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On F-theory E_6 GUTs

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Abstract

We approach the Minimum Supersymmetric Standard Model (MSSM) from an E_6 GUT by using the spectral cover construction and non-abelian gauge fluxes in F-theory. We start with an E_6 singularity unfolded from an E_8 singularity and obtain E_6 GUTs by using an $SU(3)$ spectral cover. By turning on $SU(2) \times U(1)^2$ gauge fluxes, we obtain a rank 5 model with the gauge group $SU(3) \times SU(2) \times U(1)^2$. Based on the well-studied geometric backgrounds in the literature, we demonstrate several models and discuss their phenomenology.

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1 Introduction

F-theory [1–3] is a geometrized type IIB string theory whose background is lifted to a twelve-dimensional manifold with an elliptic fibration. The singularities of the elliptic fibers correspond to the gauge groups on the seven-branes [4, 5]. Particularly, F-theory allows E-type singularities which inspired the study of constructing the Grand Unification Theory (GUT) local models admitting the down-type quark Yukawa couplings [6–9]. Recently, F-theory and spectral cover construction [8, 10] originally introduced in the heterotic string compactifications [11] have been used to build an $SU(5)$ GUT with an $SU(5)$ cover [8–10, 12–36], a flipped $SU(5)$ and an $SO(10)$ with $SU(4)$ covers [37–39], and an MSSM with an $SU(5)\times U(1)$ cover [40,41]. The studies in global models can be found in [42–44]. For a systematic review of recent progress of F-theory compactifications and model buildings, see [45].

To break the GUT symmetry in F-theory models, one can either use Wilson lines [6, 46] or introduce a supersymmetric $U(1)$ flux corresponding to a fractional line bundle [7, 18–20, 37, 38]. In local models, an abelian or a non-abelian flux of the rank higher than two may be turned on on the bulk to break the gauge group [7]. Following this idea, an MSSM model from breaking an $SU(6)$ model by an $U(1)\times U(1)$ gauge flux has been studied [47]. There are two kinds of rank three fluxes, $U(1)^3$ and $SU(2)\times U(1)^2$, both embedded in the E_6 gauge group with commutants including the Standard Model (SM) gauge structure. We are particularly interested in the second case containing a non-abelian $SU(2)$ gauge flux. In this paper, we shall study the physics of the E_6 GUT model [48] broken by the $SU(2)\times U(1)^2$ fluxes. E_6 GUT models with $U(1)_{PQ}$ symmetry in local F-theory has been explored in [49]. The detailed study of the non-abelian fluxes and the corresponding vector bundles will be presented elsewhere [50].

There are many breaking routes from E_6 to a subgroup containing the SM gauge group, such as via $SO(10)$ and then $SU(5)$, via $SU(6)$, via Pati-Salam, or via trinification. Basically, these breaking routes end up with two resulting gauge groups, $G_1 : SU(3)\times SU(2)_L\times U(1)^3$ and $G_2 : SU(3)\times SU(2)_L\times SU(2)\times U(1)^2$. These two subgroups are referred to as extended MSSM models of rank 6. By suitable rotation of the $U(1)$ gauge groups and the third component of the $SU(2)$ gauge group, we can show that these two subgroups are equivalent. It was found that the extended MSSM models can be obtained from an E_6 unification by an $SU(2)\times U(1)^2$ or $U(1)^3$

flux[1] in the heterotic string models [52]. In the literature the gauge group obtained by breaking E_6 can be rank 5 or rank 6 depending on the flux turned on [52–60]. When a non-abelian flux $SU(2) \times U(1)^2$ is turned on, E_6 is broken directly to a rank 5 model with a gauge group $SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_\eta$ after rearranging the $U(1)$s. Normally rank 6 models have more degrees of freedom with which to solve the problems in phenomenology. However, the $U(1)$ gauge groups induce additional gauge bosons and increase exotic fields. By giving a large VEV to one of the $U(1)$ gauge groups, the rank 6 models can be further reduced to the so-called effective rank 5 models. By arranging the matter assignments, one can build many interesting low energy models, such as $SU(3) \times SU(2) \times U(1)_Y \times U(1)_N$. In the rank 6 model, $U(1)_N$ is inherited from the third $U(1)$ gaining a VEV, whereas in the rank 5 model, $U(1)_\eta$ is fixed and does not possess additional symmetries.

On the other hand, one of the motivations to consider models with an additional gauge group $U(1)'$ as a gauge extension of the Standard Model (NMSSM) is for solving the μ-problem. The minimum matter content for such a model with gauge group $SU(3) \times SU(2) \times U(1)_Y \times U(1)'$ includes the MSSM fermions, two Higgs doublets H and $\bar{H}$, an SM singlet S with a non-zero $U(1)'$ charge, and exotic color triplets. The effective scale of μ-term can arise from the coupling $SH\bar{H}$ when the singlet S acquires a VEV. The radiative breaking of the $U(1)'$ gauge symmetry is usually achieved by the large Yukawa couplings between the singlet S and the exotic fields. This model can be naturally embedded in a model with the E_6 gauge group while the fields mentioned above are included in the three families of $\mathbf{27}$-plets. For the desire of gauge unification without introducing anomalies, a pair of Higgs-like doublets from one or more additional $(\mathbf{27} + \overline{\mathbf{27}})$ is also needed. Recently, the minimum MSSM from the E_6 GUT has been studied, for example, in [61–65], and phenomenology such as the neutrino physics [66], leptogenesis [67], and baryogenesis [68] were also discussed.

In this paper we construct E_6 GUT models in F-theory by using the spectral cover construction and study their breaking down to the rank 5 extended MSSM by turning on the non-abelian fluxes. We only consider the case that the Higgs multiplets are located on a different $\mathbf{27}$ due to the reasons of desiring for more degrees of freedom as well as the singularity structure of Yukawa coupling in F-theory. We represent a few examples corresponding to two spectral cover factorizations. In the example of $(2, 1)$

[1] For breaking scenarios via discrete Wilson lines in the context of orbifold constructions, please see [51] and references therein.

factorization in dP_7, all the fermions are located on one **27** curve and the introduction of fluxes for gauge breaking results in extra copies of quarks and leptons which are exotic to the conventional three-generation E_6 models. We find a better model in the $(1,1,1)$ factorization where the fermions are from two different **27** curves and there is only a pair of vector-like triplet exotic field. Both examples in dP_7 contain exotic fields on the Higgs **27** curve, and we assume they obtain zero vacuum expectation values.

The organization of the rest of the paper is as follows: in section 2, we give a brief review of the $SU(3)$ spectral cover and its factorizations. In section 3, we discuss the subgroups of E_6 and introduce non-abelian $SU(2)\times U(1)^2$ fluxes. Tadpole cancellation conditions for the model building are discussed in section 4. We demonstrate several numerical results of rank 5 models in section 5, and then conclude in the last section.

2 Spectral Cover

In this section we briefly review the construction of an $SU(3)$ spectral cover inducing an $SU(3)$ Higgs bundle breaking the gauge group E_8 down to E_6. We also construct $(2,1)$ and $(1,1,1)$ factorizations of the cover as well as universal fluxes for semi-local model building.

2.1 $SU(3)$ Spectral Cover

Let X_4 be an elliptically fibered Calabi-Yau fourfold $\pi_{X_4} : X_4 \to B_3$ with a section $\sigma_{B_3} : B_3 \to X_4$ and S be one component of the discriminant locus of X_4 with a projection $\widetilde{\pi} : X_4 \to S$, where X_4 develops an E_6 singularity[2]. To describe X_4, let us consider the following Tate model [5]:

$$y^2 = x^3 + \mathbf{b}_3 y z^2 + \mathbf{b}_2 x z^3 + \mathbf{b}_0 z^5, \tag{2.1}$$

where x, y are the coordinates of the fibration and z is the coordinate of the normal direction of S in B_3. Note that the coefficients $\mathbf{b}_k$ generically depend on the coordinate

[2]From now on, S will be assumed to be a del Pezzo surface unless otherwise stated [69, 70].

z and that Eq. (2.1) can be regarded as unfolding of an E_8 singularity[3] into an E_6 singularity. For convenience, we define the shorthand notations $c_1(S) \equiv c_1$, $t \equiv -c_1(N_{S/B_3})$, and $\eta \equiv 6c_1 - t$ where c_1 is the first Chern class and N_{S/B_3} is the normal bundle of S in B_3. To maintain the Calabi-Yau condition $c_1(X_4) = 0$, it is required that x and y in Eq. (2.1) are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$, respectively. It follows that the homological classes $[\mathbf{b}_k]$ are $\eta - kc_1$. Note that the fiber $\widetilde{\pi}^{-1}(b)$ for $b \in S$ is an ALE space [71–76]. The singularity of the fiber over S is determined by the volumes λ_k of (-2) 2-cycles of the ALE space. So unfolding a singularity corresponds to setting the volumes of some of these 2-cycles finite. In the Tate model Eq. (2.1), the fibration singularity is determined by the coefficients $\mathbf{b}_k$. Indeed the coefficients $\mathbf{b}_k$ encode the information of the volumes λ_k. In what follows, we shall introduce the spectral cover construction making the relation between the coefficients $\mathbf{b}_k$ in Eq. (2.1) and the volumes λ_k of (-2) 2-cycles manifest[4]. Before introducing the spectral cover, we would like to briefly review the BPS equations arising from the compactification of the eight-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory on S. The details could be found in [6, 8, 16].

Let us consider the eight-dimensional $\mathcal{N} = 1$ gauge theory compactified on S. To obtain unbroken $\mathcal{N} = 1$ supersymmetry in four dimensions, it was shown that the bosonic fields, a gauge connection A and an adjoint Higgs field Φ, have to satisfy the following BPS equations:

$$\begin{cases} F_A \wedge \omega_S + \frac{i}{2}[\Phi^\dagger, \Phi] = 0 \\ F_A^{2,0} = F_A^{0,2} = 0 \\ \bar{\partial}_A \Phi = 0, \end{cases} \tag{2.2}$$

where F is the curvature two-form of A and ω_S is a Kähler form of S. To solve BPS equations, one may take V as a holomorphic vector bundle over S with the connection A and Φ being holomorphic. The simplest solution for (A, Φ) is that Φ is diagonal and V is a stable bundle. In particular, let us consider a 3×3 case as follows:

$$\Phi = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \sum_{k=1}^{3} \lambda_k = 0, \tag{2.3}$$

where λ_k is holomorphic for $k = 1, 2, 3$. In this case $[\Phi^\dagger, \Phi] = 0$ and Eq. (2.2) is then reduced to the Hermitian Yang-Mills equations

$$F_A^{2,0} = F_A^{0,2} = 0, \qquad F_A \wedge \omega_S = 0. \tag{2.4}$$

[3]If $\mathbf{b}_3 = \mathbf{b}_2 = 0$, the elliptic fibration $y^2 = x^3 + \mathbf{b}_0 z^5$ possess an E_8 singularity at z=0.

[4]For more details, please see [10] and references therein.

The low energy spectrum is therefore decoupled to Φ and only depends on the Hermitian Yang-Mills connection A. The eigenvalues λ_k characterize the locations of intersecting seven-branes. Alternatively, the information of intersecting seven-branes can be encoded in the characteristic polynomial $P_\Phi(s) = \det(sI - \Phi)$ associated with a spectral cover over S. For generically diagonal Φ, the polynomial equation $P_\Phi(s) = 0$ has distinct roots and the associated spectral cover is smooth. However, it is not the case when Φ is upper triangular in the following form

$$\Phi = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

In this case $P_\Phi(s) = 0$ is singular and the spectrum is coupled to Φ due to $[\Phi^\dagger, \Phi] \neq 0$. Moreover, the polynomial $P_\Phi(s)$ may not capture the entire information of the system any more. In particular, one has to specify not only the spectral polynomial $P_\Phi(s)$ but also the Higgs field Φ to calculate the spectrum. Such configurations of seven-branes characterized by upper triangular Φ are called T-branes. For the detailed analysis of T-branes, we refer readers to [77]. In what follows, we shall focus on the case of Eq. (2.3) and its associated spectral cover. Notice that the polynomial equation

$$b_0 \det(sI - \Phi) = b_0 s^3 + b_2 s + b_3 = 0 \tag{2.6}$$

defines a three-sheeted cover of S inside the total space of the canonical bundle $K_S \to S$, a local Calabi-Yau threefold, where $b_k \equiv \mathbf{b}_k|_{z=0},\ k = 0, 2, 3$. However, this threefold is non-compact. For well-defined intersection numbers, one can compactify the non-compact threefold to the total space of projective bundle $\mathbb{P}(\mathcal{O}_S \oplus K_S)$ over S. Let us define X the total space of the projective bundle with two sections U, V and with a projection map $\pi : X \to S$. The homological classes of zero sections $\{U = 0\}$ and $\{V = 0\}$ are σ and $\sigma + c_1$, respectively. In compact threefold X, the spectral cover Eq. (2.6) can be expressed as a homogeneous polynomial as follows:

$$\mathcal{C}^{(3)} :\ b_0 U^3 + b_2 U V^2 + b_3 V^3 \equiv b_0 \prod_{k=1}^{3} (U + \lambda_k V) = 0, \tag{2.7}$$

with a projection map $p_3 : \mathcal{C}^{(3)} \to S$. The homological class of $\mathcal{C}^{(3)}$ is given by $[\mathcal{C}^{(3)}] = 3\sigma + \pi^*\eta$. The singularities get enhanced at some loci of S. Let us consider the following breaking pattern

$$\begin{array}{ccl} E_8 & \to & E_6 \times SU(3) \\ \mathbf{248} & \to & (\mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{8}) + (\mathbf{27}, \mathbf{3}) + (\overline{\mathbf{27}}, \overline{\mathbf{3}}). \end{array} \tag{2.8}$$

