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Consistent Warped-Space Kaluza-Klein Reductions, Half-Maximal Gauged Supergravities and CP^n Constructions

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ABSTRACT

We obtain new consistent Kaluza-Klein embeddings of the gauged supergravities with half of maximal supersymmetry in dimensions $D = 7$, 6, 5 and 4. They take the form of warped embeddings in type IIA, type IIB, M-theory and type IIB respectively, and are obtained by performing Kaluza-Klein circle reductions or T-duality transformations on Hopf fibres in S^3 submanifolds of the previously-known sphere reductions. The new internal spaces are in some sense "mirror manifolds" that are dual to the original internal spheres. The vacuum AdS solutions of the gauged supergravities then give rise to warped products with these internal spaces. As well as these embeddings, which have singularities, we also construct new non-singular warped Kaluza-Klein embeddings for the $D = 5$ and $D = 4$ gauged supergravities. The geometry of the internal spaces in these cases leads us to study Fubini-Study metrics on complex projective spaces in some detail.

1 Introduction

It has long been known that gravity coupled to antisymmetric tensors can allow AdS×Sphere solutions [1]. Such configurations occur for eleven-dimensional supergravity and type IIB supergravity, where they give rise to solutions $\text{AdS}_4 \times S^7$, $\text{AdS}_7 \times S^4$ and $\text{AdS}_5 \times S^5$, which preserve maximal supersymmetry [2]. At the linearised level the fluctuations around these backgrounds correspond to the fields of the associated maximal gauged supergravities [3, 4, 5, 6]. It was expected that the reductions on S^7, S^4 and S^5 would in fact be consistent at the full non-linear level, and indeed for the S^7 reduction [7] and the S^4 reduction [8] this has been demonstrated. As well as these maximally-supersymmetric reductions, the explicit non-linear reductions with half of maximal supersymmetry have been obtained for the S^7 and S^4 cases [9, 10], and in addition for the S^5 example [11]. Although no results for the complete maximally-supersymmetric reduction of type IIB supergravity on S^5 exist, further supporting evidence has been obtained by constructing the complete S^5 reduction of the $SL(2,\mathbb{R})$-singlet sector of the type IIB theory [12].

The higher-dimensional interpretation of six-dimensional gauged supergravity is more subtle. First of all, gauged supergravity in $D = 6$ can at most have half of the supersymmetry that is possible in the ungauged theory [13]. It was suggested that the gauged theory might be related to the massive type IIA supergravity [14]. Indeed, it was shown in [15] that the massive type IIA theory admits a warped product of AdS_6 and S^4, with a warp factor in front of the AdS_6 metric that depends on a coordinate of the S^4. This solution can be derived [15] as a near-horizon limit of a semi-localised D4/D8-brane intersection [16]. The fully non-linear consistent embedding of the six-dimensional $N = 2$ $SU(2)$-gauged supergravity in the massive type IIA theory was subsequently constructed in [17]. It was recently observed [18] that AdS_6 could also be embedded in type IIB theory, as the near-horizon limit of a semi-localised intersecting D3/D5/NS5-brane system. In this paper, we shall obtain the non-linear embedding of the six-dimensional $N = 2$ gauged supergravity in type IIB.

Recently, it was shown that AdS_5 can also arise in a warped spacetime solution of M-theory, as the near-horizon geometry [19, 20] of a semi-localised M5/M5-brane intersection [16]. A large class of analogous solutions involving warped products of AdS and an internal space were subsequently constructed [18], arising as semi-localised intersections of two or more p-branes. In all these cases, the warp factors that multiply the AdS metrics depend only on certain coordinates of the internal spaces. In this paper, we shall consider those warped configurations that are associated with intersecting branes that preserve half of

maximal supersymmetry in their near-horizon regions.

As in the case of the maximally-supersymmetric direct-product AdS×Sphere solutions, one might expect that the occurrence of the half-maximally supersymmetric warped AdS solutions would also presage the possibility of obtaining the associated half-maximally super-symmetric gauged supergravities by consistent Kaluza-Klein reduction on the corresponding internal spaces.

In this paper we construct a variety of examples of such half-maximally supersymmetric gauged supergravities, arising as consistent Kaluza-Klein reductions. We obtain them by starting with the previously-known maximally-supersymmetric sphere reductions, and exploiting the fact that the internal sphere S^n can itself be viewed as a foliation of $S^p \times S^q$ with $n = p + q + 1$. In all the examples $n \geq 4$, and so we can arrange that at least one of p or q is equal to 3. We then perform a standard S^1 Kaluza-Klein reduction on the $U(1)$ fibre of S^3 viewed as the Hopf bundle over S^2. The gauge groups that are compatible with the $S^p \times S^q$ structures are smaller than the $SO(n+1)$ groups of the maximally-supersymmetric theories, and in fact in all cases the subgroups that *are* compatible are precisely those of the half-maximally supersymmetric supergravities. These are the theories that whose consistent reductions were obtained in [10, 17, 11, 9].

In section 2 we carry out this procedure for the $N = 2$ $SU(2)$-gauged theory in $D = 7$, obtained as an S^4 reduction from $D = 11$. The S^4 is viewed as a foliation of S^3 surfaces, on which the $SU(2)$ gauge group acts transitively. We perform a reduction on the $U(1)$ Hopf fibres of S^3, thereby arriving at a reduction of type IIA supergravity that yields the same $N = 2$ gauged theory in $D = 7$. In section 3 we consider the warped S^4 reduction of the massive type IIA theory. Since this is already half-maximally supersymmetric, it is already compatible with the S^3 structure of the foliation. We reduce this on the Hopf fibres to $D = 9$, and after performing a T-duality transformation we arrive at a type IIB embedding of the six-dimensional $N = 2$ $SU(2)$-gauged supergravity. In section 4 we start from the half-maximally supersymmetric reduction of type IIB supergravity on S^5, whose $SU(2) \times U(1)$ gauge group is precisely compatible with the $S^3 \times S^1$ foliation of S^5. After a Hopf reduction and T-duality transformation, we obtain a type IIA embedding of the five-dimensional $N = 4$ $SU(2) \times U(1)$-gauged supergravity. In section 5 we start from the half-maximally supersymmetric reduction of eleven-dimensional supergravity on S^7, whose $SO(4) \sim SU(2) \times SU(2)$ gauge group is compatible with an $S^3 \times S^3$ foliation. Here there are two bundles, associated with the two S^3 factors, and so we are able to reduce first to a type IIA embedding, and then after a second reduction we can obtain an embedding of the

four-dimensional $N = 4$ $SO(4)$-gauged supergravity in type IIB. In all these examples, the embeddings that we obtain are of the form of warped Kaluza-Klein reductions. Indeed, the AdS vacuum solutions of the half-maximally supersymmetric lower-dimensional theories all lift to give the warped products that were found in [19, 20, 18].

A characteristic feature of these warped Kaluza-Klein reductions is that the warp-factor that multiplies the lower-dimensional spacetime metric is singular, tending to zero at one end of the range of a coordinate in the internal space.[1] In the cases of those originating from the S^5 and S^7 reductions, a slightly more general type of Hopf reduction can be performed, which again involves a warped product structure, but now with an entirely non-singular warp factor. This is possible because in these two cases the foliating surfaces are actually products of *two* odd-dimensional spheres ($S^3 \times S^1$ and $S^3 \times S^3$ respectively). Thus we have two natural $U(1)$ Killing directions in each case, associated with the Hopf fibres over S^3 or from the S^1 factor. For a single step of reduction it is now possible to take a linear combination of the two $U(1)$ Killing directions, and use this as the circle for the S^1 reduction. Since the radii of the two odd-dimensional spheres in the foliation never vanish simultaneously, this means that the radius of the circle on which the Kaluza-Klein reduction is performed never vanishes. As a consequence, the resulting warped Kaluza-Klein reduction is then entirely non-singular. In fact, for the most natural choice of linear combination of the Killing directions, it turns out that the internal space after the S^1 reduction is a complex projective space; CP^2 in the S^5 case, and CP^3 in the S^7 case. In fact, the Kaluza-Klein reductions that we obtain in these cases correspond, in the vacuum, to the $AdS_5 \times CP^2$ and $AdS_4 \times CP^3$ backgrounds that were considered in [23, 24, 25]. The non-singular warped Kaluza-Klein reductions are discussed in section 6.

Motivated by the occurrence of CP^2 and CP^3 in the non-singular warped reductions, in an appendix we study some of the related geometrical aspects of the Fubini-Study metrics on complex projective spaces. We obtain general constructions for CP^{m+n+1} in terms of a product of CP^m and CP^n spaces. Applying this to the case $m = n = 1$ gives a very simple explicit construction of the Fubini-Study metric on CP^3.

Finally we note that the subjects of sphere compactification and semi-localised intersecting p-brane solutions were extensively discussed in the literature, see additional references, *e.g.* [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 47, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56, 58, 59, 60, 61, 62]

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[1]Non-singular embedding of AdS_5 in M-theory and AdS_3 in type IIB were recently constructed in [21, 22] and [22] respectively.

2 $D = 7$ $SU(2)$-gauged $N = 2$ supergravity from type IIA

The embedding of seven-dimensional $N = 2$ $SU(2)$-gauged supergravity in $D = 11$ was obtained in [10], in a framework where the 4-sphere is described as a foliation of S^3 surfaces. In this section, we take this construction as our starting point, and then perform a Kaluza-Klein reduction on the $U(1)$ fibres of the S^3 foliations, thereby obtaining an embedding of the seven-dimensional theory in type IIA supergravity.

From [10], we have the Kaluza-Klein reduction Ansatz for the S^4 reduction from $D = 11$, where we truncate to the bosonic sector of $N = 2$ gauged $SU(2)$ supergravity in $D = 7$. The metric reduction is given by

$$d\hat{s}_{11}^2 = \Delta^{1/3} ds_7^2 + 2g^{-2} X^3 \Delta^{1/3} d\xi^2 + \tfrac{1}{2} g^{-2} \Delta^{-2/3} X^{-1} c^2 \sum_i h_i^2 , \tag{1}$$

where

$$\begin{aligned}
\Delta &= X c^2 + X^{-4} s^2 , \\
c &\equiv \cos\xi , \qquad s \equiv \sin\xi , \\
h_i &\equiv \sigma_i - g A_{(1)}^i .
\end{aligned} \tag{2}$$

The left-invariant 1-forms of $SU(2)$ are given by

$$\begin{aligned}
\sigma_1 &= \cos\psi \, d\theta + \sin\psi \, \sin\theta \, d\varphi , \\
\sigma_2 &= -\sin\psi \, d\theta + \cos\psi \, \sin\theta \, d\varphi , \\
\sigma_3 &= d\psi + \cos\theta \, d\varphi .
\end{aligned} \tag{3}$$

X is the scalar field in the seven-dimensional $N = 2$ gauged supergravity, and $A_{(1)}^i$ are the $SU(2)$ gauge fields.

