UPR-1254-T MIFPA-13-31

Electrodynamics of Black Holes in STU Supergravity

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ABSTRACT

External magnetic fields can probe the composite structure of black holes in string theory. With this motivation we study magnetised four-charge black holes in the STU model, a consistent truncation of maximally supersymmetric supergravity with four types of electromagnetic fields. We employ solution generating techniques to obtain Melvin backgrounds, and black holes in these backgrounds. For an initially electrically charged static black hole immersed in magnetic fields, we calculate the resultant angular momenta and analyse their global structure. Examples are given for which the ergoregion does not extend to infinity. We calculate magnetic moments and gyromagnetic ratios via Larmor's formula. Our results are consistent with earlier special cases. A scaling limit and associated subtracted geometry in a single surviving magnetic field is shown to lift to $AdS_3 \times S^2$. Magnetizing magnetically charged black holes give static solutions with conical singularities representing strings or struts holding the black holes against magnetic forces. In some cases it is possible to balance these magnetic forces.

Contents

1 Introduction

In recent work [1, 2] the structure and thermodynamics of charged and rotating black holes in Einstein-Maxwell theory immersed in an external magnetic field were studied. The solutions were obtained by means of a solution-generating technique pioneered by Ernst [3], starting from a seed solution with no external magnetic field present. By the well-known electric-magnetic duality of Einstein-Maxwell theory, one may just as easily consider the effects due to the immersion in an electric field of an electrically and magnetically charged rotating black hole. The solution-generating property of Einstein-Maxwell theory extends to ungauged supergravity theories, and nowadays is best seen as part of the web of dualities at the heart of the modern synthesis of supergravity and superstring theory known as M-Theory. The supergravity theories of interest contain some number $k > 1$ of generalised Maxwell fields, and black holes may thus carry k generalised electric and k generalised magnetic charges. Explicit solutions are available for all eight charges (four electric and four magnetic) [4] in a theory often referred to as the STU model, which is $\mathcal{N} = 2$ supergravity coupled to 3 additional vector multiplets. (See [5, 6] for the results for just four charges.) If this is reduced to three dimensions on a Killing symmetry, the bosonic sector of the resulting theory can be cast into the form of a scalar sigma model with a global $O(4, 4)$ symmetry coupled to gravity. By acting with an appropriate subgroup of $O(4, 4)$ on the spatial reduction of a four-charge black hole, one can introduce further parameters that acquire the interpretation of describing external electric and/or magnetic fields after lifting the solution back to four dimensions. It is therefore of interest to ask how these black holes respond to being immersed in a combination of the 4 possible electric and 4 possible magnetic fields.

Even in the Einstein-Maxwell case, the general situation is extremely complicated because a magnetic field may exert a torque on a charged black hole and cause it to rotate, even if it was originally non-rotating and static. Another potential complication is that applying, for instance, a magnetic field to a magnetically charged black hole should cause it to accelerate. In fact, in the case of Einstein-Maxwell theory, it was found that a static magnetically charged but electrically neutral Reissner-Nordström black hole seed remains static and non-accelerating upon magnetization, but the metric then exhibits a conical singularity along the axis of symmetry which represents a cosmic string whose tension is tuned so as to prevent acceleration.

It is of interest to explore these phenomena further when more than one electromagnetic field is present, since, as we shall show in this paper, new features then arise. In view of the many complications introduced by rotation, we have decided to focus in this paper just on static seed solutions.

The theory that we shall be considering, the four-dimensional STU supergravity model, can arise as a 6-torus reduction of ten-dimensional type IIA (or heterotic) supergravity to give $\mathcal{N} = 8$ ($\mathcal{N} = 4$) supergravity in four dimensions, followed by a consistent truncation. The truncation can be performed in a variety of different ways that are all equivalent under four-dimensional U-duality, but which have different ten-dimensional interpretations, depending on how the four surviving gauge fields are selected.[1]

In the bulk of the paper we shall consider the STU model in the formulation that was used in [5, 6], in which two of the gauge fields come from the reduction of the NS-NS 2-form potential of IIA supergravity, and the other two are Kaluza-Klein vectors coming from the reduction of the ten-dimensional metric. The four-dimensional Reissner-Nordström black hole lifts to a pp-wave/NUT/NS1/NS5 intersection in this description.

In an alternative description of the STU model one of the gauge potentials is the direct reduction of the Ramond-Ramond vector potential in the IIA theory, whilst the rest of the gauge potentials come from the Kaluza-Klein reduction of the Ramond-Ramond 3-form potential. In this description the Reissner-Nordström black hole lifts to a D0/D4/D4/D4 intersection in ten dimensions. (Alternatively, in M-theory this configuration is obtained from a pp-wave/M5/M5/M5 intersection.) We shall discuss the relation between the above two formulations of the STU model in appendix B.

There does not seem to be an ideal and succinct way of referring to the two basic types of solution that we shall be discussing in this paper. In essence, we wish to consider the 4-field STU model generalisations of two distinct magnetised Reissner-Nordström (RN) black holes:

(1) Magnetised RN black hole carrying electric charge

(2) Magnetised RN black hole carrying magnetic charge (1.1)

Because of the duality complexions of the four field strengths in the STU model, the 4-field generalisation of solution 1 above actually has two field strengths with electric charges and external magnetic fields, while the other two field strengths have magnetic charges and external electric fields. The situation is the opposite for the 4-field generalisation of solution 2, in the sense that the first two field strengths now have magnetic charges and external magnetic fields, while the remaining two field strengths have electric charges and external electric fields.

In order to avoid a cumbersome description of these two types of solution we shall sometimes for brevity refer to them as if we were working in a duality complexion where all

[1] A black hole solution of the STU model, specified by five independent charges, is a generating solution for general black holes of the full $\mathcal{N}=8$ ($\mathcal{N}=4$) supergravity theory with 28 electric and 28 magnetic charges, which are obtained by acting on the generating solution with a subset of U-duality (or S and T-duality) transformations (See, for example, [7]). The seed solution (in fact presented, for symmetry reasons, with eight charges rather than the minimal set of five charges) has recently been constructed in [4]. The five-charge static and BPS black holes were obtained in [8] and [9].

four field strengths carried electric charges and external magnetic fields in the generalisations of solution 1, and all four fields carried magnetic charges and external magnetic fields in the generalisations of solution 2.[2] Thus, in summary, we shall refer to the STU model generalisations of solution 1 as magnetised electric black holes, and the generalisations of solution 2 as magnetised magnetic black holes.

Before constructing the STU model generalisations of solutions 1 and 2 above, we begin in section 3 by obtaining the STU model generalisation of the pure Melvin magnetic universe. This can be constructed by starting from Minkowski spacetime as the seed solution, acting with the appropriate $O(4,4)$ global symmetries after reduction to three dimensions, and then lifting back to four dimensions. We also show how it can alternatively be constructing in a manner that generalises a procedure described in [14], as an analytic continuation of a limiting form of the four-charge static black hole solutions in the STU model.

In section 4 we construct the STU model generalisations of solution 1 above (the magnetised Reissner-Nordström solution with electric charge). The solutions we obtain have four independent charges and four independent external fields. As one might anticipate from the results of [1], they are in general stationary, even though we again start from a static seed solution (the four-charge black holes of the STU model). This is because the external fields exert torques on the charges carried by the black hole. We show that it is in fact possible, by choosing the charges and the external fields appropriately, to balance the torques and thereby obtain a static black hole solution. We also examine the asymptotic structure of the metrics in the general case, showing that generically there is an ergoregion extending out to infinity, close to the axis. We discuss the conditions on the charges and magnetic fields under which the metrics become asymptotically static at infinity, with no ergoregions.

For many purposes it is helpful to focus on the simplifications that result by taking a *near horizon limit* of the black hole metrics. In [10], the resulting *subtracted geometry* [11, 12, 13] was obtained by taking a suitable scaling limit. In section 5, we apply this idea the to the metrics considered in this paper.

In section 6 we turn to the STU model generalisation of solution 2 above (the magnetised Reissner-Nordström solution with magnetic charge). The solutions, which are all static, again have four independent charges and four independent external fields. As one might have anticipated on the basis of the results in [1], the metrics in general have a conical

[2]Unfortunately, as we discuss in appendix B, in actuality the Lagrangian of the STU model in such a duality complexion would be extremely complicated and inconvenient.

singularity on the axis, corresponding to a delta-function tension holding the black hole in place. Interestingly, this may be eliminated by imposing an appropriate condition on the charges and magnetic fields. This can be interpreted as being due to a cancellation of the forces associated with the individual charges and fields.

In addition to the concrete results obtained above, we include in the appendices some more technical material in which the explicit calculations and comparisons between different formalisms used are described in more detail than is done in the body of the text.

2 The STU Model and its Black Holes

In this paper we shall be studying some properties of black holes in the four-dimensional STU supergravity theory, which comprises $\mathcal{N}=2$ supergravity coupled to three vector multiplets. The Lagrangian for the bosonic sector of the STU model, in the notation of [6], is

$$
\begin{aligned}
\mathcal{L}_4 = & \, R\,{*\mathbb{1}} - \tfrac12 {*d\varphi_i} \wedge d\varphi_i - \tfrac12 e^{2\varphi_i}\, {*d\chi_i} \wedge d\chi_i - \tfrac12 e^{-\varphi_1}\Big(e^{\varphi_2-\varphi_3}\,{*F_{(2)1}} \wedge F_{(2)1} \\
& + e^{\varphi_2+\varphi_3}\,{*F_{(2)2}} \wedge F_{(2)2} + e^{-\varphi_2+\varphi_3}\,{*\mathcal{F}^1_{(2)}} \wedge \mathcal{F}^1_{(2)} + e^{-\varphi_2-\varphi_3}\,{*\mathcal{F}^2_{(2)}} \wedge \mathcal{F}^2_{(2)}\Big) \\
& + \chi_1\,(F_{(2)1} \wedge \mathcal{F}^1_{(2)} + F_{(2)2} \wedge \mathcal{F}^2_{(2)})\,, \qquad (2.1)
\end{aligned}
$$

where the index i labelling the dilatons φ_i and axions χ_i ranges over $1 \le i \le 3$. The four field strengths can be written in terms of potentials as

$$
\begin{aligned}
F_{(2)1} &= dA_{(1)1} - \chi_2\, d\mathcal{A}^2_{(1)}\,, \\
F_{(2)2} &= dA_{(1)2} + \chi_2\, d\mathcal{A}^1_{(1)} - \chi_3\, dA_{(1)1} + \chi_2\,\chi_3\, d\mathcal{A}^2_{(1)}\,, \\
\mathcal{F}^1_{(2)} &= d\mathcal{A}^1_{(1)} + \chi_3\, d\mathcal{A}^2_{(1)}\,, \\
\mathcal{F}^2_{(2)} &= d\mathcal{A}^2_{(1)}\,. \qquad (2.2)
\end{aligned}
$$

Note that (2.1) could be obtained by reducing the six-dimensional bosonic string action on $S^1 \times S^1$, and then dualising the 2-form potential $A_{(2)}$ to the axion that is called χ_1 here.

