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A Heterotic Multimonopole Solution †

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An exact multimonopole solution of heterotic string theory is presented. The solution is constructed by a modification of the 't Hooft ansatz for a four-dimensional instanton. An analogous solution in Yang-Mills field theory saturates a Bogomoln'yi bound and possesses the topology and far field limit of a multimonopole configuration, but has divergent action near each source. In the string solution, however, the divergences from the Yang-Mills sector are precisely cancelled by those from the gravity sector. The resultant action is finite and easily computed. The Manton metric on moduli space for the scattering of two string monopoles is found to be flat to leading order in the impact parameter, in agreement with the trivial scattering predicted by a test monopole calculation.

4/92

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1. Introduction

In recent work several classical solitonic solutions of string theory with higher-membrane structure have been presented. In [1], the tree-level axionic instanton solution of [2] is extended to an exact solution of bosonic string theory for the special case of a linear dilaton [3,4] wormhole solution. Exactness is shown by combining the metric and antisymmetric tensor in a generalized curvature [5,6], which is written covariantly in terms of the tree-level dilaton field, and rescaling the dilaton order by order in the parameter α'. An exact heterotic multi-soliton solution with instanton structure in the four dimensional transverse space can be obtained[7,8,9] by equating the curvature of the Yang-Mills gauge field with the above generalized curvature. This latter solution represents an exact extension of the tree-level fivebrane solutions of [10,11].

In this paper we present an exact heterotic multi-soliton solution which represents a multimonopole configuration. We obtain this solution via a modification of the 't Hooft ansatz for the Yang-Mills instanton. We identify an analogous multimonopole solution in field theory with divergent action and indicate how in the string solution these divergences are cancelled. We also study the dynamics of the string monopoles and find that, unlike BPS monopoles, the string monopoles scatter trivially to leading order in the impact parameter.

We first review in section 2 the basic bosonic solution with monopole-like structure discussed in [12]. A tree-level multi-soliton solution for the massless fields of the string is written. The corresponding single source wormhole solution is extended to order α'. This latter solution is noted to contain the basic outline of a stringy correction to a magnetic monopole. We then summarize the tree-level monopole solution in $N = 4$ supersymmetric low-energy string theory of [13].

We proceed in section 3 to construct an exact heterotic multimonopole solution by modifying the 't Hooft ansatz[14–18] for the Yang-Mills instanton. We note the relationship of this solution to the exact multi-instanton solution in [8]. Unlike the latter solution, however, the multimonopole solution does not lend itself easily to a CFT description.

We note in section 4 that an analogous field theory solution representing a multi-monopole configuration not in the Prasad-Sommerfield[19] limit can be immediately obtained from the modified 't Hooft ansatz independently of string theory. This solution has the topology of $Q = 1$ monopole sources, saturates the Bogomoln'yi bound[20] and exhibits the far field behaviour of multimonopole sources. However, the action for this solution diverges near each source.

We demonstrate in section 5 that the string solution, by contrast, has finite action. The divergences coming from the Yang-Mills sector are precisely cancelled by those from the gravitational sector. The resultant action reduces to the tree-level form and is easily calculated. The zero force condition for string solitons is seen to arise as a direct result of the force cancellation in the gauge sector, once the generalized connection and gauge connection are identified.

In section 6 we study the scattering of two string monopoles by two methods. The first approach computes the Manton metric on moduli space, which defines distance on the static solution manifold. A flat metric is obtained to leading order in the impact parameter. This result is consistent with a calculation of the dynamic force on a test string monopole moving in the background of a source string monopole.

We conclude in section 7 with a discussion of our results and their implications.

2. Bosonic and Tree-Level Solutions

In this section we briefly review two previously obtained solutions: the bosonic multi-soliton solution obtained in [12] and the Prasad-Sommerfield monopole[19] solution to supersymmetric low-energy superstring theory in [13]. Both classes of solutions possess three-dimensional spherical symmetry, as opposed to the four-dimensional spherical symmetry of other instanton and fivebrane solutions[1,21,10,11,8,9].

The tree-level bosonic multi-soliton solution to the string equations of motion is given by[12]

$$\begin{aligned} e^{2\phi} &= C + \sum_{i=1}^{N} \frac{m_i}{|\vec{x}-\vec{a}_i|}, \\ g_{\mu\nu} &= e^{2\phi}\delta_{\mu\nu}, \qquad\qquad \mu,\nu = 1,2,3,4, \\ g_{ab} &= \eta_{ab}, \qquad\qquad a,b = 0,5,6...25, \\ H_{\alpha\beta\gamma} &= \pm\epsilon_{\alpha\beta\gamma}{}^{\mu}\partial_\mu\phi, \qquad \alpha,\beta,\gamma,\mu = 1,2,3,4, \end{aligned} \tag{2.1}$$

where ϕ is the dilaton, g_{MN} is the string sigma model metric and $H_{MNP} = \partial_{[M}B_{NP]}$, where B_{NP} is the antisymmetric tensor. $\vec{x} = (x_1, x_2, x_3)$ is a three-dimensional coordinate vector in the (123) subspace of the four-dimensional transverse space (1234). m_i represents the charge and a_i the location in the three-space of the ith source.

Note that we have singled out a direction x_4 and projected out all the field dependence on x_4. By doing so, we destroy the $SO(4)$ invariance in the transverse space possessed by the instanton solution[1]. However, (2.1) is an equally valid solution to the string equations as the multi-instanton solution with $e^{2\phi} = 1 + \sum_{i=1}^{N} \frac{Q_i}{|\vec{x}-\vec{a}_i|^2}$, where in this case the vectors are four-dimensional, since in both cases the dilaton field satisfies the Poisson equation $e^{-2\phi}\Box\, e^{2\phi} = 0$. The projection is necessary to obtain the three-dimensional symmetry of a magnetic monopole.