It is evident that $\partial/\partial\varphi$ is a Killing direction, and so we can perform a Kaluza-Klein circle reduction on the φ coordinate. In order to do this, it is convenient to make the following redefinitions:

$$\begin{aligned}
z &= \frac{1}{\sqrt{2}\,g} \varphi , \\
A_{(1)}^{ij} &= \epsilon_{ijk} A_{(1)}^k , \\
\mu_1 &= \sin\theta \sin\psi , \qquad \mu_2 = \sin\theta \cos\psi , \qquad \mu_3 = \cos\theta .
\end{aligned} \tag{4}$$

We also define the gauge-covariant exterior derivative

$$D\mu^i = d\mu^i + g A_{(1)}^{ij} \mu^j . \tag{5}$$

Note that the coordinates μ^i satisfy

$$\mu^i \, \mu^i = 1 \,. \tag{6}$$

It now follows that the 1-forms h_i can be written as

$$h_i = -\epsilon_{ijk} \, \mu^j \, D\mu^k + \sqrt{2} \, g \, \mu^i \, (dz + \mathcal{A}_{(1)}) \,, \tag{7}$$

where

$$\mathcal{A}_{(1)} = \frac{1}{\sqrt{2} \, g} \, \cos\theta \, d\psi + \tfrac{1}{2\sqrt{2}} \, \epsilon_{ijk} \, A^{ij}_{(1)} \, \mu^k \,. \tag{8}$$

Of course the first term could be expressed in terms of an object $\omega_{(1)}$ such that $d\omega_{(1)} = \Omega_{(2)}$, the volume form of the 2-sphere, rather than using $\cos\theta \, d\psi$. It also follows that

$$\sum_i h_i^2 = \sum_i (D\mu^i)^2 + 2g^2 \, (dz + \mathcal{A}_{(1)})^2 \,, \tag{9}$$

Substituting this expression into (1), we obtain the eleven-dimensional metric in the form

$$
\begin{aligned}
d\hat{s}^2_{11} = {} & \Delta^{1/3} \, ds_7^2 + 2g^{-2} \, X^3 \, \Delta^{1/3} \, d\xi^2 + \tfrac{1}{2} g^{-2} \, \Delta^{-2/3} \, X^{-1} \, c^2 \sum_i (D\mu^i)^2 \\
& + \Delta^{-2/3} \, X^{-1} \, c^2 \, (dz + \mathcal{A}_{(1)})^2 \,.
\end{aligned}
\tag{10}
$$

Comparing this with the standard Kaluza-Klein S^1 reduction from $D = 11$ to $D = 10$,

$$d\hat{s}^2_{11} = e^{-\frac{1}{6}\phi} \, ds^2_{10} + e^{\frac{4}{3}\phi} \, (dz + \mathcal{A}_{(1)})^2 \,, \tag{11}$$

we find that the metric Ansatz describing the embedding of seven-dimensional $N = 2$ gauged $SU(2)$ supergravity in $D = 11$ can be reinterpreted as an embedding in type IIA supergravity, with the metric and dilaton given by

$$
\begin{aligned}
ds^2_{10} &= X^{-1/8} \, c^{1/4} \left[\Delta^{1/4} \, ds_7^2 + 2g^{-2} \, X^3 \, \Delta^{1/4} \, d\xi^2 + \tfrac{1}{2} g^{-2} \, X^{-1} \, \Delta^{-3/4} \, c^2 \sum_i (D\mu^i)^2 \right] \,, \\
e^\phi &= X^{-3/4} \, \Delta^{-1/2} \, c^{3/2} \,.
\end{aligned}
\tag{12}
$$

The Ansatz for the vector potential of the type IIA theory is given by (8). The metric of the vacuum solution corresponds to taking $X = 1$, $A^{ij}_{(1)} = 0$ and ds_7^2 to be AdS$_7$. This solution can be viewed as the near-horizon limit of the semi-localised NS5/D6-brane system [18].

We can now reduce the original S^4 Ansatz for the 3-form potential $\hat{A}_{(3)}$ of $D = 11$ supergravity in an analogous way. It is given in [10], and takes the form

$$\hat{A}_{(3)} = s \, A_{(3)} + \tfrac{1}{2\sqrt{2}} \, g^{-3} \, (2s + s \, c^2 \, X^{-4} \, \Delta^{-1}) \, \epsilon_{(3)} - \tfrac{1}{\sqrt{2}} g^{-2} \, s \, F^i_{(2)} \wedge h_i - \tfrac{1}{\sqrt{2}} \, g^{-1} \, s \, \omega_{(3)} \,, \tag{13}$$

where

$$\epsilon_{(3)} \equiv h_1 \wedge h_2 \wedge h_3 \,, \tag{14}$$

and

$$\omega_{(3)} \equiv A^i_{(1)} \wedge F^i_{(2)} - \tfrac{1}{6} g\, \epsilon_{ijk}\, A^i_{(1)} \wedge A^j_{(1)} \wedge A^k_{(1)} \,. \tag{15}$$

By comparing this with the standard S^1 reduction of $\hat{A}_{(3)}$ to $D = 10$, with ten-dimensional fields $\bar{A}_{(3)}$ and $\bar{A}_{(2)}$ defined by

$$\hat{A}_{(3)} = \bar{A}_{(3)} + \bar{A}_2 \wedge (dz + \mathcal{A}_{(1)}) \,, \tag{16}$$

we can read off the Asätze for $\bar{A}_{(3)}$ and $\bar{A}_{(2)}$. To do this, the following *lemmata* are useful:

$$\epsilon_{(3)} = \tfrac{1}{\sqrt{2}}\, g\, \epsilon_{ijk}\, \mu_\kappa\, D\mu^i \wedge D\mu^j \wedge (dz + \mathcal{A}_{(1)}) \,,$$
$$F^i_{(2)} \wedge h_i = -F^{ij}_{(2)} \wedge (\mu^i\, D\mu^j) + \tfrac{1}{\sqrt{2}}\, g\, \epsilon_{ijk}\, \mu^k\, F^{ij}_{(2)} \wedge (dz + \mathcal{A}_{(1)}) \,. \tag{17}$$
$$\tfrac{1}{2}\epsilon_{ijk} F^i_{(2)} \wedge h_j \wedge h_k = \tfrac{1}{2} F^{ij}_{(2)} \wedge D\mu^i \wedge D\mu^j - \tfrac{1}{\sqrt{2}}\, g\, \epsilon_{ijk}\, F^{ij}_{(2)} \wedge D\mu^k \wedge (dz + \mathcal{A}_{(1)}) \,.$$

From these results, it then follows that the Ansätze for the ten-dimensional 3-form and 2-form potentials defined by (16) are

$$\bar{A}_{(3)} = s\, A_{(3)} + \tfrac{1}{\sqrt{2}}\, g^{-2}\, s\, F^{ij}_{(2)} \wedge (\mu^i\, D\mu^j) - \tfrac{1}{\sqrt{2}}\, g^{-1}\, s\, \omega_{(3)} \,,$$
$$\bar{A}_{(2)} = \tfrac{1}{4}\, g^{-3}\, s\, \mu^k\, \epsilon_{ijk} \left((2 + c^2\, X^{-4}\, \Delta^{-1})\, D\mu^i \wedge D\mu^j - 2g\, F^{ij}_{(2)} \right) \,. \tag{18}$$

Finally, note that the expression (8) for the Kaluza-Klein vector $\mathcal{A}_{(1)}$ can be shown, after some algebra, to imply the following expression for the corresponding field strength $\mathcal{F}_{(2)} \equiv d\mathcal{A}_{(1)}$:

$$\mathcal{F}_{(2)} = \tfrac{1}{2\sqrt{2}}\, g^{-1}\, \epsilon_{ijk}\, \mu^k\, D\mu^i \wedge D\mu^j - \tfrac{1}{2\sqrt{2}}\, \epsilon_{ijk}\, \mu^k\, F^{ij}_{(2)} \,. \tag{19}$$

In fact this expression for $\mathcal{F}_{(2)}$ allows a rewriting of $\bar{A}_{(2)}$, given in (18), in a slightly more elegant way:

$$\bar{A}_{(2)} = \sqrt{2}\, g^{-2}\, s\, \mathcal{F}_{(2)} + \tfrac{1}{4}\, g^{-3}\, s\, c^2\, X^{-4}\, \Delta^{-1}\, \epsilon_{ijk}\, \mu^k\, D\mu^i \wedge D\mu^j \,. \tag{20}$$

3 $D = 6$ $SU(2)$-gauged $N = 2$ supergravity from type IIB

The embedding of six-dimensional $N = 2$ $SU(2)$-gauged supergravity in the massive type IIA theory was obtained in [17]. Here, we apply a similar $U(1)$ Hopf reduction to this local S^4 reduction, thereby obtaining a reduction of $D = 9$ supergravity to the $D = 6$ gauged theory. Then, by applying the standard T-duality rules, we can lift the nine-dimensional theory back to $D = 10$, expressed now as a consistent reduction of type IIB supergravity.

