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# Supercriticality of a Class of Critical String Cosmological Solutions

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For a class of Friedmann-Robertson-Walker-type string solutions with compact hyperbolic spatial slices formulated in the critical dimension, we find the world sheet conformal field theory which involves the linear dilaton and Wess-Zumino-Witten-type model with a compact hyperbolic target space. By analyzing the infrared spectrum, we conclude that the theory is actually supercritical due to modular invariance. Thus, taking into account previous results, we conclude that all simple non-trivial string cosmological solutions are in fact supercritical. In addition, we discuss the relationship of this background with the Supercritical String Cosmology (SSC).

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## I. INTRODUCTION

String theory is usually formulated in the critical dimension (D=26 for the bosonic string, and D=10 for the superstring ), and there are various ways to justify why this is the natural choice[1]. The idea of a critical dimension has been firmly entrenched in the way that string theory is usually formulated, and it is widely taken for granted that this should be the case in most studies. Nevertheless, it is possible to formulate string theory in a dimension other than the critical dimension, and such a way of formulating string theory finds a useful application in studying time-varying backgrounds such as those which are necesssary for cosmology [2][3].

There are several reasons to suspect that non-critical string theory is relevant to the web of string duality. The first clue comes from Cosmology. Observational data from the SnIa [4] project and WMAP $1,3$ [5] strongly suggests that the universe is currently in an accelerating phase, which may be explained by the presence of dark energy. String theory, as a theory of quantum gravity, should give an explanation of the dark energy. However, it is very difficult to find exact time-dependent solutions in critical string theory. However, it has been shown in [6][7] that if one goes beyond the critical dimension, one can find simple time-dependent Friedmann-Robertson-Walker (FRW) solutions with a time-like linear dilaton. So-called supercritical strings (for strings with dimension $D > D_{critical}$) may then also provide an explanation for the existence of dark energy[8].

From the theoretical point of view, past experience tell us that strings probe space time in a very different way than we may intuitively think. A compelling example is the emergence of M theory, where when we study the strong coupling behavior of Type IIA string theory, the theory develops a new dimension, and the theory is then formulated in 11 dimensions! The extended nature of the string reveals many amazing things about the property of space time (T duality is one of the famous example). We expect that strings experience the space-time dimensions in a novel way, and it may be possible that string theory formulated in critical dimensions actually exhibits non-critical behavior, i.e. the number of effective space time dimensions in which strings can freely oscillate is larger than the critical dimension.

In [6], the authors analyzed the conformal field theory of FRW type solutions with space curvature $\kappa = 0$ and $\kappa = +1$; it was shown that the space dimensions should be supercritical. In this paper, we will consider the case with space curvature $\kappa = -1$. This solution did not attract much attention because conformal field theory with hyperbolic target space is hard to study. For this solution, we can choose the time-like linear dilaton and the level of the hyperbolic CFT properly so that the space time dimensions are critical. This solution is trivial if the hyperbolic manifold is noncompact because the background is only part of the flat Minkowski space. The amazing thing is that if we make the hyperbolic manifold compact, (to make the solution topologically nontrivial) the actual space time dimensions of this background is supercritical. So we conclude for all the nontrivial FRW type solutions, the theory should be formulated in non-critical dimensions.

This paper is organized in the following way: In section II, we discuss the cosmological solutions derived from the string equations of motion and the quantization of the supercritical linear dilaton background. It is shown that for this simple time dependent background, the infrared (IR) and ultraviolet (UV) behaviors are changed drastically. In section III, we discuss the stability of the cosmological solutions. We find that the pseudotachyon modes do not imply the instability of the background. In section IV, we study the cosmological solutions with the compact hyperbolic spatial section in critical dimension and we find that the theory is indeed supercritical. In section V, the relation of this background with supercritical string cosmology is pointed out.

## II. COSMOLOGICAL SOLUTIONS AND THE QUANTIZATION

To study the nontrivial space time background, it is quite useful to start with the string world sheet non-linear sigma model in curved space time. The bosonic string action reads:

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\sqrt{-h}[h^{\alpha\beta}\partial_\alpha X_\mu\partial_\beta X_\nu G^{\mu\nu} + i\epsilon_{\alpha\beta}B^{\mu\nu}\partial_\alpha X_\mu\partial_\beta X_\nu + \alpha' R\Phi(X)]. \quad (1)$$

The equation of motion for the background fields can be derived from the requirement that the $\beta$ functions of this two dimensional field theory vanish to one-loop order, where the $\beta$ functions can be found in [9][10]. We are interested in finding the four dimensional FRW type solution (we assume that the four dimensional CFT is decoupled from the internal Conformal Field Theory (CFT)). We have found in [6] the following simple and *asymptotically unique* cosmological solutions (we only write the four large dimension with the internal part of the background suppressed) :

1. The Einstein static universe, with the background fields as (the metric is expressed in the Einstein frame, the relation of the metric between string frame and Einstein frame is $g^E_{\mu\nu} = e^{-2\Phi/(D-2)}G^S_{\mu\nu}$)

$$\Phi = \Phi_0, \qquad b = 2e^{-\Phi_0}\sqrt{\kappa}X^0. \quad (2)$$

Here we have a constant dilaton, while the axion $b$ is related to the NS field strength through the duality relation $H_{\lambda\mu\nu} = e^{2\phi}\epsilon_{\lambda\mu\nu\rho}\bigtriangledown^\rho b$, and the metric is given by

$$(ds)^2 = -(dX^0)^2 + [\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (3)$$

