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## HOPF STRUCTURES ON THE MULTIPLIHEDRA

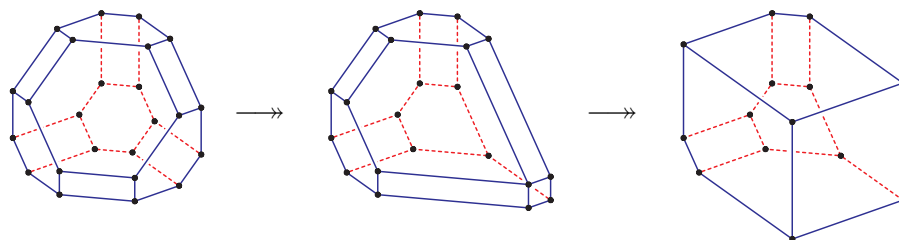
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**ABSTRACT.** We investigate algebraic structures that can be placed on vertices of the multiplihedra, a family of polytopes originating in the study of higher categories and homotopy theory. Most compelling among these are two distinct structures of a Hopf module over the Loday–Ronco Hopf algebra.

### INTRODUCTION

The permutahedra  $\mathfrak{S}$ . form a family of highly symmetric polytopes that have been of interest since their introduction by Schoute in 1911 [23]. The associahedra  $\mathcal{Y}$ . are another family of polytopes that were introduced by Stasheff as cell complexes in 1963 [25], and with the permutahedra were studied from the perspective of monoidal categories and  $H$ -spaces [17] in the 1960s. Only later were associahedra shown to be polytopes [11, 13, 18]. Interest in these objects was heightened in the 1990s, when Hopf algebra structures were placed on them in work of Malvenuto, Reutenauer, Loday, Ronco, Chapoton, and others [6, 14, 16]. More recently, the associahedra were shown to arise in Lie theory through work of Fomin and Zelevinsky on cluster algebras [7].

We investigate Hopf structures on another family of polyhedra, the multiplihedra,  $\mathcal{M}$ .. Stasheff introduced them in the context of maps preserving higher homotopy associativity [26] and described their 1-skeleta. Boardman and Vogt [5], and then Iwase and Mimura [12] described the multiplihedra as cell complexes, and only recently were they shown to be convex polytopes [8]. These three families of polytopes are closely related. For each integer  $n \geq 1$ , the permutahedron  $\mathfrak{S}_n$ , multiplihedron  $\mathcal{M}_n$ , and associahedron  $\mathcal{Y}_n$  are polytopes of dimension  $n-1$  with natural cellular surjections  $\mathfrak{S}_n \twoheadrightarrow \mathcal{M}_n \twoheadrightarrow \mathcal{Y}_n$ , which we illustrate when  $n = 4$ .



The faces of these polytopes are represented by different flavors of planar trees; permutahedra by ordered trees (set compositions), multiplihedra by bi-leveled trees (Section 2.1), and associahedra by planar trees. The maps between them forget

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the additional structure on the trees. These maps induce surjective maps of graded vector spaces spanned by the vertices, which are binary trees. The span  $\mathfrak{S}Sym$  of ordered trees forms the Malvenuto-Reutenauer Hopf algebra [16] and the span  $\mathcal{Y}Sym$  of planar binary trees forms the Loday-Ronco Hopf algebra [14]. The algebraic structures of multiplication and comultiplication on  $\mathfrak{S}Sym$  and  $\mathcal{Y}Sym$  are described in terms of geometric operations on trees and the composed surjection  $\tau: \mathfrak{S} \rightarrow \mathcal{Y}$ . gives a surjective morphism  $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$  of Hopf algebras.

We define  $\mathcal{M}Sym$  to be the vector space spanned by the vertices of all multiplihedra. The factorization of  $\tau$  induced by the maps of polytopes,  $\mathfrak{S}Sym \rightarrow \mathcal{M}Sym \rightarrow \mathcal{Y}Sym$ , does not endow  $\mathcal{M}Sym$  with the structure of a Hopf algebra. Nevertheless, some algebraic structure does survive the factorization. We show in Section 3 that  $\mathcal{M}Sym$  is an algebra, which is simultaneously a  $\mathfrak{S}Sym$ -module and a  $\mathcal{Y}Sym$ -Hopf module algebra, and the maps preserve these structures.

We perform a change of basis in  $\mathcal{M}Sym$  using Möbius inversion that illuminates its comodule structure. Such changes of basis helped to understand the coalgebra structure of  $\mathfrak{S}Sym$  [1] and of  $\mathcal{Y}Sym$  [2]. Section 4 discusses a second  $\mathcal{Y}Sym$  Hopf module structure that may be placed on the positive part  $\mathcal{M}Sym_+$  of  $\mathcal{M}Sym$ . This structure also arises from polytope maps between  $\mathfrak{S}$  and  $\mathcal{Y}$ , but not directly from the algebra structure of  $\mathfrak{S}Sym$ . Möbius inversion again reveals an explicit basis of  $\mathcal{Y}Sym$  coinvariants in this alternate setting.

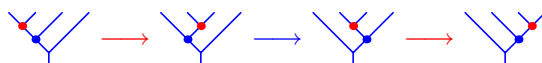
## 1. BASIC COMBINATORIAL DATA

The structures of the Malvenuto-Reutenauer and Loday-Ronco algebras are related to the weak order on ordered trees and the Tamari order on planar trees. There are natural maps between the weak and Tamari orders which induce a morphism of Hopf algebras. We first recall these partial orders and then the basic structure of these Hopf algebras. In Section 1.3 we establish a formula involving the Möbius functions of two posets related by an interval retract. This is a strictly weaker notion than that of a Galois correspondence, which was used to study the structure of the Loday-Ronco Hopf algebra.

1.1.  $\mathfrak{S}$  and  $\mathcal{Y}$ . The 1-skeleta of the families of polytopes  $\mathfrak{S}$ ,  $\mathcal{M}$ , and  $\mathcal{Y}$  are Hasse diagrams of posets whose structures are intertwined with the algebra structures we study. We use the same notation for a polytope and its poset of vertices. Similarly, we use the same notation for a cellular surjection of polytopes and the poset map formed by restricting that surjection to vertices.

For the permutahedron  $\mathfrak{S}_n$ , the corresponding poset is the (left) *weak order*, which we describe in terms of permutations. A cover in the weak order has the form  $w \triangleleft (k, k+1)w$ , where  $k$  precedes  $k+1$  among the values of  $w$ . Figure 1 displays the weak order on  $\mathfrak{S}_4$ . We let  $\mathfrak{S}_0 = \{\emptyset\}$ , where  $\emptyset$  is the empty permutation of  $\emptyset$ .

Let  $\mathcal{Y}_n$  be the set of rooted, planar binary trees with  $n$  nodes. The cover relations in the *Tamari order* on  $\mathcal{Y}_n$  are obtained by moving a child node directly above a given node from the left to the right branch above the given node. Thus



is an increasing chain in  $\mathcal{Y}_3$  (the moving vertices are marked with dots). Figure 1 shows the Tamari order on  $\mathcal{Y}_4$ .

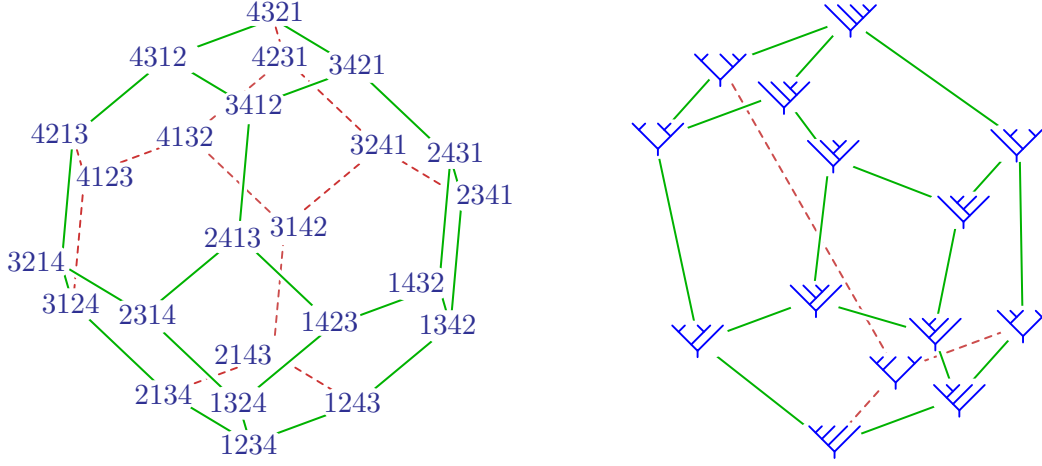
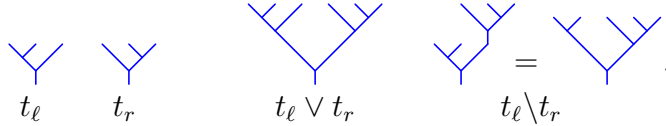


FIGURE 1. Weak order on  $\mathfrak{S}_4$  and Tamari order on  $\mathcal{Y}_4$

The unique tree in  $\mathcal{Y}_1$  is  $\Upsilon$ . Given trees  $t_\ell$  and  $t_r$ , form the tree  $t_\ell \vee t_r$  by grafting the root of  $t_\ell$  (respectively of  $t_r$ ) to the left (respectively right) leaf of  $\Upsilon$ . Form the tree  $t_\ell \setminus t_r$  by grafting the root of  $t_r$  to the rightmost leaf of  $t_\ell$ . For example,



Decompositions  $t = t_1 \setminus t_2$  correspond to pruning  $t$  along the right branches from the root. A tree  $t$  is *indecomposable* if it has no nontrivial decomposition  $t = t_1 \setminus t_2$  with  $t_1, t_2 \neq \mathbb{1}$ . Equivalently, if the root node is the rightmost node of  $t$ . Any tree  $t$  is uniquely decomposed  $t = t_1 \setminus \dots \setminus t_m$  into indecomposable trees  $t_1, \dots, t_m$ .

We define a poset map  $\tau : \mathfrak{S}_n \rightarrow \mathcal{Y}_n$ . First, given distinct integers  $a_1, \dots, a_k$ , let  $\bar{a} \in \mathfrak{S}_k$  be the unique permutation such that  $\bar{a}(i) < \bar{a}(j)$  if and only if  $a_i < a_j$ . Thus  $\overline{4726} = 2413$ . Since  $\mathfrak{S}_0, \mathcal{Y}_0, \mathfrak{S}_1$ , and  $\mathcal{Y}_1$  are singletons, we must have

$$\begin{aligned} \tau : \mathfrak{S}_0 &\longrightarrow \mathcal{Y}_0 \quad \text{with} \quad \tau : \emptyset \longmapsto \mathbb{1}, \quad \text{and} \\ \tau : \mathfrak{S}_1 &\longrightarrow \mathcal{Y}_1 \quad \text{with} \quad \tau : 1 \longmapsto \Upsilon. \end{aligned}$$

Let  $n > 0$  and assume that  $\tau$  has been defined on  $\mathfrak{S}_k$  for  $k < n$ . For  $w \in \mathfrak{S}_n$  suppose that  $w(j) = n$ , and define

$$\tau(w) := \tau(\overline{w(1), \dots, w(j-1)}) \vee \tau(\overline{w(j+1), \dots, w(n)}).$$

For example,

$$\begin{aligned} \tau(12) &= \Upsilon \vee \mathbb{1} = \Upsilon \setminus \mathbb{1}, \quad \tau(21) = \mathbb{1} \vee \Upsilon = \Upsilon \setminus \mathbb{1}, \quad \text{and} \\ \tau(3421) &= \tau(\overline{3}) \vee \tau(\overline{21}) = \tau(1) \vee \tau(21) = \Upsilon \vee \Upsilon \setminus \mathbb{1} = \Upsilon \setminus (\Upsilon \setminus \mathbb{1}). \end{aligned}$$

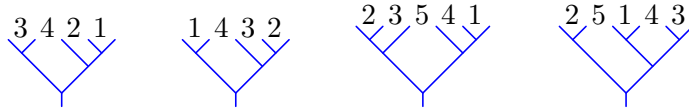
Loday and Ronco [15] show that the fibers  $\tau^{-1}(t)$  of  $\tau$  are intervals in the weak order. This gives two canonical sections of  $\tau$ . For  $t \in \mathcal{Y}_n$ ,

$$\mathbf{min}(t) := \min \{w \mid \tau(w) = t\} \quad \text{and} \quad \mathbf{max}(t) := \max \{w \mid \tau(w) = t\},$$

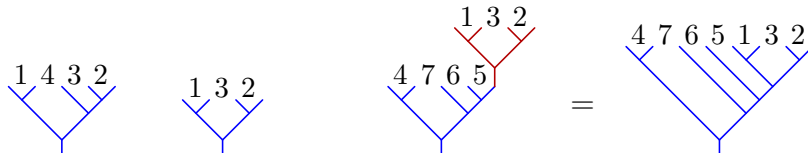
the minimum and maximum in the weak order. Equivalently,  $\mathbf{min}(t)$  is the unique 231-avoiding permutation in  $\tau^{-1}(t)$  and  $\mathbf{max}(t)$  is the unique 132-avoiding permutation. These maps are order-preserving.