First, consider the metric and dilaton ϕ_1^{IIA}. From [17] we have

$$d\hat{s}_{10}^2(IIA) \quad = \quad s^{1/12}\, X^{1/8}\left[\Delta^{3/8}\, ds_6^2 + 2g^{-2}\, X^2\, \Delta^{3/8}\, d\xi^2 + \tfrac{1}{2}g^{-2}\, X^{-1}\, \Delta^{-5/8}\, c^2 \sum_i h_i^2\right],$$

$$e^{\phi_1^{IIA}} \quad = \quad s^{-5/6}\, X^{-5/4}\, \Delta^{1/4}\,, \tag{21}$$

where in this case

$$\Delta = X\, c^2 + X^{-3}\, s^2\,. \tag{22}$$

The reduction to $D = 9$ is as follows:

$$d\hat{s}_{10}^2(IIA) = e^{-2\alpha\,\phi_2^{IIA}}\, ds_9^2 + e^{14\alpha\,\phi_2^{IIA}}\, (dz + \mathcal{A}_{(1)}^{IIA})^2\,, \tag{23}$$

where $\alpha = 1/(4\sqrt{7})$. From (9) the internal metric can be rewritten using

$$\sum_i h_i^2 = \sum_i (D\mu^i)^2 + 2g^2\, (dz + \mathcal{A}_{(1)}^{IIA})^2\,. \tag{24}$$

Substituting (24) into (21), and comparing with (23), we therefore obtain the following expressions for the nine-dimensional metric and dilatons:

$$ds_9^2 \quad = \quad s^{2/21}\, c^{2/7}\left[\Delta^{2/7}\, ds_6^2 + 2g^{-2}\, X^2\, \Delta^{2/7}\, d\xi^2 + \tfrac{1}{2}g^{-2}\, X^{-1}\, \Delta^{-5/7}\, c^2\, D\mu^i\, D\mu^i\right],$$

$$e^{\phi_1^{IIA}} \quad = \quad s^{-5/6}\, X^{-5/4}\, \Delta^{1/4}\,, \tag{25}$$

$$e^{2\alpha\,\phi_2^{IIA}} \quad = \quad s^{1/84}\, c^{2/7}\, X^{-1/8}\, \Delta^{-5/56}\,. \tag{26}$$

Under the IIA/IIB T-duality transformation in $D = 9$, the IIB dilatons are related to the IIA dilatons by the orthogonal transformation

$$\phi_1^{IIB} = \tfrac{3}{4}\,\phi_1^{IIA} - \tfrac{\sqrt{7}}{4}\,\phi_2^{IIA}\,, \qquad \phi_2^{IIB} = -\tfrac{\sqrt{7}}{4}\,\phi_1^{IIA} - \tfrac{3}{4}\,\phi_2^{IIA}\,. \tag{27}$$

The nine-dimensional metric and dilatons can now be lifted back to ten dimensions in the IIB variables, using the analogue of (23), namely

$$d\hat{s}_{10}^2(IIB) = e^{-2\alpha\,\phi_2^{IIB}}\, ds_9^2 + e^{14\alpha\,\phi_2^{IIB}}\, (dz + \mathcal{A}_{(1)}^{IIB})^2\,, \tag{28}$$

Doing this, we obtain

$$d\hat{s}_{10}^2(IIB) \quad = \quad c^{1/2}\, X^{-1/4}\left[\Delta^{1/4}\, ds_6^2 + 2g^{-2}\, X^2\, \Delta^{1/4}\, d\xi^2 + \tfrac{1}{2}g^{-2}\, X^{-1}\, \Delta^{-3/4}\, c^2\, D\mu^i\, D\mu^i\right]$$

$$+ s^{2/3}\, c^{-3/2}\, X^{7/4}\, \Delta^{1/4}\, (dz + \mathcal{A}_{(1)}^{IIB})^2\,,$$

$$e^{\phi_1^{IIB}} \quad = \quad s^{-2/3}\, c^{-1}\, X^{-1/2}\, \Delta^{1/2}\,. \tag{29}$$

(As we shall see below, after making the S^1 reduction of the field strengths, the Kaluza-Klein vector $\mathcal{A}_{(1)}^{IIB}$ in the type IIB picture, which comes from the winding vector of the

original type IIA description, is actually zero.) The vacuum solution, corresponding to taking $X = 1$, $A_{(1)}^{ij} = 0$ and ds_6^2 to be AdS_6, can be obtained as the near-horizon limit of a semi-localised D5/D7/NS5-brane system [18].

It is interesting to note that if we express the two ten-dimensional metrics in their respective string frames, related to the Einstein frames by

$$ds_{10}^2(str) = e^{\frac{1}{2}\phi_1}\, d\hat{s}_{10}^2\,, \tag{30}$$

then we get the following:

$$
\begin{aligned}
ds_{10}^2(IIA, str) &= s^{-1/3}\, X^{-1/2}\left[\Delta^{1/2}\, ds_6^2 + 2g^{-2}\, X^2\, \Delta^{1/2}\, d\xi^2\right.\\
&\quad\left. + \tfrac{1}{2}g^{-2}\, X^{-1}\, \Delta^{-1/2}\, c^2\, D\mu^i\, D\mu^i\right] + s^{-1/3}\, X^{-3/2}\, c^2\, (dz + \mathcal{A}_{(1)}^{IIA})^2\,,\\
ds_{10}^2(IIB, str) &= s^{-1/3}\, X^{-1/2}\left[\Delta^{1/2}\, ds_6^2 + 2g^{-2}\, X^2\, \Delta^{1/2}\, d\xi^2\right.\\
&\quad\left. + \tfrac{1}{2}g^{-2}\, X^{-1}\, \Delta^{-1/2}\, c^2\, D\mu^i\, D\mu^i\right] + s^{1/3}\, X^{3/2}\, c^{-2}\, (dz + \mathcal{A}_{(1)}^{IIB})^2\,. \tag{31}
\end{aligned}
$$

Thus we see that the effect of the T-duality is, as one might expect, simply to invert the prefactor in the $U(1)$ direction.

From [17], the Ansätze for the reduction to $D = 6$ of the various field strengths of the massive type IIA theory are

$$
\begin{aligned}
\hat{F}_{(4)} &= -\tfrac{\sqrt{2}}{6}\, g^{-3}\, s^{1/3}\, c^3\, \Delta^{-2}\, U\, d\xi \wedge \epsilon_{(3)} - \sqrt{2}g^{-3}\, s^{4/3}\, c^4\, \Delta^{-2}\, X^{-3}\, dX \wedge \epsilon_{(3)}\\
&\quad - \sqrt{2}g^{-1}\, s^{1/3}\, c\, X^4\, {*}F_{(3)} \wedge d\xi - \tfrac{1}{\sqrt{2}}s^{4/3}\, X^{-2}\, {*}F_{(2)}\\
&\quad + \tfrac{1}{\sqrt{2}}g^{-2}\, s^{1/3}\, c\, F_{(2)}^i \wedge h^i \wedge d\xi - \tfrac{1}{4\sqrt{2}}g^{-2}\, s^{4/3}\, c^2\, \Delta^{-1}\, X^{-3}\, F_{(2)}^i \wedge h^j \wedge h^k\, \epsilon_{ijk}\,, \tag{32}\\
\hat{F}_{(3)} &= s^{2/3}\, F_{(3)} + g^{-1}\, s^{-1/3}\, c\, F_{(2)} \wedge d\xi\,,\\
\hat{F}_{(2)} &= \tfrac{1}{\sqrt{2}}s^{2/3}\, F_{(2)}\,, \qquad e^{\hat{\phi}} = s^{-5/6}\, \Delta^{1/4}\, X^{-5/4}\,,
\end{aligned}
$$

In the Hopf reduction from $D = 10$ to $D = 9$, we follow the standard Kaluza-Klein rules, with the field strengths reduced as follows:

$$
\begin{aligned}
\hat{F}_{(4)} &= \bar{F}_{(4)} + \bar{F}_{(3)1} \wedge (dz + \mathcal{A}_{(1)}^{IIA})\,,\\
\hat{F}_{(3)} &= \bar{F}_{(3)} + \bar{F}_{(2)1} \wedge (dz + \mathcal{A}_{(1)}^{IIA})\,, \tag{33}\\
\hat{F}_{(2)} &= \bar{F}_{(2)} + \bar{F}_{(1)1} \wedge (dz + \mathcal{A}_{(1)}^{IIA})\,,
\end{aligned}
$$

Using the *lemmata* given in (17), it is now straightforward to read off the expressions for the nine-dimensional fields:

$$\bar{F}_{(4)} = -\sqrt{2}\, g^{-1}\, s^{1/3}\, c\, X^4\, {*}F_{(3)} \wedge d\xi - \tfrac{1}{\sqrt{2}}\, s^{4/3}\, X^{-2}\, {*}F_{(2)}$$

$$-\tfrac{1}{\sqrt{2}}\,g^{-2}\,s^{1/3}\,c\,F_{(2)}^{ij}\wedge(\mu^i\,D\mu^j)\wedge d\xi-\tfrac{1}{4\sqrt{2}}\,g^{-2}\,s^{4/3}\,c^2\,X^{-3}\,\Delta^{-1}\,F_{(2)}^{ij}\wedge D\mu^i\wedge D\mu^j\,,$$

$$\bar{F}_{(3)1}=-\tfrac{1}{6}g^{-2}\,s^{1/3}\,c^3\,\Delta^{-2}\,U\,\epsilon_{ijk}\,\mu^k\,D\mu^i\wedge D\mu^j\wedge d\xi$$
$$-g^{-2}\,s^{4/3}\,c^4\,X^{-3}\,\Delta^{-2}\,\epsilon_{ijk}\,\mu^k\,dX\wedge D\mu^i\wedge D\mu^j$$
$$-\tfrac{1}{2}g^{-1}\,s^{1/3}\,c\,\mu^k\,\epsilon_{ijk}\,F_{(2)}^{ij}\wedge d\xi+\tfrac{1}{4}g^{-1}\,s^{4/3}\,c^2\,X^{-3}\,\Delta^{-1}\,\epsilon_{ijk}\,F_{(2)}^{ij}\wedge D\mu^k\,,$$

$$\bar{F}_{(3)}=s^{2/3}\,F_{(3)}+g^{-1}\,s^{-1/3}\,c\,F_{(2)}\wedge d\xi\,,$$

$$\bar{F}_{(2)1}=0\,,\tag{34}$$

$$\bar{F}_{(2)}=\tfrac{1}{\sqrt{2}}\,s^{2/3}\,F_{(2)}\,,$$

$$\bar{F}_{(1)1}=0\,.$$

From these, we may note the following. Firstly, the fact that the NS-NS field $\bar{F}_{(2)1}$ is zero means that after the T-duality transformation, which maps this into the Kaluza-Klein 2-form field strength $\mathcal{F}_{(2)}^{IIB}\equiv d\mathcal{A}_{(1)}^{IIB}$ of the type IIB reduction to $D=9$, we shall have $\mathcal{A}_{(1)}^{IIB}=0$. This means that in the expressions in (29) and (31) for the type IIB metric, the contribution in the z direction involves just a pure untwisted dz^2.