The matter **27** is localized on the curve $\Sigma_{\mathbf{27}}$ given by the locus of $\{b_3 = 0\}$ where the singularity E_6 is enhanced to E_7, so it implies the homological class of $[\Sigma_{\mathbf{27}}]$ is $\eta - 3c_1$ in S. Alternatively, it follows from $\lambda_i = 0$ in Eq. (2.7) that the homological class of $[\Sigma_{\mathbf{27}}]$ can be also computed by $[\mathcal{C}^{(3)}] \cdot \sigma|_\sigma = \eta - 3c_1$. With a spectral cover $\mathcal{C}^{(3)}$, one can obtain a Higgs bundle $p_{3*}\mathcal{L}$ on S by the pushforward of a line bundle $\mathcal{L}$ on $\mathcal{C}^{(3)}$. To maintain the traceless condition $c_1(p_{3*}\mathcal{L}) = 0$, it is required that $p_{3*}\gamma^{(3)} = 0$ where $c_1(\mathcal{L}) \equiv \gamma^{(3)} + \frac{1}{2}r^{(3)} \in H_4(X, \mathbb{Z})$ and $r^{(3)}$ is the ramification divisor of the projection map $p_3 : \mathcal{C}^{(3)} \to S$. Up to a constant, the unique solution of the traceless condition $p_{3*}\gamma^{(3)} = 0$ is $\gamma^{(3)} = (3 - p_3^* p_{3*})[\mathcal{C}^{(3)}] \cdot \sigma$, and one can calculate the chiral spectrum by turning on the traceless flux $\gamma^{(3)}$. More precisely, the net chirality $N_{\mathbf{27}}$ of the matter field **27** can be computed as

$$N_{\mathbf{27}} = \gamma^{(3)} \cdot \Sigma_{\mathbf{27}} = -\eta \cdot_S (\eta - 3c_1). \tag{2.9}$$

To obtain three generations for **27**, it is required that $(6c_1 - t) \cdot_S (3c_1 - t) = -3$ which is a non-trivial constraint on embedding of S into the Calabi-Yau fourfold X_4. On the other hand, the irreducible cover $\mathcal{C}^{(3)}$ only provides a single matter curve, so we need more matter curves and more degrees of freedom on the cover flux to promise realistic models. Therefore we shall study the factorizations of the spectral cover $\mathcal{C}^{(3)}$ in what follows.

2.2 (2,1) Factorization

Let us consider the factorization $\mathcal{C}^{(3)} \to \mathcal{C}^{(a)} \times \mathcal{C}^{(b)}$:

$$b_0U^3 + b_2UV^2 + b_3V^3 = (a_0U^2 + a_1UV + a_2V^2)(d_0U + d_1V) \tag{2.10}$$

with projection maps $p_a : \mathcal{C}^{(a)} \to S$ and $p_b : \mathcal{C}^{(b)} \to S$, respectively. Let $[d_1] \equiv \xi$, one can write the homological class of remaining sections as

$$[a_n] = \eta - (n+1)c_1 - \xi, \; n = 0, 1, 2, \qquad [d_0] = c_1 + \xi. \tag{2.11}$$

It follows from Eqs. (2.10) and (2.11) that the homological classes of the covers $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(b)}$ are given by

$$[\mathcal{C}^{(a)}] = 2\sigma + \pi^*(\eta - \xi - c_1), \qquad [\mathcal{C}^{(b)}] = \sigma + \pi^*(\xi + c_1). \tag{2.12}$$

With the homological classes $[\mathcal{C}^{(a)}]$ and $[\mathcal{C}^{(b)}]$, one can compute the homological classes of matter curves $\Sigma_{\mathbf{27}}^{(a)}$ and $\Sigma_{\mathbf{27}}^{(b)}$ as

$$[\Sigma_{\mathbf{27}}^{(a)}] = [\mathcal{C}^{(a)}] \cdot \sigma|_\sigma = \eta - 3c_1 - \xi, \quad [\Sigma_{\mathbf{27}}^{(b)}] = [\mathcal{C}^{(b)}] \cdot \sigma|_\sigma = \xi. \tag{2.13}$$

The ramification divisors of the maps $p_a : \mathcal{C}^{(a)} \to S$ and $p_b : \mathcal{C}^{(b)} \to S$ are given by

$$r^{(a)} = [\mathcal{C}^{(a)}] \cdot \pi^*(\eta - 2c_1 - \xi), \quad r^{(b)} = [\mathcal{C}^{(b)}] \cdot (-\sigma + \pi^*\xi). \tag{2.14}$$

The traceless fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ is defined as $(2-p_a^* p_{a*})[\mathcal{C}^{(a)}]\cdot\sigma$ and $(1-p_b^* p_{b*})[\mathcal{C}^{(b)}]\cdot\sigma$, respectively, where $p_{a*}\gamma_0^{(a)} = 0$ and $p_{b*}\gamma_0^{(b)} = 0$. The explicit forms of the traceless fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ are given by

$$\gamma_0^{(a)} = [\mathcal{C}^{(a)}] \cdot (2\sigma - \pi^*(\eta - 3c_1 - \xi)), \quad \gamma_0^{(b)} = [\mathcal{C}^{(b)}] \cdot (\sigma - \pi^*\xi). \tag{2.15}$$

The chirality of matter **27** on each matter curve due to the fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ is then shown in Table 1.

	$\gamma_0^{(a)}$	$\gamma_0^{(b)}$
$\mathbf{27}^{(a)}$	$-(\eta - c_1 - \xi) \cdot_S (\eta - 3c_1 - \xi)$	0
$\mathbf{27}^{(b)}$	0	$-\xi \cdot_S (c_1 + \xi)$

Table 1: Chirality induced by the fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$.

Due to the factorization, one can introduce the additional fluxes $\delta^{(a)} = (1 - p_b^* p_{a*})[\mathcal{C}^{(a)}] \cdot \sigma$ and $\delta^{(b)} = (2 - p_a^* p_{b*})[\mathcal{C}^{(b)}] \cdot \sigma$. It is not difficult to obtain [20]:

$$\delta^{(a)} = [\mathcal{C}^{(a)}] \cdot \sigma - [\mathcal{C}^{(b)}] \cdot \pi^*(\eta - 3c_1 - \xi), \quad \delta^{(b)} = [\mathcal{C}^{(b)}] \cdot 2\sigma - [\mathcal{C}^{(a)}] \cdot \pi^*\xi. \tag{2.16}$$

Also for any $\rho \in H_2(S, \mathbb{R})$, one can define a non-trivial flux $\tilde{\rho}$ as

$$\tilde{\rho} = (2p_b^* - p_a^*)\rho, \tag{2.17}$$

then the chirality induced by these additional fluxes on each matter curve is summarized in Table 2.

The total flux Γ is then a linear combination of the fluxes above:

$$\Gamma = k_a\gamma_0^{(a)} + k_b\gamma_0^{(b)} + m_a\delta^{(a)} + m_b\delta^{(b)} + \tilde{\rho} \equiv \Gamma^{(a)} + \Gamma^{(b)}, \tag{2.18}$$

	$\delta^{(a)}$	$\delta^{(b)}$	$\tilde{\rho}$
$\mathbf{27}^{(a)}$	$-c_1 \cdot_S (\eta - 3c_1 - \xi)$	$-\xi \cdot_S (\eta - 3c_1 - \xi)$	$-\rho \cdot_S (\eta - 3c_1 - \xi)$
$\mathbf{27}^{(b)}$	$-\xi \cdot_S (\eta - 3c_1 - \xi)$	$-2c_1 \cdot_S \xi$	$2\rho \cdot_S \xi$

Table 2: Chirality induced by the fluxes $\delta^{(a)}$, $\delta^{(b)}$, and $\tilde{\rho}$.

where

$$\Gamma^{(a)} \equiv [\mathcal{C}^{(a)}] \cdot [\tilde{\mathcal{C}}^{(a)}] = [\mathcal{C}^{(a)}] \cdot [(2k_a + m_a)\sigma - \pi^*(k_a(\eta - 3c_1 - \xi) + m_b\xi + \rho)], \quad (2.19)$$
$$\Gamma^{(b)} \equiv [\mathcal{C}^{(b)}] \cdot [\tilde{\mathcal{C}}^{(b)}] = [\mathcal{C}^{(b)}] \cdot [(k_b + 2m_b)\sigma - \pi^*(k_b\xi + m_a(\eta - 3c_1 - \xi) - 2\rho)]. \quad (2.20)$$

The parameters k_a, k_b, m_a, m_b will be determined later by the physical and consistency conditions. In addition, by

$$p_{a*}\Gamma^{(a)} = m_a(\eta - 3c_1 - \xi) - 2m_b\xi - 2\rho, \quad (2.21)$$
$$p_{b*}\Gamma^{(b)} = -m_a(\eta - 3c_1 - \xi) + 2m_b\xi + 2\rho, \quad (2.22)$$

we find that $\Gamma^{(a)}$ and $\Gamma^{(a)}$ indeed satisfy the traceless condition $p_{a*}\Gamma^{(a)} + p_{b*}\Gamma^{(b)} = 0$. In the $(2,1)$ factorization, the quantization conditions are then given by

$$(2k_a + m_a)\sigma - \pi^*(k_a(\eta - 3c_1 - \xi) + m_b\xi + \rho - \frac{1}{2}(\eta - 2c_1 - \xi)) \in H_4(X, \mathbb{Z}), \quad (2.23)$$
$$(k_b + 2m_b - \frac{1}{2})\sigma - \pi^*(k_b\xi + m_a(\eta - 3c_1 - \xi) - 2\rho - \frac{1}{2}\xi) \in H_4(X, \mathbb{Z}). \quad (2.24)$$

In addition, the supersymmetry condition is

$$[m_a(\eta - 3c_1 - \xi) - 2m_b\xi - 2\rho] \cdot_S [\omega] = 0, \quad (2.25)$$

where $[\omega]$ is an ample divisor dual to a Kähler form of S.

2.3 (1,1,1) Factorization

Let us consider the factorization $\mathcal{C}^{(3)} \to \mathcal{C}^{(l_1)} \times \mathcal{C}^{(l_2)} \times \mathcal{C}^{(l_3)}$:

$$b_0U^3 + b_2UV^2 + b_3V^3 = (f_0U + f_1V)(g_0U + g_1V)(h_0U + h_1V), \quad (2.26)$$

with the projection maps $p_{l_1} : \mathcal{C}^{(l_1)} \to S$, $p_{l_2} : \mathcal{C}^{(l_2)} \to S$, and $p_{l_3} : \mathcal{C}^{(l_3)} \to S$. Let $[g_1] \equiv \xi_1$ and $[h_1] \equiv \xi_2$, the homological classes of the remaining sections are

$$[f_m] = \eta - (m + 2)c_1 - \xi_1 - \xi_2,\ m = 0, 1. \quad [g_0] = c_1 + \xi_1,\ [h_0] = c_1 + \xi_2. \quad (2.27)$$

It follows from Eqs. (2.26) and (2.27) that the homological classes of the covers $\mathcal{C}^{(l_1)}$, $\mathcal{C}^{(l_2)}$, and $\mathcal{C}^{(l_3)}$ are given by

$$[\mathcal{C}^{(l_1)}] = \sigma + \pi^*(\eta - 2c_1 - \xi_1 - \xi_2), \quad [\mathcal{C}^{(l_2)}] = \sigma + \pi^*(\xi_1 + c_1), \quad [\mathcal{C}^{(l_3)}] = \sigma + \pi^*(\xi_2 + c_1). \tag{2.28}$$

The homological classes of the matter curves can be obtained from the intersection $[\mathcal{C}^{(l_i)}] \cdot \sigma|_\sigma$:

$$[\Sigma_{\mathbf{27}}^{(l_1)}] = \eta - 3c_1 - \xi_1 - \xi_2, \quad [\Sigma_{\mathbf{27}}^{(l_2)}] = \xi_1, \quad [\Sigma_{\mathbf{27}}^{(l_3)}] = \xi_2. \tag{2.29}$$

In the $(1,1,1)$ factorization, the ramification divisors are given by

$$r_{l_1} = [\mathcal{C}^{(l_1)}]\cdot[-\sigma+\pi^*(\eta-3c_1-\xi_1-\xi_2)], \;\; r_{l_2} = [\mathcal{C}^{(l_2)}]\cdot(-\sigma+\pi^*\xi_1), \;\; r_{l_3} = [\mathcal{C}^{(l_3)}]\cdot(-\sigma+\pi^*\xi_2). \tag{2.30}$$

For general fluxes $\gamma^{(i)} = [\mathcal{C}^{(i)}] \cdot \sigma$, we define the traceless fluxes $\gamma_0^{(i)}$ as

$$\gamma_0^{(l_1)} = (1 - p_{l_1}^* p_{l_1 *})\gamma^{(l_1)} = [\mathcal{C}^{(l_1)}] \cdot [\sigma - \pi^*(\eta - 3c_1 - \xi_1 - \xi_2)], \tag{2.31}$$

$$\gamma_0^{(l_2)} = (1 - p_{l_2}^* p_{l_2 *})\gamma^{(l_2)} = [\mathcal{C}^{(l_2)}] \cdot (\sigma - \pi^*\xi_1), \tag{2.32}$$

$$\gamma_0^{(l_3)} = (1 - p_{l_3}^* p_{l_3 *})\gamma^{(l_3)} = [\mathcal{C}^{(l_3)}] \cdot (\sigma - \pi^*\xi_2). \tag{2.33}$$

It is easy to see that $\gamma_0^{(i)}$ satisfies the condition $p_{i*}\gamma_0^{(i)} = 0$ for all i. The chirality induced by the fluxes $\gamma_0^{(l_1)}$, $\gamma_0^{(l_2)}$, and $\gamma_0^{(l_3)}$ is summarized in Table 3.

