AN INTERIOR PENALTY METHOD WITH C^0 FINITE ELEMENTS FOR THE APPROXIMATION OF THE MAXWELL EQUATIONS IN HETEROGENEOUS MEDIA: CONVERGENCE ANALYSIS WITH MINIMAL REGULARITY

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Abstract. The present paper proposes and analyzes an interior penalty technique using C^0-finite elements to solve the Maxwell equations in domains with heterogeneous properties. The convergence analysis for the boundary value problem and the eigenvalue problem is done assuming only minimal regularity in Lipschitz domains. The method is shown to converge for any polynomial degrees and to be spectrally correct.

1. Introduction

The objectives of the present paper is to propose and analyze a nodal C^0-finite element technique to solve the Maxwell equations in domains with heterogeneous properties. More precisely, given a three-dimensional open domain Ω with boundary Γ, we want to construct an approximation of the following problem using an interior penalty technique and C^0-Lagrange finite elements:

$$\nabla\times(\kappa\nabla\times\mathbf{E}) = \varepsilon\mathbf{g}, \quad \nabla\cdot(\varepsilon\mathbf{E}) = 0, \qquad \mathbf{E}\times\mathbf{n}_{|\partial\Omega} = 0, \tag{1.1}$$

where the fields κ and ε are only piecewise smooth. This task is non-trivial on two counts: first, the solution of (1.1) is singular in general, see e.g. Bonito et al. [6]; second, it is known since the pioneering work of Costabel [15] that $\mathbf{H}^1$-conforming approximation techniques that rely on uniform L^2-stability estimates both on the curl and the divergence of the approximate field do not converge properly if Ω is non-smooth and non-convex. This defect is a consequence of $\mathbf{H}^1(\Omega)\cap\mathbf{H}_{0,\mathrm{curl}}(\Omega)$ being a closed proper subspace of $\mathbf{H}_{\mathrm{div}}(\Omega)\cap\mathbf{H}_{0,\mathrm{curl}}(\Omega)$. This is probably one reason why edge elements have been favored over C^0-Lagrange finite elements over the years. It is only recently, say since the ground-breaking "rehabilitation" work of Costabel and Dauge [16], Bramble and Pasciak [7] and Bramble et al. [8] that C^0-Lagrange finite elements have regained their status as credible approximation tools for the Maxwell equations and more generally for div-curl problems. The key idea developed in the above references is that the divergence of the discrete field approximating $\varepsilon\mathbf{E}$ must be controlled in a space that is intermediate between $L^2(\Omega)$ and $H^{-1}(\Omega)$. This

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program is carried out in Costabel and Dauge [16] by controlling the divergence of $\varepsilon\mathbf{E}$ in a weighted L^2-space where the weight is a distance to the re-entrant corners of the domain to some appropriate power depending on the strength of the singularity. The analysis of the method by Costabel and Dauge [16] requires the approximation space to contain the gradient of C^1 scalar-valued functions, which excludes low-order finite-elements spaces. This restriction on low-order elements is removed in Buffa et al. [12] by considering a mixed form of the weighted L^2-stabilization technique on special meshes. The method developed by Bramble and Pasciak [7] and Bramble et al. [8] involves a least-square approximation of a discrete problem with different test and trial spaces. The trial space is $\mathbf{L}^2(\Omega)$ and the components of the test space are subspaces of $\mathrm{H}^1(\Omega)$. The numerical method uses piecewise constant functions for the trial space and piecewise linear functions enriched with face bubbles for the test space. A technique based on a local L^2-stabilization of the divergence of $\varepsilon\mathbf{E}$ and using finite elements of order high enough so as to contain the gradient of Argyris or Hsieh-Clough-Tocher C^1-finite elements is introduced in Duan et al. [19]. The convergence analysis of the method requires the source term to be smooth enough so that $\nabla\times\mathbf{E} \in \mathbf{H}^r(\Omega)$ with $r > \frac{1}{2}$. This method is further revisited in two space dimensions in Duan et al. [18] to allow for low-order finite elements and to remove the smoothness assumption on $\nabla\times\mathbf{E}$.

The present paper is the second part of a research program started in Bonito and Guermond [5] and is part of the PhD thesis of Luddens [28]. The technique adopted in [5, 21] consists of stabilizing the divergence of the field $\varepsilon\mathbf{E}$ in a negative Sobolev norm through a mixed formulation. It is shown in [5, 21] that stabilizing the divergence in $\mathrm{H}^{-1}(\Omega)$ is sufficient to solve the boundary value problem (1.1), but it may not be sufficient in general to solve the associated eigenvalue problem if only Lipschitz regularity of the domain is assumed. In this case the divergence must be stabilized in $\mathrm{H}^{-\alpha}(\Omega)$ with $\alpha \in (\frac{\ell}{2\ell-1}, 1]$ where $\ell - 1$ is the polynomial degree of the approximation of $\mathbf{E}$, $\ell \geq 1$. Note in passing that the method introduced in [5, 21] with the particular choice $\alpha = 1$ has also been proposed in Badia and Codina [4]. The convergence analysis of the boundary value problem in [4] assumes that the right-hand side is divergence free and either the solution to (1.1) is smooth or the degree of the finite element space is large enough or the mesh is specifically constructed so as to contain the gradient of C^1 scalar-valued functions. The method proposed in Bonito and Guermond [5] converges for all $\alpha \in (\frac{\ell}{2\ell-1}, 1]$ as stated in [5, Lemma 5.4], and the convergence rate is even maximal when $\alpha = 1$ *without* the extra assumptions used in [4], provided the right-hand side of the boundary value problem is solenoidal (which is usually the case). Yet, the possibility of choosing $\alpha < 1$ has been introduced in [5] to ensure the spectral correctness of the approximation for eigenvalue problems.

The objective of the present paper is to generalize the analysis of Bonito and Guermond [5] to boundary and eigenvalue problems with coefficients κ and ε in (1.1) that are only piecewise smooth. Our analysis assumes only the natural regularity of the solution; in particular the a priori regularity of $\mathbf{E}$ may be lower than that of $\mathbf{H}^{\frac{1}{2}}(\Omega)$, see Theorem 2.1. We focus mainly our attention on the convergence analysis in the very low regularity range, $\mathbf{E} \in \mathbf{H}^s(\Omega)$, $0 < s < \frac{1}{2}$. This range is rarely investigated in the literature since it entails many technical difficulties. One purpose of the present paper is to show that these difficulties can be handled properly when using continuous finite elements; the analysis with discontinuous elements has already been done, see e.g. Buffa and Perugia [10], Buffa et al. [11].

The approximation that we propose consists of using a mixed formulation with nodal finite elements and an interior penalty method to account for the jumps in the coefficients κ and ε. The convergence analysis presented holds for any polynomial degree (greater than one). One essential argument of this paper is the construction of a smoothing operator in $\mathbf{H}_{0,\text{curl}}(\Omega)$ that commutes (almost) with the curl operator, see Lemma 3.6. In passing we correct a mistake from [5] where the smoothing operator was not constructed properly. The second important argument is Lemma A.3 in the Appendix. This is a variant of Lemma 8.2 in Buffa and Perugia [10]; however, our proof slightly differs from that in Buffa and Perugia [10] since the estimates therein do not seem to be uniform in the meshsize.

The method presented in this paper has been implemented in a three-dimensional parallel MHD code, SFEMaNS, see e.g. Guermond et al. [24]. The code is developed under an open source licence and is used to test various experimental and astrophysical dynamo scenarios, e.g. Giesecke et al. [21, 22], Hollerbach et al. [25]. A substantial part of the work presented in this paper has been motivated by the VKS experiment, where the heterogeneous distribution of magnetic permeability plays a key role on the onset of the dynamo effect, see **?**]. SFEMaNs is also used to investigate MHD instabilities in liquid metal flows, see e.g. **?**] for an application to liquid metal batteries.

The paper is organized as follows. We introduce notation and recall a priori regularity results in §2. The smoothing operator in $\mathbf{H}_{0,\text{curl}}(\Omega)$ is introduced in §3. The key properties of this operator are stated in Theorem 3.1 and Lemma 3.6 (the estimate (3.17) is particularly important). The finite element framework and the interior penalty technique are presented in §4. This section also contains stability estimates for the weak formulation of the boundary value problem. The convergence analysis for the boundary value problem is done in §5. The two important results from this section are Theorem 5.1 and Theorem 5.3. Theorem 5.1 establishes convergence in a discrete norm and Theorem 5.3 establishes convergence in $\mathbf{L}^2(\Omega)$ using a duality argument. Additional convergence estimates assuming full regularity are given in Theorem 5.2 for completeness. The spectral correctness of the approximation of the eigenvalue problem is analyzed in §6. The paper is complemented with an appendix containing technical details. Lemma A.3 is one of the key results from the Appendix.

2. Preliminaries

2.1. **Spaces.** Let D be an open connected Lipschitz domain in $\mathbb{R}^3$. (In the rest of the paper D denotes a generic open Lipschitz domain that may differ from Ω.) The space of smooth functions with compact support in D is denoted $\mathcal{D}(D)$. The norm in $\mathrm{H}^1(D)$ is defined as follows:

$$\|v\|^2_{\mathrm{H}^1(D)} := \|v\|^2_{\mathrm{L}^2(D)} + \|\nabla v\|^2_{\mathbf{L}^2(D)}. \tag{2.1}$$

The space $\mathrm{H}^s(D)$ for $s \in (0,1)$ is defined by the method of real interpolation between $\mathrm{H}^1(D)$ and $\mathrm{L}^2(D)$ (see e.g. Tartar [33]), i.e.,

$$\mathrm{H}^s(D) = [\mathrm{L}^2(D), \mathrm{H}^1(D)]_{s,2}. \tag{2.2}$$

We also define the space $\mathrm{H}^1_0(D)$ to be the completion of $\mathcal{D}(D)$ with respect to the following norm:

$$\|v\|_{\mathrm{H}^1_0(D)} := \|\nabla v\|_{\mathbf{L}^2(D)}. \tag{2.3}$$

This allows us again to define the space $\mathrm{H}^s_0(D)$ for $s \in (0,1)$ by the method of real interpolation between $\mathrm{H}^1_0(D)$ and $\mathrm{L}^2(D)$ as follows:

$$\mathrm{H}^s_0(D) = [\mathrm{L}^2(D), \mathrm{H}^1_0(D)]_{s,2}. \tag{2.4}$$

This definition is slightly different from what is usually done; the only difference occurs at $s = \frac{1}{2}$. What we hereafter denote by $\mathrm{H}^{\frac{1}{2}}_0(D)$ is usually denoted by $\mathrm{H}^{\frac{1}{2}}_{00}(D)$ elsewhere. Owing to these definitions, the spaces $\mathrm{H}^s_0(D)$ and $\mathrm{H}^s(D)$ coincide for $s \in [0, \frac{1}{2}]$ and their norms are equivalent but not uniform with respect to s as s approached $\frac{1}{2}$, (see e.g. Lions and Magenes [27, Thm 11.1], **?** , Thm. 1.4.2.4] orTartar [33, Chap. 33]). The space $\mathrm{H}^{-s}(D)$ is defined by duality with $\mathrm{H}^s_0(D)$ for $0 \leq s \leq 1$, i.e., with a slight abuse of notation we define

$$\|v\|_{\mathrm{H}^{-s}(D)} = \sup_{0 \neq w \in \mathrm{H}^s_0(D)} \frac{\int_D vw \,\mathrm{d}\mathbf{x}}{\|w\|_{\mathrm{H}^s_0(D)}}.$$

It is a standard result that $\mathrm{H}^{-s}(D) = [\mathrm{L}^2(D), \mathrm{H}^{-1}(D)]_{s,2}$, see Lions and Magenes [27, Thm. 3.1].

The above definitions are naturally extended to the vector-valued Sobolev spaces $\mathbf{H}^s(D)$ and $\mathbf{H}^s_0(D)$. We additionally introduce the following spaces of vector-valued functions:

$$\mathbf{H}_{\mathrm{curl}}(D) = \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla{\times}\mathbf{v} \in \mathbf{L}^2(D)\}, \tag{2.5}$$

$$\mathbf{H}_{0,\mathrm{curl}}(D) = \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla{\times}\mathbf{v} \in \mathbf{L}^2(D),\ \mathbf{v}{\times}\mathbf{n}|_{\partial D} = 0\}, \tag{2.6}$$

$$\mathbf{H}^r_{\mathrm{curl}}(D) = \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla{\times}\mathbf{v} \in \mathbf{H}^r(D)\}, \tag{2.7}$$

$$\mathbf{H}^r_{0,\mathrm{curl}}(D) = \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla{\times}\mathbf{v} \in \mathbf{H}^r(D),\ \mathbf{v}{\times}\mathbf{n}|_{\partial D} = 0\}, \tag{2.8}$$

all equipped with their natural norm; for instance, $\|\mathbf{v}\|^2_{\mathbf{H}_{\mathrm{curl}}(D)} = \|\mathbf{v}\|^2_{\mathbf{L}^2(D)} + \|\nabla{\times}\mathbf{v}\|^2_{\mathbf{L}^2(D)}$.

2.2. **The domain.** The domain Ω is a bounded open set in $\mathbb{R}^3$, but the analysis presented in this paper can be applied to the two-dimensional counterparts of the problem.The boundary of Ω, say Γ, is assumed to have the Lipschitz regularity and to be connected. To simplify the presentation we also assume that $0 \in \Omega$ and Ω is star-shaped with respect to an open neighborhood of 0. This assumption implies the compact embedding stated in the following lemma.

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^d$ for $d > 0$. Then the following statements are equivalent:*

(i) There exists a neighborhood of the origin $\mathcal{V}$ such that Ω is star-shaped with respect to all points in $\mathcal{V}$,
(ii) There exists $\chi > 0$ such that, for any $\delta \in (0,1)$,

$$\Omega_\delta := (1-\delta)\Omega + B(0, \delta\chi) \subset\subset \Omega, \tag{2.9}$$

where the notation $\subset\subset$ indicates a compact embedding.

Proof. Equation (2.9) immediately implies (i) with $\mathcal{V} = B(0,\chi)$. Let us prove the converse $(i) \Rightarrow (ii)$. Let $\mathcal{V}$ be a neighborhood of the origin and let us assume that Ω is star-shaped with respect to all points in $\mathcal{V}$. In particular the following holds:

$$(1-\delta)\Omega + \delta\mathcal{V} \subset \Omega \qquad \forall \quad \delta \in [0,1]. \tag{2.10}$$

Let $\chi > 0$ such that $B(0,2\chi) \subset \mathcal{V}$ so that in conjunction with (2.10), we have

$$(1-\delta)\Omega + B(0,\delta\chi) \subset (1-\delta)\Omega + \delta B(0,2\chi) \subset (1-\delta)\Omega + \delta\mathcal{V} \subset \Omega \tag{2.11}$$

for all $\delta \in (0,1)$. Hence, it remains to prove that that the embedding is compact, which is done upon showing that $\overline{\Omega_\delta} \subset \Omega$. To do this, let $\{x_n\}_{n\geq 0} \subset \Omega_\delta$ be a sequence converging to some $x \in \mathbb{R}^d$ and write $x_n = (1-\delta)y_n + \delta\chi r_n$ for some $y_n \in \Omega$ and $r_n \in B(0,1)$. Upon extracting subsequences (still indexed by n), there exists $r \in \overline{B(0,1)}$ and $y \in \overline{\Omega}$ such that $r_n \to r$, $y_n \to y$ and therefore $x = (1-\delta)y + \delta\chi r$. As $y \in \overline{\Omega}$, we deduce that the set $B(y,\delta\chi) \cap \Omega$ is non-empty, i.e., there exist $\tilde{y} \in \Omega$ and $z \in B(0,1)$ such that $y = \tilde{y} + \delta\chi z$. Rewriting $x = (1-\delta)\tilde{y} + \delta\chi(r + (1-\delta)z)$, we realize that $x \in (1-\delta)\Omega + B(0,2\delta\chi)$, which yields $x \in \Omega$ (owing to (2.11)) and $\overline{\Omega}_\delta \subset \Omega$. This proves that the embedding $\Omega_\delta \subset \Omega$ is compact. □

A key piece of the convergence analysis of the method that we propose in this paper is based on the existence of a family of smoothing operators in $\mathbf{H}_{0,\mathrm{curl}}(\Omega)$. This construction is discussed in detail in §3. The main challenge one encounters when constructing this family of operators is to make it compatible with the boundary condition and to commutes with the curl operator. The purpose of the star-shape assumption is to make this construction possible and to simplify the presentation. It can be lifted for generic bounded Lipschitz domains by invoking Proposition 4.15 (or Proposition 4.19) from **?**]. The results presented in this paper remain valid for any domain bounded Lipschitz domains.

2.3. **Mixed formulation of the problem.** It will prove convenient to reformulate the original problem (1.1) in mixed form to have a better control on the divergence of the field $\varepsilon\mathbf{E}$. More precisely, from now on we consider the following problem: Given a vector field $\mathbf{g}$, find $\mathbf{E}$ and p such that

$$\nabla\times(\kappa\nabla\times\mathbf{E}) + \varepsilon\nabla p = \varepsilon\mathbf{g}; \quad \nabla\cdot(\varepsilon\mathbf{E}) = 0, \quad \mathbf{E}\times\mathbf{n}_{|\Gamma} = 0, \quad p_{|\Gamma} = 0. \tag{2.12}$$

The two problems (2.12) and (1.1) are equivalent if $\nabla\cdot(\varepsilon\mathbf{g}) = 0$, since in this case $p = 0$ in (2.12) (apply the divergence operator to the first equation).

The scalar fields κ and ε are assumed to be piecewise smooth. More precisely we assume that Ω is partitioned into N Lipschitz open subdomains $\Omega_1, \cdots, \Omega_N$ such that the restrictions of κ and ε to these subdomains are smooth. To better formalize this assumption we define

$$\Sigma := \bigcup_{i\neq j} \partial\Omega_i \cap \partial\Omega_j, \tag{2.13}$$

$$\mathrm{W}^{1,\infty}_{\Sigma}(\Omega) := \left\{\nu \in \mathrm{L}^\infty(\Omega) \mid \nabla(\nu_{|\Omega_i}) \in \mathbf{L}^\infty(\Omega_i),\ i = 1,\cdots,N\right\}. \tag{2.14}$$

We refer to Σ as the interface between the subdomains Ω_i. In the rest of the paper we assume that the fields ε and κ satisfy the following properties: There exist $\varepsilon_{\min}$, $\kappa_{\min} > 0$ such that

$$\varepsilon, \kappa \in \mathrm{W}^{1,\infty}_{\Sigma}(\Omega), \qquad \text{and} \qquad \varepsilon \geq \varepsilon_{\min}, \quad \kappa \geq \kappa_{\min} \quad \text{æin } \Omega. \tag{2.15}$$

The following stability results proved in Bonito et al. [6] play important roles in the stability of the finite element method developed in this paper:

Theorem 2.1. *Assuming that* $\varepsilon\mathbf{g} \in \mathbf{L}^2(\Omega)$ *and* (2.15), *Problem* (2.12) *has a unique solution in* $\mathbf{H}_{0,\mathrm{curl}}(\Omega)\times\mathrm{H}^1_0(\Omega)$. *Moreover, there exist* c, τ_ε *and* τ_κ, *depending on* Ω *and the fields* ε *and* κ, *so that*

$$\|\mathbf{E}\|_{\mathbf{H}^s(\Omega)} \le c\|\varepsilon\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \qquad \forall s \in [0,\tau_\varepsilon), \tag{2.16}$$

$$\|\nabla\times\mathbf{E}\|_{\mathbf{H}^s(\Omega)} \le c\|\varepsilon\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \qquad \forall s \in [0,\tau_\kappa). \tag{2.17}$$

$$\|\nabla\times(\kappa\nabla\times\mathbf{E})\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{\mathbf{L}^2(\Omega)} \le c\|\varepsilon\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \tag{2.18}$$

Remark 2.1. *In general the regularity indices* τ_ε *and* τ_κ *are smaller than* $\frac{1}{2}$ *when the domain* Ω *is not convex and the scalar field* ε *and* κ *are discontinuous across* Σ.