Four-charge rotating black hole solutions in the STU theory were constructed in [5]. We shall use the conventions and notation of [6], in which the metric for the four-charge black holes is given by

$$
ds_4^2 = -\frac{\rho^2 - 2mr}{W}\,(dt + \mathcal{B}_{(1)})^2 + W\Big(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta\,\sin^2\theta\, d\phi^2}{\rho^2 - 2mr}\Big)\,. \qquad (2.3)
$$

where

$$\begin{aligned}
\Delta &= r^2 - 2mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta\,, \\
\mathcal{B}_{(1)} &= \frac{2ma\sin^2\theta(r\Pi_c - (r-2m)\Pi_s)}{(\rho^2 - 2mr)}\,d\phi\,, \\
W^2 &= r_1\,r_2\,r_3\,r_4 + a^4\cos^4\theta \\
&\quad + a^2\cos^2\theta\,[2r^2 + 2mr\sum_{i=1}^{4} s_i^2 + 8m^2\Pi_s\,\Pi_c - 4m^2(2\Pi_s^2 + \sum_{i=1}^{4}\Pi_s^i)]\,,
\end{aligned} \tag{2.4}$$

$r_i = r + 2ms_i^2$, $s_i = \sinh\delta_i$, $c_i = \cosh\delta_i$, and $\Pi_c = c_1c_2c_3c_4$ and $\Pi_s = s_1s_2s_3s_4$. We also define

$$\Pi_s^i = s_i^{-1}\,\Pi_s\,, \qquad \Pi_c^i = c_i^{-1}\,\Pi_c\,. \tag{2.5}$$

The expressions for the gauge potentials, axions and dilatons can be found [6].

The mass physical M, angular momentum J, charges Q_i and dipole moments μ_i were calculated in [5]. In the notation and conventions of [6] that we are using here, they are given by

$$\begin{aligned}
M &= \tfrac{1}{4}m\sum_{i=1}^{4}(c_i^2 + s_i^2)\,, \qquad J = ma\,(\Pi_c - \Pi_s)\,, \\
Q_i &= 2ms_i\,c_i\,, \qquad \mu_i = 2ma\,(s_i\,\Pi_c^i - c_i\,\Pi_s^i)\,.
\end{aligned} \tag{2.6}$$

In standard Maxwell electrodynamics, the magnetic moment of a particle of mass M and angular momentum J carrying a charge Q is given by $\mu = \mathbf{g}JQ/(2M)$, where $\mathbf{g}$ is the gyromagnetic ratio. Generically, for the four-charge black holes in the STU model, we can expect a relation of the form

$$\mu_i = \frac{J}{2M}\sum_{j=1}^{4}\mathbf{g}_{ij}\,Q_j\,. \tag{2.7}$$

From the quantities (2.6) given above it is not possible, in the absence of additional criteria, to derive a unique form for the "gyromagnetic matrix" $\mathbf{g}_{ij}$. However, if we impose the additional requirements that it be a symmetric matrix, and furthermore that it exhibit the same symmetries as the metric under permutation of the four charge parameters δ_i, then we are led to the following result:

$$\begin{aligned}
i = j: &\qquad \mathbf{g}_{ii} = \frac{1}{2c_i^2}\sum_{k=1}^{4}(c_k^2 + s_k^2)\,, \\
i \neq j: &\qquad \mathbf{g}_{ij} = -\frac{\Pi_s}{6c_ic_js_is_j}\,\frac{\sum_{k=1}^{4}(c_k^2 + s_k^2)}{\Pi_c - \Pi_s}\,.
\end{aligned} \tag{2.8}$$

In special cases the expression (2.8) for the gyromagnetic ratio reduces to previously-known results. For example, if we consider the single-charge case where $\delta_2 = \delta_3 = \delta_4 = 0$

then we obtain the "Kaluza-Klein" result [16, 17]

$$\mathbf{g} = \mathbf{g}_{11} = 2 - \tanh^2 \delta_1 \,. \tag{2.9}$$

If two or more of the charges are non-zero, the gyromagnetic matrix has off-diagonal components. If we take all four charges to be equal, then

$$\mathbf{g}_{ij} = \frac{2(c^2 + s^2)}{c^2} \,, \ i = j \,, \qquad \mathbf{g}_{ij} = -\frac{2s^2}{3c^2} \,, \ i \neq j \,, \tag{2.10}$$

and so with $Q_i = Q$ we have $\mathbf{g}_{ij} Q_j = 2Q$, implying the standard result [18] that $\mathbf{g} = 2$ for the Kerr-Newman black hole.

In the case of two non-zero equal charges, say, $Q_1 = Q_2 = Q$ and $Q_3 = Q_4 = 0$, we obtain the following nonzero gyromagnetic matrix coefficients:

$$\mathbf{g}_{11} = \mathbf{g}_{22} = 2 \,, \ \mathbf{g}_{33} = \mathbf{g}_{44} = 2\,c^2 \,, \ \mathbf{g}_{34} = \mathbf{g}_{43} = -\frac{2}{3} s^2 \,. \tag{2.11}$$

Thus, $\mathbf{g}_{1j} Q_j = \mathbf{g}_{2j} Q_j = 2Q$ which implies $\mathbf{g} = 2$ for Q.

In the case of three non-zero equal charges, say, $Q_1 = Q_2 = Q_3 = Q$ and $Q_4 = 0$, we get the following nonzero gyromagnetic matrix coefficients:

$$\begin{aligned} &\mathbf{g}_{11} = \mathbf{g}_{22} = \mathbf{g}_{33} = 2 + \tanh^2 \delta \,, \ \mathbf{g}_{44} = 3c^2 - 1 \,, \\ &\mathbf{g}_{i4} = \mathbf{g}_{4i} = -\frac{1}{3} \tanh^2 \delta (2 + \tanh^2 \delta) \,, \ i = 1, 2, 3 \,. \end{aligned} \tag{2.12}$$

In this case $\mathbf{g}_{ij} Q_j = (2 + \tanh^2 \delta) Q$ for $i = 1, 2, 3$, and thus $\mathbf{g} = 2 + \tanh^2 \delta$. Furthermore, even though $Q_4 = 0$, a nonzero μ_4 is induced, since $\mathbf{g}_{4j} Q_j = -\tanh^2 \delta (2 + \tanh^2 \delta)$ and thus $\mathbf{g}_4 = -\tanh^2 \delta (2 + \tanh^2 \delta)$.

Another explicit example can be obtained with pair-wise equal charges, say, $Q_1 = Q_3$ and $Q_2 = Q_3$. In this case the pair-wise equal magnetic moments $\mu_1 = \mu_3$ and $\mu_2 = \mu_4$ are related to the pair-wise equal charges as:

$$\mu_I = \frac{J}{2M} \sum_{J=1}^{2} \mathbf{G}_{IJ} Q_J \,, \quad I = 1, 2 \,, \tag{2.13}$$

where the coefficients of the gyromagnetic matrix $\mathbf{G}$ are

$$\mathbf{G}_{11} = \frac{2(3c_1^2 - 2 + 2c_2^2)}{3c_1^2} \,, \ \mathbf{G}_{22} = \frac{2(3c_1^2 - 2 + 2c_2^2)}{3c_2^2} \,, \ \mathbf{G}_{12} = \mathbf{G}_{21} = -\frac{4}{3} \frac{s_1 s_2}{c_1 c_2} \tag{2.14}$$

The matrix $\mathbf{G}$ has eigenvalues 2 and $2 + \frac{4}{3}(\tanh^2 \delta_1 + \tanh^2 \delta_2)$.

3 Pure Melvin-type Solution in the STU Model

Later in the paper, we shall be constructing solutions in the STU model describing four-charge black holes immersed in external magnetic fields. These solutions will, under appropriate circumstances, be asymptotic to the STU model generalisations of the Melvin universe of Einstein-Maxwell theory. It is useful, therefore, first to consider the simpler case of these pure Melvin-type solutions, where there is no black hole but just the external magnetic fields. (To be precise, as explained in the introduction, when we use the expression "external magnetic fields" we mean that the fields numbered 1 and 3 carry external electric fields, while those numbered 2 and 4 carry external magnetic fields.) The STU model in the conventions we are using is given in appendix A. Melvin-type solutions can be found using the results presented in appendix A, starting from a purely Minkowski seed solution. They can also be read off from the expressions for magnetised black holes presented in section 3, by setting the black hole mass and charges to zero. Thus the metric is given by (4.1) and (4.5) with $\omega = 0$ and

$$\Delta = \prod_{i=1}^{4} \Delta_i \,, \qquad \Delta_i = 1 + \beta_i^2\, r^2\, \sin^2\theta \,, \tag{3.1}$$

and so

$$ds_4^2 = \sqrt{\Delta}\,(-dt^2 + dr^2 + r^2\, d\theta^2) + \frac{1}{\sqrt{\Delta}}\, r^2\, \sin^2\theta\, d\phi^2 \,. \tag{3.2}$$

Note that here, and throughout the rest of the paper, we use the notation that

$$\beta_i = \tfrac{1}{2} B_i \,, \tag{3.3}$$

where B_i is the physical asymptotic strength of the i'th field on the symmetry axis at large distance. This is done in order to avoid many cumbersome factors of $\frac{1}{2}$ and powers of $\frac{1}{2}$ in subsequent formulae. In the pure Melvin case under discussion here, where there is no black hole, the field strengths are in fact constant along the axis.

The scalar fields are given by

$$e^{2\varphi_1} = \frac{\Delta_1\Delta_3}{\Delta_2\Delta_4}\,, \qquad e^{2\varphi_2} = \frac{\Delta_2\Delta_3}{\Delta_1\Delta_4}\,, \qquad e^{2\varphi_3} = \frac{\Delta_1\Delta_2}{\Delta_3\Delta_4}\,, \tag{3.4}$$

with the axions all vanishing. The four electromagnetic potentials $\{A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)}\}$ are given by

$$A_{(1)1} = -2\beta_1\, r\, \cos\theta\, dt \,, \qquad \mathcal{A}^1_{(1)} = -2\beta_3\, r\, \cos\theta\, dt \,,$$

$$A_{(1)2} = \frac{\beta_2\, r^2\, \sin^2\theta}{\Delta_2} d\phi \,, \qquad \mathcal{A}^2_{(1)} = \frac{\beta_4\, r^2\, \sin^2\theta}{\Delta_4} d\phi \,. \tag{3.5}$$

In terms of cylindrical coordinates (ρ, z) defined by $\rho = r\sin\theta$ and $z = r\cos\theta$, we have

$$ds_4^2 = \sqrt{\Delta}\,(-dt^2 + d\rho^2 + dz^2) + \frac{\rho^2}{\sqrt{\Delta}}\,d\phi^2 \tag{3.6}$$

with Δ_i in (3.1) now given by $\Delta_i = 1+\beta_i^2\,\rho^2$. Making the further coordinate transformations to $x = \rho\,\cos\phi$ and $y = \rho\,\sin\phi$, the metric near the axis approaches Minkowski spacetime $ds_4^2 \to -dt^2 + dx^2 + dy^2 + dz^2$, and near the axis the field strengths approach

$$F_{(2)1} \to B_1\,dt\wedge dz\,, \quad F_{(2)2} \to B_2\,dx\wedge dy\,, \quad \mathcal{F}^1_{(2)} \to B_3\,dt\wedge dz\,, \quad \mathcal{F}^2_{(2)} \to B_4\,dx\wedge dy\,. \tag{3.7}$$

Thus, as mentioned above, the electric and magnetic field strengths have magnitude B_i on the axis for all values of z, in this pure Melvin case.