	$\gamma_0^{(l_1)}$	$\gamma_0^{(l_2)}$	$\gamma_0^{(l_3)}$
$\mathbf{27}^{(l_1)}$	$-(\eta - 2c_1 - \xi_1 - \xi_2) \cdot_S (\eta - 3c_1 - \xi_1 - \xi_2)$	0	0
$\mathbf{27}^{(l_2)}$	0	$-\xi_1 \cdot_S (c_1 + \xi_1)$	0
$\mathbf{27}^{(l_3)}$	0	0	$-\xi_2 \cdot_S (c_1 + \xi_2)$

Table 3: Chirality induced by the fluxes $\gamma_0^{(l_1)}$, $\gamma_0^{(l_2)}$, and $\gamma_0^{(l_3)}$.

There are many choices of the additional fluxes, for simplicity, we consider

$$\begin{aligned}\delta^{(l_1)} &= [(1 - p_{l_2}^* p_{l_1 *}) + (1 - p_{l_3}^* p_{l_1 *})]\gamma^{(l_1)} \\ &= [\mathcal{C}^{(l_1)}] \cdot 2\sigma - ([\mathcal{C}^{(l_2)}] + [\mathcal{C}^{(l_3)}]) \cdot \pi^*(\eta - 3c_1 - \xi_1 - \xi_2),\end{aligned} \tag{2.34}$$

$$\begin{aligned}\delta^{(l_2)} &= [(1 - p_{l_1}^* p_{l_2 *}) + (1 - p_{l_3}^* p_{l_2 *})]\gamma^{(l_2)} \\ &= [\mathcal{C}^{(l_2)}] \cdot 2\sigma - [\mathcal{C}^{(l_1)}] \cdot \pi^*\xi_1 - [\mathcal{C}^{(l_3)}] \cdot \pi^*\xi_1,\end{aligned} \tag{2.35}$$

$$\begin{aligned}\delta^{(l_3)} &= [(1 - p_{l_1}^* p_{l_3 *}) + (1 - p_{l_2}^* p_{l_3 *})]\gamma^{(l_3)} \\ &= [\mathcal{C}^{(l_3)}] \cdot 2\sigma - [\mathcal{C}^{(l_1)}] \cdot \pi^*\xi_2 - [\mathcal{C}^{(l_2)}] \cdot \pi^*\xi_2.\end{aligned} \tag{2.36}$$

$$\widehat{\rho} = (p_{l_2}^* - p_{l_1}^*)\rho_1 + (p_{l_3}^* - p_{l_2}^*)\rho_2 + (p_{l_1}^* - p_{l_3}^*)\rho_3, \tag{2.37}$$

where $\rho_i \in H_2(S,\mathbb{R}),\ \forall i$. The chirality induced by these additional fluxes on each matter curve is summarized in Table 4.

	$\delta^{(l_1)}$	$\delta^{(l_2)}$	$\delta^{(l_3)}$	$\widehat{\rho}$
$\mathbf{27}^{(l_1)}$	$-2c_1 \cdot_S [f_1]$	$-\xi_1 \cdot_S [f_1]$	$-\xi_2 \cdot_S [f_1]$	$(\rho_3 - \rho_1) \cdot_S [f_1]$
$\mathbf{27}^{(l_2)}$	$-\xi_1 \cdot_S [f_1]$	$-2c_1 \cdot_S \xi_1$	$-\xi_1 \cdot_S \xi_2$	$(\rho_1 - \rho_2) \cdot_S \xi_1$
$\mathbf{27}^{(l_3)}$	$-\xi_2 \cdot_S [f_1]$	$-\xi_1 \cdot_S \xi_2$	$-2c_1 \cdot_S \xi_2$	$(\rho_2 - \rho_3) \cdot_S \xi_2$

Table 4: Chirality induced by the fluxes $\delta^{(l_1)}$, $\delta^{(l_2)}$, $\delta^{(l_3)}$ and $\widehat{\rho}$.

The total flux Γ with the parameters k_{l_1}, k_{l_2} k_{l_3}, m_{l_1}, m_{l_2}, and m_{l_3} is [20]

$$\Gamma = k_{l_1}\gamma_0^{(l_1)} + k_{l_2}\gamma_0^{(l_2)} + k_{l_3}\gamma_0^{(l_3)} + m_{l_1}\delta^{(l_1)} + m_{l_2}\delta^{(l_2)} + m_{l_3}\delta^{(l_3)} + \widehat{\rho} \equiv \Gamma^{(l_1)} + \Gamma^{(l_2)} + \Gamma^{(l_3)}, \tag{2.38}$$

where

$$\Gamma^{(l_1)} \equiv [\mathcal{C}^{(l_1)}] \cdot [\tilde{\mathcal{C}}^{(l_1)}] = [\mathcal{C}^{(l_1)}] \cdot [(k_{l_1} + 2m_{l_1})\sigma - \pi^*(k_{l_1}[f_1] + m_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_1 - \rho_3)], \tag{2.39}$$
$$\Gamma^{(l_2)} \equiv [\mathcal{C}^{(l_2)}] \cdot [\tilde{\mathcal{C}}^{(l_2)}] = [\mathcal{C}^{(l_2)}] \cdot [(k_{l_2} + 2m_{l_2})\sigma - \pi^*(m_{l_1}[f_1] + k_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_2 - \rho_1)], \tag{2.40}$$
$$\Gamma^{(l_3)} \equiv [\mathcal{C}^{(l_3)}] \cdot [\tilde{\mathcal{C}}^{(l_3)}] = [\mathcal{C}^{(l_3)}] \cdot [(k_{l_3} + 2m_{l_3})\sigma - \pi^*(m_{l_1}[f_1] + m_{l_2}\xi_1 + k_{l_3}\xi_2 + \rho_3 - \rho_2)]. \tag{2.41}$$

It is then straightforward to compute

$$p_{l_1*}\Gamma^{(l_1)} = 2m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) - m_{l_2}\xi_1 - m_{l_3}\xi_2 - \rho_1 + \rho_3, \tag{2.42}$$
$$p_{l_2*}\Gamma^{(l_2)} = -m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) + 2m_{l_2}\xi_1 - m_{l_3}\xi_2 - \rho_2 + \rho_1, \tag{2.43}$$
$$p_{l_3*}\Gamma^{(l_3)} = -m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) - m_{l_2}\xi_1 + 2m_{l_3}\xi_2 - \rho_3 + \rho_2. \tag{2.44}$$

The sum is zero, as it should be for the traceless condition. In this case, the quantization conditions are given by

$$(k_{l_1} + 2m_{l_1} - \frac{1}{2})\sigma - \pi^*\{(k_{l_1} - \frac{1}{2})[f_1] + m_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_1 - \rho_3\} \in H_4(X,\mathbb{Z}), \tag{2.45}$$
$$(k_{l_2} + 2m_{l_2} - \frac{1}{2})\sigma - \pi^*\{m_{l_1}[f_1] + (k_{l_2} - \frac{1}{2})\xi_1 + m_{l_3}\xi_2 + \rho_2 - \rho_1\} \in H_4(X,\mathbb{Z}), \tag{2.46}$$
$$(k_{l_3} + 2m_{l_3} - \frac{1}{2})\sigma - \pi^*\{m_{l_1}[f_1] + m_{l_2}\xi_1 + (k_{l_3} - \frac{1}{2})\xi_2 + \rho_3 - \rho_2\} \in H_4(X,\mathbb{Z}), \tag{2.47}$$

and the supersymmetry conditions are as follows:

$$[2m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) - m_{l_2}\xi_1 - m_{l_3}\xi_2 - \rho_1 + \rho_3] \cdot_S [\omega] = 0, \tag{2.48}$$
$$[-m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) + 2m_{l_2}\xi_1 - m_{l_3}\xi_2 - \rho_2 + \rho_1] \cdot_S [\omega] = 0, \tag{2.49}$$
$$[-m_{l_1}(\eta - 3c_1 - \xi_1 - \xi_2) - m_{l_2}\xi_1 + 2m_{l_3}\xi_2 - \rho_3 + \rho_2] \cdot_S [\omega] = 0. \tag{2.50}$$

3 Breaking E_6

The MSSM fermion and electroweak Higgs fields can be included in the same **27** multiplet of a three-family E_6 GUT model. On the other hand, it is possible to assign the Higgs fields to a different $\mathbf{27}_H$ multiplet where only the Higgs doublets and singlets obtain the electroweak scale energy. The Yukawa coupling for these two cases can be written as

$$\mathcal{W} \supset \mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27} \text{ (Case A)} \quad \text{or} \quad \mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27}_H \text{ (Case B)}. \tag{3.1}$$

The Yukawa coupling of Case A is either a triple-intersection of one **27** curve or an intersection of three different curves in F-theory model building. It is difficult to obtain a three family model from a single curve and the geometry of a triple-intersection is generally complicated. On other hand, it is not easy to achieve the mass hierarchy of the third generation in the three-curve model. Therefor we do not consider Case A in this paper. In case B, there are two possible constructions from spectral cover factorizations. In the $(2,1)$ factorization, the fermions are assigned to $\mathbf{27}^{(a)}$ curve and the Higgs fields come from the other $\mathbf{27}^{(b)}$ curve. The Yukawa coupling then turns out

$$\mathcal{W}^{(2,1)} \supset \mathbf{27}^{(a)} \cdot \mathbf{27}^{(a)} \cdot \mathbf{27}^{(b)}. \tag{3.2}$$

In the $(1,1,1)$ factorization, the matter fields are assigned to curve $\mathbf{27}^{(a)}$ and $\mathbf{27}^{(b)}$ while the Higgs fields come from the $\mathbf{27}^{(c)}$ curve. In this case the Yukawa coupling is then

$$\mathcal{W}^{(1,1,1)} \supset \mathbf{27}^{(a)} \cdot \mathbf{27}^{(b)} \cdot \mathbf{27}^{(c)}. \tag{3.3}$$

In order to realize the MSSM in the E_6 GUT models, it is useful to study the subgroups of E_6. In our F-theory model building we consider the picture that the E_6 gauge group is broken by the $SU(2) \times U(1)^2$ flux on the seven-branes. This flux may tilt the chirality of the matter on the curve after E_6 is broken.

3.1 Subgroups of E_6

The subgroups of E_6 including the Standard Model gauge group can be denoted $E_6 \supset SU(3) \times SU(2)_L \times G_c$. Here G_c marks a rank 3 group which is a product of $U(1)$ or $SU(2)$. It has been shown (for example, [48,54,60,78]) that by suitable assignments of the hypercharge of the SM and the $B-L$ symmetry, these E_6 subgroups with different G_c are equivalent to different matter content arrangements. This property would be useful for the analysis of the non-abelian fluxes of type G_c. In this section we will briefly review the subgroups of E_6.

Let us consider the following breaking patterns of E_6:

$$(1a)\quad E_6 \to SO(10)\times U(1) \to SU(5)\times U(1)^2, \tag{3.4}$$

$$(1b)\quad E_6 \to SO(10)\times U(1) \to SU(4)\times SU(2)\times SU(2)\times U(1), \tag{3.5}$$

$$(2a)\quad E_6 \to SU(6)\times SU(2) \to SU(5)\times U(1)\times SU(2), \tag{3.6}$$

$$(2b)\quad E_6 \to SU(6)\times SU(2) \to SU(4)\times SU(2)\times U(1)\times SU(2), \tag{3.7}$$

$$(2c)\quad E_6 \to SU(6)\times SU(2) \to SU(3)\times SU(3)\times U(1)\times SU(2), \tag{3.8}$$

$$(3)\quad E_6 \to SU(3)\times SU(3)\times SU(3). \tag{3.9}$$

In all of these cases, there are two possible outcomes when E_6 is broken down to the subgroups containing the Standard Model group. Case $(1a)$ turns out to be

$$E_6 \to SU(3)\times SU(2)_L\times U(1)_Y\times U(1)_\chi\times U(1)_\psi, \tag{3.10}$$

and the other cases become

$$E_6 \to SU(3)\times SU(2)\times SU(2)\times U(1)_U\times U(1)_W. \tag{3.11}$$

Note that the assignments of $U(1)_U$ and $U(1)_W$ groups of the cases $(1b)$, $(2a)$, $(2b)$, $(2c)$ and (3) are different, but they are equivalent up to linear transformations and the details can be found in the appendix. Take case (3) as an example, the breaking is through a trinification model, therefore we can write

$$E_6 \supset SU(3)\times SU(2)_L\times SU(2)_{(R)}\times U(1)_{Y_L}\times U(1)_{Y_{(R)}}. \tag{3.12}$$

The parenthesis on R in $SU(2)_{(R)}$ indicates that it has three different assignments denoted by $SU(2)_R$, $SU(2)_{R'}$, and $SU(2)_E$ [78]. The third component $I_{3(R)}$ of $SU(2)_{(R)}$ along with the quantum numbers of $U(1)_{Y_L}$ and $U(1)_{Y_{(R)}}$ can have a linear relation to the quantum numbers of $U(1)_Y$, $U(1)_\chi$ and $U(1)_\psi$ of case $(1a)$ in (3.10), i.e.,

$$Y = a_1Y_L + a_2Y_{(R)} + a_3I_{3(R)},\ \chi = b_1Y_L + b_2Y_{(R)} + b_3I_{3(R)},\ \psi = c_1Y_L + c_2Y_{(R)} + c_3I_{3(R)}, \tag{3.13}$$

where a_i, b_i and c_i are coefficients of the transformation. These three different kinds of $SU(2)_{(R)}$ assignments also confine the three different embedding of SM matter representations into the $SU(5)$ multiplets belonging to **27** of E_6, as well as the corresponding assignments of the hypercharge. The three assignments of $U(1)_Y$ should be orthogonal to the three $SU(2)_{(R)}$, respectively.