2. The second class of solutions involves both a non-trivial dilaton and the non-trivial axion field. The solution is

$$\Phi = -2QX^0, \qquad b = 2Q^2\sqrt{\kappa}(\frac{e^{QX^0}}{Q}), \quad (4)$$

while the metric is given by

$$(ds)^2 = -(dt)^2 + t^2[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (5)$$

The $\kappa$ is the curvature of the spatial slice, and $X^0$ is the world sheet time coordinate, it is related to the Einstein time coordinate as $t = \frac{1}{Q}e^{QX^0}$. The central charge deficit is $\delta c = c_I - 22 \propto (1+\kappa)$, so only for $\kappa = -1$, can we find the cosmological solutions in critical dimension, while for the flat and positive curvature the theory is supercritical. However, for $\kappa = -1$, the solution is trivial due to the fact that this background is only part of the flat space. We can take a quotient of this space to make the solution non-trivial; the remarkable thing is that this background with the compact hyperbolic section is indeed supercritical. Before we get into the discussion of the hyperbolic case, it is very useful to give a review of the well-studied solutions with $\kappa = 0$ and $\kappa = 1$, we will find some generic features of time-dependent solutions.

### A. Spectrum of Cosmological Solutions with $\kappa = 0$ and $\kappa = 1$

The simplest time dependent background is the so-called supercritical linear dilaton background (SCLD), this solution can be derived if we take $\kappa = 0$ in our second class of solutions. A discussion of this background may also be found in[11]. Regardless of its simplicity, we can definitely learn a lot from studying this background due to the nontrivial time dependence of the solution. The background reads:

$$G_{\mu\nu} = \eta_{\mu\nu}, \qquad \Phi = -2QX^0. \quad (6)$$

The central charge of this CFT is

$$c = D - 12Q^2. \quad (7)$$

The anomaly cancelation requires that $c = 26$ for bosonic string, so for $Q$ real, it is clear that the space time dimension is larger than the critical dimension.

The effect of the linear dilaton is to change the world sheet energy-momentum tensor to the form (where we have set $\alpha' = 2$)

$$T_{zz} = -\frac{1}{2}\partial_z X^\mu\partial_z X_\mu - Q\partial_z^2 X^0. \quad (8)$$

We can quantize the theory by using the conventional mode expansion, whic is possible, because the theory is Weyl invariant and reparameterization invariant, and because we can gauge fix the world-sheet metric as $h_{ij} = \delta_{ij}$).

The Virasoro generators are [6]

$$L_n =: \frac{1}{2}\sum_k a^\mu_{n-k}a_{k\mu} : + iQ(n+1)a^0_n, \quad (9)$$

where we consider the left moving modes only; the right moving modes have the similar form. The commutation relation for the creation and annihilation operators are familiar:

$$[a^\mu_m, a^\nu_n] = m\eta^{\mu\nu}\delta_{m+n}. \quad (10)$$

The hermiticity relations of the Virasoro generators $L^+_n = L_{-n}$ dictate that

$$a^{\mu+}_n = a^\mu_{-n} + i2Q\eta^{\mu 0}\delta_n. \quad (11)$$

This relation means that the zeroth component has a fixed imaginary part:

$$p^0 = E + iQ. \quad (12)$$

The spectrum can be constructed by acting on the tachyon state with the creation operators

$$| \phi >= a^{\alpha_1}_{-n_1} ......a^{\alpha_k}_{-n_k} | p; 0 >. \quad (13)$$

Using the method of OCQ quantization, we find the mass shell condition to be

$$m^2 = -\vec{p}^2 = -2 + N + \bar{N} + i2Qp^0 + Q^2 - i2EQ = -2 + N + \bar{N} - Q^2. \quad (14)$$

We see that the effect of the linear dilaton is to give a shift $Q^2$ to the mass spectrum of the critical string. Now the spectrum can be divided into three classes. Those states whose unshifted mass lie in the range $m^2 < 0$ are the true tachyons. Those states whose unshifted mass lies in the range $0 < m^2 < Q^2$ have negative mass square now, we call them pseudotachyons as in [18]. Those states which have unperturbed mass square $m^2 > Q^2$ are still massive in the linear dilaton theory. Intuitively one might think that the background is highly unstable due to the new emerging negative mass squared modes. However, we will see in the following section that those pseudo-tachyon modes are not so harmful as one might think.

It is straightforward to extend the solution to the heterotic and type II superstrings. The energy-momentum tensor and supercharge become:

$$T_B = -\frac{1}{2}\partial X_\mu \partial X^\mu + Q\partial^2 X^0 - \frac{1}{2}\psi_\mu \partial \psi^\mu, \quad (15)$$

$$T_F = -\psi_\mu \partial X^\mu + 2Q\partial\psi^0. \quad (16)$$

The anomaly cancelation condition becomes

$$D - 8Q^2 = 10. \quad (17)$$

For the bosonic sector of the heterotic string, the anomaly cancelation remains unchanged.