The 1-skeleta of  $\mathfrak{S}_n$  and  $\mathcal{Y}_n$  form the Hasse diagrams of the weak and Tamari orders, respectively. Since  $\tau$  is an order-preserving surjection, it induces a cellular map between the 1-skeleta of these polytopes. Tonks [27] extended  $\tau$  to the faces of  $\mathfrak{S}_n$ , giving a cellular surjection.

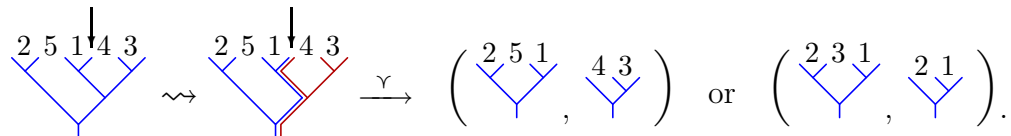
The nodes and internal edges of a tree are the Hasse diagram of a poset with the root node maximal. Labeling the nodes (equivalently, the gaps between the leaves) of  $\tau(w)$  with the values of the permutation  $w$  gives a linear extension of the node poset of  $\tau(w)$ , and all linear extensions of a tree  $t$  arise in this way for a unique permutation in  $\tau^{-1}(t)$ . Such a linear extension  $w$  of a tree is an *ordered tree* and  $\tau(w)$  is the corresponding unordered tree. In this way,  $\mathfrak{S}_n$  is identified with the set of ordered trees with  $n$  nodes. Here are some ordered trees,



Given ordered trees  $u, v$ , form the ordered tree  $u \setminus v$  by grafting the root of  $v$  to the rightmost leaf of  $u$ , where the nodes of  $u$  are greater than the nodes of  $v$ , but the relative orders within  $u$  and  $v$  are maintained. Thus we may decompose an ordered tree  $w = u \setminus v$  whenever  $\tau(w) = r \setminus s$  with  $\tau(u) = r$ ,  $\tau(v) = s$ , and the nodes of  $r$  in  $w$  precede the nodes of  $s$  in  $w$ . An ordered tree  $w$  is *indecomposable* if it has no nontrivial such decompositions. Here are ordered trees  $u, v$  and  $u \setminus v$ ,



We may *split* an ordered tree  $w$  along a leaf to obtain either an ordered forest (where the nodes in the forest are totally ordered) or a pair of ordered trees,



Write  $w \overset{\Upsilon}{\rightarrow} (w_0, w_1)$  to indicate that the ordered forest  $(w_0, w_1)$  (or pair of ordered trees) is obtained by splitting  $w$  along some leaf. (Context will determine how to interpret the result.) More generally, we may split an ordered tree  $w$  along a multiset of  $m \geq 0$  of its leaves to obtain an ordered forest, or tuple of ordered

trees, written  $w \xrightarrow{\gamma} (w_0, \dots, w_m)$ . For example,

$$(1.1) \quad \begin{array}{c} \text{Diagram 1: A tree with root at the bottom and four children. The children are labeled 3, 2, 7, 5, 1, 6, 4 from left to right. The root is at the bottom, and the children are at the top. The root is connected to 3 and 2, 3 and 2 are connected to 7 and 5, 7 and 5 are connected to 1 and 6, 1 and 6 are connected to 4.} \\ \rightsquigarrow \\ \text{Diagram 2: The same tree as Diagram 1, but with the children 3, 2, 7, 5, 1, 6, 4 colored in different colors: 3 (blue), 2 (blue), 7 (green), 5 (green), 1 (red), 6 (red), 4 (purple).} \\ \xrightarrow{\gamma} \left( \begin{array}{c} \text{Diagram 3: A tree with root at the bottom and two children. The children are labeled 3, 2.} \\ \text{Diagram 4: A vertical line with a root at the bottom and a child at the top labeled 1.} \\ \text{Diagram 5: A tree with root at the bottom and three children. The children are labeled 7, 5, 1.} \\ \text{Diagram 6: A tree with root at the bottom and two children. The children are labeled 6, 4.} \end{array} \right). \end{array}$$

Given  $v \in \mathfrak{S}_m$  and an ordered forest  $(w_0, \dots, w_m)$ , let  $(w_0, \dots, w_m)/v$  be the ordered tree obtained by grafting the root of  $w_i$  to the  $i$ th leaf of  $v$ , where the nodes of  $v$  are greater than all nodes of  $w$ , but the relative orders within the  $w_i$  and  $v$  are maintained. When  $v$  is the ordered tree corresponding to 1432 and  $w \xrightarrow{\gamma} (w_0, \dots, w_m)$  is the splitting (1.1), this grafting is

$$\begin{array}{c} \text{Diagram 1: A tree with root at the bottom and four children. The children are labeled 3, 2, 8, 11, 7, 5, 1, 10, 6, 9, 4 from left to right. The root is at the bottom, and the children are at the top. The root is connected to 3 and 2, 3 and 2 are connected to 8 and 11, 8 and 11 are connected to 7 and 5, 7 and 5 are connected to 1 and 10, 1 and 10 are connected to 6 and 9, 6 and 9 are connected to 4.} \\ = \\ \text{Diagram 2: A tree with root at the bottom and four children. The children are labeled 3, 2, 8, 11, 7, 5, 1, 10, 6, 9, 4 from left to right. The root is at the bottom, and the children are at the top. The root is connected to 3 and 2, 3 and 2 are connected to 8 and 11, 8 and 11 are connected to 7 and 5, 7 and 5 are connected to 1 and 10, 1 and 10 are connected to 6 and 9, 6 and 9 are connected to 4.} \end{array}$$

The notions of splitting and grafting also make sense for the unordered trees  $\mathcal{Y}_n$  and we use the same notation,  $\bullet \xrightarrow{\gamma} \bullet$  and  $\bullet/\bullet$ . (Simply delete the labels in the constructions above.) These operations of splitting and grafting are compatible with the map  $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$ : if  $w \xrightarrow{\gamma} (w_0, \dots, w_m)$  then  $\tau(w) \xrightarrow{\gamma} (\tau(w_0), \dots, \tau(w_m))$  and all splittings in  $\mathcal{Y}_\bullet$  are induced in this way from splittings in  $\mathfrak{S}_\bullet$ . The same is true for grafting,  $\tau((w_0, \dots, w_m)/v) = (\tau(w_0), \dots, \tau(w_m))/\tau(v)$ .

**1.2.  $\mathfrak{S}Sym$  and  $\mathcal{Y}Sym$ .** For basics on Hopf algebras, see [19]. Let  $\mathfrak{S}Sym := \bigoplus_{n \geq 0} \mathfrak{S}Sym_n$  be the graded  $\mathbb{Q}$ -vector space whose  $n^{\text{th}}$  graded piece has basis  $\{F_w \mid w \in \mathfrak{S}_n\}$ . Malvenuto and Reutenauer [16] defined a Hopf algebra structure on  $\mathfrak{S}Sym$ . For  $w \in \mathfrak{S}_\bullet$ , define the coproduct

$$\Delta F_w := \sum_{w \xrightarrow{\gamma} (w_0, w_1)} F_{w_0} \otimes F_{w_1},$$

where  $(w_0, w_1)$  is a pair of ordered trees. If  $v \in \mathfrak{S}_m$ , define the product

$$F_w \cdot F_v := \sum_{w \xrightarrow{\gamma} (w_0, \dots, w_m)} F_{(w_0, \dots, w_m)/v}.$$

The counit is the projection  $\varepsilon: \mathfrak{S}Sym \rightarrow \mathfrak{S}Sym_0$  onto the 0th graded piece, which is spanned by the unit,  $1 = F_\emptyset$ , for this multiplication.

**Proposition 1.1** ([16]). *With these definitions of coproduct, product, counit, and unit,  $\mathfrak{S}Sym$  is a graded, connected cofree Hopf algebra that is neither commutative nor cocommutative.*

Let  $\mathcal{Y}Sym := \bigoplus_{n \geq 0} \mathcal{Y}Sym_n$  be the graded  $\mathbb{Q}$ -vector space whose  $n^{\text{th}}$  graded piece has basis  $\{F_t \mid t \in \mathcal{Y}_n\}$ . Loday and Ronco [14] defined a Hopf algebra structure on  $\mathcal{Y}Sym$ . For  $t \in \mathcal{Y}_.$ , define the coproduct

$$\Delta F_t := \sum_{t \overset{\mathcal{Y}}{\rightarrow} (t_0, t_1)} F_{t_0} \otimes F_{t_1},$$

and if  $s \in \mathcal{Y}_m$ , define the product

$$F_t \cdot F_s := \sum_{t \overset{\mathcal{Y}}{\rightarrow} (t_0, \dots, t_m)} F_{(t_0, \dots, t_m)/s}.$$

The counit is the projection  $\varepsilon: \mathcal{Y}Sym \rightarrow \mathcal{Y}Sym_0$  onto the 0th graded piece, which is spanned by the unit,  $1 = F_1$ , for this multiplication. The map  $\tau$  extends to a linear map  $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$ , defined by  $\tau(F_w) = F_{\tau(w)}$ .

**Proposition 1.2** ([14]). *With these definitions of coproduct, product, counit, and unit,  $\mathcal{Y}Sym$  is a graded, connected cofree Hopf algebra that is neither commutative nor cocommutative and the map  $\tau$  a morphism of Hopf algebras.*

Some structures of the Hopf algebras  $\mathfrak{S}Sym$  and  $\mathcal{Y}Sym$ , particularly their primitive elements and coradical filtrations are better understood with respect to a second basis. The Möbius function  $\mu$  (or  $\mu_P$ ) of a poset  $P$  is defined for pairs  $(x, y)$  of elements of  $P$  with  $\mu(x, y) = 0$  if  $x \not\leq y$ ,  $\mu(x, x) = 1$ , and, if  $x < y$ , then

$$(1.2) \quad \mu(x, y) = - \sum_{x \leq z < y} \mu(x, z) \quad \text{so that} \quad 0 = \sum_{x \leq z \leq y} \mu(x, z).$$

For  $w \in \mathfrak{S}_.$  and  $t \in \mathcal{Y}_.$ , set

$$(1.3) \quad M_w := \sum_{w \leq v} \mu(w, v) F_v \quad \text{and} \quad M_t := \sum_{t \leq s} \mu(t, s) F_s,$$

where the first sum is over  $v \in \mathfrak{S}_.$ , the second sum over  $s \in \mathcal{Y}_.$ , and  $\mu(\cdot, \cdot)$  is the Möbius function in the weak and Tamari orders.

**Proposition 1.3** ([1, 2]). *If  $w \in \mathfrak{S}_.$ , then*

$$(1.4) \quad \tau(M_w) = \begin{cases} M_{\tau(w)}, & \text{if } w = \max(\tau(w)), \\ 0, & \text{otherwise} \end{cases}$$

and

$$(1.5) \quad \Delta M_w = \sum_{w = u \setminus v} M_u \otimes M_v.$$

If  $t \in \mathcal{Y}_.$ , then

$$(1.6) \quad \Delta M_t = \sum_{t = r \setminus s} M_r \otimes M_s.$$

This implies that the set  $\{M_w \mid w \in \mathfrak{S}_.\}$  is indecomposable is a basis for the primitive elements of  $\mathfrak{S}Sym$  (and the same for  $\mathcal{Y}Sym$ ), thereby explicitly realizing the cofree-ness of  $\mathfrak{S}Sym$  and  $\mathcal{Y}Sym$ .

