,\partial_\rho]\epsilon=-\tfrac{\mathrm{i}}{2}\big[\Gamma_z\partial_zf-(e^{g\rho}f)'+\tfrac{1}{2}ge^{-g\rho}S\big]\Gamma_z\,\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_z]\epsilon=-\tfrac{1}{2}\big[\mathrm{i}(e^{g\rho}\partial_+f+\tfrac{1}{2}e^{-g\rho}\partial_+S)\Gamma_z-(\mathrm{i}e^{g\rho}\,\Gamma_+-g^{-1}e^{-g\rho}S)\partial_zf\\
&\quad +(e^{g\rho}f+\tfrac{1}{2}e^{-g\rho}S)^2\,\Gamma_z+\tfrac{1}{2}(e^{g\rho}\partial_zH'+\Gamma_z\partial_z^2H)\big]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_\rho]\epsilon=-\tfrac{1}{2}\big[\mathrm{i}(\partial_+f-\tfrac{1}{2}e^{-2g\rho}\partial_+S)+g^{-1}S(e^{-g\rho}f)'\\
&\quad -\mathrm{i}\big((e^{g\rho}f)'-\tfrac{1}{2}ge^{-g\rho}S\big)\Gamma_++\tfrac{1}{2}e^{g\rho}(H''+gH')+\tfrac{1}{2}\Gamma_z\partial_zH'\\
&\quad +e^{g\rho}(f^2-\tfrac{1}{4}e^{-4g\rho}S^2)\big]\Gamma_-\epsilon\,. \qquad (33)
\end{aligned}
$$

It is clear from these expressions that if we want more supersymmetry than $\frac{1}{4}$ we need again to impose $\partial_zf=0=\partial_zH'$. The vanishing of the integrability conditions in this case then yields the equations

$$
\begin{aligned}
&(e^{g\rho}f)' - \tfrac{1}{2}ge^{-g\rho}S = 0\,,\\
&\mathrm{i}(\partial_+ f + \tfrac{1}{2}e^{-2g\rho}\partial_+ S) + \tfrac{1}{2}e^{-g\rho}\partial_z^2 H + e^{g\rho}(f + \tfrac{1}{2}e^{-2g\rho}S)^2 = 0\,, \qquad (34)\\
&\mathrm{i}(\partial_+ f - \tfrac{1}{2}e^{-2g\rho}\partial_+ S) + g^{-1}S(e^{-g\rho}f)' + \tfrac{1}{2}e^{g\rho}(H'' + gH') + e^{g\rho}f^2 - \tfrac{1}{4}e^{-3g\rho}S^2 = 0\,.
\end{aligned}
$$

From the first of eqs.(35) we obtain

$$
f = -\tfrac{1}{2}e^{-2g\rho}S + e^{-g\rho}U\,, \qquad (35)
$$

where $U = U(x^+)$ is in general a complex function. Note that S is a real function. Using the solution for f and the equation for H the remaining two equations in (35) gives

$$
\begin{aligned}
&\mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}\partial_z^2 H = 0\,,\\
&\mathrm{i}\frac{dS}{dx^+} - e^{-g\rho}S^2 + 3S\,U + ge^{3g\rho}H' + e^{g\rho}\partial_z^2 H = 0\,. \qquad (36)
\end{aligned}
$$

Since the functions S and U depend only on x^+ we need to check that the ρ dependence in the equation for S drops out before we can proceed. For this we need to make use of the solution for H, which is given by (25). Setting $H_0 = 0$, and substituting H into eqs.(36) we have[1]

$$
\mathrm{i}\frac{dS}{dx^+} - 3S(b\,S - U) = 0\,, \qquad \mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}c\,S^2 = 0\,. \qquad (37)
$$

In order to solve these equations we rewrite U into an real and imaginary part $U = u + \mathrm{i}v$. Eqs.(37) then yield the following set of equations:

$$
\begin{aligned}
&\frac{dS}{dx^+} + 3v\,S = 0\,, \qquad \frac{du}{dx^+} + 2u\,v = 0\,, \qquad S(u - b\,S) = 0\,,\\
&\frac{dv}{dx^+} + v^2 - u^2 - \tfrac{1}{2}c\,S^2 = 0\,. \qquad (38)
\end{aligned}
$$

We have four equations for the five functions S, u, v, b and c, and so one function will be left arbitrary. We present the solution to eqs.(38) in terms of the function b. The solution is given by

$$
\begin{aligned}
S &= \frac{k}{b^3}\,, \qquad u = \frac{k}{b^2}\,, \qquad v = b^{-1}\frac{db}{dx^+}\,,\\
c &= \frac{2b^5}{k^2}\Big[\frac{d^2b}{dx^{+2}} - \frac{k^2}{b^3}\Big]\,, \qquad (39)
\end{aligned}
$$

where k is an arbitrary constant and we have taken $S \neq 0$. (The case with $S = 0$ was considered in section 2.) Note that the original generic $\frac{1}{4}$ supersymmetric solution depending

[1] It is straightforward to verify that in general supernumeary supersymmetry requires that H_0 be given by (17), which is not the most general solution for $\Box H_0 = 0$.

on the three functions b, c and S now only have one independent function in order for the solution to have the enhanced $\frac{1}{2}$ supersymmetry.

We next turn to presenting the explicit Killing spinors. The Killing spinor equations are

$$\begin{aligned}
&[\partial_+ - \tfrac{\mathrm{i}}{2}S - \tfrac{1}{2}g^{-1}e^{-g\rho}f\,S\,\Gamma_- + \tfrac{\mathrm{i}}{2}U\,\Gamma_+\,\Gamma_- - \tfrac{1}{4}(e^{g\rho}H' + c\,z\,S^2\,\Gamma_z)\Gamma_-]\epsilon = 0\,,\\
&\partial_-\,\epsilon = 0\,, \qquad [\partial_z + \tfrac{\mathrm{i}}{2}U\,\Gamma_z\,\Gamma_-]\epsilon = 0\,, \qquad [\partial_\rho - \tfrac{\mathrm{i}}{2}g^{-1}f'\,\Gamma_- - \tfrac{1}{2}g]\epsilon = 0\,,
\end{aligned} \tag{40}$$

where f is given by (35). The third equation of the above implies $\epsilon = (1 - \frac{\mathrm{i}}{2}z\,U\,\Gamma_z\,\Gamma_-) \times \chi(\rho, x^+)$. Substituting this into the fourth equation yields the solution $\chi = e^{\frac{1}{2}g\rho}(1 + \frac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\eta(x^+)$. The equation for η can be obtained from the first equation of (40) after making use of eqs.(37). We have

$$\frac{d\eta}{dx^+} - \tfrac{\mathrm{i}}{2}[S - U\,\Gamma_+\,\Gamma_-]\eta = 0\,. \tag{41}$$

Note that it requires conspiracy for the z and ρ dependent terms to drop out. Finally, we arrive at the Killing spinor, given by

$$\begin{aligned}
\epsilon &= e^{\frac{1}{2}g\rho + \frac{\mathrm{i}}{2}\int S\,dx^+}(1 - \tfrac{\mathrm{i}}{2}z\,U\,\Gamma_z\,\Gamma_-)(1 + \tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\\
&\quad \times\Big[1 - \tfrac{1}{2}(1 - e^{-\mathrm{i}\int U dx^+})\Gamma_+\,\Gamma_-\Big]\epsilon_0\,,
\end{aligned} \tag{42}$$

where ϵ_0 is a constant spinor, satisfying the projection

$$(\Gamma_\rho + 1)\epsilon_0 = 0\,. \tag{43}$$

There are two special cases that are worth considering. The first case is that b is set to a constant, implying that $v = 0$. It follows then that the functions S and u are constants as well, and $c = -2b^2$. Assuming $S = \mu$ the Killing spinor in this case is given by

$$\begin{aligned}
\epsilon &= e^{\frac{1}{2}g\rho + \frac{\mathrm{i}}{2}\mu x^+}(1 - \tfrac{\mathrm{i}}{2}\mu b\,z\,\Gamma_z\,\Gamma_-)(1 + \tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\\
&\quad \times\Big[1 - \tfrac{1}{2}(1 - e^{-\mathrm{i}\,\mu b x^+})\Gamma_+\,\Gamma_-\Big]\epsilon_0\,,
\end{aligned} \tag{44}$$

where ϵ_0 is a constant spinor, satisfying the projection $(\Gamma_\rho + 1)\epsilon_0 = 0$. Thus after imposing the condition $c = -2b^2$, the solution has $\frac{1}{2}$ of the supersymmetry instead of the $\frac{1}{4}$ for a generic pp-wave solution. The standard Killing spinors are those with an additional projection $\Gamma_-\epsilon_0 = 0$, in which case, ϵ of (44) becomes that in (29). The supernumerary Killing spinors are the remaining half with $\Gamma_-\epsilon_0 \neq 0$.