Although the above bosonic multi-soliton solution (2.1) lacks the gauge and Higgs fields normally attributed to a magnetic monopole in field theory, one can think of the dual field in the transverse four-space $H^*_\mu \equiv \frac{1}{6}\epsilon_{\alpha\beta\gamma\mu}H^{\alpha\beta\gamma}$ as the magnetic field strength of a multimonopole configuration in the space (123) (note that $H^*_4 = 0$).

Since the dilaton equation is essentially unaffected when we try to obtain a tree-level supersymmetric solution, we can follow the derivation of Duff and Lu's fivebrane solution[10], but assume that the fields are independent of one coordinate (say x_4), and again obtain a $D = 10$ multi-fivebrane solution which breaks half the spacetime supersymmetries, but with monopole-like structure.

Unlike the four-dimensional (instanton) solutions, the three-dimensional solutions do not easily lend themselves to a CFT description, and it is therefore difficult to go beyond $O(\alpha')$ in obtaining stringy corrections to the tree-level fields. In [1], the $O(\alpha')$ correction was worked out for the special case of a single source with $C = 0$. The metric and antisymmetric tensor were unchanged to $O(\alpha')$, but the dilaton is corrected:

$$e^{2\phi} = \frac{m}{r}\left(1 - \frac{\alpha'}{8mr}\right). \tag{2.2}$$

Note that, unlike the $O(\alpha')$ correction to the four-dimensional solution in [1], the dilaton correction is not a simple rescaling of the power of r to order α'. This fact is intimately connected with the difficulty in formulating a CFT description of the three-dimensional solution.

We now briefly summarize the tree-level monopole solution of [13]. Starting with $N = 1, D = 10$ supergravity coupled to super Yang-Mills, Harvey and Liu find a solution to the equations of motion with background fermi fields set to zero. Supersymmetry requires that there exists a positive chirality Majorana-Weyl spinor ϵ satisfying

$$\delta\psi_M = \left(\nabla_M - \tfrac{1}{4}H_{MAB}\Gamma^{AB}\right)\epsilon = 0, \tag{2.3}$$

$$\delta\lambda = \left(\Gamma^A\partial_A\phi - \tfrac{1}{6}H_{AMC}\Gamma^{ABC}\right)\epsilon = 0, \tag{2.4}$$

$$\delta\chi = F_{AB}\Gamma^{AB}\epsilon = 0, \tag{2.5}$$

where ψ_M, λ and χ are the gravitino, dilatino and gaugino fields. The Bianchi identity is given by

$$dH = \alpha'\left(\mathrm{tr}R\wedge R - \tfrac{1}{30}\mathrm{Tr}F\wedge F\right). \tag{2.6}$$

Choose the spacetime indices to be $0,1,2,3$ and the internal indices to be $4,5...9$. The $(9+1)$-dimensional Majorana-Weyl fermions decompose down to chiral spinors according to $SO(9,1)\supset SO(3,1)\otimes SO(6)$ for the $M^{9,1}\to M^{3,1}\times M^6$ decomposition. Again if we single out a direction in internal space (say x_4), the above supersymmetry equations and Bianchi identity are solved by a constant chiral spinor[13] $\epsilon_\pm = \pm\Gamma^{1234}\epsilon_\pm$ and the ansatz

$$\begin{aligned}
F_{\mu\nu} =& \pm\frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\sigma}F_{\lambda\sigma},\\
H_{\mu\nu\lambda} =& \mp\,\epsilon_{\mu\nu\lambda}{}^{\sigma}\partial_\sigma\phi,\\
g_{MN} =& \mathrm{diag}(-1, e^{2\phi}, e^{2\phi}, e^{2\phi}, e^{2\phi}, 1, 1, 1, 1, 1),\\
\nabla_\rho\nabla^\rho =& \mp\frac{1}{4}\alpha'\epsilon^{\mu\nu\lambda\sigma}\mathrm{tr}F_{\mu\nu}F_{\lambda\sigma},
\end{aligned} \tag{2.7}$$

where $\mu,\nu,\lambda,\sigma = 1,2,3,4$. The BPS monopole solution for the gauge and Higgs fields is given by[19,20]

$$\begin{aligned}
A_i^a =& \epsilon_{iab}\frac{x^b}{r^2}(K-1),\\
\Phi^a =& \frac{x^a}{r^2}H,
\end{aligned} \tag{2.8}$$

where $H = Cr\coth Cr - 1$, $K = \frac{Cr}{\sinh Cr}$ and C is the vacuum expectation value of the Higgs. Making the identification $A_4^a \equiv \Phi^a$, replacing (2.8) into (2.7) and solving the dilaton equation yields

$$e^{2\phi} = e^{2\phi_0} + 2\alpha'\frac{1}{r^2}\left[1 - K^2 + 2H\right], \tag{2.9}$$

which is nonsingular at $r=0$ and represents a single monopole source.

Since (2.7) can be solved by any (anti) self-dual configuration, we can in principle write down a multimonopole solution. While this solution is supersymmetric, it is only tree-level in α', and not necessarily an exact solution (*i.e.* in principle, we would have to obtain corrections to the fields to higher order in α').

3. Exact Heterotic Multimonopole Solution

In this section we construct an exact multimonopole solution of heterotic string theory. The derivation of this solution closely parallels that of the multi-instanton solution presented in [8,9], but in this case, the solution possesses three-dimensional spherical symmetry near each source, which turns out to represent a magnetic monopole of topological charge $Q = 1$. Again the reduction is effected by singling out a direction in the transverse space.