3. Smooth approximation in $\mathbf{H}_{0,\mathrm{curl}}(\Omega)$

We introduce in this section a smoothing operator in $\mathbf{H}_{0,\mathrm{curl}}(\Omega)$ that will be used to prove the convergence of the finite element approximation. The key difficulty that we are facing is to find an approximation that is compatible with the boundary condition in $\mathbf{H}_{0,\mathrm{curl}}(\Omega)$ and commutes with the curl operator. We essentially proceed as in Bonito and Guermond [5] but modify the argument to correct an incorrect statement made therein. When invoking $\mathcal{C}_h(A\mathbf{E})_\varepsilon$ in the proof of Lemma 3.3 in Bonito and Guermond [5] it was incorrectly assumed that $(A\mathbf{E})_\varepsilon$ is in $\mathbf{H}_{0,\mathrm{curl}}(\Omega)$, which is not the case. We resolve this issue in the present construction by introducing an additional scaling operator, S^δ_D. Some of the tools introduced in this section are similar in spirit to those developed in Schöberl [32], Christiansen and Winther [13], Arnold et al. [2]

3.1. Extension operator.

Let D be an open Lipschitz domain in $\mathbb{R}^3$. For any $\mathbf{F} \in \mathbf{L}^1(D)$, we denote $E_D\mathbf{F}$ the extension of $\mathbf{F}$ by 0, i.e.,

$$E_D\mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}(\mathbf{x}) & \text{if } \mathbf{x} \in D, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1}$$

Let $\delta \in [0,\frac{1}{2}]$, define $\bar{\delta} := 1-\delta$ and set $D_\delta := \bar{\delta}D$. Note that the assumption on δ means that $\bar{\delta} \in [\frac{1}{2},1]$, i.e., the quantity $\bar{\delta}^{-1}$ is uniformly bounded with respect to δ; this observation will be used repeatedly. We define the scaling operator $S^\delta_D : \mathbf{L}^1(D) \longmapsto \mathbf{L}^1(D_\delta)$ by

$$\forall \mathbf{F} \in \mathbf{L}^1(D),\ \forall \mathbf{x} \in D_\delta, \quad (S^\delta_D\mathbf{F})(\mathbf{x}) := \mathbf{F}\left(\mathbf{x}\bar{\delta}^{-1}\right). \tag{3.2}$$

Lemma 3.1. *The following commuting properties hold:*

$$S^\delta_{\mathbb{R}^3}E_D = E_{D_\delta}S^\delta_D \tag{3.3}$$

$$\partial_{x_i}(S^\delta_D\mathbf{F}) = \bar{\delta}^{-1}S^\delta_D(\partial_{x_i}\mathbf{F}), \qquad \forall \mathbf{F} \in \mathbf{L}^1(D),\ \forall i = 1,\ldots,d, \tag{3.4}$$

$$\nabla\times(E_D\mathbf{F}) = E_D(\nabla\times\mathbf{F}), \qquad \forall \mathbf{F} \in \mathbf{H}_{0,\mathrm{curl}}(D), \tag{3.5}$$

$$\nabla(E_D\mathbf{F}) = E_D(\nabla\mathbf{F}), \qquad \forall \mathbf{F} \in \mathbf{H}^1_0(D). \tag{3.6}$$

Proof. (3.3) is evident and (3.4) is just the chain rule. We only prove (3.5) since the proof of (3.6) is similar. Let $\mathbf{F}$ be a member of $\mathbf{H}_{0,\mathrm{curl}}(D)$, then the following holds:

$$\langle \nabla\times(E_D\mathbf{F}), \boldsymbol{\phi}\rangle = \int_{\mathbb{R}^3} (E_D\mathbf{F})\cdot\nabla\times\boldsymbol{\phi} = \int_D \mathbf{F}\cdot\nabla\times\boldsymbol{\phi} = \int_D \nabla\times\mathbf{F}\cdot\boldsymbol{\phi}, \quad \forall\boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^3),$$

where the last equality holds owing to $\mathbf{F}$ being in $\mathbf{H}_{0,\text{curl}}(D)$. Then

$$\langle \nabla\times(E_D\mathbf{F}), \boldsymbol{\phi}\rangle = \int_{\mathbb{R}^3} E_D(\nabla\times\mathbf{F})\cdot\boldsymbol{\phi}, \quad \forall\boldsymbol{\phi}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^3),$$

which proves the statement. □

Lemma 3.2. *The following holds for all $r\in[0,1]$: (i) the linear operator $E_D : \mathbf{H}^r_0(D)\to\mathbf{H}^r_0(\mathbb{R}^3)$ is bounded; (ii) the family of operators $\{S^\delta_D\} : \mathbf{H}^r(D)\to\mathbf{H}^r(D_\delta)$ is uniformly bounded with respect to $\delta\in[0,\frac{1}{2}]$.*

Proof. It is a standard result that E_D is a continuous operator from $\mathbf{L}^2(D)$ to $\mathbf{L}^2(\mathbb{R}^3)$ and from $\mathbf{H}^1_0(D)$ to $\mathbf{H}^1_0(\mathbb{R}^3)$, see Adams and Fournier [1]. Then the first assertion follows directly from the interpolation theory. For the second part, a scaling argument ensures that S^δ_D is a continuous operator from $\mathbf{L}^2(D)$ to $\mathbf{L}^2(D_\delta)$. Using (3.4), we infer that it is also a continuous operator from $\mathbf{H}^1(D)$ to $\mathbf{H}^1(D_\delta)$. The conclusion follows from the interpolation theory. □

Taking $r\in\left[0,\frac{1}{2}\right)$ and using the fact that the spaces $\mathbf{H}^r_0(\Omega)$ and $\mathbf{H}^r(\Omega)$ coincide (with equivalent norms), we infer that there exists c such that,

$$\forall\mathbf{F}\in\mathbf{H}^r(\Omega), \qquad \|E_\Omega\mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} \le c\,\|\mathbf{F}\|_{\mathbf{H}^r(\Omega)}. \tag{3.7}$$

Moreover, using this inequality and the second part of Lemma 3.2 with $D=\mathbb{R}^3$, we infer that $S^\delta_{\mathbb{R}^3}E_\Omega : \mathbf{H}^r(\Omega)\to\mathbf{H}^r(\mathbb{R}^3)$ is a linear continuous operator, and there exists c, uniform in δ, such that

$$\forall\mathbf{F}\in\mathbf{H}^r(\Omega), \qquad \|S^\delta_{\mathbb{R}^3}E_\Omega\mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} \le c\,\|\mathbf{F}\|_{\mathbf{H}^r(\Omega)}. \tag{3.8}$$

Lemma 3.3. *The following holds:*

$$\forall\mathbf{F}\in\mathbf{H}_{0,\text{curl}}(\Omega), \qquad \nabla\times(S^\delta_{\mathbb{R}^3}E_\Omega\mathbf{F}) = \bar\delta^{-1}S^\delta_{\mathbb{R}^3}E_\Omega(\nabla\times\mathbf{F}). \tag{3.9}$$

Proof. Let $\mathbf{F}\in\mathbf{H}_{0,\text{curl}}(\Omega)$. By (3.4) we infer that

$$\nabla\times(S^\delta_{\mathbb{R}^3}E_\Omega\mathbf{F}) = \bar\delta^{-1}S^\delta_{\mathbb{R}^3}\nabla\times(E_\Omega\mathbf{F}).$$

Then (3.5) from Lemma 3.1 implies

$$\nabla\times(S^\delta_{\mathbb{R}^3}E_\Omega\mathbf{F}) = \bar\delta^{-1}S^\delta_{\mathbb{R}^3}E_\Omega(\nabla\times\mathbf{F}),$$

since $\mathbf{F}\in\mathbf{H}_{0,\text{curl}}(\Omega)$. This concludes the proof. □

Lemma 3.4. *The linear operator $S^\delta_{\mathbb{R}^3}E_\Omega : \mathbf{H}^r_{0,\text{curl}}(\Omega)\longrightarrow\mathbf{H}^r_{0,\text{curl}}\left(\mathbb{R}^3\right)$ is bounded for all $r\in[0,\frac{1}{2})$. More precisely there is c, uniform with respect to δ, so that the following holds:*

$$\|\nabla\times(S^\delta_{\mathbb{R}^3}E_\Omega\mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} \le c\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r(\Omega)}. \tag{3.10}$$

Proof. The identity (3.9) implies that $S^\delta_{\mathbb{R}^3}E_\Omega$ is a continuous map from $\mathbf{H}_{0,\text{curl}}(\Omega)$ to $\mathbf{H}_{0,\text{curl}}\left(\mathbb{R}^3\right)$. Let r$\in[0,\frac{1}{2})$ and let $\mathbf{F}$ be a member of $\mathbf{H}^r_{0,\text{curl}}(\Omega)$. A simple scaling argument implies that $S^\delta_\Omega\mathbf{F}$ is a member of $\mathbf{H}^r_{0,\text{curl}}(\Omega_\delta)$. Since $\nabla\times S^\delta_\Omega\mathbf{F}$ is in $\mathbf{H}^r(\Omega)$ and $r\in[0,\frac{1}{2})$, the extension by zero is stable in $\mathbf{H}^r(\mathbb{R}^3)$, i.e., $E_{\Omega_\delta}\nabla\times S^\delta_\Omega\mathbf{F}$ is a member of $\mathbf{H}^r(\mathbb{R}^3)$ and there is a constant c, uniform with respect to $\mathbf{F}$ and δ, so that

$$\begin{aligned}\|E_{\Omega_\delta}\nabla\times S_\Omega^\delta\mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} &\le c'\|\nabla\times S_\Omega^\delta\mathbf{F}\|_{\mathbf{H}^r(\Omega_\delta)} = c'\bar{\delta}^{-1}\|S_\Omega^\delta\nabla\times\mathbf{F}\|_{\mathbf{H}^r(\Omega_\delta)}\\ &\le c\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r(\Omega)}.\end{aligned}$$

Then, applying (3.3) and (3.5) to the above inequality gives

$$\begin{aligned}\|\nabla\times(S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} &= \|\nabla\times(E_{\Omega_\delta}S_\Omega^\delta\mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} = \|E_{\Omega_\delta}\nabla\times S_\Omega^\delta\mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)}\\ &\le c\,\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r(\Omega)},\end{aligned}$$

which concludes the proof. □

We now state a lemma that gives some important approximation properties of the operator $\mathbf{F}\mapsto S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}$.

Lemma 3.5. *There exists K_1 such that the following holds for every $\mathbf{F}\in\mathbf{H}_0^r(\Omega)$ and for all $r\in[0,1]$:*

$$\|\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}\|_{\mathbf{H}_0^s(\Omega)} \le K_1\delta^{r-s}\|\mathbf{F}\|_{\mathbf{H}_0^r(\Omega)} \qquad 0\le s\le r\le 1, \tag{3.11}$$

and for all $r\in[0,\frac{1}{2})$ there exists K_2, such that the following holds every $\mathbf{F}\in\mathbf{H}_{0,\mathrm{curl}}^r(\Omega)$:

$$\|\nabla\times(\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F})\|_{\mathbf{H}^s(\Omega)} \le K_2\delta^{r-s}\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r(\Omega)} \qquad 0\le s\le r<\frac{1}{2}. \tag{3.12}$$

Proof. We prove the first inequality (3.11) by means of an interpolation technique. Let $\mathbf{F}\in\mathbf{H}_0^1(\Omega)$, then usingLemma 3.1 together with $d=3$, we have

$$\|\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \|S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}\|_{\mathbf{L}^2(\Omega)} \le \left(1+\bar{\delta}^{\frac{d}{2}}\right)\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)} \le 2\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}.$$

$$\begin{aligned}\|\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}\|_{\mathbf{H}_0^1(\Omega)} &= \|\nabla(\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F})\|_{\mathbf{L}^2(\Omega)} \le \|\nabla\mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \|\nabla(S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F})\|_{\mathbf{L}^2(\Omega)}\\ &= \|\nabla\mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \bar{\delta}^{-1}\|S_{\mathbb{R}^3}^\delta\nabla(E_\Omega\mathbf{F})\|_{\mathbf{L}^2(\Omega)} = \|\nabla\mathbf{F}\|_{\mathbf{L}^2(\Omega)} + \bar{\delta}^{\frac{d}{2}-1}\|E_\Omega\nabla\mathbf{F}\|_{\mathbf{L}^2(\Omega)}\\ &= \left(1+\bar{\delta}^{\frac{d}{2}-1}\right)\|\nabla\mathbf{F}\|_{\mathbf{L}^2(\Omega)} \le 2\|\mathbf{F}\|_{\mathbf{H}_0^1(\Omega)}.\end{aligned}$$

We now derive an estimate for the mapping $\mathbf{H}_0^1(\Omega)\ni\mathbf{F}\mapsto\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}\in\mathbf{L}^2(\Omega)$. The definition of $S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}$ implies that

$$\begin{aligned}\|\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}\|_{\mathbf{L}^2(\Omega)}^2 &= \int_\Omega \left|(E_\Omega\mathbf{F})(\mathbf{x}) - (E_\Omega\mathbf{F})\left(\mathbf{x}\bar{\delta}^{-1}\right)\right|^2\,\mathrm{d}\mathbf{x}\\ &= \int_\Omega\left|\int_0^1 \nabla(E_\Omega\mathbf{F})\left((1-t)\mathbf{x}+t\mathbf{x}\bar{\delta}^{-1}\right)\cdot\frac{\delta}{\bar{\delta}}\mathbf{x}\,\mathrm{d}t\right|^2\,\mathrm{d}\mathbf{x}\\ &\le \int_\Omega \frac{\delta^2}{\bar{\delta}^2}|\mathbf{x}|^2\int_0^1\left|\nabla(E_\Omega\mathbf{F})\left((1-t)\mathbf{x}+t\mathbf{x}\bar{\delta}^{-1}\right)\right|^2\,\mathrm{d}t\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Then, we introduce $M:=\sup_{\mathbf{x}\in\Omega}|\mathbf{x}|$, and we apply Fubini's theorem:

$$\|\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}\|_{\mathbf{L}^2(\Omega)}^2 \le M^2\frac{\delta^2}{\bar{\delta}^2}\int_0^1\int_\Omega\left|\nabla(E_\Omega\mathbf{F})\left((1-t)\mathbf{x}+t\mathbf{x}\bar{\delta}^{-1}\right)\right|^2\,\mathrm{d}\mathbf{x}\mathrm{d}t$$

Using a change of variable in the inner integral, we finally obtain

$$\|\mathbf{F}-S_{\mathbb{R}^3}^\delta E_\Omega\mathbf{F}\|_{\mathbf{L}^2(\Omega)}^2 \le M^2\frac{\delta^2}{\bar{\delta}^2}\|\nabla(E_\Omega\mathbf{F})\|_{\mathbf{L}^2(\Omega_\delta)}^2\int_0^1\left(\frac{\bar{\delta}}{\bar{\delta}+\delta t}\right)^3\mathrm{d}t \le M^2\delta^2\bar{\delta}^{-2}\|\nabla(E_\Omega\mathbf{F})\|_{\mathbf{L}^2(\mathbb{R}^3)}^2.$$

Since $\mathbf{F} \in \mathbf{H}^1_0(\Omega)$, we have $\|\nabla(E_\Omega \mathbf{F})\|_{\mathbf{L}^2(\mathbb{R}^3)} = \|E_\Omega \nabla \mathbf{F}\|_{\mathbf{L}^2(\mathbb{R}^3)} = \|\nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)}$. Using now the assumption $\delta \leq \frac{1}{2}$, i.e., $\bar{\delta}^{-1} \leq 2$, we finally deduce that

$$\|\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} \leq 2M\delta \|\nabla \mathbf{F}\|_{\mathbf{L}^2(\Omega)} = 2M\delta \|\mathbf{F}\|_{\mathbf{H}^1_0(\Omega)}. \tag{3.13}$$

We now set $K_1 := \max(2, 2M)$ and we have proven that

$$\begin{aligned} \|\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} &\leq K_1 \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}, \\ \|\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}\|_{\mathbf{L}^2(\Omega)} &\leq K_1 \delta \|\mathbf{F}\|_{\mathbf{H}^1_0(\Omega)}, \\ \|\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}\|_{\mathbf{H}^1_0(\Omega)} &\leq K_1 \|\mathbf{F}\|_{\mathbf{H}^1_0(\Omega)}. \end{aligned}$$

We conclude that (3.11) holds by using the Lions-Peetre Reiteration Theorem.

We now turn our attention to (3.12). Let $r \in [0, \frac{1}{2})$ and consider $s \in [0, r]$. Let $\mathbf{F}$ be a member of $\mathbf{H}^r_{0,\mathrm{curl}}(\Omega)$. We observe first that $S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}$ is also in $\mathbf{H}^r_{0,\mathrm{curl}}(\Omega)$ owing to Lemma 3.4. Using (3.9) gives

$$\begin{aligned} \|\nabla\times(\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F})\|_{\mathbf{H}^s_0(\Omega)} &= \|\nabla\times\mathbf{F} - \bar{\delta}^{-1} S^\delta_{\mathbb{R}^3} E_\Omega \nabla\times\mathbf{F}\|_{\mathbf{H}^s_0(\Omega)} \\ &\leq \|\nabla\times\mathbf{F} - \bar{\delta}^{-1}\nabla\times\mathbf{F}\|_{\mathbf{H}^s_0(\Omega)} + \bar{\delta}^{-1}\|\nabla\times\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega(\nabla\times\mathbf{F})\|_{\mathbf{H}^s_0(\Omega)} \\ &\leq \delta\bar{\delta}^{-1}\|\nabla\times\mathbf{F}\|_{\mathbf{H}^s_0(\Omega)} + K_1\bar{\delta}^{-1}\delta^{r-s}\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r_0(\Omega)}. \end{aligned}$$

Using $\delta < \frac{1}{2}$, i.e., $\bar{\delta}^{-1} \leq 2$, we have

$$\|\nabla\times(\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F})\|_{\mathbf{H}^s_0(\Omega)} \leq 2(K_1 + \delta^{1-r+s})\delta^{r-s}\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r_0(\Omega)},$$

Remembering that $\mathbf{H}^s(\Omega)$ and $\mathbf{H}^s_0(\Omega)$ coincide for $0 \leq s \leq r < \frac{1}{2}$ and that their norm are equivalent, say $\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r_0(\Omega)} \leq c(r)\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r(\Omega)}$, the above inequality yields (3.12) with $K_2 = 2(K_1 + 1)c(r)$since $1 - r + s \geq 1 - r > 0$ and $\delta \leq \frac{1}{2}$. □

3.2. **Smooth approximation.** We now use the above extension operator $S^\delta_{\mathbb{R}^3} E_\Omega$ together with a mollification to construct a smooth approximation operator. For $\delta \in (0, \frac{1}{2})$, we set

$$\rho_\delta(\mathbf{x}) := \delta^{-3}\rho(\mathbf{x}/\delta), \text{ where } \rho(\mathbf{x}) := \begin{cases} \eta \exp\left(-\frac{1}{1-|\mathbf{x}|^2}\right), & \text{if } |\mathbf{x}| < 1, \\ 0, & \text{if } |\mathbf{x}| \geq 1, \end{cases} \tag{3.14}$$

where η is chosen so that $\int_{\mathbb{R}^3} \rho(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$. We define a family of approximation operators $\{\mathcal{K}_\delta\}_{\delta>0}$ in the following way:

$$\mathcal{K}_\delta \mathbf{F} = \rho_{\delta\chi} \star (S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}), \qquad \forall \mathbf{F} \in \mathbf{L}^1(\Omega) \tag{3.15}$$

where χ is the constant introduced in (2.9).

Theorem 3.1. *$\mathcal{K}_\delta \mathbf{F}|_\Omega$ is in $\boldsymbol{\mathcal{C}}^\infty_0(\Omega)$ for all $\mathbf{F} \in \mathbf{L}^1(\Omega)$. Let ℓ be a positive integer. There exists a constant K, possibly depending on ℓ,such that the following estimates hold for any $0 < \delta < \frac{1}{2}$:*

$$\|\mathbf{F} - \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^s_0(\Omega)} \leq K\delta^{r-s}\|\mathbf{F}\|_{\mathbf{H}^r_0(\Omega)} \qquad 0 \leq s \leq r \leq 1 \tag{3.16}$$

$$\|\nabla\times\mathbf{F} - \nabla\times\mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^s(\Omega)} \leq K\delta^{r-s}\|\nabla\times\mathbf{F}\|_{\mathbf{H}^r(\Omega)} \qquad 0 \leq s \leq r < \tfrac{1}{2} \tag{3.17}$$

$$\|\mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^r(\Omega)} \leq K\chi^{s-r}\delta^{s-r}\|\mathbf{F}\|_{\mathbf{H}^s(\Omega)} \qquad 0 \leq s \leq r \leq \ell,\ s < \tfrac{1}{2} \tag{3.18}$$

and all $\mathbf{F} \in \mathbf{H}^r_0(\Omega)$, all $\mathbf{F} \in \mathbf{H}^r_{0,\mathrm{curl}}(\Omega)$, and all $\mathbf{F} \in \mathbf{H}^r(\Omega)$, respectively.