The $U(1)_{B-L}$ symmetry is conserved in SUSY E_6 models, which is not difficult to see from the gauge breaking via the Pati-Salam gauge group. $U(1)_{B-L}$ has a linear relation with $U(1)_{Y_L}$, $U(1)_{Y_{(R)}}$, and the third component of $SU(2)_{(R)}$. There are also three $U(1)_{B-L}$ assignments orthogonal to the three $SU(2)_{(R)}$, respectively. For consistency with the SM structure, $U(1)_{B-L}$ and $U(1)_Y$ are not orthogonal to the same $SU(2)_{(R)}$. Therefore, there are totally six different charge assignments of the SM multiplets, in other words six different embedding of SM multiplets in **27** of E_6. For the detailed analysis, we refer readers to [78].

The E_6 subgroups listed in Eqs. (3.10) and (3.11) are rank 6. In heterotic string compactifications, E_6 can be broken by a non-abelian flux down to a rank 5 subgroup [52–55]:

$$E_6 \to SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_\eta. \tag{3.14}$$

This model is usually marked as the η-model. Rank 6 models [56, 58, 59] have more symmetries, but it is common practice to give a large VEV to one $U(1)$ gauge group to reduce them to the so called *effective* rank 5 models. For instance, from Eq. (3.10) the remaining abelian gauge group $U(1)_\theta$ is a reduction

$$U(1)_\theta = \cos\theta U(1)_\chi + \sin\theta U(1)_\psi. \tag{3.15}$$

Particularly, the rank 5 η-model can be regarded as a special case of this setup by

$$U(1)_\eta = \sqrt{\frac{3}{8}}U(1)_\chi - \sqrt{\frac{5}{8}}U(1)_\psi. \tag{3.16}$$

In our F-theory models, a non-abelian flux $SU(2) \times U(1)^2$ is turned on to break the E_6 gauge group into $SU(3) \times SU(2) \times U(1)^2$ taken to be the η-model. However, since $U(1)_\eta$ is only determined by the two $U(1)$s while the $SU(2)$ is integrated out, the η-model does not possess the degrees of freedom from the mixing angle θ preserving some symmetries such as the $B-L$ symmetry [60]. The corresponding phenomenology of the F-theory rank 5 model will basically follow the properties of the η-model.

The particle content of the E_6 model we will consider is conventional. It includes three copies of **27**-plets, each copy includes an SM ordinary family, two Higgs-type doublets, two SM singlets, and two exotic $SU(2)$-singlet quarks. The **27** matter content of the $SU(3)\times SU(2)\times U(1)_Y\times U(1)_\eta$ model with the corresponding charges are

$$\begin{aligned}\mathbf{27} \ &\rightarrow Q(\mathbf{3},\mathbf{2})_{\frac{1}{3},2}+u^c(\bar{\mathbf{3}},\mathbf{1})_{-\frac{4}{3},2}+e^c(\mathbf{1},\mathbf{1})_{2,2}\\ &+\ L(\mathbf{1},\mathbf{2})_{-1,-1}+d^c(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3},-1}+\nu^c(\mathbf{1},\mathbf{1})_{0,5}\\ &+\ \bar{D}(\mathbf{3},\mathbf{1})_{-\frac{2}{3},-4}+\bar{h}(\mathbf{1},\mathbf{2})_{1,-4}\\ &+\ D(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3},-1}+h(\mathbf{1},\mathbf{2})_{-1,-1}+S(\mathbf{1},\mathbf{1})_{0,5},\end{aligned} \tag{3.17}$$

where the first subscription denotes the $U(1)_Y$ charge and the second indicates the $U(1)_\eta$ charge. The superpotential for the $\mathbf{27}\cdot\mathbf{27}\cdot\mathbf{27}$ coupling can be expanded as

$$\mathcal{W}=\mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2+\mathcal{W}_3+\cdots\,, \tag{3.18}$$

$$\mathcal{W}_0=\lambda_1\bar{h}Qu^c+\lambda_2 hQd^c+\lambda_3 hLe^c+\lambda_4 h\bar{h}S+\lambda_5 D\bar{D}S, \tag{3.19}$$

$$\mathcal{W}_1=\lambda_6\bar{D}u^ce^c+\lambda_7 DQL+\lambda_8\bar{D}\nu^cd^c, \tag{3.20}$$

$$\mathcal{W}_2=\lambda_9\bar{D}QQ+\lambda_{10}Du^cd^c, \tag{3.21}$$

$$\mathcal{W}_3=\lambda_{11}\bar{h}L\nu^c. \tag{3.22}$$

To avoid the terms that may cause serious phenomenological problems, additional symmetries such as discrete symmetry should be considered. The exotic fields are only confined by the charge, isospin, and hypercharge assignments while their baryon and lepton numbers remain unspecified. By assigning baryon and lepton numbers to D, it is possible to forbid some of the interactions in $\mathcal{W}$ by the conservation of baryon and lepton numbers. For example, if the baryon number $B(\bar{D})=\frac{1}{3}$ and the lepton number $L(\bar{D})=1$, $\mathcal{W}_2=0$; if $B(\bar{D})=-\frac{2}{3}$ and $L(\bar{D})=0$, then $\mathcal{W}_1=0$. In the case $B(\bar{D})=\frac{1}{3}$ and $L(\bar{D})=0$, $\bar{D}$ is regarded as a conventional quark able to mix with the d-quarks, and then decay via flavor changing neutral currents (FCNC) or charged currents (CC) [60]. By setting $B(h,\bar{h})=L(h,\bar{h})=0$ and $B(S)=L(S)=0$, h and $\bar{h}$ are the usual MSSM Higgs doublets, and the VEV of S provides a mass for D. See [60] for a detailed review.

Another possibility is considering the MSSM Higgs fields coming from a different $\mathbf{27}_H$ (or $\overline{\mathbf{27}}_H$). In this case the exotics of the matter **27**-plet are taken as the ordinary quarks and leptons, $B(\bar{D})=\frac{1}{3}$ and $L(\bar{D})=0$, as well as $B(h,\bar{h},\nu^c,S)=0$ and $L(h,\bar{h},\nu^c,S)=\pm1$. The doublets $H_1(\mathbf{1},\mathbf{2})_{-1,-1}$, $H_2(\mathbf{1},\mathbf{2})_{-1,-1}$ and $\bar{H}_2(\mathbf{1},\mathbf{2})_{1,-4}$, and the singlets $H_3(\mathbf{1},\mathbf{1})_{0,5}$ and $H_4(\mathbf{1},\mathbf{1})_{0,5}$ of $\mathbf{27}_H$ develop VEVs so that the superpotential takes the form

$$\begin{aligned}\mathcal{W}'\ &\supset\ \bar{H}_2Qu^c+H_2Qd^c+H_2Le^c+H_1he^c+\bar{h}hH_4\\ &+\ \bar{H}_2hS+H_2\bar{h}S+\bar{D}DH_4+H_1QD+H_3\bar{D}d^c+\bar{H}_2L\nu^c+\cdots\,.\end{aligned} \tag{3.23}$$

We can see the mixing terms between the ordinary fermions and their corresponding exotic fields. These kinds of mixings allow the exotics to decay via FCNC or CC. For example, the coupling to W of charged currents (CC) for electric charge $Q_e = \frac{2}{3}, -\frac{1}{3}$ sector can be [60]

$$\mathcal{L}^{CC} \sim \frac{g}{\sqrt{2}} (\bar{u}, 0)_L \begin{pmatrix} U_L^u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} U_L^{d\dagger} \gamma_\mu (1-\gamma_5) \begin{pmatrix} d \\ \bar{D} \end{pmatrix} W^\mu + h.c. \,, \tag{3.24}$$

where U_L^u and I are $n \times n$ matrices and U_L^d is a $2n \times 2n$ matrix for n generations. U_L^u and U_L^d are transformations from weak eigenstates to mass eigenstates. On the other hand for $Q_e = 0, -1$ sector, if the two components of the doublet h are $h = (N, E)$, the coupling is [60]

$$\mathcal{L}^{CC} \sim \frac{g}{2\sqrt{2}} (\bar{\nu}, \bar{N})_L \gamma_\mu (V - A\gamma_5) \begin{pmatrix} e \\ E \end{pmatrix} W^\mu + h.c. \,, \tag{3.25}$$

where V and A are $2n \times 2n$ matrices composed of the left and right weak-mass transformations:

$$V = U_L^\nu U_L^{e\dagger} + U_R^\nu \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} U_R^{e\dagger}, \quad A = U_L^\nu U_L^{e\dagger} - U_R^\nu \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} U_R^{e\dagger}. \tag{3.26}$$

Therefore the CC couplings allow the decays $\bar{D} \to u + W$ and $E \to \nu + W$. Similarly, for the fermions couple to the neutral gauge bosons Z_a^μ, $a = Y, \eta$, the couplings can be written as [60]

$$\mathcal{L}^{NC} \sim \sum_{i,a} (\bar{f}_{L,i} \gamma_u C_L^{i,a} f_{L,i} + \text{L} \leftrightarrow \text{R}) Z_a^\mu, \tag{3.27}$$

where $C_{L,R}^{i,a} = U_{L,R}^i P_{L,R}^{i,a} {U_{L,R}^i}^\dagger$, $P_{L,R}^{i,a}$ are coupling matrices, and f_i present fermions u, d, $\bar{D}$, e, $\ldots$ etc. This will allow the decays $\bar{D} \to d + Z$ and $E \to e + Z$. In addition, the Z_η boson can mix with Z boson through the conventional $Z - Z'$ mixing mechanism and decay into either fermion pairs, SUSY partners, W bosons, higgsinos and gauginos, or Z boson with Higgses. More phenomenology details can be referred to [60][5].

There can be one or more additional Higgs-like doublets from $(\mathbf{27} + \overline{\mathbf{27}})$ vector-like pairs preserving the gauge unification without introducing anomalies. In summary, with the picture of electroweak Higgs fields from a different $\mathbf{27}_H$, the minimum

[5]We only briefly discussed the phenomenology of the scenario that fermions and Higgs fields are from two different **27** multiplets (curves) due to the F-theory construction. We also focused on the case of h-leptons and D-quarks assignments. There could be additional conditions from F-theory or geometry to confine the degrees of freedom left for the exotic fields. We leave this topic for our future study.

spectrum at low energy is

$$3 \times \mathbf{27} + (\mathbf{27}_H) + (\mathbf{27} + \overline{\mathbf{27}}). \tag{3.28}$$

3.2 Non-abelian Gauge Fluxes

In what follows, we shall analyze the effects on the chirality after the $SU(2) \times U(1)^2$ flux is turned on. We choose the breaking chain ($1b$) in Eq. (3.5) via $SO(10)$ and $SU(4) \times SU(2) \times SU(2)$. When the flux is turned on, the matter on the bulk is decomposed as

$$\begin{array}{lll}
E_6 & \xrightarrow[U(1)_a]{} & SO(10) \times [U(1)_a] \\
 & \xrightarrow[SU(2)]{} & SU(4) \times SU(2)_1 \times [SU(2)_2 \times U(1)_a] \\
 & \xrightarrow[U(1)_b]{} & SU(3) \times SU(2)_1 \times [SU(2)_2 \times U(1)_a \times U(1)_b] \\
 & & \\
\mathbf{78} & \to & \mathbf{45}_0 + \mathbf{1}_0 + \mathbf{16}_{-3} + \overline{\mathbf{16}}_3 \\
 & \to & (\mathbf{15},\mathbf{1},\mathbf{1})_0 + (\mathbf{6},\mathbf{2},\mathbf{2})_0 + (\mathbf{1},\mathbf{3},\mathbf{1})_0 + (\mathbf{1},\mathbf{1},\mathbf{3})_0 + (\mathbf{1},\mathbf{1},\mathbf{1})_0 \\
 & & [(\mathbf{4},\mathbf{2},\mathbf{1})_{-3} + (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{-3} + c.c.] \\
 & \to & (\mathbf{8},\mathbf{1},\mathbf{1})_{0,0} + (\mathbf{3},\mathbf{1},\mathbf{1})_{0,-4} + (\bar{\mathbf{3}},\mathbf{1},\mathbf{1})_{0,4} + (\mathbf{1},\mathbf{1},\mathbf{1})_{0,0} \\
 & & +(\mathbf{3},\mathbf{2},\mathbf{2})_{0,2} + (\bar{\mathbf{3}},\mathbf{2},\mathbf{2})_{0,-2} + (\mathbf{1},\mathbf{3},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{1},\mathbf{3})_{0,0} + (\mathbf{1},\mathbf{1},\mathbf{1})_{0,0} \\
 & & +[(\mathbf{3},\mathbf{2},\mathbf{1})_{-3,-1} + (\mathbf{1},\mathbf{2},\mathbf{1})_{-3,3} + (\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{-3,1} + (\mathbf{1},\mathbf{1},\mathbf{2})_{-3,-3} + c.c.].
\end{array} \tag{3.29}$$

The SM hypercharge is defined as

$$U(1)_Y = \frac{1}{2}[U(1)_a + \frac{1}{3}U(1)_b]. \tag{3.30}$$

Under the breaking pattern (3.29), the gauge group E_6 can be broken down to $SU(3)\times SU(2)_1 \times U(1)_a \times U(1)_b$ by turning on a gauge bundle on S with the structure group $SU(2)_2 \times U(1)_a \times U(1)_b$. Let us define L_1 and L_2 to be the line bundles associated with $U(1)_a$ and $U(1)_b$, respectively. V_2 is defined as a vector bundle of rank two with the structure group $SU(2)$. To preserve supersymmtry, the connection of the gauge bundle $W = V_2 \oplus L_1 \oplus L_2$ has to satisfy the Hermitian Yang-Mills equations (2.4)[6]. It was shown in [79,80] that the bundle W has to be poly-stable with $\mu_{[\omega]}(V_2) = \mu_{[\omega]}(L_1) = \mu_{[\omega]}(L_2) = 0$, where slope $\mu_{[\omega]}(E)$ of a bundle E on S is defined by $\mu_{[\omega]}(E) = \frac{1}{\mathrm{rank}(E)} c_1(E) \cdot_S [\omega]$ and $[\omega]$ is an ample divisor of S. The poly-stability also requires that V_2 is a $[\omega]$-stable bundle. Since S is a del Pezzo surface, it was shown in [6] that for any