The supersymmetry equations (2.3), (2.4) and (2.5) are unchanged at tree-level in heterotic string theory. In this case, however, the $(9+1)$-dimensional Majorana-Weyl fermions decompose down to chiral spinors according to $SO(9,1) \supset SO(5,1) \otimes SO(4)$ for the $M^{9,1} \to M^{5,1} \times M^4$ decomposition. Let $\mu, \nu, \lambda, \sigma = 1,2,3,4$ and $a, b = 0,5,6,7,8,9$. Then the ansatz

$$\begin{aligned} g_{\mu\nu} &= e^{2\phi}\delta_{\mu\nu}, \\ g_{ab} &= \eta_{ab}, \\ H_{\mu\nu\lambda} &= \pm\epsilon_{\mu\nu\lambda\sigma}\partial^\sigma\phi \end{aligned} \tag{3.1}$$

with constant chiral spinors $\epsilon_\pm$ again solves the supersymmetry equations (again with zero background fermi fields) provided the YM gauge field satisfies the instanton (anti)self-duality condition

$$F_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\sigma}F_{\lambda\sigma}. \tag{3.2}$$

An exact solution is obtained as follows. Define a generalized connection by

$$\Omega^{AB}_{\pm M} = \omega^{AB}_M \pm H^{AB}_M \tag{3.3}$$

embedded in an SU(2) subgroup of the gauge group, and equate it to the gauge connection A_μ[22] so that $dH = 0$ and the corresponding curvature $R(\Omega_\pm)$ cancels against the Yang-Mills field strength F. The crucial point is that for $e^{-2\phi}\Box\, e^{2\phi} = 0$ with the above ansatz, the curvature of the generalized connection can be written in the covariant form[1]

$$\begin{aligned} R(\Omega_\pm)^{mn}_{\mu\nu} =& \delta_{n\nu}\nabla_m\nabla_\mu\phi - \delta_{n\mu}\nabla_m\nabla_\nu\phi + \delta_{m\mu}\nabla_n\nabla_\nu\phi - \delta_{m\nu}\nabla_n\nabla_\mu\phi \\ &\pm \epsilon_{\mu mn\alpha}\nabla_\alpha\nabla_\nu\phi \mp \epsilon_{\nu mn\alpha}\nabla_\alpha\nabla_\mu\phi, \end{aligned} \tag{3.4}$$

from which it easily follows that

$$R(\Omega_\pm)^{mn}_{\mu\nu} = \mp\tfrac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\sigma}R(\Omega_\pm)^{mn}_{\lambda\sigma}. \tag{3.5}$$

Thus we have a solution with the ansatz (3.1) such that

$$F^{mn}_{\mu\nu} = R(\Omega_\pm)^{mn}_{\mu\nu}, \tag{3.6}$$

where both F and R are (anti)self-dual. This solution becomes exact since $A_\mu = \Omega_{\pm\mu}$ implies that all the higher order corrections vanish[23,24]. The self-dual solution for the gauge connection is then given by the 't Hooft ansatz for the four-dimensional instanton

$$A_\mu = i\overline{\Sigma}_{\mu\nu}\partial_\nu \ln f, \tag{3.7}$$

where $\overline{\Sigma}_{\mu\nu} = \overline{\eta}^{i\mu\nu}(\sigma^i/2)$ for $i = 1, 2, 3$ (σ^i, $i = 1, 2, 3$ are the 2×2 Pauli matrices), where

$$\begin{aligned} \overline{\eta}^{i\mu\nu} = -\overline{\eta}^{i\nu\mu} &= \epsilon^{i\mu\nu}, \qquad && \mu, \nu = 1, 2, 3, \\ &= -\delta^{i\mu}, && \nu = 4 \end{aligned} \tag{3.8}$$

and where $f^{-1}\Box\, f = 0$. The ansatz for the anti-self-dual solution is similar, with the δ-term in (3.8) changing sign.

To obtain a multi-instanton solution, one solves for f in the four-dimensional space to obtain

$$f = e^{-2\phi_0}e^{2\phi} = 1 + \sum_{i=1}^{N} \frac{\rho_i^2}{|\vec{x} - \vec{a}_i|^2}, \tag{3.9}$$

where ρ_i^2 is the instanton scale size and $\vec{a}_i$ the location in four-space of the ith instanton.

To obtain a multimonopole solution, we modify the 't Hooft ansatz as follows. We again single out a direction in the transverse four-space (say x_4) and assume all fields are independent of this coordinate. Then the solution for f can be written as

$$f = e^{-2\phi_0}e^{2\phi} = 1 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|}, \tag{3.10}$$

where m_i is the charge and $\vec{a}_i$ the location in the three-space (123) of the ith source. If we make the identification $\Phi \equiv A_4$, then the gauge and Higgs fields may be simply written in terms of the dilaton as

$$\begin{aligned} \Phi^a &= -\frac{2}{g}\delta^{ia}\partial_i\phi, \\ A^a_k &= -\frac{2}{g}\epsilon^{akj}\partial_j\phi \end{aligned} \tag{3.11}$$

for the self-dual solution. For the anti-self-dual solution, the Higgs field simply changes sign. Here g is the YM coupling constant. Note that ϕ_0 drops out in (3.11). The solution in (3.10) can be thought of as a multi-line source instanton solution, each monopole being interpreted as an "instanton string"[25].

The above solution (with the gravitational fields obtained directly from (3.1) and (3.10)) represents an exact multimonopole solution of heterotic string theory. In order to more clearly see the monopole structure of this solution, we first consider in the next section an analogous solution in field theory and study its properties, which then carry over directly into the string solution.