Proof. Owing to the properties of the mollification operator, we have $\mathcal{K}_\delta \mathbf{F}|_\Omega \in \boldsymbol{\mathcal{C}}^\infty(\Omega)$. We now prove that the support of $\mathcal{K}_\delta \mathbf{F}$ is compact in Ω. The definition of the convolution operation implies that the following holds for all $\mathbf{x} \in \mathbb{R}^3$:

$$\mathcal{K}_\delta \mathbf{F}(\mathbf{x}) = \int_{\mathbb{R}^3} (S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F})(\mathbf{y}) \rho_{\delta\chi}(\mathbf{x}-\mathbf{y})\, \mathrm{d}\mathbf{y} = \int_{\bar{\delta}\Omega} \mathbf{F}(\mathbf{y}/\bar{\delta}) \rho_{\delta\chi}(\mathbf{x}-\mathbf{y})\, \mathrm{d}\mathbf{y}.$$

If $\mathbf{x} \notin \bar{\delta}\Omega + B(0,\delta\chi)$, then for all $\mathbf{y} \in \bar{\delta}\Omega$, we have $\rho_{\delta\chi}(\mathbf{x}-\mathbf{y}) = 0$ and then $\mathcal{K}_\delta \mathbf{F}(\mathbf{x}) = 0$. As a result, $\mathcal{K}_\delta \mathbf{F}$ is supported in $\bar{\delta}\Omega + B(0,\delta\chi)$ which is compactly embedded in Ω owing to the assumption (2.9). Hence, $\mathcal{K}_\delta \mathbf{F} \in \boldsymbol{\mathcal{C}}^\infty_0(\Omega)$; in particular, we have $\mathcal{K}_\delta \mathbf{F} \in \mathbf{H}_{0,\mathrm{curl}}(\Omega)$. We now prove the estimates (3.16) to (3.18). Let us first consider $\mathbf{F} \in \mathbf{H}^r_0(\Omega)$. Using that $S^\delta_{\mathbb{R}^3} E_\Omega$ is stable from $\mathbf{H}^s_0(\Omega)$ to $\mathbf{H}^s_0(\mathbb{R}^3)$ together with standardapproximation properties of the mollification operator we deduce that there exists a uniform constant $K_3 > 0$ so that

$$\|S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F} - \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^s_0(\Omega)} \le K_3 (\delta\chi)^{r-s} \|S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}\|_{\mathbf{H}^r_0(\mathbb{R}^3)}, \qquad 0 \le s \le r \le 1.$$

Using the triangle inequality and Lemma 3.5 we have

$$\begin{aligned}\|\mathbf{F} - \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^s_0(\Omega)} &\le \|\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}\|_{\mathbf{H}^s_0(\Omega)} + \|S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F} - \mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^s_0(\Omega)} \\ &\le K_1 \delta^{r-s} \|\mathbf{F}\|_{\mathbf{H}^r_0(\Omega)} + K_3 \chi^{r-s} \delta^{r-s} \|S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}\|_{\mathbf{H}^r_0(\mathbb{R}^3)} \\ &\le (K_1 + 2K_3 \chi^{r-s}) \delta^{r-s} \|\mathbf{F}\|_{\mathbf{H}^r_0(\Omega)}.\end{aligned}$$

This proves (3.16) with $K = K_1 + 2K_3$ since $\chi \le 1$ and $s \le r$. Let us now consider $\mathbf{F} \in \mathbf{H}^r_{0,\mathrm{curl}}(\Omega)$. Using that $\nabla\times \mathcal{K}_\delta \mathbf{F} = \rho_{\delta\chi} \star \nabla\times (S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F})$, we infer that

$$\|\nabla\times (S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F} - \mathcal{K}_\delta \mathbf{F})\|_{\mathbf{H}^s(\Omega)} \le K_3 (\delta\chi)^{r-s} \|\nabla\times (S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} \quad 0 \le s \le r$$

Using the triangle inequality together with (3.10), Lemma 3.5, and assuming that $r < \frac{1}{2}$ we have

$$\begin{aligned}\|\nabla\times (\mathbf{F} - \mathcal{K}_\delta \mathbf{F})\|_{\mathbf{H}^s(\Omega)} &\le \|\nabla\times (\mathbf{F} - S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F})\|_{\mathbf{H}^s(\Omega)} + \|\nabla\times (S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F} - \mathcal{K}_\delta \mathbf{F})\|_{\mathbf{H}^s(\Omega)} \\ &\le K_2 \delta^{r-s} \|\nabla\times \mathbf{F}\|_{\mathbf{H}^r(\Omega)} + K_3 (\delta\chi)^{r-s} \|\nabla\times (S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F})\|_{\mathbf{H}^r(\mathbb{R}^3)} \\ &\le \delta^{r-s} (K_2 + K_3 \chi^{r-s}) \|\nabla\times \mathbf{F}\|_{\mathbf{H}^r(\Omega)},\end{aligned}$$

which proves (3.17) with $K = K_2 + K_3$ since $\chi \le 1$ and $s \le r$. Let us finally assume that $\mathbf{F} \in \mathbf{H}^r(\Omega)$. Using again the properties of the mollification operator, we infer that there exists $K_4(\ell)$ such that

$$\|\mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^r(\Omega)} \le \|\mathcal{K}_\delta \mathbf{F}\|_{\mathbf{H}^r(\mathbb{R}^3)} \le K_4 (\delta\chi)^{s-r} \|S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F}\|_{\mathbf{H}^s(\mathbb{R}^3)} \quad 0 \le s \le r \le \ell.$$

Applying (3.8), we obtain (3.18). Note that the assumption $s < \frac{1}{2}$ is required in order to ensure that $S^\delta_{\mathbb{R}^3} E_\Omega \mathbf{F} \in \mathbf{H}^s(\mathbb{R}^3)$. □

Remark 3.1. In the rest of the paper we will use (3.18) without mentioning the coefficient χ^{s-r} in the right hand sides. Indeed, we will use the inequality with r bounded from above by the polynomial degree of the approximation; as a result, χ^{s-r} is uniformly bounded.

We end this section by mentioning a key commuting property on $\mathcal{K}_\delta$.

Lemma 3.6. *The following holds for any* $\mathbf{F} \in \mathbf{H}_{0,\mathrm{curl}}(\Omega)$*:*

$$\bar{\delta} \nabla\times \mathcal{K}_\delta \mathbf{F} = \mathcal{K}_\delta (\nabla\times \mathbf{F}). \tag{3.19}$$

Proof. Owing to the properties of the convolution, the following holds for any $\mathbf{F} \in \mathbf{H}_{0,\mathrm{curl}}(\Omega)$:

$$\nabla\times\mathcal{K}_\delta\mathbf{F} = \rho_{\delta\chi} \star \left(\nabla\times\left(S^\delta_{\mathbb{R}^3} E_\Omega\mathbf{F}\right)\right).$$

Applying (3.9), we infer that

$$\begin{aligned}\nabla\times\mathcal{K}_\delta\mathbf{F} &= \rho_{\delta\chi} \star \left(\bar{\delta}^{-1} S^\delta_{\mathbb{R}^3} E_\Omega(\nabla\times\mathbf{F})\right)\\ &= \bar{\delta}^{-1}\rho_{\delta\chi} \star \left(S^\delta_{\mathbb{R}^3} E_\Omega(\nabla\times\mathbf{F})\right) = \bar{\delta}^{-1}\mathcal{K}_\delta(\nabla\times\mathbf{F}).\end{aligned}$$

This completes the proof. □

4. Finite Element Approximation of the boundary value problem

We introduce and study the stability properties of a Lagrange finite element technique for solving the boundary value problem (2.12).

4.1. **Finite Element Spaces.** We assume that the sub-domains Ω_i, $i = 1, \ldots, N$ are polyhedra. Let $\{\mathcal{T}_h\}_{h>0}$ be a shape regular sequence of affine meshes that we assume to be conforming in each sub-domain Ω_i, i.e., Σ is partitioned by a set of interface cells. We additionally assume that either it is possible to extract from each mesh $\mathcal{T}_h$ another one, say $\mathcal{G}_h$, that is globally conforming and of equivalent typical mesh size (this assumption is obviously true if $\mathcal{T}_h$ is globally conforming or if $\mathcal{T}_h$ is obtained from $\mathcal{G}_h$ after a few refinement step consisting of subdivisions), or each interface cell on one side of Σ is the union of interface cells from the other side of Σ, the cardinal number of this union being a priori bounded by a fixed number. An example of triangulation satisfying both geometric assumptions above is shown in Figure 1. We finally assume that the mesh sequence is quasi-uniform.

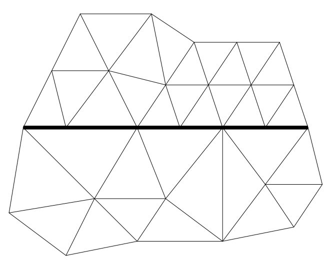

Figure 1. Example of an admissible mesh. The interface Σ is materialized by the thick line.

This assumption is non-essential and could be lifted by localizing some estimates; it is adopted here to simplify the presentation. The typical mesh size is denoted h.We introduce the following discrete space:

$$\mathbf{X}_h := \left\{ \mathbf{F} \in \prod_{i=1}^{N} \boldsymbol{\mathcal{C}}^0(\bar{\Omega}_i), \mid \forall K \in \mathcal{T}_h,\ \mathbf{F}_{|K} \in \mathbb{P}_{\ell-1} \right\} \tag{4.1}$$

where $\mathbb{P}_{\ell-1}$ denotes the vector space of vector-valued polynomial of total degree at most $\ell - 1$, $\ell \geq 2$. Note that the approximation space is non-conforming, i.e., $\mathbf{X}_h \not\subset \mathbf{H}_{0,\mathrm{curl}}(\Omega)$ and $\mathbf{X}_h \not\subset \mathbf{H}_{\mathrm{div}}(\Omega, \epsilon)$. We assume that the mesh sequence is such that there exists a family of local approximation operators $\mathcal{C}_h^l : \prod_{i=1}^N \mathbf{H}^\ell(\Omega_i) \to \mathbf{X}_h$ satisfying the following properties: there exists c uniform in h such that

$$\|\mathcal{C}_h^l \mathbf{F}\|_{\mathbf{H}^r(\Omega_i)} \le c\,\|\mathbf{F}\|_{\mathbf{H}^r(\Omega_i)}, \qquad 0 \le r < \tfrac{3}{2}, \tag{4.2}$$

$$\|\mathcal{C}_h^l \mathbf{F} - \mathbf{F}\|_{\mathbf{H}^t(\Omega_i)} \le c\,h^{r-t}\|\mathbf{F}\|_{\mathbf{H}^r(\Omega_i)}, \qquad 0 \le t \le r \le \ell,\ t < \tfrac{3}{2}, \tag{4.3}$$

for every $\mathbf{F} \in \prod_{i=1}^N \mathbf{H}^\ell(\Omega_i)$. We introduce $\|.\|^2_{\mathbf{L}^2(\mathcal{T}_h)} := \sum_{K\in\mathcal{T}_h} \|.\|^2_{\mathbf{L}^2(K)}$ and $\|.\|^2_{\mathbf{H}^s(\mathcal{T}_h)} := \sum_{K\in\mathcal{T}_h} \|.\|^2_{\mathbf{H}^s(K)}$. Owing to the quasi-uniformity assumption of the mesh sequence, we are going to regularly invoke various inverse inequalities like the following:

$$h\|\nabla\times\kappa\nabla\times\mathbf{F}_h\|_{\mathbf{L}^2(\mathcal{T}_h)} \le c\|\kappa\nabla\times\mathbf{F}_h\|_{\mathbf{L}^2(\mathcal{T}_h)}, \qquad \forall \mathbf{F}_h \in \mathbf{X}_h \tag{4.4}$$

$$h^s\|\kappa\nabla\times\mathbf{F}_h\|_{\mathbf{H}^s(\mathcal{T}_h)} \le c\|\kappa\nabla\times\mathbf{F}_h\|_{\mathbf{L}^2(\mathcal{T}_h)}, \qquad \forall \mathbf{F}_h \in \mathbf{X}_h, \tag{4.5}$$

$$h^{\frac{1}{2}}\|\mathbf{F}_h\|_{\mathbf{L}^2(\Sigma\cup\Gamma)} \le c\|\mathbf{F}_h\|_{\mathbf{L}^2(\mathcal{T}_h)}, \qquad \forall \mathbf{F}_h \in \mathbf{X}_h. \tag{4.6}$$

The assumptions adopted above for the mesh sequence imply that there exists a family of discrete subspaces $\mathbf{Y}_h$ such that $\mathbf{Y}_h \subset \mathbf{X}_h \cap \mathbf{H}_0^1(\Omega)$ and a family of global approximation operators $\mathcal{C}_h^g : \boldsymbol{\mathcal{C}}_0^\infty(\Omega) \longrightarrow \mathbf{Y}_h$ so that

$$\|\mathcal{C}_h^g \mathbf{F} - \mathbf{F}\|_{\mathbf{H}^t(\Omega)} \le c\,h^{r-t}\|\mathbf{F}\|_{\mathbf{H}^r(\Omega)}, \qquad 0 \le t \le r \le \ell,\ t < \tfrac{3}{2}, \tag{4.7}$$

for every $\mathbf{F} \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$. We additionally introduce the scalar-valued discrete space

$$M_h := \left\{q \in \mathrm{C}^0(\bar{\Omega}),\ |\ \forall K \in \mathcal{T}_h,\ q_{|K} \in \mathbb{P}_{\ell-1},\ q_{|\Gamma} = 0\right\} \subset \mathrm{H}_0^1(\Omega). \tag{4.8}$$

Again, the approximation theory of finite elements ensures that there exists an approximation operator $\mathcal{C}_h^p : \mathrm{H}_0^1(\Omega) \longrightarrow M_h$ satisfying the scalar counterparts of (4.2) and (4.3) for all $q \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^l(\Omega)$.

$$\|\mathcal{C}_h^p q\|_{\mathrm{H}^l(\Omega)} \le c\|q\|_{\mathrm{H}^l(\Omega)} \qquad 0 \le l \le \tfrac{3}{2} \tag{4.9}$$

$$\|\mathcal{C}_h^p q - q\|_{\mathrm{H}^t(\Omega)} \le ch^{l-t}\|q\|_{\mathrm{H}^l(\Omega)} \qquad 0 \le t \le l \le \ell,\ t < \tfrac{3}{2}. \tag{4.10}$$

Note that both $\mathbf{Y}_h$ and M_h can be constructed either by invoking the existence of the mesh sequence $\{\mathcal{G}_h\}_{h>0}$, or by constraining the possible hanging nodes on the interface Σ.

We denote $\mathcal{F}_h^i$ the set of the mesh interfaces: F is an interface if there are two elements in $\mathcal{T}_h$, say K_m and K_n so that $F = K_m \cap K_n$ and F is a $d-1$ manifold. We denote $\mathcal{F}_h^\partial$ the set of the boundary faces: F is a boundary face if there is an element in $\mathcal{T}_h$, say K_m so that $F = K_m \cap \Gamma$ and F is a $d-1$ manifold. To simplify the notation we also introduce $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. For any mesh interface $F \in \mathcal{F}_h^i$, $F = K_m \cap K_n$, and any function $\mathbf{v}$ whose restrictions over K_m and K_n are continuous, we define the tangent and normal jump of $\mathbf{v}$ across F by

$$[\![\mathbf{v}\times\mathbf{n}]\!](\mathbf{x}) := (\mathbf{v}_{|K_m}\times\mathbf{n}_m)(\mathbf{x}) + (\mathbf{v}_{|K_n}\times\mathbf{n}_n)(\mathbf{x}), \qquad \forall \mathbf{x} \in F, \tag{4.11}$$

$$[\![\mathbf{v}\cdot\mathbf{n}]\!](\mathbf{x}) := (\mathbf{v}_{|K_m}\cdot\mathbf{n}_m)(\mathbf{x}) + (\mathbf{v}_{|K_n}\cdot\mathbf{n}_n)(\mathbf{x}), \qquad \forall \mathbf{x} \in F, \tag{4.12}$$

where $\mathbf{n}_l$ is the unit outer normal to K_l. The average of $\mathbf{v}$ across across F is defined by

$$\{\!\{\mathbf{v}\}\!\}(\mathbf{x}) := \frac{1}{2}\left(\mathbf{v}_{|K_m}(\mathbf{x}) + \mathbf{v}_{|K_n}(\mathbf{x})\right), \qquad \forall \mathbf{x} \in F. \tag{4.13}$$

Whenever F is a boundary face we set $[\![\mathbf{v}\times\mathbf{n}]\!](\mathbf{x}) := \mathbf{v}_{|K_m}\times\mathbf{n}_m(\mathbf{x})$, $[\![\mathbf{v}\cdot\mathbf{n}]\!](\mathbf{x}) := \mathbf{v}_{|K_m}\cdot\mathbf{n}_m(\mathbf{x})$ and $\{\!\{\mathbf{v}\}\!\}(\mathbf{x}) := \mathbf{v}_{|K_m}(\mathbf{x})$.

Remark 4.1. Note that for any $\mathbf{F} \in \boldsymbol{\mathcal{C}}_0^\infty(\Omega)$, $\mathcal{C}_h^g\mathbf{F} \in \mathbf{H}_0^1(\Omega)$; in particular, we have $[\![\mathcal{C}_h^g\mathbf{F}\times\mathbf{n}]\!] = 0$ across all the interfaces in $\mathcal{F}_h^i$.

4.2. **Discrete formulation.** It will be useful to work with broken norms; for instance, we introduce $\Omega_\Sigma := \cup_{1\le i\le N}\Omega_i$ (recall that the domains Ω_i are open) together withthe following notation:

$$\|v\|^2_{\mathrm{H}^s(\Omega_\Sigma)} := \sum_{i=1}^N \|v\|^2_{\mathrm{H}^s(\Omega_i)}, \qquad (v,w)_{\Omega_\Sigma} := \sum_{i=1}^N \int_{\Omega_i} vw, \tag{4.14}$$

$$\|v\|^2_{\mathrm{L}^2(\Sigma\cup\Gamma)} := \|v\|^2_{\mathrm{L}^2(\Sigma)} + \|v\|^2_{\mathrm{L}^2(\Gamma)}, \qquad (v,w)_{\Sigma\cup\Gamma} := \int_\Sigma vw + \int_\Gamma vw. \tag{4.15}$$

We construct a discrete formulation of (2.12) by proceeding as in Bonito and Guermond [5]. Let $\alpha \in [0,1]$ be a parameter yet to be chosen. We define the following bilinear form $a_h : \mathbf{X}_h\times M_h \longrightarrow \mathbb{R}$,

$$\begin{aligned} a_h((\mathbf{E}_h,p_h),(\mathbf{F}_h,q_h)) :=\ & (\kappa\nabla\times\mathbf{E}_h,\nabla\times\mathbf{F}_h)_{\Omega_\Sigma} + (\{\!\{\kappa\nabla\times\mathbf{E}_h\}\!\},[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma} \\ & + \theta\,(\{\!\{\kappa\nabla\times\mathbf{F}_h\}\!\},[\![\mathbf{E}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma} + \gamma h^{-1}\,(\{\!\{\kappa\}\!\}\,[\![\mathbf{E}_h\times\mathbf{n}]\!],[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma} \\ & + (\varepsilon\nabla p_h,\mathbf{F}_h)_\Omega - (\varepsilon\mathbf{E}_h,\nabla q_h)_\Omega + c_\alpha\Big(h^{2\alpha}\,(\nabla\cdot(\varepsilon\mathbf{E}_h),\nabla\cdot(\varepsilon\mathbf{F}_h))_{\Omega_\Sigma} \\ & + h^{2(1-\alpha)}\,(\varepsilon\nabla p_h,\nabla q_h)_\Omega + h^{(2\alpha-1)}\,([\![\varepsilon\mathbf{E}_h\cdot\mathbf{n}]\!],[\![\varepsilon\mathbf{F}_h\cdot\mathbf{n}]\!])_\Sigma\Big), \end{aligned} \tag{4.16}$$

where γ, $c_\alpha > 0$, and $\theta \in \{-1,0,+1\}$ are user-defined parameters. We say that the formulation is anti-symmetric, incomplete, or symmetric depending whether θ is equal to -1, 0, or 1, respectively. The choice $\theta = 1$ ensures the adjoint consistency of the method. The term proportional to γ enforces the weak continuity of the tangent component of $\mathbf{E}$. The purpose of the term proportional to c_α is to penalize $\nabla\cdot(\varepsilon\mathbf{E}_h)$ in $\mathbf{H}^{-\alpha}(\Omega)$. The exponent α is somewhat similar to the exponent that is used in Costabel and Dauge [16] to define the L^2-weighted space that controls $\nabla\cdot(\varepsilon\mathbf{E}_h)$.

The discrete formulation considered in the rest of the paper consists of looking for $(\mathbf{E}_h,p_h) \in \mathbf{X}_h\times M_h$ such that the following holds for all $(\mathbf{F}_h,q_h) \in \mathbf{X}_h\times M_h$:

$$a_h\left((\mathbf{E}_h,p_h),(\mathbf{F}_h,q_h)\right) = (\varepsilon\mathbf{g},\mathbf{F}_h)_\Omega + c_\alpha h^{2(1-\alpha)}\,(\varepsilon\mathbf{g},\nabla q_h)_\Omega\,, \tag{4.17}$$

where $(\cdot,\cdot)_D$ henceforth denotes the scalar product in $\mathrm{L}^2(D)$.

To perform the consistency analysis of the method we are led to introduce

$$\mathbf{Z}^s(\Omega) = \{\mathbf{F} \in \mathbf{H}^s_{0,\mathrm{curl}}(\Omega)\,;\, \nabla\times(\kappa\nabla\times\mathbf{F}) \in \mathbf{L}^2(\Omega),\ \nabla\cdot(\varepsilon\mathbf{F}) \in \mathbf{L}^2(\Omega)\}. \tag{4.18}$$

Owing to Theorem 2.1, it is a priori known that there exists $s > 0$ such that the solution to the boundary value problem (2.12) is in $\mathbf{Z}^s(\Omega)\cap\mathbf{H}^s(\Omega)$. We shall use the notation $\mathbf{Z}^s$ instead of $\mathbf{Z}^s(\Omega)$ when the context is unambiguous.

Proposition 4.1. *Assuming* (2.15)*, it is possible to extend the bilinear form* $a_h(.,.)$ *to* $\left[(\mathbf{Z}^s + \mathbf{X}_h)\times\mathrm{H}^1_0(\Omega)\right]^2$ *for all* $s > 0$.