After lifting the various nine-dimensional fields given in (34) to $D=10$ in the type IIB variables (see, for example [26, 27]), we therefore find that the Ansätze for the self-dual 5-form, and the R-R and NS-NS three forms, are

$$\widetilde{F}_{(5)}=\ast\bar{F}_{(4)}+\bar{F}_{(4)}\wedge dz\,,$$
$$\widetilde{F}_{(3)}^{RR}=-\bar{F}_{(3)1}+\bar{F}_{(2)}\wedge dz\,,\tag{35}$$
$$\widetilde{F}_{(3)}^{NS}=\bar{F}_{(3)}+\mathcal{F}_{(2)}^{IIA}\wedge dz\,.$$

Since $\bar{F}_{(1)1}$ is zero, there is no axionic field excitation. However, since the T-duality that relates the massive IIA theory to the type IIB theory involves a generalised Scherk-Schwarz reduction [28], the ten-dimensional axion $\hat{\chi}$ of the type IIB theory is given by

$$\hat{\chi}=m\,z\tag{36}$$

in the reduction to gauged six-dimensional supergravity.

4 $D=5$ $SU(2)\times U(1)$ gauged $N=4$ supergravity from type IIA and $D=11$

Five-dimensional $N=4$ $SU(2)\times U(1)$-gauged supergravity was obtained as a consistent S^5 reduction of type IIB supergravity in [11]. The 5-sphere can be viewed as a foliation of $S^3\times S^1$ surfaces. In this section, we perform an S^1 reduction on the $U(1)$ Hopf fibres

over the S^3, thereby obtaining a reduction Ansatz for a nine-dimensional embedding of the five-dimensional theory. After a T-duality transformation, we can then express this as a consistent embedding of the five-dimensional gauged supergravity in the type IIA theory, and eleven-dimensional supergravity.

The Kaluza-Klein S^5 reduction Ansatz from the type IIB theory is given by [11]:

$$
\begin{aligned}
d\hat{s}_{10}^2 &= \Delta^{1/2}\, ds_5^2 + 2g^{-2}\, \Delta^{1/2}\, X\, d\xi^2 + \Delta^{-1/2}\, X^2\, s^2\, (d\tau + B_{(1)})^2 \\
&\quad + \tfrac{1}{2} g^{-2}\, \Delta^{-1/2}\, X^{-1}\, c^2 \sum_i h_i^2\,, \\
\hat{G}_{(5)} &= \sqrt{2}\, g\, U\, \varepsilon_5 - \frac{3\sqrt{2}\, sc}{g} X^{-1} *dX \wedge d\xi + \frac{c^2}{4\sqrt{2}\, g^2} X^{-2} *F_{(2)}^i \wedge h^j \wedge h^k\, \varepsilon_{ijk} \\
&\quad - \frac{sc}{\sqrt{2}\, g^2} X^{-2} *F_{(2)}^i \wedge h^i \wedge d\xi + \frac{2sc}{g^2} X^4 *G_{(2)} \wedge d\xi \wedge (d\tau + gB_{(1)}), \\
\hat{A}_{(2)} &\equiv \hat{A}_{(2)}^{RR} + \mathrm{i}\, \hat{A}_{(2)}^{NS} = -s\, g^{-1}\, e^{\mathrm{i}\, g\, \tau/\sqrt{2}}\, A_{(2)}\,, \\
\hat{\phi} &= 0,\quad \hat{\chi} = 0,
\end{aligned}
\tag{37}
$$

where the self-dual 5-form is given by $\hat{H}_{(5)} = \hat{G}_{(5)} + \hat{*}\hat{G}_{(5)}$, $U \equiv X^2\, c^2 + X^{-1}\, s^2 + X^{-1}$, and ϵ_5 is the volume form in the five-dimensional spacetime metric ds_5^2, and

$$
\Delta = X\, c^2 + X^{-2}\, s^2\,.
\tag{38}
$$

Note that we have defined the complex 2-form potential $\hat{A}_{(2)} \equiv \hat{A}_{(2)}^1 + \mathrm{i}\, \hat{A}_{(2)}^2$ in the type IIB theory. The ten-dimensional dilaton and the axion are constants, which without loss of generality we have set to zero. The conventions that we are using here are related to those in [11] by making the following replacements on the quantities in [11]: $g \longrightarrow g/\sqrt{2}$, $\tau \longrightarrow -g\,\tau/\sqrt{2}$. Note that the scalar X and the gauge fields $A_{(1)}^{ij}$ parameterise deformations of a 5-sphere. It is foliated by $S^3 \times S^1$, where τ is the coordinate on the S^1 factor.

Following an analogous strategy to that of the previous section, we substitute (9) and (17) into the Ansatz, and perform a T-duality transformation on the z coordinate. We find that in the string frame, the resulting type IIA metric Ansatz is given by

$$
\begin{aligned}
ds_{10}^2(IIA, str) &= \Delta^{1/2}\, ds_5^2 + 2g^{-2}\, \Delta^{1/2}\, X\, d\xi^2 + \Delta^{-1/2}\, X^2\, s^2\, (d\tau + B_{(1)})^2 \\
&\quad + \tfrac{1}{2} g^{-2}\, \Delta^{-1/2}\, X^{-1}\, c^2 \sum_i (D\mu^i)^2 + \Delta^{1/2}\, X\, c^{-2}\, dz_2^2\,,
\end{aligned}
\tag{39}
$$

and the dilaton of the type IIA theory is given by

$$
e^{\phi_1^{IIA}} = \Delta^{1/4}\, X^{1/2}\, c^{-1}\,.
\tag{40}
$$

(We are naming the reduction coordinate z_2 here, in anticipation of performing a further oxidation to $D = 11$ presently.)

The field strengths of the type IIA theory turn out to be as follows:

$$
\begin{aligned}
F^{IIA}_{(4)} &= \widetilde{F}_{(4)} - F^{RR}_{(3)} \wedge dz_2\,, \\
F^{IIA}_{(3)} &= F^{NS}_{(3)} + \mathcal{F}^{IIB}_{(2)} \wedge dz_2\,, \\
F^{IIA}_{(2)} &= 0\,,
\end{aligned}
\tag{41}
$$

where

$$
\begin{aligned}
\widetilde{F}_{(4)} = \Big[&\tfrac{1}{2}s\,c^3\,g^{-2}\,U\,\Delta^{-2}\,\mu^k\,d\xi \wedge D\mu^i \wedge D\mu^j \wedge (d\tau + B_{(1)}) \\
&-\tfrac{3}{4}s^2\,c^4\,g^{-2}\,\Delta^{-2}\,X^{-2}\,\mu^k\,dX \wedge D\mu^i \wedge D\mu^j \wedge (d\tau + B_{(1)}) \\
&+\tfrac{1}{4\sqrt{2}}\,g^{-1}\,s^2\,c^2\,\Delta^{-1}\,X^{-2}\,F^{ij}_{(2)} \wedge D\mu^k \wedge (d\tau + B_{(1)}) \\
&-\tfrac{1}{2\sqrt{2}}\,g^{-1}\,s\,c\,\mu^k\,d\xi \wedge F^{ij}_{(2)} \wedge (d\tau + B_{(1)}) - \tfrac{1}{4}g^{-1}\,c^2\,X^{-2}\,{*F}^{ij}_{(2)} \wedge D\mu^k \\
&+\tfrac{1}{2}g^{-1}\,s\,c\,\mu^k\,X^{-2}\,{*F}^{ij}_{(2)} \wedge d\xi - \tfrac{1}{4}g^{-2}\,c^4\,\mu^k\,\Delta^{-1}\,X\,G_{(2)} \wedge D\mu^i \wedge D\mu^j \Big] \epsilon_{ijk}\,, \\
F^{RR}_{(3)} + &\,\mathrm{i}\,F^{NS}_{(3)} = d\hat{A}_{(2)}\,, \\
\mathcal{F}^{IIB}_{(2)} = &\tfrac{1}{2\sqrt{2}}\,g^{-1}\,\epsilon_{ijk}\,\mu^k\,D\mu^i \wedge D\mu^j - \tfrac{1}{2\sqrt{2}}\,\epsilon_{ijk}\,\mu^k\,F^{ij}_{(2)}\,.
\end{aligned}
\tag{42}
$$

The embedding of $D = 5$ $SU(2) \times U(1)$ gauged $N = 4$ supergravity in type IIA supergravity that we have just derived can be lifted further, to $D = 11$ supergravity. For the eleven-dimensional metric, we find

$$
\begin{aligned}
d\hat{s}^2_{11} = \; & X^{-1/3}\,c^{2/3}\Big[\Delta^{1/3}\,ds^2_5 + 2g^{-2}\,\Delta^{1/3}\,X\,d\xi^2 + \Delta^{-2/3}\,X^2\,s^2\,(d\tau + B_{(1)})^2 \\
&+\tfrac{1}{2}g^{-2}\,\Delta^{-2/3}\,X^{-1}\,c^2\sum_i (D\mu^i)^2\Big] + \Delta^{1/3}\,X^{2/3}\,c^{-4/3}\,(dz_1^2 + dz_2^2)\,,
\end{aligned}
\tag{43}
$$

where z_1 is the coordinate on the additional S^1. The 4-form field strength in $D = 11$ is given by

$$
\hat{F}_{(4)} = \widetilde{F}_{(4)} + \mathcal{I}m[\widetilde{F}_{(3)} \wedge (dz_1 - \mathrm{i}\,dz_2)] - \mathcal{F}^{IIB}_{(2)} \wedge dz_1 \wedge dz_2\,.
\tag{44}
$$

The vacuum solution, corresponding to setting $X = 1$, $A^{ij}_{(1)} = 0$, $B_{(1)} = 0$ and taking ds^2_5 to be AdS$_5$, can be viewed as the near-horizon limit of a semi-localised M5/M5-brane system [19, 20]. Note that the Hopf T-duality has the effect of untwisting the S^3 into $S^2 \times S^1$. The effect of this procedure on AdS$_3 \times S^3$ was extensively studied in [63].

5 $SO(4)$-gauged $N = 4$ supergravity in $D = 4$ from type IIB

The $SO(4)$-gauged $N = 4$ supergravity in $D = 4$ was explicitly obtained as an S^7 reduction from $D = 11$ supergravity [9]. In this reduction, the S^7 has a natural description in terms of a foliation of $S^3 \times S^3$ surfaces. The two copies of $SU(2)$ in the $SO(4) \sim SU(2) \times SU(2)$

gauge group come from left-invariant actions on the two copies of S^3. Since there are two Hopf circles, one from each S^3, we can perform two steps of Kaluza-Klein S^1 reduction. The first gives an embedding of the $N = 4$ gauged theory in type IIA supergravity, and the second, combined with a T-duality transformation, gives the embedding of the $N = 4$ gauged theory in type IIB supergravity.