**1.3. Möbius functions and interval retracts.** A pair  $f: P \rightarrow Q$  and  $g: Q \rightarrow P$  of poset maps is a *Galois connection* if  $f$  is left adjoint to  $g$  in that

$$\forall p \in P \text{ and } q \in Q, \quad f(p) \leq_Q q \iff p \leq_P g(q).$$

When this occurs, Rota [21, Theorem 1] related the Möbius functions of  $P$  and  $Q$ :

$$\forall p \in P \text{ and } q \in Q, \quad \sum_{f(y)=q} \mu_P(p, y) = \sum_{g(x)=q} \mu_Q(x, q).$$

Rota's formula was used in [2] to establish the coproduct formulas (1.4) and (1.6), as the maps  $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$  and  $\max: \mathcal{Y}_\bullet \rightarrow \mathfrak{S}_\bullet$  form a Galois connection [4, Section 9].

We do not have a Galois connection between  $\mathfrak{S}_\bullet$  and  $\mathcal{M}_\bullet$ , and so cannot use Rota's formula. Nevertheless, there is a useful relation between the Möbius functions of  $\mathfrak{S}_\bullet$  and  $\mathcal{M}_\bullet$  that we establish here in a general form. A surjective poset map  $f: P \rightarrow Q$  from a finite lattice  $P$  is an *interval retract* if the fibers of  $f$  are intervals and if  $f$  admits an order-preserving section  $g: Q \rightarrow P$  with  $f \circ g = \text{id}$ .

**Theorem 1.4.** *Let the poset map  $f: P \rightarrow Q$  is an interval retract, then the Möbius functions  $\mu_P$  and  $\mu_Q$  of  $P$  and  $Q$  are related by the formula*

$$(1.7) \quad \mu_Q(x, y) = \sum_{\substack{f(a)=x \\ f(b)=y}} \mu_P(a, b) \quad (\forall x, y \in Q).$$

In Section 2, we define an interval retract  $\beta: \mathfrak{S}_n \rightarrow \mathcal{M}_n$ .

We evaluate each side of (1.7) using Hall's formula, which expresses the Möbius function in terms of *chains*. A linearly ordered subset  $C: x_0 < \dots < x_r$  of a poset is a *chain* of length  $\ell(C) = r$  from  $x_0$  to  $x_r$ . Given a poset  $P$ , let  $\mathcal{C}(P)$  be the set of all chains in  $P$ . A poset  $P$  is an *interval* if it has a unique maximum element and a unique minimum element. If  $P = [x, y]$  is an interval, let  $\mathcal{C}'(P)$  denote the chains in  $P$  beginning in  $x$  and ending in  $y$ . Hall's formula states that

$$\mu(x, y) = \sum_{C \in \mathcal{C}'[x, y]} (-1)^{\ell(C)}.$$

Our proof rests on the following two lemmas.

**Lemma 1.5.** *If  $P$  is an interval, then  $\sum_{C \in \mathcal{C}(P)} (-1)^{\ell(C)} = 1$ .*

*Proof.* Suppose that  $P = [x, y]$  and append new minimum and maximum elements to  $P$  to get  $\hat{P} := P \cup \{\hat{0}, \hat{1}\}$ . Then the definition of Möbius function (1.2) gives

$$\mu(\hat{0}, \hat{1}) = - \sum_{\hat{0} \leq z \leq y} \mu(\hat{0}, z),$$

which is zero by (1.2). By Hall's formula,

$$0 = \mu(\hat{0}, \hat{1}) = \sum_{C \in \mathcal{C}'[\hat{0}, \hat{1}]} (-1)^{\ell(C)} = -1 + \sum_{C \in \mathcal{C}(P)} (-1)^{\ell(C)+2},$$

where the term  $-1$  comes from the chain  $\hat{0} < \hat{1}$ . This proves the lemma.  $\square$

Call a partition  $P = K_0 \sqcup \cdots \sqcup K_r$  of  $P$  into subposets  $K_i$  *monotone* if  $x < y$  with  $x \in K_i$  and  $y \in K_j$  implies that  $i \leq j$ . Given  $\emptyset \subsetneq I \subseteq [0, r]$ , write  $\mathfrak{U}_I(P)$  for the subset of chains  $C$  in  $\mathfrak{U}(P)$  such that  $C \cap K_i \neq \emptyset$  if and only if  $i \in I$ .

**Lemma 1.6.** *Let  $P = K_0 \sqcup \cdots \sqcup K_r$  be a monotonic partition of a poset  $P$ . If  $\bigcup_{i \in I} K_i$  is an interval for all  $I \subseteq [0, r]$ , then*

$$(1.8) \quad \sum_{C \in \mathfrak{U}_{[0,r]}(P)} (-1)^{\ell(C)} = (-1)^r.$$

*Proof.* We argue by induction on  $r$ . Lemma 1.5 is the case  $r = 0$  (wherein  $K_0 = P$ ), so we consider the case  $r \geq 1$ .

Form the poset  $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$  as in the proof of Lemma 1.5. Since  $P$  is an interval, we have  $\sum_{C \in \mathfrak{U}'[\hat{0}, \hat{1}]} (-1)^{\ell(C)} = 0$ . As  $\mathfrak{U}'[\hat{0}, \hat{1}] = \bigsqcup_I \mathfrak{U}_I(P)$  we have,

$$0 = -1 + \sum_{\emptyset \subsetneq I \subsetneq [0,r]} \left( \sum_{C \in \mathfrak{U}_I(P)} (-1)^{\ell(C)} \right) + \sum_{C \in \mathfrak{U}_{[0,r]}(P)} (-1)^{\ell(C)},$$

where the term  $-1$  counts the chain  $\hat{0} < \hat{1}$ . Applying induction, we have

$$0 = \sum_{k=0}^r \binom{r+1}{k} (-1)^{k-1} + \sum_{C \in \mathfrak{U}_{[0,r]}(P)} (-1)^{\ell(C)}.$$

Comparing this to the binomial expansion of  $(1-1)^{r+1}$  completes the proof.  $\square$

*Proof of Theorem 1.4.* Fix  $x < y$  in  $Q$ . We use Hall's formula to rewrite the right-hand side of (1.7) as

$$(1.9) \quad \sum_{\substack{f(x)=a \\ f(y)=b}} \sum_{C \in \mathfrak{U}'[a,b]} (-1)^{\ell(C)}.$$

Fix a chain  $D: q_0 < \cdots < q_r$  in  $\mathfrak{U}'[x, y]$  and let  $P|_D$  be the subposet of  $P$  consisting of elements that occur in some chain of  $P$  that maps to  $D$  under  $f$ . This is nonempty as  $f$  has section. Furthermore, the sets  $K_i := f^{-1}(q_i) \cap P|_D$ , for  $i = 0, \dots, r$ , form a monotonic partition of  $P|_D$ . We claim that  $\bigcup_{i \in I} K_i$  is an interval for all  $I \subseteq [0, r]$ . If so, let us first rewrite (1.9) as a sum over chains  $D$  in  $Q$ ,

$$\sum_{D \in \mathfrak{U}'[x,y]} \sum_{C \in \mathfrak{U}_{[0,\ell(D)]}(P|_D)} (-1)^{\ell(C)}.$$

By Lemma 1.6, the inner sum becomes  $\sum_D (-1)^{\ell(D)}$ , which completes the proof.

To prove the claim, suppose that  $I = \{i_0 < \cdots < i_s\}$ . Each set  $K_i$  ( $i \in I$ ) is an interval, as it is the intersection of two intervals in the lattice  $P$ . Thus  $K_{i_0}$  and  $K_{i_s}$  are intervals with minimum and maximum elements  $m$  and  $M$ , respectively. Any chain in  $\bigcup_{i \in I} K_i$  can be extended to a chain beginning with  $m$  and ending at  $M$ , so  $\bigcup_{i \in I} K_i$  is an interval.  $\square$



2. THE MULTIPLIHEDRA  $\mathcal{M}_\bullet$ .

The map  $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$  forgets the linear ordering of the node poset of an ordered tree, and it induces a morphism of Hopf algebras  $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$ . In fact, one may take the (ahistorical) view that the Hopf structure on  $\mathcal{Y}Sym$  is induced from that on  $\mathfrak{S}Sym$  via the map  $\tau$ . Forgetting some, but not all, of the structure on a tree in  $\mathfrak{S}_\bullet$  factorizes the map  $\tau$ . Here, we study combinatorial consequences of one such factorization, and later treat its algebraic consequences.

**2.1. Bi-leveled trees.** A *bi-leveled tree*  $(t; \mathbb{T})$  is a planar binary tree  $t \in \mathcal{Y}_n$  together with an (upper) order ideal  $\mathbb{T}$  of its node poset, where  $\mathbb{T}$  contains the leftmost node of  $t$  as a minimal element. Thus  $\mathbb{T}$  contains all nodes along the path from the leftmost leaf to the root, and none above the leftmost node. Numbering the gaps between the leaves of  $t$  by  $1, \dots, n$  from left to right,  $\mathbb{T}$  becomes a subset of  $\{1, \dots, n\}$ .

Saneblidze and Umble [22] introduced bi-leveled trees to describe a cellular projection from the permutahedra to Stasheff’s multiplihedra  $\mathcal{M}_\bullet$ , with the bi-leveled trees on  $n$  nodes indexing the vertices  $\mathcal{M}_n$ . Stasheff used a different type of tree for the vertices of  $\mathcal{M}_\bullet$ . These alternative trees lead to a different Hopf structure which we explore in a forthcoming paper [9]. We remark that  $\mathcal{M}_0 = \{|\}$ .

The partial order on  $\mathcal{M}_n$  is defined by  $(s; \mathbb{S}) \leq (t; \mathbb{T})$  if  $s \leq t$  in  $\mathcal{Y}_n$  and  $\mathbb{S} \supseteq \mathbb{T}$ . The Hasse diagrams of the posets  $\mathcal{M}_n$  are 1-skeleta for the multiplihedra. We represent a bi-leveled tree by drawing the underlying tree  $t$  and circling the nodes in  $\mathbb{T}$ . The Hasse diagram of  $\mathcal{M}_4$  appears in Figure 2.

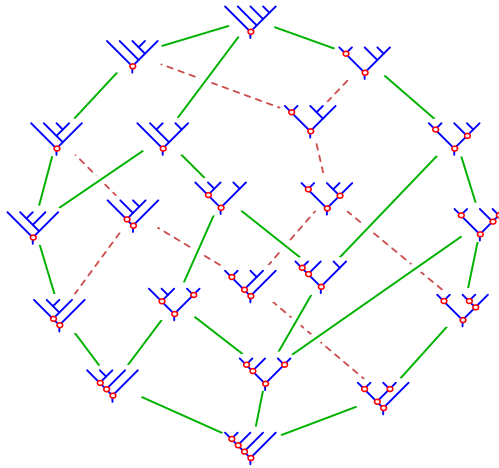


FIGURE 2. The 1-skeleton of the multiplihedron  $\mathcal{M}_4$ .

**2.2. Poset maps.** Forgetting the order ideal in a bi-leveled tree,  $(t; \mathbb{T}) \mapsto t$ , is a poset map  $\phi: \mathcal{M}_\bullet \rightarrow \mathcal{Y}_\bullet$ . We define a map  $\beta: \mathfrak{S}_\bullet \rightarrow \mathcal{M}_\bullet$  so that

$$\mathfrak{S}_\bullet \xrightarrow{\beta} \mathcal{M}_\bullet \xrightarrow{\phi} \mathcal{Y}_\bullet$$

factors the map  $\tau: \mathfrak{S}_\bullet \rightarrow \mathcal{Y}_\bullet$ , and we define a right inverse (section)  $\iota$  of  $\beta$ .

Let  $w \in \mathfrak{S}_n$  be an ordered tree. Define the set

$$(2.1) \quad \mathbb{T}(w) := \{i \mid w(i) \geq w(1)\}.$$

Observe that  $(\tau(w); \mathbb{T}(w))$  is a bi-leveled tree. Indeed, as  $w$  is a linear extension of  $\tau(w)$ ,  $\mathbb{T}(w)$  is an upper order ideal which by definition (2.1) contains the leftmost node as a minimal element. Since covers in the weak order can only decrease the subset  $\mathbb{T}(w)$  and  $\tau$  is also a poset map, we see that  $\beta$  is a poset map.

