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Master symmetry in the $AdS_5 \times S^5$ pure spinor string

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Abstract: We lift the set of classical non-local symmetries recently studied by Klose, Loebbert, and Münkler in the context of $\mathbb{Z}_2$ cosets to the pure spinor description of the superstring in the $AdS_5 \times S^5$ background.

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1 Introduction

Finding and studying integrable structures in the context of the AdS/CFT correspondence has been one of the most active areas of research in high energy physics. The theories on both sides of the conjecture enjoy a large number of symmetries that make it possible to obtain impressive results and checks of the conjecture. Although it lacks more recent updates, a good review with an extensive list of references is [1]. A more recent development not covered in [1] is the the Quantum Spectral Curve method [2, 3]. For some of its applications, including higher loop computations, see [4–9].

In the famous work of Bena, Polchinski, and Roiban [10], it was shown that the Green-Schwarz superstring in $AdS_5 \times S^5$ [11] has an infinite set of classical conserved currents. The existence of an analogous set of currents in the context of the $AdS_5 \times S^5$ pure spinor superstring was demonstrated in reference [12]. Since this string is a generalization of the usual $\mathbb{Z}_2$ coset to a super-coset with $\mathbb{Z}_4$ symmetry, the ability to lift this symmetry to the super-coset is non-trivial. In this note, we go one step further and show that the pure spinor string in $AdS_5 \times S^5$ admits an extension of the master symmetry $\hat{\delta}$ described by Klose, Loebbert, and Münkler [13]. This symmetry complements the Yangian symmetry, acting as a raising operator on the classical Yangian charges. The master symmetry is not essentially new, however it provides a unifying picture containing all local and non-local symmetries of a coset model. In particular, it is interesting that the conserved charge associated with $\hat{\delta}$ is the Casimir of the global symmetry algebra. Furthermore, this description is expected to be of practical use, for example, in applications to the study of supersymmetry Wilson loops in the $AdS_5 \times S^5$ super-coset.

The work presented here extends this structure to its super-analogue, specifically, to the $\mathbb{Z}_4$ super-coset description of the $AdS_5 \times S^5$ pure spinor string. In a sense the ghosts present in the pure spinor string make the $\mathbb{Z}_4$ symmetry manifest with the ghosts' Lorentz current playing the role of a gauge covariant current with vanishing $\mathbb{Z}_4$ charge.

The classical and quantum integrability of the string in this background has been explored much more for the GS string (see e.g. [14]) than for the pure spinor version. Some interesting results concerning the classical and quantum integrability in the pure spinor formalism are given in references [15–18]. A possible application of integrability techniques to the quantum pure spinor string is to study its worldsheet dilatation operator [19]. It has been shown that semi-classical computations in the pure spinor string give the same results as the GS string for a set of classical solutions [20, 21], but very little is known about solutions dual to Wilson loops. This is an interesting line of research to which the results presented in this work may have suitable applications.

This paper is organized as follows: in section 2, we give a short review of the pure spinor string in $AdS_5 \times S^5$ including its flat current using a notation that will be useful in the subsequent sections. In section 3, we extend the master symmetry discussed in [13] to the pure spinor string. In section 4, we derive how the existence of the first Yangian charge is a consequence of the master symmetry and the global $\mathfrak{psu}(2,2|4)$ symmetry. We then give a general derivation of all higher non-local and non-abelian charges the superstring has. We conclude the paper and discuss directions for future research in section 5.

2 Pure spinor string in $AdS_5 \times S^5$

The pure spinor string in the $AdS_5 \times S^5$ background is described in terms of the supercoset $\mathrm{PSU}(2,2|4)/\mathrm{SO}(1,4) \times \mathrm{SO}(5)$. The Lie algebra $\mathfrak{g} = \mathfrak{psu}(2,2|4)$ is decomposed as $\mathfrak{g} = \bigoplus_{i=0}^{3} \mathfrak{g}_i$ with the projections satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j \,\mathrm{mod}\, 4}. \tag{2.1}$$