The quantization is similar to the bosonic case. One important observation is that there is no mass shift to the fermionic modes. This can be seen from several points of view. First the mass-shell condition for a lowest-lying Ramond state is

$$(E^2 + Q^2 - \vec{p}^2) = -1 + \frac{(D-2)}{8}, \quad (18)$$

Using the anomaly cancelation condition, we find $E^2 - \vec{p}^2 = 0$. Alternatively, this follows from the supercharge, whose moments are

$$G_n = i\sum_k \psi^\mu_{n-k} X_{\mu,k} - 2Q(k + \frac{1}{2})\psi^0_n. \quad (19)$$

When acting on a highest weight state the zeroth moment becomes

$$G_0 = -i(\gamma_0 E - \gamma\vec{p}), \quad (20)$$

which is precisely the massless Dirac operator. From the field-theory point of view, the quadratic piece of the lagrangian of a fermion in the linear dilaton background is

$$L = e^{2QX^0}(\bar{\psi}\partial_\mu\gamma^\mu\psi + m\bar{\psi}\psi). \quad (21)$$

We need to rescale the fields so that it has the canonical kinetic terms; the rescaled field $\tilde{\psi} = e^{QX^0}\psi$, however, obeys the free Dirac equation in flat space-time without a mass shift.

The quantization is similar for the $\kappa = 1$ solution; the corresponding world-sheet theory is a linear dilaton with a Wess-Zumino-Witten (WZW) $SO(3)$ model, and the central charge of this system is

$$c = 1 - 12Q^2 + \frac{3k}{k+1} = 4 - 12Q^2 - \frac{3}{k+1}. \quad (22)$$

Thus, the theory is non-critical. A subtle point is that the WZW curvature is different from the curvature in the Einstein frame: the WZW curvature is given by $Q^2\kappa$ [6]. A similar quantization of this solution gives the same mass-shift to the whole spectrum [6].

### B. Partition Function and Modular Invariance

The partition function for the SCLD reads [6]:

$$Z_D = V_D \int \frac{d^D k}{(2\pi)^D}(q\bar{q})^{-c/24-(-k^2+Q^2)/2} \sum_{oscillators} q^N \bar{q}^{\bar{N}}. \quad (23)$$

Setting $c = D - 12Q^2$, the partition function can be easily calculated:

$$Z(\tau) = iV_D(Z_X(\tau))^D, \quad (24)$$

where

$$Z_X(\tau) = (8\pi^2\tau_2)^{-1/2} | \eta(\tau) |^{-2}, \quad (25)$$

and $\eta(\tau)$ is the Dedekind $\eta$ function. Now, including the ghost part, the one-loop amplitude becomes:

$$Z_{T^2} = iV \int_F \frac{d\tau d\bar{\tau}}{32\pi^2\tau_2^2}(8\pi^2\tau_2)^{(2-D)/2}(\eta\bar{\eta})^{2-D}. \quad (26)$$

Using the modular transformation for the $\eta$ function, it is easy to show that the amplitude is modular invariant.

Next, we will analyze the IR and UV behaviors of the partition function. The general form for the partition function looks like

$$iV_D \int^\infty \frac{d\tau_2}{2\tau_2}(8\pi^2\tau_2)^{-D/2} \sum_i \exp(-2\pi m_i^2 \tau_2). \quad (27)$$

In the IR limit, which corresponds to $\tau_2 \to \infty$, we see that there is divergence due to the presence of a mode with negative squared mass. For the bosonic string, we have a tachyon in the spectrum, which signals an instability of our theory, and we are expanding around the wrong vacua. For the above cosmological solutions, we have new pseudotachyon modes besides the original tachyons. One may conclude that those modes signal the more severe instability of our background. However, we will show below that this is not the case due to the decreasing string coupling.

In the UV limit, since the dimensions that the string can oscillate in are greater than that of the critical string, we expect the high energy behavior to be modified, and the high energy density of states of the theory to be changed. This is true as we can see from direct calculations. The asymptotic density of states is (this can be calculated by using the similar method in the critical string [12][13])

$$\rho(m) \to e^{2\sqrt{2}\pi\sqrt{(D-2)/6}m} = e^{4\pi\sqrt{2+Q^2}m}. \tag{28}$$

The density of states for the critical string is reproduced when we set $Q = 0$.

Even if we did not know how to calculate the high energy density of states directly, the supercritical behavior can be seen from the modular invariance of the one-loop amplitude. The IR limit of the partition function is controlled by the lowest lying negative mass squared state with mass-squared $m^2 = -2 - Q^2$. The IR limit is

$$Z_{IR} \to e^{2\pi\tau_2(2+Q^2)}, \tag{29}$$

and due to the modular invariance under $\tau_2 \to \frac{1}{\tau_2}$, the UV limit should be

$$Z_{UV} \to e^{2\pi(2+Q^2)\frac{1}{\tau_2}}, \tag{30}$$

which shows that the number of dimensions in which string can oscillate is supercritical.