Proof. Note first that $M_h \subset \mathrm{H}^1_0(\Omega)$ and the extension of the bilinear form to scalar fields in $\mathrm{H}^1_0(\Omega)$ does not pose any difficulty. We decompose a_h into three pieces:

$$
\begin{aligned}
a_{0h}((\mathbf{E}_h,p_h),(\mathbf{F}_h,q_h)) &:= (\{\!\{\kappa\nabla\times\mathbf{E}_h\}\!\}, [\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma} + \theta\,(\{\!\{\kappa\nabla\times\mathbf{F}_h\}\!\}, [\![\mathbf{E}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}\\
a_{1h}((\mathbf{E}_h,p_h),(\mathbf{F}_h,q_h)) &:= (\kappa\nabla\times\mathbf{E}_h, \nabla\times\mathbf{F}_h)_{\Omega_\Sigma} + \gamma h^{-1}\,(\{\!\{\kappa\}\!\}\,[\![\mathbf{E}_h\times\mathbf{n}]\!], [\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}\\
a_{2h}((\mathbf{E}_h,p_h),(\mathbf{F}_h,q_h)) &:= c_\alpha\Big(h^{2\alpha}\,(\nabla\cdot(\varepsilon\mathbf{E}_h), \nabla\cdot(\varepsilon\mathbf{F}_h))_{\Omega_\Sigma} + h^{2(1-\alpha)}\,(\varepsilon\nabla p_h, \nabla q_h)_\Omega\\
&\quad + h^{(2\alpha-1)}\,([\![\varepsilon\mathbf{E}_h\cdot\mathbf{n}]\!], [\![\varepsilon\mathbf{F}_h\cdot\mathbf{n}]\!])_\Sigma\Big) + (\varepsilon\nabla p_h, \mathbf{F}_h)_\Omega - (\varepsilon\mathbf{E}_h, \nabla q_h)_\Omega\,.
\end{aligned}
$$

The bilinear form a_{1h} can clearly be extended to $\big[(\mathbf{Z}^s+\mathbf{X}_h)\times\mathrm{H}^1_0(\Omega)\big]^2$, since every function $\mathbf{E}$ in $\mathbf{Z}^s$ is such that $[\![\mathbf{E}\times\mathbf{n}]\!]_{\Sigma\cup\Gamma}$ is zero. Hence, if either $(\mathbf{E},\mathbf{F})\in\mathbf{Z}^s\times(\mathbf{Z}^s+\mathbf{X}_h)$ or $(\mathbf{E},\mathbf{F})\in(\mathbf{Z}^s+\mathbf{X}_h)\times\mathbf{Z}^s$, we set

$$a_{1h}((\mathbf{E},p),(\mathbf{F},q)) := (\kappa\nabla\times\mathbf{E}, \nabla\times\mathbf{F})_{\Omega_\Sigma}\,,$$

for all $(p,q)\in\mathrm{H}^1_0(\Omega)$. The bilinear form a_{2h} can also be extended to $\big[(\mathbf{Z}^s+\mathbf{X}_h)\times\mathrm{H}^1_0(\Omega)\big]^2$, since every function $\mathbf{E}$ in $\mathbf{Z}^s$ is such that $[\![\varepsilon\mathbf{E}\cdot\mathbf{n}]\!]_\Sigma$ is zero. Hence, if either $(\mathbf{E},\mathbf{F})\in\mathbf{Z}^s\times(\mathbf{Z}^s+\mathbf{X}_h)$ or $(\mathbf{E},\mathbf{F})\in(\mathbf{Z}^s+\mathbf{X}_h)\times\mathbf{Z}^s$, we set

$$
\begin{aligned}
a_{2h}((\mathbf{E},p),(\mathbf{F},q)) := c_\alpha\Big(&h^{2\alpha}\,(\nabla\cdot(\varepsilon\mathbf{E}), \nabla\cdot(\varepsilon\mathbf{F}))_{\Omega_\Sigma} + h^{2(1-\alpha)}\,(\varepsilon\nabla p, \nabla q)_\Omega\Big)\\
&+ (\varepsilon\nabla p, \mathbf{F})_\Omega - (\varepsilon\mathbf{E}, \nabla q)_\Omega\,.
\end{aligned}
$$

for all $(p,q)\in\mathrm{H}^1_0(\Omega)$.

The question of the extension of a_{0h} is more subtle, and we must now distinguish the trial and test spaces. We are going to use Lemma A.3 to show that the bilinear form $(\mathbf{H}^s(\Omega)\cap\mathbf{H}_{\mathrm{curl}}(\Omega))\times\mathbf{X}_h \ni (\boldsymbol{\phi},\mathbf{F}_h)\longmapsto(\int_F\boldsymbol{\phi}\cdot(\mathbf{F}_{h|K_m}\times\mathbf{n}_m), \int_F\boldsymbol{\phi}\cdot(\mathbf{F}_{h|K_n}\times\mathbf{n}_n))\in\mathbb{R}^2$ is well defined for all $F=K_m\cap K_n\in\mathcal{F}_h^i$, with the obvious equivalent statement if $F\in\mathcal{F}_h^\partial$.Let $\mathbf{E}$ be a member of $\mathbf{Z}^s$, then $\nabla\times\mathbf{E}\in\mathbf{H}^s(\Omega)$, $s>0$ and in particular, $\nabla\times\mathbf{E}\in\mathbf{H}^\sigma(\Omega)$ for some $\sigma\in(0,\frac{1}{2})$. Owing to (2.15), $\kappa\in\mathrm{W}^{1,\infty}_\Sigma(\Omega)$ so that $\kappa\nabla\times\mathbf{E}\in\mathbf{H}^\sigma(\Omega)$, see e.g. Bonito et al. [6]. Note in addition that $\mathbf{E}$ being a member of $\mathbf{Z}^s$ implies that $\nabla\times(\kappa\nabla\times\mathbf{E})\in\mathbf{L}^2(\Omega)$, which in turn also implies that $\{\!\{\kappa\nabla\times\mathbf{E}\}\!\}_{|\Sigma}=\kappa\nabla\times\mathbf{E}_{|\Sigma}$. Hence, Lemma A.3 shows that the expressions $\int_F\kappa\nabla\times\mathbf{E}\cdot(\mathbf{F}_{h|K_m}\times\mathbf{n}_m)$, $\int_F\kappa\nabla\times\mathbf{E}\cdot(\mathbf{F}_{h|K_n}\times\mathbf{n}_n)$ are meaningfulfor all $F\in\mathcal{F}_h$ and for all $(\mathbf{E},\mathbf{F}_h)\in\mathbf{Z}^s\times\mathbf{X}_h$. The extension of a_{0h} for $(\mathbf{E}_h,\mathbf{F})\in\mathbf{X}_h\times\mathbf{Z}^s$ is justified similarly. The extension of a_{0h} for $(\mathbf{E},\mathbf{F})\in\mathbf{Z}^s\times\mathbf{Z}^s$ is trivial since the tangent jumps of $\mathbf{E}$ and $\mathbf{F}$ across F are zero. Summing up, a_{0h} can be extended to $\big[(\mathbf{Z}^s+\mathbf{X}_h)\times\mathrm{H}^1_0(\Omega)\big]^2$ by setting

$$
\begin{aligned}
a_{0h}((\mathbf{E}+\mathbf{E}_h,p),(\mathbf{F}+\mathbf{F}_h,q)) := {}&(\kappa\nabla\times\mathbf{E}, [\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma} + (\{\!\{\kappa\nabla\times\mathbf{E}_h\}\!\}, [\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}\\
&+ \theta\,(\kappa\nabla\times\mathbf{F}, [\![\mathbf{E}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma} + \theta\,(\{\!\{\kappa\nabla\times\mathbf{F}_h\}\!\}, [\![\mathbf{E}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}\,,
\end{aligned}
$$

for all $(\mathbf{E},\mathbf{E}_h)\in\mathbf{Z}^s\times\mathbf{X}_h$, all $(\mathbf{F},\mathbf{F}_h)\in\mathbf{Z}^s\times\mathbf{X}_h$, and all $(p,q)\in\mathrm{H}^1_0(\Omega)$. This ends the proof. □

Remark 4.1. *One could avoid invoking Lemma A.3 in the above proof by using instead a result from Buffa and Ciarlet [9] where it is shown that the bilinear form* $\mathbf{H}_{\mathrm{curl}}(\Omega)\times\mathbf{H}_{\mathrm{curl}}(\Omega)\ni(\boldsymbol{\phi},\mathbf{F})\longmapsto\int_F\boldsymbol{\phi}\cdot(\mathbf{F}\times\mathbf{n})\in\mathbb{R}$ *is well defined and continuous for all* $F\in\mathcal{F}_h$

Remark 4.2 (Continuous Approximation of p). Observe that the approximation of the Lagrange multiplier p is globally continuous. This is critical to derive a global control of $\nabla\cdot(\varepsilon\mathbf{E})$ in $\mathbf{H}^{-\alpha}(\Omega)$ (encoded in the bilinear form a_{2h} in the above proof) instead of $\prod_{i=1}^{N}\mathbf{H}^{-\alpha}(\Omega_i)$. We refer to Bonito and Guermond [5] for more precisions.

Lemma 4.1. *Assume* (2.15) *and let* $(\mathbf{E},p)$ *be the solution of* (2.12). *Let* $s>0$ *be such that* $\mathbf{E}\in\mathbf{Z}^s$. *The following holds for any* $(\mathbf{F}+\mathbf{F}_h,q)\in(\mathbf{Z}^s+\mathbf{X}_h)\times\mathrm{H}^1_0(\Omega)$*:*

$$a_h\left((\mathbf{E},p),(\mathbf{F}+\mathbf{F}_h,q)\right)=\left(\varepsilon\mathbf{g},\mathbf{F}+\mathbf{F}_h\right)_\Omega+c_\alpha h^{2(1-\alpha)}\left(\varepsilon\mathbf{g},\nabla q\right)_\Omega.$$

Proof. Let us first observe that

$$a_h\left((\mathbf{E},p),(\mathbf{F}+\mathbf{F}_h,q)\right)=(\kappa\nabla\times\mathbf{E},\nabla\times(\mathbf{F}+\mathbf{F}_h))_{\Omega_\Sigma}+(\kappa\nabla\times\mathbf{E},[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}$$
$$+(\varepsilon\nabla p,\mathbf{F}+\mathbf{F}_h)_\Omega+c_\alpha h^{2(1-\alpha)}(\varepsilon\nabla p,\nabla q)_\Omega,$$

where all the terms make sense owing to the extension properties of a_h stated in Proposition 4.1. We now test (2.12) with $\mathbf{F}+\mathbf{F}_h\in(\mathbf{Z}^s+\mathbf{X}_h)$,

$$(\nabla\times(\kappa\nabla\times\mathbf{E}),\mathbf{F})_\Omega+\sum_{i=1}^{N}(\nabla\times(\kappa\nabla\times\mathbf{E}),\mathbf{F}_h)_{\Omega_i}+(\varepsilon\nabla p,\mathbf{F}+\mathbf{F}_h)_\Omega=(\varepsilon\mathbf{g},\mathbf{F}+\mathbf{F}_h)_\Omega,$$

and we perform the integration by parts over Ω when the test function is $\mathbf{F}$ and over each sub-domain when the test function is $\mathbf{F}_h$,

$$(\kappa\nabla\times\mathbf{E},\nabla\times\mathbf{F})_\Omega+\sum_{i=1}^{N}(\kappa\nabla\times\mathbf{E},\nabla\times\mathbf{F}_h)_{\Omega_i}+(\kappa\nabla\times\mathbf{E},[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}+(\varepsilon\nabla p,\mathbf{F}+\mathbf{F}_h)_\Omega=(\varepsilon\mathbf{g},\mathbf{F}+\mathbf{F}_h)_\Omega.$$

Note that the term $(\kappa\nabla\times\mathbf{E},[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}$ is meaningful owing to Lemma A.3 and $\mathbf{E}$ being a member of $\mathbf{Z}^s$. This implies that

$$a_h\left((\mathbf{E},p),(\mathbf{F}+\mathbf{F}_h,q)\right)=(\varepsilon\mathbf{g},\mathbf{F}+\mathbf{F}_h)_\Omega+c_\alpha h^{2(1-\alpha)}(\varepsilon\nabla p,\nabla q)_\Omega.$$

Upon testing again (2.12) with ∇q, $q\in\mathrm{H}^1_0(\Omega)$, we infer that $(\varepsilon\nabla p,\nabla q)_\Omega=(\varepsilon\mathbf{g},\nabla q)_\Omega$, which in turn implies the desired result. □

4.3. **Well posedness of the discrete formulation.** We discuss in this section the existence and uniqueness of a solution $(\mathbf{E}_h,p_h)$ to (4.17). This issue is addressed by equipping $\mathbf{X}_h\times M_h$ with the following discrete norm:

$$\begin{aligned}\|\mathbf{F}_h,q_h\|_h^2:=&\|\kappa^{\frac12}\nabla\times\mathbf{F}_h\|^2_{\mathbf{L}^2(\Omega_\Sigma)}+\gamma h^{-1}\|\{\!\{\kappa\}\!\}^{\frac12}[\![\mathbf{F}_h\times\mathbf{n}]\!]\|^2_{\mathrm{L}^2(\Sigma\cup\Gamma)}\\&+c_\alpha\Big(h^{2\alpha}\|\nabla\cdot(\varepsilon\mathbf{F}_h)\|^2_{\mathrm{L}^2(\Omega_\Sigma)}+h^{2(1-\alpha)}\|\varepsilon^{\frac12}\nabla q_h\|^2_{\mathbf{L}^2(\Omega)}\\&+h^{(2\alpha-1)}\|[\![\varepsilon\mathbf{F}_h\cdot\mathbf{n}]\!]\|^2_{\mathbf{L}^2(\Sigma)}\Big),\end{aligned}\tag{4.19}$$

by proving a coercivity property, uniform in h, and by establishing some continuity estimates for the bilinear form $a_h(.,.)$. Notice that we do not include the L^2-norm in the discrete norm since this quantity is better handled by a duality argument. We postpone this discussion to Section 5.2.

We first establish the coercivity of a_h.

Proposition 4.2 (Coercivity)**.** *If* $\theta\in\{0,1\}$, *there exists* $\gamma_0>0$ *and* $c(\gamma_0)>0$, *uniform with respect to* h, *so that the following holds for all* $\gamma\geq\gamma_0$ *and for any* $0\leq\alpha\leq1$*:*

$$a_h((\mathbf{E}_h,p_h),(\mathbf{E}_h,p_h))\geq c(\gamma_0)\|\mathbf{E}_h,p_h\|_h^2,\qquad\forall(\mathbf{E}_h,p_h)\in\mathbf{X}_h\times M_h,\tag{4.20}$$

and this inequality holds for all $\gamma > 0$ with $c(\gamma_0) = 1$ if $\theta = -1$.

Proof. We first observe that

$$a_h((\mathbf{E}_h, p_h), (\mathbf{E}_h, p_h)) = \|\mathbf{E}_h, p_h\|_h^2 + (1+\theta)\left(\{\!\{\kappa\nabla\times\mathbf{E}_h\}\!\}, [\![\mathbf{E}_h\times\mathbf{n}]\!]\right)_{\Sigma\cup\Gamma}.$$

The conclusion is evident if $\theta = -1$. Otherwise we have to control the term $(\{\!\{\kappa\nabla\times\mathbf{E}_h\}\!\}, [\![\mathbf{E}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}$. Invoking the inverse trace inequality (4.6) and the inequality $ab \leq \frac{1}{4}a^2 + b^2$, we deduce that there exists a constant c_0 only depending on the trace inequality constant and the ratio $\kappa_{\max}/\kappa_{\min}$ such that

$$\left|\left(\{\!\{\kappa\nabla\times\mathbf{F}_h\}\!\}, [\![\mathbf{F}_h\times\mathbf{n}]\!]\right)_{\Sigma\cup\Gamma}\right| \leq \frac{1}{4}\|\kappa^{\frac{1}{2}}\nabla\times\mathbf{F}_h\|^2_{\mathrm{L}^2(\Omega_\Sigma)} + c_0 h^{-1}\|\{\!\{\kappa\}\!\}^{\frac{1}{2}}[\![\mathbf{F}_h\times\mathbf{n}]\!]\|^2_{\mathrm{L}^2(\Sigma\cup\Gamma)}.$$

Hence, if $\gamma \geq \gamma_0 := 4c_0$, we infer that the following holds:

$$a_h((\mathbf{E}_h, p_h), (\mathbf{E}_h, p_h)) \geq \frac{1}{2}\|\mathbf{E}_h, p_h\|_h^2 \geq 0. \tag{4.21}$$

This completes the proof. □

We now establish the uniform boundedness of the bilinear form a_h.

Proposition 4.3 (Continuity)**.** *For any $s \in \left(0, \frac{1}{2}\right)$, there is $c > 0$, uniform in h such that the following holds for any $0 \leq \alpha \leq 1$ and for every $(\mathbf{E}, p) \in \mathbf{Z}^s\times\mathrm{H}^1_0(\Omega)$ and $(\mathbf{G}_h, d_h), (\mathbf{F}_h, q_h) \in \mathbf{X}_h\times M_h$:*

$$\begin{aligned} c\,\frac{|a_h((\mathbf{E}-\mathbf{G}_h, p-d_h), (\mathbf{F}_h, q_h))|}{\|\mathbf{F}_h, q_h\|_h} &\leq \mathbf{a}\|\mathbf{E}-\mathbf{G}_h, p-d_h\|_h + h^{\alpha-1}\|\mathbf{E}-\mathbf{G}_h\|_{\mathbf{L}^2(\Omega)} \\ &\quad + h^s\|\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\|_{\mathbf{H}^s(\mathcal{T}_h)} + h\|\nabla\times\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\|_{\mathbf{L}^2(\mathcal{T}_h)} \\ &\quad + h^{-\alpha}\|p-d_h\|_{\mathrm{L}^2(\Omega)} + h^{(\frac{1}{2}-\alpha)}\|p-d_h\|_{\mathrm{L}^2(\Sigma)}. \end{aligned} \tag{4.22}$$

Proof. Upon applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} &(\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h), \nabla\times\mathbf{F}_h)_{\Omega_\Sigma} + \gamma h^{-1}\left(\{\!\{\kappa\}\!\}[\![(\mathbf{E}-\mathbf{G}_h)\times\mathbf{n}]\!], [\![\mathbf{F}_h\times\mathbf{n}]\!]\right)_{\Sigma\cup\Gamma} \\ &\quad + c_\alpha\Big(h^{2\alpha}\left(\nabla\cdot(\varepsilon(\mathbf{E}-\mathbf{G}_h)), \nabla\cdot(\varepsilon\mathbf{F}_h)\right)_{\Omega_\Sigma} + h^{2(1-\alpha)}\left(\varepsilon\nabla(p-d_h), \nabla q_h\right)_\Omega \\ &\quad + h^{(2\alpha-1)}\left([\![\varepsilon(\mathbf{E}-\mathbf{G}_h)\cdot\mathbf{n}]\!], [\![\varepsilon\mathbf{F}_h\cdot\mathbf{n}]\!]\right)_\Sigma\Big) \\ &\quad \leq \|\mathbf{F}_h, q_h\|_h\|\mathbf{E}-\mathbf{G}_h, p-d_h\|_h. \end{aligned}$$

We now bound separately the remaining terms appearing in the definition (4.16) of $a_h(.,.)$:

$$\begin{aligned} |-(\varepsilon(\mathbf{E}-\mathbf{G}_h), \nabla q_h)_\Omega| &\leq \|\varepsilon\|_{L^\infty(\Omega)}h^{\alpha-1}\|\nabla q_h\|_{\mathbf{L}^2(\Omega)}h^{1-\alpha}\|\mathbf{E}-\mathbf{G}_h\|_{\mathbf{L}^2(\Omega)}, \\ (\varepsilon\nabla(p-d_h), \mathbf{F}_h)_\Omega &\leq h^\alpha\|\nabla\cdot(\varepsilon\mathbf{F}_h)\|_{\mathrm{L}^2(\Omega_\Sigma)}h^{-\alpha}\|p-d_h\|_{\mathrm{L}^2(\Omega)} \\ &\quad + h^{(\alpha-\frac{1}{2})}\|[\![\varepsilon\mathbf{F}_h\cdot\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma)}h^{(\frac{1}{2}-\alpha)}\|p-d_h\|_{\mathrm{L}^2(\Sigma)}, \end{aligned}$$

where we used an integration by parts for the second estimate. We are now left with the consistency terms

$$\left(\{\!\{\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\}\!\}, [\![\mathbf{F}_h\times\mathbf{n}]\!]\right)_{\Sigma\cup\Gamma} + \theta\left(\{\!\{\kappa\nabla\times\mathbf{F}_h\}\!\}, [\![(\mathbf{E}-\mathbf{G}_h)\times\mathbf{n}]\!]\right)_{\Sigma\cup\Gamma}. \tag{4.23}$$

For the first term in (4.23), we apply Lemma A.3 with $\mathbf{v} = [\![\mathbf{F}_h\times\mathbf{n}]\!]$, which is a polynomial of degree $\ell - 1$, and $\boldsymbol{\phi} = \{\!\{\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\}\!\}$. Then for any $F \in \mathcal{F}_h$, we infer that

$$
\begin{aligned}
|\left(\{\!\!\{\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\}\!\!\}, [\![\mathbf{F}_h\times\mathbf{n}]\!]\right)_F| \le c h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(F)} \\
\times\sum_{i=1}^{2}\Big(h^s\|\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\|_{\mathbf{H}^s(K_i)} + h\|\nabla\times\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\|_{\mathbf{L}^2(K_i)} \\
+ \|\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\|_{\mathbf{L}^2(K_i)}\Big),
\end{aligned}
$$

where $K_1, K_2 \in \mathcal{T}_h$ such that $F = \overline{K_1}\cap\overline{K_2}$. Hence, summing over all the faces we arrive at

$$
\begin{aligned}
|(\{\!\!\{\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\}\!\!\}, [\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}| \le c\, h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}\Big(h^s\|\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\|_{\mathbf{H}^s(\mathcal{T}_h)} \\
+ h\|\nabla\times\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\|_{\mathbf{L}^2(\mathcal{T}_h)} + \|\kappa\nabla\times(\mathbf{E}-\mathbf{G}_h)\|_{\mathbf{L}^2(\mathcal{T}_h)}\Big).
\end{aligned}
$$

For the second term in (4.23) we notice that $[\![(\mathbf{E}-\mathbf{G}_h)\times\mathbf{n}]\!] = -[\![\mathbf{G}_h\times\mathbf{n}]\!]$ owing to the regularity of $\mathbf{E}$. Then by using Lemma A.3 again, we arrive at

$$
\begin{aligned}
|\left(\{\!\!\{\kappa\nabla\times\mathbf{F}_h\}\!\!\}, [\![(\mathbf{E}-\mathbf{G}_h)\times\mathbf{n}]\!]\right)_{\Sigma\cup\Gamma}| &\le c h^{-\frac{1}{2}}\|[\![\mathbf{G}_h\times\mathbf{n}]\!]\|_{\mathrm{L}^2(\Sigma\cup\Gamma)}\|\kappa\nabla\times\mathbf{F}_h\|_{\mathrm{L}^2(\mathcal{T}_h)} \\
&\le c h^{-\frac{1}{2}}\|[\![(\mathbf{E}-\mathbf{G}_h)\times\mathbf{n}]\!]\|_{\mathrm{L}^2(\Sigma\cup\Gamma)}\|\kappa\nabla\times\mathbf{F}_h\|_{\mathrm{L}^2(\mathcal{T}_h)},
\end{aligned}
$$

where we used the inverse inequalities (4.4), (4.5). The desired result is obtained by gathering the above estimates. □

The following result will be instrumental to apply the Nitsche-Aubin duality argument and derive a convergence result in $\mathbf{L}^2(\Omega)$.