Since the expression in [9] for the Kaluza-Klein S^7 reduction of the $D = 11$ 4-form is very complicated, we shall not present explicit formulae here for the field strengths in the type IIA and type IIB pictures. It is completely straightforward to obtain them, by following the same steps as we did in previous sections. Thus we shall just present the Kaluza-Klein Ansatz for the metric reductions here.

The Kaluza-Klein Ansatz for the reduction of the eleven-dimensional metric is [9]

$$d\hat{s}_{11}^2 = \Delta^{\frac{2}{3}} ds_4^2 + 2g^{-2}\Delta^{\frac{2}{3}} d\xi^2 + \tfrac{1}{2}g^{-2}\Delta^{\frac{2}{3}}\left[c^2\,\Omega^{-1}\sum_i (h^i)^2 + s^2\,\widetilde{\Omega}^{-1}\sum_i (\tilde{h}^i)^2\right], \qquad (45)$$

where

$$\begin{aligned}
\widetilde{X} &\equiv X^{-1}\,q\,, \qquad q^2 \equiv 1 + \chi^2\,X^4\,, \\
\Omega &\equiv c^2\,X^2 + s^2\,, \qquad \widetilde{\Omega} \equiv s^2\,\widetilde{X}^2 + c^2\,, \\
\Delta &\equiv \left[(c^2\,X^2 + s^2)(s^2\,\widetilde{X}^2 + c^2)\right]^{\frac{1}{2}}\,, \\
c &\equiv \cos\xi\,, \qquad s \equiv \sin\xi\,, \\
h^i &\equiv \sigma_i - g\,A_{(1)}^i\,, \qquad \tilde{h}^i \equiv \tilde{\sigma}_i - g\,\widetilde{A}_{(1)}^i\,.
\end{aligned} \qquad (46)$$

Here $X = e^{\frac{1}{2}\phi}$, and (ϕ, χ) are the dilatonic and axionic scalars of the four-dimensional gauged theory.

As a first step, we make a Hopf reduction on the untilded S^3, using the expression (9). Comparing with the standard S^1 reduction in (11), this gives the type IIA ten-dimensional metric and dilaton:

$$\begin{aligned}
ds_{10}^2(IIA) &= \Delta^{3/4}\,c^{1/4}\,\Omega^{-1/8}\left[ds_4^2 + 2g^{-2}\,d\xi^2 + \tfrac{1}{2}g^{-2}\,c^2\,\Omega^{-1}\sum_i (D\mu^i)^2\right. \\
&\qquad\qquad \left. +\tfrac{1}{2}g^{-2}\,s^2\,\widetilde{\Omega}^{-1}\sum_i (\tilde{h}^i)^2\right], \\
e^{\phi_1^{IIA}} &= \Delta^{1/2}\,c^{3/2}\,\Omega^{-3/4}\,.
\end{aligned} \qquad (47)$$

The next step is to perform a Hopf reduction on the second S^3 factor. Denoting all relevant quantities with tildes, we use the same result (9), and reduce the metric according to the standard S^1 reduction (23). This gives the nine-dimensional metric, and second

dilaton:

$$ds_9^2(IIA) = \Delta^{4/7}\,(s\,c)^{2/7}\left[ds_4^2 + 2g^{-2}\,d\xi^2 + \tfrac{1}{2}g^{-2}\,c^2\,\Omega^{-1}\sum_i(D\mu^i)^2\right.$$

$$\left. +\tfrac{1}{2}g^{-2}\,s^2\,\widetilde{\Omega}^{-1}\sum_i(\widetilde{D}\tilde\mu^i)^2\right],$$

$$e^{2\alpha\,\phi_2^{IIA}} = \Delta^{3/28}\left(\frac{c^2}{\Omega}\right)^{1/56}\left(\frac{s^2}{\widetilde{\Omega}}\right)^{1/7}. \tag{48}$$

After transforming to type IIB variables, including the dilaton transformation (27), the metric can be oxidised back to $D = 10$, as an embedding now in the type IIB theory. This metric, and the corresponding type IIA metric before the Hopf T-duality transformation, are most usefully expressed in the string frame. The expressions are as follows:

$$ds_{10}^2(IIA, str) = \Delta\,c\,\Omega^{-1/2}\left[ds_4^2 + 2g^{-2}\,d\xi^2 + \tfrac{1}{2}g^{-2}\,c^2\,\Omega^{-1}\sum_i(D\mu^i)^2\right.$$

$$\left. +\tfrac{1}{2}g^{-2}\,s^2\,\widetilde{\Omega}^{-1}\sum_i(\widetilde{D}\tilde\mu^i)^2\right] + s^2\,c^1\,\widetilde{\Omega}^{-1/2}\,(dz_2 + \mathcal{A}_{(1)})^2\,,$$

$$ds_{10}^2(IIB, str) = \Delta\,c\,\Omega^{-1/2}\left[ds_4^2 + 2g^{-2}\,d\xi^2 + \tfrac{1}{2}g^{-2}\,c^2\,\Omega^{-1}\sum_i(D\mu^i)^2\right.$$

$$\left. +\tfrac{1}{2}g^{-2}\,s^2\,\widetilde{\Omega}^{-1}\sum_i(\widetilde{D}\tilde\mu^i)^2\right] + s^{-2}\,c^{-1}\,\widetilde{\Omega}^{1/2}\,dz_2^2\,. \tag{49}$$

Note that there is no "twist" involving the z_2 coordinate in the type IIB ten-dimensional metric. This is a reflection of the fact that there is no winding vector in $D = 9$ in the type IIA reduction. Such a vector would have come from the reduction of the 3-form field strength in the $D = 10$ type IIA theory. This, in turn comes from the reduction of the 4-form of $D = 11$. But the 3-form in $D = 10$ comes from the Hopf reduction of the first S^3 factor in the $S^3 \times S^3$ foliation of S^7. Consequently, it has no terms involving the directions in the second S^3 factor, and so no winding vector emerges in $D = 9$.

For a similar reason, the axion $\bar\chi$ of the type IIB theory is zero in the reduction Ansatz. It would correspond, in the type IIA picture, to the axion that would come from the reduction of the Kaluza-Klein vector in $D = 10$. But this lives in the directions of the first S^3 (see (8)), and so it does not give rise to any axion when the further reduction to $D = 9$ on the Hopf fibres of the second S^3 is performed.

The vacuum solution, corresponding to taking $X = 1$, $\chi = 0$, $A_{(1)}^{ij} = 0$, $\widetilde{A}_{(1)}^{ij} = 0$ and ds_4^2 to be AdS$_4$, can be viewed as the near-horizon limit of a semi-localised D2/D6-brane system in type IIA supergravity or a D3/D5/NS5-system in type IIB supergravity [18].

6 Hopf reduction on non-singular fibres

All the examples that we have considered so far in this paper involve performing Hopf reductions on circles whose radius goes to zero for some value of the azimuthal coordinate ξ on the internal spherical manifold. For example, when we reinterpreted the S^4 reduction of $D = 11$ supergravity in section 2 as a reduction of type IIA supergravity, the circle parameterised by z in (10) reduced to zero radius at $\xi = \frac{1}{2}\pi$. In certain cases a more general type of Hopf reduction can be performed, in which the radius of the $U(1)$ fibres remains non-zero for all values of ξ. Specifically, this can be done for the S^5 and S^7 reductions. The reason for this is that in each of these examples, there are in fact two $U(1)$ Killing vectors in the higher-dimensional metric, corresponding to the fact that the foliating surfaces at constant ξ are the product of two odd-dimensional spheres ($S^3 \times S^1$ and $S^3 \times S^3$ respectively). In each case when the radius of one of the spheres goes to zero (at $\xi = 0$ or $\xi = \frac{1}{2}\pi$), the other has non-zero radius. Thus by taking a linear combination of the two Killing directions for the S^1 Kaluza-Klein reduction, a non-singular embedding can be achieved.

To see how this works, consider the relevant two-dimensional factor in the higher-dimensional metric, namely the part involving the two $U(1)$ directions. We shall write this as

$$d\bar{s}^2 = \alpha^2 \, (d\tau_1 + \mathcal{A}^1_{(1)})^2 + \beta^2 \, (d\tau_2 + \mathcal{A}^2_{(1)})^2 \,. \tag{50}$$

Now, we make the coordinate redefinitions

$$\tau_1 = a\,x + b\,y \,, \qquad \tau = -b\,x + a\,y \,, \tag{51}$$

where a and b are constants. It is straightforward to establish the following lemma:

$$d\bar{s}^2 = (\alpha^2 \, b^2 + \beta^2 \, a^2) \left[dy + \frac{a\,b\,(\alpha^2 - \beta^2)\,dx + \alpha^2\,b\,\mathcal{A}^1_{(1)} + \beta^2\,a\,\mathcal{A}^2_{(1)}}{\alpha^2\,b^2 + \beta^2\,a^2} \right]^2$$
$$+ \frac{\alpha^2 \, \beta^2}{\alpha^2 \, b^2 + \beta^2 \, a^2} \left[(a^2 + b^2)\,dx + a\,\mathcal{A}^1_{(1)} - b\,\mathcal{A}^2_{(1)} \right]^2 \,. \tag{52}$$

If we now perform an S^1 reduction on the y coordinate we see that the $U(1)$ fibres will always have non-zero length, provided that the functions α and β do not vanish simultaneously. Since, in both our examples one of them is proportional to $\sin\xi$, while the other is proportional to $\cos\xi$, this condition for non-singularity of the y fibres is satisfied.

6.1 Non-singular Hopf reduction for the S^5 embedding

Using (9), the metric Ansatz for the S^5 reduction given in (37) can be rewritten as

$$d\hat{s}^2_{10} \;=\; \Delta^{1/2}\,ds^2_5 + 2g^{-2}\,\Delta^{1/2}\,X\,d\xi^2 + \Delta^{-1/2}\,X^2\,s^2\,(d\tau + B_{(1)})^2$$

$$+\tfrac{1}{2}g^{-2}\,\Delta^{-1/2}\,X^{-1}\,c^2\sum_i(D\mu^i)^2+\Delta^{-1/2}\,X^{-1}\,c^2\,(dz+\mathcal{A}_{(1)})^2\,. \tag{53}$$

As far as this metric reduction Ansatz is concerned, we see that there are two $U(1)$ Killing directions, namely z and τ. Accordingly, we can choose a more general reduction scheme, in which we take a linear combination of these two coordinates for our circle reduction.