[6]More precisely, L_1 and L_2 are fractional line bundles [6–9].

non-trivial holomorphic vector bundle E satisfies Eq. (2.4), $h^0(S,E) = h^2(S,E) = 0$. This vanishing theorem dramatically simplifies the calculation of the chiral spectrum. It turns out that the matter spectrum can be calculated by the holomorphic Euler characteristic [81, 82]. By the decomposition Eq. (3.29) and the vanishing theorem, the spectrum is given by

$$n_{(\mathbf{3},\mathbf{1},\mathbf{1})_{0,-4}} = -\chi(S, G^{-1}) \equiv \gamma_1, \tag{3.31}$$
$$n_{(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})_{0,4}} = -\chi(S, G) \equiv \gamma_2, \tag{3.32}$$
$$n_{(\mathbf{3},\mathbf{2},\mathbf{2})_{0,2}} = -\chi(S, U_2) \equiv \gamma_3, \tag{3.33}$$
$$n_{(\bar{\mathbf{3}},\mathbf{2},\mathbf{2})_{0,-2}} = -\chi(S, U_2^\vee) \equiv \gamma_4, \tag{3.34}$$
$$n_{(\mathbf{3},\mathbf{2},\mathbf{1})_{-3,-1}} = -\chi(S, F) \equiv \gamma_5, \tag{3.35}$$
$$n_{(\bar{\mathbf{3}},\mathbf{2},\mathbf{1})_{3,1}} = -\chi(S, F^{-1}) \equiv \gamma_6, \tag{3.36}$$
$$n_{(\mathbf{3},\mathbf{1},\mathbf{2})_{3,-1}} = -\chi(S, U_2^\vee \otimes F^{-1}) \equiv \gamma_7, \tag{3.37}$$
$$n_{(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{-3,1}} = -\chi(S, U_2 \otimes F) \equiv \gamma_8, \tag{3.38}$$
$$n_{(\mathbf{1},\mathbf{1},\mathbf{2})_{-3,-3}} = -\chi(S, U_2^\vee \otimes F) \equiv \delta_1, \tag{3.39}$$
$$n_{(\mathbf{1},\mathbf{1},\mathbf{2})_{3,3}} = -\chi(S, U_2 \otimes F^{-1}) \equiv \delta_2, \tag{3.40}$$
$$n_{(\mathbf{1},\mathbf{2},\mathbf{1})_{-3,3}} = -\chi(S, G \otimes F) \equiv \delta_3, \tag{3.41}$$
$$n_{(\mathbf{1},\mathbf{2},\mathbf{1})_{3,-3}} = -\chi(S, G^{-1} \otimes F^{-1}) \equiv \delta_4, \tag{3.42}$$

where $\vee$ stands for the dual bundle, χ is the holomorphic Euler characteristic defined by $\chi(S,E) = \sum_i h^{0,i}(S,E)$, $U_2 = V_2 \otimes L_2^2$, $F = L_1^{-3} \otimes L_2^{-1}$, $G = L_2^4$, and $\gamma_i,\ \delta_i \in \mathbb{Z}_{\geqslant 0}$.

After some algebra, Eqs. (3.31)-(3.42) can be recast as

$$c_1(G)^2 = -2 - \gamma_1 - \gamma_2, \tag{3.43}$$

$$c_1(F)^2 = -2 - \gamma_5 - \gamma_6, \tag{3.44}$$

$$c_1(S) \cdot c_1(G) = \gamma_1 - \gamma_2, \tag{3.45}$$

$$c_1(S) \cdot c_1(F) = \gamma_6 - \gamma_5, \tag{3.46}$$

$$c_2(V_2) = \frac{1}{4}(6 - \gamma_1 - \gamma_2 + 2\gamma_3 + 2\gamma_4), \tag{3.47}$$

$$c_1(G) \cdot c_1(F) = \frac{1}{2}(4 + \gamma_3 + \gamma_4 + 2\gamma_5 + 2\gamma_6 - \gamma_7 - \gamma_8), \tag{3.48}$$

$$\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0, \tag{3.49}$$

$$\gamma_1 - \gamma_2 - 2\gamma_5 + 2\gamma_6 - \gamma_7 + \gamma_8 = 0, \tag{3.50}$$

$$\delta_1 = \frac{1}{2}(8 + \gamma_1 - \gamma_2 + 2\gamma_3 + 2\gamma_4 + 6\gamma_5 + 2\gamma_6 - \gamma_7 - \gamma_8), \tag{3.51}$$

$$\delta_2 = \frac{1}{2}(8 - \gamma_1 + \gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 6\gamma_6 - \gamma_7 - \gamma_8), \tag{3.52}$$

$$\delta_3 = -\frac{1}{2}(2 - 2\gamma_2 + \gamma_3 + \gamma_4 + 2\gamma_6 - \gamma_7 - \gamma_8), \tag{3.53}$$

$$\delta_4 = -\frac{1}{2}(2 - 2\gamma_1 + \gamma_3 + \gamma_4 + 2\gamma_5 - \gamma_7 - \gamma_8). \tag{3.54}$$

Note that given $\gamma_k,\ k = 1, 2, ..., 8$ satisfying the constraints Eqs. (3.49) and (3.50), (F, G, V_2) are constrained by Eqs. (3.43)-(3.48) and $(\delta_1, \delta_2, \delta_3, \delta_4)$ are then given by Eqs. (3.51)-(3.54). In particular, we are interested in the configurations of the vector-like pairs, namely $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \delta_1, \delta_2, \delta_3, \delta_4) = (a, a, b, b, c, c, d, d, e, e, f, f)$, where a, b, c, d, e are all non-negative integers. Then Eqs. (3.43)-(3.54) reduce to

$$\begin{cases} c_1(G)^2 = -2 - 2a \\ c_1(F)^2 = -2 - 2c \\ c_1(S) \cdot c_1(G) = 0 \\ c_1(S) \cdot c_1(F) = 0 \\ c_2(V_2) = \frac{1}{2}(3 + 2b - a) \\ c_1(G) \cdot c_1(F) = 2 + b + 2c - d \\ e = 4 + 2b + 4c - d \\ f = -1 + a - b - c + d. \end{cases} \tag{3.55}$$

It was proven in [83] that for an algebraic surface S with a given $n \geqslant 4([h^0(S, K_S)/2] + 1)$, there exists a $[\omega]$-stable bundle V of rank two with $c_1(V) = 0$ and $c_2(V) = n$. When S is a del Pezzo surface, $h^0(S, K_S) = 0$ and this theorem implies that for any given number $m \geqslant 4$, there exists a $[\omega]$-stable bundle of rank two with $c_1(V) = 0$ and $c_2(V) = m$. To apply this theorem to our case, we require that $c_2(V_2) \geqslant 4$. In general,

$c_1(V)$ and $c_2(V)$ of a stable bundle V over a compact Kähler surface S with $c_1(S) > 0$ satisfy the inequality $2rc_2(V) - (r-1)c_1(V)^2 \geqslant (r^2-1)$, where r is the rank of V [84]. When $r = 2$ and $c_1(V) = 0$, one can obtain the lower bound $c_2(V) \geqslant 2$. It is possible to obtain a $[\omega]$-stable bundle V of rank two with $c_1(V) = 0$ and $c_2(V) \leqslant 4$ for S being a del Pezzo surface. One can start with V defined by the following extension:

$$0 \to L \to V \to M \to 0. \tag{3.56}$$

To obtain vanishing $c_1(V)$, one can set $M = L^{-1}$ and compute $c_2(V) = -c_1(L)^2$. The extension is classified by $\mathrm{Ext}^1(L, M) = H^1(S, L \otimes M^*)$. When $M = L^{-1}$, the obstruction of the non-trivial extension is $h^1(S, L^2) \neq 0$. Let L be a non-trivial line bundle and S be a del Pezzo surface. By the vanishing theorem, one can obtain

$$h^1(S, L^2) = -1 - c_1(S) \cdot c_1(L) - 2c_1(L)^2. \tag{3.57}$$

If $c_1(S) \cdot c_1(L) = 0$ with negative $c_1(L)^2$, it is easy to see that $h^1(S, L^2) \geqslant 1$. The simple example for such a line bundle is $L = \mathcal{O}_S(e_i - e_j),\ i \neq j$, where $\{e_1, ..., e_8\}$ is a set of the exceptional divisors of S. With non-trivial extensions, one may construct a $[\omega]$-stable bundle V with $(r, c_1(V), c_2(V)) = (2, 0, 2)$ and with the structure group $SU(2)$ [50]. In what follows, we shall focus on the case of $c_2(V_2) \geqslant 4$. We summarize the constraints for (a, b, c, d) as follows:

$$\begin{cases} 2b + 4c - d \geqslant -4 \\ a - b - c + d \geqslant 1 \\ a - 2b \leqslant -5 \\ a, b, c, d \in \mathbb{Z}_{\geqslant 0}. \end{cases} \tag{3.58}$$

Note that a must be odd otherwise $c_2(V_2)$ cannot be integral. It follows from the condition $c_2(V_2) \geqslant 4$ that $b \geqslant 3$. Let us consider the case $(a, b, c) = (1, 3, 0)$. Then Eq. (3.55) becomes

$$\begin{cases} c_1(G)^2 = -4 \\ c_1(F)^2 = -2 \\ c_1(S) \cdot c_1(G) = 0 \\ c_1(S) \cdot c_1(F) = 0 \\ c_2(V_2) = 4 \\ c_1(G) \cdot c_1(F) = 5 - d \\ e = 10 - d \\ f = -3 + d. \end{cases} \tag{3.59}$$

Note that for the case $(a, b, c) = (1, 3, 0)$, the necessary condition for d is $3 \leqslant d \leqslant 10$. From the conditions $c_1(G)^2 = -4$ and $c_1(F)^2 = -2$, we set $G = \mathcal{O}_S(e_i - e_j + e_k -$

$e_l)$, $i \neq j \neq k \neq l$ and $F = \mathcal{O}_S(e_m - e_n)$, $m \neq n$. Clearly, G and F also satisfy the conditions $c_1(S) \cdot c_1(G) = 0$ and $c_1(S) \cdot c_1(F) = 0$. We shall not attempt to explore all solutions (G, F) and only list some solutions as follows [50]:

$$(G,F) = \begin{cases} (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_i - e_j)), \ (d,e,f) = (7,3,4) \\ (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_m - e_j)), \ (d,e,f) = (6,4,3) \\ (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_i - e_k)), \ (d,e,f) = (5,5,2) \\ (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_j - e_n)), \ (d,e,f) = (4,6,1) \\ (\mathcal{O}_S(e_i - e_j + e_k - e_l), \mathcal{O}_S(e_j - e_k)), \ (d,e,f) = (3,7,0). \end{cases} \tag{3.60}$$

Let us consider another example, $(a, b, c) = (3, 4, 0)$. In this case Eq. (3.55) reduces to

$$\begin{cases} c_1(G)^2 = -8 \\ c_1(F)^2 = -2 \\ c_1(S) \cdot c_1(G) = 0 \\ c_1(S) \cdot c_1(F) = 0 \\ c_2(V_2) = 4 \\ c_1(G) \cdot c_1(F) = 6 - d \\ e = 12 - d \\ f = -2 + d. \end{cases} \tag{3.61}$$

When $(a, b, c) = (3, 4, 0)$, it follows from Eq. (3.61) that the necessary condition for d is $2 \leqslant d \leqslant 12$. From the conditions $c_1(G)^2 = -8$ and $c_1(F)^2 = -2$, we set $G = \mathcal{O}_S(2e_i - 2e_j)$, $i \neq j$ and $F = \mathcal{O}_S(e_m - e_n)$, $m \neq n$. It is not difficult to see that G and F satisfy the conditions $c_1(S) \cdot c_1(G) = 0$ and $c_1(S) \cdot c_1(F) = 0$. Some solutions of (G, F) are as follows:

$$(G,F) = \begin{cases} (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_i - e_j)), \ (d,e,f) = (10,2,8) \\ (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_m - e_j)), \ (d,e,f) = (8,4,6) \\ (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_m - e_n)), \ (d,e,f) = (6,6,4) \\ (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_m - e_i)), \ (d,e,f) = (4,8,2) \\ (\mathcal{O}_S(2e_i - 2e_j), \mathcal{O}_S(e_j - e_i)), \ (d,e,f) = (2,10,0). \end{cases} \tag{3.62}$$

Let us turn to the chiral spectrum on the matter curves. The breaking pattern of the presentation $\mathbf{27}$ is

$$\begin{aligned} E_6 &\rightarrow SU(3) \times SU(2)_1 \times [SU(2)_2 \times U(1)_a \times U(1)_b] \\ \mathbf{27} &\rightarrow (\mathbf{3},\mathbf{2},\mathbf{1})_{1,-1} + (\mathbf{1},\mathbf{2},\mathbf{1})_{1,3} + (\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{1,1} + (\mathbf{1},\mathbf{1},\mathbf{2})_{1,-3} \\ &\quad +(\mathbf{3},\mathbf{1},\mathbf{1})_{-2,2} + (\bar{\mathbf{3}},\mathbf{1},\mathbf{1})_{-2,-2} + (\mathbf{1},\mathbf{2},\mathbf{2})_{-2,0} + (\mathbf{1},\mathbf{1},\mathbf{1})_{4,0}. \end{aligned} \tag{3.63}$$

Let us define $V_{\mathbf{27}} \otimes L_1^4|_{\Sigma_{\mathbf{27}}^{(k)}} = \Gamma|_{\Sigma_{\mathbf{27}}^{(k)}} = M^{(k)}$, $F|_{\Sigma_{\mathbf{27}}^{(k)}} = N_1^{(k)}$, and $G|_{\Sigma_{\mathbf{27}}^{(k)}} = N_2^{(k)}$. The chirality of matter localized on matter curves $\Sigma_{\mathbf{27}}^{(k)}$ is determined by the restrictions of the cover flux Γ and gauge fluxes to the curves. The spectrum induced by the cover flux and gauge fluxes is summarized in Table 5.