Proposition 4.4 (Adjoint continuity)**.** *For any $s \in \left(0, \frac{1}{2}\right)$, there is $c > 0$, uniform in h such that for any $0 \le \alpha \le 1$, the following holds for every $(\mathbf{E}, p), (\mathbf{F}, q) \in \mathbf{Z}^s\times\mathrm{H}^1_0(\Omega)$, $\mathbf{F}_h \in \mathbf{Y}_h$, $q_h \in M_h$ and $(\mathbf{G}_h, d_h) \in \mathbf{X}_h\times M_h$:*

$$
\begin{aligned}
c\,\frac{|a_h\left((\mathbf{E}-\mathbf{G}_h, p-d_h), (\mathbf{F}-\mathbf{F}_h, q-q_h)\right)|}{\|\mathbf{E}-\mathbf{G}_h, p-d_h\|_h} &\le \|\mathbf{F}-\mathbf{F}_h, q-q_h\|_h + h^{\alpha-1}\|\mathbf{F}-\mathbf{F}_h\|_{\mathbf{L}^2(\Omega)} \\
&\quad + h^s\|\kappa\nabla\times(\mathbf{F}-\mathbf{F}_h)\|_{\mathrm{H}^s(\mathcal{T}_h)} \\
&\quad + h\|\nabla\times\kappa\nabla\times(\mathbf{F}-\mathbf{F}_h)\|_{\mathrm{L}^2(\mathcal{T}_h)} \\
&\quad + h^{-\alpha}\|q-q_h\|_{\mathrm{L}^2(\Omega)} + h^{(\frac{1}{2}-\alpha)}\|q-q_h\|_{\mathrm{L}^2(\Sigma)}.
\end{aligned}
\tag{4.24}
$$

Proof. The proof proceeds similarly as in the proof of Proposition 4.3. The only difference here is that we have $(\{\!\!\{\nabla\times(\mathbf{E}-\mathbf{G}_h)\}\!\!\}, [\![(\mathbf{F}-\mathbf{F}_h)\times\mathbf{n}]\!])_{\Sigma\cup\Gamma} = 0$, owing to the assumption that $\mathbf{F}_h \in \mathbf{Y}_h \subset \mathbf{X}_h \cap \mathbf{H}^1_0(\Omega)$. This identity makes the analysis of the consistency term (4.23) tractable. □

5. Convergence analysis for the boundary value problem

In the first part of this section, we prove two convergence results for the discrete problem (4.17) using the discrete norm $\|\cdot\|_h$, one assuming minimal regularity and the other assuming full smoothness. In the second part of the section we use a Nitsche-Aubin duality argument to establish convergence in $\mathbf{L}^2(\Omega)$. The performance of the method is numerically illustrated at the end of the section.

5.1. **Convergence in the discrete norm.** We assume first that the solution to the boundary value problem (2.12) has minimal regularity properties, and we start with the Galerkin orthogonality.

Lemma 5.1 (Galerkin Orthogonality)**.** *Assume* (2.15)*, then the Galerkin orthogonality holds, i.e., let* $(\mathbf{E}, p)$ *be the solution of* (2.12) *and* $(\mathbf{E}_h, p_h)$ *be the solution of* (4.17)*, then for any* $(\mathbf{F}_h, q_h) \in \mathbf{X}_h \times M_h$

$$a_h\left((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{F}_h, q_h)\right) = 0. \tag{5.1}$$

Proof. This is a direct consequence of Lemma 4.1 and formulation (4.17). □

Theorem 5.1. *Let* $\mathbf{g} \in \mathbf{L}^2(\Omega)$ *and* $\tau \in (0, \min(\tau_\varepsilon, \tau_\kappa))$ *where* τ_ε *and* τ_κ *are defined in Theorem 2.1. Let* $(\mathbf{E}, p)$ *and* $(\mathbf{E}_h, p_h)$ *be the solution of* (2.12) *and* (4.17)*, respectively. Then, for any* $\alpha \in \left(\frac{\ell(1-\tau)}{\ell-\tau}, 1\right]$*, there exists* $c > 0$*, uniform in* h*, such that*

$$\|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h \leq c h^r \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \tag{5.2}$$

where $r = \alpha - 1 + \tau\left(1 - \frac{\alpha}{\ell}\right)$ *if* $\nabla\cdot(\varepsilon\mathbf{g}) = 0$ *and* $r = \min\left(1 - \alpha, \alpha - 1 + \tau\left(1 - \frac{\alpha}{\ell}\right)\right)$ *otherwise.*

Proof. We first recall that, owing to Theorem 2.1, we have $\mathbf{E} \in \mathbf{H}^\tau(\Omega) \cap \mathbf{H}^\tau_{0,\mathrm{curl}}(\Omega)$, together with the estimates

$$\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} + \|\nabla\times\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} + \|\nabla\times(\kappa\nabla\times\mathbf{E})\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{\mathbf{L}^2(\Omega)} \leq c\,\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

We establish (5.2) by using the triangular inequality

$$\begin{aligned}\|\mathbf{E} - \mathbf{E}_h, p - p_h\|_h \leq \|\mathbf{E} - \mathcal{K}_\delta\mathbf{E}, 0\|_h + \|\mathcal{K}_\delta\mathbf{E} - \mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}, p - \mathcal{C}_h^p p\|_h \\ + \|\mathcal{C}_h\mathcal{K}_\delta\mathbf{E} - \mathbf{E}_h, \mathcal{C}_h^p p - p_h\|_h,\end{aligned}$$

for some $\delta > 0$ to be defined later, and by bounding from above the three terms separately.

Using the definition of $\|\cdot\|_h$ together with the approximation properties of $\mathcal{K}_\delta$, cf. (3.16)-(3.17)-(3.18), we conclude that

$$\|\mathbf{E} - \mathcal{K}_\delta\mathbf{E}, 0\|_h \leq c\left(\delta^\tau\|\nabla\times\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} + h^\alpha\delta^{\tau-1}\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} + h^{\alpha-\frac{1}{2}}\|\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{L}^2(\Sigma)}\right).$$

Note that the estimate (3.17) is critical to obtain a bound that depends only on $\|\nabla\times\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}$ instead of $\|\mathbf{E}\|_{\mathbf{H}^{1+\tau}(\Omega)}$. To estimate the last term in the above inequality, we apply (A.5) with $\Theta = \frac{1-2\tau}{2(1-\tau)}$,

$$\begin{aligned}h^{\alpha-\frac{1}{2}}\|\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{L}^2(\Sigma)} &\leq c h^{\alpha-\frac{1}{2}}\|\mathcal{K}_\delta\mathbf{E}\|^{1-\Theta}_{\mathbf{H}^\tau(\Omega)}\|\mathcal{K}_\delta\mathbf{E}\|^\Theta_{\mathbf{H}^1(\Omega)} \\ &\leq c h^{\alpha-\frac{1}{2}}\delta^{\Theta(\tau-1)}\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)} \leq c h^{\alpha-\frac{1}{2}}\delta^{\tau-\frac{1}{2}}\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}.\end{aligned}$$

Finally, we arrive at

$$\|\mathbf{E} - \mathcal{K}_\delta\mathbf{E}, 0\|_h \leq c\left(\delta^\tau + h^\alpha\delta^{\tau-1} + h^{\alpha-\frac{1}{2}}\delta^{\tau-\frac{1}{2}}\right)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{5.3}$$

Let us now turn our attention to $\|\mathcal{K}_\delta\mathbf{E} - \mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}, p - \mathcal{C}_h^p p\|_h$. Owing to the definition of $\mathcal{C}_h^g$ and the regularity of $\mathcal{K}_\delta\mathbf{E}$, we have $\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E} \in \mathbf{H}^1_0(\Omega) \subset \mathbf{H}_{0,\mathrm{curl}}(\Omega)$, so that we only have four terms to bound (the jumps of $\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}$ across the mesh interfaces and the tangent trace on Γ are zero, cf. Remark 4.1). Using the properties of $\mathcal{K}_\delta$ and $\mathcal{C}_h^g$ together with (A.4) we deduce that

$$\begin{aligned}
\|\kappa^{\frac{1}{2}}\nabla\times(\mathcal{K}_\delta\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E})\|_{\mathbf{L}^2(\Omega)} &\le c\,h^{\ell-1}\|\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{H}^\ell(\Omega)} \le ch^{\ell-1}\delta^{\tau-\ell}\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)},\\
h^\alpha\|\nabla\cdot\left(\varepsilon(\mathcal{K}_\delta\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E})\right)\|_{\mathrm{L}^2(\Omega_\Sigma)} &\le c\,h^{\alpha+\ell-1}\|\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{H}^\ell(\Omega_\Sigma)} \le ch^{\alpha+\ell-1}\delta^{\tau-\ell}\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)},\\
h^{1-\alpha}\|\varepsilon^{\frac{1}{2}}\nabla(p-\mathcal{C}_h^p p)\|_{\mathbf{L}^2(\Omega)} &\le c\,h^{1-\alpha}\|p\|_{\mathrm{H}_0^1(\Omega)},\\
h^{\alpha-\frac{1}{2}}\|[\![\varepsilon(\mathcal{K}_\delta\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E})\cdot\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma)} &\le c\,h^{\alpha-\frac{1}{2}}\|\mathcal{K}_\delta\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{L}^2(\Sigma)}\\
&\le c\,h^{\alpha-\frac{1}{2}}\|\mathcal{K}_\delta\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{L}^2(\Omega)}^{1-\frac{1}{2\alpha}}\|\mathcal{K}_\delta\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{H}^\alpha(\Omega)}^{\frac{1}{2\alpha}}\\
&\le c\,h^{\alpha-\frac{1}{2}}h^{\ell\left(1-\frac{1}{2\alpha}\right)}h^{(\ell-\alpha)\frac{1}{2\alpha}}\|\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{H}^\ell(\Omega)}\\
&\le c\,h^{\alpha+\ell-1}\delta^{\tau-\ell}\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}.
\end{aligned}$$

When combining the above estimates, we obtain

$$\|\mathcal{K}_\delta\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}, p-\mathcal{C}_h^p p\|_h \le c\left(h^{\ell-1}\delta^{\tau-\ell}+\xi h^{1-\alpha}\right)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \tag{5.4}$$

where $\xi=0$ if $\nabla\cdot(\varepsilon\mathbf{g})=0$ and $\xi=1$ otherwise (note that $p=0$ when $\nabla\cdot(\varepsilon\mathbf{g})=0$).

The last term, $\|\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E}_h, \mathcal{C}_h^p p-p_h\|_h$, is a little more subtle to handle. We start from the coercivity of a_h, (4.21), and use both the Galerkin orthogonality (5.1) and the continuity of a_h, (4.22), with $s=1-\alpha$ to get the following estimate:

$$\begin{aligned}
&\|\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E}_h, \mathcal{C}_h^p p-p_h\|_h\\
&\quad\le c\,\frac{a_h\left((\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E}_h,\mathcal{C}_h^p p-p_h),(\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E}_h,\mathcal{C}_h^p p-p_h)\right)}{\|\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E}_h,\mathcal{C}_h^p p-p_h\|_h}\\
&\quad\le c\,\frac{a_h\left((\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E},\mathcal{C}_h^p p-p),(\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E}_h,\mathcal{C}_h^p p-p_h)\right)}{\|\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E}_h,\mathcal{C}_h^p p-p_h\|_h}\\
&\quad\le c\left(\|\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E},\mathcal{C}_h^p p-p\|_h+h^{\alpha-1}\|\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{L}^2(\Omega)}\right.\\
&\qquad+h^{1-\alpha}\|\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E})\|_{\mathbf{H}^{1-\alpha}(\Omega)}+h^{-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Omega)}\\
&\qquad\left.+h\|\nabla\times\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)}+h^{\frac{1}{2}-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Sigma)}\right).
\end{aligned}$$

We now handle each term in the right hand side separately. Using the triangle inequality $\|\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E},\mathcal{C}_h^p p-p\|_h \le \|\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathcal{K}_\delta\mathbf{E},\mathcal{C}_h^p p-p\|_h+\|\mathcal{K}_\delta\mathbf{E}-\mathbf{E},0\|_h$ and the estimates (5.3)-(5.4), we obtain

$$\|\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}-\mathbf{E},\mathcal{C}_h^p p-p\|_h \le c\left(\delta^\tau+h^\alpha\delta^{\tau-1}+h^{\alpha-\frac{1}{2}}\delta^{\tau-\frac{1}{2}}+h^{\ell-1}\delta^{\tau-\ell}+\xi h^{1-\alpha}\right)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Similarly, we obtain

$$h^{\alpha-1}\|\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{L}^2(\Omega)} \le c\left(h^{\alpha-1}\delta^\tau+h^{\alpha+\ell-1}\delta^{\tau-\ell}\right)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Now using that $\kappa\nabla\times\mathbf{E}\in\mathbf{H}^\tau(\Omega)$ and $1-\alpha\le\tau$, owing to the assumption $\alpha\in\left(\frac{\ell(1-\tau)}{\ell-\tau},1\right]$, we infer that

$$h^{1-\alpha}\|\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E})\|_{\mathbf{H}^{1-\alpha}(\Omega)} \le c\left(h^{1-\alpha}\delta^{\tau+\alpha-1}+h^{\ell-1}\delta^{\tau-\ell}\right)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

For the last term involving $\mathbf{E}$ we use the commuting property $\bar{\delta}\nabla\times\mathcal{K}_\delta\mathbf{E} = \mathcal{K}_\delta\nabla\times\mathbf{E}$, see (3.19), to derive

$$\begin{aligned} h\|\nabla\times(\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}))\|_{\mathbf{L}^2(\mathcal{T}_h)} &\le h\|\nabla\times(\kappa\nabla\times\mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} + h\|\nabla\times(\kappa\nabla\times\mathcal{K}_\delta\mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} \\ &\quad + h\|\nabla\times(\kappa\nabla\times(\mathcal{K}_\delta\mathbf{E}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{E}))\|_{\mathbf{L}^2(\mathcal{T}_h)} \\ &\le c\left(h\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + h\|\nabla\times\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{H}^1(\Omega)} + h^{\ell-1}\|\mathcal{K}_\delta\mathbf{E}\|_{\mathbf{H}^\ell(\Omega)}\right) \\ &\le c\left(h\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + h\|\mathcal{K}_\delta\nabla\times\mathbf{E}\|_{\mathbf{H}^1(\Omega)} + h^{\ell-1}\delta^{\tau-\ell}\|\mathbf{E}\|_{\mathbf{H}^\tau(\Omega)}\right) \\ &\le c\left(h + h\delta^{\tau-1} + h^{\ell-1}\delta^{\tau-\ell}\right)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

For the remaining terms involving p, we use (A.4) together with the approximation properties of $\mathcal{C}_h^p$:

$$\begin{aligned} h^{-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Omega)} &\le c\,h^{1-\alpha}\|p\|_{\mathrm{H}_0^1(\Omega)} \le c\xi h^{1-\alpha}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \\ h^{\frac12-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Sigma)} &\le h^{\frac12-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Omega)}^{1-\frac{1}{2\alpha}}\|p-\mathcal{C}_h^p p\|_{\mathrm{H}^\alpha(\Omega)}^{\frac{1}{2\alpha}} \\ &\le c\,h^{\frac12-\alpha}h^{1-\frac{1}{2\alpha}}h^{(1-\alpha)\frac{1}{2\alpha}}\|p\|_{\mathrm{H}_0^1(\Omega)} \le c\xi h^{1-\alpha}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Gathering all the above estimates together with (5.3) and (5.4), we finally obtain

$$\begin{aligned} \|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h \le & c\big(\delta^\tau + \xi h^{1-\alpha} + h + h\delta^{\tau-1} + h^{\ell-1}\delta^{\tau-\ell} + h^{\alpha-1}\delta^\tau \\ & + h^{1-\alpha}\delta^{\tau+\alpha-1} + h^\alpha\delta^{\tau-1} + h^{\alpha-\frac12}\delta^{\tau-\frac12} + h^{\alpha+\ell-1}\delta^{\tau-\ell}\big)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{5.5}$$

We want to use $\delta = h^\beta$ for some $\beta\in(0,1)$, i.e., $\delta h^{-1}\to+\infty$ as $h\to 0$. Once the negligible terms are removed in (5.5), we derive the following estimate:

$$\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h \le c\big(h^{\alpha-1}\delta^\tau + \xi h^{1-\alpha} + h^{\ell-1}\delta^{\tau-\ell}\big)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Using $\delta = h^{1-\frac{\alpha}{\ell}}$ implies that $h^{\alpha-1}\delta^\tau = h^{\ell-1}\delta^{\tau-\ell}$ and we arrive at

$$\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h \le c(h^{\alpha-1+\tau\left(1-\frac{\alpha}{\ell}\right)} + \xi h^{1-\alpha})\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)},$$

which leads to (5.2) with $r := \min\left(1-\alpha, \alpha-1+\tau\left(1-\frac{\alpha}{\ell}\right)\right)$ if $\nabla\cdot(\varepsilon\mathbf{g}) \neq 0$ and $r = \alpha-1+\tau\left(1-\frac{\alpha}{\ell}\right)$ otherwise. Note that the assumed lower bound on α ensures that we have a convergence result as $h\to 0$. □

Remark 5.1 ($\alpha = 1$). Note that the best choice for α when $\nabla\cdot(\varepsilon\mathbf{g}) = 0$ is $\alpha = 1$; the convergence rate is then $\tau\left(1-\frac{1}{\ell}\right)$ and it approaches the optimal rate τ as ℓ increases. When $\nabla\cdot(\varepsilon\mathbf{g}) \neq 0$, the best choice for α is such that $1-\alpha = \alpha-1+\tau\left(1-\frac{\alpha}{\ell}\right)$. This choice gives the following convergence rate $\frac{\tau}{2}(1-\frac{1}{\ell}) < r = \tau\frac{\ell-1}{2\ell-\tau} < \frac{\tau}{2}$.

We now derive a convergence estimate assuming that the solution of (2.12) is smooth. In the next theorem we allow the parameter α to be any number in the interval $[0,1]$.

Theorem 5.2. *Let $\mathbf{g}\in\mathbf{L}^2(\Omega)$ and let $(\mathbf{E},p)$ and $(\mathbf{E}_h,p_h)$ be the solution of* (2.12) *and* (4.17)*, respectively. Assume moreover that $\mathbf{E}\in\mathbf{H}^{k+1}(\Omega_\Sigma)$ and $p\in\mathrm{H}^{k+\alpha}(\Omega_\Sigma)$ for some $0<k\le\ell-1$. Then there exists $c>0$, uniform in h, such that*

$$\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h \le c\,h^k\left(\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|p\|_{\mathrm{H}^{k+\alpha}(\Omega_\Sigma)}\right). \tag{5.6}$$

Proof. The proof is similar to that of Theorem 5.1. We start from the triangular inequality

$$\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h \le \|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}, p-\mathcal{C}_h^p p\|_h + \|\mathcal{C}_h^l\mathbf{E}-\mathbf{E}_h, \mathcal{C}_h^p p - p_h\|_h,$$

We bound the two terms in the right hand side separately. For the first one, we use the local approximation properties of the operators $\mathcal{C}_h^l$ and $\mathcal{C}_h^p$ to derive

$$\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}, p-\mathcal{C}_h^p p\|_h \le c\Big(h^k\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + h^{k+\alpha}\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + h^{-\frac{1}{2}}\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}$$
$$+ h^{1-\alpha}h^{k+\alpha-1}\|p\|_{\mathrm{H}^{k+\alpha}(\Omega_\Sigma)} + h^{\alpha-\frac{1}{2}}\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{L}^2(\Sigma)}\Big).$$

Using (A.4) for any $\sigma\in\left(\frac{1}{2},1\right)$, we have

$$\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{L}^2(\Sigma\cup\Gamma)} \le c\,\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{L}^2(\Omega_\Sigma)}^{1-\frac{1}{2\sigma}}\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{H}^\sigma(\Omega_\Sigma)}^{\frac{1}{2\sigma}} \le c\,h^{k+\frac{1}{2}}\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)}.$$

As a result, we obtain

$$\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}, p-\mathcal{C}_h^p p\|_h \le ch^k\left(\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|p\|_{\mathrm{H}^{k+\alpha}(\Omega_\Sigma)}\right). \tag{5.7}$$

Now we turn our attention to $\|\mathcal{C}_h^l\mathbf{E}-\mathbf{E}_h, \mathcal{C}_h^p p-p_h\|_h$. We use the coercivity of a_h, the Galerkin orthogonality and the continuity of a_h (for any $\sigma\in\left(0,\frac{1}{2}\right)$) to get

$$\|\mathcal{C}_h^l\mathbf{E}-\mathbf{E}_h, \mathcal{C}_h^p p-p_h\|_h \le\ c\,\big(\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}, p-\mathcal{C}_h^p p\|_h + h^{\alpha-1}\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{L}^2(\Omega)}$$
$$+ h^\sigma\|\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^l\mathbf{E})\|_{\mathbf{H}^\sigma(\mathcal{T}_h)}$$
$$+ h\|\nabla\times\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^l\mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)}$$
$$+ h^{-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Omega)} + h^{\frac{1}{2}-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Sigma)}\big).$$

Using the approximation properties of $\mathcal{C}_h^l$ together with (5.7), we infer

$$h^{\alpha-1}\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{L}^2(\Omega)} \le ch^{k+\alpha}\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)},$$
$$h^\sigma\|\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^l\mathbf{E})\|_{\mathbf{H}^\sigma(\mathcal{T}_h)} \le ch^k\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)},$$
$$h^{-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Omega)} \le ch^k\|p\|_{\mathrm{H}^{k+\alpha}(\Omega_\Sigma)}.$$

For the last term involving p, we use (A.4) for some $\sigma\in\left(\frac{1}{2},1\right)$:

$$h^{\frac{1}{2}-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Sigma)} \le ch^{\frac{1}{2}-\alpha}\|p-\mathcal{C}_h^p p\|_{\mathrm{L}^2(\Omega_\Sigma)}^{1-\frac{1}{2\sigma}}\|p-\mathcal{C}_h^p p\|_{\mathrm{H}^\sigma(\Omega_\Sigma)}^{\frac{1}{2\sigma}}$$
$$\le ch^{\frac{1}{2}-\alpha}h^{k+\alpha-\frac{1}{2}}\|p\|_{\mathrm{H}^{k+\alpha}(\Omega_\Sigma)} = ch^k\|p\|_{\mathrm{H}^{k+\alpha}(\Omega_\Sigma)}.$$

For the last term involving $\mathbf{E}$, we distinguish two cases depending whether $k<1$ or $k\ge 1$. If $k<1$, we use an inverse inequality together with the approximation properties of $\mathcal{C}_h^l$ to deduce that

$$h\|\nabla\times\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^l\mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} \le h\|\nabla\times\kappa\nabla\times\mathbf{E}\|_{\mathbf{L}^2(\Omega)} + ch\|\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{H}^2(\mathcal{T}_h)}$$
$$\le h\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + h^k\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)}.$$

If $k\ge 1$, we use the local approximation properties of $\mathcal{C}_h^l$ to get

$$h\|\nabla\times\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^l\mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} \le ch\|\mathbf{E}-\mathcal{C}_h^l\mathbf{E}\|_{\mathbf{H}^2(\mathcal{T}_h)} \le ch^k\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)}.$$

In both cases, we have:

$$h\|\nabla\times\kappa\nabla\times(\mathbf{E}-\mathcal{C}_h^l\mathbf{E})\|_{\mathbf{L}^2(\mathcal{T}_h)} \le ch^k\left(\|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}\right).$$

Gathering all the above estimates and using (5.7) gives the desired result (5.6). $\square$

Remark 5.2. Note that the error estimate (5.6) is optimal since it implies that $\|\nabla\times(\mathbf{E}-\mathbf{E}_h)\|_{\mathbf{L}^2(\Omega_\Sigma)} \leq c\,h^k$, which is the best that can be expected from piecewise polynomial approximation of degree k. Note also that there is no lower bound on α to get convergence when the solution of (2.12) is smooth, i.e., any α in the range $[0,1]$ is acceptable.