**Theorem 2.1.** *The maps  $\beta: \mathfrak{S}_n \rightarrow \mathcal{M}_n$  and  $\phi: \mathcal{M}_n \rightarrow \mathcal{Y}_n$  are surjective poset maps with  $\tau = \phi \circ \beta$ .*

The fibers of the map  $\beta$  are intervals (indeed, products of intervals); see Figure 3. We prove this using an equivalent representation of a bi-leveled tree and

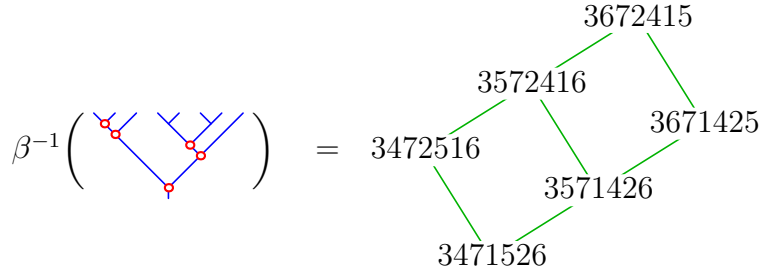


FIGURE 3. The preimages of  $\beta$  are intervals.

a description of the map  $\beta$  in that representation. If we prune a bi-leveled tree  $b = (t; \mathbb{T})$  above the nodes in  $\mathbb{T}$  (but not on the leftmost branch) we obtain a tree  $t'_0$  (the order ideal) on  $r$  nodes and a planar forest  $\mathbf{t} = (t_1, \dots, t_r)$  of  $r$  trees. If we prune  $t'_0$  just below its leftmost node, we obtain the tree  $\mathbb{Y}$  (from the pruning) and a tree  $t_0$ , and  $t'_0$  is obtained by grafting  $\mathbb{Y}$  onto the leftmost leaf of  $t_0$ . We may recover  $b$  from this tree  $t_0$  on  $r-1$  nodes and the planar forest  $\mathbf{t} = (t_1, \dots, t_r)$ , and so we also write  $b = (t_0, \mathbf{t})$ . We illustrate this correspondence in Figure 4.

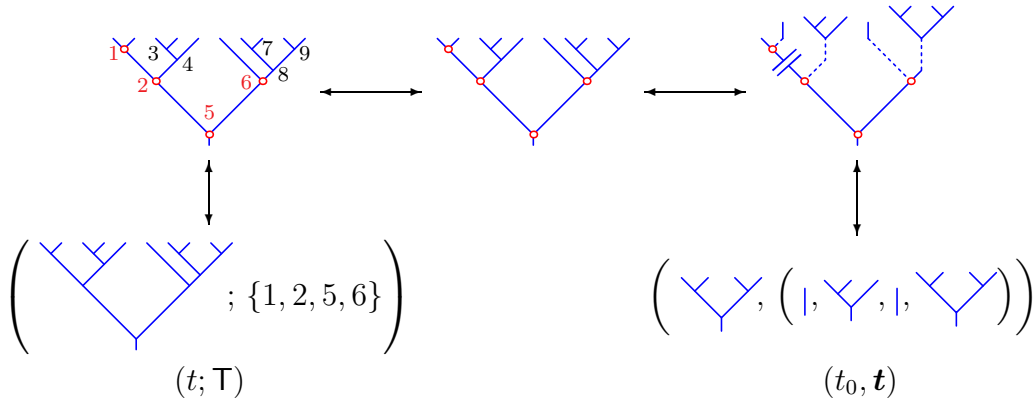


FIGURE 4. Two representations of bi-leveled trees.

We describe the map  $\beta$  in terms of this second representation of bi-leveled trees. Given a permutation  $w$  with  $\beta(w) = (t; \mathbb{T})$  and  $|\mathbb{T}| = r$ , let  $u_1 u_2 \dots u_r$  be the

restriction of  $w$  to the set  $\mathbb{T}$ . We may write the values of  $w$  as  $w = u_1v^1u_2 \cdots u_rv^r$ , where  $v^i$  is the (possibly empty) subword of  $w$  between the numbers  $u_i$  and  $u_{i+1}$  and  $v^r$  is the word after  $u_r$ . Call this the *bi-leveled factorization* of  $w$ . For example,

$$654789132 \longmapsto (\underbrace{6, 54, 7, \emptyset, 8, \emptyset, 9, 132}_{\substack{u_1 \quad v^1 \quad u_2 \quad v^2 \quad u_3 \quad v^3 \quad u_4 \quad v^4}}).$$

Note that  $\beta(w) = (\tau(\overline{u_2 \dots u_r}), (\tau(\overline{v^1}), \dots, \tau(\overline{v^r})))$ .

**Theorem 2.2.** *For any  $b \in \mathcal{M}_n$  the fiber  $\beta^{-1}(b) \subseteq \mathfrak{S}_n$  is a product of intervals.*

*Proof.* Let  $b = (t_0, (t_1, \dots, t_r)) = (t; \mathbb{T}) \in \mathcal{M}_n$  be a bi-leveled tree. A permutation  $w \in \beta^{-1}(b) \in \mathfrak{S}_n$  has a bi-leveled factorization  $w = u_1v^1u_2 \dots u_rv^r$  with

$$(2.2) \quad \begin{aligned} (i) \quad & w|_{\mathbb{T}} = u_1u_2 \dots u_r, \quad u_1 = n+1-r, \quad \tau(\overline{u_2 \dots u_r}) = t_0, \quad \text{and} \\ (ii) \quad & \tau(\overline{v^i}) = t_i, \quad \text{for } i = 1, \dots, r. \end{aligned}$$

Since  $u_1 < u_2, \dots, u_r$  are the values of  $w$  in the positions of  $\mathbb{T}$ , and  $u_1 = n+1-r$  exceeds all the letters in  $v^1, \dots, v^r$ , which are the values of  $w$  in the positions in the complement of  $\mathbb{T}$ , these two parts of the bi-leveled factorization may be chosen independently to satisfy (2.2), which shows that  $\beta^{-1}(b)$  is a product.

To see that the factors are intervals, and thus  $\beta^{-1}(b)$  is an interval, we examine the conditions (i) and (ii) separately. Those  $u_1 \dots u_r = w|_{\mathbb{T}}$  for  $w$  in the fiber  $\beta^{-1}(b)$  are exactly the set of  $n+1-r, u_2, \dots, u_r$  with  $\{u_2, \dots, u_r\} = \{n+2-r, \dots, n\}$  and  $\tau(\overline{u_2 \dots u_r}) = t_0$ . This is a poset under the restriction of the weak order, and it is in natural bijection with the interval  $\tau^{-1}(t_0) \subset \mathfrak{S}_{r-1}$ . Its minimal element is  $\min_0(b) = u_1u_2 \dots u_r$ , where  $u_2 \dots u_r$  is the unique 231-avoiding word on  $\{n+1-r, \dots, n\}$  satisfying (i), and its maximal element is  $\max_0(b) = u_1u_2 \dots u_r$ , where now  $u_2 \dots u_r$  is the unique 132-avoiding word on  $\{n+1-r, \dots, n\}$  satisfying (i).

Now consider sequences of words  $v^1, \dots, v^r$  on distinct letters  $\{1, \dots, n-r\}$  satisfying (ii). This is also a poset under the restriction of the weak order. It has a minimal element, which is the unique such sequence  $\min(b)$  satisfying (ii) where the letters of  $v^i$  precede those of  $v^j$  whenever  $i < j$ , and where each  $v^i$  is 231-avoiding. Its maximal element is the unique sequence  $\max(b)$  satisfying (ii) where the letters of  $v^i$  are greater than those of  $v^j$  when  $i < j$  and  $v^i$  is 132-avoiding.  $\square$

The fibers of  $\beta$  are intervals so that consistently choosing the minimum or maximum in a fiber gives two set-theoretic sections. These are not order-preserving as may be seen from Figure 5. We have  $\begin{smallmatrix} \swarrow \\ \downarrow \\ \searrow \end{smallmatrix} < \begin{smallmatrix} \swarrow \\ \downarrow \\ \searrow \end{smallmatrix}$  but the maxima in their fibers

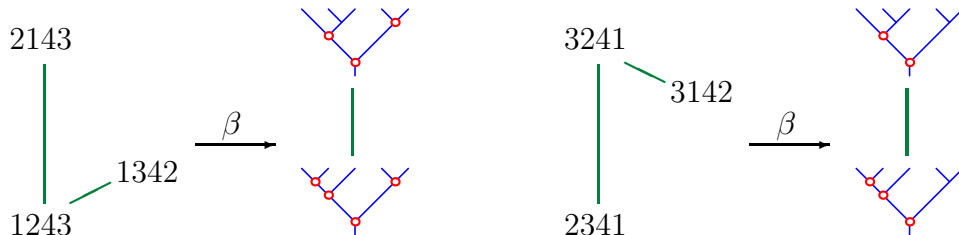


FIGURE 5. Fibers of  $\beta$ .

under  $\beta$ , 1342 and 2143, are incomparable. Similarly,  $\begin{smallmatrix} \swarrow \\ \downarrow \\ \searrow \end{smallmatrix} < \begin{smallmatrix} \swarrow \\ \downarrow \\ \swarrow \end{smallmatrix}$  but the minima in their fibers under  $\beta$ , 2341 and 3142, are incomparable. This shows that the map  $\beta: \mathfrak{S}_n \rightarrow \mathcal{M}_n$  is not a lattice congruence (unlike the map  $\tau: \mathfrak{S}_n \rightarrow \mathcal{Y}_n$  [20]).

In the notation of the proof, given a bi-leveled tree  $b = (t_0, (t_1, \dots, t_r))$ , let  $\iota(b)$  be the permutation  $w \in \beta^{-1}(b)$  with bi-leveled factorization  $w = u_1 v^1 u_2 \dots u_r v^r$  where  $u_1 u_2 \dots u_r = \min_0(b)$  and  $(v^1, \dots, v^r) = \underline{\max}(b)$ . This defines a map  $\iota: \mathcal{M}_n \rightarrow \mathfrak{S}_n$  that is a section of the map  $\beta$ . For example,

$$\iota \left( \begin{array}{c} \begin{array}{ccc} \swarrow & \swarrow & \swarrow \\ \downarrow & \downarrow & \downarrow \\ \searrow & \searrow & \searrow \\ \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow \end{array} \\ \downarrow \\ \downarrow \end{array} \right) = \begin{array}{c} 7 \ 8 \ 6 \ 11 \ 4 \ 5 \ 9 \ 10 \ 2 \ 3 \ 1 \\ \swarrow \ \downarrow \ \swarrow \ \downarrow \ \swarrow \ \downarrow \ \swarrow \ \downarrow \ \swarrow \\ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \\ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \end{array} = 786 \ 11 \ 459 \ 10 \ 231.$$

**Remark 2.3.** This map  $\iota$  may be characterized in terms of pattern avoidance: the permutation  $\iota(b)$  is the unique  $w \in \beta^{-1}(b)$  avoiding the pinned patterns

$$\{\underline{2}031, \underline{0}231, \underline{3}021\},$$

where the underlined letter must be the first letter of a permutation. To see this, note that the first pattern forces the letters in  $v^i$  to be larger than those in  $v^{i+1}$  for  $1 \leq i < r$ , the second pattern forces  $u_2 \dots u_r$  to be 231-avoiding, and the last pattern forces each  $v^i$  to be 132-avoiding.

**Theorem 2.4.** *The map  $\iota$  is injective, right-inverse to  $\beta$ , and order-preserving. That is,  $\beta: \mathfrak{S}_n \rightarrow \mathcal{M}_n$  is an interval retract.*

Since  $\mathfrak{S}_n$  is a lattice [10], the fibers of  $\beta$  are intervals, and  $\iota$  is a section of  $\beta$ . That is, we need only verify that  $\iota$  is order-preserving. We begin by describing the covers in  $\mathcal{M}_n$ . Since  $\beta$  is a surjective poset map, every cover in  $\mathcal{M}_n$  is the image of some cover  $w \lessdot w'$  in  $\mathfrak{S}_n$ .