The function H, for the pp-wave with supernumerary supersymmetry, is given by

$$\begin{aligned}
H &= -\mu^2\,b^2 z^2 - g^{-2}f^2 = -\mu^2\Big(b^2 z^2 + g^{-2}(b^2\,e^{-2g\rho} + \tfrac{1}{4}e^{-4g\rho} - b\,e^{-3g\rho})\Big)\,,\\
f &= -\tfrac{1}{2}\mu(e^{-2g\rho} - 2b\,e^{-g\rho})\,.
\end{aligned} \tag{45}$$

If we set $b = \frac{1}{2}$, we have $H = -\mu^2 \,[\frac{1}{4} z^2 + g^{-2} \sinh^2(\frac{1}{2} g\rho)\, e^{-3g\rho}]$. We can then take the $g \to 0$ limit and obtain a pp-wave in ungauged $D = 4$, $\mathcal{N} = 2$ Einstein Maxwell supergravity. The solution is given by

$$ds^2 = -4dx^+ \, dx^- - \tfrac{1}{4}\mu^2 (z^2 + \rho^2)\,(dx^+)^2 + dz^2 + d\rho^2, \tag{46}$$

$$F_{(2)} = -\mu \, dx^+ \wedge d\rho \,. \tag{47}$$

This is precisely the pp-wave arising from the Penrose limit of $\mathrm{AdS}_2 \times S^2$, which is known to have supernumerary supersymmetries [8, 9].

Note that in the ansatz (23), we could instead have used $A_{(1)} = \mu z \, dx^+$. The metric in this case is identical to that with $A_{(1)}$ given in (23). However, we verified that the solution would be non-supersymmetric, because of the explicit $A_{(1)}$ dependence in the supersymmetry transformation rule.

Charged pp-waves with $c = 0$ were also obtained in [23], by performing an infinite boost of the AdS charged black holes. It can be deduced from the above analysis that the solution with $c = 0$ has only the standard supersymmetry. We can also obtain pure gravitational $\frac{1}{2}$ supersymmetric pp-waves by setting $b = \tilde{b}/\mu$ and then sending $\mu \to 0$.

In [24] a general class of pp-waves that preserve $\frac{1}{4}$ of the supersymmetry were given. PP-waves with $\frac{1}{2}$ of the supersymmetry were also obtained in [25], where the Killing spinors were given in component language, whilst ours are presented in an elegant form, in terms of constant spinors satisfying a single gamma matrix projection.

The second special case corresponds to the absence of the Kaigorodov component b which can be achieved by taking a degenerate limit of (39). It is worth examing on its own. In this case we have the coupled system

$$\frac{dS}{dx^+} + 3v\, S = 0\,, \qquad \frac{dv}{dx^+} + v^2 - \tfrac{1}{2} c\, S^2 = 0\,. \tag{48}$$

This implies a relation between the metric functions c and S, given by

$$c = -\tfrac{2}{3} S^{-3} \frac{d^2 S}{dx^{+2}} + \tfrac{8}{9} S^{-4} \Big(\frac{dS}{dx^+}\Big)^2. \tag{49}$$

Making use of these equations together with the solutions for H and f the Killing spinor equations (40) yield the solution

$$\epsilon = e^{\frac{1}{2} g\rho}\, e^{\frac{\mathrm{i}}{2}\int S\, dx^+} (1 + \tfrac{1}{2} z\, v\, \Gamma_z\, \Gamma_-)(1 + \tfrac{\mathrm{i}}{2} g^{-1} f\, \Gamma_-) \times \Big[1 - \tfrac{1}{2}(1 - e^{\int v\, dx^+})\Gamma_+\, \Gamma_-\Big] \epsilon_0\,, \tag{50}$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$. For the functions H and f we have

$$
\begin{aligned}
H &= \tfrac{1}{2} S^2 \Big[c\, z^2 + \tfrac{1}{2} g^{-2} e^{-2g\rho}(2c - e^{-2g\rho}) \Big] , \\
f &= -\tfrac{1}{2} e^{-2g\rho} S + \mathrm{i} e^{-g\rho} v \,. \qquad (51)
\end{aligned}
$$

We can consider a special case of eqs.(48) by setting $c \equiv$ constant and $v = \tilde{k} S$ where $\tilde{k}$ is a (real) constant. In this case the equations fixes $\tilde{k}$ to $\tilde{k}^2 = -\frac{1}{4} c$ with $c < 0$. The equation for S is

$$
\frac{dS}{dx^+} + \tilde{k} S^2 = 0 \,, \qquad (52)
$$

with the solution given by $S(x^+) = 1/(1 + \tilde{k}\, x^+)$.

4 PP-waves in $D = 5$ gauged supergravity

4.1 The solution

For simplicity, we consider simple gauged supergravity in $D = 5$. The Lagrangian for the bosonic sector is given by [26]

$$
e^{-1} \mathcal{L}_5 = R - \tfrac{1}{4} F_{(2)}^2 + \tfrac{1}{12\sqrt{3}} \epsilon^{MNPQR} F_{MN} F_{PQ} A_R + 12 g^2 . \qquad (53)
$$

Analogous to the $D = 4$ discussion, we use the following pp-wave ansatz

$$
\begin{aligned}
ds^2 &= e^{2g\rho}(-4 dx^+\, dx^- + H (dx^+)^2 + dz_1^2 + dz_2^2) + d\rho^2 , \\
A_{(1)} &= \tfrac{1}{2} g^{-1} S (1 - e^{-2g\rho})\, dx^+ , \qquad (54)
\end{aligned}
$$

where $S = S(x^+)$. The supergravity equations of motion then reduce to the following

$$
\Box H \equiv H'' + 4 g H' + e^{-2g\rho} \sum_{i=1}^{2} \partial_i^2 H = -e^{-6g\rho} S^2 . \qquad (55)
$$

The solution is given by

$$
H = S^2 \big[\tfrac{1}{2}(c_1 z_1^2 + c_2 z_2^2) + g^{-2} (\tfrac{1}{4}(c_1 + c_2)\, e^{-2g\rho} - \tfrac{1}{12} e^{-6g\rho} + b\, e^{-4g\rho}) \big] + H_0 \,, \qquad (56)
$$

where c_i and b are functions of x^+ and $\Box H_0 = 0$. The generalised Kaigorodov-type metric is obtained by setting $S = 0$ and $H_0 = c_0 + b\, e^{-4g\rho}$ with c_0 and b now being constants.

4.2 Supersymmetry

The supersymmetry transformation on the gravitino is given by

$$
\delta\Psi_M = [\nabla_M - \tfrac{3\,\mathrm{i}}{2\sqrt{3}}\, g A_M - \tfrac{\mathrm{i}}{16\sqrt{3}} F_{AB} \,(\Gamma_M \,\Gamma^{AB} - 3\, \Gamma^{AB}\, \Gamma_M) + \tfrac{1}{2} g\, \Gamma_M]\epsilon \,, \qquad (57)
$$

where ϵ is a complex symplectic spinor. For our pp-wave background, the Killing spinor equations are given by

$$
\begin{aligned}
&[\partial_+ + \tfrac{1}{2} g e^{g\rho}\,\Gamma_+(\Gamma_\rho + 1) + \tfrac{1}{4} g e^{g\rho} H\,\Gamma_-(\Gamma_\rho + 1) + \tfrac{1}{4} e^{g\rho} H'\,\Gamma_{-\rho} \\
&\qquad + \tfrac{1}{4}\sum_{i=1}^{2} \Gamma_{-i}\partial_i H + \tfrac{3\,\mathrm{i}}{4\sqrt{3}} S(e^{-2g\rho} - 1) \\
&\qquad + \tfrac{\mathrm{i}}{8\sqrt{3}} e^{-2g\rho} S\,\Gamma_\rho(\Gamma_+\,\Gamma_- + 3\Gamma_-\,\Gamma_+)]\epsilon = 0\,, \\
&[\partial_- - g e^{g\rho}\,\Gamma_-(\Gamma_\rho + 1)]\epsilon = 0\,, \\
&[\partial_i + \tfrac{1}{2} g e^{g\rho}\,\Gamma_i(\Gamma_\rho + 1) + \tfrac{\mathrm{i}}{4\sqrt{3}} e^{-2g\rho} S\,\Gamma_{i\rho}\,\Gamma_-]\epsilon = 0\,, \qquad i = 1, 2, \\
&[\partial_\rho - \tfrac{\mathrm{i}}{2\sqrt{3}} e^{-3g\rho} S\,\Gamma_- + \tfrac{1}{2} g\,\Gamma_\rho]\epsilon = 0\,. \qquad (58)
\end{aligned}
$$

As in the case of $D = 4$, the standard Killing spinors, which exist for all H satisfying (55), arise with the following projections $(\Gamma_\rho + 1)\epsilon = 0$ and $\Gamma_-\epsilon = 0$. The Killing spinor equations become

$$
[\partial_+ - \mathrm{i}\tfrac{\sqrt{3}}{4} S]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \qquad \partial_i\,\epsilon = 0\,, \qquad [\partial_\rho - \tfrac{1}{2} g]\epsilon = 0\,. \qquad (59)
$$

Thus, the generic pp-waves we considered preserve $\frac{1}{4}$ of the standard supersymmetry. In [27], a general class of null solutions with $\frac{1}{4}$ of the supersymmetry were obtained, however, the issue of supernumerary supersymmetry was not addressed. We demonstrate below that, as in the case of $D = 4$, supernumerary Killing spinors can also arise.