The Killing form $\mathrm{Str}(\cdot)$ also respects this symmetry in the sense that

$$\mathrm{Str}(\mathfrak{g}_i \mathfrak{g}_j) \neq 0 \quad \text{iff} \quad i + j = 0 \text{ mod } 4. \tag{2.2}$$

This is a $\mathbb{Z}_4$ generalization of the usual $\mathbb{Z}_2$ symmetry present in any symmetric space and, in particular, in the bosonic coset construction. For comparison and general convenience, we will define

$$\mathfrak{h} := \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{m} := \bigoplus_{i=1}^{3} \mathfrak{g}_i. \tag{2.3}$$

Note that $\mathfrak{h} = \mathfrak{so}(1,4) \oplus \mathfrak{so}(5)$. An element g of the coset defines the left-invariant currents

$$J = g^{-1} dg = K + A, \tag{2.4}$$

where $d = dz\partial + d\bar{z}\bar{\partial}$,[1] $K \in \mathfrak{m}$, and $A \in \mathfrak{h}$. We will also decompose $K = K_1 + K_2 + K_3$ with $K_i \in \mathfrak{g}_i$ when convenient. The gauge field A is used in worldsheet covariant derivatives $\nabla = d + [A, \cdot\,]$. The Maurer-Cartan identity

$$dJ + J \wedge J = 0 \tag{2.5}$$

will also decompose into four independent identities along each $\mathfrak{g}_i$.

---

[1] We will also use the notation $J = g^{-1}\partial g$ and $\bar{J} = g^{-1}\bar{\partial} g$. We hope the difference between the 1-form current and its dz component can be understood from context.

In addition to the geometric part, the pure spinor string is defined with pure spinor ghosts and their conjugate momenta. These are invariant under global PSU(2, 2|4) transformations. The ghosts are fermionic elements of the algebra

$$\lambda \in \mathfrak{g}_1 \quad \text{and} \quad \bar{\lambda} \in \mathfrak{g}_3 \tag{2.6}$$

that satisfy

$$\{\lambda, \lambda\} = 0 = \{\bar{\lambda}, \bar{\lambda}\}. \tag{2.7}$$

This is the coset generalization of the pure spinor condition in flat space. The momenta conjugate to the pure spinor variables are denoted

$$\omega \in \mathfrak{g}_3 \quad \text{and} \quad \bar{\omega} \in \mathfrak{g}_1. \tag{2.8}$$

They suffer the gauge transformations

$$\delta\omega = [A, \lambda] \quad \text{and} \quad \delta\bar{\omega} = [B, \bar{\lambda}], \tag{2.9}$$

where A and B are any two local bosonic elements of $\mathfrak{g}_2$. We will also define

$$N = -\{\lambda, \omega\} \quad \text{and} \quad \bar{N} = -\{\bar{\lambda}, \bar{\omega}\} \tag{2.10}$$

which are the Lorentz generators for the ghosts. Note that they have zero $\mathbb{Z}_4$ charge. The pure spinor condition implies

$$[\lambda, N] = 0 = [\bar{\lambda}, \bar{N}]. \tag{2.11}$$

Having all the ingredients, we can write the pure spinor action [22–24]

$$S = \frac{1}{4}\int d^2z \,\mathrm{Str}\left(K_1\bar{K}_3 + 2K_2\bar{K}_2 + 3K_3\bar{K}_1 - 4N\bar{N} + 4\omega\bar{\nabla}\lambda + 4\bar{\omega}\nabla\bar{\lambda}\right). \tag{2.12}$$

The geometric part of this action is the standard kinetic term of a coset model plus a Wess-Zumino term

$$S_{\mathrm{WZ}} = -\frac{1}{4}\int d^2z \,\mathrm{Str}\left(K_1\bar{K}_3 - K_3\bar{K}_1\right). \tag{2.13}$$

This particular coefficient of the Wess-Zumino term is fundamental for BRST symmetry and integrability [12, 16].

By construction, the action has global PSU(2, 2|4) invariance and local SO(1, 4) × SO(5) invariance. Global transformations act on g by left multiplication and the local transformations act on g by right multiplication. The current J is invariant under the global symmetry. On the other hand, K tranforms in the adjoint representation of $\mathfrak{h}$ if $\delta g = gM$, where $M \in \mathfrak{h}$ and A transforms as a connection. The ghosts and their conjugate momenta transform in the adjoint representation of $\mathfrak{h}$ as well.