Comparing (52) with (53), we see that the functions α and β are given by

$$\alpha^2=\Delta^{-1/2}\,X^{-1}\,c^2,,\qquad \beta^2=\Delta^{-1/2}\,X^2\,s^2\,. \tag{54}$$

Reducing on the y coordinate, following the standard procedure, we arrive at the nine-dimensional metric

$$\begin{aligned}ds_9^2&=&\widetilde{\Delta}^{1/7}\,\Delta^{3/7}\left[ds_5^2+2g^{-2}\,X\,d\xi^2+\tfrac{1}{2}g^{-2}\,\Delta^{-1}\,X^{-1}\,c^2\sum_i(D\mu^i)^2\right.\\&&\left.+\widetilde{\Delta}^{-1}\,\Delta^{-1}\,X\,s^2\,c^2\,[(a^2+b^2)\,dx+a\,\mathcal{A}_{(1)}-b\,B_{(1)}]^2\right],\end{aligned} \tag{55}$$

where we have defined

$$\widetilde{\Delta}=b^2\,X^{-1}\,c^2+a^a\,X^2\,s^2\,. \tag{56}$$

Note that the dilaton ϕ_2^{IIB} resulting from the Kaluza-Klein reduction from $D=10$ to $D=9$ is given by

$$e^{2\alpha\,\phi_2^{IIB}}=\Delta^{-1/14}\,\widetilde{\Delta}^{1/7}\,, \tag{57}$$

where as usual $\alpha=1/(4\sqrt{7})$.

Since the type IIB dilaton ϕ_1^{IIB} in the original S^5 reduction is zero, it follows from (27) that after performing a T-duality transformation in $D=9$, and lifting back to the type IIA theory, the metric becomes

$$\begin{aligned}d\hat{s}_{10}^2(IIA)&=&\widetilde{\Delta}^{1/4}\,\Delta^{3/8}\left[ds_5^2+2g^{-2}\,X\,d\xi^2+\tfrac{1}{2}g^{-2}\,\Delta^{-1}\,X^{-1}\,c^2\sum_i(D\mu^i)^2\right.\\&&\left.+\widetilde{\Delta}^{-1}\,\Delta^{-1}\,X\,s^2\,c^2\,[(a^2+b^2)\,dx+a\,\mathcal{A}_{(1)}-b\,B_{(1)}]^2+\widetilde{\Delta}^{-1}\,dy^2\right].\end{aligned} \tag{58}$$

(Note that as usual there is no longer any "twist" in the y direction of the S^1, after the T-duality transformation.) This embedding can be further lifted to $D=11$, with the metric given by

$$\begin{aligned}d\hat{s}_{11}^2&=&(\widetilde{\Delta}\,\Delta)^{1/3}\left[ds_5^2+2g^{-2}\,X\,d\xi^2+\tfrac{1}{2}g^{-2}\,\Delta^{-1}\,X^{-1}\,c^2\sum_i(D\mu^i)^2\right.\\&&\left.+\widetilde{\Delta}^{-1}\,\Delta^{-1}\,X\,s^2\,c^2\,[(a^2+b^2)\,dx+a\,\mathcal{A}_{(1)}-b\,B_{(1)}]^2+\widetilde{\Delta}^{-1}\,(dz_1^2+dy^2)\right]\end{aligned} \tag{59}$$

where z_1 is the eleventh coordinate.

It is instructive to examine the metric (59) in the "vacuum" case where the lower-dimensional scalar and gauge fields are set to zero, in which case we can take ds_5^2 to be AdS$_5$. Bearing in mind that $\mathcal{A}_{(1)}$ is still non-zero, and given by the first term in (8), we see that the eleven-dimensional metric becomes

$$d\hat{s}_{11}^2 = \widetilde{\Delta}^{1/3}\, ds_5^2 + 2g^{-2}\, \widetilde{\Delta}^{1/3}\, d\bar{s}^2 + \widetilde{\Delta}^{-2/3}\left(dz_1^2 + dy^2\right), \tag{60}$$

where

$$
\begin{aligned}
d\bar{s}^2 &= d\xi^2 + \tfrac{1}{4}c^2\left(d\theta^2 + \sin^2\theta\, d\psi^2\right) + \tfrac{1}{4}s^2\, c^2\left(d\sigma + \cos\theta\, d\psi\right)^2, \\
\widetilde{\Delta} &= b^2\, c^2 + a^2\, s^2, \\
\sigma &= \frac{\sqrt{2}\, g\,(a^2 + b^2)}{a}\, x.
\end{aligned}
\tag{61}
$$

In order for the level surfaces at constant ξ to be globally non-singular, the angular coordinate σ should be chosen to have period $4\pi/p$, where p is an integer. The level surfaces are then cyclic lens spaces S^3/Z_p. As ξ approaches 0, the metric $d\bar{s}^2$ approaches

$$d\bar{s}^2 = d\xi^2 + \frac{a^2}{4b^2}\, s^2\left(d\sigma + \cos\theta\, d\psi\right)^2 + \tfrac{1}{4}(d\theta^2 + \sin^2\theta\, d\psi^2). \tag{62}$$

In general, there will be a conical singularity at $\xi = 0$, but this is avoided if

$$\frac{a}{b} = p. \tag{63}$$

As ξ approaches the other endpoint, at $\xi = \tfrac{1}{2}\pi$, the metric $d\bar{s}^2$ approaches

$$d\bar{s}^2 = d\xi^2 + \tfrac{1}{4}c^2\left[d\theta^2 + \sin^2\theta\, d\psi^2 + (d\sigma + \cos\theta\, d\psi)^2\right]. \tag{64}$$

This is locally $\mathbb{R}^4$, but there will be a conical singularity at $\xi = 0$ unless the lens space S^3/Z_p is just S^3 itself; in other words $p = 1$.

Thus we see that if $a = b$ and $p = 1$, we have a completely non-singular embedding of AdS$_5$ in eleven-dimensional supergravity. In this case the warp factor $\widetilde{\Delta}$ is simply equal to the constant a^2. For other choices of p and the constants a and b, we have an M-theory embedding of AdS$_5$ that is "almost" non-singular, with relatively mild orbifold-like conical singularities at $\xi = 0$ and $\xi = \tfrac{1}{2}\pi$. In these more general cases the warp factor $\widetilde{\Delta}$ given in (61) is a function of the azimuthal coordinate ξ. It is, however, always non-singular, provided that $a\, b \neq 0$.

It is worth remarking that in the non-singular case $a = b$, $p = 1$, the metric $d\bar{s}^2$ is precisely the Fubini-Study metric on CP^2. This and other geometrical aspects of the complex projective spaces are discussed in the appendix.

At the level of the $AdS_5 \times S^5$ vacuum solution, the untwisting of the fibres to give an $AdS_5 \times CP^2 \times S^1$ solution in the Hopf-duality related type IIA framework was already seen in [23]. Here, we have gone further, and obtained the Kaluza-Klein reduction Ansatz for the $SU(2) \times U(1)$ gauged five-dimensional supergravity, viewed now as a $CP^2 \times S^1$ reduction of type IIA supergravity. As was discussed in [23], there are some peculiarities associated with the type IIA description, resulting from the fact that CP^2 does not admit an ordinary spin structure. This means that at the level of the low-energy supergravities, there will be no fermions at all in the type IIA description. They will only be restored when the T-duality is considered at the level of the full string theories, with the fermions that carry charges with respect to the winding-mode vector in the type IIB picture ending up in the Kaluza-Klein spectrum in the type IIA picture.

In fact this type of phenomenon is not confined to the fermionic sector. A bosonic example can be seen by looking at the reduction Ansatz for the NS-NS and R-R 2-form potentials of the type IIB theory. In the S^5 reduction Ansatz in (37), we saw that the 2-forms reduce as [11]

$$\hat{A}^{NS}_{(2)} + \mathrm{i}\, \hat{A}^{RR}_{(2)} = -s\, g^{-1}\, e^{\mathrm{i}\, g\, \tau/\sqrt{2}}\, A_{(2)}\,. \tag{65}$$

Thus we see that although the metric reduction Ansatz (53) is invariant under both the $\partial/\partial z$ and $\partial/\partial \tau$ $U(1)$, Killing symmetries, the 2-form Ansatz (65) is not invariant under the $\partial/\partial \tau$ symmetry. Thus we would have to truncate A_2 from the five-dimensional theory in order to carry out the previously-discussed S^1 reduction. In a similar fashion, one would find that the reduction Ansätze for all the fermion fields would involve τ-dependent complex exponential factors, and thus would have to be truncated from the theory.

6.2   Non-singular Hopf reduction of the S^7 embedding

We can also perform a non-singular Hopf reduction of the S^7 embedding of $N = 4$ gauged $SO(4)$ four-dimensional supergravity. Here, we may take linear combinations of the two S^1 Hopf fibres in the two S^3 factors in the foliation of S^7, so that for one of the two combinations the circle never degenerates to zero radius, for any value of ξ. This allows us to perform a non-singular reduction of the S^7 Ansatz to give an Ansatz for the non-singular embedding of $N = 4$ gauged $SO(4)$ supergravity into type IIA supergravity. As we shall see, in the limit where all the scalar and gauged fields in $D = 4$ are set to zero, this reduces to the $AdS_4 \times CP^3$ solution, which was discussed from a string-theoretic viewpoint in [25]. A Kaluza-Klein reduction at the level of linearised fluctuations around this background was discussed in [24]. Using our procedure here, we can obtain the fully non-linear reduction

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Ansatz for the $N = 4$ gauged $SO(4)$ theory.