To obtain the supernumerary Killing spinor and the corresponding conditions on H, we impose the following projection on the spinors

$$
g(\Gamma_\rho + 1)\epsilon = \mathrm{i} f\,\Gamma_-\epsilon\,. \qquad (60)
$$

The Killing spinor equations become

$$
\begin{aligned}
&\big[\partial_+ - \tfrac{3\mathrm{i}}{4\sqrt{3}} S + \tfrac{\mathrm{i}}{2}(e^{g\rho} f + \tfrac{1}{2\sqrt{3}} e^{-2g\rho} S)\Gamma_+\,\Gamma_- - \tfrac{1}{4}\sum_i \Gamma_i\,\Gamma_-\partial_i H \\
&\qquad - \tfrac{1}{4}(e^{g\rho} H' + \sqrt{3} g^{-1} e^{-2g\rho} f\,S)\Gamma_-\big]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \\
&[\partial_i + \tfrac{\mathrm{i}}{2}(e^{g\rho} f + \tfrac{1}{2\sqrt{3}} e^{-2g\rho} S)\Gamma_i\,\Gamma_-]\epsilon = 0\,, \\
&[\partial_\rho + \tfrac{\mathrm{i}}{2}(f - \tfrac{1}{\sqrt{3}} e^{-3g\rho} S)\Gamma_- - \tfrac{1}{2} g]\epsilon = 0\,. \qquad (61)
\end{aligned}
$$

The integrability conditions among these equations are

$$
\begin{aligned}
0 &= [\partial_i\,,\partial_\rho]\epsilon = -\tfrac{\mathrm{i}}{2}\big[\partial_i f - (e^{g\rho}f)'\,\Gamma_i + \tfrac{1}{\sqrt{3}}g e^{-2g\rho} S\,\Gamma_i\big]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_i]\epsilon = -\tfrac{1}{2}\big[\mathrm{i}(e^{g\rho}\partial_+ f + \tfrac{1}{2\sqrt{3}}e^{-2g\rho}\partial_+ S)\Gamma_i - (\mathrm{i}e^{g\rho}\,\Gamma_+ - \tfrac{3}{2\sqrt{3}}g^{-1}e^{-2g\rho}S)\partial_i f\\
&\quad +\tfrac{1}{2}e^{g\rho}\partial_i H' + \tfrac{1}{2}\sum_j \Gamma_j\partial_j\partial_i H + (e^{g\rho}f + \tfrac{1}{2\sqrt{3}}e^{-2g\rho}S)^2\,\Gamma_i\big]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_\rho]\epsilon = -\tfrac{1}{2}\big[\mathrm{i}(\partial_+ f - \tfrac{1}{\sqrt{3}}e^{-3g\rho}\partial_+ S) + \tfrac{3}{2\sqrt{3}}g^{-1}S(e^{-2g\rho}f)' + \tfrac{1}{2}\sum_i \Gamma_i\partial_i H'\\
&\quad -\mathrm{i}\big((e^{g\rho}f)' - \tfrac{1}{\sqrt{3}}g e^{-2g\rho}S\big)\Gamma_+ + \tfrac{1}{2}e^{g\rho}(H'' + gH')\\
&\quad +(f - \tfrac{1}{\sqrt{3}}e^{-3g\rho}S)(e^{g\rho}f + \tfrac{1}{2\sqrt{3}}e^{-2g\rho}S)\big]\Gamma_-\epsilon\,.
\end{aligned}
\tag{62}
$$

To have more supersymmetry than the $\frac{1}{4}$ we need to set

$$
\partial_i f = 0 = \partial_i H' \quad \text{and} \quad \partial_j\partial_i H = 0\,, \quad i \neq j\,. \tag{63}
$$

The integrability conditions then imply

$$
\begin{aligned}
&f = -\tfrac{1}{2\sqrt{3}}e^{-3g\rho}S + e^{-g\rho}U\,,\\
&\mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}\partial_i^2 H = 0\,, \qquad i = 1,2,\\
&\mathrm{i}\Big(\frac{dS}{dx^+} - \tfrac{2}{\sqrt{3}}e^{2g\rho}\frac{dU}{dx^+}\Big) - g^{-1}e^{3g\rho}S(e^{-2g\rho}f)' - \tfrac{1}{\sqrt{3}}e^{4g\rho}(H'' + gH')\\
&\qquad\qquad -\tfrac{2}{\sqrt{3}}e^{3g\rho}U(f - \tfrac{1}{\sqrt{3}}e^{-3g\rho}S) = 0\,,
\end{aligned}
\tag{64}
$$

where $U = U(x^+)$. Substituting in the solution for H, given by (56), we find that it is necessary to have that $c_1 = c_2 \equiv c$, and that H_0 is given by (17). For simplicity, we set $H_0 = 0$ here since the H_0 represents the pure gravitational component, which was discussed in section 2. The equations for S and U are then given by

$$
\mathrm{i}\frac{dS}{dx^+} - 4S(\sqrt{3}\,b\,S - U) = 0\,, \qquad \mathrm{i}\frac{dU}{dx^+} + U^2 + \tfrac{1}{2}c\,S^2 = 0\,. \tag{65}
$$

Substituting $U = u + \mathrm{i}v$ into the above yields the equations

$$
\begin{aligned}
&\frac{dS}{dx^+} + 4v\,S = 0\,, \qquad \frac{du}{dx^+} + 2u\,v = 0\,, \qquad S(u - \sqrt{3}\,b\,S) = 0\,,\\
&\frac{dv}{dx^+} + v^2 - u^2 - \tfrac{1}{2}c\,S^2 = 0\,.
\end{aligned}
\tag{66}
$$

The solution to these equations is

$$
\begin{aligned}
S &= \frac{k}{b^2}\,, \qquad u = \frac{\sqrt{3}\,k}{b}\,, \qquad v = \frac{1}{2b}\frac{db}{dx^+}\,,\\
c &= \frac{b^3}{k^2}\Big[\frac{d^2 b}{dx^{+2}} - \frac{1}{2b}\Big(\frac{db}{dx^+}\Big)^2\Big] - 6\,b^2,
\end{aligned}
\tag{67}
$$

where k is an arbitrary constant and we have taken $S \neq 0$. Note that as in the case of $D = 4$, the original generic $\frac{1}{4}$-supersymmetric metric depending on the four functions b, c_1, c_2 and S now only have one independent function in order for the solution to have the enhanced $\frac{1}{2}$ supersymmetry.

The Killing spinor is calculated from the equations

$$
\begin{aligned}
&[\partial_+ - \tfrac{3\,\mathrm{i}}{4\sqrt{3}} S - \tfrac{3}{4\sqrt{3}} g^{-1} e^{-2g\rho} f S\, \Gamma_- + \tfrac{\mathrm{i}}{2} U\, \Gamma_+ \Gamma_- \\
&\qquad - \tfrac{1}{4}\big(e^{g\rho} H' + c\, S^2 (z_1\, \Gamma_1 + z_2\, \Gamma_2)\big)\Gamma_-]\epsilon = 0\,, \\
&\partial_-\, \epsilon = 0\,, \qquad [\partial_i + \tfrac{\mathrm{i}}{2} U\, \Gamma_i\, \Gamma_-]\epsilon = 0\,, \qquad [\partial_\rho - \tfrac{\mathrm{i}}{2} g^{-1} f'\, \Gamma_- - \tfrac{1}{2} g]\epsilon = 0\,.
\end{aligned}
\tag{68}
$$