The next fundamental symmetry is BRST invariance defined by[2]

$$\delta g = g(\lambda + \bar{\lambda}), \quad \delta\lambda = 0, \quad \delta\bar{\lambda} = 0, \quad \delta\omega = -K_3, \quad \delta\bar{\omega} = -\bar{K}_1. \tag{2.14}$$

[2]These transformations are nilpotent only up to local SO(1, 4) × SO(5) transformations and equations of motion. There are ways to fix both these issues [25–28], however, they will not be needed here.

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The conserved current associated with BRST symmetry is given by

$$j_{\mathrm{BRST}} = \mathrm{Str}(\lambda K_3)dz + \mathrm{Str}(\bar\lambda \bar K_1)d\bar z. \tag{2.15}$$

It is not only conserved $d * j_{\mathrm{BRST}} = 0$, but its components are holomorphic and anti-holomorphic

$$\bar\partial\big(\mathrm{Str}(\lambda K_3)\big) = 0 \quad \text{and} \quad \partial\big(\mathrm{Str}(\bar\lambda \bar K_1)\big) = 0 \tag{2.16}$$

after using the equations of motion which will be discussed below. This fact means that the charges defined by

$$Q_\epsilon = \oint dz\, \epsilon(z)\mathrm{Str}(\lambda K_3) \quad \text{and} \quad Q_{\bar\epsilon} = \oint d\bar z\, \bar\epsilon(\bar z)\mathrm{Str}(\bar\lambda \bar K_1) \tag{2.17}$$

also generate symmetries for any two independent holomorphic and anti-holomorphic functions $\epsilon(z)$ and $\bar\epsilon(\bar z)$. In this case the BRST transformations above generalize to[3]

$$\delta g = g\big[\epsilon(z)\lambda + \bar\epsilon(\bar z)\bar\lambda\big], \quad \delta\lambda = 0, \quad \delta\bar\lambda = 0, \quad \delta\omega = -\epsilon(z)K_3, \quad \delta\bar\omega = -\bar\epsilon(\bar z)\bar K_1. \tag{2.18}$$

We now compute the current associated with the global PSU$(2,2|4)$ symmetry. The coset element transforms as $\delta g = \Omega g$. We will let Ω be a local parameter and use the Noether method. The left invariant currents transform as

$$\delta K_i = g^{-1}(d\Omega)g\Big|_{\mathfrak{g}_i}, \quad \delta A = g^{-1}(d\Omega)g\Big|_{\mathfrak{h}}. \tag{2.19}$$

When inserting this transformation into the action, we can drop the restriction on the subspaces since the transformations will always come together with a dual algebra element inside a supertrace. The action transforms as

$$\delta S = \frac{1}{4}\int d^2z\, \mathrm{Str}\left(g^{-1}\partial\Omega g\bar K_3 + g^{-1}\bar\partial\Omega g K_1 + \cdots + 4g^{-1}\partial\Omega g\bar N + 4g^{-1}\bar\partial\Omega g N\right), \tag{2.20}$$

from which we read off the Noether current

$$\mathrm{j} = g\left(K_1 + 2K_2 + 3K_3 + 4N\right)g^{-1}dz + g\left(3\bar K_1 + 2\bar K_2 + \bar K_3 + 4\bar N\right)g^{-1}d\bar z. \tag{2.21}$$

Conservation of the current $d * \mathrm{j} = 0$ implies the equations of motion which can be written compactly as

$$[\bar\nabla + \bar K, K_1 + 2K_2 + 3K_3 + 4N] + [\nabla + K, 3\bar K_1 + 2\bar K_2 + \bar K_3 + 4\bar N] = 0. \tag{2.22}$$