It is interesting to note that the 4-field Melvin solution can be obtained instead by means of a limiting procedure and analytic continuation from the four-charge static black hole solution in the STU model, generalising the procedure described in [14] for the Melvin solution in the Einstein-Maxwell theory. The four-charge black hole metric, which can be read off from the magnetised black holes in section 3 by sending the magnetic fields B_i to zero, is given by

$$ds^2 = -\frac{r(r-2m)}{\sqrt{r_1r_2r_3r_4}}\,dt^2 + \sqrt{r_1r_2r_3r_4}\,\Big[\frac{dr^2}{r(r-2m)} + d\theta^2 + \sin^2\theta\,d\phi^2\Big]\,, \tag{3.8}$$

where $r_i = r + 2ms_i^2$. We then write the 2-sphere metric in the form $d\theta^2 + \sin^2\theta\,d\phi^2 = 4(1+|\zeta|^2)^{-2}\,d\zeta d\bar\zeta$, where $\zeta = \tan\frac{1}{2}\theta\,e^{\mathrm{i}\phi}$, and perform the scalings

$$r = \tilde r\,\lambda^{-1}\,, \qquad t = \tilde t\,\lambda\,, \qquad m = \tilde m\,\lambda^{-3}\,, \qquad s_i = \tilde s_i\,\lambda\,, \qquad \zeta = \tilde\zeta\,\lambda\,. \tag{3.9}$$

Sending $\lambda \to 0$ gives the metric

$$ds^2 = \frac{2\tilde m\tilde r}{\sqrt{\tilde r_1\tilde r_2\tilde r_3\tilde r_4}}\,d\tilde t^2 + \sqrt{\tilde r_1\tilde r_2\tilde r_3\tilde r_4}\Big(-\frac{d\tilde r^2}{\tilde m\tilde r} + 4d\tilde\zeta d\bar{\tilde\zeta}\Big)\,. \tag{3.10}$$

Defining

$$\tilde r = -\tfrac{1}{2}\tilde m\,\rho^2\,, \qquad \tilde\zeta = x + \mathrm{i}y\,, \tag{3.11}$$

and taking

$$x = \tfrac{1}{2}\mathrm{i}\,\hat t\,, \qquad y = \tfrac{1}{2}z\,, \qquad \tilde t = \frac{\mathrm{i}}{\tilde m}\,\tilde\phi\,, \qquad \tilde s_i = \frac{\mathrm{i}}{2\beta_i}\,, \qquad \tilde m = 2\sqrt{\beta_1\beta_2\beta_3\beta_4}\,, \tag{3.12}$$

we obtain the 4-field Melvin metric

$$ds^2 = \sqrt{\Delta}\,(-d\hat t^2 + d\rho^2 + dz^2) + \frac{\rho^2}{\sqrt{\Delta}}\,d\tilde\phi^2\,, \tag{3.13}$$

where $\Delta = \prod_i \Delta_i$ with $\Delta_i = 1+\beta_i^2\,\rho^2$. We see that this metric coincides with (3.6), after a minor change of notation. Applying the same scalings and analytic continuations to the scalar fields and gauge fields in the four-charge black hole solutions, one reproduces the results given in (3.4) and (3.5).

4 Magnetised Electrically Charged Black Holes

Here, we consider the magnetisation of the four-charge solution of the STU model that reduces, when the charges are set equal, to the magnetisation of the electrically-charged Reissner-Nordström solution (i.e. it reduces to solution 1 in (1.1)). Using the notation and conventions of [6], this is achieved when the field strengths numbered 1 and 3 carry magnetic charges, while the field strengths numbered 2 and 4 carry electric charges. In order to be able to present the magnetised solution in the most compact way, we shall denote the four charge parameters by (q_1, q_2, q_3, q_4).

Applying the procedure described in appendix A, we find that the metric is given by

$$ds_4^2 = H\,[-r(r-2m)dt^2 + \frac{r_1 r_2 r_3 r_4}{r(r-2m)}\,dr^2 + r_1 r_2 r_3 r_4 d\theta^2\,] + H^{-1}\,\sin^2\theta\,(d\phi - \omega dt)^2\,, \quad (4.1)$$

where

$$r_i = r + 2m s_i^2\,, \quad (4.2)$$

and we shall use the notation $s_i = \sinh\delta_i$ and $c_i = \cosh\delta_i$. The function ω is given by

$$\omega = \sum_{i=1}^{4}\Big[-\frac{q_i\,\beta_i}{r_i} + \frac{q_i\,\Xi_i\,[r_i + (r-2m)\cos^2\theta]r}{r_i}\Big]\,, \quad (4.3)$$

where

$$q_i = 2m s_i c_i\,, \qquad \Xi_i = \frac{\beta_1\beta_2\beta_3\beta_4}{\beta_i}\,, \qquad \beta_i = \tfrac{1}{2}B_i\,, \quad (4.4)$$

and B_i denotes the external magnetic field strengths for each of the four gauge fields. Finally, the function H is given in this case by

$$H = \frac{\sqrt{\Delta}}{\sqrt{r_1 r_2 r_3 r_4}}\,, \quad (4.5)$$

where

$$\begin{aligned}\Delta\, =\, & 1 + \sum_i \frac{\beta_i^2 r_1 r_2 r_3 r_4}{r_i^2}\,\sin^2\theta + 2[\beta_3\beta_4 q_1 q_2 + \cdots]\cos^2\theta + [\beta_3^2\,\beta_4^2\,R_1^2\,R_2^2 + \cdots]\\ & -2(\prod_j \beta_j r_j)\sum_i \frac{q_i^2}{r_i^2}\,\sin^2\theta\cos^2\theta + [2\beta_2\beta_3\beta_4^2 q_2 q_3\,R_1^2 + \cdots]\cos^2\theta + \prod_i \beta_i^2\,R_i^2\\ & + r_1 r_2 r_3 r_4 \sum_i \frac{\Xi_i^2\,R_i^2}{r_i^2}\,\sin^2\theta + [2\beta_1\beta_2\beta_3^2\beta_4^2 q_3 q_4\,R_1^2\,R_2^2 + \cdots]\cos^2\theta\,, \end{aligned} \quad (4.6)$$

and we have defined

$$R_i^2 = r_i^2\,\sin^2\theta + q_i^2\,\cos^2\theta\,. \quad (4.7)$$

Note that in each of the square-bracketed terms, the ellipses denote all the analogous terms that arise by taking all inequivalent permutations of the indices 1, 2, 3 and 4.

The periodicity $\Delta\phi$ of the azimuthal coordinate ϕ is determined by the requirement that there should be no conical singularity at the north and south poles of the sphere. Since Δ is an even function of $\cos\theta$, the requirements at the north and the south poles are identical, and they imply that ϕ should have period given by

$$\Delta\phi = 2\pi\alpha\,, \qquad \alpha = \Big(1 + [\beta_1\beta_2 q_3 q_4 + \cdots] + \prod_i \beta_i q_i\Big)\,, \tag{4.8}$$

where the ellipses in the square brackets represent the five additional terms that follow from the indicated term by taking all inequivalent permutations of the labels 1, 2, 3 and 4.

The physical charges carried by the four gauge fields can be calculated easily using the expressions in appendix A.3. The non-zero ones are (P_1, Q_2, P_3, Q_4). For the sake of uniformity we shall change the notation and call these $(\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3, \widetilde{Q}_4)$ respectively. They turn out to be given by

$$\widetilde{Q}_i = \frac{(q_i - \beta_i^2 q_1 q_2 q_3 q_4/q_i)}{\alpha}\, \frac{\Delta\phi}{2\pi}\,, \tag{4.9}$$

where α is defined in (4.8). We therefore have

$$\widetilde{Q}_i = q_i - \frac{\beta_i^2\, q_1\, q_2\, q_3\, q_4}{q_i}\,. \tag{4.10}$$

The solutions for the gauge potentials are given by

$$\begin{aligned}
A_{(1)1} &= \beta_1\, r(r-2m)\cos\theta\, \Big[\frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_3} - \frac{1}{r_4}\Big]\, dt + \sigma_1\,(d\phi - \omega dt)\,,\\
A_{(1)2} &= \Big[-\frac{q_2}{r_2} + \sum_{i=1,3,4} \frac{r\, q_i\, \beta_1\beta_3\beta_4\, [r_i + (r-2m)\cos^2\theta]}{\beta_i\, r_i}\Big]\, dt + \sigma_2\,(d\phi - \omega dt)\,,\\
\mathcal{A}^1_{(1)} &= \beta_3\, r(r-2m)\cos\theta\, \Big[\frac{1}{r_3} - \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{r_4}\Big]\, dt + \sigma_3\,(d\phi - \omega dt)\,,\\
\mathcal{A}^2_{(1)} &= \Big[-\frac{q_4}{r_4} + \sum_{i=1}^{3} \frac{r\, q_i\, \beta_1\beta_2\beta_3\, [r_i + (r-2m)\cos^2\theta]}{\beta_i\, r_i}\Big]\, dt + \sigma_4\,(d\phi - \omega dt)\,,
\end{aligned} \tag{4.11}$$

where $\sigma_i \equiv \tilde{\sigma}_i/\Delta$, and

$$\begin{aligned}
\tilde{\sigma}_1 &= -q_1 \cos\theta + \beta_1 \cos\theta \Big[\frac{\beta_2}{r_2}\,(r_1 r_3 r_4 q_2 \sin^2\theta - q_1 q_3 q_4 r_2 \cos^2\theta) + \cdots\Big] - (\beta_3\beta_4 q_2 + \cdots) R_1^2 \cos\theta \\
&\quad -\beta_1^2 \cos\theta \Big[\frac{\beta_3\beta_4}{r_2}\,(r_1 r_3 r_3 q_2 \sin^2\theta - q_1 q_3 q_4 r_2 \cos^2\theta) R_2^2 + \cdots\Big] + \beta_1^3(\beta_2 q_2 R_3^2 R_4^2 + \cdots) \cos\theta \\
&\quad +\frac{\beta_1\beta_2\beta_3\beta_4 \cos\theta}{r_1}(r_2 r_3 r_4 q_1 \sin^2\theta - q_2 q_3 q_4 r_1 \cos^2\theta) R_1^2 + \beta_1^3\beta_2\beta_3\beta_4 q_1 \cos\theta R_2^2 R_3^2 R_4^2\,, \qquad (4.12) \\
\tilde{\sigma}_2 &= \frac{\beta_2 r_1 r_3 r_4}{r_2} \sin^2\theta + (\beta_1 q_3 q_4 + \cdots) \cos^2\theta + \beta_2(\beta_1^2 R_3^2 R_4^2 + \cdots) + 2\beta_2(\beta_3\beta_4 q_3 q_4 R_1^2 + \cdots) \cos^2\theta \\
&\quad +q_2[\beta_1^2(\beta_3 q_3 R_4^2 + \beta_4 q_4 R_3^2) + \cdots] \cos^2\theta + 4\beta_1\beta_3\beta_4 q_2 q_1 q_3 q_4 \cos^4\theta \\
&\quad -\frac{\beta_1\beta_3\beta_4 q_2^2 r_1 r_3 r_4}{r_2} \sin^2\theta\cos^2\theta - \beta_1\beta_3\beta_4\, r_2 \Big(\frac{q_1^2 r_3 r_4}{r_1} + \cdots\Big) \sin^2\theta \cos^2\theta \\
&\quad +\beta_1\beta_3\beta_4(\beta_3\beta_4 q_3 q_4 R_1^2 + \cdots) R_2^2 \cos^2\theta + \beta_2 r_2 \Big[\frac{\beta_3^2\beta_4^2 r_3 r_4}{r_1}\, R_1^4 + \cdots\Big] \sin^2\theta \\
&\quad +2\beta_1\beta_2\beta_3\beta_4 q_2(\beta_1 q_1 R_3^2 R_4^2 + \cdots) \cos^2\theta + \beta_2\beta_1^2\beta_3^2\beta_4^2 R_1^2 R_2^2 R_3^2 R_4^2\,, \qquad (4.13) \\
\tilde{\sigma}_3 &= (\tilde{\sigma}_1 \text{ with } 1 \leftrightarrow 3)\,, \qquad (4.14) \\
\tilde{\sigma}_4 &= (\tilde{\sigma}_2 \text{ with } 2 \leftrightarrow 4)\,. \qquad (4.15)
\end{aligned}$$

When ellipses occur within a bracketed expression, they denote the two additional terms obtained by cycling the three index values taken from the set $\{1, 2, 3, 4\}$ that are not equal to i.