4. Multimonopole Solution in Field Theory

We now turn to an analogous multimonopole solution in field theory. Consider the four-dimensional Euclidean action

$$S = -\frac{1}{2g^2}\int d^4x \mathrm{Tr} G_{\mu\nu}G^{\mu\nu}, \qquad\qquad \mu,\nu = 1,2,3,4. \tag{4.1}$$

For gauge group $SU(2)$, the fields may be written as $A_\mu = (g/2i)\sigma^a A_\mu^a$ and $G_{\mu\nu} = (g/2i)\sigma^a G_{\mu\nu}^a$. The equation of motion derived from this action is solved by the modified 't Hooft ansatz shown in the previous section:

$$A_\mu = i\overline{\Sigma}_{\mu\nu}\partial_\nu \ln f, \tag{4.2}$$

where again

$$f = 1 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x}-\vec{a}_i|}, \tag{4.3}$$

where m_i is the charge and $\vec{a}_i$ the location in the three-space (123) of the ith source. To obtain a multimonopole solution, we again identify the scalar field $\Phi \equiv A_4$ (we loosely refer to this field as a Higgs field in this paper, although there is no apparent symmetry breaking mechanism). The Lagrangian density for the above ansatz can be rewritten as

$$\begin{aligned} G_{\mu\nu}^a G_{\mu\nu}^a =& G_{ij}^a G_{ij}^a + 2G_{k4}^a G_{k4}^a \\ =& G_{ij}^a G_{ij}^a + 2D_k\Phi^a D_k\Phi^a, \end{aligned} \tag{4.4}$$

which has the same form as the Lagrangian density for YM + massless scalar field in three dimensions.

We now go to $3+1$ dimensions with the Lagrangian density (signature $(-+++)$)

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G^{\mu\nu a} - \frac{1}{2}D_\mu\Phi^a D^\mu\Phi^a, \tag{4.5}$$

and show that the above multi-soliton ansatz is a static solution with $A_0^a = 0$ and all time derivatives vanish. The equations of motion in this limit are given by

$$\begin{aligned} D_i G^{jia} &= g\epsilon^{abc}(D^j\Phi^b)\Phi^c, \\ D_i D^i \Phi^a &= 0. \end{aligned} \tag{4.6}$$

It is then straightforward to verify that the above equations are solved by

$$\begin{aligned} \Phi^a &= \mp\frac{1}{g}\delta^{ai}\partial_i\omega, \\ A_k^a &= \epsilon^{akj}\partial_j\omega, \end{aligned} \tag{4.7}$$

where $\omega \equiv \ln f$. This solution represents a multimonopole configuration with sources at $\vec{a}_i = 1, 2...N$. A simple observation of far field and near field behaviour shows that this solution does not arise in the Prasad-Sommerfield[19] limit. In particular, the fields are singular near the sources and vanish as $r \to \infty$.

The topological charge of each source is easily computed ($\hat{\Phi}^a \equiv \Phi^a/|\Phi|$) to be

$$Q = \int d^3x k_0 = \frac{1}{8\pi}\int d^3x \epsilon_{ijk}\epsilon^{abc}\partial_i\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c = 1. \tag{4.8}$$

The magnetic charge of each source is then given by $m_i = Q/g = 1/g$. It is also straightforward to show that the Bogomoln'yi[20] bound

$$G_{ij}^a = \epsilon_{ijk}D_k\Phi^a \tag{4.9}$$

is saturated by this solution. Finally, it is easy to show that the magnetic field $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$ (where $F_{\mu\nu} \equiv \hat{\Phi}^a G_{\mu\nu}^a - (1/g)\epsilon^{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c$ is the gauge-invariant electromagnetic field tensor defined by 't Hooft[26]) has the the far field limit behaviour of a multimonopole configuration:

$$B(\vec{x}) \to \sum_{i=1}^{N}\frac{m_i(\vec{x}-\vec{a}_i)}{|\vec{x}-\vec{a}_i|^3}, \qquad \text{as} \quad r \to \infty. \tag{4.10}$$

As usual, the existence of this static multimonopole solution owes to the cancellation of the gauge and Higgs forces of exchange–the "zero-force" condition.

We have presented all the monopole properties of this solution. Unfortunately, this solution as it stands has divergent action near each source, and this singularity cannot be simply removed by a unitary gauge transformation. This can be seen for a single source by noting that as $r \to 0$, $A_k \to \frac{1}{2}\left(U^{-1}\partial_k U\right)$, where U is a unitary 2×2 matrix. The expression in parentheses represents a pure gauge, and there is no way to get around the 1/2 factor in attempting to "gauge away" the singularity[27]. The field theory solution is therefore not very interesting physically. As we shall see in the next section, however, the string theory solution has far greater potential.

5. Finiteness of String Solution

The string solution presented in section 3 has the same structure in the four-dimensional transverse space as the multimonopole solution of the YM + scalar field action of section 4. If we identify the (123) subspace of the transverse space as the space part of the four-dimensional spacetime (with some toroidal compactification, similar to that used in [13]) and take the timelike direction as the usual X^0, then the monopole properties described in the previous section carry over directly into the string solution.

The string action contains a term $-\alpha' F^2$ which also diverges as in the field theory solution. This divergence, however, is precisely cancelled by the term $\alpha' R^2(\Omega_\pm)$ in the $O(\alpha')$ action. This result follows from the exactness condition $A_\mu = \Omega_{\pm\mu}$ which leads to $dH = 0$ and the vanishing of all higher order corrections in α'. Another way of seeing this is to consider the higher order corrections to the bosonic action shown in [23,24]. All such terms contain the tensor T_{MNPQ}, a generalized curvature incorporating both $R(\Omega_\pm)$ and F. The ansatz is contructed precisely so that this tensor vanishes identically[1,7]. The action thus reduces to its lowest order form and can be calculated directly for a multi-source solution from the expressions for the massless fields in the gravity sector.