5.2. **Convergence in the $\mathbf{L}^2$-norm.** Before proving that the discrete solution converges to the exact solution in the $\mathbf{L}^2$-norm, we prove a global version of Lemma A.3 that will be useful in the proof of Theorem 5.3.

Lemma 5.2. *Let $s \in \big(0,\frac{1}{2}\big)$. Then there exists $c>0$, uniform in h, such that the following holds, for any $\boldsymbol{\psi} \in \mathbf{H}_{\text{curl}}(\Omega)\cap\mathbf{H}^s(\Omega)$ and any $\mathbf{F}_h \in \mathbf{X}_h$:*
(5.8)
$$|(\boldsymbol{\psi},[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}| \leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}\ \left(h^s\|\boldsymbol{\psi}\|_{\mathbf{H}^s(\Omega)} + h\|\nabla\times\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)}\right).$$

Proof. Let us consider $\boldsymbol{\psi} \in \mathbf{H}_{\text{curl}}(\Omega)\cap\mathbf{H}^s(\Omega)$ and $\mathbf{F}_h \in \mathbf{X}_h$. Notice that the left hand side is well defined owing to Lemma A.3. We start from
$$\big|(\boldsymbol{\psi},[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}\big| \leq \underbrace{|(\boldsymbol{\psi}-\mathcal{K}_\delta\boldsymbol{\psi},[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}|}_{:=I_1} + \underbrace{|(\mathcal{K}_\delta\boldsymbol{\psi},[\![\mathbf{F}_h\times\mathbf{n}]\!])_{\Sigma\cup\Gamma}|}_{:=I_2},$$
for some δ to be defined later. We handle the two terms I_1, I_2 separately. For the first one, we apply Lemma A.3 with $\mathbf{v}=[\![\mathbf{F}_h\times\mathbf{n}]\!]$, $\boldsymbol{\phi}=\boldsymbol{\psi}-\mathcal{K}_\delta\boldsymbol{\psi}$ and $\sigma=s$, and we sum over all the faces $F\in\Sigma\cup\Gamma$. This leads to
$$\begin{aligned} I_1 &\leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}\big(h^s\|\boldsymbol{\psi}-\mathcal{K}_\delta\boldsymbol{\psi}\|_{\mathbf{H}^s(\Omega_\Sigma)} \\ &\quad + h\|\nabla\times(\boldsymbol{\psi}-\mathcal{K}_\delta\boldsymbol{\psi})\|_{\mathbf{L}^2(\Omega_\Sigma)} + \|\boldsymbol{\psi}-\mathcal{K}_\delta\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega_\Sigma)}\big) \\ &\leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}\big(h^s\|\boldsymbol{\psi}-\mathcal{K}_\delta\boldsymbol{\psi}\|_{\mathbf{H}^s(\Omega_\Sigma)} \\ &\quad + h\|\nabla\times\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega_\Sigma)} + h\|\nabla\times\mathcal{K}_\delta\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega_\Sigma)} + \|\boldsymbol{\psi}-\mathcal{K}_\delta\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega_\Sigma)}\big). \end{aligned}$$
Using the approximation properties of $\mathcal{K}_\delta$ (3.16) and (3.18), we arrive at
$$\begin{aligned} I_1 &\leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}\big(h^s\|\boldsymbol{\psi}\|_{\mathbf{H}^s(\Omega_\Sigma)} \\ &\quad + h\|\nabla\times\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega_\Sigma)} + \delta^s\|\boldsymbol{\psi}\|_{\mathbf{H}^s(\Omega_\Sigma)} + h\|\mathcal{K}_\delta\boldsymbol{\psi}\|_{\mathbf{H}^1(\Omega_\Sigma)}\big) \\ &\leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}\big((h^s+\delta^s+h\delta^{s-1})\|\boldsymbol{\psi}\|_{\mathbf{H}^s(\Omega_\Sigma)} + h\|\nabla\times\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega_\Sigma)}\big). \end{aligned}$$
We handle I_2 by using the Cauchy-Schwarz inequality on every $\partial\Omega_i$, $i=1,\cdots,N$.
$$I_2 \leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}\sum_{i=1}^N h^{\frac{1}{2}}\|\mathcal{K}_\delta\boldsymbol{\psi}\|_{\mathbf{L}^2(\partial\Omega_i)}.$$
We use (A.5) on every Ω_i with $\Theta := \frac{1-2s}{2(1-s)}$ to obtain
$$\begin{aligned} I_2 &\leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}\sum_{i=1}^N h^{\frac{1}{2}}\|\mathcal{K}_\delta\boldsymbol{\psi}\|^{1-\Theta}_{\mathbf{H}^s(\Omega_i)}\|\mathcal{K}_\delta\boldsymbol{\psi}\|^{\Theta}_{\mathbf{H}^1(\Omega_i)} \\ &\leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}h^{\frac{1}{2}}\|\mathcal{K}_\delta\boldsymbol{\psi}\|^{1-\Theta}_{\mathbf{H}^s(\Omega_\Sigma)}\|\mathcal{K}_\delta\boldsymbol{\psi}\|^{\Theta}_{\mathbf{H}^1(\Omega_\Sigma)}, \end{aligned}$$
where the constant c depends on N, which we recall is a fixed number. Using again the approximation properties of $\mathcal{K}_\delta$ we infer that
$$\begin{aligned} I_2 &\leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}h^{\frac{1}{2}}\delta^{(s-1)\Theta}\|\boldsymbol{\psi}\|_{\mathbf{H}^s(\Omega_\Sigma)} \\ &\leq c\,h^{-\frac{1}{2}}\|[\![\mathbf{F}_h\times\mathbf{n}]\!]\|_{\mathbf{L}^2(\Sigma\cup\Gamma)}h^{\frac{1}{2}}\delta^{s-\frac{1}{2}}\|\boldsymbol{\psi}\|_{\mathbf{H}^s(\Omega_\Sigma)}. \end{aligned}$$

Then (5.8) is obtained by gathering the above estimates and setting $\delta = h$. □

Remark 5.3 (Alternative Decomposition). Estimate (5.8) can alternatively be derived using the decomposition $\boldsymbol{\psi} = \boldsymbol{\psi} - \mathcal{C}_h^l \boldsymbol{\psi} + \mathcal{C}_h^l \boldsymbol{\psi}$ instead of $\boldsymbol{\psi} = \boldsymbol{\psi} - \mathcal{K}_\delta \boldsymbol{\psi} + \mathcal{K}_\delta \boldsymbol{\psi}$.

Theorem 5.3. *Let $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and let $(\mathbf{E}, p)$ be the solution of (2.12). Let $\tau < \min(\tau_\varepsilon, \tau_\kappa)$ where τ_ε and τ_κ are defined in Theorem 2.1. Let $(\mathbf{E}_h, p_h)$ be solution of (4.17). For any $\alpha \in \left(\frac{\ell(1-\tau)}{\ell-\tau}, 1\right)$, there exists $c > 0$, uniform in h, such that*

$$\|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \le c\, h^{r_1+r_2} \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \tag{5.9}$$

with $r_1 := \min\left(1-\alpha, \alpha - 1 + \tau\left(1 - \frac{\alpha}{\ell}\right)\right)$ and $r_2 = r_1$ if $\nabla\cdot(\varepsilon \mathbf{g}) \neq 0$ and $r_2 = \alpha - 1 + \tau\left(1 - \frac{\alpha}{\ell}\right)$ if $\nabla\cdot(\varepsilon\mathbf{g}) = 0$. If in addition $\mathbf{E} \in \mathbf{H}^{k+1}(\Omega_\Sigma)$ and $p \in \mathrm{H}^{k+\alpha}(\Omega_\Sigma)$ for some $0 < k < \ell - 1$, then the following holds:

$$\|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \le c\, h^{k+r_1} \left(\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}\|_{\mathbf{H}^{k+1}(\Omega_\Sigma)} + \|p\|_{\mathrm{H}^{k+\alpha}(\Omega_\Sigma)}\right). \tag{5.10}$$

Proof. We are going to use a duality argument à la Nitsche-Aubin. In the following we denote a_h^1 the extension to $\left[(\mathbf{Z}^\tau(\Omega) + \mathbf{X}_h)\times \mathrm{H}_0^1(\Omega)\right]^2$ of the bilinear form defined on $[\mathbf{X}_h \times M_h]^2$ in (4.16) with $\theta = 1$. Then the following symmetry property holds:

$$a_h^1\left((\mathbf{F}, q), (\mathbf{G}, r)\right) = a_h^1\left((\mathbf{G}, -r), (\mathbf{F}, -q)\right).$$

for all $((\mathbf{F}, q), (\mathbf{G}, r)) \in \left[(\mathbf{Z}^\tau(\Omega) + \mathbf{X}_h)\times \mathrm{H}_0^1(\Omega)\right]^2$. Let $(\mathbf{w}, q) \in \mathbf{H}_{0,\mathrm{curl}}(\Omega)\times \mathrm{H}_0^1(\Omega)$ be the solution of the following (adjoint) problem:

$$\nabla\times(\kappa\nabla\times\mathbf{w}) - \varepsilon\nabla q = \varepsilon\left(\mathbf{E} - \mathbf{E}_h\right), \qquad \nabla\cdot(\varepsilon\mathbf{w}) = 0. \tag{5.11}$$

Recall that Theorem 2.1 implies that $\mathbf{w} \in \mathbf{Z}^\tau(\Omega) \cap \mathbf{H}^\tau(\Omega)$ and that

$$\|\mathbf{w}\|_{\mathbf{H}^\tau(\Omega)} + \|\kappa\nabla\times\mathbf{w}\|_{\mathbf{H}^\tau(\Omega)} + \|\nabla\times\kappa\nabla\times\mathbf{w}\|_{\mathbf{L}^2(\Omega)} \le c\,\|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}. \tag{5.12}$$

Upon testing (5.11) with $\mathbf{E} - \mathbf{E}_h$, using the definition of a_h^1 in (4.16), and recalling that $\nabla\cdot(\varepsilon\mathbf{w}) = 0$ and both the tangential jump of $\mathbf{w}$ across Σ and the tangential trace of $\mathbf{w}$ on Γ are zero, we obtain the following identity:

$$\|\varepsilon^{\frac{1}{2}}(\mathbf{E} - \mathbf{E}_h)\|^2_{\mathbf{L}^2(\Omega)} = a_h^1\left((\mathbf{w}, -q), (\mathbf{E} - \mathbf{E}_h, p_h - p)\right) + c_\alpha h^{2(1-\alpha)}\left(\varepsilon\nabla q, \nabla(p_h - p)\right)_\Omega$$

The definition of the pair $(\mathbf{w}, q)$ implies that $(\varepsilon\nabla q, \nabla\varphi)_\Omega = -(\varepsilon(\mathbf{E} - \mathbf{E}_h, \nabla\varphi))_\Omega$ for all $\varphi \in \mathrm{H}_0^1(\Omega)$; hence,

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}(\mathbf{E} - \mathbf{E}_h)\|^2_{\mathbf{L}^2(\Omega)} &= a_h^1\left((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{w}, q)\right) + c_\alpha h^{2(1-\alpha)}\left(\varepsilon(\mathbf{E} - \mathbf{E}_h), \nabla(p - p_h)\right)_\Omega \\ &= a_h\left((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{w}, q)\right) + c_\alpha h^{2(1-\alpha)}\left(\varepsilon(\mathbf{E} - \mathbf{E}_h), \nabla(p - p_h)\right)_\Omega \\ &\quad + (1-\theta)\left(\{\!\{\kappa\nabla\times\mathbf{w}\}\!\}, [\![-\mathbf{E}_h\times\mathbf{n}]\!]\right)_{\Sigma\cup\Gamma}.\end{aligned}$$

We now use the Galerkin orthogonality and we introduce the global approximation $\mathcal{C}_h^g \mathcal{K}_\delta \mathbf{w}$, with $\delta = h^{1-\frac{\alpha}{\ell}}$, and the pressure approximation $\mathcal{C}_h^p q$:

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}(\mathbf{E} - \mathbf{E}_h)\|^2_{\mathbf{L}^2(\Omega)} &= a_h\left((\mathbf{E} - \mathbf{E}_h, p - p_h), (\mathbf{w} - \mathcal{C}_h^g\mathcal{K}_\delta\mathbf{w}, q - \mathcal{C}_h^p q)\right) \\ &\quad + c_\alpha h^{2(1-\alpha)}\left(\varepsilon(\mathbf{E} - \mathbf{E}_h), \nabla(p - p_h)\right)_\Omega - (1-\theta)\left(\kappa\nabla\times\mathbf{w}, [\![\mathbf{E}_h\times\mathbf{n}]\!]\right)_{\Sigma\cup\Gamma}.\end{aligned} \tag{5.13}$$

Note that we replaced $\{\!\{\kappa\nabla\times\mathbf{w}\}\!\}$ by $\kappa\nabla\times\mathbf{w}$ since the tangent component of $\kappa\nabla\times\mathbf{w}$ is continuous across the interfaces owing to $\nabla\times(\kappa\nabla\times\mathbf{w}) \in \mathbf{L}^2(\Omega)$.

We now handle the three terms in the right hand side separately. For the first one, we use Proposition 4.4 with $s = 1-\alpha$, $\mathbf{F} = \mathbf{w}$ and $\mathbf{F}_h = \mathcal{C}_h^g\mathcal{K}_\delta\mathbf{w}$ (note that $\mathbf{F}_h \in \mathbf{Y}_h \subset \mathbf{X}_h \cap \mathbf{H}_{0,\mathrm{curl}}(\Omega)$); we then infer that

$$\begin{aligned}\big|a_h\big((\mathbf{E}-\mathbf{E}_h, p-p_h), (\mathbf{w}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{w}, q-\mathcal{C}_h^p q)\big)\big| \leq \\ c\,\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h\big(\|\mathbf{w}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{w}, q-\mathcal{C}_h^p q\|_h \\ + h^{\alpha-1}\|\mathbf{w}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + h^{-\alpha}\|q-\mathcal{C}_h^p q\|_{\mathrm{L}^2(\Omega)} + h^{\frac12-\alpha}\|q-\mathcal{C}_h^p q\|_{\mathrm{L}^2(\Sigma)} \\ + h\|\nabla\times\kappa\nabla\times(\mathbf{w}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{w})\|_{\mathbf{L}^2(\mathcal{T}_h)} + h^{1-\alpha}\|\nabla\times(\mathbf{w}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{w})\|_{\mathbf{H}^{1-\alpha}(\Omega)}\big).\end{aligned}$$

The term in parentheses on the right-hand side is estimated asin the proof of Theorem 5.1. We then have

$$\begin{aligned}\big|a_h\big((\mathbf{E}-\mathbf{E}_h, p-p_h),(\mathbf{w}-\mathcal{C}_h^g\mathcal{K}_\delta\mathbf{w}, \mathcal{C}_h^p q-q)\big)\big| \\ \leq c\,\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h h^{r_1}\|\mathbf{E}-\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{5.14}$$

The second term in (5.13) is estimated by using the Cauchy-Schwarz inequality, the definition of the norm $\|\cdot\|_h$ and the inequality $r_1 \leq 1-\alpha$,

$$\begin{aligned}\left|h^{2(1-\alpha)}\left(\varepsilon(\mathbf{E}-\mathbf{E}_h), \nabla(p-p_h)\right)_\Omega\right| &\leq c\,h^{1-\alpha}\|\nabla(p-p_h)\|_{\mathbf{L}^2(\Omega)}h^{1-\alpha}\|\mathbf{E}-\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \\ &\leq c\,\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h h^{r_1}\|\mathbf{E}-\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{5.15}$$

The last term in (5.13) is estimated by using Lemma 5.2 with $\boldsymbol{\psi} := \kappa\nabla\times\mathbf{w}$ and $s := \tau$:

$$\begin{aligned}\big|(1-\theta)\big(\kappa\nabla\times\mathbf{w}, [\![\mathbf{E}_h\times\mathbf{n}]\!]\big)_{\Sigma\cup\Gamma}\big| \\ \leq c\,\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h\left(h^\tau\|\kappa\nabla\times\mathbf{w}\|_{\mathbf{H}^\tau(\Omega)} + h\|\nabla\times(\kappa\nabla\times\mathbf{w})\|_{\mathbf{L}^2(\Omega)}\right) \\ \leq c\,\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h\, h^{r_1}\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_{\mathbf{L}^2(\Omega)},\end{aligned} \tag{5.16}$$

where we have used (5.12) and $r_1 \leq \frac{\tau}{2} < \tau$. Upon inserting (5.14)-(5.15)-(5.16) in (5.13) we obtain

$$\|\varepsilon^{\frac12}(\mathbf{E}-\mathbf{E}_h)\|^2_{\mathbf{L}^2(\Omega)} \leq ch^{r_1}\|\mathbf{E}-\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h.$$

Owing to the uniform positivity of ε, this leads to:

$$\|\mathbf{E}-\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)} \leq ch^{r_1}\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h.$$

Now we consider two cases. Assuming only minimal regularity, Theorem 5.1 gives a bound on $\|\mathbf{E}-\mathbf{E}_h, p-p_h\|_h$ that leads to (5.9). If $\mathbf{E}$ and p are piecewise smooth, then we can apply Theorem 5.2 and we obtain (5.10). □

Remark 5.4. Let $\tau \in (0, \frac12)$ and denote $(\mathbf{E}, p)$ the solution of (2.12). Assume that $\mathbf{E} \in \mathbf{H}^\tau(\Omega)$ and $\mathbf{E} \notin \mathbf{H}^{\tau^+}(\Omega)$ for all $\tau^+ > \tau$. Then, irrespective of the value of $\nabla\cdot(\varepsilon\mathbf{g})$, the best choice for α is $\alpha = \frac{\ell(2-\tau)}{2\ell-\tau}$, which gives the convergence rate $r_1 + r_2 = \tau\frac{\ell-1}{\ell-\frac{\tau}{2}}$; this convergence rate approaches the optimal rate, τ, when the approximation degree ℓ is large. Note also that α is close to 1 when ℓ is large.

Remark 5.5. Note that the degree of the polynomials used for M_h is not involved in the convergence rate when minimal regularity is assumed. This means that we can use different degrees of polynomials for $\mathbf{X}_h$ and M_h, and that it is sufficient to take polynomials of degree 1 for M_h to get convergence.