The solution is

$$
\begin{aligned}
\epsilon \;&=\; e^{\frac{1}{2}g\rho + \mathrm{i}\frac{\sqrt{3}}{4}\int S\, dx^+} \big(1 - \tfrac{\mathrm{i}}{2} U\, (z_1\, \Gamma_1 + z_2\, \Gamma_2)\Gamma_-\big)(1 + \tfrac{\mathrm{i}}{2} g^{-1} f\, \Gamma_-) \\
&\qquad \times \Big[1 - \tfrac{1}{2}(1 - e^{-\mathrm{i}\int U dx^+})\Gamma_+\, \Gamma_-\Big]\epsilon_0\,,
\end{aligned}
\tag{69}
$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$. As in $D = 4$ we consider two special cases. The first corresponds to $v = 0$, which implies that b, c and S are all constants, with $c = -6b^2$. Letting $S = \mu$ the Killing spinor in this case is given by

$$
\begin{aligned}
\epsilon \;&=\; e^{\frac{1}{2}g\rho + \mathrm{i}\frac{\sqrt{3}}{4}\mu x^+} \Big(1 - \mathrm{i}\tfrac{\sqrt{3}}{2}\, \mu b (z_1\, \Gamma_1 + z_2\, \Gamma_2)\, \Gamma_-\Big)(1 + \tfrac{\mathrm{i}}{2} g^{-1} f\, \Gamma_-) \\
&\qquad \times \Big[1 - \tfrac{1}{2}(1 - e^{-\mathrm{i}\sqrt{3}\,\mu b x^+})\Gamma_+\, \Gamma_-\Big]\epsilon_0\,,
\end{aligned}
\tag{70}
$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho + 1)\epsilon_0 = 0$. Thus the solution preserves half of the supersymmetry. Among all the Killing spinors, the standard ones are those with $\Gamma_-\epsilon_0 = 0$, whilst the remaining half with $\Gamma_-\epsilon_0 \neq 0$ are the supernumerary ones. The function H for the pp-waves with supernumerary supersymmetry is given by

$$
\begin{aligned}
H \;&=\; -3\mu^2 b^2 (z_1^2 + z_2^2) - g^{-2} f^2 \\
&=\; -\mu^2 [3b^2 (z_1^2 + z_2^2) + g^{-2}(3b^2\, e^{-2g\rho} + \tfrac{1}{12} e^{-6g\rho} - b\, e^{-4g\rho})]\,, \\
f \;&=\; -\tfrac{1}{2\sqrt{3}} \mu (e^{-3g\rho} - 6b\, e^{-g\rho})\,.
\end{aligned}
\tag{71}
$$

If we further let $b = \frac{1}{6}$, we have $H = -\frac{1}{12}\mu^2 (z_1^2 + z_2^2 + 4g^{-2}\sinh^2(g\rho)\, e^{-4g\rho})$. This enables us to take the limit $g \to 0$, giving rise to a pp-wave in the corresponding ungauged $D = 5$ supergravity, given by

$$
\begin{aligned}
ds^2 \;&=\; -4dx^+ dx^- - \tfrac{1}{12}\mu^2\, (z_1^2 + z_2^2 + 4\rho^2)\, (dx^+)^2 + dz_1^2 + dz_2^2 + d\rho^2, \\
F_{(2)} \;&=\; -\mu dx^+ \wedge d\rho\,.
\end{aligned}
\tag{72}
$$

This pp-wave can also arise from the Penrose limit of $\mathrm{AdS}_3\times S^2$ or $\mathrm{AdS}_2\times S^3$, which have supernumerary supersymmetries.

The second case is that of $b=0$, and hence eqs.(66) reduce to

$$\frac{dS}{dx^+}+4v\,S=0\,,\qquad \frac{dv}{dx^+}+v^2-\tfrac12 c\,S^2=0\,. \tag{73}$$

The Killing spinor is then given by

$$\begin{aligned}\epsilon &= e^{\frac12 g\rho}\, e^{\mathrm{i}\frac{\sqrt3}{4}\int S\,dx^+}\big(1+\tfrac12 v(z_1\,\Gamma_1+z_2\,\Gamma_2)\Gamma_-\big)(1+\tfrac{\mathrm{i}}{2}g^{-1}f\,\Gamma_-)\\ &\qquad\times\Big[1-\tfrac12(1-e^{\int v\,dx^+})\Gamma_+\,\Gamma_-\Big]\epsilon_0\,,\end{aligned} \tag{74}$$

where ϵ_0 is a constant spinor satisfying $(\Gamma_\rho+1)\epsilon_0=0$ and

$$\begin{aligned}H &= \tfrac12 S^2\Big[c\,(z_1^2+z_2^2)+2g^{-2}e^{-2g\rho}(c-\tfrac{1}{12}e^{-4g\rho})\Big]\,,\\ f &= -\tfrac{1}{2\sqrt3}e^{-3g\rho}S+\mathrm{i}e^{-g\rho}v\,.\end{aligned} \tag{75}$$

If we specialise to $v=\tilde kS$ and $c=-6\tilde k^2$ where $\tilde k$ is a constant, the system (73) simplifies to

$$\frac{dS}{dx^+}+4\tilde kS^2=0\,. \tag{76}$$

5 PP-waves in $D=6$ and $D=7$

5.1 $D=6$

Our next example is in the Romans six-dimensional gauged $\mathcal{N}=(1,1)$ supergravity [28]. The bosonic field content comprises the metric, a dilaton ϕ, a 2-form potential, a $U(1)$ potential and the gauge potentials $A^i_{(1)}$ of $SU(2)$ Yang-Mills. The Lagrangian describing the bosonic sector is [29]

$$\begin{aligned}\mathcal{L} &= R*\mathbb{1}-\tfrac12{*d\phi}\wedge d\phi+(2g_1^2X^2+\tfrac83 g_1g_2X^{-2}-\tfrac29 g_2^2X^{-6})*\mathbb{1}\\ &- \tfrac12 X^4*F_{(3)}\wedge F_{(3)}-\tfrac12 X^{-2}\Big(*G_{(2)}\wedge G_{(2)}+*F^a_{(2)}\wedge F^a_{(2)}\Big)\\ &- A_{(2)}\wedge(\tfrac12 dB_{(1)}\wedge dB_{(1)}+\tfrac13 g_2A_{(2)}\wedge dB_{(1)}+\tfrac{2}{27}g_2^2A_{(2)}\wedge A_{(2)}+\tfrac12 F^a_{(2)}\wedge F^a_{(2)})\,,\end{aligned} \tag{77}$$

where $X\equiv e^{-\frac{1}{2\sqrt2}\phi}$, $F_{(3)}=dA_{(2)}$, $G_{(2)}=dB_{(1)}+\tfrac23 g_2A_{(2)}$, $F^a_{(2)}=dA^a_{(1)}+\tfrac12 g_1\epsilon_{abc}\,A^b_{(1)}\wedge A^c_{(1)}$. The fermions of this theory comprise symplectic-Majorana gravitinos Ψ_{Mi} and dilatinos λ_i

where $i=1,2$ is an $SP(1)$ index. The supersymmetry transformations are given by [30]

$$\begin{aligned}
\delta\Psi_{Mi} &= [D_M - \tfrac{1}{48}X^2 F_{ABC}\,\Gamma^{ABC}\,\Gamma_M\Gamma^7 - \tfrac{1}{4\sqrt{2}}(g_1 X + \tfrac{1}{3}g_2 X^{-3})\,\Gamma_M]\epsilon_i \\
&\qquad + \tfrac{1}{16\sqrt{2}}(\Gamma_M\,\Gamma^{AB} - 2\Gamma^{AB}\,\Gamma_M)X^{-1}(G_{AB}\,\delta_i{}^j - \mathrm{i}\,\Gamma^7 F_{AB\,i}{}^j)\Gamma^7\epsilon_j\,, \\
\delta\lambda_i &= [-\tfrac{1}{2\sqrt{2}}\Gamma^M\partial_M\phi + \tfrac{1}{24}X^2 F_{ABC}\,\Gamma^{ABC}\,\Gamma^7 + \tfrac{1}{2\sqrt{2}}(g_1 X - g_2 X^{-3})]\epsilon_i \\
&\qquad + \tfrac{1}{8\sqrt{2}}X^{-1}(G_{AB}\,\delta_i^j - \mathrm{i}\,\Gamma^7 F_{AB\,i}{}^j)\,\Gamma^{AB}\,\Gamma^7\epsilon_j\,. \qquad (78)
\end{aligned}$$

The gauge covariant derivative is defined as $D_M\epsilon_i = \nabla_M\epsilon_i + \frac{\mathrm{i}}{2}g_1 A_{M\,i}{}^j\epsilon_j$ where $A_{M\,i}{}^j \equiv A^a_M(-\sigma^a)_i{}^j$ with the field strength given by $F_{MNi}{}^j = \partial_M A_{Ni}{}^j + \frac{\mathrm{i}}{2}g_1 A_{Mi}{}^k A_{Nk}{}^j - (M \leftrightarrow N)$ and σ^a are the usual Pauli matrices.

In this paper, we consider pp-wave solutions supported by only one field strength. Owing to the Chern-Simons modifications to various field strengths, we find that this can only be done with a $U(1)$ vector field coming from the $SU(2)$ Yang-Mills. Thus we consistently set all the remaining form fields to zero, and also without loss of generality (while insisting on AdS background) take $g_1 = g_2 = -3g/\sqrt{2}$. This leads to the pp-wave ansatz

$$\begin{aligned}
ds^2 &= e^{2g\rho}(-4dx^+\,dx^- + H(dx^+)^2 + dz_1^2 + dz_2^2 + dz_3^2) + d\rho^2, \\
A_{(1)} &= \tfrac{1}{3}g^{-1}S(1 - e^{-3g\rho})\,dx^+, \qquad (79)
\end{aligned}$$

where $S = S(x^+)$. The equations of motion reduce to

$$\Box H \equiv H'' + 5gH' + e^{-2g\rho}\sum_{i=1}^{3}\partial_i^2 H = -e^{-8g\rho}S^2, \qquad (80)$$

and the solution for H is given by

$$H = S^2\Big[\tfrac{1}{2}\sum_{i=1}^{3}c_i z_i + g^{-2}\big(\tfrac{1}{6}(c_1+c_2+c_3)e^{-2g\rho} - \tfrac{1}{24}e^{-8g\rho} + b\,e^{-5g\rho}\big)\Big] + H_0\,, \qquad (81)$$

where $\Box H_0 = 0$. The b and c_i are functions of x^+.