The divergences in the gravitational sector in heterotic string theory thus serve to cancel the divergences stemming from the field theory solution. This solution thus provides an interesting example of how this type of cancellation can occur in string theory, and supports the promise of string theory as a finite theory of quantum gravity. Another point of interest is that the string solution represents a supersymmetric multimonopole solution coupled to gravity, in which the zero-force condition in the gravitational sector (*i.e.* the cancellation between the attractive gravitational force and repulsive antisymmetric tensor force) arises as a direct result of the zero-force condition in the gauge sector (cancellation between gauge and Higgs exchange forces) once the gauge connection and generalized connection are identified.

We now calculate the mass of the heterotic multimonopole configuration. Naively, the mass can be calculated from the tree-level action (since the higer order terms drop out)

$$S = -\frac{1}{2\kappa^2}\int d^3x\sqrt{g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{H^2}{12}\right). \tag{5.1}$$

There is one subtlety we must consider, however (see [28]). From the term $\sqrt{g}e^{-2\phi}R$ in the integrand of the action, the action density in (5.1) contains double derivative terms of the metric component fields. In general, one would like to work with an action which depends only on the fields and their first derivatives. This problem was solved in general relativity by Gibbons and Hawking[29,30], who added a surface term which precisely cancelled the double derivative terms in the action in general relativity. The addition of a surface term does not, of course, affect the equations of motion.

It turns out that there is a relatively straightforward generalization of the Gibbons-Hawking surface term (GHST) to string theory[31,32]. By antisymmetry, the axion field does not contribute to the GHST and the surface term in this case can be written in the simple form

$$S_{GHST} = -\frac{1}{\kappa^2}\int_{\partial M}\left(e^{-2\phi}K - K_0\right), \tag{5.2}$$

where ∂M is the surface boundary and K and K_0 are the traces of the fundamental form of the boundary surface embedded in the metric g and the Minkowskian metric η respectively. The correct effective action is thus obtained by adding the surface term of (5.2) to the volume term of (5.1):

$$S = -\frac{1}{2\kappa^2}\left[\int d^3x\sqrt{g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{H^2}{12}\right) + 2\int_{\partial M}\left(e^{-2\phi}K - K_0\right)\right]. \tag{5.3}$$

By using the equations of motion, the volume term S_V can be written as a surface term (see [28]):

$$S_V = -\frac{1}{\kappa^2}\int_{\partial M}\hat{n}\cdot\vec{\nabla}e^{-2\phi}. \tag{5.4}$$

Note that $\sqrt{g}$ has been absorbed into the surface measure of ∂M. Since we have separability of sources in the limit of surfaces of infinite radius, we may therefore compute S_V for a single monopole configuration in three-space

$$\begin{aligned} e^{2\phi} &= 1 + \frac{m}{r}, \\ g_{ij} &= e^{2\phi}\delta_{ij}, \end{aligned} \tag{5.5}$$

and simply add the contributions of an arbitrary number of sources. The contribution of a single monopole to the static volume action is given by

$$\begin{aligned} S_V &= -\frac{1}{\kappa^2}(\frac{\partial}{\partial r}e^{-2\phi})A(M) \\ &= -\frac{4\pi m}{\kappa^2} \end{aligned} \tag{5.6}$$

in the $r \to \infty$ limit, where $A(M) = 4\pi r^2(1 + m/r)$ is the area of the boundary surface.

We now turn to the GHST. A simple calculation of the extrinsic curvature K for a single monopole configuration (5.5) gives

$$K = \frac{2}{r^2} e^{-3\phi}(r + m/2). \tag{5.7}$$

When the surface ∂M is embedded in flat space, the radius of curvature R is given by $R = re^{\phi}$. The extrinsic curvature K_0 is then given by

$$K_0 = \frac{2}{R} = \frac{2}{r} e^{-\phi}. \tag{5.8}$$

The GHST is therefore given by

$$S_{GHST} = -\frac{2}{\kappa^2 r}\left(e^{-5\phi}(1 + \frac{m}{2r}) - e^{-\phi}\right) A(M) = \frac{12\pi m}{\kappa^2} \tag{5.9}$$

in the $r \to \infty$ limit.

The total static action for a multi-soliton configuration, equal to the total mass of the solitons, can then be obtained by adding the static contributions to the action of the volume part and the GHST. The result is

$$M_T = \frac{8\pi}{\kappa^2} \sum_{n=1}^{N} m_n. \tag{5.10}$$

For our multimonopole configuration, however, it should be noted that $m_n = 1/g$ for $n = 1, 2...N$.

6. Dynamics of String Monopoles

We now consider the dynamics of the string monopoles. For this purpose, we adopt two different methods. The first computes the Manton metric on moduli space for the scattering of two string monopoles, while the second studies the motion of a test string monopole in the background of a source string monopole. We will find that the two methods yield consistent results.

Manton's prescription[33] for the study of soliton scattering may be summarized as follows. We first invert the constraint equations of the system. The resultant time dependent field configuration does not in general satisfy the full time dependent field equations, but provides an initial data point for the fields and their time derivatives. Another way of saying this is that the initial motion is tangent to the set of exact static solutions. The kinetic action obtained by replacing the solution to the constraints into the action defines a metric on the parameter space of static solutions. This metric defines geodesic motion on the moduli space[33].