Using the standard formulae

$$
\begin{aligned}
\sum_i h_i^2 &= \sum_i (D\mu^i)^2 + 2g^2 (dz + \mathcal{A}_{(1)})^2, \\
\sum_i \tilde{h}_i^2 &= \sum_i (\widetilde{D}\tilde{\mu}^i)^2 + 2g^2 (d\tilde{z} + \widetilde{\mathcal{A}}_{(1)})^2,
\end{aligned}
\tag{66}
$$

and then defining the linear combinations $z = x+y$, $\tilde{z} = -x+y$ as the new S^1 coordinates, the eleven-dimensional metric Ansatz (45) becomes

$$
\begin{aligned}
d\hat{s}_{11}^2 &= \Delta^{\frac{2}{3}} ds_4^2 + 2g^{-2} \Delta^{\frac{2}{3}} d\xi^2 + \tfrac{1}{2}g^{-2} \Delta^{\frac{2}{3}} \Big[c^2 \Omega^{-1} \sum_i (D\mu^i)^2 + s^2 \widetilde{\Omega}^{-1} \sum_i (\widetilde{D}\tilde{\mu}^i)^2 \Big] \\
&\quad + \Delta^{-4/3} \widetilde{\Delta} \Big[dy + \frac{(c^2\, \widetilde{\Omega} - s^2\, \Omega)\, dx + c^2\, \widetilde{\Omega}\, \mathcal{A}_{(1)} + s^2\, \Omega\, \widetilde{\mathcal{A}}_{(1)}}{\widetilde{\Delta}} \Big]^2 \\
&\quad + \Delta^{2/3} \widetilde{\Delta}^{-1} s^2\, c^2\, [(a^2 + b^2)\, dx + a\, \mathcal{A}_{(1)} - b\, \widetilde{\mathcal{A}}_{(1)}]^2,
\end{aligned}
\tag{67}
$$

where

$$
\widetilde{\Delta} \equiv b^2\, c^2\, \widetilde{\Omega} + a^2\, s^2\, \Omega.
\tag{68}
$$

We can now perform the Hopf reduction on the y coordinate. This gives

$$
\begin{aligned}
ds_{10}^2 &= \Delta^{1/2}\, \widetilde{\Delta}^{1/8} \Big\{ ds_4^2 + 2g^{-2}\, d\xi^2 + \tfrac{1}{2}g^{-2}\, c^2\, \Omega^{-1} \sum_i (D\mu^i)^2 \\
&\quad + \tfrac{1}{2}g^{-2}\, s^2\, \widetilde{\Omega}^{-1} \sum_i (\widetilde{D}\tilde{\mu}^i)^2 + s^2\, c^2\, \widetilde{\Delta}^{-1} [(a^2 + b^2)\, dx + a\, \mathcal{A}_{(1)} - b\, \widetilde{\mathcal{A}}_1]^2 \Big\}, \\
e^\phi &= \Delta^{-1}\, \widetilde{\Delta}^{3/4}.
\end{aligned}
\tag{69}
$$

If we look at the vacuum solution where $X = 1$, $\chi = 0$, $A_{(1)}^{ij} = 0$, $\widetilde{A}_{(1)}^{ij} = 0$ and the metric ds_4^2 is taken to be AdS$_4$, th type IIA metric in (69) takes the form

$$
ds_{10}^2 = \widetilde{\Delta}^{1/8}\, ds_4^2 + 2g^{-2}\, \widetilde{\Delta}^{1/8}\, d\bar{s}^2,
\tag{70}
$$

where

$$
\begin{aligned}
d\bar{s}^2 &= d\xi^2 + \tfrac{1}{4}c^2\, (d\theta^2 + \sin^2\theta\, d\psi^2) + \tfrac{1}{4}s^2\, (d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\psi}^2) \\
&\quad + s^2\, c^2\, \widetilde{\Delta}^{-1} [(a^2 + b^2)\, dx + \frac{a}{\sqrt{2}g} \cos\theta\, d\psi - \frac{b}{\sqrt{2}g} \cos\tilde{\theta}\, d\tilde{\psi}]^2.
\end{aligned}
\tag{71}
$$

To avoid conical singularities on the level surfaces at constant ξ we must require that the period δx of the angular coordinate x should be such that

$$
\frac{\sqrt{2}\, g\, (a^2 + b^2)}{a}\, \delta x = \frac{4\pi}{p}, \qquad \frac{\sqrt{2}\, g\, (a^2 + b^2)}{b}\, \delta x = \frac{4\pi}{q},
\tag{72}
$$

where p and q are integers, and so the ratio a/b must be rational:

$$\frac{a}{b} = \frac{p}{q}\,. \tag{73}$$

Furthermore, if we wish to avoid conical singularities at the points $\xi = 0$ and $\xi = \frac{1}{2}\pi$ where the level surfaces degenerate, we must require that $p = q = 1$. (The discussion is analogous to that in the previous subsection.) Thus if $p = q = 1$, implying that $a = b$, we obtain a completely non-singular embedding of AdS$_4$ in the type IIA theory. In this case the warp factor $\widetilde{\Delta}$ is simply equal to the constant a^2. For more general choices of p and q, there are mild orbifold-like conical singularities at $\xi = 0$ and $\xi = \frac{1}{2}\pi$, and the warp factor $\widetilde{\Delta}$ becomes dependent on ξ. (It is, however, non-singular, provided that a and b are both non-zero.)

Note that in the case where $p = q = 1$ and $a = b$, the metric $d\bar{s}^2$ is in fact the Fubini-Study metric on CP^3, written in a particularly simple coordinate system. This is discussed in more detail in the appendix. It should perhaps be emphasised that unlike the CP^2 reduction of the previous subsection, here the CP^3 reduction does not lead to any loss of the (half-maximal) supersymmetry. This is related to the fact that CP^3, unlike CP^2, does admit an ordinary spin structure.

Although our principal purpose in this section was to examine non-singular embeddings, we can also entertain the idea of making a further S^1 reduction on the x coordinate in the type IIA metric (69). This can be used in order to obtain another embedding of the four-dimensional $N = 4$ $SO(4)$-gauged supergravity in type IIB supergravity. Since the radius of the circle parameterised by x vanishes both at $\xi = 0$ and $\xi = \frac{1}{2}\pi$ this IIB embedding will be a singular one. In the string frame, the type IIB metric is

$$
\begin{aligned}
ds_{10}^2(IIB, str) \;=\; & \widetilde{\Delta}^{1/2}\Big\{ ds_4^2 + 2g^{-2}\,d\xi^2 + \tfrac{1}{2}g^{-2}\,c^2\,\Omega^{-1}\sum_i (D\mu^i)^2 \\
& +\tfrac{1}{2}g^{-2}\,s^2\,\widetilde{\Omega}^{-1}\sum_i (\widetilde{D}\tilde{\mu}^i)^2 + \frac{s^{-2}\,c^{-2}}{a^2 + b^2}\,dx^2 \Big\}\,.
\end{aligned}
\tag{74}
$$

7 Conclusions

In this paper, we obtained warped Kaluza-Klein embeddings of the $D = 7$, 6, 5 and 4 gauged supergravities with half-maximal supersymmetry. The characteristic feature of these warped embeddings is that the vacuum solution where the lower-dimensional spacetime is AdS has a non-trivial warp-factor multiplying the AdS metric, which depends on one of the coordinates of the internal reduction manifold. We constructed these warped embeddings by starting from the previously-known spherical reductions that give rise to these supergravities. After performing circle reductions or T-duality transformations on the Hopf fibres of

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S^3 submanifolds of the original internal spheres, we obtained the new embeddings which can be viewed as reductions on "mirror manifolds" dual to the original spheres. For all cases except $D = 6$, the original Kaluza-Klein reductions give non-warped solutions in the case of a pure AdS vacuum solution. Table 1 summarises the half-maximally supersymmetric supergravities, their previously-known embeddings, and the new ones that we obtained in this paper.

D	7	6	5	4
N	2	2	4	4
G	$SU(2)$	$SU(2)$	$SU(2) \times U(1)$	$SO(4)$
Previously Known Embedding	M-theory	Massive IIA	Type IIB	M-theory
New Warped Embedding	Type IIA	Type IIB	M-theory	Type IIB

Table 1: The half-maximally supersymmetric gauged supergravities in dimension D, with N supersymmetries and gauge group G. Their previously known Kaluza-Klein embeddings and the new warped embeddings obtained in this paper are listed.

In the above warped embeddings, the warp factors can become singular at the limits of the range of the internal azimuthal coordinate ξ, since the Hopf fibres on which we performed S^1 reductions can approach zero radius at these endpoints. However, we also showed that in the cases $D = 5$ and $D = 4$, it is possible to perform an S^1 Hopf reduction on a $U(1)$ fibre whose length remains non-zero everywhere. Thus in such cases the resulting warped embedding can be non-singular. The non-singular embeddings give rise, in the case of a pure vacuum solution, to $\text{AdS}_5 \times CP^2 \times T^2$ in M-theory [23] and $\text{AdS}_4 \times CP^3$ in type IIA supergravity [24, 25]. These solutions themselves are in fact not warped. We also obtained warped generalisations, at the price of introducing rather mild orbifold-like singularities in the internal manifolds.

Recently, non-singular warped embeddings of AdS_5 in M-theory [21, 22] and AdS_3 in type IIB supergravity [22] were obtained. The construction in [22] consists of finding an $\text{AdS}_d \times \Sigma_g$ solution in $(d + 2)$-dimensional gauged supergravity, where Σ_g is a Riemann surface, and then lifting this to M-theory or type IIB using the Kaluza-Klein Ansätze

constructed in [33]. These warped solutions are inequivalent to any of the ones that we have obtained in this paper. The construction leads to solutions with non-singular warp factors, but its applicability is restricted to AdS_d with $d = 3$ and 5. Clearly it cannot be applied to $d = 7$, since there is no suitable gauged supergravity in $D = 9$. The singularity of the warped embedding of AdS_7 in type IIA that was discussed in this paper and in [18] may therefore be unavoidable. In the case of AdS_4, the analogous construction starting from $AdS_4 \times \Sigma_g$ would require a normal (unwarped) embedding of the six-dimensional $N = 2$ $SU(2)$-gauged supergravity in a higher dimension. However, as far as is known, no such unwarped embedding exists [17]. The fact that the only way to embed AdS_6 appears to be through a singular warped configuration possibly suggests that singular warped embeddings should not be overlooked.

Acknowledgements

A CP^{m+n+1} from $CP^m \times CP^n$

The metric on the unit $(p + q + 1)$-sphere can be written in terms of a foliation of $S^p \times S^q$ for any p and q, as

$$d\Omega_{p+q+1}^2 = d\xi^2 + c^2\, d\Omega_p^2 + s^2\, d\Omega_q^2\,, \tag{75}$$

where as usual $c \equiv \cos\xi$, $s = \sin\xi$, and the angle ξ lies in the interval $0 \leq \xi \leq \frac{1}{2}\pi$ (see, for example, [18]). Furthermore, we know that if p and q are odd, $p = 2m + 1$, $q = 2n + 1$, the metrics $d\Omega_p^2$ and $d\Omega_q^2$ on the unit S^p and S^q spheres can each be written in terms of "unit" CP^m and CP^n Fubini-Study metrics[2] $d\Sigma_m^2$ and $d\Sigma_n^2$ as

$$\begin{aligned} d\Omega_p^2 &= (d\tau_1 + \mathcal{A}_{(1)})^2 + d\Sigma_m^2\,, \\ d\Omega_q^2 &= (d\tau_2 + \widetilde{\mathcal{A}}_{(1)})^2 + d\Sigma_n^2\,, \end{aligned} \tag{76}$$

where $d\mathcal{A}_{(1)} = 2J$ and $d\widetilde{\mathcal{A}}_{(1)} = 2\widetilde{J}$, with J and $\widetilde{J}$ being the Kähler forms on CP^m and CP^n.