We now investigate the supersymmetry of the pp-waves. This is more conveniently done if we rewrite the symplectic-Majorana spinors using a Dirac notation. (See [31] for details.) The Killing spinor equations from the gravitino transformation rule are given by

$$\begin{aligned}
&\big[\partial_+ - \tfrac{\mathrm{i}}{2\sqrt{2}}S + \tfrac{\mathrm{i}}{2}(e^{g\rho}f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)\Gamma_+\Gamma_- - \tfrac{1}{4}\sum_i \Gamma_i\partial_i H\,\Gamma_- \\
&\qquad - \tfrac{1}{2\sqrt{2}}(g^{-1}e^{-3g\rho}fS + \tfrac{1}{\sqrt{2}}e^{g\rho}H')\Gamma_-\big]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \\
&[\partial_i + \tfrac{\mathrm{i}}{2}(e^{g\rho}f + \tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)\Gamma_i\,\Gamma_-]\epsilon = 0\,, \qquad i = 1,2,3, \\
&[\partial_\rho + \tfrac{\mathrm{i}}{2}(f - \tfrac{3}{4\sqrt{2}}e^{-4g\rho}S)\Gamma_- - \tfrac{1}{2}g]\epsilon = 0\,, \qquad (82)
\end{aligned}$$

where we have made use of the projection condition $g(\Gamma_\rho+1)\epsilon=\mathrm{i}f\,\Gamma_-\epsilon$ and where $f=f(x^+,\rho,z_i)$. The integrability conditions $[\partial_M\,,\partial_N]\epsilon=0$ among these projected Killing spinor equations are

$$
\begin{aligned}
0 &= [\partial_i\,,\partial_\rho]\epsilon=-\tfrac{\mathrm{i}}{2}\big[\partial_i f-(e^{g\rho}f+\tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)'\,\Gamma_i\big]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_i]\epsilon=-\tfrac{1}{2}\big[\mathrm{i}(e^{g\rho}\partial_+f+\tfrac{1}{4\sqrt{2}}e^{-3g\rho}\partial_+S)\Gamma_i+\tfrac{1}{2}\sum_j\Gamma_j\partial_j\partial_iH\\
&\quad +\tfrac{1}{2}e^{g\rho}\partial_iH'+(e^{g\rho}f+\tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)^2\,\Gamma_i-(\mathrm{i}e^{g\rho}\,\Gamma_+-\tfrac{1}{\sqrt{2}}g^{-1}e^{-3g\rho}S)\partial_if\big]\Gamma_-\epsilon\,,\\
0 &= [\partial_+\,,\partial_\rho]\epsilon=-\tfrac{1}{2}\big[\mathrm{i}(\partial_+f-\tfrac{3}{4\sqrt{2}}e^{-4g\rho}\partial_+S)+\tfrac{1}{2}\sum_i\Gamma_i\partial_iH'\\
&\quad -\mathrm{i}(e^{g\rho}f+\tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)'\,\Gamma_++\tfrac{1}{\sqrt{2}}g^{-1}S(e^{-3g\rho}f)'\\
&\quad +(f-\tfrac{3}{4\sqrt{2}}e^{-4g\rho}S)(e^{g\rho}f+\tfrac{1}{4\sqrt{2}}e^{-3g\rho}S)+\tfrac{1}{2}e^{g\rho}(H''+gH')\big]\Gamma_-\epsilon\,.
\end{aligned}
\tag{83}
$$

As before it is required that we set

$$
\partial_i f=0=\partial_iH'\qquad\text{and}\qquad\partial_j\partial_iH=0\,,\qquad i\neq j\,,
\tag{84}
$$

and $c_i=c$. The integrability conditions yield after using the solution for H the following results

$$
\begin{aligned}
& f=-\tfrac{1}{4\sqrt{2}}e^{-4g\rho}S+e^{-g\rho}\,U\,,\\
& \mathrm{i}\frac{dU}{dx^+}+U^2+\tfrac{1}{2}c\,S^2=0\,,\\
& \mathrm{i}\frac{dS}{dx^+}+\tfrac{1}{12\sqrt{2}}e^{-3g\rho}S\big[S(7-240b\,e^{3g\rho})+60\sqrt{2}\,e^{3g\rho}\,U\big]=0\,.
\end{aligned}
\tag{85}
$$

In the case of $S=0$, corresponding to purely gravitational waves, discussed in section 2, the last equation is trivially satisfied. When $S\neq0$, due to the ρ dependence, we conclude that no supersymmetry enhancement can occur here. This is expected, since in ungauged $D=6$, $\mathcal{N}=(1,1)$ supergravity, the pp-waves supported by a 2-form field strength also have no supernumerary supersymmetry. The solution does have standard supersymmetry though. The Killing spinor is given by

$$
\epsilon=e^{\frac{1}{2}g\rho+\frac{\mathrm{i}}{2\sqrt{2}}\int S\,dx^+}\epsilon_0\,,
\tag{86}
$$

where $(\Gamma_\rho+1)\epsilon_0=0=\Gamma_-\epsilon_0$. It is easy to verify that the Killing spinor equations associated with both the gravitino and dilatino transformation rules are satisfied. Thus the solution preserves $\frac{1}{4}$ of the supersymmetry.

5.2 $D = 7$

The Lagrangian for the bosonic sector of half-maximum supergravity in seven dimensions [32] can be written as follows [33]

$$\begin{aligned} \mathcal{L} &= R * 1\!\!1 - \tfrac{1}{2}{*d\phi} \wedge d\phi - \tfrac{1}{2} X^4 * F_{(4)} \wedge F_{(4)} - \tfrac{1}{2} X^{-2} * F^a_{(2)} \wedge F^a_{(2)} \\ &\quad + \tfrac{1}{2} F^a_{(2)} \wedge F^a_{(2)} \wedge A_{(3)} - \tfrac{1}{2\sqrt{2}} g_2 F_{(4)} \wedge A_{(3)} \\ &\quad + (2 g_1^2 X^2 + 2 g_1 g_2 X^{-3} - \tfrac{1}{4} g_2^2 X^{-8}) * 1\!\!1 \,, \end{aligned} \tag{87}$$

where $X = e^{-\frac{1}{\sqrt{10}}\phi}$, $F_{(4)} = dA_{(3)}$ and $F^a_{(2)} = dA^a_{(1)} + \frac{1}{2} g_1 \epsilon_{abc} A^b_{(1)} \wedge A^c_{(1)}$. In addition there is a "self-duality" condition that must be imposed, given by

$$X^4 * F_{(4)} = -\tfrac{1}{\sqrt{2}} g_2 A_{(3)} + \tfrac{1}{2}\omega_{(3)} \,, \tag{88}$$

where $\omega_{(3)}$ is defined as $\omega_{(3)} = A^a_{(1)} \wedge F^a_{(2)} - \frac{1}{6} g_1\, \epsilon_{abc}\, A^a_{(1)} \wedge A^b_{(1)} \wedge A^c_{(1)}$. This theory has a pair of symplectic-Majorana gravitinos ψ_{Mi} and a pair of dilatinos λ_i, where $i = 1, 2$ is an $SP(1)$ index. The fermionic supersymmetry transformations are given by [30]

$$\begin{aligned} \delta\psi_{Mi} &= \nabla_M \epsilon_i + \tfrac{\mathrm{i}}{2} g_1 A_{Mi}{}^j \epsilon_j + \tfrac{1}{960} X^2 F_{ABCD} (\Gamma_M\, \Gamma^{ABCD} + 5 \Gamma^{ABCD}\, \Gamma_M) \epsilon_i \\ &\quad - \tfrac{\mathrm{i}}{40\sqrt{2}} X^{-1} (3 \Gamma_M\, \Gamma^{AB} - 5 \Gamma^{AB}\, \Gamma_M) F_{ABi}{}^j \epsilon_j - \tfrac{1}{5\sqrt{2}} (g_1 X + \tfrac{1}{4} g_2 X^{-4}) \Gamma_M \epsilon_i \,, \\ \delta\lambda_i &= [-\tfrac{1}{2\sqrt{2}} \Gamma^M \partial_M \phi + \tfrac{1}{48\sqrt{5}} X^2 F_{ABCD}\, \Gamma^{ABCD}] \epsilon_i - \tfrac{\mathrm{i}}{4\sqrt{10}} X^{-1} F_{ABi}{}^j\, \Gamma^{AB} \epsilon_j \\ &\quad + \tfrac{1}{\sqrt{10}} (g_1 X - g_2 X^{-4}) \epsilon_i \,, \end{aligned} \tag{89}$$

where $A_{Mi}{}^j \equiv A^a_M (-\sigma^a)_i{}^j$. Owing to the odd-dimensional self-duality condition for the $A_{(3)}$, our standard ansatz for the pp-wave metric does not work for $A_{(3)}$. We thus consider the pp-wave supported only by the $U(1)$ subsector of the $SU(2)$ Yang-Mills. The pp-wave solution is given by

$$\begin{aligned} ds^2 &= e^{2g\rho} (-4 dx^+\, dx^- + H (dx^+)^2 + dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) + d\rho^2, \\ A_{(1)} &= \tfrac{1}{4} g^{-1} S (1 - e^{-4g\rho})\, dx^+, \end{aligned} \tag{90}$$

where $S = S(x^+)$ and H satisfies

$$\Box H \equiv H'' + 6 g H' + e^{-2g\rho} \sum_{i=1}^{4} \partial_i^2 H = -e^{-10g\rho} S^2. \tag{91}$$