A calculation of the metric on moduli space for the scattering of BPS monopoles and a description of its geodesics was worked out by Atiyah and Hitchin[34]. Several interesting properties of monopole scattering were found, such as the conversion of monopoles into dyons and the right angle scattering of two monopoles on a direct collision course[34,35]. The configuration space is found to be a four-dimensional manifold M_2 with a self-dual Einstein metric.

In this section, we adapt Manton's prescription to study the dynamics of heterotic string monopoles. A similar procedure was followed in [28] for the Manton scattering of heterotic instantons. Indeed, many of the formal computations carry over from the instanton computation. For the monopoles, however, the divergences plagueing the instanton calculation are absent, thus rendering our task far simpler. In both cases, we follow essentially the same steps that Manton outlined for monopole scattering, but take into account the peculiar nature of the string effective action. Since we work in the low-velocity limit, our kinematic analysis is nonrelativistic.

We first solve the constraint equations for the soliton solutions. These equations are simply the $(0j)$ components of the equations of motion (see [1,28])

$$\begin{aligned} R_{0j} - \frac{1}{4}H_{0j}^2 + 2\nabla_0\nabla_j\phi = 0, \\ -\frac{1}{2}\nabla_k H^k{}_{0j} + H_{0j}{}^k\partial_k\phi = 0. \end{aligned} \tag{6.1}$$

Note that we use the tree-level equations of motion, as the higher order corrections in α' automatically vanish. We wish to find an $O(\beta)$ solution to the above equations which represents a quasi-static version of (3.1) (i.e. a solution of the form (3.1) but with time dependent $\vec{a}_i$). In other words, we would like to give each source an arbitrary transverse velocity $\vec{\beta}_n$ in the (123) subspace of the four-dimensional transverse space and see what corrections to the fields are required by the constraints. The vector $\vec{a}_n$ representing the position of source n in the three-space (123) is given by

$$\vec{a}_n(t) = \vec{A}_n + \vec{\beta}_n t, \tag{6.2}$$

where $\vec{A}_n$ is the initial position of the nth source. Note that at $t = 0$ we have an exact static multi-soliton solution. Our solution to the constraints will adjust our quasi-static approximation so that the initial motion in the parameter space is tangent to the initial exact solution at $t = 0$.

The $O(\beta)$ solution to the constraints is given by

$$
\begin{aligned}
e^{2\phi(\vec{x},t)} &= 1 + \sum_{n=1}^{N} \frac{m_n}{|\vec{x} - \vec{a}_n(t)|}, \\
g_{00} &= -1, \qquad g^{00} = -1, \qquad g_{ij} = e^{2\phi}\delta_{ij}, \qquad g^{ij} = e^{-2\phi}\delta_{ij}, \\
g_{0i} &= -\sum_{n=1}^{N} \frac{m_n \vec{\beta}_n \cdot \hat{x}_i}{|\vec{x} - \vec{a}_n(t)|}, \qquad g^{0i} = e^{-2\phi} g_{0i}, \\
H_{ijk} &= \epsilon_{ijkm}\partial_m e^{2\phi}, \\
H_{0ij} &= \epsilon_{ijkm}\partial_m g_{0k} = \epsilon_{ijkm}\partial_k \sum_{n=1}^{N} \frac{m_n \vec{\beta}_n \cdot \hat{x}_m}{|\vec{x} - \vec{a}_n(t)|},
\end{aligned}
\tag{6.3}
$$

where $i, j, k, m = 1, 2, 3, 4$, the $\vec{a}_n(t)$ are given by (6.2) and we use a flat space ϵ-tensor. Note that g_{00}, g_{ij} and H_{ijk} are unaffected to order β. Also note that we can interpret the solitons as either line sources in the four-dimensional space (1234) or point sources in the three-dimensional subspace (123).

The kinetic Lagrangian is obtained by replacing the expressions for the fields in (6.3) into (5.3). Since (6.3) is a solution to order β, the leading order terms in the action (after the quasi-static part) are of order β^2. In the volume term of the action, $O(\beta)$ terms in the solution give $O(\beta^2)$ terms in the kinetic action. As explained in [28], the contribution of the GHST to the kinetic action can be written in the form $m_s\beta^2/2$ for each source, and the contributions of the sources can be simply added. The GHST does not therefore play an important role in the dynamics of the string monopoles, but merely serves to give the correct total mass. Collecting all $O(\beta^2)$ terms in S_V we get the following kinetic Lagrangian density for the volume term:

$$
\begin{aligned}
\mathcal{L}_{kin} = -\frac{1}{2\kappa^2}\Bigg(& 4\dot{\phi}\vec{M} \cdot \vec{\nabla}\phi - e^{-2\phi}\partial_i M_j \partial_i M_j - e^{-2\phi} M_k \partial_j \phi \left(\partial_j M_k - \partial_k M_j\right) \\
& + 4M^2 e^{-2\phi}(\vec{\nabla}\phi)^2 + 2\partial_t^2 e^{2\phi} - 4\partial_t(\vec{M} \cdot \vec{\nabla}\phi) - 4\vec{\nabla} \cdot (\dot{\phi}\vec{M})\Bigg),
\end{aligned}
\tag{6.4}
$$

where $\vec{M} \equiv -\sum_{n=1}^{N} \frac{m_n \vec{\beta}_n}{|\vec{x} - \vec{a}_n(t)|}$. Henceforth let $\vec{X}_n \equiv \vec{x} - \vec{a}_n(t)$. The last three terms in (6.4) are time-surface or space-surface terms which vanish when integrated. Note that the kinetic Lagrangian has the same form as in [28]. The contributions of the GHST are again simply flat kinetic terms.