The above asymptotic form of the UV partition function can be checked by using the explicit form of the high energy density of states $\rho(m)$ we get above:

$$Z_{UV} \to \sum_m \rho(m) e^{-2\pi\tau_2 m^2}. \tag{31}$$

Summing over m by use of the saddle point approximation, we get that

$$Z_{UV} \to e^{2\pi(2+Q^2)\frac{1}{\tau_2}}, \tag{32}$$

which is exactly the same as we derived by using the modular invariance.

Some comments are needed here. We have seen that the time dependence of the background change the critical theory dramatically, in the IR limit, there is a mass shift to the lowest mass modes; in the UV limit, the high energy density of states is changed because the string lives in the supercritical dimensions. There is no mystery here. However, the relation between the IR behavior and UV supercriticality may provide a very novel way to generate new dimensions.

The idea is that if we start from certain critical time dependent backgrounds, we expect that there is mass shift to the critical spectrum, i.e. the pseudotachyon modes appear in the spectrum. Then, the background is actually supercritical from the requirement of modular invariance of partition function.

Before finding an example, there are two questions that we should answer. Since the pseudotachyons play a very important role in our construction, the natural questions are, does our background with the pseudotachyons make sense? Is it stable? This question will be answered in the next section; answers are positive and the background is stable at least in the late time period!

Another question is that where the supercritical behavior comes from, or other words where are the hidden dimensions? This is essence of our construction. Recall the famous example of T-duality, where we compactify string theory on a circle. When we shrink the radius of the circle, instead of losing a dimension, the winding modes will restore the dimension and so in the dual theory we have the equivalent theory with the same number of large dimensions. This provides a clue to the source of the supercriticality: The winding modes will provide the necessary degrees of freedom.

## III. PSEUDOTACHYON AND ITS RESOLUTION

The first question we mentioned in last section was discussed first in [6] and [14] (see also the discussion in [15], in which the pseudotachyon is called "good tachyon", for recent discussion, see [18]). We will give a short review of the result. We have seen that in our time dependent background, the time dependence will induce a mass shift to the various string modes, so that there will be some negative mass- squared particles if originally $0 < m^2 < Q^2$. Intuitively, these modes may signal the instability of our background, but we will show that these modes will not induce the large back-reaction to our background. This can be seen first from the effective field theory point of view. The quadratic piece of the lagrangian of a scalar field in the linear dilaton background is

$$L = e^{2QX^0}(-\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2). \tag{33}$$

Then we rescale the field $\tilde{\phi} = e^{QX^0}\phi$ so that the the kinetic term takes the canonical form, and $\tilde{\phi}$ obey the free wave equation in the flat space and with a rescaled physical mass $(m^2 - Q^2)$. Then for the pseudotachyon with $(m^2 - Q^2) < 0$, the modes $\tilde{\phi}$ is growing exponentially, but the fields $\phi$ is coupled to the dilaton, $\phi = e^{-QX^0}\tilde{\phi}$. We can then see that the exponentially decreasing factor of the dilaton will compensate the growing of the rescaled field $\tilde{\phi}$. So the pseudotachyons do not necessarily signal classical instability.

The real problem with the tachyon is not the fact that they grow exponentially with time, but that their back-reaction on the originally configuration grows exponentially with time, so that including this back-reaction moves us far away from our starting point. However this is not the case for our pseudotachyons , because our pseudotachyons couple to the other fields through the string coupling $g_s = e^{<\phi>}$,, and though the rescaled field grows exponentially, taking into account of the string coupling, the back reaction is in fact decays exponentially with time.

We can also consider this from the tachyon condensation point of view. The one-loop divergence in the partition function of SCLD shows that there is an instability toward producing exponentially-growing pseudotachyon fields, as has shown in [18], in the large $Q$ limit(this corresponds to the weakly coupled region), the tachyon condensation process is in exact agreement with the world-sheet RG flow, and since the corresponding deformation operators for the RG flow are constructed from the marginal and irrelevant operators. This fact means that the condensation process will not change the central charge. Therefore, the SCLD are consistent stable string backgrounds, even when the loop corrections are taken into account.

## IV. SUPERCRITICALITY FROM COSMOLOGICAL SOLUTIONS IN CRITICAL DIMENSION

We have examined the solutions we outlined in the section II, and we see that there is only one possibility where we can find a string solution which is of the critical dimension, namely if we take $\kappa = -1$, though we need to allow an imaginary B field. The metric of this background is

$$ds^2 = -dt^2 + t^2 ds^2_{H_3} + ds^2_{\perp}, \tag{34}$$

$$\Phi = -2QX^0, b = i2Q^2(\frac{e^{QX^0}}{Q}), \tag{35}$$

where $H_3$ is the 3 dimensional non-compact hyperbolic space with $\kappa = -1$, and $ds^2_{\perp}$ is a 6 dimensional internal compact space. The curvature of the WZW model is $-Q^2$ which is similar to the positive curvature case. In reference [17][18], they consider a time dependent string frame background which is rather different from our Einstein frame cosmological background here; their backgrounds correspond to a accelerating expanding version of our cosmological background in the Einstein frame.