Here we have set $g_1 = g_2 = -2\sqrt{2}\, g$. The function H can be solved, given by

$$H = S^2 \Big[\tfrac{1}{2} \sum_{i=1}^{4} c_i z_i^2 + g^{-2} \big(\tfrac{1}{8} \sum_{i=1}^{4} c_i\, e^{-2g\rho} - \tfrac{1}{40} e^{-10g\rho} + b\, e^{-6g\rho} \big) \Big] + H_0 \,, \tag{92}$$

with $\Box H_0 = 0$ and b and c_i are functions of x^+.

The projected Killing spinor equations from the gravitino transformation rule are given by

$$\begin{aligned}
&\big[\partial_+ - \tfrac{\mathrm{i}}{2\sqrt{2}} S + \tfrac{\mathrm{i}}{2}(e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)\Gamma_+\Gamma_- - \tfrac{1}{4}\sum_i \Gamma_i \partial_i H\, \Gamma_- \\
&\quad - \tfrac{1}{2\sqrt{2}}(g^{-1} e^{-4g\rho} f S + \tfrac{1}{\sqrt{2}} e^{g\rho} H')\Gamma_-\big]\epsilon = 0\,, \qquad \partial_-\,\epsilon = 0\,, \\
&[\partial_i + \tfrac{\mathrm{i}}{2}(e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)\Gamma_i\,\Gamma_-]\epsilon = 0\,, \qquad i = 1,2,3,4, \\
&[\partial_\rho + \tfrac{\mathrm{i}}{2}(f - \tfrac{4}{5\sqrt{2}} e^{-5g\rho} S)\Gamma_- - \tfrac{1}{2} g]\epsilon = 0\,.
\end{aligned} \tag{93}$$

The integrability conditions

$$\begin{aligned}
0 &= [\partial_i\,, \partial_\rho]\epsilon = -\tfrac{\mathrm{i}}{2}\big[\partial_i f - (e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)'\,\Gamma_i\big]\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,, \partial_i]\epsilon = -\tfrac{1}{2}\big[\mathrm{i}(e^{g\rho}\partial_+ f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho}\partial_+ S)\Gamma_i + \tfrac{1}{2}\sum_j \Gamma_j \partial_j \partial_i H \\
&\quad + \tfrac{1}{2} e^{g\rho}\partial_i H' + (e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)^2\,\Gamma_i - (\mathrm{i} e^{g\rho}\,\Gamma_+ - \tfrac{1}{\sqrt{2}} g^{-1} e^{-4g\rho} S)\partial_i f\big]\Gamma_-\epsilon\,, \\
0 &= [\partial_+\,, \partial_\rho]\epsilon = -\tfrac{1}{2}\big[\mathrm{i}(\partial_+ f - \tfrac{4}{5\sqrt{2}} e^{-5g\rho}\partial_+ S) + \tfrac{1}{2}\sum_i \Gamma_i \partial_i H' \\
&\quad - \mathrm{i}(e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S)'\,\Gamma_+ + \tfrac{1}{\sqrt{2}} g^{-1} S (e^{-4g\rho} f)' \\
&\quad + (f - \tfrac{4}{5\sqrt{2}} e^{-5g\rho} S)(e^{g\rho} f + \tfrac{1}{5\sqrt{2}} e^{-4g\rho} S) + \tfrac{1}{2} e^{g\rho}(H'' + g H')\big]\Gamma_-\epsilon\,,
\end{aligned} \tag{94}$$

imply that there is no supernumerary Killing spinors in this case. This should be expected since in $D = 7$, even in ungauged supergravities, there is no pp-wave supported by a 2-form field strength that has supernumerary supersymmetry. The solution does have $\frac{1}{4}$ of standard supersymmetry, with the Killing spinor given by

$$\epsilon = e^{\frac{1}{2} g\rho + \frac{\mathrm{i}}{2\sqrt{2}}\int S\, dx^+}\,\epsilon_0\,, \tag{95}$$

where $(\Gamma_\rho + 1)\epsilon_0 = 0 = \Gamma_-\epsilon_0$.

6 Conclusions

In this paper, we have constructed $U(1)$-charged pp-wave solutions in AdS gauged supergravities in $4 \le D \le 7$ dimensions. Generically these solutions preserve $\frac{1}{4}$ of the supersymmetry. In $D = 4$ and $D = 5$, with an appropriate choice for the integration constants, we have shown that supernumerary supersymmetry can arise so that the solutions instead preserve $\frac{1}{2}$ of the supersymmetry. These solutions can take a limit to become the pp-waves that are the Penrose limits of AdS$\times$sphere of the corresponding ungauged supergravities.

In $D = 6$ and $D = 7$, we find that there can be no supernumerary supersymmetry for the $U(1)$-charged pp-waves. We also considered a general class of purely gravitational pp-waves in Einstein gravity with a cosmological constant in diverse dimensions. We showed that supernumerary supersymmetry could arise and obtained explicitly the $\frac{1}{2}$-BPS gravitational pp-waves.

The introduction of a pp-wave in the AdS background can be viewed as performing an infinite boost in the strong coupled dual conformal field theory with a finite momentum density. The non-vanishing momentum breaks the original supersymmetry and superconformal symmetry, and hence the supersymmetry is now $\frac{1}{4}$ of the unboosted theory. We have shown in the supergravity side that the supersymmetry can be doubled when the pp-wave is $U(1)$ charged, corresponding to an R charge in the dual field theory. This indicates a novel supersymmetry enhancement associated with the R charges in the dual three- and four-dimensional field theories. It is of interest to discover such a phenomenon in the dual quantum field theory in the infinite-momentum frame.

Acknowledgments

We are grateful to Mirjam Cvetič and Chris Pope for useful discussions.

APPENDICES

A Uplifting to M/String theory

In this appendix we uplift the supersymmetric solutions supported by the U(1) charge to ten and eleven dimensions. In the case of the four- and five-dimensional solutions we uplift those with S being a constant. The four- and seven-dimensional solutions are embedded in M-theory and the solutions in $D = 5$ and $D = 6$ are uplifted to type-IIB supergravity and to Romans massive theory, respectively.

A.1 $D = 4$ oxidised to $D = 11$

The embedding formulae to eleven dimensions were obtained in [34] or we can also use the ansatz in [35] after truncating to our case. We obtain

$$\begin{aligned} d\hat{s}_{11} &= e^{2g\rho}[-4dx^+dx^- - \mu^2(\tfrac{1}{4}z^2 + g^{-2}\sinh^2(\tfrac{1}{2}g\rho)e^{-3g\rho})(dx^+)^2 + dz^2] + d\rho^2 \\ &\quad +4g^{-2}d\xi + g^{-2}[c^2\,(\sigma_1^2+\sigma_2^2+h_3^2) + s^2\,(\tilde\sigma_1^2+\tilde\sigma_2^2+\tilde h_3^2)]\,, \\ \hat{F}_{(4)} &= -6ge^{3g\rho}dx^+\wedge dx^-\wedge d\rho\wedge dz - \mu g^{-2}[s\,c\,d\xi\wedge\sigma_3 + \tfrac{1}{2}c^2\sigma_1\wedge\sigma_2 \\ &\quad -s\,c\,d\xi\wedge\tilde\sigma_3 + \tfrac{1}{2}s^2\,\tilde\sigma_1\wedge\tilde\sigma_2]\wedge dx^+\wedge dz\,, \end{aligned} \tag{96}$$

where σ_i are the three left-invariant 1-forms on S^3 satisfying $d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j\wedge\sigma_k$. They are given by $\sigma_1 + \mathrm{i}\sigma_2 = e^{-\mathrm{i}\psi}(d\theta + \mathrm{i}\sin\theta\,d\varphi)$ and $\sigma_3 = d\psi + \cos\theta\,d\varphi$ in terms of the Euler angles. The $\tilde\sigma_i$ are left-invariant 1-forms on a second S^3. We have also defined

$$\begin{aligned} &c\equiv\cos\xi\,, \qquad s\equiv\sin\xi\,, \\ &h_3\equiv\sigma_3 - \tfrac{1}{2}\mu(1-e^{-g\rho})dx^+, \qquad \tilde h_3\equiv\tilde\sigma_3 - \tfrac{1}{2}\mu(1-e^{-g\rho})dx^+, \\ &\epsilon_{(3)} = \sigma_1\wedge\sigma_2\wedge h_3\,, \qquad \tilde\epsilon_{(3)} = \tilde\sigma_1\wedge\tilde\sigma_2\wedge\tilde h_3\,. \end{aligned} \tag{97}$$

In this pp-wave, the internal S^7 is twisted but not flattened. Analogous solution but with untwisted round S^7 can be found in [36].