Curve	Matter	Bundle	Chirality
$\mathbf{27}^{(k)}$	$(\mathbf{3},\mathbf{2},\mathbf{1})_{1,-1}$	$V_{\mathbf{27}}\otimes L_1\otimes L_2^{-1}\|_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}+N_1^{(k)}$
	$(\mathbf{1},\mathbf{2},\mathbf{1})_{1,3}$	$V_{\mathbf{27}}\otimes L_1\otimes L_2^{3}\|_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}+N_1^{(k)}+N_2^{(k)}$
	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{2})_{1,1}$	$V_{\mathbf{27}}\otimes V_2\otimes L_1\otimes L_2\|_{\Sigma_{\mathbf{27}}^{(k)}}$	$2(M^{(k)}+N_1^{(k)})+N_2^{(k)}$
	$(\mathbf{1},\mathbf{1},\mathbf{2})_{1,-3}$	$V_{\mathbf{27}}\otimes V_2\otimes L_1\otimes L_2^{-3}\|_{\Sigma_{\mathbf{27}}^{(k)}}$	$2(M^{(k)}+N_1^{(k)})-N_2^{(k)}$
	$(\mathbf{3},\mathbf{1},\mathbf{1})_{-2,2}$	$V_{\mathbf{27}}\otimes L_1^{-2}\otimes L_2^{2}\|_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}+2N_1^{(k)}+N_2^{(k)}$
	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1})_{-2,-2}$	$V_{\mathbf{27}}\otimes L_1^{-2}\otimes L_2^{-2}\|_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}+2N_1^{(k)}$
	$(\mathbf{1},\mathbf{2},\mathbf{2})_{-2,0}$	$V_{\mathbf{27}}\otimes V_2\otimes L_1^{-2}\|_{\Sigma_{\mathbf{27}}^{(k)}}$	$2(M^{(k)}+2N_1^{(k)})+N_2^{(k)}$
	$(\mathbf{1},\mathbf{1},\mathbf{1})_{4,0}$	$V_{\mathbf{27}}\otimes L_1^{4}\|_{\Sigma_{\mathbf{27}}^{(k)}}$	$M^{(k)}$

Table 5: Chirality of matter localized on matter curve $\mathbf{27}^{(k)}$.

4 Tadpole Cancellation

The cancellation of tadpoles is crucial for consistent compactifications. In general, there are induced tadpoles from 7-brane, 5-brane, and 3-brane charges in F-theory. The 7-brane tadpole cancellation in F-theory is automatically satisfied since X_4 is a Calabi-Yau manifold. The cancellation of the $D5$-brane tadpole in the spectral cover construction follows from the topological condition that the overall first Chern class of the Higgs bundle vanishes. Therefore, the non-trivial tadpole cancellation in F-theory needed to be satisfied is the $D3$-brane tadpole which can be calculated by the Euler characteristic $\chi(X_4)$. The cancellation condition is of the form [85]

$$N_{D3}=\frac{\chi(X_4)}{24}-\frac{1}{2}\int_{X_4}G\wedge G, \tag{4.1}$$

where N_{D3} is the number of $D3$-branes and G is the four-form flux on X_4. For a non-singular elliptically fibered Calabi-Yau fourfold X_4, it was shown in [85] that the Euler characteristic $\chi(X_4)$ can be expressed as

$$\chi(X_4)=12\int_{B_3}c_1(B_3)[c_2(B_3)+30c_1(B_3)^2], \tag{4.2}$$

where $c_k(B_3)$ are the Chern classes of B_3. It follows from Eq. (4.2) that $\chi(X_4)/24$ is at least half-integral[7]. When X_4 admits non-abelian singularities, the Euler characteristic of X_4 is replaced by a refined Euler characteristic, the Euler characteristic of the smooth fourfold obtained from a suitable resolution of X_4. On the other hand, G-flux encodes the two-form gauge fluxes on the 7-branes. It was shown in [86] that

$$\int_{X_4} G \wedge G = -\Gamma^2, \tag{4.3}$$

where Γ is the universal cover flux defined in section 2 and Γ^2 is defined as the self-intersection number of Γ inside the spectral cover. It is a challenge to find compactifications with non-vanishing G-flux and non-negative N_{D_3} to satisfy the tadpole cancellation condition (4.1). In the next two subsections, we shall derive the formulae of the refined Euler characteristic $\chi(X_4)$ and the self-intersection of the universal cover fluxes Γ^2 for the $(2,1)$ and $(1,1,1)$ factorizations.

4.1 Geometric Contribution

In the presence of non-abelian singularities, X_4 becomes singular and the Euler characteristic $\chi(X_4)$ needs to be modified by resolving the singularities. To be more concrete, let us define H to be the gauge group corresponding to the non-abelian singularity over S and G to be the complement of H in E_8. Then the Euler characteristic is modified to

$$\chi(X_4) = \chi^*(X_4) + \chi_G - \chi_{E_8}, \tag{4.4}$$

where $\chi^*(X_4)$ is the Euler characteristic for a smooth fibration over B_3 given by Eq. (4.2) and the characteristic χ_{E_8} is given by [25, 86, 87]

$$\chi_{E_8} = 120 \int_S (3\eta^2 - 27\eta c_1 + 62c_1^2). \tag{4.5}$$

For the case of $G = SU(n)$, the characteristic $\chi_{SU(n)}$ is computed as

$$\chi_{SU(n)} = \int_S (n^3 - n)c_1^2 + 3n\eta(\eta - nc_1). \tag{4.6}$$

When the group G splits into a product of two groups G_1 and G_1, χ_G in Eq. (4.4) is then replaced by $\chi_{G_1}^{(k)} + \chi_{G_2}^{(l)}$ where η in χ_G is split into the classes $\eta^{(m)}$ as shown

[7] For a generic Calabi-Yau manifold X_4, $\chi(X_4)/24$ takes value in $\mathbb{Z}_4$ [85].

in the footnote below. It turns out that the refined Euler characteristic of the $(2,1)$ factorization is given by

$$\begin{aligned}\chi(X_4) &= \chi^*(X_4) + \chi^{(a)}_{SU(2)} + \chi^{(b)}_{SU(1)} - \chi_{E_8} \\ &= \chi^*(X_4) + \int_S 3[c_1(32c_1 - 16t - 15\xi) + (2t^2 + 4t\xi + 3\xi^2)] - \chi_{E_8}. \quad (4.7)\end{aligned}$$

In the $(1,1,1)$ factorization, the refined Euler characteristic[8] is

$$\begin{aligned}\chi(X_4) &= \chi^*(X_4) + \chi^{(l_1)}_{SU(1)} + \chi^{(l_2)}_{SU(1)} + \chi^{(l_3)}_{SU(1)} - \chi_{E_8} \\ &= \chi^*(X_4) + \int_S 3\{c_1[12c_1 - 7t - 6(\xi_1 + \xi_2)] + [t^2 + 2t(\xi_1 + \xi_2) + 2(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)]\} \\ &- \chi_{E_8}. \quad (4.8)\end{aligned}$$

4.2 Cover Flux Contribution

Under cover factorizations, the universal cover flux is of the form

$$\Gamma = \sum_k \Gamma^{(k)}, \quad (4.9)$$

where the fluxes $\Gamma^{(k)}$ satisfy the traceless condition $\sum_k p_{k*}\Gamma^{(k)} = 0$. In what follows, we shall compute the self-intersection Γ^2 of the universal fluxes for the (2,1) and (1,1,1) factorizations.

4.2.1 $(2,1)$ Factorization

Let us recall that in the $(2,1)$ factorization, the universal cover flux is given by

$$\Gamma = k_a\gamma_0^{(a)} + k_b\gamma_0^{(b)} + m_a\delta^{(a)} + m_b\delta^{(b)} + \tilde{\rho} = \Gamma^{(a)} + \Gamma^{(b)}, \quad (4.10)$$

where $\Gamma^{(a)}$ and $\Gamma^{(b)}$ are

$$\Gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [(2k_a + m_a)\sigma - \pi^*(k_a[a_2] + m_b[d_1] + \rho)] \equiv [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}], \quad (4.11)$$

$$\Gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot [(k_b + 2m_b)\sigma - \pi^*(k_b[d_1] + m_a[a_2] - 2\rho)] \equiv [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}]. \quad (4.12)$$

[8]For the $(2,1)$ factorization, $\eta^{(a)} = (\eta - c_1 - \xi)$ and $\eta^{(b)} = (c_1 + \xi)$. For the $(1,1,1)$ factorization, $\eta^{(l_1)} = (\eta - 2c_1 - \xi_1 - \xi_2)$, $\eta^{(l_2)} = (c_1 + \xi_1)$, and $\eta^{(l_3)} = (c_1 + \xi_2)$.

Then the self-intersection Γ^2 is calculated by [20]

$$\Gamma^2 = [\mathcal{C}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] \cdot [\widetilde{\mathcal{C}}^{(a)}] + [\mathcal{C}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}] \cdot [\widetilde{\mathcal{C}}^{(b)}]. \tag{4.13}$$

Recall that in the $(2,1)$ factorization, $[\mathcal{C}^{(a)}] = 2\sigma + \pi^*(\eta - c_1 - \xi)$ and $[\mathcal{C}^{(b)}] = \sigma + \pi^*(c_1 + \xi)$. With Eqs. (4.11) and (4.12), it is straightforward to compute

$$\begin{aligned}\Gamma^2 &= [\mathcal{C}_2^{(a)}] \cdot [\tilde{\mathcal{C}}_2^{(a)}]^2 + [\mathcal{C}_1^{(b)}] \cdot [\tilde{\mathcal{C}}_1^{(b)}]^2 \\ &= -\frac{1}{2}(2k_a + m_a)^2 [a_2] \cdot [a_0] - (k_b + 2m_b)^2 [d_1] \cdot [d_0] \\ &+ \frac{3}{2}(m_a[a_2] - 2m_b[d_1] - 2\rho)^2. \end{aligned} \tag{4.14}$$

4.2.2 $(1,1,1)$ Factorization

In the $(1,1,1)$ factorization, the universal flux is given by

$$\Gamma = k_{l_1}\gamma_0^{(l_1)} + k_{l_2}\gamma_0^{(l_2)} + k_{l_3}\gamma_0^{(l_3)} + m_{l_1}\delta^{(l_1)} + m_{l_2}\delta^{(l_2)} + m_{l_3}\delta^{(l_3)} + \tilde{\rho} \equiv \Gamma^{(l_1)} + \Gamma^{(l_2)} + \Gamma^{(l_3)}, \tag{4.15}$$

where $\Gamma^{(l_1)}$, $\Gamma^{(l_2)}$, and $\Gamma^{(l_3)}$ are

$$\Gamma^{(l_1)} \equiv [\mathcal{C}^{(l_1)}] \cdot [\tilde{\mathcal{C}}^{(l_1)}] = [\mathcal{C}^{(l_1)}] \cdot [(k_{l_1} + 2m_{l_1})\sigma - \pi^*(k_{l_1}[f_1] + m_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_1 - \rho_3)], \tag{4.16}$$
$$\Gamma^{(l_2)} \equiv [\mathcal{C}^{(l_2)}] \cdot [\tilde{\mathcal{C}}^{(l_2)}] = [\mathcal{C}^{(l_2)}] \cdot [(k_{l_2} + 2m_{l_2})\sigma - \pi^*(m_{l_1}[f_1] + k_{l_2}\xi_1 + m_{l_3}\xi_2 + \rho_2 - \rho_1)], \tag{4.17}$$
$$\Gamma^{(l_3)} \equiv [\mathcal{C}^{(l_3)}] \cdot [\tilde{\mathcal{C}}^{(l_3)}] = [\mathcal{C}^{(l_3)}] \cdot [(k_{l_3} + 2m_{l_3})\sigma - \pi^*(m_{l_1}[f_1] + m_{l_2}\xi_1 + k_{l_3}\xi_2 + \rho_3 - \rho_2)]. \tag{4.18}$$

In this case the self-intersection Γ^2 is computed as

$$\Gamma^2 = [\mathcal{C}^{(l_1)}] \cdot [\widetilde{\mathcal{C}}^{(l_1)}] \cdot [\widetilde{\mathcal{C}}^{(l_1)}] + [\mathcal{C}^{(l_2)}] \cdot [\widetilde{\mathcal{C}}^{(l_2)}] \cdot [\widetilde{\mathcal{C}}^{(l_2)}] + [\mathcal{C}^{(l_2)}] \cdot [\widetilde{\mathcal{C}}^{(l_3)}] \cdot [\widetilde{\mathcal{C}}^{(l_3)}]. \tag{4.19}$$

Recall that $[\mathcal{C}^{(l_1)}] = \sigma + \pi^*(\eta - 2c_1 - \xi_1 - \xi_2)$, $[\mathcal{C}^{(l_2)}] = \sigma + \pi^*(c_1 + \xi_1)$, and $[\mathcal{C}^{(l_3)}] = \sigma + \pi^*(c_1 + \xi_2)$. It follows from Eqs. (4.16)-(4.18) that

$$\begin{aligned}\Gamma^2 &= [\mathcal{C}^{(l_1)}] \cdot [\tilde{\mathcal{C}}^{(l_2)}]^2 + [\mathcal{C}^{(l_2)}] \cdot [\tilde{\mathcal{C}}^{(l_2)}]^2 + [\mathcal{C}^{(l_3)}] \cdot [\tilde{\mathcal{C}}^{(l_3)}]^2 \\ &= -(k_{l_1} + 2m_{l_1})^2 [f_1] \cdot [f_0] - (k_{l_2} + 2m_{l_2})^2 [g_1] \cdot [g_0] - (k_{l_3} + 2m_{l_3})^2 [h_1] \cdot [h_0] \\ &\quad + (\rho_1 - \rho_3 - 2m_{l_1}[f_1] + m_{l_2}[g_1] + m_{l_3}[h_1])^2 \\ &\quad + (\rho_2 - \rho_1 + m_{l_1}[f_1] - 2m_{l_2}[g_1] + m_{l_3}[h_1])^2 \\ &\quad + (\rho_3 - \rho_2 + m_{l_1}[f_1] + m_{l_2}[g_1] - 2m_{l_3}[h_1])^2. \end{aligned} \tag{4.20}$$