Our background is a time dependent critical background with an expanding spacial slices. Actually it is nothing but the inside of a light cone in flat space $M_{1,3}$. We can formulate the world sheet theory in analogy with the positive curvature case. The corresponding world sheet sigma model is a non-compact $SU(2,C)/SU(2)$ WZW model with the linear dilaton. This non-compact WZW model has been studied in many papers in the context of AdS/CFT correspondence [19], [20], and for the Euclidean case ( which is of our interest here), in [21][22].

In the case of the SCLD background, the linear dilaton induces a mass shift to the whole spectrum; we might expect that there also are pseudotachyons for this background. However, there is a mass gap of the Laplacian on the hyperbolic manifold, so there are no pseudotachyons in the noncompact theory. This is reasonable, otherwise, the background is only part of the flat space. The linear dilaton and the curved geometry is a description of flat Minkowski background in a different gauge fixing procedure. Also if there are pseudotachyon modes, the theory is inconsistent since there are no extra central charge sources which can provide the necessary high energy density of states.

This background is trivial; we take a subgroup $\Gamma$ (to make the quotient space to be a compact manifold, there must be some constraints on the $\Gamma$ [23]) of the isometry group of the Hyperbolic 3 manifold $PSL(2,C)$. The metric then is

$$ds^2 = -dt^2 + t^2 ds^2_{M_3 = H_3/\Gamma} + ds^2_{\perp}. \tag{36}$$

In the IR limit, the compactness of the manifold removes the gap of the spectrum and we have the pseudotachyon; the similar analysis of the modular invariance dictates that the actual number of dimensions is supercrticial. The question is, what is the source for the extra dimensions? Luckily, there is a theorem by Milnor [24] which states that the compact manifold of negative sectional curvature has a fundamental group of exponential growth. Therefore the winding modes can probably provide enough degrees of freedom to make the theory supercritical. We will prove in the following section that it is indeed the case.

### A. Hyperbolic manifold and the IR spectrum

To simplify the discussion, we study the bosonic string in detail, and the result can be easily generalized to the superstring. The metric tensor of the hyperbolic manifold $H_n$ with $R = 1$ takes the form:

$$ds^2 = [dy^2 + \sinh^2 y d\Omega^2_{n-1}]. \tag{37}$$

The Laplace-Beltrami operator for this manifold reads:

$$\triangle = \frac{\partial^2}{\partial \sigma^2} + (n-1)\coth y \frac{\partial}{\partial y} + (\sinh y)^2 \triangle_{S^{N-1}}. \tag{38}$$

The eigenvalue equation is then $-\triangle\phi = \lambda\phi$, The solution of this equation can be solved by separation of variables, and solutions can be found in [23] :

$$\phi_\lambda = f_\lambda(y) Y_{lm} = \Gamma(\frac{n}{2})(\frac{\sinh y}{2})^{1-n/2} P^{\mu}_{-1/2+ir}(\cosh y) Y_{lm}. \tag{39}$$

Where $Y_{lm}$ is the spherical harmonic on $S_{n-1}$ and $P^{\mu}_{\nu}(x)$ is the associated Legendre functions of the first kind. The constant $r$ is related to the eigenvalue through the relation $\rho_n = (n-1)/2, r = \sqrt{(\lambda - \rho_n^2)}$. The asymptotic behavior for the radial wave function is

$$f_\lambda(y) \simeq \frac{2^n\Gamma(n/2)\Gamma(ir)}{4\pi^{1/2}\Gamma(\rho_n + ir)} e^{-\rho_n y + iry} + c.c. \qquad (40)$$

The radial functions will remain bounded at infinity provided the parameter $r$ is real, so we find that there is a gap in the spectrum i.e. $\lambda \geq \rho_n^2 = (n-1)^2/4$, $\lambda \geq 1$ for $n = 3$.

The coset $H_3^+ = SL(2,C)/SU(2)$ is the set of all hermitian two-by-two matrices $h$ with determinant one. We start with the action [21],

$$S[hh^+] = \frac{k}{\pi}\int d^2z(\partial_z\psi\partial_{\bar{z}}\psi + (\partial_z + \partial_z\psi)\bar{v}(\partial_{\bar{z}} + \partial_{\bar{z}}\psi)v)), \qquad (41)$$

where $hh^+$ is the parametrization of the $H_3$:

$$hh^+ = \begin{pmatrix} e^{\psi}(1+\mid v\mid^2)^{1/2} & v \\ \bar{v} & e^{-\psi}(1+\mid v\mid^2)^{1/2} \end{pmatrix}.$$

This is a conformal field theory with the central charge $c_1 = 3k/(k-2)$, and the central charge of our system is then

$$c = 1 - 12Q^2 + \frac{3k}{k-2} + c_I. \qquad (42)$$

This separation makes sense only when the hyperbolic-radius is larger than the plank length, i.e. $k \gg 1$. The level $k$ is related to the Einstein frame space curvature as $1 = 1/(2Q^2k)$ [6]; so the radius of the WZW target space is $R = \sqrt{1/Q^2} = \sqrt{2k}$.