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[2]We define the "unit" Fubini-Study metric on CP^n to be the one whose scale size is such that its Hopf bundle gives the unit-radius $(2n + 1)$-sphere.

We can now follow the same procedure as in earlier sections, where a $U(1)$ fibre coordinate is selected that is a linear combination of those from the two spheres S^p and S^q. (We had $(p, q) = (3, 1)$ for the S^5 reduction of type IIB, and $(p, q) = (3, 3)$ for the S^7 reduction of $D = 11$ supergravity.) To avoid unnecessary factors of 2, we can just define $\tau_1 = \frac{1}{2}\psi + y$, $\tau_2 = -\frac{1}{2}\psi + y$ here, and hence we get

$$d\Omega_{2m+2n+3}^2 = d\xi^2 + c^2\,d\Sigma_m^2 + s^2\,d\Sigma_n^2 + s^2\,c^2\,(d\psi + \mathcal{A}_{(1)} - \widetilde{\mathcal{A}}_{(1)})^2 + (dy + B_{(1)})^2\,, \quad (77)$$

where

$$B_{(1)} \equiv \tfrac{1}{2}(c^2 - s^2)\,d\psi + c^2\,\mathcal{A}_{(1)} + s^2\,\widetilde{\mathcal{A}}_{(1)}\,. \quad (78)$$

Since this metric on $S^{2m+2n+3}$ is now written as a $U(1)$ Hopf fibration (with unit radius for the fibres, whose coordinate is y), it follows that the part of the metric orthogonal to $\partial/\partial y$ must be the unit Fubini-Study metric on CP^{m+n+1}. Thus we must have that

$$d\Sigma_{m+n+1}^2 = d\xi^2 + c^2\,d\Sigma_m^2 + s^2\,d\Sigma_n^2 + s^2\,c^2\,(d\psi + \mathcal{A}_{(1)} - \widetilde{\mathcal{A}}_{(1)})^2\,. \quad (79)$$

Since the construction of the unit sphere as the $U(1)$ Hopf fibration over a complex projective space goes as in (76), it furthermore follows that the Kähler form $\hat{J}$ on CP^{m+n+1} will be given by

$$\hat{J} \equiv \tfrac{1}{2}dB_{(1)} = -s\,c\,d\xi \wedge (d\psi + \mathcal{A}_{(1)} - \widetilde{\mathcal{A}}_{(1)}) + c^2\,J + s^2\,\widetilde{J}\,. \quad (80)$$

Thus if we define the natural vielbeins $\hat{e}^A$ for the metric $d\Sigma_{m+n+1}^2$ in (79), namely

$$\hat{e}^0 = d\xi\,, \qquad \hat{e}^1 = s\,c\,(d\psi + \mathcal{A}_{(1)} - \widetilde{\mathcal{A}}_{(1)})\,, \qquad \hat{e}^a = c\,e^a\,, \qquad \hat{e}^{\tilde{a}} = s\,e^{\tilde{a}}\,, \quad (81)$$

where e^a and $e^{\tilde{a}}$ are vielbeins for $d\Sigma_m^2$ and $d\Sigma_n^2$, then we have

$$\hat{J} = -\hat{e}^0 \wedge \hat{e}^1 + c^2\,J + s^2\,\widetilde{J}\,. \quad (82)$$

In other words, the non-vanishing vielbein components $\hat{J}_{AB}$ of the Kähler form on CP^{m+n+1} are given by

$$\hat{J}_{01} = -1\,, \qquad \hat{J}_{ab} = J_{ab}\,, \qquad \hat{J}_{ij} = \widetilde{J}_{ij}\,. \quad (83)$$

A simple calculation shows that the curvature 2-form for the metric (79) on CP^{m+n+1} indeed has the form,

$$\hat{\Theta}_{AB} = \hat{e}^A \wedge \hat{e}^B + \hat{J}_{AC}\,\hat{J}_{BD}\,\hat{e}^C \wedge \hat{e}^D + \hat{J}_{AB}\,\hat{J}_{CD}\,\hat{e}^C \wedge \hat{e}^D\,, \quad (84)$$

which one expects for the unit Fubini-Study metric. In terms of vielbein components, we see that the Riemann tensor has the characteristic structure for a space of constant holomorphic sectional curvature [66],

$$\hat{R}_{ABCD} = \delta_{AC}\,\delta_{BD} - \delta_{AD}\,\delta_{BC} + \hat{J}_{AC}\,\hat{J}_{BD} - \hat{J}_{AD}\,\hat{J}_{BC} + 2\hat{J}_{AB}\,\hat{J}_{CD}\,. \quad (85)$$

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This study of Fubini-Study metrics encompasses various previously-known results, as well as providing many new ones. For example, if we take $m = 0$, $n = 1$, we get the metric on CP^2 written (after sending $\psi \longrightarrow \psi/2$ for convenience) as[3]

$$d\Sigma_2^2 = d\xi^2 + \tfrac{1}{4}s^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) + \tfrac{1}{4}s^2\,c^2\,(d\psi + \cos\theta\,d\varphi)^2\,, \tag{86}$$

which was first obtained in this form in [64] (with the coordinate r of that paper related to ξ by $r = \tan\xi$). The general class of cases $m = 0$, with n arbitrary, was obtained in [65]; it gives an iterative expression for the Fubini-Study metric on CP^{n+1} in terms of that on CP^n.

As a new example, we may obtain the following expression for the Fubini-Study metric on CP^3, by taking $m = n = 1$ (and sending $\psi \longrightarrow \psi/2$ for convenience):

$$\begin{aligned}
d\Sigma_3^2 &= d\xi^2 + \tfrac{1}{4}c^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) + \tfrac{1}{4}s^2\,(d\tilde{\theta}^2 + \sin^2\tilde{\theta}\,d\tilde{\varphi}^2) \\
&\quad + \tfrac{1}{4}s^2\,c^2\,(d\psi - \cos\theta\,d\varphi + \cos\tilde{\theta}\,d\tilde{\varphi})^2\,.
\end{aligned} \tag{87}$$

The Kähler form is given by

$$\hat{J} = -\tfrac{1}{2}s\,c\,d\xi \wedge (d\psi - \cos\theta\,d\varphi + \cos\tilde{\theta}\,d\tilde{\varphi}) + \tfrac{1}{4}c^2\,\sin\theta\,d\theta \wedge d\varphi + \tfrac{1}{4}s^2\,\sin\tilde{\theta}\,d\tilde{\theta} \wedge d\tilde{\varphi}\,. \tag{88}$$

The metric (87) reveals some interesting features of the geometry of CP^3. At each end of the ξ coordinate range, $0 \le \xi \le \tfrac{1}{2}\pi$, the metric approaches a product of a smooth $\mathbb{R}^4 \times S^2$; for example at $\xi \approx 0$ we have

$$ds^2 \approx d\xi^2 + \sin^2\xi\,d\Omega_3'^2 + \tfrac{1}{4}d\Omega_2^2\,, \tag{89}$$

with

$$d\Omega_3'^2 = \tfrac{1}{4}\Big[d\tilde{\theta}^2 + \sin^2\tilde{\theta}\,d\tilde{\varphi}^2 + (d\psi + \cos\tilde{\theta}\,d\tilde{\varphi} - \cos\theta\,d\varphi)^2\Big]\,. \tag{90}$$

Thus the terms $d\xi^2 + \sin^2\xi\,d\Omega_3'^2$ approach $\mathbb{R}^4$ as ξ tends to zero, described in hyperspherical polar coordinates. There is a "twist" in the $U(1)$ fibres of the S^3 metric $d\Omega_3'^2$, involving the topologically non-trivial Dirac monopole bundle over the S^2 factor $d\Omega_2^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$. An analogous phenomenon occurs at the other endpoint, at $\xi = \tfrac{1}{2}\pi$. This form of the CP^3 metric is precisely the one that arises in the non-singular embedding of the four-dimensional $N = 4$ $SO(4)$-gauged supergravity in section 6.2.

Another interesting aspect of the geometry of CP^3 that can be seen from (87) is that each foliating surface at constant ξ has the structure of the manifold $Q(1,1)$ (sometimes

---

[3]CP^1 is the same as S^2. Note, however, that the *unit* CP^1, which we have defined to be such that its Hopf bundle gives the unit-radius 3-sphere, is consequently a 2-sphere of radius $\tfrac{1}{2}$, whose metric is $d\Sigma_1^2 = \tfrac{1}{4}(d\theta^2 + \sin^2\theta\,d\varphi^2)$.

known as T^{11}), which is defined as the $U(1)$ bundle over $S^2 \times S^2$ where the fibres have winding number 1 with respect to both of the S^2 factors in the base. The metric

$$ds_5^2 = \frac{1}{\Lambda}\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) + \frac{1}{\widetilde{\Lambda}}\left(d\tilde{\theta}^2 + \sin^2\tilde{\theta}\,d\tilde{\varphi}^2\right) + c^2\left(d\psi - \cos\theta\,d\varphi + \cos\tilde{\theta}\,d\tilde{\varphi}\right)^2 \quad (91)$$

on this manifold is homogeneous for any choice of the constants Λ $\widetilde{\Lambda}$ and c, and it is Einstein if $\Lambda = \widetilde{\Lambda} = 2/(3c^2)$ (see, for example, [23]). Thus as the coordinate ξ ranges over the interval $0 < \xi < \frac{1}{2}\pi$ in (87), the foliating surfaces correspond to $Q(1,1)$ with varying non-singular homogeneous "squashings." None of the foliating surfaces corresponds to the Einstein metric on $Q(1,1)$.

Note that in general, the level surfaces at constant ξ in the metric (79) are the higher-dimensional generalisations of the $Q(1,1)$ space, namely $U(1)$ bundles over $CP^m \times CP^n$, with winding number 1 over each factor.

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