A.2 $D = 5$ oxidised to type IIB

Using the uplifting formulae to type IIB in [34, 37] we obtain for the metric

$$\begin{aligned} d\hat{s}_{10}^2 &= e^{2g\rho}[-4dx^+dx^- - \tfrac{1}{12}\mu^2(z_1^2+z_2^2+4g^{-2}\sinh^2(g\rho)\,e^{-4g\rho})(dx^+)^2 \\ &\quad +dz_1^2+dz_2^2] + d\rho^2 + g^{-2}\sum_{i=1}^{3}\big[d\mu_i^2 + \mu_i^2(d\phi_i + \tfrac{1}{2\sqrt{3}}\mu(1-e^{-2g\rho})dx^+)^2\,\big]\,, \end{aligned} \tag{98}$$

and for the 5-form field strength $F_{(5)} = G_{(5)} + {*G_{(5)}}$,

$$G_{(5)} = -8ge^{4g\rho}dx^+\wedge dx^-\wedge d\rho\wedge d^2z - \tfrac{1}{2\sqrt{3}}\mu g^{-2}\sum_{i=1}^{3} d(\mu_i^2)\wedge d\phi_i\wedge dx^+\wedge d^2z\,. \tag{99}$$

The μ_i are parameterised as

$$\mu_1 = \sin\theta\,, \qquad \mu_2 = \cos\theta\,\sin\psi\,, \qquad \mu_3 = \cos\theta\,\cos\psi\,, \tag{100}$$

in terms of the angles on a 2-sphere.

A.3 $D = 6$ oxidised to Romans massive theory

The bosonic sector of Romans massive theory [38] is described by the Lagrangian

$$\mathcal{L}_{10} = \hat{R}\hat{*}\mathbb{1} - \tfrac{1}{2}\hat{*}d\hat{\phi}\wedge d\hat{\phi} - \tfrac{1}{2}e^{\frac{3}{2}\hat{\phi}}\hat{*}\hat{F}_{(2)}\wedge\hat{F}_{(2)} - \tfrac{1}{2}e^{-\hat{\phi}}\hat{*}\hat{F}_{(3)}\wedge\hat{F}_{(3)} - \tfrac{1}{2}e^{\frac{1}{2}\hat{\phi}}\hat{*}\hat{F}_{(4)}\wedge\hat{F}_{(4)}$$
$$-\tfrac{1}{2}d\hat{A}_{(3)}\wedge d\hat{A}_{(3)}\wedge\hat{A}_{(2)} - \tfrac{1}{6}m\, d\hat{A}_{(3)}\wedge(\hat{A}_{(2)})^3 - \tfrac{1}{40}m^2(\hat{A}_{(2)})^5 - \tfrac{1}{2}m^2 e^{\frac{5}{2}\hat{\phi}}\hat{*}\mathbb{1}\,, \qquad (101)$$

where the field strengths are defined as

$$\hat{F}_{(2)} = d\hat{A}_{(1)} + m\hat{A}_{(2)}\,, \qquad \hat{F}_{(3)} = d\hat{A}_{(2)}\,,$$
$$\hat{F}_{(4)} = d\hat{A}_{(3)} + \hat{A}_{(1)}\wedge d\hat{A}_{(2)} + \tfrac{1}{2}m\hat{A}_{(2)}\wedge\hat{A}_{(2)}\,. \qquad (102)$$

Note that the Bianchi identities in this theory are given by

$$d\hat{F}_{(4)} = \hat{F}_{(2)}\wedge\hat{F}_{(3)}\,, \qquad d\hat{F}_{(3)} = 0\,, \qquad d\hat{F}_{(2)} = m\hat{F}_{(3)}\,. \qquad (103)$$

Using the embedding formulae obtained in [29] we can lift our six dimensional solution to a solution of the above theory. It is given by (with $m = g$)

$$d\hat{s}_{10}^2 = s^{\frac{1}{12}}\,[ds_6^2 + \tfrac{4}{9}g^{-2}d\xi^2 + \tfrac{1}{9}g^{-2}c^2\,(\sigma_1^2+\sigma_2^2+h_3^2)]\,,$$
$$\hat{F}_{(4)} = \tfrac{10}{81}g^{-3}s^{1/3}c^3\,d\xi\wedge\epsilon_{(3)} - \tfrac{2}{9\sqrt{2}}g^{-2}e^{-3g\rho}S[s^{1/3}c\,\sigma_3\wedge d\xi - \tfrac{1}{2}s^{4/3}c^2\,\sigma_1\wedge\sigma_2]\wedge dx^+\wedge d\rho\,,$$
$$\hat{F}_{(3)} = 0\,, \qquad \hat{F}_{(2)} = 0\,, \qquad e^{\hat{\phi}} = s^{-5/6}, \qquad (104)$$

where ds_6^2 is given by (79) and (81), and $s, c, \epsilon_{(3)}$ and σ_i have the same definitions as before and $h_3 = \sigma_3 - \frac{1}{\sqrt{2}}S(1-e^{-3g\rho})dx^+$.

A.4 $D = 7$ oxidised to $D = 11$

Using the embedding ansatz in [33] we obtain

$$d\hat{s}_{11} = ds_7^2 + \tfrac{1}{4}g^{-2}d\xi^2 + \tfrac{1}{16}g^{-2}c^2(\sigma_1^2+\sigma_2^2+h_3^2)\,, \qquad (105)$$
$$\hat{A}_{(3)} = \tfrac{1}{64}g^{-3}(2s + s\,c^2)\epsilon_{(3)} + \tfrac{1}{8\sqrt{2}}g^{-2}S\,e^{-4g\rho}s\,dx^+\wedge d\rho\wedge\sigma_3\,,$$

where ds_7^2 is given by (90) and (92). The field strength $\hat{F}_{(4)} = d\hat{A}_{(3)}$ is

$$\hat{F}_{(4)} = \tfrac{3}{64}g^{-3}c^3\,d\xi\wedge\epsilon_{(3)} + \tfrac{1}{8\sqrt{2}}g^{-2}S\,e^{-4g\rho}c\,dx^+\wedge d\rho\wedge d\xi\wedge\sigma_3$$
$$+\tfrac{1}{16\sqrt{2}}g^{-2}S\,e^{-4g\rho}\,s\,c^2\,dx^+\wedge d\rho\wedge\sigma_1\wedge\sigma_2\,, \qquad (106)$$

where $s, c, \epsilon_{(3)}$ and σ_i have the same definitions as before and $h_3 = \sigma_3 - \frac{1}{\sqrt{2}}S(1-e^{-4g\rho})dx^+$.