To satisfy the anomaly cancelation condition, the dilaton can be chosen as

$$\Phi = -2QX^0, \qquad Q = \sqrt{\frac{1}{2(k-2)}}, \qquad (43)$$

Notice that the central charge of the linear dilaton is $c_2 = 1 - 12Q^2$, and the central charge of the four dimensional cosmological solutions is $c_1 + c_2 = 4$, the theory is indeed critical.

We now analyze the spectrum of the theory. The mass-shell condition of the this CFT reads:

$$(L_0 + \bar{L_0} - 2)\mid \Psi, N, \tilde{N}, p >= 0, \qquad (44)$$

Where $L_0$ includes both the Euclidean $AdS$ part and the internal part; for the $AdS$ part, the conformal weight of the states [25] is

$$L_0 \mid \psi >= -\frac{j(j-1)}{k-2}|\psi >, \qquad (45)$$

where $j$ parametrizes the representations of the $SL(2,C)$, with the value $j = 1/2 + is$, so the conformal weight of the states becomes

$$\Delta(j) = \frac{1/4 + s^2}{k-2}. \qquad (46)$$

The minimum of the conformal weight is $\Delta(j) = \frac{1}{4(k-2)}$, and similar analysis on the right hand modes give the same result. The mass squared of this state is therefore

$$m^2 = \frac{1}{4(k-2)} + \frac{1}{4(k-2)} - 2 - Q^2. \qquad (47)$$

Since $Q^2 = \frac{1}{2(k-2)}$, we see that the minimum mass squared of the spectrum is $m^2 = -2$, which is the same as that of the critical bosonic string.

This result can also be seen from the harmonic analysis of $H_3^+$ space. The standard Sugawara construction gives the Hamiltonian $L_0 = -\frac{1}{4(k-2)}\triangle$ where $\triangle$ denotes the Laplace-Beltrami operator on $H_3^+$. We have proved that there is a gap in the spectrum for the Laplace-Beltrami operator, so $L_0$ is bounded from below with the minimum value $\frac{1}{4(k-2)}$. Including the right hand modes and the linear dilaton, the result is the same as we found above. We conclude that the minimum mass does not change. This result is reasonable since our background is only part of the flat space time.

Now we take a quotient of the $H_3$ and get a compact manifold. The background is still the solution to the string equations of motion, and our world-sheet theory involves the quotient of the previous sigma model. It seems that the theory is still living in the critical dimension, but careful analysis of IR and UV behavior implies that our background is supercritical.

It will give us some insight about what is happening to our system if we recall what the theory changes when we compactify sting theory on a circle. One the one hand, the momentum of the center of motion is quantized; on the other hand, the modular invariance of the partition function requires the new contributions to the spectrum, and those winding strings which wrap on the circle give the desired excitations.

For the compact daughter space $M_3 = H_3/\Gamma$, the situation is rather similar: due to the compactness of the manifold, the spectrum is discrete and the new novel feature is that the original gap has been removed! The spectrum begins from $\lambda = 0$ [26](since there is no renormalizable conditions on the wavefunction for the compact manifolds).

The quotient changes both the IR behavior and UV behavior of the theory. In the IR limit, due to the disappearance of gap, the minimum mass becomes $m^2 = -2 - \frac{1}{2(k-2)}$; The mass-shift of the minimum mass signals the emergence of new dimensions as we see from the modular invariance condition; but where are the new dimensions? The analogy with the circle case implies that the new dimensions may hide as the winding strings. One clue indicates that it is indeed the case. In the UV limit, due to the rich topology of the hyperbolic manifold, i.e. the fundamental group is increasing exponentially, the density of states of the theory is changed drastically and the effect dimensions of the theory is indeed supercritical. We will confirm this in the next section.

While we have only studied the bosonic string in the above analysis, the same analysis can be easily applied to the bosons of the superstring. The conclusion is that the minimum mass-squared shifts from zero to $m^2 = -\frac{1}{2(k-2)}$, and as the result of the fermionic mode in SCLD background, there is no mass shift to the fermions, so the minimal mass squared of the theory is $m^2 = -\frac{1}{2(k-2)}$.

### B. Modular Invariance and Supercriticality

Modular invariance is a very important consistency condition for the string theory, and it also plays a important role in studying the high energy density of states of the theory [27].

Recall that, for our particular CFT, in the IR limit, the minimum mass squared is

$$m^2_{min} = -\frac{1}{2(k-2)}. \tag{48}$$

We are interested in the large $k$ limit where we can have the four large macroscopic dimensions; The minimum is approximately $m^2_{min} = -\frac{1}{2k\alpha'}$ (the $\alpha'$ is included from the dimensional analysis). The IR limit of the partition function looks like

$$Z_{IR} \sim \exp\left(\frac{\pi\tau_2}{2k}\right), \tag{49}$$

. Due to the modular invariance, the UV limit should look like

$$Z_{UV} \sim \exp\left(\frac{\pi}{2k\tau_2}\right). \tag{50}$$

From our study of the SCLD partition function, we can conclude that the actual number of dimensions in which the string can freely oscillate is supercritical.