**Lemma 2.5.** *If a cover  $w \lessdot w' \in \mathfrak{S}_n$  does not collapse under  $\beta$ , i.e.,  $\beta(w) \neq \beta(w')$ , then it yields one of three types of covers  $\beta(w) \lessdot \beta(w')$  in  $\mathcal{M}_n$ .*

- (i) *In exactly one tree  $t_i$  in  $\beta(w) = (t_0, (t_1, \dots, t_r))$ , a node is moved from left to right across its parent to obtain  $\beta(w')$ . That is,  $t_i \lessdot t'_i$ .*
- (ii) *If  $\beta(w) = (t; \mathbb{T})$ , the leftmost node of  $t$  is moved across its parent, which has no other child in the order ideal  $\mathbb{T}$ , and is deleted from  $\mathbb{T}$  to obtain  $\beta(w')$ .*
- (iii) *If  $\mathbb{T}(w) = \{1 = T_1 < \dots < T_r\}$ , then  $\tau(w') = \tau(w)$  and  $\mathbb{T}(w') = \mathbb{T}(w) \setminus \{T_j\}$  for some  $j > 2$ .*

*Proof.* Put  $w' = (k, k+1)w$ , with  $k, k+1$  appearing in order in  $w$ . Let  $(t; \mathbb{T})$  and  $(t_0, (t_1, \dots, t_r))$  be the two representations of  $\beta(w)$ . Write  $\mathbb{T} = \{T_1 < \dots < T_r\}$  (with  $T_1 = 1$ ) and  $w|_{\mathbb{T}} = u_1 u_2 \dots u_r$ . If  $w \lessdot w'$  and  $\beta(w) \lessdot \beta(w')$ , then  $k$  appears within  $w$  in one of three ways: (i)  $u_1 \neq k$ , (ii)  $u_1 = k$  and  $u_2 = k+1$ , or (iii)  $u_1 = k$  and  $u_j = k+1$  for some  $j > 2$ . These yield the corresponding descriptions in the statement of the lemma. (Note that in type (i),  $\mathbb{T}(w') = \mathbb{T}$ , so if we set  $\beta(w') = (t'_0, (t'_1, \dots, t'_r))$ , then  $t_i = t'_i$ , except for one index  $i$ , where  $t_i \lessdot t'_i$ .)  $\square$

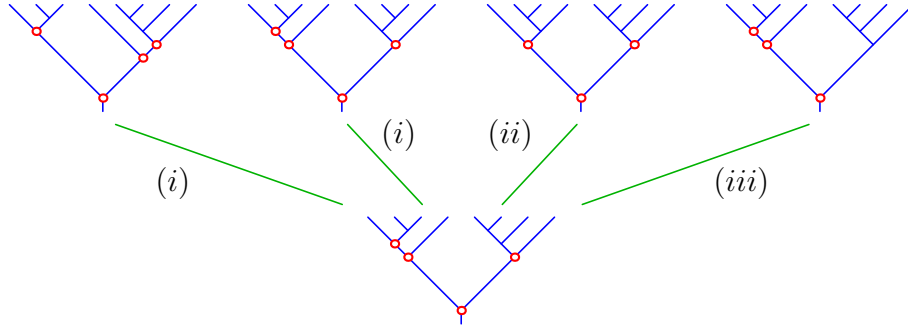

 FIGURE 6. Some covers in  $\mathcal{M}_7$ .

Figure 6 illustrates these three types of covers, labeled by their type.

For  $\mathbb{T} \subset \{1, \dots, n\}$  with  $1 \in \mathbb{T}$ , let  $\mathfrak{S}_n(\mathbb{T}) := \{w \in \mathfrak{S}_n \mid \mathbb{T}(w) = \mathbb{T}\}$ . Let  $\mathcal{M}_n(\mathbb{T})$  be those bi-leveled trees whose order ideal consists of the nodes in  $\mathbb{T}$ . Note that  $\beta(\mathfrak{S}_n(\mathbb{T})) = \mathcal{M}_n(\mathbb{T})$  and  $\beta^{-1}(\mathcal{M}_n(\mathbb{T})) = \mathfrak{S}_n(\mathbb{T})$ .

**Lemma 2.6.** *The map  $\iota: \mathcal{M}_n(\mathbb{T}) \rightarrow \mathfrak{S}_n(\mathbb{T})$  is a map of posets.*

*Proof.* Let  $\mathbb{T} = \{1 = T_1 < \dots < T_r\}$ . Setting  $T_{r+1} = n+1$ , define  $a_i := T_{i+1} - T_i - 1$  for  $i = 1, \dots, r$ . Then  $b \mapsto (t_0, (t_1, \dots, t_r))$  gives an isomorphism of posets,

$$\mathcal{M}_n(\mathbb{T}) \xrightarrow{\sim} \mathcal{Y}_{r-1} \times \mathcal{Y}_{a_1} \times \dots \times \mathcal{Y}_{a_r}.$$

As the maps  $\min, \max: \mathcal{Y}_a \rightarrow \mathfrak{S}_a$  are order-preserving, the proof of Theorem 2.2 gives the desired result.  $\square$

*Proof of Theorem 2.4.* Let  $b \triangleleft c$  be a cover in  $\mathcal{M}_n$ . We will show that  $\iota(b) \leq \iota(c)$  in  $\mathfrak{S}_n$ . Suppose that  $b = (t; \mathbb{T})$ , with  $\mathbb{T} = \{1 = T_1 < \dots < T_r\}$ . Let  $\iota(b)$  have bi-leveled factorization  $\iota(b) = u_1 v^1 u_2 v^2 \dots u_r v^r$ , and set  $k := n + 1 - |\mathbb{T}|$ .

The result is immediate if the cover  $b \triangleleft c$  is of type (i), for then  $b, c \in \mathcal{M}_n(\mathbb{T})$  and  $\iota: \mathcal{M}_n(\mathbb{T}) \rightarrow \mathfrak{S}_n$  is order-preserving, as observed in Lemma 2.6.

Now suppose that  $b \triangleleft c$  is a cover of type (ii). Set  $w := \iota(b)$ . We claim that  $w \triangleleft (k, k+1)w$  and  $\iota(c) = (k, k+1)w$ . Now,  $u_1 = k$  labels the leftmost node of  $b$ , so the first claim is immediate. Note that  $u_2$  labels the parent of the node labeled  $b$ . This parent has no other child in  $\mathbb{T}$ , so we must have  $u_2 < u_3$ . As  $u_2 u_3 \dots u_r$  is 231-avoiding and contains  $k+1$ , we must have  $u_2 = k+1$ . This shows that

$$\iota(c) = (k, k+1)w = u_2 (v^1 u_1 v^2) u_3 \dots u_r v^r.$$

Indeed,  $u_2$  is minimal among  $u_2, \dots, u_r$  and  $u_3 \dots u_r$  is 231-avoiding, thus  $\min_0(c) = u_2 \dots u_r$ . The bi-leveled factorization of  $(k, k+1)w$  gives  $(v^1 u_1 v^2, v^3, \dots, v^r)$ , which we claim is  $\max(c)$ . As  $u_1$  is the largest letter in the sequence, we need only check that  $v^1 u_1 v^2$  is 132-avoiding. But this is true for  $v^1$  and  $v^2$  and there can be no 132-pattern involving  $u_1$  as the letters in  $v^1$  are all greater than those in  $v^2$ .

Finally, suppose that  $b \triangleleft c$  is of type (iii). Then  $c = (t; \mathbb{T} \setminus \{T_j\})$  for some  $j > 2$ . We will find a permutation  $w' \in \beta^{-1}(b)$  satisfying  $(k, k+1)w' \in \beta^{-1}(c)$  and

$$(2.3) \quad \iota(b) \leq w' \triangleleft (k, k+1)w' \leq \iota(c).$$

Let  $w' \in \beta^{-1}(b)$  be the minimal permutation having bi-leveled factorization

$$w' = u'_1 v^1 u'_2 \dots u'_r v^r, \quad \text{with } u'_j = k+1.$$

Here  $(v^1, \dots, v^r) = \underline{\max}(b)$  is the same sequence as in  $\iota(b)$ . The structure of  $\beta^{-1}(b)$  implies that  $\iota(b) \leq w'$ . We also have

$$w' \prec (k, k+1)w' \quad \text{and} \quad \beta((k, k+1)w') = c.$$

While  $\iota(c)$  and  $(k, k+1)w'$  are not necessarily equal, we do have that

$$(k, k+1)w'|_{\tau \setminus \{T_j\}} = u'_j u'_2 \dots u'_{j-1} u'_{j+1} \dots u'_r$$

and  $u'_2 \dots u'_{j-1} u'_{j+1} \dots u'_r$  is 231-avoiding. That is,  $(k, k+1)w'|_{\tau \setminus \{T_j\}} = \iota(c)|_{\tau \setminus \{T_j\}}$ . Otherwise,  $w'$  would not be minimal. The bi-leveled factorization of  $(k, k+1)w'$  is

$$u'_j v^1 u'_2 \dots u'_{j-1} (v^{j-1} u'_1 v^j) u'_{j+1} \dots u'_r v^r,$$

and we necessarily have  $(v^1, \dots, v^{j-1} u'_1 v^j, \dots, v^r) \leq \underline{\max}(c)$ , which implies that  $(k, k+1)w' \leq \iota(c)$ . We thus have the chain (2.3) in  $\mathfrak{S}_n$ , completing the proof.  $\square$

If  $b \prec c$  is the cover of type (iii) in Figure 6, the chain (2.3) from  $\iota(b)$  to  $\iota(c)$  is

$$4357126 \leq 4367125 \prec 5367124 \leq 5467123.$$

### 2.3. Tree enumeration. Let

$$\mathbf{S}(q) := \sum_{n \geq 0} n! q^n = 1 + q + 2q^2 + 6q^3 + 24q^4 + 120q^5 + \dots$$

be the enumerating series of permutations, and define  $\mathbf{M}(q)$  and  $\mathbf{Y}(q)$  similarly

$$(2.4) \quad \begin{aligned} \mathbf{M}(q) &:= \sum_{n \geq 0} A_n q^n = 1 + q + 2q^2 + 6q^3 + 21q^4 + 80q^5 + \dots, \\ \mathbf{Y}(q) &:= \sum_{n \geq 0} C_n q^n = 1 + q + 2q^2 + 5q^3 + 14q^4 + 42q^5 + \dots, \end{aligned}$$

where  $A_n := |\mathcal{M}_n|$  and  $C_n := |\mathcal{Y}_n|$  are the Catalan numbers  $\frac{1}{n+1} \binom{2n}{n}$ , whose enumerating series satisfies

$$\mathbf{Y}(q) = \frac{1 - \sqrt{1 - 4q}}{2q} = \frac{2}{1 + \sqrt{1 - 4q}}.$$

Bi-leveled trees are Catalan-like [8, Theorem 3.1]: for  $n \geq 1$ ,  $A_n = C_{n-1} + \sum_{i=1}^{n-1} A_i A_{n-i}$ . See also [24, A121988]. Their enumerating series satisfies

$$\mathbf{M}(q) = 1 + q\mathbf{Y}(q) \cdot \mathbf{Y}(q\mathbf{Y}(q)).$$

We will also be interested in  $\mathbf{M}_+(q) := \sum_{n > 0} A_n q^n = q\mathbf{Y}(q) \cdot \mathbf{Y}(q\mathbf{Y}(q))$ .

**Theorem 2.7.** *The only nontrivial quotients of the enumerating series  $\mathbf{S}(q)$ ,  $\mathbf{M}(q)$ ,  $\mathbf{M}_+(q)$ , and  $\mathbf{Y}(q)$  whose expansions have nonnegative coefficients are*

$$\mathbf{S}(q)/\mathbf{M}(q), \quad \mathbf{S}(q)/\mathbf{Y}(q), \quad \mathbf{M}_+(q)/\mathbf{Y}(q), \quad \text{and} \quad \mathbf{M}(q)/\mathbf{Y}(q).$$

*Proof.* We prove the positivity of the quotient  $\mathbf{S}(q)/\mathbf{M}(q)$  in Section 4.2. The positivity of  $\mathbf{S}(q)/\mathbf{Y}(q)$  was established after [2, Theorem 7.2], which shows that  $\mathfrak{S}Sym$  is a smash product over  $\mathcal{Y}Sym$ .