The axions and dilatons are given by

$$\chi_i = \frac{Z_i \cos\theta}{Y_i}\,, \qquad e^{2\varphi_i} = \frac{Y_i^2}{\Delta\, r_1 r_2 r_3 r_4}\,, \qquad i = 1, 2, 3\,, \qquad (4.16)$$

where

$$\begin{aligned}
Z_1 &= r_2 r_4[(\beta_1 q_3 + \beta_3 q_1) + \beta_2\beta_4(\beta_1 q_1 R_3^2 + \beta_3 q_3 R_1^2)] \\
&\quad -r_1 r_3[(\beta_2 q_4 + \beta_4 q_2) + \beta_1\beta_3(\beta_2 q_2 R_4^2 + \beta_4 q_4 R_2^2)]\,, \qquad (4.17) \\
Y_1 &= r_1 r_3(1 + 2\beta_1\beta_3 q_2 q_4 \cos^2\theta + \beta_1^2\beta_3^2 R_2^2 R_4^2) \\
&\quad +r_2 r_4(\beta_1^2 R_3^2 + \beta_3^2 R_1^2 + 2\beta_1\beta_3 q_1 q_3 \cos^2\theta)\,, \qquad (4.18) \\
(Z_2, Y_2) &= (-Z_1, Y_1) \text{ with } 1 \leftrightarrow 2\,, \qquad (4.19) \\
(Z_3, Y_3) &= (Z_1, Y_1) \text{ with } 2 \leftrightarrow 3\,. \qquad (4.20)
\end{aligned}$$

4.1 Angular momentum

The angular momentum can be calculated using the standard procedure developed by Wald. The details of this calculation, and, in particular, the evaluation of the angular momentum in terms of the quantities in the dimensionally-reduced three-dimensional theory, are given in [2]. A subtlety in the calculation concerns the different boundary conditions that arise depending upon whether a gauge field carries an electric charge or a magnetic charge. If the charges were all electric, then the conserved angular momentum corresponding to the Killing vector $\xi = \partial/\partial\tilde{\phi}$, where $\tilde{\phi} = \phi/\alpha$ is the rescaled azimuthal coordinate that has period 2π and α is defined in (4.8), would be [2]

$$J = \frac{\alpha}{16\pi} \int_{S^2} d(\chi_4 + \sigma_i\,\psi_i) \wedge d\phi = \frac{(\Delta\phi)^2}{32\pi^2} \Big[\chi_4 + \sigma_i\,\psi_i\Big]_{\theta=0}^{\theta=\pi} . \tag{4.21}$$

As discussed in [2], this expression is invariant under the $U(1)^4$ abelian gauge transformations of the four gauge potentials that preserve the condition that the Lie derivatives of the gauge potentials with respect to the azimuthal Killing vector $\partial/\partial\phi$ vanish.

In our case, however, the fields $A_{(1)1}$ and $\mathcal{A}^1_{(1)}$ carry magnetic, rather than electric, charges. A simple way to evaluate the angular momentum is to perform dualisations of these two fields. Although rather involved in the four-dimensional theory itself, the dualisations can be easily implemented in the reduced three-dimensional theory, since then they amount to exchanging the roles of the σ_i and ψ_i axions for the fields in question. As can be seen from (A.3), since the the Kaluza-Klein vector $\bar{\mathcal{B}}_{(1)}$ must be invariant under duality it follows that the required duality transformations require also sending

$$\chi_4 + \sigma_i\,\psi_i \longrightarrow \chi_4 + \sigma_i\,\psi_i - \sigma_1\,\psi_1 - \sigma_3\,\psi_3 . \tag{4.22}$$

The conserved angular momentum for the four-charge black holes is therefore given by

$$J = \frac{(\Delta\phi)^2}{32\pi^2} \Big[\chi_4 + \sigma_2\,\psi_2 + \sigma_4\,\psi_4\Big]_{\theta=0}^{\theta=\pi} . \tag{4.23}$$

Evaluating this, we find

$$J = \tfrac{1}{2}[\beta_1\,q_2q_3q_4 + \cdots] + \tfrac{1}{2}q_1q_2q_3q_4\,[q_1\beta_2\beta_3\beta_4 + \cdots] , \tag{4.24}$$

where the ellipses in each case denote the additional three symmetry-related terms.

4.2 Pairwise equal charges

A considerable simplification arises in the function Δ if we set the fields pairwise equal, so that

$$B_3 = B_1 , \qquad B_4 = B_2 , \qquad \delta_3 = \delta_1 , \qquad \delta_4 = \delta_2 . \tag{4.25}$$

We then find that

$$\Delta = \Big[1 + \sum_{i=1}^2 \beta_i^2 (r_i^2 \sin^2\theta + q_i^2 \cos^2\theta) + 4\beta_1\beta_2 q_1 q_2 \cos^2\theta + \prod_{i=1}^2 \beta_i^2 (r_i^2 \sin^2\theta + q_i^2 \cos^2\theta)\Big]^2 . \tag{4.26}$$

With the fields set pairwise equal, i.e. $q_3 = q_1$, $q_4 = q_2$ and $\beta_3 = \beta_1$ and $\beta_4 = \beta_2$. We then find

$$\begin{aligned}
\mathcal{A}^2_{(1)} &= \Big[-\frac{q_2}{r_2} + \beta_1^2 q_2\, r\, \Big(1 + \frac{(r-2m)}{r_2}\, \cos^2\theta\Big) + 2\beta_1\beta_2 q_1\, r\, \Big(1 + \frac{(r-2m)}{r_1}\, \cos^2\theta\Big)\Big]\, dt \\
&\quad + \sigma_4\, (d\phi - \omega dt)\,, \\
\mathcal{A}^1_{(1)} &= -\frac{2\beta_1 r(r-2m)}{r_2}\, \cos\theta\, dt + \sigma_3\, (d\phi - \omega dt)\,,
\end{aligned} \tag{4.27}$$

with analogous expressions for $A_{(1)1}$ and $\mathcal{A}^1_{(1)}$. The fields σ_3 and σ_4 are given by

$$\begin{aligned}
\sigma_3 &= -q_1 \cos\theta\, (1 - \beta_1^2 R_2^2)\, Y^{-1}\,, \\
\sigma_4 &= \Big[\beta_2 R_1^2 + 2\beta_1 q_1 q_2 \cos^2\theta + \beta_1^2 \beta_2 R_1^2 R_2^2\Big]\, Y^{-1}\,,
\end{aligned} \tag{4.28}$$

where

$$\begin{aligned}
R_i^2 &= r_i^2\, \sin^2\theta + q_i^2\, \cos^2\theta\,, \\
Y &= 1 + \beta_1^2 R_2^2 + \beta_2^2 R_1^2 + 4\beta_1\beta_2 q_1 q_2 \cos^2\theta + \beta_1^2\beta_2^2 R_1^2 R_2^2\,.
\end{aligned} \tag{4.29}$$

A different specialisation arises if we instead reverse the sign of the fields B_3 and B_4 before the pairwise identification, in other words, if we set

$$B_3 = -B_1\,, \qquad B_4 = -B_2\,, \qquad \delta_3 = \delta_1\,, \qquad \delta_4 = \delta_2\,. \tag{4.30}$$

Now, the function Δ becomes instead

$$\begin{aligned}
\Delta = {} & [1 + 2\beta_1 q_2 \cos\theta + \beta_1^2 (r_1^2 \sin^2\theta + q_1^2 \cos^2\theta)][1 - 2\beta_1 q_2 \cos\theta + \beta_1^2 (r_1^2 \sin^2\theta + q_1^2 \cos^2\theta)] \times \\
& [1 + 2\beta_2 q_1 \cos\theta + \beta_2^2 (r_2^2 \sin^2\theta + q_2^2 \cos^2\theta)][1 - 2\beta_2 q_1 \cos\theta + \beta_2^2 (r_2^2 \sin^2\theta + q_2^2 \cos^2\theta)]
\end{aligned} \tag{4.31}$$

Note that in this case the function ω now vanishes, and so the metric is purely static. In fact it is not hard to show that all the possible ways of making ω vanish involve making one or another of the following choices

$$\begin{array}{lllll}
(1) & q_i = q_j\,, & q_k = q_\ell\,, & B_i = -B_j\,, & B_k = -B_\ell \\
(2) & q_i = q_j\,, & q_k = -q_\ell\,, & B_i = -B_j\,, & B_k = B_\ell \\
(3) & q_i = -q_j\,, & q_k = q_\ell\,, & B_i = B_j\,, & B_k = -B_\ell \\
(4) & q_i = -q_j\,, & q_k = -q_\ell\,, & B_i = B_j\,, & B_k = B_\ell\,,
\end{array} \tag{4.32}$$

where i, j, k and ℓ are all different and are chosen from 1, 2, 3 and 4. It can easily be seen that, as one would expect, the angular momentum (4.24) vanishes in all of these cases.

4.3 Asymptotic structure and ergoregions

It was observed in [1] that the metric component g_{tt} in the magnetised electrically charged Reissner-Nordström solution becomes arbitrarily large and positive at large distances near to the z axis, thus indicating the presence of an ergoregion extending to infinity. Not surprisingly, the same is in general true in the STU model generalisations of this solution that we are considering here. Specifically, if we introduce cylindrical coordinates $\rho = r\,\sin\theta$ and $z = r\,\cos\theta$, then it is easily seen from (4.1), (4.3) and (4.5) that to leading order in large z and small ρ we shall in general have

$$g_{tt} \sim +z^2\,\rho^2\,\Big(\sum_i \beta_i\,\Xi_i\Big)^2\,, \tag{4.33}$$

and thus an ergoregion extending to infinity. The reason for this metric behaviour is that the function ω given in (4.3) has the large-z expansion

$$\omega = 2z\,\sum_{i=1}^4 q_i\,\Xi_i - 2m\sum_{i=1}^4 q_i\,\Xi_i\,(1+s_i^2) + \mathcal{O}(\frac{1}{z})\,. \tag{4.34}$$

The ergoregion is avoided if one imposes the condition $\sum_i q_i\,\Xi_i = 0$ on the charges and magnetic fields, i.e. if

$$\beta_1\beta_2\beta_3\beta_4\sum_{i=1}^4 \frac{q_i}{\beta_i} = 0\,. \tag{4.35}$$

One way to achieve this is if one (or more) of the four field strengths is set to zero; for example, by taking $q_4 = 0$ and $\beta_4 = 0$. Under these circumstances the metric is still stationary, as opposed to static, but is asymptotically non-rotating at infinity. It can be seen from (4.24) that the angular momentum also vanishes in such a case.