We want to check this relation by direct calculation as we have done in the SCLD case. There is a powerful tool that we can use to calculate the asymptotical behavior of the partition function, the Selberg trace formula (see [23] for a detailed explanation of the formula). The formula relates the quantum quantity to the counting of the classical orbits. The Selberg trace formula is

$$\sum_{j=0}^{\infty} h(r_j) = \Omega(F_n)\int_0^{\infty} h(r)\Phi_n(r)dr + \sum_{\{\gamma\}}\sum_{n=1}^{\infty}\frac{\chi^n(\gamma)l_\gamma}{S_3(n;l_\gamma)}\hat{h}(nl_\gamma), \tag{51}$$

where $h$ is a function defined on the compact hyperbolic manifold with some analytic constraints; $l_\gamma$ is the length of the nontrivial cycle. $S_3(n;l_\gamma)$ is a definite function of $n$ and $l_\gamma$. The summation over $n$ is a sum over winding numbers and summation over $\gamma$ is a sum over all the nontrivial cycles.

We can use this trace formula to compute $Tr\exp(\Delta_\Gamma)$ over the $L^2(H_3/\Gamma)$, which is essential in the calculation of the partition function. Then $\hat{h}(nl_\gamma) = e^{-(nl_\gamma)^2\tau_2/4\pi\alpha'}$. In the UV limit, the summation over $n$ and $\gamma$ can be transformed to a integral over the length of nontrivial circle. In this limit, the function $S_3(n,l_\gamma)$ is simply

$$\frac{1}{S_3(n,l_\gamma)} = \left(\frac{1}{\sinh(l/2\sqrt{2k\alpha'})}\right)^2, \tag{52}$$

where we have used the fact that $R = \sqrt{2k\alpha'}$; and now $\hat{h}(nl_\gamma) = e^{-\tau_2 l^2/4\pi\alpha'}$. According to the Milnor's theorem, the density of periodic geodesics is given by [28][29][30]:

$$\rho(l) = \frac{1}{l}e^{2l/\sqrt{2k\alpha'}}. \tag{53}$$

Altogether, the UV partition function is then

$$\int dl e^{2l/\sqrt{2k\alpha'}}\left(\frac{1}{\sinh(l/2\sqrt{2k\alpha'})}\right)^2 e^{-l^2\tau_2/(4\pi\alpha')}. \tag{54}$$

Using the saddle point approximation, this integral gives the result

$$Z_{UV} \sim \exp\left(\frac{\pi}{2k\tau_2}\right). \tag{55}$$

. This is in exact agreement with the result from the modular invariance.

So far we only consider the bosonic sector. For the fermions, as we have seen in the Heterotic and Type II case, there is no mass shift to the fermionic modes due to the linear dilaton. So we would expect that there is no exponential contributions in the UV limit, so that there is no boson fermion cancelation (See also the discussion in [17] ). It is interesting to find a specific co-compact subgroup of $PSL(2,C)$ and calculate the exact partition function.

We have shown that for this specific critical background with compact negative curvature slice the theory is indeed supercritical. One may wonder whether we can find a dual description in which the supercriticality can be manifest in a geometrical description like we usually see in the linear dilaton theory. We may find a dual conformal field theory in which the supercriticality is manifest by using different kinds of gauge fixing procedure along the line [31]. For the accelerating expanding background [17], there is a paper [32] which appeared recently, which shows that the "D-dual" description of a Riemann surface of genus $h$ can be expressed in terms of its $2h$ dimensional Jacobian torus, perturbed by a closed string tachyon arising as a potential energy term in the worldsheet sigma model (see also the discussion on the dualized D brane in [33]). Since our manifold contains a $AdS$ factor (the Euclidean version), we may study the dual theory from the $AdS/CFT$ correspondence as discussed also in [32].

Another interesting observation of the accelerating expanding solution is that we have the dynamical dimension reduction. Since the central charge is decreasing with time (the dilaton is $\phi = -2QX^0$, and $Q(t) = \frac{1}{t}$ [17], here $t$ is the Einstein time and it is rescaled so that $t = R$, where $R$ is the radius of the hyperbolic space; note that for our background, $Q = \sqrt{\frac{1}{2(k-2)}} = \frac{1}{R}$ in the large k limit), and from the dual picture, the geometric dimensions are decreasing due to the tachyon condensation. For detailed description of dimension changing solutions in the supercritical string theory, see [34],[35]. One of important questions about string cosmology is to understand why our universe can reduce down to four dimensions while the string theory is usually formulated in ten dimensions. The solution discussed above demonstrates that the cosmological solution can dynamically reduce the space dimension. This may give a dynamical explanation of why we are living in four dimensions. A natural question is why the transition stops at $D = 4$. A possible explanation is provided in [13]. As noted in section II, the high energy density of states is

$$\rho(m) \propto m^{-D} \exp^{4\pi\sqrt{Q^2+2}m}. \tag{56}$$

A level density of this type leads to a phase transition at a temperature $T = \frac{1}{4\pi\sqrt{Q^2+2}}$ [36]. As the universe is cooling down, there are a couple of phase transitions which dynamically reduce the space time dimensions. The dynamical dimension reduction must stop at $D = 4$ as shown in [13], because for this level density, the pressure and density is infinite for $D < 4$ near the transition temperature, while the pressure and density for $D \geq 4$ is finite near the transition point, so the state with $D < 4$ can not be obtained from a phase with finite pressure and density i.e. $D = 4$. The universe stops at $D = 4$.