5.3. **Numerical illustrations.** In this section we illustrate numerically the performance of the method on a boundary value problem on the L-shaped domain

$$\Omega = (-1,1)^2 \setminus ([0,+1]\times[-1,0]).$$

We assume that Ω is composed of three subdomains:

$$\Omega_1 = (0,1)^2, \qquad \Omega_2 = (-1,0)\times(0,1), \qquad \Omega_3 = (-1,0)^2.$$

We use $\kappa \equiv 1$ in Ω, $\varepsilon_{|\Omega_2} = 1$ and $\varepsilon_{|\Omega_1} = \varepsilon_{|\Omega_3} =: \varepsilon_r$. Denoting $\lambda > 0$ a real number such that $\tan\left(\frac{\lambda\pi}{4}\right)\tan\left(\frac{\lambda\pi}{2}\right) = \varepsilon_r$, we define the scalar potential $S_\lambda(r,\vartheta) = r^\lambda \phi_\lambda(\vartheta)$, where (r,ϑ) are the polar coordinates, and ϕ_λ is defined by

$$\phi_\lambda(\vartheta) = \begin{cases} \sin(\lambda\vartheta) & \text{if } 0 \le \vartheta < \frac{\pi}{2}, \\ \frac{\sin\left(\frac{\lambda}{2}\pi\right)}{\cos\left(\frac{\lambda}{4}\pi\right)} \cos\left(\lambda\left(\vartheta - \frac{3}{4}\pi\right)\right) & \text{if } \frac{\pi}{2} \le \vartheta < \pi, \\ \sin\left(\lambda\left(\frac{3}{2}\pi - \vartheta\right)\right) & \text{if } \pi \le \vartheta \le \frac{3\pi}{2}. \end{cases}$$

Then we solve the problem

$$\nabla\times\nabla\times\mathbf{E} = 0, \qquad \nabla\cdot(\varepsilon\mathbf{E}) = 0, \qquad \mathbf{E}\times\mathbf{n}_{|\partial\Omega} = \nabla S_\lambda\times\mathbf{n}. \tag{5.17}$$

The exact solution is $\mathbf{E} = \nabla S_\lambda \in \mathbf{H}^\lambda(\Omega)$. We present in Table 1 two series of simulations done with the two-dimensional version of SFEMaNS, see e.g. Guermond et al. [24], with $\mathbb{P}_1$ finite elements on quasi-uniform triangular Delaunay meshes; i.e., $\ell = 2$ in (4.1). All the technical assumptions made in the paper are met: (2.15) hold and the meshes are quasi-uniform and H^1-conforming.We use $\lambda = 0.535$ in Table 1(a) and $\lambda = 0.24$ in Table 1(b), which gives $\varepsilon_r \simeq 0.5$ and $\varepsilon_r \simeq 7.55\ 10^{-2}$, respectively. The relative error in the $\mathbf{L}^2$-norm is reported in the column "rel. err." and the convergence rate is reported in the column "coc". Several values of α are used to evaluate the effect of λ and α on the convergence rates. We observe that the convergence rate is quasi-optimal when α is close to 1, which is consistent with Remark 5.1, since (5.17) can be re-written in the form (2.12) with $\nabla\cdot(\varepsilon\mathbf{g}) = 0$.

It has been pointed out in the literature (see e.g. Costabel and Dauge [16, §8.3.1], Duan et al. [19], Badia and Codina [4]) that it is possible to build special meshes allowing the existence of C^1 interpolation operators, i.e., it is possible to represent gradients on these meshes with optimal approximation properties. We now investigate theses possibilities with $\mathbb{P}_1$ and $\mathbb{P}_2$ finite elements. We solve again the above boundary value problem with $\lambda = 0.535$ and $\alpha = 0.9$. For the $\mathbb{P}_1$ approximation, we construct Powell-Sabin type meshes (see Powell and Sabin [31]) and compare the results obtained on these meshes with those obtained on generic Delaunay meshes (see Table 2(a)). We indeed observe an improvement since now the convergence rate is optimal, i.e., close to 0.535. For the $\mathbb{P}_2$ approximation we construct Hsieh-Clough-Tocher meshes, see Clough and Tocher [14, item 4, p. 520]. It is possible to construct on these meshes $\mathbb{P}_3$ finite element spaces containing C^1 functions with optimal approximation properties. Then, the standard vector-valued $\mathbb{P}_2$ finite element spaces constructed on these meshes contains enough gradients. We compare the results obtained on Hsieh-Clough-Tocher meshes with those obtained on generic Delaunay meshes (see Table 2(b)). We do not observe any significant improvement, since the optimal order was already numerically achieved on the generic Delaunay meshes.

TABLE 1. $\mathbf{L}^2$-errors and convergence rates with $\ell = 2$. The convergence rates are almost optimal for $\alpha = 0.9$ in both cases.

(a) $\mathbf{L}^2$-errors and convergence rates for $\lambda = 0.535$

h	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.9$		$\alpha = 1.0$	
	rel. err.	coc	rel. err.	coc	rel. err.	coc	rel. err.	coc
0.2	2.332E-1	-	1.444E-1	-	1.249E-1	-	1.297E-1	-
0.1	2.473E-1	-0.08	1.168E-1	0.31	8.846E-2	0.50	9.167E-2	0.50
0.05	2.631E-1	-0.09	9.452E-2	0.31	6.186E-2	0.52	6.392E-2	0.52
0.025	2.797E-1	-0.09	7.700E-2	0.30	4.289E-2	0.53	4.427E-2	0.53
0.0125	2.968E-1	-0.09	6.312E-2	0.29	2.962E-2	0.53	3.059E-2	0.53

(b) $\mathbf{L}^2$-errors and convergence rates for $\lambda = 0.24$

h	$\alpha = 0.4$		$\alpha = 0.6$		$\alpha = 0.9$		$\alpha = 1.0$	
	rel. err.	coc	rel. err.	coc	rel. err.	coc	rel. err.	coc
0.2	5.773E-1	-	4.739E-1	-	4.426E-1	-	4.495E-1	-
0.1	6.209E-1	-0.11	4.507E-1	0.07	3.801E-1	0.22	3.838E-1	0.23
0.05	6.711E-1	-0.11	4.413E-1	0.03	3.259E-1	0.22	3.272E-1	0.23
0.025	7.180E-1	-0.10	4.452E-1	-0.01	2.788E-1	0.23	2.788E-1	0.23
0.0125	7.564E-1	-0.08	4.602E-1	-0.05	2.380E-1	0.23	2.376E-1	0.23

TABLE 2. $\mathbf{L}^2$-errors and convergence rates for $\lambda = 0.535$, $\alpha = 0.9$ on different kinds of meshes

(a) $\mathbf{L}^2$-errors and convergence rates for $\ell = 1$

h	Delaunay mesh		Powell-Sabin mesh	
	rel.err	coc	rel. err	coc
0.2	2.166E-1	-	1.742E-1	-
0.1	1.652E-1	0.39	1.246E-1	0.48
0.05	1.268E-1	0.38	8.711E-2	0.52
0.025	9.821E-2	0.37	6.052E-2	0.53
0.0125	7.758E-2	0.34	4.200E-2	0.53

(b) $\mathbf{L}^2$-errors and convergence rates for $\ell = 2$

h	Delaunay mesh		Hsieh-Clough-Tocher mesh	
	rel.err	coc	rel. err	coc
0.2	1.297E-1	-	1.359E-1	-
0.1	9.167E-2	0.50	9.446E-2	0.53
0.05	6.392E-2	0.52	6.535E-2	0.53
0.025	4.427E-2	0.53	4.515E-2	0.53
0.0125	3.059E-2	0.53	3.117E-2	0.53

6. Eigenvalue problem

We extend in this section the theory introduced above to eigenvalue problems. We want to establish an approximation result for the solutions to the following problem: Find $(\mathbf{E},\lambda)\in[\mathbf{H}_{0,\mathrm{curl}}(\Omega)\cap\mathbf{H}_{\mathrm{div}}(\Omega,\epsilon)]\times\mathbb{R}$ such that

$$\nabla\times\kappa\nabla\times\mathbf{E}=\lambda\varepsilon\mathbf{E}. \tag{6.1}$$

We restrict ourselves in the rest of this section to the symmetric variant of the bilinear form a_h defined in (4.16), i.e., we set $\theta=1$. We finally assume from now on that α is chosen as in Theorem 5.1, i.e.,

$$\alpha\in\left(\frac{\ell(1-\tau)}{\ell-\tau},1\right), \tag{6.2}$$

where τ is the minimal regularity index of the problem (2.12) as defined in Theorem 2.1. In the following we set $r:=\min\left(1-\alpha,\alpha-1+\tau\left(1-\frac{\alpha}{\ell}\right)\right)$.

6.1. **Framework.** Let us equip $\mathbf{L}^2(\Omega)$ with the inner product $(\mathbf{f},\mathbf{g})_\varepsilon:=\int_\Omega\varepsilon\mathbf{f}\cdot\mathbf{g}$. This inner product is equivalent to the usual $\mathbf{L}^2$-inner product owing to (2.15). The associated norm is denoted $\|\cdot\|_\varepsilon$.

For any $\mathbf{g}\in\mathbf{L}^2(\Omega)$, we denote $(\mathbf{E},p)$ the solution of (2.12) and we set $A\mathbf{g}:=\mathbf{E}$. This defines an operator $A:\mathbf{L}^2(\Omega)\to\mathbf{L}^2(\Omega)$ that is self-adjoint and compact (cf. Theorem 2.1). We now define two families of discrete operators $\mathcal{E}_h:\mathbf{L}^2(\Omega)\longrightarrow\mathbf{X}_h$ and $\mathcal{P}_h:\mathbf{L}^2(\Omega)\longrightarrow M_h$ so that for any $\mathbf{g}\in\mathbf{L}^2(\Omega)$, the pair $(\mathcal{E}_h\mathbf{g},\mathcal{P}_h\mathbf{g})$ solves (4.17). Then we finally define

$$\begin{aligned} A_h:\mathbf{L}^2(\Omega)&\longrightarrow\mathbf{X}_h+\nabla M_h\subset\mathbf{L}^2(\Omega)\\ \mathbf{g}&\longmapsto\mathcal{E}_h\mathbf{g}-c_\alpha h^{2(1-\alpha)}\nabla\mathcal{P}_h\mathbf{g}. \end{aligned} \tag{6.3}$$

We want to study whether the eigenvalues and eigenspaces spaces of A_h converges to those of A. For this purpose we are going to use the following result:

Theorem 6.1 (Spectral correctness Babuška and Osborn [3], Osborn [30])**.** *Let X be an Hilbert space and $A:X\to X$ be a self-adjoint compact operator. Let $\Theta=\{h_n;\ n\in\mathbb{N}\}$ be a discrete subset of $\mathbb{R}$ such that $h_n\to0$ as $n\to+\infty$. Assume that there exists a family of operators $A_h:X\to X$, $h\in\Theta$, such that:*

- *A_h is a linear self-adjoint operator, for all $h\in\Theta$.*
- *A_h converges pointwise to A.*
- *The family is collectively compact.*

Let μ be an eigenvalue of A of multiplicity m and let $\{\phi_j\}$, $j=1,\cdots,m$ be a set of associated orthonormal eigenvectors.

(i) For any $\epsilon>0$ such that the disk $B(\mu,\epsilon)$ contains no other eigenvalues of A, there exists h_ϵ such that, for all $h<h_\epsilon$, A_h has exactly m eigenvalues (repeated according to their multiplicity) in the disk $B(\mu,\epsilon)$.

(ii) In addition, for $h<h_\epsilon$, if we denote $\mu_{h,j}$, $j=1,\cdots,m$ the set of the eigenvalues of A_h in $B(\mu,\epsilon)$, there exists $c>0$ such that

$$c|\mu-\mu_{h,j}|\le\sum_{j,l=1}^{m}|\left((A-A_h)\phi_j,\phi_l\right)_X|+\sum_{j=1}^{m}\|(A-A_h)\phi_j\|_X^2. \tag{6.4}$$

6.2. **Approximation result.** We start by proving that the operators $\{A_h\}$ are self-adjoint, then we prove the pointwise convergence, and we finally establish the collective compactness.

Lemma 6.1. *For any h, $A_h : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ is a self-adjoint operator, i.e., for any $\mathbf{e}, \mathbf{f} \in \mathbf{L}^2(\Omega)$, the following holds*

$$(A_h\mathbf{e}, \mathbf{f})_\varepsilon = (\mathbf{e}, A_h\mathbf{f})_\varepsilon . \tag{6.5}$$

Proof. Let $\mathbf{e}, \mathbf{f} \in \mathbf{L}^2(\Omega)$. By definition we have

$$a_h\left((\mathcal{E}_h\mathbf{e}, \mathcal{P}_h\mathbf{e}), (\mathcal{E}_h\mathbf{f}, -\mathcal{P}_h\mathbf{f})\right) = (\mathbf{e}, \mathcal{E}_h\mathbf{f})_\varepsilon - c_\alpha h^{2(1-\alpha)} (\mathbf{e}, \nabla\mathcal{P}_h\mathbf{f})_\varepsilon = (\mathbf{e}, A_h\mathbf{f})_\varepsilon .$$

Using the symmetry properties of a_h, we infer

$$\begin{aligned} a_h\left((\mathcal{E}_h\mathbf{e}, \mathcal{P}_h\mathbf{e}), (\mathcal{E}_h\mathbf{f}, -\mathcal{P}_h\mathbf{f})\right) &= a_h\left((\mathcal{E}_h\mathbf{f}, \mathcal{P}_h\mathbf{f}), (\mathcal{E}_h\mathbf{e}, -\mathcal{P}_h\mathbf{e})\right) \\ &= (\mathbf{f}, \mathcal{E}_h\mathbf{e})_\varepsilon - c_\alpha h^{2(1-\alpha)} (\mathbf{f}, \nabla\mathcal{P}_h\mathbf{e})_\varepsilon = (\mathbf{f}, A_h\mathbf{e})_\varepsilon , \end{aligned}$$

thereby proving that the operator A_h is self-adjoint on the Hilbert space $\mathbf{L}^2(\Omega)$ equipped with the inner product $(\cdot,\cdot)_\varepsilon$. ☐

Lemma 6.2. *Under the above assumptions, there exists $c > 0$, uniform with respect to h such that,*

$$\forall \mathbf{e} \in \mathbf{L}^2(\Omega), \qquad \|A_h\mathbf{e} - A\mathbf{e}\|_\varepsilon \le ch^{2r}\|\mathbf{e}\|_\varepsilon. \tag{6.6}$$

Proof. Let $A\mathbf{e} \in \mathbf{L}^2(\Omega)$ and $p \in \mathrm{H}^1_0(\Omega)$ such that $\nabla\times(\kappa\nabla\times A\mathbf{e}) + \varepsilon\nabla p = \varepsilon\mathbf{e}$. Using the triangular inequality, Theorems 5.1 and 5.3, the equivalence between the norms on $\mathbf{L}^2(\Omega)$ and the fact that $r \le 1-\alpha$, we infer that

$$\begin{aligned} \|A\mathbf{e} - A_h\mathbf{e}\|_\varepsilon &\le \|A\mathbf{e} - \mathcal{E}_h\mathbf{e}\|_\varepsilon + c_\alpha h^{2(1-\alpha)}\|\nabla\mathcal{P}_h\mathbf{e} - \nabla p\|_\varepsilon + h^{2(1-\alpha)}\|\nabla p\|_\varepsilon \\ &\le c(h^{2r}\|\mathbf{e}\|_\varepsilon + h^{1-\alpha}\|A\mathbf{e} - \mathcal{E}_h\mathbf{e}, p - \mathcal{P}_h\mathbf{e}\|_h + h^{2(1-\alpha)}\|\mathbf{e}\|_\varepsilon) \le c\, h^{2r}\|\mathbf{e}\|_\varepsilon, \end{aligned}$$

which concludes the proof. ☐

Note that the above result is stronger than the pointwise convergence hypothesis, i.e., A_h converges in norm to A. Now let us turn our attention to the question of collective compactness. Recall that a set $\mathcal{A} := \{A_h \in \mathcal{L}(X;X),\ h \in \Theta\}$ is said to be collectively compact if, for each bounded set $U \subset X$, the image set $\mathcal{A}U := \{A_h\mathbf{g},\ \mathbf{g} \in U,\ A_h \in \mathcal{A}\}$ is relatively compact in X.

Lemma 6.3. *The family $\{A_h\}_{h>0}$ is collectively compact under the above assumptions provided $\alpha \in \left(\frac{\ell(1-\tau)}{\ell-\tau}, 1\right)$.*

Proof. Of course, Lemma 6.2 implies the result, but we are now going to provide an alternative proof.Owing to the compact embedding $\mathbf{H}^s(\Omega) \subset \mathbf{L}^2(\Omega)$ for any $s > 0$, it is sufficient to prove that there exists $s > 0$ and $c > 0$ such that, for any $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and any $h > 0$,

$$\|A_h\mathbf{g}\|_{\mathbf{H}^s(\Omega)} \le c\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Let us take $\mathbf{g} \in \mathbf{L}^2(\Omega)$. Owing to the definition of $\mathbf{X}_h$ and M_h, we know that $A_h\mathbf{g} \in \mathbf{H}^s(\Omega)$ for any $s \in \left(0, \frac{1}{2}\right)$. Moreover, there exists c, only depending on s and the shape regularity of the mesh sequence, such that the following inverse inequality holds:

$$\|A_h\mathbf{g}\|_{\mathbf{H}^s(\Omega)} \le ch^{-s}\|A_h\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

Let us consider $s < \min(r, \frac{\tau}{2})$. Using the triangular inequality, interpolation results, the above inverse inequality together with Theorems 5.3 and 2.1 and Lemma 6.2leads to:

$$\begin{aligned}\|A_h\mathbf{g}\|_{\mathbf{H}^s(\Omega)} &\le \|A_h\mathbf{g} - A\mathbf{g}\|_{\mathbf{H}^s(\Omega)} + \|A\mathbf{g}\|_{\mathbf{H}^s(\Omega)} \\ &\le c\,\|A_h\mathbf{g} - A\mathbf{g}\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega)}\|A_h\mathbf{g} - A\mathbf{g}\|^{\frac{1}{2}}_{\mathbf{H}^{2s}(\Omega)} + c\,\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \\ &\le c\,h^r\|\mathbf{g}\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega)}\left(h^{-s}\|A_h\mathbf{g}\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega)} + \|A\mathbf{g}\|^{\frac{1}{2}}_{\mathbf{H}^{2s}(\Omega)}\right) + c\,\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \\ &\le c\,\left(h^{r-s}+1\right)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

This implies the collective compactness of $\{A_h\}$ since $r > s$. □

We conclude that the approximation is spectrally correct, i.e., we can apply Theorem 6.1 by combining Lemmas 6.1, 6.2, 6.3. Note finally that the convergence rate on the eigenvalues is at least $\mathcal{O}(h^{2r})$ owing to (6.4) and (6.6).

6.3. **Numerical illustration for $\alpha < 1$.** In this section, we present some eigenvalues computations. We consider the square $\Omega = (-1,1)^2$ in the plane. We divide Ω into four subdomains

$$\Omega_1 = (0,1)^2, \qquad \Omega_2 = (-1,0)\times(0,1), \qquad \Omega_3 = (-1,0)^2, \qquad \Omega_4 = (0,1)\times(-1,0).$$

We use $\kappa \equiv 1$ in Ω, $\varepsilon_{|\Omega_1} = \varepsilon_{|\Omega_3} = 1$ and $\varepsilon_{|\Omega_2} = \varepsilon_{|\Omega_4} = \varepsilon_r$. Benchmark results for this checkerboard problem are available in Dauge [17] for $\varepsilon_r^{-1} \in \{2, 10, 100, 10^8\}$. Tables 3 and 4 show results for $\varepsilon_r = 0.5$ and $\varepsilon_r = 0.1$ respectively. The ratio $\frac{|\lambda_c - \lambda_r|}{\lambda_r}$ is reported in column "rel. err.", where λ_c and λ_r are the computed and reference eigenvalues, respectively. The reference values are those from the benchmark. The computed order of convergence is shown in the column "coc". The computations have been done using ARPACK (cf. Lehoucq et al. [26]) with tolerance 10^{-8}. Note that the computed order of convergence seems to reach a constant value for sufficiently small h, for every eigenvalue, as expected.

TABLE 3. Approximation of the first four eigenvalues for $\varepsilon_r = 0.5$. We used $\alpha = 0.7$ in the simulations.

h	$\lambda_r \simeq 3.3175$		$\lambda_r \simeq 3.3663$		$\lambda_r \simeq 6.1863$		$\lambda_r \simeq 13.926$	
	rel. err.	coc	rel. err.	coc	rel. err.	coc	rel. err.	coc
0.2	9.364E-4	-	3.943E-3	-	1.439E-1	-	6.104E-1	-
0.1	1.833E-4	2.35	2.147E-3	0.88	1.734E-4	9.70	4.484E-1	0.44
0.05	3.751E-5	2.29	1.188E-3	0.85	2.241E-5	2.95	1.599E-1	1.49
0.025	8.405E-6	2.16	6.463E-4	0.88	2.833E-6	2.98	1.120E-5	13.8
0.0125	2.081E-6	2.01	3.439E-4	0.91	3.667E-7	2.95	1.478E-6	2.92

6.4. **The case $\alpha = 1$.** We have shown that the numerical method is optimally convergent with $\alpha = 1$ for the boundary value problem (2.12) if $\nabla\cdot(\varepsilon\mathbf{g}) = 0$. It is then reasonable to investigate the convergence properties of the method for the eigenvalue problem with $\alpha = 1$ even though the theoretical analysis seems to show that there might be a loss of compactness in this case; i.e., we cannot apply Theorem 6.1. We investigate this issue by solving again the checkerboard problem introduced in the previous section and by comparing the results obtained with $\alpha = 0.7$ and $\alpha = 1$. We compute the first 10 eigenvalues for $\varepsilon_r = 0.5$ and report the results in Table 5

TABLE 4. Approximation of the first four eigenvalues for $\varepsilon_r = 0.1$. We used $\alpha = 0.8$ in the simulations.

h	$\lambda_r \simeq 4.5339$		$\lambda_r \simeq 6.2503$		$\lambda_r \simeq 7.0371$		$\lambda_r \simeq 22.342$	
	rel. err.	coc	rel. err.	coc	rel. err.	coc	rel. err.	coc
0.2	4.559E-1	-	6.052E-1	-	6.410E-1	-	8.869E-1	-
0.1	2.859E-1	0.67	4.731E-1	0.36	5.310E-1	0.27	8.512E-1	0.06
0.05	3.306E-2	3.11	2.982E-1	0.67	3.763E-1	0.50	8.033E-1	0.08
0.025	2.154E-6	13.9	7.748E-2	1.94	1.772E-1	1.09	7.406E-1	0.12
0.0125	2.608E-7	3.05	3.258E-3	4.57	5.946E-7	18.2	6.602E-1	0.17

for $\mathbb{P}_1$ finite elements and Table 6 for $\mathbb{P}_2$ finite elements. The typical meshsize in these simulations is 0.025. Inspection of these tables show that the approximation with $\alpha = 1$ is not spectrally correct. Other results on meshes with different meshsizes or structure (Delaunay, Powell-Sabin or HCT), not reported here, show the same type of behavior, i.e., there are spurious eigenvalues when $\alpha = 1$. This series of numerical tests confirms the sharpness on the upper bound on α stated in Lemma 6.3.