Clearly there are also more general ways to satisfy (4.35), where all four fields are non-vanishing. If we assume that (4.35) is satisfied then it follows from (4.34) that the asymptotic metric near the axis is rotating with an angular velocity

$$\Omega_\infty = 2m\sum_{i=1}^4 q_i\,\Xi_i\,s_i^2 = 4m^2\beta_1\beta_2\beta_3\beta_4\,\sum_{i=1}^4 \frac{\sinh^3\delta_i\,\cosh\delta_i}{\beta_i}\,. \tag{4.36}$$

It can also be seen from (4.3) that if (4.35) holds then on the black hole horizon at $r = 2m$, the angular velocity will be

$$\Omega_H = \sum_{i=1}^4 \frac{q_i\,\beta_i}{2mc_i^2} = \sum_{i=1}^4 \beta_i\,\tanh\delta_i\,. \tag{4.37}$$

Note that in general, the angular momentum (4.24) is non-vanishing if (4.35) is satisfied.

Of course if any of the conditions enumerated in (4.32) holds, then not merely is (4.35) satisfied but the metric is non-rotating everywhere, and also $J = 0$.

5 Scaling Limit, and Lift to Five Dimensions

The scaling limits of our magnetised non-extremal black holes, which will be parameterised by $\tilde{m}$, $\tilde{\Pi}_s$, $\tilde{\Pi}_c$ and $\tilde{\beta}_i$ $(i = 1, \cdots, 4)$, can be obtained by taking a specific scaling limit [10] of the magnetised electric black holes of section 3 parameterised by m, δ_i, β_i with $\delta_1 = \delta_2 = \delta_3$. After taking the limit, the solution can then be lifted to five dimensions, where it can be seen to be $\mathrm{AdS}_3 \times S^2$.

The limit can be implemented by setting $\delta_1 = \delta_2 = \delta_3$ and making the scaling [10]:

$$\begin{aligned} m &= \tilde{m}\,\epsilon\,, \qquad r = \tilde{r}\,\epsilon\,, \qquad t = \tilde{t}\,\epsilon^{-1}\,, \qquad \beta_i = \tilde{\beta}_i\,\epsilon\,, \quad i = 1,2,3,4\,, \\ \sinh^2\delta_4 &= \frac{\widetilde{\Pi}_s^2}{\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2}\,, \qquad \sinh^2\delta_i = (\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^{1/3}\,\epsilon^{-4/3}\,, \quad i = 1,2,3\,, \end{aligned} \tag{5.1}$$

where ϵ is then sent to zero.

The implementation of the scaling limit (5.1) gives

$$(d\phi - \omega dt) \longrightarrow d\phi - (\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_3)d\tilde{t} - \frac{2\tilde{m}\tilde{\beta}_4\,\widetilde{\Pi}_c\widetilde{\Pi}_s}{(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)\tilde{r} + 2\tilde{m}\widetilde{\Pi}_s^2}\,d\tilde{t}\,, \tag{5.2}$$

and

$$\Delta \longrightarrow 1 + \frac{8\tilde{m}^3\tilde{\beta}_4^2(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^2\sin^2\theta}{(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)\tilde{r} + 2\tilde{m}\widetilde{\Pi}_s^2}\,. \tag{5.3}$$

The quantities $\tilde{\beta}_1$, $\tilde{\beta}_2$ and $\tilde{\beta}_3$ drop out completely in the scaling limit if we send $\phi \longrightarrow \phi + (\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_3)\tilde{t}$. We shall assume from now on that this redefinition has been performed. Therefore, the obtained scaling limit of magnetised non-extremal black holes depend only on *four* independent parameters: $\tilde{m}$, $\tilde{\Pi}_c$, $\tilde{\Pi}_s$ and $\tilde{\beta}_4$.

In the case of vanishing magnetic fields, $\beta_i = 0$ it was possible [10] to identify the scaling limits with the *subtracted geometry* [12] of a non-extreme black hole parameterised by $\tilde{m}, \tilde{\delta}_i$. In that case we have $\tilde{\Pi}_s \equiv \Pi_{i=1}^4 \sinh\tilde{\delta}_i$ and $\tilde{\Pi}_c \equiv \Pi_{i=1}^4 \cosh\tilde{\delta}_i$, determined by (5.1). In our case we have no independent derivation of a subtracted geometry and so no unique identification of $\tilde{\delta}_i$ is possible.

The lifting of the subtracted geometry solution to five dimensions is given by

$$ds_5^2 = e^{\varphi_1}\,ds_4^2 + e^{-2\varphi_1}\,(dz + \mathcal{A}_{(1)}^2)^2\,. \tag{5.4}$$

Applying the scaling limit (5.1) here, together with $z = \tilde{z}\,\epsilon^{-1}$, we find that the five-dimensional metric ds_5^2 scales as $\epsilon^{-2/3}$, and defining $ds_5^2 = \epsilon^{-2/3}\, d\tilde{s}_5^2$ we have

$$\begin{aligned} d\tilde{s}_5^2 &= 4\tilde{m}^2(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^{2/3}\,[d\theta^2 + \sin^2\theta\,(d\phi + \tilde{\beta}_4 d\tilde{z})^2] \\ &\quad + \frac{(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)\tilde{r} + 2\tilde{m}\widetilde{\Pi}_s^2}{2\tilde{m}(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^{4/3}}\, d\tilde{z}^2 - \frac{(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)\tilde{r} - 2\tilde{m}\widetilde{\Pi}_c^2}{2\tilde{m}(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^{4/3}}\, d\tilde{t}^2 \\ &\quad - \frac{2\widetilde{\Pi}_c\widetilde{\Pi}_s}{(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^{4/3}}\, d\tilde{t}d\tilde{z} + \frac{4\tilde{m}^2(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^{2/3}}{\tilde{r}(\tilde{r} - 2\tilde{m})}\, d\tilde{r}^2\,. \end{aligned} \tag{5.5}$$

It can be seen that $\tilde{\beta}_4$ disappears from the five-dimensional metric if we make the further coordinate redefinition

$$\phi = \tilde{\phi} - \tilde{\beta}_4\,\tilde{z}\,, \tag{5.6}$$

This is a reflection of the fact that the magnetisation of the four-dimensional gauge field associated with the Kaluza-Klein vector $\mathcal{A}^2_{(1)}$ of the five-dimensional reduction can be implemented (or, in the above calculation, undone) by performing a rotation in the $(\phi, \tilde{z})$ plane [3]. This transformation is related to a spectral flow in a dual conformal field theory interpretation of AdS_3 geometries.

Finally, if we define new coordinates ρ, σ and τ by

$$\tilde{r} = 2\tilde{m}\,\cosh^2\rho\,, \qquad \tilde{z} = 2\mathrm{i}\,(2\tilde{m})^{3/2}\,(\widetilde{\Pi}_c\,\tau + \widetilde{\Pi}_s\,\sigma)\,, \qquad \tilde{t} = 2\mathrm{i}\,(2\tilde{m})^{3/2}\,(\widetilde{\Pi}_c\,\sigma + \widetilde{\Pi}_s\,\tau)\,, \tag{5.7}$$

the five-dimensional metric can be seen to become

$$d\tilde{s}_5^2 = 16\tilde{m}^2(\widetilde{\Pi}_c^2 - \widetilde{\Pi}_s^2)^{2/3}\left[(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\sigma^2) + \tfrac{1}{4}(d\theta^2 + \sin^2\theta d\tilde{\phi}^2)\right], \tag{5.8}$$

which is the metric on $\mathrm{AdS}_3 \times S^2$.

6 Magnetostatic Black Holes

6.1 Magnetised magnetically charged black holes

Here we exchange the roles of the electric and the magnetic charges in the original four-charge seed solution. That is, the charges numbered 1 and 3 are now electric, while those numbered 2 and 4 are magnetic, in the conventions of [6]. (Some of the properties of the resulting metrics were discussed previously in [19, 20, 21, 22].) We shall denote the four charge parameters by (p_1, p_2, p_3, p_4) in this case. In the case that the charges are set equal, the solution reduces to the magnetised magnetically-charge Reissner-Nordström black hole.

[3] The role of the specific Melvin transformation as a coordinate transformation in the $(\phi, \tilde{z})$ plane of the lifted geometry was first observed for dilatonic black holes in [15].

Concretely, in the original seed solution, reduced to three dimensions, we replace (A.9) and (A.10) by

$$e^{2\varphi_1} = \frac{r_2\, r_4}{r_1\, r_3}\,, \qquad e^{2\varphi_2} = \frac{r_1\, r_4}{r_2\, r_3}\,, \qquad e^{2\varphi_3} = \frac{r_3\, r_4}{r_1\, r_2}\,, \qquad e^{2\varphi_4} = r_1\, r_2\, r_3\, r_4\, \sin^4\theta\,, \tag{6.1}$$

and

$$\begin{aligned}
&\chi_1 = 0 \qquad \chi_2 = 0\,, \qquad \chi_3 = 0\,, \qquad \chi_4 = 0\,,\\
&\sigma_1 = 0\,, \qquad \sigma_2 = p_2 \cos\theta\,, \qquad \sigma_3 = 0\,, \qquad \sigma_4 = p_4 \cos\theta\,,\\
&\psi_1 = p_1 \cos\theta\,, \qquad \psi_2 = 0\,, \qquad \psi_3 = p_3 \cos\theta\,, \qquad \psi_4 = 0\,,
\end{aligned} \tag{6.2}$$

The metric of the magnetised solution will still be given by (4.1), but now we have $\omega = 0$ and the function Δ in (4.5) is given by

$$\Delta = \prod_{i=1}^{4} \Delta_i\,, \qquad \Delta_i = (1 + \beta_i p_i \cos\theta)^2 + \beta_i^2\, r_i^2\, \sin^2\theta\,. \tag{6.3}$$

Because Δ is not an even function of $\cos\theta$ in this case, the periodicity conditions on ϕ for the metric to be free of conical singularities are different at the north and south poles of the sphere. Specifically, we find that the required periodicities are

$$\begin{aligned}
\theta = 0: &\qquad \Delta\phi = 2\pi \prod_i (1 + \beta_i p_i)\,,\\
\theta = \pi: &\qquad \Delta\phi = 2\pi \prod_i (1 - \beta_i p_i)\,.
\end{aligned} \tag{6.4}$$

The metric can be rendered free of conical singularities if the charges and magnetic fields satisfy the "no-force condition"

$$\prod_i (1 + \beta_i p_i) = \prod_i (1 - \beta_i p_i)\,. \tag{6.5}$$

Using the expressions given in section A.3, we can calculate the physical electric and magnetic charges carried by the four gauge fields. In this case, the non-vanishing ones are (Q_1, P_2, Q_3, P_4). For the sake of uniformity, we shall relabel these as $(\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3, \widetilde{P}_4)$ respectively. They turn out to be given by

$$\widetilde{P}_i = \frac{p_i}{(1 - \beta_i^2\, p_i^2)}\, \frac{\Delta\phi}{2\pi}\,. \tag{6.6}$$