5 Models

In this section we give some numerical examples in the geometric backgrounds dP_2 studied in [18] and dP_7 in [25]. The basic geometric data of dP_2 in X_4 is

$$c_1 = 3h - e_1 - e_2, \quad t = h, \quad \eta = 17h - 6e_1 - 6e_2. \tag{5.1}$$

It follows from Eqs. (4.7) and (4.8) that the refined Euler characteristic $\chi(X_4)$ for the $(2,1)$ and $(1,1,1)$ factorizations are

$$\chi(X_4)_{(2,1)} = 10662 + \int_S 3[-15\xi c_1 + 4t\xi + 3\xi^2], \tag{5.2}$$

$$\chi(X_4)_{(1,1,1)} = 10320 + \int_S 6\left[(t - 3c_1)(\xi_1 + \xi_2) + (\xi_1^2 + \xi_1\xi_2 + \xi_2^2)\right], \tag{5.3}$$

where $\chi^*(X_4) = 13968$ has been used. The ample divisor $[\omega]_{dP_2}$ is chosen to be

$$[\omega]_{dP_2} = \alpha(e_1 + e_2) + \beta(h - e_1 - e_2), \quad 2\alpha > \beta > \alpha > 0. \tag{5.4}$$

For the dP_7 studied in [25], the basic geometric data is

$$\begin{aligned} c_1 &= 3h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7, \\ t &= 2h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6, \\ \eta &= 16h - 5e_1 - 5e_2 - 5e_3 - 5e_4 - 5e_5 - 5e_6 - 6e_7. \end{aligned} \tag{5.5}$$

with $\chi^*(X_4) = 1728$. By Eqs. (4.7) and (4.8), the refined Euler characteristic $\chi(X_4)$ for the $(2,1)$ and $(1,1,1)$ factorizations are

$$\chi(X_4)_{(2,1)} = 708 + \int_S 3[-15\xi c_1 + 4t\xi + 3\xi^2], \tag{5.6}$$

$$\chi(X_4)_{(1,1,1)} = 594 + \int_S 6\left[(t - 3c_1)(\xi_1 + \xi_2) + (\xi_1^2 + \xi_1\xi_2 + \xi_2^2)\right]. \tag{5.7}$$

In this case we choose the ample divisor $[\omega]_{dP_7}$ to be

$$[\omega]_{dP_7} = 14\beta h - (5\beta - \alpha)\sum_{i=1}^{7} e_i, \quad 5\beta > \alpha > 0. \tag{5.8}$$

We shall discuss the models of the (2,1) and (1,1,1) factorizations. In each case the trivial and non-trivial restrictions of the $U(1)$ fluxes to the matter curves will be discussed. Non-trivial restriction leads to the modification of the chirality of each matter on the curve after E_6 is broken according to the calculation in section 3. In addition, there could exist vector-like pairs on each curve since we only know the net chirality. The Higgs vector-like pair ($\mathbf{27} + \overline{\mathbf{27}}$) needed for the gauge unification is therefore assigned to one of these pairs, though the machinery to calculate the exact number of these vector-like fields is not clear yet.

5.1 Examples of the $(2, 1)$ Factorization

In the (2,1) factorization the matter fields are assigned to $\mathbf{27}^{(a)}$ curve and the Higgs fields come from the other $\mathbf{27}^{(b)}$ curve. The Yukawa coupling then turns out to be

$$\mathcal{W} \supset \mathbf{27}^{(a)} \cdot \mathbf{27}^{(a)} \cdot \mathbf{27}^{(b)}. \tag{5.9}$$

Since the fermion and Higgs fields are not on the same **27** curve, the exotic fields in $\mathbf{27}^{(a)}$ can be taken as exotic quarks and leptons which are able to mix with the ordinary ones by suitable discrete symmetries and to decay via mechanisms such as FCNC after E_6 is broken mentioned in section 3.

5.1.1 A three-family E_6 model in dP_2

The parameters of the model are listed in Table 6.

k_a	k_b	m_a	m_b	ρ	ξ	α	β
0.5	-1.5	-1	-1	$-\frac{5}{2}h + \frac{3}{2}e_1 - \frac{3}{2}e_2$	e_1	2	3

Table 6: Parameters of an example of a three-generation E_6 GUT.

These parameters give the spectrum $N_{\mathbf{27}^{(a)}} = 3$ and $N_{\mathbf{27}^{(b)}} = 3$ with $N_{D3} = 415$ as shown in Table 7. The dP_2 surface is probably too limited for the fluxes to break the E_6 gauge group. Therefore, we stop at a three-generation E_6 GUT model in this example.

5.1.2 An example of three-generation without flux restriction in dP_7

The parameters of the model with $N_{D3} = 12$ are listed in the Table 8.

Curve	Class	Gen.
$\mathbf{27}^{(a)}$	$8h - 4e_1 - 3e_2$	3
$\mathbf{27}^{(b)}$	e_1	3

Table 7: The **27** curves of the three-generation E_6 example in dP_2.

k_a	k_b	m_a	m_b	ρ	ξ	α	β
-0.5	1.5	0	-0.5	$\frac{1}{2}(3e_1 + e_2 + e_3 + e_4)$	$h - e_5 - e_6 + e_7$	3	1

Table 8: Parameters of an example of the (2,1) factorization in dP_7.

The matter contents on the curves are listed in Table 9. If the line bundles G and F associated to $SU(2) \times U(1)_a \times U(1)_b$ flux are chosen to have trivial restrictions[9] to both matter **27** curves, for example, $F = \mathcal{O}_S(e_5 - e_6)$ and $G = \mathcal{O}_S(e_1 - e_2 + e_3 - e_4)$,[10] then the chirality on each matter curve remains the same after E_6 is broken down to $SU(3) \times SU(2) \times U(1)_a \times U(1)_b$. After suitably transforming the $U(1)$ gauge groups, the corresponding matter content and phenomenology at low energy is a conventional rank 5 model discussed in section 3.

Curve	Class	Gen.
$\mathbf{27}^{(a)}$	$6h - 2e_1 - 2e_2 - 2e_3 - 2e_4 - e_5 - e_6 - 4e_7$	3
$\mathbf{27}^{(b)}$	$h - e_5 - e_6 + e_7$	2

Table 9: The **27** curves of the example of the (2,1) factorization without flux restrictions in dP_7.

5.1.3 An example with non-trivial flux restrictions in dP_7

In this example we consider a model with non-trivial flux restrictions to the matter curves in dP_7. From the chirality formulae discussed in section 3 and listed in Table 5, we find that it is unavoidable to have exotic fields under this construction. To maintain at least three copies for the MSSM matter after the gauge group E_6 is

[9] To avoid receiving a Green-Schwarz mass, it is required that $[H] \cdot_S c_1{=}0$ and $[H] \cdot_S \eta = 0$, for $H = F,\ G$ [6–9, 19].

[10] G can be chosen also as $G = \mathcal{O}_S(2(e_3 - e_4))$ from Eq. (3.62).

broken, we may have to start from a model with more chirality on the **27** curves. The parameters of an example of this scenario are listed in Table 10.

k_a	k_b	m_a	m_b	ρ	ξ	α	β
0.5	-0.5	-1	-0.5	$-h+\frac{1}{2}(e_1-2e_2+e_3+e_4+e_6)$	$h-e_2+e_5-e_7$	13	11

Table 10: Parameters of an example with non-trivial flux restrictions in dP_7.

It follows from Eq. (4.1) and the parameters in Table 10 that $N_{D3}=14$. We choose chirality-three curve for the matter fields and a chirality-four curve for the Higgs fields to make sure that there are enough MSSM matter after the gauge group E_6 is broken. From Eq. (3.60), we can turn on the fluxes $F=\mathcal{O}_S(e_1-e_2)$ and $G=\mathcal{O}_S(e_2-e_3+e_4-e_5)$ in dP_7.[11] The detailed information of the curves and the restrictions of fluxes to each curve are listed in Table 11.

Curve	Class	M	N_1	N_2
$\mathbf{27}^{(a)}$	$6h-2e_1-e_2-2e_3-2e_4-3e_5-2e_6-2e_7$	3	1	-2
$\mathbf{27}^{(b)}$	$h-e_2+e_5-e_7$	4	-1	2

Table 11: The **27** curves with non-trivial flux restrictions in dP_7.

The low energy spectrum is listed in Table 12. One can see that there are exotic fields including extra generations of quarks. One possible solution to these exotic fields is including them in the FCNC and CC mechanisms discussed in section 3 so that they could gain large masses and decay after mixing with ordinary generations. The detailed low energy physics is dedicated to future study.

5.2 Examples of the $(1,1,1)$ Factorization

The Yukawa coupling of the **27** curves in the $(1,1,1)$ factorization is $\mathbf{27}^{(l_1)}\mathbf{27}^{(l_2)}\mathbf{27}^{(l_3)}$. The fermions are assigned on the two **27** curves while the Higgs fields are located on the third **27** curve. For instance,

$$\mathcal{W}\supset \mathbf{27}_M^{(l_1)}\cdot\mathbf{27}_M^{(l_2)}\cdot\mathbf{27}_H^{(l_3)}. \tag{5.10}$$

[11] G can be chosen also as $G=\mathcal{O}_S(2(e_4-e_5))$ from Eq. (3.62).

Rep.	Gen. on $\mathbf{27}^{(a)}$	Gen. on $\mathbf{27}^{(b)}$
$(\mathbf{3},\mathbf{2})_{1,-1}$	$3\times Q+1\times(\mathbf{3},\mathbf{2})_{1,-1}$	3
$(\bar{\mathbf{3}},\mathbf{1})_{-2,-2}$	$3\times u^c+2\times(\bar{\mathbf{3}},\mathbf{1})_{-2,-2}$	2
$(\bar{\mathbf{3}},\mathbf{1})_{1,1}$	$3\times d^c+3\times D$	4+4
$(\mathbf{1},\mathbf{2})_{-2,0}$	$3\times L+5\times h$	$3\times(H_1+H_2)$
$(\mathbf{1},\mathbf{1})_{4,0}$	$3\times e^c$	4
$(\mathbf{1},\mathbf{1})_{1,-3}$	$3\times \nu^c+7\times S$	$2\times(H_3+H_4)$
$(\mathbf{3},\mathbf{1})_{-2,2}$	$3\times \bar{D}$	4
$(\mathbf{1},\mathbf{2})_{1,3}$	$2\times \bar{h}$	$5\times \bar{H}_2$

Table 12: The MSSM spectrum of the $(2,1)$ factorization in dP_7.

In this scenario the fermions are separated on different matter curves and the sum of the generations should accomplish a three-family model, for example, two families on $\mathbf{27}^{(l_1)}$ and one family on $\mathbf{27}^{(l_2)}$, or vice versa. However, this construction generally results in some problems in the mass matrices. With the assistance from the flux restrictions, the method studied in [88] can be applied to obtain a more reasonable Yukawa structure. However, again from the chirality given in Table 5 we expect exotic fields to remain in the spectrum after this mechanism. In what follows, we demonstrate one example for each case in the $(1,1,1)$ factorization.

5.2.1 An example of three-generation without flux restriction in dP_7

The parameters of the model are listed in Table 13.

k_{l_1}	k_{l_2}	k_{l_3}	m_{l_1}	m_{l_2}	m_{l_3}	ρ_1	ξ_1	ξ_2	α	β
-1.5	-0.5	1.5	0	0	0	$-h+e_1+2e_2$	e_1	$2h-2e_1-e_2+e_3-e_7$	1	3

Table 13: Parameters of a three family model in dP_7 with $\rho_2=2\rho_1$ and $\rho_3=0$.

These parameters give the spectrum shown in Table 14 with $N_{D3}=10$. Let us choose the line bundles to be $F=\mathcal{O}_S(e_5-e_6)$ and $G=\mathcal{O}_S(e_2-e_4+e_3-e_6)$,[12] having trivial restrictions to each $\mathbf{27}$ curve. Then the chirality remains the same after E_6

[12] G can be chosen also as $G=\mathcal{O}_S(2(e_4-e_5))$ from Eq. (3.62).

Curve	Class	Gen.	Matter
$\mathbf{27}^{(l_1)}$	$5h - e_1 - e_2 - 3e_3 - 2e_4 - 2e_5 - 2e_6 - 2e_7$	2	Fermion
$\mathbf{27}^{(l_2)}$	e_1	1	Fermion
$\mathbf{27}^{(l_3)}$	$2h - 2e_1 - e_2 + e_3 - e_7$	4	Higgs

Table 14: The spectrum of the three-generation model in dP_7.

is broken down to $SU(3) \times SU(2) \times U(1)_a \times U(1)_b$. After suitably transforming the $U(1)$ charges, the corresponding matter content and phenomenology at low energy is again a conventional rank 5 model.

5.2.2 An Example of non-trivial flux restrictions in dP_7

The parameters of the model are listed in Table 15.

k_{l_1}	k_{l_2}	k_{l_3}	m_{l_1}	m_{l_2}	m_{l_3}	ρ_1	ξ_1	ξ_2	α	β
-0.5	-0.5	-0.5	0	0	-1	e_2	$2h - 2e_1 - e_3 - e_7$	$h - e_1 - e_2$	1	3

Table 15: Parameters of a three family model in dP_7 with $\rho_2 = 2\rho_1$ and $\rho_3 = 0$.