In contrast to the instanton case, the kinetic Lagrangian $L_{kin} = \int d^3x \mathcal{L}_{kin}$ for monopole scattering converges everywhere. This can be seen simply by studying the limiting behaviour of L_{kin} near each source. For a single source at $r = 0$ with magnetic charge m and velocity β, we collect the logarithmically divergent pieces and find that they cancel:

$$\frac{m\beta^2}{2} \int r^2 dr d\theta \sin\theta d\phi \left(-\frac{1}{r^3} + \frac{3\cos^2\theta}{r^3} \right) = 0. \tag{6.5}$$

So unlike the instanton case, in which we were compelled to extract information from the convergent interaction terms, in this case we can use the self-terms directly.

We now specialize to the case of two heterotic monopoles of magnetic charge $m_1 = m_2 = m = 1/g$ and velocities $\vec{\beta}_1$ and $\vec{\beta}_2$. Let the monopoles be located at $\vec{a}_1$ and $\vec{a}_2$. Our moduli space consists of the configuration space of the relative separation vector $\vec{a} \equiv \vec{a}_2 - \vec{a}_1$. The most general kinetic Lagrangian can be written as

$$\begin{aligned} L_{kin} =& h(a)(\vec{\beta}_1 \cdot \vec{\beta}_1 + \vec{\beta}_2 \cdot \vec{\beta}_2) + p(a)\left((\vec{\beta}_1 \cdot \hat{a})^2 + (\vec{\beta}_2 \cdot \hat{a})^2 \right) \\ &+ 2f(a)\vec{\beta}_1 \cdot \vec{\beta}_2 + 2g(a)(\vec{\beta}_1 \cdot \hat{a})(\vec{\beta}_2 \cdot \hat{a}). \end{aligned} \tag{6.6}$$

Now suppose $\vec{\beta}_1 = \vec{\beta}_2 = \vec{\beta}$, so that (6.6) reduces to

$$L_{kin} = (2h + 2f)\beta^2 + (2p + 2g)(\vec{\beta} \cdot \hat{a})^2. \tag{6.7}$$

This configuration, however, represents the boosted solution of the two-static soliton solution. The kinetic energy should therefore be simply

$$L_{kin} = \frac{M_T}{2}\beta^2, \tag{6.8}$$

where $M_T = M_1 + M_2 = 2M = 16\pi m/\kappa^2$ is the total mass of the two soliton solution. It then follows that the anisotropic part of (6.7) vanishes and we have

$$\begin{aligned} g + p &= 0, \\ 2(h + f) &= \frac{M_T}{2}. \end{aligned} \tag{6.9}$$

It is therefore sufficient to compute h and p. This can be done by setting $\vec{\beta}_1 = \vec{\beta}$ and $\vec{\beta}_2 = 0$. The kinetic Lagrangian then reduces to

$$L_{kin} = h(a)\beta^2 + p(a)(\vec{\beta} \cdot \hat{a})^2. \tag{6.10}$$

Suppose for simplicity also that $\vec{a}_1 = 0$ and $\vec{a}_2 = \vec{a}$ at $t = 0$. The Lagrangian density of the volume term in this case is given by

$$\mathcal{L}_{kin} = \frac{-1}{2\kappa^2}\Bigg(\frac{3m^3e^{-4\phi}}{2r^4}(\vec{\beta}\cdot\vec{x})\left[\frac{\vec{\beta}\cdot\vec{x}}{r^3} + \frac{\vec{\beta}\cdot(\vec{x}-\vec{a})}{|\vec{x}-\vec{a}|^3}\right] - \frac{e^{-2\phi}m^2\beta^2}{r^4} \\ - \frac{e^{-4\phi}m^3\beta^2}{2r^4}\left(\frac{1}{r} + \frac{\vec{x}\cdot(\vec{x}-\vec{a})}{|\vec{x}-\vec{a}|^3}\right) + \frac{e^{-6\phi}m^4\beta^2}{r^2}\left(\frac{1}{r^4} + \frac{1}{|\vec{x}-\vec{a}|^4} + \frac{2\vec{x}\cdot(\vec{x}-\vec{a})}{r^3|\vec{x}-\vec{a}|^3}\right)\Bigg). \tag{6.11}$$

The GHST contribution to the kinetic Lagrangian can be simply added after integration and will not affect the analysis below.

The integration of the kinetic Lagrangian density in (6.11) over three-space yields the kinetic Lagrangian from which the metric on moduli space can be read off. For large a, the nontrivial leading order behaviour of the components of the metric, and hence for the functions $h(a)$ and $p(a)$, is generically of order $1/a$. In fact, for Manton scattering of YM monopoles, the leading order scattering angle is $2/b$[36], where b is the impact parameter. In this paper, we restrict our computation to the leading order metric in moduli space. A tedious but straightforward collection of $1/a$ terms in the Lagrangian yields

$$\frac{-1}{2\kappa^2}\frac{1}{a}\int d^3x\left[-\frac{3m^4e^{-6\phi_1}}{r^7}(\vec{\beta}\cdot\vec{x})^2 + \frac{m^3e^{-4\phi_1}}{r^4}\beta^2 + \frac{m^4e^{-6\phi_1}}{r^5}\beta^2 - \frac{3m^5e^{-8\phi_1}}{r^6}\beta^2\right], \tag{6.12}$$

where $e^{2\phi_1} \equiv 1 + m/r$. The first and third terms clearly cancel after integration over three-space. The second and fourth terms are spherically symmetric. A simple integration yields

$$\int_0^\infty r^2dr\left(\frac{e^{-4\phi_1}}{r^4} - \frac{3m^2e^{-8\phi_1}}{r^6}\right) = \int_0^\infty \frac{dr}{(r+m)^2} - 3m^2\int_0^\infty \frac{dr}{(r+m)^4} = 0. \tag{6.13}$$

The $1/a$ terms therefore cancel, and the leading order metric on moduli space is flat. This implies that the leading order scattering is trivial. In other words, there is no deviation from the initial trajectories to leading order in the impact parameter.