The electromagnetic potentials are given by

$$\begin{aligned}
\hat{A}_{(1)1} &= \Big[-\frac{p_1}{r_1} + \frac{2\beta_1 r(r-2m)\cos\theta}{r_1} - \frac{\beta_1^2\, p_1\,[r_1^2 + r(r-2m)\,\cos^2\theta]}{r_1}\Big]\, dt\,,\\
\hat{A}_{(1)2} &= \frac{p_2\cos\theta + \beta_2\, R_2^2}{\Delta_2}\, d\phi\,,\\
\hat{\mathcal{A}}^1_{(1)} &= \Big[-\frac{p_3}{r_3} + \frac{2\beta_3 r(r-2m)\cos\theta}{r_3} - \frac{\beta_3^2\, p_3\,[r_3^2 + r(r-2m)\,\cos^2\theta]}{r_3}\Big]\, dt\,,\\
\hat{\mathcal{A}}^2_{(1)} &= \frac{p_4\cos\theta + \beta_4\, R_4^2}{\Delta_4}\, d\phi\,,
\end{aligned} \tag{6.7}$$

where $R_i^2 = r_i^2\sin^2\theta + p_i^2\cos^2\theta$. The scalar fields are given by

$$\begin{aligned}
&e^{2\varphi_1} = \frac{r_2\, r_4\, \Delta_1\, \Delta_3}{r_1\, r_3\, \Delta_2\, \Delta_4}\,, \qquad e^{2\varphi_2} = \frac{r_1\, r_4\, \Delta_2\, \Delta_3}{r_2\, r_3\, \Delta_1\, \Delta_4}\,, \qquad e^{2\varphi_3} = \frac{r_3\, r_4\, \Delta_1\, \Delta_2}{r_1\, r_2\, \Delta_3\, \Delta_4}\,,\\
&\chi_1 = 0\,, \qquad \chi_2 = 0\,, \qquad \chi_3 = 0\,.
\end{aligned} \tag{6.8}$$

6.2 $SL(2,R)^4$ truncations of the sigma model

The three-dimensional scalar sigma model associated with the timelike or spacelike reduction of the four-dimensional STU model has an $O(4,4)$ global symmetry. The Lagrangian in the case of the timelike reduction can be found in section 2.1 of [6]. The sixteen scalars comprise the original three dilatons $(\varphi_1, \varphi_2, \varphi_3)$ and three axions (χ_1, χ_2, χ_3) of the STU model; the Kaluza-Klein scalar φ_4 and the axion χ_4 dual to the Kaluza-Klein vector; the four axions σ_i coming from the direct dimensional reductions of the four gauge potentials; and finally the four axions ψ_i coming from the dualisations of the four gauge potentials in the dimensionally-reduced theory.

If we restrict attention to purely static configurations then χ_4 will vanish. If we furthermore restrict to configurations where the axions (χ_1, χ_2, χ_3) of the STU model vanish, then it can be seen from the sigma-model Lagrangian in eqn (7) of [6] that there are two possible disjoint truncations of the remaining scalar fields for which the vanishing of the four χ_i axions is consistent with their equations of motion.[4] Specifically, we can have either

$$\sigma_1 = \psi_2 = \sigma_3 = \psi_4 = 0 \tag{6.9}$$

[4]In [6] a *timelike* reduction to three dimensions was performed. Here, we are instead reducing on the spacelike azimuthal Killing vector $\partial/\partial\phi$ rather than the timelike Killing vector $\partial/\partial t$. The formulae in [6] can be repurposed to the spacelike reduction with very straightforward modifications. In particular, the three-dimensional sigma-model Lagrangian in eqn (7) of [6] will take the same form in the case of the spacelike reduction, except that the kinetic terms for all the scalar fields will now have the standard negative sign appropriate to a Minkowski-signature theory.

or

$$\psi_1 = \sigma_2 = \psi_3 = \sigma_4 = 0\,. \tag{6.10}$$

In the truncation (6.9), if we define

$$\begin{aligned}
&u_1 = \tfrac12(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4)\,, \qquad u_2 = \tfrac12(-\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4)\,,\\
&u_3 = \tfrac12(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)\,, \quad u_4 = \tfrac12(-\varphi_1 - \varphi_2 - \varphi_3 - \varphi_4)\,,\\
&\alpha_1 = \psi_1 \qquad \alpha_2 = \sigma_2\,, \qquad \alpha_3 = \psi_3\,, \qquad \alpha_4 = \sigma_4\,,
\end{aligned} \tag{6.11}$$

then the three-dimensional sigma-model Lagrangian in equation (7) of [6], after the appropriate sign-changes because we are making a spacelike reduction, becomes

$$\mathcal{L}_{\rm scal} = \sum_{i=1}^4 \Big(-\tfrac12 (\partial u_i)^2 - \tfrac12 e^{2u_i}\, (\partial\alpha_i)^2\Big)\,. \tag{6.12}$$

This can be recognised as describing the coset $[SL(2,R)/O(2)]^4$. Similarly, if we consider instead the truncations (6.10), then defining instead

$$\begin{aligned}
&u_1 = \tfrac12(-\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)\,, \quad u_2 = \tfrac12(\varphi_1 - \varphi_2 - \varphi_3 - \varphi_4)\,,\\
&u_3 = \tfrac12(-\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4)\,, \quad u_4 = \tfrac12(\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4)\,,\\
&\alpha_1 = \sigma_1 \qquad \alpha_2 = \psi_2\,, \qquad \alpha_3 = \sigma_3\,, \qquad \alpha_4 = \psi_4
\end{aligned} \tag{6.13}$$

gives again an $[SL(2,R)/O(2)]^4$ sigma model with Lagrangian (6.11).

(Note that if we considered a timelike reduction on the coordinate t rather than a spacelike reduction on the coordinate ϕ, we would end up with a Lagrangian like (6.12) except with a minus sign in front of the exponential terms. The coset in this case would be $[SL(2,R)/O(1,1)]^4$.)

The truncation described by (6.11) corresponds to the case where the gauge fields numbered 1 and 3 are purely electric, while those numbered 2 and 4 are purely magnetic. Since in this paper we always consider Melvin backgrounds where fields 1 and 3 carry external electric fields, while 2 and 4 carry external magnetic fields, this means that we can remain within the truncation if we additionally allow fields 1 and 3 to carry electric charges, and fields 2 and 4 to carry magnetic charges. This is precisely the situation we considered in section 4, namely the STU model generalisations of the magnetically-charged Reissner-Nordström black hole in an external magnetic field. It can indeed be seen from equations (6.7) and (6.8), together with the staticity of the metric, that the solutions fall within the class described by the truncation (6.9) and (6.11).

By contrast, although the *charges* carried by the gauge fields in the solutions in section 3 are compatible with the truncation described by (6.10) and (6.13), the external fields are still appropriate for the other truncation, (6.9) and (6.11), and so the solutions in section 3 are not described by either of the truncated theories. And indeed, the axions χ_i are non-zero and the metric is not static.

6.3 Multi-centre BPS black holes in external magnetic fields

Returning to the truncation (6.9) and (6.11), we can in fact use it to describe more general situations than the "magnetised magnetically charged" black holes obtained in section 4. In particular, we can consider the case of multi-centre BPS black holes that are then immersed in external fields, provided that we align them all along a line so that we can apply the "Melvinising" transformation. For these purposes, it is useful first to present the general expressions for the transformations of the scalar fields under the "Melvinising" transformations. If we start with a seed solution for which the fields are denoted by bars, then after the transformation we will have

$$e^{u_i} = e^{\bar{u}_i}\,[(1+\beta_i\,\bar{\alpha}_i)^2 + \beta_i^2\,e^{-2\bar{u}_i}]\,,\qquad \alpha_i = \frac{\bar{\alpha}_i\,(1+\beta_i\,\bar{\alpha}_i)+\beta_i\,e^{-2\bar{u}_i}}{(1+\beta_i\,\bar{\alpha}_i)^2+\beta_i^2\,e^{-2\bar{u}_i}}\,. \tag{6.14}$$

In particular this means that the transformed function φ_4 that appears in the metric ansatz (A.1) is given by

$$e^{-2\varphi_4} = e^{-2\bar{\varphi}_4}\prod_{i=1}^4 [(1+\beta_i\,\bar{\alpha}_i)^2 + \beta_i^2\,e^{-2\bar{u}_i}]\,. \tag{6.15}$$

The multi-centre black holes in the STU model have metrics given by

$$ds^2 = -\Big(\prod_{i=1}^4 H_i\Big)^{-1/2} dt^2 + \Big(\prod_{i=1}^4 H_i\Big)^{1/2} d\vec{y}^2\,, \tag{6.16}$$

where the functions H_i are harmonic in the 3-dimensional Euclidean space with metric $d\vec{y}^2$. For black holes aligned along an axis we can conveniently use cylindrical coordinates in which

$$d\vec{y}^2 = d\rho^2 + \rho^2\,d\phi^2 + dz^2\,. \tag{6.17}$$

We shall take the harmonic functions to be given by

$$H_i = 1 + \sum_a \frac{p_i^{(a)}}{\sqrt{\rho^2+(z-z_a)^2}}\,, \tag{6.18}$$

where the charges $p_i^{(a)}$ are constants and the black holes are located at the point z_a on the z axis. The metric is free of conical singularities on the z axis provided that ϕ has period 2π.