## V. THE RELATION WITH THE SUPERCRITICAL STRING COSMOLOGY

If the universe is described by critical string theory, the total dimensions of the flat spatial coordinates and the internal space must add up to $c_{tot} = 25(\hat{c}_{tot} = 9$ in superstring) at the present stage of evolution. As we mentioned as in the Introduction, it is very hard to find a critical solution describing the accelerating universe.

However, one may speculate that the total number of space-time dimensions has been larger or smaller than 25 (for clarity, we take the bosonic string as the example) in the early universe, [6]. Moreover, one may propose that the present universe is in a supercritical phase [37][38]. With this assumption, we can find the accelerating solutions. This proposal is supported in [39], which proves that the space of general time-dependent solutions of classical bosonic string theory contains attractors which are static solutions with spatial CFT parts which are minima of $\mid \bar{c} - 25 \mid$, where $\bar{c}$ is a generalized Zamolodchikoc's c-function.

Our particular model supports this proposal, and corresponds to the case $c_{tot} = 25$ or $\hat{c} = 9$ for the superstring, where the critical limit ($t \longrightarrow \infty$, flat space limit) is the fixed point of our time-dependent solution, and at the present time the universe is in a supercritical phase with central charge $c > 25$.

For the general case with fixed point $c = 25$, the equations at the vicinity of this fixed point are [2][3][39][38]:

$$\ddot{\vec{\lambda}} + Q(t)\dot{\vec{\lambda}} + O(\dot{\lambda}^2) = -\vec{\beta}, \tag{57}$$

where $\vec{\lambda}$ is the background fields and $\beta$ is the world-sheet $\beta$ function, i.e. the time evolution of the solutions is the same as the world sheet renormalization group flow. It has been shown that for the accelerating expanding model $Q(t) = 1/t$ [17], and one interesting thing is that in [37][38][39] it has been shown that for the $SO(3)$ WZW model with $c_{tot} = 25$, $Q(t)$ is proportional to $1/t$.

## VI. DISCUSSION AND CONCLUSION

In this paper, we find that the simple non-trivial cosmological solutions with the compact hyperbolic spatial slices, although formulated in critical dimension, are indeed supercritical. With the previous results on the $\kappa = 0$ and $\kappa = 1$ case, we may conclude that for all the simple non-trivial cosmological solutions, the theory is indeed supercritical. In the framework of the non-critical string theory, the space time dimensions can be reduced dynamically through the tachyon condensation, and we found a possible explanation of why we are living in $D = 4$: the transition must stop at $D = 4$ because the temperature and pressure is infinite near the transition point for $D < 3$ while those thermodynamical quantities are finite for $D \geq 4$. A state with infinite temperature and pressure cannot be obtained from a state with finite thermodynamical quantities.

We also found an interesting way to generate the supercritical behavior from the critical string, namely to produce the pseudotachyon. We analyzed an example that exhibits the desired behavior. The simplicity and solidity of our treatment suggests that supercritical behavior is very important in studying the time dependent background in string theory. It is interesting to see whether the similar phenomenon appears in the conventional compactification like flux compactification.

The above phenomenon is very interesting, from the duality point of view. We have found that degrees of freedom coming from the winding modes give new contributions to the effective central charge, and the theory is indeed supercritical. There may exist a dual theory in which the geometric dimensions is different from the original theory. The main difference of this duality is, the two dual theories may have different space time dimensions. This is a new type of duality: D-duality [32].

Some generalization are in order. First, it is not necessary that the manifold must be compact; as long as there is a compact section in the manifold, it is possible to have the supercritical behavior. Second is the question of whether we can derive the conventional duality like the heterotic $K3$ duality from this mechanism. If we can think of $K3$ surface as a manifold containing a hyperbolic section and we take a suitable dilaton profile, we may give a semi-classical explanation of the $K3$ duality.

This model may also provide a possible solution to supersymmetry breaking in string theory. In [6][40], it is proposed that the charge deficit coming from the internal manifold can generate a tree level cosmological constant, but the problem is that to get a small cosmological constant since the internal manifold cannot be kept at the small size, the internal dimensions will decompactify. Now if we take the compact hyperbolic manifold $H^3/\Gamma$ as the spatial slice of our Universe. Since the theory is supercritical and we have the potential term coming from the winding string, supersymmetry is broken (this mechanism may signal the high energy scale SUSY breaking). For this kind of model we do not have the decompactification problem. See also [37][38].

From the cosmological point of view, since the central charge of our system is changing with the time, to restore the full conformal invariance, the dilaton is sourced, we have the time varying central charge deficit $Q^2(t)$. This is just the description of Q cosmology introduced in the reference [38]. One may worry that the space manifold must be compact so that we have the supercritical behavior. This is not the necessary condition, localized psuedotachyon can produce the supercritical behavior as well, and those psuedotachyons can possibly be produced in the early universe, for instance, in the inflationary scenario. In string theory, those modes can be produced from the catastrophic events like big bang, brane collisions, etc.

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