The above result is rather surprising and suggests that, in addition to the static force, the leading order dynamic force also vanishes. For pure YM monopoles, this is certainly not the case. For the string monopoles, however, the dynamic YM force is precisely cancelled by the dynamic gravity sector force.

To confirm this result, we employ the test-soliton approach of [37,38] to compute the dynamic force exerted on a test string monopole moving in the background of a source string monopole. Again only the massless fields in the gravitational sector come in to play at tree-level. Since the monopoles have fivebrane structure, we adopt the fivebrane action of Duff and Lu[10,11]

$$\begin{aligned} S_{\sigma_5} = &- T_6 \int d^6\xi \Bigg(\frac{1}{2}\sqrt{-\gamma}\gamma^{mn}\partial_m X^M \partial_n X^N g_{MN} e^{-\phi/6} - 2\sqrt{-\gamma} \\ &+ \frac{1}{6!}\epsilon^{mnpqrs}\partial_m X^M \partial_n X^N \partial_p X^P \partial_q X^Q \partial_r X^R \partial_s X^S A_{MNPQRS} \Bigg), \end{aligned} \tag{6.14}$$

where $m, n, p, q, r, s = 0, 5, 6, 7, 8, 9$ are fivebrane indices and $M, N, P, Q, R, S = 0, 1, ...9$ are spacetime indices (transverse indices are denoted by $i, j = 1, 2, 3, 4$). γ_{mn} is a $5+1$-dimensional worldsheet metric, g_{MN} is the canonical spacetime metric and A_{MNPQRS} is the antisymmetric six-form potential whose curl $K = dA$ is dual to the antisymmetric field strength $H_{\alpha\beta\gamma}$.

The multimonopole solution written in this frame is given by

$$\begin{aligned} ds^2 &= e^{2A}\eta_{mn}dx^m dx^n + e^{2B}\delta_{ij}dx^i dx^j, \\ A_{056789} &= -e^C, \end{aligned} \tag{6.15}$$

where all other components of A_{MNPQRS} are set to zero and the dilaton ϕ and the scalar functions A, B and C are given by

$$\begin{aligned} A &= -\frac{(\phi - \phi_0)}{4}, \\ B &= \frac{3(\phi - \phi_0)}{4}, \\ C &= -2\phi + \frac{3\phi_0}{2}, \end{aligned} \tag{6.16}$$

where ϕ_0 is the value of the dilaton field at infinity and

$$e^{2\phi} = e^{2\phi_0}\left(1 + \sum_{n=1}^{N} \frac{m_n}{|\vec{x} - \vec{a}_n|}\right), \tag{6.17}$$

where $\vec{x}$ and $\vec{a}_n$ are again vectors in the three-dimensional subspace (123) of the transverse space (1234).

The Lagrangian for a test monopole moving in a background of identical static source monopoles is given by substituting (6.15) in (6.14) and then eliminating the worldbrane metric. The result is

$$\mathcal{L}_6 = -T_6\left[\sqrt{-\det(e^{-2\phi/3+\phi_0/2}\eta_{mn} + e^{4\phi/3-3\phi_0/2}\partial_m X^M \partial_n X_M)} - e^{-2\phi+3\phi_0/2}\right]. \tag{6.18}$$

Since the test-monopole moves only in the (123) subspace of the transverse space (there is no motion along or field dependence on the direction x_4), (6.18) reduces in the low-velocity limit to

$$\begin{aligned}\mathcal{L}_6 &\simeq -T_6\left[e^{-2\phi+3\phi_0/2}\left(1-\tfrac{1}{2}e^{2(\phi-\phi_0)}(\dot{X}^i)^2\right)-e^{-2\phi+3\phi_0/2}\right]\\ &=\frac{T_6}{2}e^{-\phi_0/2}(\dot{X}^i)^2\ ,\end{aligned}\tag{6.19}$$

where $i=1,2,3$. Again both the static force and the nontrivial $O(v^2)$ velocity-dependent force vanish. Hence this result also predicts trivial scattering, in direct agreement with the flat Manton metric calculation.

7. Conclusion

In this paper, we have presented an exact multimonopole solution of heterotic string theory. This solution represents a supersymmetric extension of the bosonic string multimonopole solution outlined in [12], and is obtained by a modification of the 't Hooft ansatz for a four-dimensional instanton. Exactness is shown by the generalized curvature method used in [1,7,8,9] to obtain exact instanton solutions in bosonic and heterotic string theory. Unlike the instanton solutions, however, the monopole solutions do not seem to be easily describable in terms of conformal field theories, an unfortunate state of affairs from the point of view of string theory.

An analogous multimonopole solution of the four dimensional field theory of YM + massless scalar field can be immediately written down. This solution possesses the properties of a multimonopole solution (topology, far-field limit and Bogomoln'yi bound) but has divergent action near each source. In the string solution, however, these divergences in the YM sector are cancelled by similar divergences in the gravity sector, thus resulting in a finite action solution. This finding is significant in that it represents an example of how string theory incorporates gravity in such a way as to cancel infinities inherent in gauge theories, thus supporting its promise as a theory of quantum gravity.

The cancellation between the gauge and gravitational sectors also influences the dynamics of the string monopoles. Indeed, we find from both a Manton metric on moduli space calculation and a test string monopole calculation that the leading order dynamic force between two string monopoles vanishes. This result implies trivial scattering between string monopoles to leading order.

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