TABLE 5. Approximation of the first ten eigenvalues with $\mathbb{P}_1$ elements and $\varepsilon_r = 0.5$. Comparison between $\alpha = 0.7$ and $\alpha = 1.0$.

λ	$\alpha = 0.7$		$\alpha = 1.0$	
	app. value	rel. error	app. value	rel. error
3.31755	3.31844	2.70E-4	3.31790	1.06E-4
3.36632	3.37816	3.51E-3	3.36786	4.56E-4
6.18639	6.18732	1.50E-4	3.91497	3.67E-1
13.9263	13.9321	4.14E-4	3.91616	7.18E-1
15.0830	15.0888	3.88E-4	4.14335	7.25E-1
15.7789	15.7859	4.48E-4	4.29445	7.27E-1
18.6433	18.6555	6.53E-4	4.30863	7.68E-1
25.7975	25.8163	7.29E-4	15.0191	4.17E-1
29.8524	29.8684	5.36E-4	35.7192	1.96E-1
30.5379	30.5643	8.66E-4	305.349	9.00E0

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Appendix A. Technical Lemmas

Let $\{\mathcal{T}_h\}_{h>0}$ be an affine shape-regular mesh sequence in $\mathbb{R}^3$. Let $T_K : \widehat{K} \longrightarrow K$ be the affine mapping that maps the reference element $\widehat{K}$ to K and let J_K be the Jacobian of T_K. It is a standard result that there are constants that depend only on $\widehat{K}$ and the shape regularity constants of the mesh sequence so that

$$\|J_K\| \le ch_K, \quad \|J_K^{-1}\| \le ch_K^{-1}, \qquad |\det(J_K)| \le ch_K^3, \quad |\det(J_K^{-1})| \le ch_K^{-3}, \tag{A.1}$$

where h_K is the diameter of K.

TABLE 6. Approximation of the first ten eigenvalues with $\mathbb{P}_2$ elements and $\varepsilon_r = 0.5$. Comparison between $\alpha = 0.7$ and $\alpha = 1.0$.

λ	$\alpha = 0.7$		$\alpha = 1.0$	
	app. value	rel. error	app. value	rel. error
3.31755	3.31758	8.55E-6	3.31756	2.30E-6
3.36632	3.36857	6.68E-4	3.36634	3.62E-6
6.18639	6.18641	3.14E-6	4.28879	3.07E-1
13.9263	13.9265	1.05E-5	4.29153	6.92E-1
15.0830	15.0832	1.14E-5	4.30113	7.15E-1
15.7789	15.7791	1.36E-5	4.30145	7.27E-1
18.6433	18.6436	1.52E-5	4.30683	7.69E-1
25.7975	25.7979	1.36E-5	12.8213	5.03E-1
29.8524	29.8530	2.04E-5	37.1980	2.46E-1
30.5379	30.5395	5.43E-5	1308.73	4.19E+1

Lemma A.1. *For all $s \in [0,1]$, there is a constant c, uniform with respect to the mesh sequence, so that the following holds for all cells $K \in \mathcal{T}_h$ and all $\psi \in \mathrm{H}^s(K)$ with zero average over K:*

$$\|\widehat{\psi}\|_{\mathrm{H}^s(\widehat{K})} \le c h_K^{s-\frac{d}{2}} \|\psi\|_{\mathrm{H}^s(K)}, \qquad \textit{where} \qquad \widehat{\psi}(\mathbf{x}) := \psi(T_K(\mathbf{x})) \tag{A.2}$$

Proof. Upon making the change of variable $\mathbf{x} = T_K(\widehat{\mathbf{x}})$ we obtain

$$\|\widehat{\psi}\|_{\mathrm{L}^2(\widehat{K})} = |\det(J_K)|^{-\frac{1}{2}} \|\psi\|_{\mathrm{L}^2(K)} \le c h_K^{-\frac{d}{2}} \|\psi\|_{\mathrm{L}^2(K)}.$$

Likewise, using the fact that $\widehat{\psi}$ is of zero average, the Poincaré inequality implies

$$\begin{aligned}\|\widehat{\psi}\|_{\mathrm{H}^1(\widehat{K})} &= \left(\|\widehat{\psi}\|^2_{\mathrm{L}^2(\widehat{K})} + \|\widehat{\nabla}\widehat{\psi}\|^2_{\mathrm{L}^2(\widehat{K})}\right)^{\frac{1}{2}} \le (c_p(\widehat{K})+1)^{\frac{1}{2}} \|\widehat{\nabla}\widehat{\psi}\|_{\mathrm{L}^2(\widehat{K})} \\ &\le c|\det(J_K)|^{-\frac{1}{2}} \|J_K\| \|\nabla\psi\|_{\mathrm{L}^2(K)} \le c h_K^{-\frac{d}{2}+1} \|\psi\|_{\mathrm{H}^1(K)}.\end{aligned}$$

Then, the interpolationtheorem implies that

$$\|\widehat{\psi}\|_{\dot{\mathrm{H}}^s(\widehat{K})} \le c\, h_K^{s-\frac{3}{2}} \|\psi\|_{\dot{\mathrm{H}}^s(K)},$$

where we defined $\dot{\mathrm{H}}^s(E) := [\dot{\mathrm{L}}^2(E), \dot{\mathrm{H}}^1(E)]_{s,2}$ with $\dot{\mathrm{L}}^2(E)$ and $\dot{\mathrm{H}}^1(E)$ being the subspaces of the functions of zero average in $\mathrm{L}^2(E)$ and $\mathrm{H}^1(E)$, respectively. We conclude using Lemma A.2 □

Lemma A.2. *The spaces $[\dot{\mathrm{L}}^2(E), \dot{\mathrm{H}}^1(E)]_s$ and $[\mathrm{L}^2(E), \mathrm{H}^1(E)]_s \cap \dot{\mathrm{L}}^2(E)$ are identical and the induced norms are identical, i.e., $\|v\|_{\dot{\mathrm{H}}^s(E)} = \|v\|_{\mathrm{H}^s(E)}$ for all $v \in [\mathrm{L}^2(E), \mathrm{H}^1(E)]_s \cap \dot{\mathrm{L}}^2(E)$.*

Proof. One can use Lemma A1 from Guermond [23] with T being the projection onto $\dot{\mathrm{L}}^2(\Omega)$. □

We now state the main result of this section. It is a variant of Lemma 8.2 in Buffa and Perugia [10] with the extra term $\|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)}$. Our proof slightly differs from that in Buffa and Perugia [10] since the proof therein did not appear convincing to us (actually, the embedding inequality at line 9, page 2224 in Buffa and Perugia [10] has a constant that depends on the size of the cell; for instance, using a constant vector field for $\boldsymbol{\phi}$in this inequality yields a contradiction. As result the estimate (8.11) in [10] is not uniform with respect to h).

Lemma A.3. *For all $k \in \mathbb{N}$ and all $\sigma \in (0, \frac{1}{2})$ there is c, uniform with respect to the mesh sequence, so that the following holds for all faces $F \in \mathcal{F}_h$ in the mesh, all polynomial function $\mathbf{v}$ of degree at most k, and all function $\boldsymbol{\phi} \in \mathbf{H}^{\sigma}(K) \cap \mathbf{H}(\text{curl}, K)$*

$$\left| \int_F (\mathbf{v} \times \mathbf{n}) \cdot \boldsymbol{\phi} \right| \le c \|\mathbf{v}\|_{\mathbf{L}^2(F)} h_F^{-\frac{1}{2}} (h_K^{\sigma} \|\boldsymbol{\phi}\|_{\mathbf{H}^{\sigma}(K)} + h_K \|\nabla \times \boldsymbol{\phi}\|_{\mathbf{L}^2(K)} + \|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)}), \tag{A.3}$$

where K is either one of the two elements sharing the face F.

Proof. We restrict ourselves to three space dimensions. In two space dimensions $\boldsymbol{\phi}$ is scalar-valued and the proof must be modified accordingly. Let K be either one of the two elements sharing the face F. Let $\overline{\boldsymbol{\phi}}$ be the average of $\boldsymbol{\phi}$ over K and let us denote $\boldsymbol{\psi} := \boldsymbol{\phi} - \overline{\boldsymbol{\phi}}$. Upon denoting $\widehat{\mathbf{v}}(\widehat{\mathbf{x}}) = J_K^T \mathbf{v}(T_K(\widehat{\mathbf{x}}))$ and $\widehat{\boldsymbol{\psi}}(\widehat{\mathbf{x}}) = J_K^T \boldsymbol{\psi}(T_K(\widehat{\mathbf{x}}))$, it is a standard result (see Monk [29, 3.82]) that

$$\int_F (\mathbf{v} \times \mathbf{n}) \cdot \boldsymbol{\psi} = \int_{\widehat{F}} (\widehat{\mathbf{v}} \times \widehat{\mathbf{n}}) \cdot \widehat{\boldsymbol{\psi}},$$

where $\widehat{\mathbf{n}}$ is one of the two unit normals on $\widehat{F}$. Let us extend $\widehat{\mathbf{v}}$ by zero on $\partial\widehat{K} \backslash \widehat{F}$; then $\widehat{\mathbf{v}} \in \mathbf{H}^{\frac{1}{2}-\sigma}(\partial\widehat{K})$ for all $\sigma > 0$, since the extension by zero is stable in the H^s-norm for all $s \in [0, \frac{1}{2})$, see e.g. Lions and Magenes [27, Thm. 11.4] for smooth domains and **?** , Thm. 1.4.2.4 or Cor. 1.4.4.5] for Lipschitz domains.Note that it is not possible to have $\sigma = 0$. Now let $R : \mathbf{H}^{\frac{1}{2}-\sigma}(\partial\widehat{K}) \longrightarrow \mathbf{H}^{1-\sigma}(\widehat{K})$ be a standard lifting operator. There is a constant depending only on $\widehat{K}$ and σ so that

$$\|R\widehat{\mathbf{v}}\|_{\mathbf{L}^2(\widehat{K})} + \|\widehat{\nabla} \times R\widehat{\mathbf{v}}\|_{\mathbf{H}^{-\sigma}(\widehat{K})} \le c(\widehat{K}, \sigma) \|R\widehat{\mathbf{v}}\|_{\mathbf{H}^{1-\sigma}(\widehat{K})} \le c' c(\widehat{K}, \sigma) \|\widehat{\mathbf{v}}\|_{\mathbf{H}^{\frac{1}{2}-\sigma}(\widehat{F})},$$

where $\widehat{\nabla}\times$ is the curl operator in the coordinate system of $\widehat{K}$. Then, slightly abusing the notation by using integrals instead of duality products, we have

$$\begin{aligned}
\left| \int_{\widehat{F}} (\widehat{\mathbf{v}} \times \widehat{\mathbf{n}}) \cdot \widehat{\boldsymbol{\psi}} \right| &= \left| \int_{\widehat{K}} \left((R\widehat{\mathbf{v}}) \cdot \widehat{\nabla} \times \widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}} \cdot \widehat{\nabla} \times (R\widehat{\mathbf{v}}) \right) \right| \\
&\le c \left(\|(R\widehat{\mathbf{v}})\|_{\mathbf{L}^2(\widehat{K})} \|\widehat{\nabla} \times \widehat{\boldsymbol{\psi}}\|_{\mathbf{L}^2(\widehat{K})} + \|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}_0^{\sigma}(\widehat{K})} \|\widehat{\nabla} \times (R\widehat{\mathbf{v}})\|_{\mathbf{H}^{-\sigma}(\widehat{K})} \right) \\
&\le c \left(\|\widehat{\nabla} \times \widehat{\boldsymbol{\psi}}\|_{\mathbf{L}^2(\widehat{K})} + \|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}_0^{\sigma}(\widehat{K})} \right) \|\widehat{\mathbf{v}}\|_{\mathbf{H}^{\frac{1}{2}-\sigma}(\widehat{F})} \\
&\le c \left(\|\widehat{\nabla} \times \widehat{\boldsymbol{\psi}}\|_{\mathbf{L}^2(\widehat{K})} + \|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}^{\sigma}(\widehat{K})} \right) \|\widehat{\mathbf{v}}\|_{\mathbf{H}^{\frac{1}{2}-\sigma}(\widehat{F})},
\end{aligned}$$

where we used that $\mathbf{H}^{\sigma}(\widehat{K}) = \mathbf{H}_0^{\sigma}(\widehat{K})$ for $\sigma \in [0, \frac{1}{2})$. Due to norm equivalence for discrete functions over $\widehat{K}$ and using that $\|J_K\| \le ch_K$, $h_K/h_F \le c$ and $|F| \le ch_F^2$ in three space dimensions, where c depends of the shape-regularity constant of the mesh sequence and the polynomial degree k, we have

$$\|\widehat{\mathbf{v}}\|_{\mathbf{H}^{\frac{1}{2}-\sigma}(\widehat{F})} \le c \|\widehat{\mathbf{v}}\|_{\mathbf{L}^2(\widehat{F})} \le c \|J_K\| |F|^{-\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(F)} \le ch_K h_F^{-1} \|\mathbf{v}\|_{\mathbf{L}^2(F)} \le c' \|\mathbf{v}\|_{\mathbf{L}^2(F)}.$$

Using the identity (see Monk [29, Cor. 3.58])

$$(\nabla \times \boldsymbol{\psi})(T_K(\widehat{\mathbf{x}})) = \frac{1}{\det(J_K)} J_K (\widehat{\nabla} \times \widehat{\boldsymbol{\psi}})(\widehat{\mathbf{x}}),$$

we obtain

$$\|\widehat{\nabla} \times \widehat{\boldsymbol{\psi}}\|_{\mathbf{L}^2(K)} \le c |\det(J_K)|^{\frac{1}{2}} \|J_K^{-1}\| \|\nabla \times \boldsymbol{\psi}\|_{\mathbf{L}^2(K)} \le ch_K^{\frac{1}{2}} \ \|\nabla \times \boldsymbol{\psi}\|_{\mathbf{L}^2(K)}.$$

Since the average of $\boldsymbol{\psi}$ over K is zero, we can use Lemma A.1 (with an extra scaling by $\|J_K\|$ for $\widehat{\boldsymbol{\psi}} = J_K^T\boldsymbol{\psi}(T_K)$) to deduce

$$\|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}^\sigma(\widehat{K})} \le c h_K^{\sigma-\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}^\sigma(K)}.$$

In conclusion we have obtained the following estimate:

$$\int_F (\mathbf{v}\times\mathbf{n})\cdot(\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}) \le c\left(h_K\ \|\nabla\times\boldsymbol{\phi}\|_{\mathbf{L}^2(K)} + h_K^\sigma \|\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}\|_{\mathbf{H}^\sigma(K)}\right) h_K^{-\frac{1}{2}} \|\mathbf{v}\|_{\mathbf{L}^2(F)}.$$

Observing that $\|1\|_{\mathbf{H}^\sigma(K)} \le \|1\|^{1-\sigma}_{\mathbf{L}^2(K)}\|1\|^{\sigma}_{\mathbf{H}^1(K)} = \|1\|_{\mathbf{L}^2(K)} = |K|^{\frac{1}{2}}$, we infer that

$$\|\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}\|_{\mathbf{H}^\sigma(K)} \le \|\boldsymbol{\phi}\|_{\mathbf{H}^\sigma(K)} + |\overline{\boldsymbol{\phi}}||K|^{\frac{1}{2}}$$

The Cauchy-Schwarz inequality yields $|\overline{\boldsymbol{\phi}}| \le |K|^{-\frac{1}{2}}\|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)}$; as a result,

$$\|\boldsymbol{\phi}-\overline{\boldsymbol{\phi}}\|_{\mathbf{H}^\sigma(K)} \le \|\boldsymbol{\phi}\|_{\mathbf{H}^\sigma(K)} + \|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)} \le 2\|\boldsymbol{\phi}\|_{\mathbf{H}^\sigma(K)}.$$

Now we evaluate a bound from above on $\int_F(\mathbf{v}\times\mathbf{n})\cdot\overline{\boldsymbol{\phi}}$ as follows:

$$\begin{aligned}\left|\int_F(\mathbf{v}\times\mathbf{n})\cdot\overline{\boldsymbol{\phi}}\right| &\le |\overline{\boldsymbol{\phi}}||F|^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(F)} \le |K|^{-\frac{1}{2}}\|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)}|F|^{\frac{1}{2}}\|\mathbf{v}\|_{\mathbf{L}^2(F)}\\ &\le c\|\mathbf{v}\|_{\mathbf{L}^2(F)} h_F^{-\frac{1}{2}}\|\boldsymbol{\phi}\|_{\mathbf{L}^2(K)}.\end{aligned}$$

The result follows by combining all the above estimates. □

Lemma A.4. *Let $\alpha \in (\frac{1}{2}, 1)$. There is exists a constant $c(\alpha)$ so that*

$$\|u\|_{\mathrm{L}^2(\Gamma)} \le c(\alpha)\|u\|^{1-\frac{1}{2\alpha}}_{\mathrm{L}^2(\Omega)}\|u\|^{\frac{1}{2\alpha}}_{\mathrm{H}^\alpha(\Omega)}, \qquad \forall u \in \mathrm{H}^\alpha(\Omega). \tag{A.4}$$

Similarly, for $s \in \left(0, \frac{1}{2}\right)$, there exists a constant $c(s)$ so that, for $\Theta := \frac{1-2s}{2(1-s)}$,

$$\|u\|_{\mathrm{L}^2(\Gamma)} \le c(s)\|u\|^{1-\Theta}_{\mathrm{H}^s(\Omega)}\|u\|^{\Theta}_{\mathrm{H}^1(\Omega)}, \qquad \forall u \in \mathrm{H}^1(\Omega). \tag{A.5}$$

Proof. We start with the standard estimate

$$\|u\|_{\mathrm{L}^2(\Gamma)} \le c\|u\|^{\frac{1}{2}}_{\mathrm{L}^2(\Omega)}\|u\|^{\frac{1}{2}}_{\mathrm{H}^1(\Omega)}, \qquad \forall u \in \mathrm{H}^1(\Omega),$$

which allows us to apply Lemma A.5. This implies that the trace operator is a continuous linear mapping from $[\mathrm{L}^2(\Omega), \mathrm{H}^1(\Omega)]_{\frac{1}{2},1}$ to $\mathrm{L}^2(\Gamma)$. Then the Lions-Petree reiteration theorem Tartar [33, Thm. 26.3]implies that

$$\begin{aligned}[\mathrm{L}^2(\Omega), \mathrm{H}^\alpha(\Omega)]_{\frac{1}{2\alpha},1} &= [\mathrm{L}^2(\Omega), [\mathrm{L}^2(\Omega), \mathrm{H}^1(\Omega)]_{\alpha,2}]_{\frac{1}{2\alpha},1} = [\mathrm{L}^2(\Omega), \mathrm{H}^1(\Omega)]_{\frac{1}{2},1}\\ [\mathrm{H}^s(\Omega), \mathrm{H}^1(\Omega)]_{\Theta,1} &= [[\mathrm{L}^2(\Omega), \mathrm{H}^1(\Omega)]_{s,2}, \mathrm{H}^1(\Omega)]_{\Theta,1} = [\mathrm{L}^2(\Omega), \mathrm{H}^1(\Omega)]_{\frac{1}{2},1}\end{aligned}$$

The norms being equivalent, we can eventually write:

$$\begin{aligned}\|u\|_{\mathrm{L}^2(\Gamma)} &\le c\|u\|_{[\mathrm{L}^2(\Omega),\mathrm{H}^1(\Omega)]_{\frac{1}{2},1}} \le c(\alpha)\|u\|_{[\mathrm{L}^2(\Omega),\mathrm{H}^\alpha(\Omega)]_{\frac{1}{2\alpha},1}} \le c(\alpha)\|u\|^{1-\frac{1}{2\alpha}}_{\mathrm{L}^2(\Omega)}\|u\|^{\frac{1}{2\alpha}}_{\mathrm{H}^\alpha(\Omega)},\\ \|u\|_{\mathrm{L}^2(\Gamma)} &\le c\|u\|_{[\mathrm{L}^2(\Omega),\mathrm{H}^1(\Omega)]_{\frac{1}{2},1}} \le c(s)\|u\|_{[\mathrm{H}^s(\Omega),\mathrm{H}^1(\Omega)]_{\Theta,1}} \le c(s)\|u\|^{1-\Theta}_{\mathrm{H}^s(\Omega)}\|u\|^{\Theta}_{\mathrm{H}^1(\Omega)}.\end{aligned}$$

This concludes the proof. □

Lemma A.5 (Lions-Petree). *Let $E_1 \subset E_0$ be two Banach spaces, with continuous embedding. Let L be a linear mapping $E_1 \to F$ with F another Banach space. For $s \in (0,1)$, L extends to a linear mapping from $[E_0, E_1]_{s,1}$ to F if and only if there exists $C > 0$ such that*

$$\forall u \in E_1, \qquad \|Lu\|_F \leq C\|u\|_{E_0}^{1-s}\|u\|_{E_1}^{s}.$$

Proof. See Lemma 25.3 in Tartar [33]. □

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