These parameters confine the spectrum of E_6 shown in Table 16 with $N_{D3} = 10$. If the line bundles associated to $SU(2) \times U(1)_a \times U(1)_b$ flux are chosen as $F = \mathcal{O}_S(e_3 - e_5)$ and $G = \mathcal{O}_S(e_1 - e_2 + e_4 - e_6)$,[13] then the chirality of MSSM matter after E_6 is broken will be modified by numbers N_1 and N_2 shown in Table 16.

Curve	Class	M	N_1	N_2	Matter
$\mathbf{27}^{(l_1)}$	$4h + e_1 - e_2 - e_3 - 2e_4 - 2e_5 - 2e_6 - 2e_7$	3	-1	-2	Fermion
$\mathbf{27}^{(l^2)}$	$2h - 2e_1 - e_3 - e_7$	0	1	2	Fermion
$\mathbf{27}^{(l_3)}$	$h - e_1 - e_2$	4	0	0	Higgs

Table 16: The spectrum of the three-generation model in dP_7.

Originally, there is no chirality on curve $\mathbf{27}^{(l_2)}$ so it does not look realistic before the E_6 gauge group is broken. However after the fluxes are turned on, the chirality is

[13] G can be chosen also as $G = \mathcal{O}_S(2(e_3 - e_4))$ from Eq. (3.62).

“reshuffled” and shared between curves $\mathbf{27}^{(l_1)}$ and $\mathbf{27}^{(l_2)}$. Therefore, we can interpret the model in the way studied in [88] that is able to give a rich structure to the mass matrices via the Yukawa couplings. We demonstrate the corresponding MSSM spectrum in Table 17.

Rep.	Gen. on $\mathbf{27}^{(l_1)}$	Gen. on $\mathbf{27}^{(l_2)}$	Gen. on $\mathbf{27}^{(l_3)}$
$(\mathbf{3},\mathbf{2})_{1,-1}$	$2\times Q$	$1\times Q$	4
$(\bar{\mathbf{3}},\mathbf{1})_{-2,-2}$	$1\times u^c$	$2\times u^c$	4
$(\bar{\mathbf{3}},\mathbf{1})_{1,1}$	$1\times d^c+1\times D$	$2\times d^c+2\times D$	8
$(\mathbf{1},\mathbf{2})_{-2,0}$	0	$3\times L+3\times h$	$4\times(H_1+H_2)$
$(\mathbf{1},\mathbf{1})_{4,0}$	$3\times e^c$	0	4
$(\mathbf{1},\mathbf{1})_{1,-3}$	$3\times \nu^c+3\times S$	0	$4\times(H_3+H_4)$
$(\mathbf{3},\mathbf{1})_{-2,2}$	$1\times(\bar{\mathbf{3}},\mathbf{1})_{2,-2}$	$3\times\bar{D}+1\times(\mathbf{3},\mathbf{1})_{-2,2}$	4
$(\mathbf{1},\mathbf{2})_{1,3}$	0	$3\times\bar{h}$	$4\times\bar{H}_2$

Table 17: The MSSM matter shared by two curves in dP_7.

6 Conclusions

In this paper we discuss the E_6 GUT models where the gauge group is broken by the non-abelian flux $SU(2)\times U(1)^2$ in F-theory. The non-abelian part $SU(2)$ of the flux is not commutative with E_6 so the gauge group after breaking is $SU(3)\times SU(2)_L\times U(1)_a\times U(1)_b$ which is equivalent to a rank-5 model with $SU(3)\times SU(2)_L\times U(1)_Y\times U(1)_\eta$. We start building models from the $SU(3)$ spectral cover and then factorize it into $(2,1)$ and $(1,1,1)$ structures to obtain enough curves and degrees of freedom to construct models with minimum MSSM matter contents. The restrictions of the line bundles associated with two $U(1)$ gauge groups to the matter curves can modify the chirality of matter localized on the curves. This modification generally results in plenty of exotic fields that may cause troubles in the phenomenological interpretation of the models.

One way to arrange the matter content in the conventional E_6 GUT model building is that all the MSSM matter and Higgs fields are included in the same **27**-plet with three copies and the Yukawa coupling is $\mathbf{27}\cdot\mathbf{27}\cdot\mathbf{27}$. Such kind of interaction implies a structure of either one curve intersecting itself twice or three curves intersecting, which causes difficulties in geometry or the mass hierarchy structure in F-theory model building. Therefore, we adopt an alternate way that the weak scale Higgs particles are assigned to another **27** curve while the representations of their original assignments in the matter **27** curve are taken as exotic leptons. By additional symmetries such as baryon and lepton numbers, we can rule out the undesired interactions coupled to the exotic fields. The $(2,1)$ factorization providing two curves $\mathbf{27}^{(a)}$ and $\mathbf{27}^{(b)}$ with the interaction $\mathbf{27}^{(a)} \cdot \mathbf{27}^{(a)} \cdot \mathbf{27}^{(b)}$ satisfies the basic requirements of this picture. One the other hand, the $(1,1,1)$ factorization confines three curves to the interaction $\mathbf{27}^{(l_1)} \cdot \mathbf{27}^{(l_2)} \cdot \mathbf{27}^{(l_3)}$. In this case we have to distribute the MSSM matter to both $\mathbf{27}^{(l_1)}$ and $\mathbf{27}^{(l_2)}$ curves while the electroweak Higgs fields are assigned on the third curve. The fermion mass matrices are generally not able to admit the hierarchical structures except they are tuned by appropriate flux restrictions. As mentioned before, the additional one or more $(\mathbf{27} + \overline{\mathbf{27}})$ pairs can be included to make sure that the gauge unification occurs. These vector-like pairs generically exist on the curves in F-theory and can be assigned to the same curve containing the electroweak Higgs fields. However, the exact number of the vector-like pairs on a matter curve is still unclear in the present construction, so we assume that there exits at least one pair.

We demonstrate several models both in the $(2,1)$ and $(1,1,1)$ factorizations with geometric backgrounds dP_2 and dP_7 studied in [18] and [25], respectively. We also discuss the cases that the restrictions of the line bundles associated with $U(1)$s to the curves are trivial or non-trivial. Due to the chirality constraints to the fields on the bulk, it is hard to construct consistent $U(1)$ fluxes in dP_2. Therefore, we only demonstrate a three-family E_6 GUT model without gauge breaking in the dP_2 geometry. On the other hand, the dP_7 geometry has more degrees of freedom for the parameters to build realistic models. We therefore show in the $(2,1)$ case an example of a three-generation model without $U(1)$ flux restrictions, and an example with non-trivial $U(1)$ flux restrictions which gives rise to exotic particles. In the $(1,1,1)$ factorization, we also present an example of three-family model without flux restriction. In that case there are two flavors on one matter curve and the third flavor on the other. In the model with non-trivial flux restrictions, we adjust the parameters so that the total chirality of each representation on the two matter curves remain three while the hierarchies of the mass matrices can be maintained. Regardless of the exotic fields, the matter contents of our examples are conventional and the corresponding phenomenology has been discussed in the literature. Giving an appropriate interpretation for the exotic fields remains a challenge in the semi-local/global F-theory model building.

There are several interesting subgroups of the E_6 gauge group and we only discuss the rank 5 scenario in this paper. It would be interesting to construct rank 6 models with $U(1)^3$ fluxes, as well as the Pati-Salam-like and trinification-like models with appropriate non-abelian gauge fluxes in F-theory. We leave these possibilities for future work.

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A Breaking via $E_6 \to SU(6) \times SU(2)$

We list other possibilities of the subgroups after breaking E_6 by the $SU(2) \times U(1)^2$ flux. The full matter content of **27** and the corresponding $U(1)$ charges are presented.

Case 1. $SU(6) \to SU(5) \times U(1)$

$$
\begin{aligned}
E_6 &\xrightarrow[SU(2)]{} SU(6) \times [SU(2)] \\
&\xrightarrow[U(1)_c]{} SU(5) \times [SU(2) \times U(1)_c] \\
&\xrightarrow[U(1)_d]{} SU(3) \times SU(2) \times [SU(2) \times U(1)_c \times U(1)_d]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{27} \;\;\to\;\; & (\bar{\mathbf{6}}, \mathbf{2}) + (\mathbf{15}, \mathbf{1}) \\
\to\;\; & (\bar{\mathbf{5}}, \mathbf{2})_{-1} + (\mathbf{1}, \mathbf{2})_{5} + (\mathbf{10}, \mathbf{1})_{2} + (\mathbf{5}, \mathbf{1})_{-4} \\
\to\;\; & (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{2})_{-1,2} + (\mathbf{1}, \bar{\mathbf{2}}, \mathbf{2})_{-1,-3} + (\mathbf{1}, \mathbf{1}, \mathbf{2})_{5,0} \\
& +(\mathbf{3}, \mathbf{2}, \mathbf{1})_{2,1} + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})_{2,-4} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_{2,6} + (\mathbf{3}, \mathbf{1}, \mathbf{1})_{-4,-2} + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-4,3}
\end{aligned} \tag{A.1}
$$

$$U(1)_c = \frac{1}{2}U(1)_a - \frac{3}{2}U(1)_b, \quad U(1)_d = \frac{3}{2}U(1)_a + \frac{1}{2}U(1)_b. \tag{A.2}$$

Case 2. $SU(6) \to SU(4) \times SU(2) \times U(1)$

$$\begin{aligned}
E_6 &\xrightarrow[SU(2)]{} SU(6) \times [SU(2)] \\
&\xrightarrow[U(1)_e]{} SU(4) \times SU(2) \times [SU(2) \times U(1)_e] \\
&\xrightarrow[U(1)_f]{} SU(3) \times SU(2) \times [SU(2) \times U(1)_e \times U(1)_f]
\end{aligned}$$

$$\begin{aligned}
\mathbf{27} \quad &\to \quad (\bar{\mathbf{6}}, \mathbf{2}) + (\mathbf{15}, \mathbf{1}) \\
&\to \quad (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_1 + (\mathbf{1}, \bar{\mathbf{2}}, \mathbf{2})_{-2} + (\mathbf{6}, \mathbf{1}, \mathbf{1})_{-2} + (\mathbf{4}, \mathbf{2}, \mathbf{1})_1 + (\mathbf{1}, \mathbf{1}, \mathbf{1})_4 \\
&\to \quad (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{2})_{1,1} + (\mathbf{1}, \mathbf{1}, \mathbf{2})_{1,-3} + (\mathbf{1}, \bar{\mathbf{2}}, \mathbf{2})_{-2,0} + (\mathbf{3}, \mathbf{1}, \mathbf{1})_{-2,2} + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})_{-2,-2} \\
&\qquad + (\mathbf{3}, \mathbf{2}, \mathbf{1})_{1,-1} + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{1,3} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_{4,0}
\end{aligned} \tag{A.3}$$

$$U(1)_e = U(1)_a, \quad U(1)_f = U(1)_b. \tag{A.4}$$

Case 3. $SU(6) \to SU(3) \times SU(3) \times U(1)$

$$\begin{aligned}
E_6 &\xrightarrow[SU(2)]{} SU(6) \times [SU(2)] \\
&\xrightarrow[U(1)_g]{} SU(3) \times SU(3) \times [SU(2) \times U(1)_g] \\
&\xrightarrow[U(1)_h]{} SU(3) \times SU(2) \times [SU(2) \times U(1)_g \times U(1)_h]
\end{aligned}$$

$$\begin{aligned}
\mathbf{27} \quad &\to \quad (\bar{\mathbf{6}}, \mathbf{2}) + (\mathbf{15}, \mathbf{1}) \\
&\to \quad (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{2})_{-1} + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{2})_1 + (\mathbf{3}, \mathbf{3}, \mathbf{1})_0 + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})_2 + (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{1})_{-2} \\
&\to \quad (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{2})_{-1,0} + (\mathbf{1}, \bar{\mathbf{2}}, \mathbf{2})_{1,-1} + (\mathbf{1}, \mathbf{1}, \mathbf{2})_{1,2} + (\mathbf{3}, \mathbf{2}, \mathbf{1})_{0,1} + (\mathbf{3}, \mathbf{1}, \mathbf{1})_{0,-2} \\
&\qquad + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})_{2,0} + (\mathbf{1}, \bar{\mathbf{2}}, \mathbf{1})_{-2,-1} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_{-2,2}
\end{aligned} \tag{A.5}$$

$$U(1)_g = -\frac{1}{2}U(1)_a - \frac{1}{2}U(1)_b, \quad U(1)_h = \frac{1}{2}U(1)_a - \frac{1}{2}U(1)_b. \tag{A.6}$$

B Breaking via Trinification

$$\begin{aligned}
E_6 \quad &\longrightarrow \quad SU(3) \times SU(3) \times SU(3) \\
&\xrightarrow[SU(2)\times U(1)]{} \quad SU(3) \times SU(2) \times [SU(2) \times U(1)_i] \times U(1)_j \\
\mathbf{27} \quad &\to \quad (\mathbf{3}, \mathbf{2}, \mathbf{1})_{-1,0} + (\mathbf{3}, \mathbf{1}, \mathbf{1})_{2,0} \\
&\qquad + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{2})_{0,1} + (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})_{0,-2} \\
&\qquad + (\mathbf{1}, \mathbf{2}, \mathbf{2})_{1,-1} + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{1,2} + (\mathbf{1}, \mathbf{1}, \mathbf{2})_{-2,-1} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_{-2,2}
\end{aligned} \tag{B.1}$$

$$U(1)_i = -\frac{1}{2}U(1)_a + \frac{1}{2}U(1)_b, \quad U(1)_j = \frac{1}{2}U(1)_a + \frac{1}{2}U(1)_b. \tag{B.2}$$

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