For the positivity of  $\mathbf{M}_+(q)/\mathbf{Y}(q)$ , we use [3, Proposition 3], which computes  $\mathbf{Y}(q\mathbf{Y}(q)) = \sum_{n>0} B_n q^{n-1}$ , where

$$(2.5) \quad B_1 := C_0 \quad \text{and} \quad B_n := \sum_{k=0}^{n-1} \frac{k}{n-1} \binom{2n-k-3}{n-k-1} C_k \quad \text{for } n > 1.$$

In particular,  $B_n \geq 0$  for all  $n \geq 0$ . Returning to the quotient, we have

$$\frac{\mathbf{M}_+(q)}{\mathbf{Y}(q)} = \frac{q\mathbf{Y}(q) \cdot \mathbf{Y}(q\mathbf{Y}(q))}{\mathbf{Y}(q)} = q\mathbf{Y}(q\mathbf{Y}(q)),$$

so  $\mathbf{M}_+(q)/\mathbf{Y}(q) = \sum_{n>0} B_n q^n$  has nonnegative coefficients.

For  $\mathbf{M}(q)/\mathbf{Y}(q)$ , use the identity  $1/\mathbf{Y}(q) = 1 - q\mathbf{Y}(q)$  to obtain

$$\frac{\mathbf{M}(q)}{\mathbf{Y}(q)} = \mathbf{M}_+(q) + 1 - q\mathbf{Y}(q) = 1 + \sum_{n>0} (B_n - C_{n-1}) q^n.$$

Positivity is immediate as  $B_n - C_{n-1} \geq 0$  for  $n > 0$ .

We leave the proof that the remaining quotients have negative coefficients to the reader's computer.  $\square$

**Remark 2.8.** Up to an index shift, the quotient  $\mathbf{M}_+(q)/\mathbf{Y}(q)$  corresponds to the sequence [24, A127632] beginning with (1, 1, 3, 11, 44, 185, 804). We give a new combinatorial interpretation of this sequence in Corollary 4.3.

### 3. THE ALGEBRA $\mathcal{MSym}$

Let  $\mathcal{MSym} := \bigoplus_{n \geq 0} \mathcal{MSym}_n$  denote the graded  $\mathbb{Q}$ -vector space whose  $n^{\text{th}}$  graded piece has the basis  $\{F_b \mid b \in \mathcal{M}_n\}$ . The maps  $\beta: \mathfrak{S} \rightarrow \mathcal{M}$  and  $\phi: \mathcal{M} \rightarrow \mathcal{Y}$  of graded sets induce surjective maps of graded vector spaces

$$(3.1) \quad \mathfrak{S}Sym \xrightarrow{\beta} \mathcal{MSym} \xrightarrow{\phi} \mathcal{Y}Sym \quad F_w \mapsto F_{\beta(w)} \mapsto F_{\phi(\beta(w))},$$

which factor the Hopf algebra map  $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$ , as  $\phi(\beta(w)) = \tau(w)$ . We will show how the maps  $\beta$  and  $\tau$  induce on  $\mathcal{MSym}$  the structures of an algebra, of a  $\mathfrak{S}Sym$ -module, and of a  $\mathcal{Y}Sym$ -comodule so that the composition (3.1) factors the map  $\tau$  as maps of algebras, of  $\mathfrak{S}Sym$ -modules, and of  $\mathcal{Y}Sym$ -comodules.

**3.1. Algebra structure on  $\mathcal{MSym}$ .** For  $b, c \in \mathcal{M}$ , define

$$(3.2) \quad F_b \cdot F_c = \beta(F_w \cdot F_v),$$

where  $w, v$  are permutations in  $\mathfrak{S}$  with  $b = \beta(w)$  and  $c = \beta(v)$ .

**Theorem 3.1.** *The operation  $F_b \cdot F_c$  defined by (3.2) is independent of choices of  $w, v$  with  $\beta(w) = b$  and  $\beta(v) = c$  and it endows  $\mathcal{MSym}$  with the structure of a graded connected algebra such that the map  $\beta: \mathfrak{S}Sym \rightarrow \mathcal{MSym}$  is a surjective map of graded connected algebras.*

If the expression  $\beta(F_w \cdot F_v)$  is independent of choice of  $w \in \beta^{-1}(b)$  and  $v \in \beta^{-1}(c)$ , then the map  $\beta$  is automatically multiplicative. Associative and unital properties for  $\mathcal{MSym}$  are then inherited from those for  $\mathfrak{S}\mathcal{Sym}$ , and the theorem follows. To prove independence (in Lemma 3.2), we formulate a description of (3.2) in terms of splittings and graftings of bi-leveled trees.

Let  $s \xrightarrow{\gamma} (s_0, \dots, s_m)$  be a splitting on the underlying tree of a bi-leveled tree  $b = (s; \mathbf{S}) \in \mathcal{M}_n$ . Then the nodes of  $s$  are distributed among the nodes of the partially ordered forest  $(s_0, \dots, s_m)$  so that the order ideal  $\mathbf{S}$  gives a sequence of order ideals in the trees  $s_i$ . Write  $b \xrightarrow{\gamma} (b_0, \dots, b_m)$  for the corresponding splitting of the bi-leveled tree  $b$ , viewing  $b_i$  as  $(s_i; \mathbf{S}|_{s_i})$ . (Note that only  $b_0$  is guaranteed to be a bi-leveled tree.) Given  $c = (t; \mathbf{T}) \in \mathcal{M}_m$  and a splitting  $b \xrightarrow{\gamma} (b_0, \dots, b_m)$  of  $b \in \mathcal{M}_n$ , form a bi-leveled tree  $(b_0, \dots, b_m)/c$  whose underlying tree is  $(s_0, \dots, s_m)/t$  and whose order ideal is either

$$(3.3) \quad \begin{array}{ll} (i) & \mathbf{T}, \text{ if } b_0 \in \mathcal{M}_0, \text{ or} \\ (ii) & \mathbf{S} \cup \{\text{the nodes of } t\}, \text{ if } b_0 \notin \mathcal{M}_0. \end{array}$$

**Lemma 3.2.** *The product (3.2) is independent of choices of  $w, v$  with  $\beta(w) = b$  and  $\beta(v) = c$ . For  $b \in \mathcal{M}_n$  and  $c \in \mathcal{M}_m$ , we have*


$$F_b \cdot F_c = \sum_{b \xrightarrow{\gamma} (b_0, \dots, b_m)} F_{(b_0, \dots, b_m)/c}.$$

*Proof.* Fix any  $w \in \beta^{-1}(b)$  and  $v \in \beta^{-1}(c)$ . The bi-leveled tree  $\beta((w_0, \dots, w_m)/v)$  associated to a splitting  $w \xrightarrow{\gamma} (w_0, \dots, w_m)$  has underlying tree  $(s_0, \dots, s_m)/t$ , where  $s \xrightarrow{\gamma} (s_0, \dots, s_m)$  is the induced splitting on the underlying tree  $s = \tau(w) = \phi(b)$ . Each node of  $(w_0, \dots, w_m)/v$  comes from a node of either  $w$  or  $v$ , with the labels of nodes from  $w$  all smaller than the labels of nodes from  $v$ . Consequently, the leftmost node of  $(w_0, \dots, w_m)/v$  comes from either

- (i)  $v$ , and then  $\mathbf{T}((w_0, \dots, w_m)/v) = \mathbf{T}(v) = \mathbf{T}(c)$ , or
- (ii)  $w$ , and then  $\mathbf{T}((w_0, \dots, w_m)/v) = \mathbf{T}(w) = \mathbf{T}(b) \cup \{\text{the nodes of } v\}$ .

The first case is when  $w_0 \in \mathfrak{S}_0$  and the second case is when  $w_0 \notin \mathfrak{S}_0$ .  $\square$

Here is the product  $F_{\swarrow} \cdot F_{\swarrow}$ , together with the corresponding splittings of  $\swarrow$ ,

$$F_{\swarrow} \cdot F_{\swarrow} = F_{\swarrow \swarrow} + F_{\swarrow \searrow} + F_{\searrow \swarrow} + F_{\searrow \searrow} + F_{\swarrow \swarrow} + F_{\swarrow \searrow}.$$


**3.2.  $\mathfrak{S}\mathcal{Sym}$  module structure on  $\mathcal{MSym}$ .** Since  $\beta$  is a surjective algebra map,  $\mathcal{MSym}$  becomes a  $\mathfrak{S}\mathcal{Sym}$ -bimodule with the action

$$F_w \cdot F_b \cdot F_v = F_{\beta(w)} \cdot F_b \cdot F_{\beta(v)}.$$

The map  $\tau$  likewise induces on  $\mathcal{YSym}$  the structure of a  $\mathfrak{S}\mathcal{Sym}$ -bimodule, and the maps  $\beta$ ,  $\phi$ , and  $\tau$  are maps of  $\mathfrak{S}\mathcal{Sym}$ -bimodules.



Curiously, we may use the map  $\iota: \mathcal{M}_\bullet \rightarrow \mathfrak{S}_\bullet$  to define the structure of a right  $\mathfrak{S}Sym$ -comodule on  $\mathcal{M}Sym$ ,

$$F_b \longmapsto \sum_{\iota(b) \xrightarrow{\vee} (w_0, w_1)} F_{\beta(w_0)} \otimes F_{w_1}.$$

This induces a right comodule structure, because if  $\iota(b) \xrightarrow{\vee} (w_0, w_1)$ , then  $w_0 = \iota(\beta(w_0))$ , which may be checked using the characterization of  $\iota$  in terms of pattern avoidance, as explained in Remark 2.3.

While  $\mathcal{M}Sym$  is both a right  $\mathfrak{S}Sym$ -module and right  $\mathfrak{S}Sym$ -comodule, it is not an  $\mathfrak{S}Sym$ -Hopf module. For if it were a Hopf module, then the fundamental theorem of Hopf modules (see Remark 4.4) would imply that the series  $\mathbf{M}(q)/\mathbf{S}(q)$  has positive coefficients, which contradicts Theorem 2.7.

**3.3.  $\mathcal{Y}Sym$ -comodule structure on  $\mathcal{M}Sym$ .** For  $b \in \mathcal{M}_\bullet$ , define the linear map  $\rho: \mathcal{M}Sym \rightarrow \mathcal{M}Sym \otimes \mathcal{Y}Sym$  by

$$(3.4) \quad \rho(F_b) = \sum_{b \xrightarrow{\vee} (b_0, b_1)} F_{b_0} \otimes F_{\phi(b_1)}.$$

By  $\phi(b_1)$ , we mean the tree underlying  $b_1$ .

**Example 3.3.** In the fundamental bases of  $\mathcal{M}Sym$  and  $\mathcal{Y}Sym$ , we have

$$\rho(F_{\begin{array}{c} \diagdown \\ \diagup \end{array}}) = F_{\begin{array}{c} \diagdown \\ \diagup \end{array}} \otimes 1 + F_{\begin{array}{c} \diagdown \\ \diagup \end{array}} \otimes F_{\begin{array}{c} \diagdown \\ \diagup \end{array}} + F_{\begin{array}{c} \diagdown \\ \diagup \end{array}} \otimes F_{\begin{array}{c} \diagdown \\ \diagup \end{array}} + F_{\begin{array}{c} \diagdown \\ \diagup \end{array}} \otimes F_{\begin{array}{c} \diagdown \\ \diagup \end{array}} + 1 \otimes F_{\begin{array}{c} \diagdown \\ \diagup \end{array}}.$$

**Theorem 3.4.** *Under  $\rho$ ,  $\mathcal{M}Sym$  is a right  $\mathcal{Y}Sym$ -comodule.*

*Proof.* This is counital as  $(b, \mathbf{l})$  is a splitting of  $b$ . Coassociativity is also clear as both  $(\rho \otimes 1)\rho$  and  $(1 \otimes \Delta)\rho$  applied to  $F_b$  for  $b \in \mathcal{M}_\bullet$  are sums of terms  $F_{b_0} \otimes F_{\phi(b_1)} \otimes F_{\phi(b_2)}$  over all splittings  $b \xrightarrow{\vee} (b_0, b_1, b_2)$ .  $\square$

Careful bookkeeping of the terms in  $\rho(F_b \cdot F_c)$  show that it equals  $\rho(F_b) \cdot \rho(F_c)$  for all  $b, c \in \mathcal{M}_\bullet$  and thus  $\mathcal{M}Sym$  is a  $\mathcal{Y}Sym$ -comodule algebra. Hence,  $\phi$  is a map of  $\mathcal{Y}Sym$ -comodule algebras, and in fact  $\beta$  is also a map of  $\mathcal{Y}Sym$ -comodule algebras. We leave this to the reader, and will not pursue it further.