A field strength carrying an electric charge is described by a potential of the form

$$A^i_{\rm elec} = -H_i^{-1}\, dt\,, \tag{6.19}$$

while a field strength carrying a magnetic charge is described by a potential of the form

$$A^i_{\rm mag} = \sum_a \frac{p_i^{(a)}\,(z-z_a)}{\sqrt{\rho^2+(z-z_a)^2}}\, d\phi\,. \tag{6.20}$$

In our case, therefore, the potentials for fields 1 and 3 are of the form (6.19), while those for fields 2 and 4 are of the form (6.20). In the dimensionally-reduced three-dimensional language this implies that the axionic scalars α_i defined in (6.11) are all given in this seed solution by

$$\bar\alpha_i = \sum_a \frac{p_i^{(a)}\,(z-z_a)}{\sqrt{\rho^2+(z-z_a)^2}}\,. \tag{6.21}$$

The dilatonic scalar fields $\vec\varphi = (\varphi_1, \varphi_2, \varphi_3)$ in this multi-centre seed solution are given by

$$\vec\varphi = \tfrac12 \sum_i \epsilon_i\, \vec c_i\, \log H_i\,, \tag{6.22}$$

where

$$\mathcal{L} = \sqrt{-g}(R - \tfrac12 (\partial\vec\varphi)^2 - \tfrac14 \sum_i e^{\vec c_i\cdot\vec\varphi}\,(F^i)^2) \tag{6.23}$$

and ϵ_i is $+1$ if field i carries an electric charge and -1 if it carries a magnetic charge (see, for example, section 2.2 of [23]). Comparing the multi-centre metric given by (6.16) and (6.17) with the reduction ansatz (A.1), we see that in the multi-centre seed solution we shall have

$$e^{\bar\varphi_4} = \rho^2\, \Big(\prod_i H_i\Big)^{1/2}\,, \tag{6.24}$$

and hence, from (6.11),

$$e^{-\bar u_i} = \rho\, H_i\,. \tag{6.25}$$

Applying the Melvinising transformations (6.14), we obtain the "magnetised magnetic" multi-centre black holes with metrics

$$ds^2 = e^{-\varphi_4}\,\rho^2\,\Big[-dt^2 + \Big(\prod_i H_i\Big)\,(d\rho^2+dz^2)\Big] + e^{\varphi_4}\, d\phi^2\,, \tag{6.26}$$

where φ_4 is given by (6.15). Thus the metric is given by

$$ds^2 = Z^{1/2}\,\Big[-\Big(\prod_{i=1}^4 H_i\Big)^{-1/2} dt^2 + \Big(\prod_{i=1}^4 H_i\Big)^{1/2}\,(d\rho^2+dz^2+Z^{-1}\,\rho^2\,d\phi^2)\Big]\,, \tag{6.27}$$

where

$$Z = \prod_{i=1}^{4} [(1+\beta_i\,\bar{\alpha}_i)^2 + \beta_i^2\, e^{-2\bar{u}_i}]\,. \tag{6.28}$$

There will in general now be conical singularities along the z axis. This can be seen by looking at the form of the metric in the $(\rho,\phi))$ plane as ρ tends to zero. From (6.21) and (6.25) we see that as ρ tends to zero we shall have

$$Z \rightarrow \prod_{i=1}^{4} (1+\beta_i\,\bar{\alpha}_i)^2\,, \qquad \bar{\alpha}_i \rightarrow \sum_a p_i^{(a)}\,\mathrm{sign}(z-z_a)\,. \tag{6.29}$$

In the case of a single-centre black hole, the periodicity conditions for ϕ in order to avoid a conical singularity can be seen to reduce to those in equation (6.5).

7 Conclusions

In string theory charged black holes may be regarded as having a composite structure arising from their microscopic description in terms of intersecting D-branes/M-branes. This composite structure is reflected in the interactions of the black holes. In this paper we have demonstrated this by using as external probes the various types of magnetic fields capable of exciting each of these constituents. We have found that the behaviour of black holes is indeed rather sensitive to which type of magnetic field is applied. By far the simplest case is that of Kaluza-Klein black holes, which are made up of a single constituent. Somewhat counterintuitively it turns out that the Maxwell-Einstein case is the most complex, which may be ascribed to the fact that all the constituents and probes are turned on.

Utilising the composite structure of charges and magnetic fields allows for a balance of different forces and torques and the taming of the extent of ergoregions. This work samples only a restricted subset of static four-charge generating black hole solutions. We anticipate that further studies of rotating five-charge generating solutions will reveal an even richer structure.

Acknowledgements

We are grateful to Yi Pang for discussions. M.C. is supported in part by DOE grant de-sc0007901, the Fay R. and Eugene L. Langberg Endowed Chair and the Slovenian Research Agency (ARRS). C.N.P. is supported in part by DOE grant DE-FG02-95ER40917.

A The STU Model

A.1 Reduction of the STU model to $D = 3$

We can "magnetise" the black hole solutions by performing a spacelike reduction to three dimensions on the azimuthal Killing vector $\partial/\partial\phi$, and then acting with the appropriate $O(4,4)$ transformations. This is analogous to the discussion in [6], except that there the reduction was performed on the timelike Killing vector $\partial/\partial t$[5]. Thus we make a standard Kaluza-Klein reduction with

$$ds_4^2 = e^{-\varphi_4}\, d\bar{s}_3^2 + e^{\varphi_4}\,(d\phi + \bar{\mathcal{B}}_{(1)})^2\,, \tag{A.1}$$

and

$$\begin{aligned} A_{(1)1} &= \bar{A}_{(1)1} + \sigma_1\,(d\phi + \bar{\mathcal{B}}_{(1)})\,, \qquad A_{(1)2} = \bar{A}_{(1)2} + \sigma_2\,(d\phi + \bar{\mathcal{B}}_{(1)})\,, \\ \mathcal{A}^1_{(1)} &= \bar{\mathcal{A}}^1_{(1)} + \sigma_3\,(d\phi + \bar{\mathcal{B}}_{(1)})\,, \qquad \mathcal{A}^2_{(1)} = \bar{\mathcal{A}}^2_{(1)} + \sigma_4\,(d\phi + \bar{\mathcal{B}}_{(1)})\,. \end{aligned} \tag{A.2}$$

where, when necessary, we place bars on three-dimensional quantities in order to distinguish them from four-dimensional ones. Note that throughout, we use the ordering $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)})$ for the potentials, with σ_i being the axionic scalar coming from the direct Kaluza-Klein reduction of the i'th potential, and so on.

In three dimensions we then dualise 1-form potentials to scalars, in a fashion that is precisely analogous to the one described for the timelike reduction in [6]. The upshot is that the Kaluza-Klein 1-form $\bar{\mathcal{B}}_{(1)}$, whose field strength is $\bar{\mathcal{G}}_{(2)} = d\bar{\mathcal{B}}_1$, is replaced by the axion χ_4 with

$$e^{2\varphi_4}\, \bar{*}\bar{\mathcal{G}}_{(2)} = d\chi_4 + \sigma_1\, d\psi_1 + \sigma_2\, d\psi_2 + \sigma_3\, d\psi_3 + \sigma_4\, d\psi_4\,, \tag{A.3}$$

and the 1-form potentials in three dimensions coming from the reduction of the four 1-form potentials in four dimensions are dualised to axions ψ_i where

$$\begin{aligned} -\, e^{-\varphi_1+\varphi_2-\varphi_3+\varphi_4}\, \bar{*}\bar{F}_{(2)1} &= d\psi_1 + \chi_3\, d\psi_2 - \chi_1\, d\sigma_3 - \chi_1\,\chi_3\, d\sigma_4\,, \\ -e^{-\varphi_1+\varphi_2+\varphi_3+\varphi_4}\, \bar{*}\bar{F}_{(2)2} &= d\psi_2 - \chi_1\, d\sigma_4\,, \\ -e^{-\varphi_1-\varphi_2+\varphi_3+\varphi_4}\, \bar{*}\bar{\mathcal{F}}^1_{(2)} &= d\psi_3 - \chi_2\, d\psi_2 - \chi_1\, d\sigma_1 + \chi_1\,\chi_2\, d\sigma_4\,, \\ -e^{-\varphi_1-\varphi_2-\varphi_3+\varphi_4}\, \bar{*}\bar{\mathcal{F}}^2_{(2)} &= d\psi_4 + \chi_2\, d\psi_1 - \chi_3\, d\psi_3 - \chi_1\, d\sigma_2 + \chi_2\,\chi_3\, d\psi_2 \\ &\quad -\chi_1\,\chi_2\, d\sigma_3 + \chi_1\,\chi_3\, d\sigma_1 - \chi_1\,\chi_2\,\chi_3\, d\sigma_4\,. \end{aligned} \tag{A.4}$$

[5]One can also employ a seed solution with analytically continued coordinates: $t \to i\phi$ and $\phi \to it$, perform the reduction on the the timelike Killing vector of the analytically continued solution, act on it with the appropriate generators of $O(4,4)$ transformations defined in [6], and finally, analytically continue the obtained solution back to original coordinates (t, ϕ).

The three-dimensional Lagrangian in terms of the dualised fields is a non-linear sigma model coupled to gravity, and can be written as

$$\bar{\mathcal{L}}_3 = \sqrt{-\bar{g}}\,[\bar{R} - \tfrac{1}{2}{\rm tr}(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})]\,, \tag{A.5}$$

where $\mathcal{M} = \mathcal{V}^T\mathcal{V}$ and

$$\mathcal{V} = e^{\frac{1}{2}\varphi_i H_i}\,\mathcal{U}_\chi\,\mathcal{U}_\sigma\,\mathcal{U}_\psi\,. \tag{A.6}$$

Here

$$\begin{aligned}
\mathcal{U}_\chi &= e^{\chi_1\,E_{\chi_1}}\,e^{\chi_2\,E_{\chi_2}}\,e^{\chi_3\,E_{\chi_3}}\,e^{\chi_4\,E_{\chi_4}}\,,\\
\mathcal{U}_\sigma &= e^{\sigma_1\,E_{\sigma_1}}\,e^{\sigma_2\,E_{\sigma_2}}\,e^{\sigma_3\,E_{\sigma_3}}\,e^{\sigma_4\,E_{\sigma_4}}\,,\\
\mathcal{U}_\psi &= e^{\psi_1\,E_{\psi_1}}\,e^{\psi_2\,E_{\psi_2}}\,e^{\psi_3\,E_{\psi_3}}\,e^{\psi_4\,E_{\psi_4}}\,.
\end{aligned} \tag{A.7}$$

H_i are the Cartan generators of $O(4,4)$, whilst E_{χ_i}, E_{σ_i} and E_{ψ_i} are the positive-root generators. (See [6] for a detailed description of the notation we are using here.)

A.2 Magnetisation of the four-charge static black hole

The usual four-charge black hole carries electric charges (Q) and magnetic charges (P) in the order (P_1, Q_2, P_3, Q_4), where we use our standard ordering $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)})$ for the gauge fields. The static four-charge solution corresponds, in three dimensions, to

$$\begin{aligned}
d\bar{s}_3^2 &= [-r(r-2m)dt^2 + \frac{r_1r_2r_3r_4}{r(r-2m)}\,dr^2 + r_1r_2r_3r_4d\theta^2\,]\,\sin^2\theta\,,\\
r_i &= r + 2ms_i^2\,,
\end{aligned} \tag{A.8}$$

with

$$e^{2\varphi_1} = \frac{r_1\,r_3}{r_2\,r_4}\,,\qquad e^{2\varphi_2} = \frac{r_2\,r_3}{r_1\,r_4}\,,\qquad e^{2\varphi_3} = \frac{r_1\,r_2}{r_3\,r_4}\,,\qquad e^{2\varphi_4} = r_1\,r_2\,r_3\,r_4\,\sin^4\theta\,, \tag{A.9}$$

and

$$\begin{aligned}
\chi_1 &= 0 \qquad \chi_2 = 0\,, \qquad \chi_3 = 0\,, \qquad \chi_4 = 0\,,\\
\sigma_1 &= -q_1\cos\theta\,, \qquad \sigma_2 = 0\,, \qquad \sigma_3 = -q_3\cos\theta\,, \qquad \sigma_4 = 0\,,\\
\psi_1 &= 0\,, \qquad \psi_2 = q_2\cos\theta\,, \qquad \psi_3 = 0\,, \qquad \psi_4 = q_4\cos\theta\,,
\end{aligned} \tag{A.10}$$

The magnetisation of the four-charge solution can be implemented by transforming the coset representative $\mathcal{M}$ defined above according to

$$\mathcal{M} \longrightarrow S\mathcal{M}S^T\,, \tag{A.11}$$

where S is the $O(4,4)$ matrix

$$S = \exp(\tfrac{1}{2}B_1\, E_{\psi_1} + \tfrac{1}{2}B_2\, E_{\sigma_2} + \tfrac{1}{2}B_3\, E_{\psi_3} + \tfrac{1}{2}B_4\, E_{\sigma_4})\,, \tag{A.12}$$

with (constant) parameters B_i being the asymptotic values of the magnetic fields of the four field strengths. One then retraces the steps of dualisation and lifts the transformed solution back to four dimensions to obtain the magnetised black hole.[6] The results are presented in section 3.