B A general class of pp-waves

In this appendix we present the AdS pp-waves supported by an arbitrary n-form field strength in any dimensions D. The Lagrangian for such a system is given by

$$e^{-1}\mathcal{L} = R - \frac{1}{2n!}F_{(n)}^2 + (D-1)(D-2)g^2, \tag{107}$$

where the field strength is defined as $F_{(n)} = dA_{(n-1)}$. Our pp-wave ansatz is

$$\begin{aligned} ds^2 &= e^{2g\rho}(-4dx^+\,dx^- + H\,(dx^+)^2 + dz^2) + d\rho^2, \\ A_{(n-1)} &= \Big(zS_1(x^+) - \frac{S_2(x^+)}{g(D-2n+1)}(e^{-(D-2n+1)g\rho} - 1)\Big)\,dx^+ \wedge d^{\,n-2}z\,. \end{aligned} \tag{108}$$

The field strength and its dual are

$$\begin{aligned} F_{(n)} &= -S_1\,dx^+ \wedge dz^{n-1} + S_2 e^{-(D-2n+1)g\rho} d\rho \wedge dx^+ \wedge d^{n-2}z\,, \\ {*F_{(n)}} &= S_1 e^{(D-2n-1)g\rho} d\rho \wedge dx^+ \wedge d^{\,D-n-2}z - S_2 dx^+ \wedge d^{\,D-n-1}z\,. \end{aligned} \tag{109}$$

Thus the equation of motion $d{*F_{(n)}} = 0$ is trivially satisfied. The Einstein equation implies

$$\begin{aligned} \Box H &= -S_1^2 e^{-2ng\rho} - S_2^2 e^{-2(D-n)g\rho}\,, \\ \Box &= \partial_\rho^2 + g(D-1)\partial_\rho + e^{-2g\rho}\sum_{i=1}^{D-3}\partial_i^2\,, \end{aligned} \tag{110}$$

with the solution given by

$$\begin{aligned} H(x^+,\rho,z_i) &= a + b\,e^{-(D-1)g\rho} + \frac{e^{-2g\rho}}{2g^2(D-3)}\sum_{i=1}^{D-3} c_i + \frac{S_1^2\,e^{-2ng\rho}}{2g^2 n(D-2n-1)} \\ &- \frac{S_2^2\,e^{-2(D-n)g\rho}}{2g^2(D-n)(D-2n+1)} + \tfrac{1}{2}\sum_{i=1}^{D-3} c_i z_i^2\,. \end{aligned} \tag{111}$$

The a, b and c_i are functions of x^+. This solution is not valid for $D = 2n-1$ or $D = 2n+1$ which have to be considered separately. We find that

$$\begin{aligned} H(D=2n+1,x^+,\rho,z_i) &= a + b\,e^{-2ng\rho} + \frac{e^{-2g\rho}}{4g^2(n-1)}\sum_{i=1}^{2(n-1)} c_i + \frac{(2ng\rho+1)S_1^2}{4n^2g^2}\,e^{-2ng\rho} \\ &- \frac{S_2^2}{4g^2(n+1)}\,e^{-2(n+1)g\rho} + \tfrac{1}{2}\sum_{i=1}^{2(n-1)} c_i z_i^2\,, \end{aligned} \tag{112}$$

and $H(D = 2n-1)$ can be obtained from $H(D = 2n+1)$ by making the substitution

$$n \to n-1 \qquad \text{and} \qquad S_1 \longleftrightarrow S_2\,. \tag{113}$$

(This substitution is not performed on the field strength.)

References

[1] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *"A new maximally supersymmetric background of IIB superstring theory,* JHEP **0201**, 047 (2002), hep-th/0110242.

[2] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, Phys. Lett. **B134**, 194 (1984).

[3] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp-waves from $N = 4$ super Yang Mills,* JHEP **0204**, 013 (2002), hep-th/0202021.

[4] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 231 (1998); Int. J. Theor. Phys. **38** 1113 (1999), hep-th/9711200.

[5] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. **B428** 105 (1998), hep-th/9802109.

[6] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** 253 (1998), hep-th/9802150.

[7] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background,* Nucl. Phys. **B625**, 70 (2002), hep-th/0112044.

[8] M. Cvetič, H. Lü and C.N. Pope, *Penrose limits, pp-waves and deformed M2-branes,* Phys. Rev. **D69**, 046003 (2004), hep-th/0203082.

[9] M. Cvetič, H. Lü and C.N. Pope, *M-theory pp-waves, Penrose limits and supernumerary supersymmetries,* Nucl. Phys. B **644**, 65 (2002), hep-th/0203229.

[10] J.P. Gauntlett and C.M. Hull, *pp-waves in 11-dimensions with extra supersymmetry,* JHEP **0206**, 013 (2002), hep-th/0203255.

[11] I. Bena and R. Roiban, *Supergravity pp-wave solutions with 28 and 24 supercharges,* Phys. Rev. **D67**, 125014 (2003), hep-th/0206195.

[12] J. Michelson, *(Twisted) toroidal compactification of pp-waves,* Phys. Rev. **D66**, 066002 (2002), hep-th/0203140.

[13] J. Michelson, *A pp-wave with 26 supercharges,* Class. Quant. Grav. **19**, 5935 (2002), hep-th/0206204.

[14] H. Lü and J.F. Vázquez-Poritz, *Penrose limits of non-standard brane intersections,* Class. Quant. Grav. **19**, 4059 (2002), hep-th/0204001.

[15] M. Cvetič, H. Lü and C.N. Pope, *Spacetimes of boosted p-branes, and CFT in infinite-momentum frame,* Nucl. Phys. **B545**, 309 (1999), hep-th/9810123.

[16] D. Brecher, A. Chamblin and H.S. Reall, *AdS/CFT in the infinite momentum frame,* Nucl. Phys. **B607**, 155 (2001), hep-th/0012076.

[17] V.R. Kaigorodov, *Einstein spaces of maximum mobility*, Dokl. Akad. Nauk. SSSR **146** 793 (1962); Sov. Phys. Doklady **7**, 893 (1963).

[18] S.T.C. Siklos, *Lobatchevski plane gravitational waves,* in Galaxies, Axisymmetric Systems and Relativity, ed. M.A.H. MacCallum, Cambridge University Press, Cambridge, England, 1985.

[19] I. Ozsvath, I. Robinson and K. Rozga, *Plane-fronted gravitational and electromagnetic waves in spaces with cosmological constant,* J. Math. Phys. **26** (1985) 1755.

[20] G.W. Gibbons and P.J. Ruback, *Classical gravitons and their stability in higher dimensions,* Phys. Lett. **B171** (1986) 390.

[21] D.Z. Freedman and A. Das, *Gauge internal symmetry in extended supergravity,* Nucl. Phys. **B120**, 221 (1977).

[22] E.S. Fradkin and M.A Vasiliev, *Model of supergravity with minimal electromagnetic interaction,* Lebedev Institute preprint N 197 (1976).

[23] R.G. Cai, *Boosted domain wall and charged Kaigorodov space,* Phys. Lett. **B572**, 75 (2003), hep-th/0306140.

[24] M.M. Caldarelli and D. Klemm, *All supersymmetric solutions of* $N = 2$*,* $D = 4$ *gauged supergravity,* JHEP **0309**, 019 (2003), hep-th/0307022.

[25] S.L. Cacciatori, M.M. Caldarelli, D. Klemm and D.S. Mansi, *More on BPS solutions of* $N = 2$*,* $D = 4$ *gauged supergravity,* JHEP **0407**, 061 (2004), hep-th/0406238.

[26] M. Günaydin, G. Sierra and P.K. Townsend, *The geometry of N=2 Mawxwell-Einstein supergravity and Jordan algebras,* Nucl. Phys. **B242**, 244 (1984).

[27] J.P. Gauntlett and J.B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five dimensions,* Phys. Rev. **D68**, 105009 (2003), erratum-ibid. **D70**, 089901 (2004), hep-th/0304064.

[28] L.J. Romans, *The F(4) gauged supergravity in six-dimensions,* Nucl. Phys. **B269**, 691 (1986).

[29] M. Cvetič, H. Lü and C.N. Pope, *Gauged six-dimensional supergravity from massive type IIA,* Phys. Rev. Lett. **83**, 5226 (1999), hep-th/9906221.

[30] J.T. Liu, H. Lü and C.N. Pope, *The radion mode in consistent brane-world reductions,* hep-th/0212037.

[31] J. Kerimo, J.T. Liu, H. Lü and C.N. Pope, *Variant* $\mathcal{N} = (1,1)$ *supergravity and (Minkowski)*$_4 \times S^2$ *vacua,* Class. Quant. Grav. **21**, 3287 (2004), hep-th/0401001.

[32] P.K. Townsend and P. van Nieuwenhuizen, *Gauged seven dimensional supergravity,* Phys. Lett. **125B**, 41 (1983).

[33] H. Lü and C.N Pope, *Exact embedding of* $N = 1, D = 7$ *gauged supergravity in* $D = 11$, Phys. Lett. **B467**, 67 (1999), hep-th/9906168.

[34] M. Cvetič, M.J. Duff, P. Hoxa, J.T. Liu, H. Lü, J.X. Lu, R. Martinez-Acosta, C.N. Pope, H. Sati and T.A. Tran, *Embedding AdS black holes in ten and eleven dimensions* Nucl. Phys. **B558**, 96 (1999), hep-th/9903214.

[35] M. Cvetič, H. Lü and C.N. Pope, *Four-dimensional N*$= 4$*,* $SO(4)$ *gauged supergravity from* $D = 11$ Nucl. Phys. **B574**, 761 (2000), hep-th/9910252.

[36] A. Kumar and H.K. Kunduri, *Gravitational wave solutions in string and M-theory AdS backgrounds,* hep-th/0405261.

[37] H. Lü, C.N. Pope and T.A. Tran, *Five-dimensional* $\mathcal{N} = 4$*,* $SU(2) \times U(1)$ *gauged supergravity from type IIB,* Phys. Lett. **B475**, 261 (2000), hep-th/9909203.

[38] L.J. Romans, *Massive N=2a supergravity in ten dimensions,* Phys. Lett. **B169**, 374 (1986).