Since  $\tau: \mathfrak{S}Sym \rightarrow \mathcal{Y}Sym$  is a map of Hopf algebras,  $\mathfrak{S}Sym$  is naturally a right  $\mathcal{Y}Sym$ -comodule where the comodule map is the composition

$$\mathfrak{S}Sym \xrightarrow{\Delta} \mathfrak{S}Sym \otimes \mathfrak{S}Sym \xrightarrow{1 \otimes \tau} \mathfrak{S}Sym \otimes \mathcal{Y}Sym.$$

With these definitions, the following lemma is immediate.

**Lemma 3.5.** *The maps  $\tau$  and  $\phi$  are maps of right  $\mathcal{Y}Sym$ -comodules.*

In particular, we have the equality of maps  $\mathfrak{S}Sym \rightarrow \mathcal{M}Sym \otimes \mathcal{Y}Sym$ ,

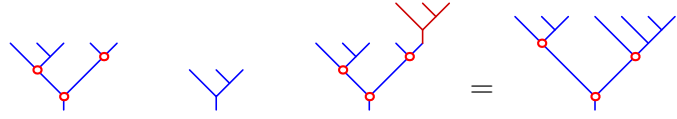
$$(3.5) \quad \rho \circ \beta = (\beta \otimes \tau) \circ \Delta.$$

**3.4. Coaction in the monomial basis.** The coalgebra structures of  $\mathfrak{S}Sym$  and  $\mathcal{Y}Sym$  were elucidated by considering a second basis related to the fundamental basis via Möbius inversion. For  $b \in \mathcal{M}_n$ , define

$$(3.6) \quad M_b := \sum_{b \leq c} \mu(b, c) F_c,$$

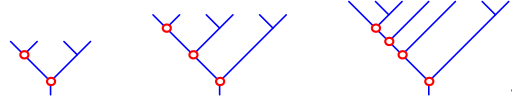
where  $\mu(\cdot, \cdot)$  is the Möbius function on the poset  $\mathcal{M}_n$ .

Given  $b \in \mathcal{M}_m$  and  $s \in \mathcal{Y}_q$ , write  $b \setminus s$  for the bi-leveled tree with  $p + q$  nodes whose underlying tree is formed by grafting the root of  $s$  onto the rightmost leaf of  $b$ , but whose order ideal is that of  $b$ . Here is an example of  $b$ ,  $s$ , and  $b \setminus s$ ,



Observe that we cannot have  $b = \mid$  in this construction.

The maximum bi-leveled tree with a given underlying tree  $t$  is  $\beta(\max(t))$ , which has order ideal  $\mathbb{T}$  consisting only of the nodes of  $t$  along its leftmost branch. Here are three such trees of the form  $\beta(\max(t))$ ,



**Theorem 3.6.** *Given  $b = (t; \mathbb{T}) \in \mathcal{M}_\bullet$ , we have*

$$\rho(M_b) = \begin{cases} \sum_{b=c \setminus s} M_c \otimes M_s & \text{if } b \neq \beta(\max(t)) \\ \sum_{b=c \setminus s} M_c \otimes M_s + 1 \otimes M_t & \text{if } b = \beta(\max(t)) \end{cases}.$$

For example,

$$\begin{aligned} \rho(M_{\setminus}) &= M_{\setminus} \otimes 1 \\ \rho(M_{\setminus \setminus}) &= M_{\setminus \setminus} \otimes 1 + M_{\setminus} \otimes M_{\setminus} \\ \rho(M_{\setminus \setminus \setminus}) &= M_{\setminus \setminus \setminus} \otimes 1 + M_{\setminus \setminus} \otimes M_{\setminus} + M_{\setminus} \otimes M_{\setminus \setminus} + 1 \otimes M_{\setminus \setminus \setminus}. \end{aligned}$$

Our proof of Theorem 3.6 uses Proposition 1.3 and the following results.

**Lemma 3.7.** *For any bi-leveled tree  $b \in \mathcal{M}_\bullet$ , we have*

$$\beta\left(\sum_{\beta(w)=b} M_w\right) = M_b.$$

*Proof.* Expand the left hand side in terms of the fundamental bases to get

$$\beta\left(\sum_{\beta(w)=b} \sum_{w \leq v} \mu_{\mathfrak{S}}(w, v) F_v\right) = \sum_{\beta(w)=b} \sum_{w \leq v} \mu_{\mathfrak{S}}(w, v) F_{\beta(v)}.$$

As  $\beta$  is surjective, we may change the index of summation to  $b \leq c$  in  $\mathcal{M}_\bullet$  to obtain

$$\sum_{b \leq c} \left( \sum_{\substack{\beta(w)=b \\ \beta(v)=c}} \mu_{\mathfrak{S}}(w, v) \right) F_c.$$

By Theorems 1.4 and 2.4, the inner sum is  $\mu_{\mathcal{M}}(b, c)$ , so this sum is  $M_b$ .  $\square$

Recall that  $w = u \setminus v$  only if  $\tau(w) = \tau(u) \setminus \tau(v)$  and the values of  $w$  in the nodes of  $u$  exceed the values in the nodes of  $v$ . We always have the trivial decomposition  $w = (\emptyset, w)$ . Suppose that  $w = u \setminus v$  with  $u \neq \emptyset$  a nontrivial decomposition. If  $\beta(w) = b = (t; \mathbb{T})$ , then  $\mathbb{T}$  is a subset of the nodes of  $u$  so that  $\beta(u) = (\tau(u); \mathbb{S})$  and  $b = \beta(u) \setminus \tau(v)$ . Moreover, for every decomposition  $b = c \setminus s$  and every  $u, v$  with  $\beta(u) = c$  and  $\tau(v) = s$ , we have  $b = \beta(u \setminus v)$ . Thus, for  $b \in \mathcal{M}_\bullet$ , we have

$$(3.7) \quad \bigsqcup_{\beta(w)=b} \bigsqcup_{\substack{w=u \setminus v \\ u \neq \emptyset}} (u, v) = \bigsqcup_{b=c \setminus t} \bigsqcup_{\beta(u)=c} \bigsqcup_{\tau(v)=t} (u, v).$$

*Proof of Theorem 3.6.* Let  $b = (t; \mathbb{T})$  with  $t \neq \mathbb{1}$ . Using Lemma 3.7, we have

$$\rho(M_b) = \rho\beta \left( \sum_{\beta(w)=b} M_w \right) = \sum_{\beta(w)=b} \rho\beta M_w.$$

By (3.5), (3.7), and (1.5), this equals

$$\begin{aligned} & \sum_{\beta(w)=b} \sum_{\substack{w=u \setminus v \\ u \neq \emptyset}} \beta(M_u) \otimes \tau(M_v) + \sum_{\beta(w)=b} \beta(M_\emptyset) \otimes \tau(M_w) \\ &= \sum_{b=c \setminus s} \left( \sum_{\beta(u)=c} \beta(M_u) \right) \otimes \left( \sum_{\tau(v)=s} \tau(M_t) \right) + \sum_{\beta(w)=b} 1 \otimes \tau(M_w). \end{aligned}$$

By Lemma 3.7 and (1.4), the first sum becomes  $\sum_{b=c \setminus s} M_c \otimes M_s$  and the second sum vanishes unless  $b = \beta(\max(t))$ . This completes the proof.  $\square$

## 4. HOPF VARIATIONS

**4.1. The  $\mathcal{Y}Sym$ -Hopf module  $\mathcal{M}Sym_+$ .** Let  $\mathcal{M}_+ := (\mathcal{M}_n)_{n \geq 1}$  be the bi-leveled trees with at least one internal node and define  $\mathcal{M}Sym_+$  to be the positively graded part of  $\mathcal{M}Sym$ , which has bases indexed by  $\mathcal{M}_+$ . A *restricted splitting* of  $b \in \mathcal{M}_+$  is a splitting  $b \xrightarrow{\Upsilon_+} (b_0, \dots, b_m)$  with  $b_0 \in \mathcal{M}_+$ , i.e.,  $b_0 \neq \mathbb{1}$ . Given  $b \xrightarrow{\Upsilon_+} (b_0, \dots, b_m)$  and  $t \in \mathcal{Y}_m$ , form the bi-leveled tree  $(b_0, \dots, b_m)/t$  by grafting the ordered forest  $(b_0, \dots, b_m)$  onto the leaves of  $t$ , with order ideal consisting of the nodes of  $t$  together with the nodes of the forest coming from the order ideal of  $b$ , as in (3.3)(ii).

We define an action and coaction of  $\mathcal{Y}Sym$  on  $\mathcal{M}Sym_+$  that are similar to the product and coaction on  $\mathcal{M}Sym$ . They come from a second collection of polytope

maps  $\mathcal{M}_n \rightarrow \mathcal{Y}_{n-1}$  arising from viewing the vertices of  $\mathcal{M}_n$  as *painted trees* on  $n-1$  nodes (see [5, 8]). For  $b \in \mathcal{M}_+$  and  $t \in \mathcal{Y}_m$ , set

$$(4.1) \quad \begin{aligned} F_b \cdot F_t &= \sum_{b \xrightarrow{\mathcal{Y}_+} (b_0, \dots, b_m)} F_{(b_0, \dots, b_m)/t}, \\ \rho_+(F_b) &= \sum_{b \xrightarrow{\mathcal{Y}_+} (b_0, b_1)} F_{b_0} \otimes F_{\phi(b_1)}. \end{aligned}$$

For example, in the fundamental bases of  $\mathcal{MSym}_+$  and  $\mathcal{YSym}$ , we have

$$\begin{aligned} F_{\swarrow} \cdot F_{\swarrow} &= F_{\swarrow \swarrow} + F_{\swarrow \searrow} + F_{\searrow \swarrow}, \\ \rho_+(F_{\swarrow \swarrow}) &= F_{\swarrow \swarrow} \otimes 1 + F_{\swarrow \swarrow} \otimes F_{\swarrow} + F_{\swarrow \swarrow} \otimes F_{\searrow} + F_{\swarrow \swarrow} \otimes F_{\searrow}. \end{aligned}$$

**Theorem 4.1.** *The operations in (4.1) define a  $\mathcal{YSym}$ -Hopf module structure on  $\mathcal{MSym}_+$ .*

*Proof.* The unital and counital properties are immediate. We check only that the action is associative, the coaction is coassociative, and the two structures commute with each other.

*Associativity.* Fix  $b = (t; \mathbb{T}) \in \mathcal{M}_+$ ,  $r \in \mathcal{Y}_m$ , and  $s \in \mathcal{Y}_n$ . A term in the expression  $(F_b \cdot F_r) \cdot F_s$  corresponds to a restricted splitting and grafting  $b \xrightarrow{\mathcal{Y}_+} (b_0, \dots, b_m) \rightsquigarrow (b_0, \dots, b_m)/r = c$ , followed by another  $c \xrightarrow{\mathcal{Y}_+} (c_0, \dots, c_n) \rightsquigarrow (c_0, \dots, c_n)/t$ . The order ideal for this term equals  $\mathbb{T} \cup \{\text{the nodes of } r \text{ and } s\}$ . Note that restricted splittings of  $c$  are in bijection with pairs of splittings

$$\left( b \xrightarrow{\mathcal{Y}_+} (b_0, \dots, b_{m+n}), r \xrightarrow{\mathcal{Y}} (r_0, \dots, r_n) \right).$$

Terms of  $F_b \cdot (F_r \cdot F_s)$  also correspond to these pairs of splittings. The order ideal for this term is again  $\mathbb{T} \cup \{\text{the nodes of } r \text{ and } s\}$ . That is,  $(F_b \cdot F_r) \cdot F_s$  and  $F_b \cdot (F_r \cdot F_s)$  agree term by term.

*Coassociativity.* Fix  $b = (t; \mathbb{T}) \in \mathcal{M}_+$ . Terms  $F_c \otimes F_r \otimes F_s$  in  $(\rho_+ \otimes \mathbb{1})\rho_+(F_b)$  and  $(\mathbb{1} \otimes \Delta)\rho_+(F_b)$  both correspond to restricted splittings  $b \xrightarrow{\mathcal{Y}_+} (c, c_1, c_2)$ , where  $\phi(c_1) = r$  and  $\phi(c_2) = s$ . In either case, the order ideal on  $c$  is  $\mathbb{T}|_c$ .