A.3 Magnetic and electric charges

The physical charges can be calculated very easily using the dimensionally-reduced quantities in three dimensions. Using the standard ordering of the $U(1)$ gauge fields, namely $\{A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)}\}$, the magnetic charges are given by

$$\begin{aligned}
P_1 &= \frac{1}{4\pi}\int_{S^2} dA_{(1)1} = \frac{1}{4\pi}\int_{S^2} d\sigma_1\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\sigma_1\Big]_{\theta=0}^{\theta=\pi}\,,\\
P_2 &= \frac{1}{4\pi}\int_{S^2} dA_{(1)2} = \frac{1}{4\pi}\int_{S^2} d\sigma_2\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\sigma_2\Big]_{\theta=0}^{\theta=\pi}\,,\\
P_3 &= \frac{1}{4\pi}\int_{S^2} d\mathcal{A}^1_{(1)} = \frac{1}{4\pi}\int_{S^2} d\sigma_3\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\sigma_3\Big]_{\theta=0}^{\theta=\pi}\,,\\
P_4 &= \frac{1}{4\pi}\int_{S^2} d\mathcal{A}^2_{(1)} = \frac{1}{4\pi}\int_{S^2} d\sigma_4\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\sigma_4\Big]_{\theta=0}^{\theta=\pi}\,,
\end{aligned} \tag{A.13}$$

where $\Delta\phi$ is the period of the azimuthal coordinate ϕ.

The electric charges are given by integrating the equations of motion of the four fields

[6]We remind the reader that, as discussed in the introduction, when we speak, for the sake of brevity, of the "magnetised electrically-charged black hole" in the STU model we mean the one for which the field strengths numbered 1 and 3 carry magnetic charges and external electric fields, while those numbered 2 and 4 carry electric charges and external magnetic fields.

$\{A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)}\}$. These give

$$
\begin{aligned}
Q_1 &= \frac{1}{4\pi}\int_{S^2} e^{-\varphi_1+\varphi_2-\varphi_3} *F_{(2)1} + \cdots = \frac{1}{4\pi}\int_{S^2} d\psi_1\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\psi_1\Big]_{\theta=0}^{\theta=\pi}\,,\\
Q_2 &= \frac{1}{4\pi}\int_{S^2} e^{-\varphi_1+\varphi_2+\varphi_3} *F_{(2)2} + \cdots = \frac{1}{4\pi}\int_{S^2} d\psi_2\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\psi_2\Big]_{\theta=0}^{\theta=\pi}\,,\\
Q_3 &= \frac{1}{4\pi}\int_{S^2} e^{-\varphi_1-\varphi_2+\varphi_3} *\mathcal{F}^1_{(2)} + \cdots = \frac{1}{4\pi}\int_{S^2} d\psi_3\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\psi_3\Big]_{\theta=0}^{\theta=\pi}\,,\\
Q_4 &= \frac{1}{4\pi}\int_{S^2} e^{-\varphi_1-\varphi_2-\varphi_3} *\mathcal{F}^2_{(2)} + \cdots = \frac{1}{4\pi}\int_{S^2} d\psi_4\wedge d\phi = \frac{\Delta\phi}{4\pi}\Big[\psi_4\Big]_{\theta=0}^{\theta=\pi}\,.
\end{aligned}
\tag{A.14}
$$

(The ellipses here denote the additional terms in the equations of motion. In each case, the full set of terms conspire to give just the simple expressions presented here in terms of the fields ψ_i.)

B STU Model in Other Duality Complexions

As we discussed before, the in the formulation [6] that we are using in this paper for the STU model, the usual four-charge black hole carries electric charges (Q) and magnetic charges (P) in the order (P_1, Q_2, P_3, Q_4), where we use our standard ordering $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_1)$ for the gauge fields. To convert into the parameterisation used, for example, in [13], we need to dualise the potential $A_{(1)2}$ to $B_{(1)}$, whose field strength is the dual of $F_{(2)2}$. To do this, we start from the Lagrangian (2.1) and then add a Lagrange multiplier

$$
\mathcal{L}_{LM} = 4dB_{(1)}\wedge(F_{(2)2} - \chi_2\, d\mathcal{A}^1_{(1)} + \chi_3\, dA_{(1)1} - \chi_2\chi_3\, d\mathcal{A}^2_{(1)})\,, \tag{B.1}
$$

treating $F_{(2)2}$ now as an independent field that we solve for algebraically and substitute back into the total Lagrangian. This leads to the dualised Lagrangian

$$
\begin{aligned}
\widetilde{\mathcal{L}}_4 \;=\; & R*\mathbb{1} - \tfrac12 *d\varphi_i\wedge d\varphi_i - \tfrac12 e^{2\varphi_i} *d\chi_i\wedge d\chi_i - 2e^{\varphi_1-\varphi_2-\varphi_3} *G_{(2)}\wedge G_{(2)}\\
& -2e^{-\varphi_1}\Big(e^{\varphi_2-\varphi_3} *F_{(2)1}\wedge F_{(2)1} + e^{-\varphi_2+\varphi_3} *\mathcal{F}^1_{(2)}\wedge\mathcal{F}^1_{(2)} + e^{-\varphi_2-\varphi_3} *\mathcal{F}^2_{(2)}\wedge\mathcal{F}^2_{(2)}\Big)\\
& -4\chi_1\, F_{(2)1}\wedge\mathcal{F}^1_{(2)} + 4dB_{(1)}\wedge(\chi_3\, dA_{(1)1} - \chi_2\, d\mathcal{A}^1_{(1)} - \chi_2\chi_3\, d\mathcal{A}^2_{(1)})\,,
\end{aligned}
\tag{B.2}
$$

where $G_{(2)} = e^{-\varphi_1+\varphi_2+\varphi_3} *F_{(2)2}$, which is written in terms of the potential $B_{(1)}$ as

$$
G_{(2)} = dB_{(1)} - \chi_1\, d\mathcal{A}^2_{(1)}\,. \tag{B.3}
$$

If we now define

$$
\begin{aligned}
\widetilde\chi^1 &= -\chi_1\,, \qquad \widetilde\chi^2 = -\chi_2\,, \qquad \widetilde\chi^3 = \chi_3\,,\\
h^I &= f^{-1}\, e^{-\varphi_i}\,, \qquad f^3 = e^{-\varphi_1-\varphi_2-\varphi_3}\,, \qquad G_{IJ} = \mathrm{diag}\{(h^1)^{-2},(h^2)^{-2},(h^3)^{-2}\}\,,\\
A^{[0]}_{(1)} &= \mathcal{A}^2_{(1)}\,, \qquad A^{[1]}_{(1)} = B_{(1)}\,, \qquad A^{[2]}_{(1)} = A_{(1)1}\,, \qquad A^{[3]}_{(1)} = \mathcal{A}^1_{(1)}\,,
\end{aligned}
\tag{B.4}
$$

then (B.2) can be written precisely in the form of equation (A.20) of [13] (where the axions $\widetilde\chi_i$ are those in [13]):

$$
\begin{aligned}
\widetilde{\mathcal{L}} = {}& R{*\mathbb{1}} - \tfrac12 G_{IJ}\,{*dh^I}\wedge dh^J - \tfrac32 f^{-2}\,{*df}\wedge df - \tfrac12 f^3\,{*F^{[0]}_{(2)}}\wedge F^{[0]}_{(2)}\\
&-\tfrac12 f^{-2}\,G_{IJ}\,{*d\widetilde\chi^I}\wedge d\widetilde\chi^J - \tfrac12 f\,G_{IJ}\,({*F^{[I]}_{(2)}} + \widetilde\chi^I\,{*F^{[0]}_{(2)}})\wedge(F^{[J]}_{(2)} + \widetilde\chi^J\,F^{[0]}_{(2)})\\
&+\tfrac12 C_{IJK}\Big[\widetilde\chi^I\,F^{[J]}_{(2)}\wedge F^{[K]}_{(2)} + \widetilde\chi^I\,\widetilde\chi^J\,F^{[0]}_{(2)}\wedge F^{[K]}_{(2)} + \tfrac13\widetilde\chi^I\,\widetilde\chi^J\,\widetilde\chi^K\,F^{[0]}_{(2)}\wedge F^{[0]}_{(2)}\Big]\,,
\end{aligned}
\tag{B.5}
$$

where $F^{[\Lambda]}_{(2)} = dA^{[\Lambda]}_{(1)}$ and $C_{IJK} = |\epsilon_{IJK}|$. The charges carried by the four-charge black hole in [6] will now be of the form (Q,P,P,P), where the fields are ordered ($A^{[0]}_{(1)}$, $A^{[1]}_{(1)}$, $A^{[2]}_{(1)}$, $A^{[3]}_{(1)}$).

Note that we can in principle perform a further transformation on the Lagrangian (B.2), and dualise the gauge potential $\mathcal{A}^2_{(1)}$ also. This would result in a formulation where the standard four-charge black hole in [6] would be supported by four gauge fields that all carried magnetic charge. This dualisation can be achieved by adding a Lagrange multiplier $4d\widetilde B_{(1)}\wedge\mathcal{F}^2_{(2)}$ to (B.2), and then solving algebraically for $\mathcal{F}^2_{(2)}$ and substituting back into the total Lagrangian. The equation for $\mathcal{F}^2_{(2)}$ is quite complicated, taking the form

$$
\alpha\,{*\mathcal{F}^2_{(2)}} = H_{(2)} + \beta\,\mathcal{F}^2_{(2)}\,,
\tag{B.6}
$$

where

$$
\begin{aligned}
\alpha &= e^{-\varphi_1-\varphi_2-\varphi_3} + \chi_1^2\,e^{\varphi_1-\varphi_2-\varphi_3} + \chi_2^2\,e^{-\varphi_1+\varphi_2+\varphi_3} + \chi_3^2\,e^{-\varphi_1-\varphi_2+\varphi_3}\,, \qquad \beta = 2\chi_1\,\chi_2\,\chi_3\,,\\
H_{(2)} &= d\widetilde B_{(1)} - \chi_2\,\chi_3\,dB_{(1)} - \chi_1\,\chi_3\,dA_{(1)1} + \chi_1\,\chi_2\,d\mathcal{A}^1_{(1)} + \chi_1\,e^{\varphi_1-\varphi_2-\varphi_3}\,{*dB_{(1)}}\\
&\quad +\chi_2\,e^{-\varphi_1+\varphi_2+\varphi_3}\,{*dA_{(1)1}} - \chi_3\,e^{-\varphi_1-\varphi_2+\varphi_3}\,{*d\mathcal{A}^1_{(1)}}\,.
\end{aligned}
\tag{B.7}
$$

Equation (B.6) can be solved for $\mathcal{F}^2_{(2)}$, giving

$$
\mathcal{F}^2_{(2)} = -\,\frac{\alpha\,{*H_{(2)}} + \beta\,H_{(2)}}{\alpha^2+\beta^2}\,,
\tag{B.8}
$$

but the result seems to be rather too complicated to be useful.

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