*Commuting structures.* Fix  $b = (s; \mathbb{S}) \in \mathcal{M}_+$  and  $t \in \mathcal{Y}_m$ . A term  $F_{c_0} \otimes F_{\phi(c_1)}$  in  $\rho_+(F_b \cdot F_t)$  corresponds to a choice of a restricted splitting and grafting  $b \xrightarrow{\mathcal{Y}_+} (b_0, \dots, b_m) \rightsquigarrow (b_0, \dots, b_r)/t = c$ , followed by a restricted splitting  $c \xrightarrow{\mathcal{Y}_+} (c_0, c_1)$ . The order ideal on  $c_0$  equals the nodes of  $c_0$  inherited from  $\mathbb{S}$ , together with the nodes of  $c_0$  inherited from  $t$ . The restricted splittings of  $c$  are in bijection with pairs of splittings  $(b \xrightarrow{\mathcal{Y}_+} (b_0, \dots, b_{m+1}), t \xrightarrow{\mathcal{Y}} (t_0, t_1))$ . If  $t_0 \in \mathcal{Y}_n$ , then the pair of graftings  $c_0 = (b_0, \dots, b_n)/t_0$  and  $c_1 = (b_{n+1}, \dots, b_m)/t_1$  are precisely the terms appearing in  $\rho_+(F_b) \cdot \Delta(F_t)$ .  $\square$

The similarity of (4.1) to the coaction (3.4) of  $\mathcal{YSym}$  on  $\mathcal{MSym}$  gives the following result, whose proof we leave to the reader.

**Corollary 4.2.** *For  $b \in \mathcal{M}_+$ , we have*

$$\rho_+(M_b) = \sum_{b=c \setminus s} M_c \otimes M_s.$$

This elucidates the structure of  $\mathcal{MSym}_+$ . Let  $\mathcal{B} \subset \mathcal{M}_+$  be the indecomposable bi-leveled trees—those with only trivial decompositions,  $b = b \setminus |$ . Then  $(t; \mathbb{T}) \in \mathcal{B}$  if and only if  $\mathbb{T}$  contains the rightmost node of  $t$ . Every tree  $c$  in  $\mathcal{M}_+$  has a unique decomposition  $c = b \setminus s$  where  $b \in \mathcal{B}$  and  $s \in \mathcal{Y}_\bullet$ . Indeed, pruning  $c$  immediately above the rightmost node in its order ideal gives a decomposition  $c = b \setminus s$  where  $b \in \mathcal{B}$  and  $s \in \mathcal{Y}_\bullet$ . This induces a bijection of graded sets,

$$\mathcal{M}_+ \longleftrightarrow \mathcal{B} \times \mathcal{Y}_\bullet.$$

Moreover, if  $b \in \mathcal{B}$  and  $s \in \mathcal{Y}_\bullet$ , then Corollary 4.2 and (1.6) together imply that

$$(4.2) \quad \rho_+(M_{b \setminus s}) = \sum_{s=r \setminus t} M_{b \setminus r} \otimes M_t.$$

Note that  $\mathbb{Q}\mathcal{B} \otimes \mathcal{YSym}$  is a graded right  $\mathcal{YSym}$ -comodule with structure map,

$$b \otimes M_s \longmapsto b \otimes (\Delta M_s),$$

for  $b \in \mathcal{B}$  and  $s \in \mathcal{Y}_\bullet$ . Comparing this with (4.2), we deduce the following algebraic and combinatorial facts.

**Corollary 4.3.** *The map  $\mathbb{Q}\mathcal{B} \otimes \mathcal{YSym} \rightarrow \mathcal{MSym}_+$  defined by  $b \otimes M_s \mapsto M_{b \setminus s}$  is an isomorphism of graded right  $\mathcal{YSym}$  comodules.*

*The quotient of enumerating series  $\mathbf{M}(q)_+/\mathbf{Y}(q)$  is equal to the enumerating series of the graded set  $\mathcal{B}$ .*

In particular, if  $\mathcal{B}_n := \mathcal{B} \cap \mathcal{M}_n$ , then  $|\mathcal{B}_n| = B_n$  by (2.5).

**Remark 4.4.** The *coinvariants* in a right comodule  $M$  over a coalgebra  $C$  are  $M^{\text{co}} := \{m \in M \mid \rho(m) = m \otimes 1\}$ . We identify the vector space  $\mathbb{Q}\mathcal{B}$  with  $\mathcal{MSym}_+^{\text{co}}$  via  $b \mapsto M_b$ . The isomorphism  $\mathbb{Q}\mathcal{B} \otimes \mathcal{YSym} \rightarrow \mathcal{MSym}_+$  is a special case of the Fundamental Theorem of Hopf Modules [19, Theorem 1.9.4]: If  $M$  is a Hopf module over a Hopf algebra  $H$ , then  $M \simeq M^{\text{co}} \otimes H$  as Hopf modules.

**4.2. Hopf module structure on  $\mathcal{MSym}$ .** We use Theorem 3.6 to identify the  $\mathcal{YSym}$ -coinvariants in  $\mathcal{MSym}$ . Let  $\mathcal{B}'$  be those indecomposable bi-leveled trees which are not of the form  $\beta(\max(t))$ , for some  $t \in \mathcal{Y}_+$ , together with  $\{|\}$ .

**Corollary 4.5.** *The  $\mathcal{YSym}$ -coinvariants of  $\mathcal{MSym}$  have a basis  $\{M_b \mid b \in \mathcal{B}'\}$ .*

For  $n > 0$ , the difference  $\mathcal{B}_n \setminus \mathcal{B}'_n$  consists of indecomposable bi-leveled trees with  $n$  nodes of the form  $\beta(\max(t))$ . If  $\beta(\max(t)) \in \mathcal{B}_n$ , then  $t = s \vee |$ , for some  $s \in \mathcal{Y}_{n-1}$ , and so  $|\mathcal{B}'_n| = B_n - C_{n-1}$ , which we saw in the proof of Theorem 2.7.

For  $t \in \mathcal{Y}_\bullet$ , set  $|\setminus t := \beta(\max(t))$ , and if  $|\neq b \in \mathcal{B}'$ , set  $b \setminus\setminus t := b \setminus t$ . Every bi-leveled tree uniquely decomposes as  $b \setminus\setminus t$  with  $b \in \mathcal{B}'$  and  $t \in \mathcal{Y}_\bullet$ . By Theorem 3.6,  $M_b \otimes M_t \mapsto M_{b \setminus\setminus t}$  induces an isomorphism of right  $\mathcal{YSym}$ -comodules,

$$(4.3) \quad \mathcal{MSym}^{\text{co}} \otimes \mathcal{YSym} \longrightarrow \mathcal{MSym},$$

where the structure map on  $\mathcal{MSym}^{\text{co}} \otimes \mathcal{YSym}$  is  $M_b \otimes M_t \mapsto M_b \otimes \Delta(M_t)$ . Treating  $\mathcal{MSym}^{\text{co}}$  as a trivial  $\mathcal{YSym}$ -module,  $M_b \cdot M_t = \varepsilon(M_t)M_b$ ,  $\mathcal{MSym}^{\text{co}} \otimes \mathcal{YSym}$  becomes a right  $\mathcal{YSym}$ -module. As explained in [19, Example 1.9.3], this makes  $\mathcal{MSym}^{\text{co}} \otimes \mathcal{YSym}$  into a  $\mathcal{YSym}$ -Hopf module.

We express this structure on  $\mathcal{MSym}$ . Let  $b \setminus t \in \mathcal{M}$ . and  $s \in \mathcal{Y}$ , then

$$(4.4) \quad M_{b \setminus t} \cdot M_s = \sum_{r \in t \cdot s} M_{b \setminus r} \quad \text{and} \quad \rho(M_{b \setminus t}) = \sum_{t=r \setminus s} M_{b \setminus r} \otimes M_s,$$

where  $t \cdot s$  is the set of trees  $r$  indexing the product  $M_t \cdot M_s$  in  $\mathcal{YSym}$ . The coaction is as before, but the product is new. It is not positive in the fundamental basis,

$$F_{\setminus} \cdot F_{\setminus} = F_{\setminus \setminus} - F_{\setminus \setminus} + F_{\setminus \setminus} + 2F_{\setminus \setminus}.$$

We complete the proof of Theorem 2.7.

**Corollary 4.6.** *The power series  $\mathbf{S}(q)/\mathbf{M}(q)$  has nonnegative coefficients.*

*Proof.* Observe that

$$\mathbf{S}(q)/\mathbf{M}(q) = \left( \mathbf{S}(q)/\mathbf{Y}(q) \right) / \left( \mathbf{M}(q)/\mathbf{Y}(q) \right).$$

Since both  $\mathfrak{S}\text{Sym}$  and  $\mathcal{MSym}$  are right  $\mathcal{YSym}$ -Hopf modules, the two quotients of enumerating series on the right are generating series for their coinvariants, by the Fundamental Theorem of Hopf modules. Thus

$$\mathbf{S}(q)/\mathbf{M}(q) = \mathbf{S}^{\text{co}}(q)/\mathbf{M}^{\text{co}}(q),$$

where  $\mathbf{S}^{\text{co}}(q)$  and  $\mathbf{M}^{\text{co}}(q)$  are the enumerating series for  $\mathfrak{S}\text{Sym}^{\text{co}}$  and  $\mathcal{MSym}^{\text{co}}$ . To show that  $\mathbf{S}^{\text{co}}(q)/\mathbf{M}^{\text{co}}(q)$  is nonnegative, we index bases for these spaces by graded sets  $\mathcal{S}$  and  $\mathcal{B}'$ , then establish a bijection  $\mathcal{B}' \times \mathcal{S}' \rightarrow \mathcal{S}$  for some graded subset  $\mathcal{S}' \subset \mathcal{S}$ .

The set  $\mathcal{B}'$  was identified in Corollary 4.5. The coinvariants  $\mathfrak{S}\text{Sym}^{\text{co}}$  were given in [2, Theorem 7.2] as a *left Hopf kernel*. The basis was identified as follows. Recall that permutations  $u \in \mathfrak{S}$ . may be written uniquely in terms of indecomposables,

$$(4.5) \quad u = u_1 \setminus \cdots \setminus u_r$$

(taking  $r = 0$  for  $u = \emptyset$ ). Let  $\mathcal{S} \subset \mathfrak{S}$ . be those permutations  $u$  whose rightmost indecomposable component has a 132-pattern, and thus  $u \neq \max(t)$  for any  $t \in \mathcal{Y}_+$ . (Note that  $u = \emptyset \in \mathcal{S}$ .) Then  $\{M_u \mid u \in \mathcal{S}\}$  is a basis for  $\mathfrak{S}\text{Sym}^{\text{co}}$ .

Fix a section  $g: \mathcal{M} \rightarrow \mathfrak{S}$ . of the map  $\beta: \mathfrak{S} \rightarrow \mathcal{M}$ . and define a subset  $\mathcal{S}' \subset \mathcal{S}$  as follows. Given the decomposition  $u = u_1 \setminus \cdots \setminus u_r$  in (4.5) with  $r \geq 0$ , consider the length  $\ell \geq 0$  of the maximum initial sequence  $u_1 \setminus \cdots \setminus u_\ell$  of indecomposables belonging to  $g(\mathcal{B}')$ . Put  $u \in \mathcal{S}'$  if  $\ell$  is even. Define the map of graded sets

$$\kappa: \mathcal{B}' \times \mathcal{S}' \longrightarrow \mathcal{S} \quad \text{by} \quad (b, v) \longmapsto g(b) \setminus v.$$

The image of  $\kappa$  lies in  $\mathcal{S}$  as the last component of a nontrivial  $g(b) \setminus v$  is either  $g(b)$  or the last component of  $v$ , neither of which can be  $\max(t)$  for  $t \in \mathcal{Y}_+$ .

We claim that  $\kappa$  is bijective. If  $u \in \mathcal{S}'$ , then  $u = \kappa(l, u)$ . If  $u \in \mathcal{S} \setminus \mathcal{S}'$ , then  $u$  has an odd number of initial components from  $g(\mathcal{B}')$ . Letting its first factor be  $g(b)$ ,

we see that  $u = g(b) \setminus u' = \kappa(b, u')$  with  $u' \in \mathcal{S}'$ . This surjective map is injective as the expressions  $\kappa(l, u')$  and  $\kappa(b, u')$  with  $b \in \mathcal{B}'_+$  and  $u' \in \mathcal{S}'$  are unique.

This isomorphism of graded sets identifies the enumerating series of the graded set  $\mathcal{S}'$  as the quotient  $\mathbf{S}^{\text{co}}(q)/\mathbf{M}^{\text{co}}(q)$ , which completes the proof.  $\square$

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