These equations are calculated by varying the action (2.12) with respect to a variation of the coset element given by $\delta g = gX$ with $X \in \mathfrak{m}$. Using the $\mathbb{Z}_4$ decomposition, the Maurer-Cartan identity for J, and

$$\nabla\bar N - \bar\nabla N - 2[N, \bar N] = 0, \tag{2.23}$$

[3]It may seem surprising that the BRST invariance in the pure spinor superstring implies a much larger symmetry than the usual BRST symmetry in field theory. However, we should remember that the pure spinor BRST should also imply Virasoro symmetry which is an infinite-dimensional chiral symmetry.

we can separate (2.22) into eight equations of motion. To derive this last equation we use the equations of motion for the ghosts coming from (2.12):

$$[\bar{\nabla} - \bar{N}, \lambda + \omega] = [\nabla - N, \bar{\lambda} + \bar{\omega}] = 0. \tag{2.24}$$

We can use an operator Σ defined in [29] to write very compact expressions for the action and other observables. We define the action of Σ on the basic currents as

$$\Sigma(A) = 0, \quad \Sigma(K_1) = K_1, \quad \Sigma(K_2) = 2K_2, \quad \Sigma(K_3) = 3K_3, \quad \Sigma(N) = 4N, \tag{2.25}$$
$$\Sigma(\bar{A}) = 0, \quad \Sigma(\bar{K}_1) = 3\bar{K}_1, \quad \Sigma(\bar{K}_2) = 2\bar{K}_2, \quad \Sigma(\bar{K}_3) = \bar{K}_3, \quad \Sigma(\bar{N}) = 4\bar{N}. \tag{2.26}$$

Then, the action can be written as

$$S = \int d^2z \Big(\frac{1}{4} \bar{K}\, \Sigma(K) + \omega \bar{\nabla} \lambda + \bar{\omega} \nabla \bar{\lambda} - N\bar{N} \Big), \tag{2.27}$$

and the components of the Noether current can be written as

$$j_z = g\Sigma(K+N)g^{-1}, \quad j_{\bar{z}} = g\Sigma(\bar{K}+\bar{N})g^{-1}. \tag{2.28}$$

For the supertrace, we have

$$\mathrm{Str}\big(O_i \Sigma(\bar{O}_j)\big) = \mathrm{Str}\big(\Sigma(O_i)\bar{O}_j\big), \tag{2.29}$$

where O_i is any current with a defined action of Σ. We note, however, that the usefulness of Σ in computations is limited by the fact that it is not a Lie algebra homomorphism (e.g. it does not preserve the Lie bracket).

2.1 The flat current

In contrast to the Noether current of the bosonic cosets, the conserved current (2.21) of the $\mathbb{Z}_4$ super-coset is not flat. Instead, it was shown in reference [12] that the pure spinor string in $AdS_5 \times S^5$ has a family of flat currents depending on a complex parameter μ:

$$\mathcal{L}_\mu = l_\mu dz + \bar{l}_\mu d\bar{z}, \tag{2.30}$$

with

$$\begin{aligned} l_\mu &= g\left[(e^{2\mu}-1)K_2 + (e^{\mu}-1)K_1 + (e^{3\mu}-1)K_3 + (e^{4\mu}-1)N\right]g^{-1} \\ \bar{l}_\mu &= g\left[(e^{-2\mu}-1)\bar{K}_2 + (e^{-3\mu}-1)\bar{K}_1 + (e^{-\mu}-1)\bar{K}_3 + (e^{-4\mu}-1)\bar{N}\right]g^{-1}. \end{aligned} \tag{2.31}$$

Using the Σ operator defined above, we can write this compactly as

$$l_\mu = g\left[e^{\mu\Sigma}-1\right](J+N)g^{-1} \quad \text{and} \quad l_\mu = g\left[e^{-\mu\Sigma}-1\right](\bar{J}+\bar{N})g^{-1}. \tag{2.32}$$

The current is flat

$$d\mathcal{L}_\mu + \mathcal{L}_\mu \wedge \mathcal{L}_\mu = 0 \tag{2.33}$$

as a consequence of the equations of motion.

The existence of this current is remarkable given that, as just mentioned, the conserved current of a $\mathbb{Z}_4$ super-coset is generally not flat. In particular, there is no value of μ for which the flat current (2.30) reduces to (2.21). However, note that

$$\mathcal{L}_0' = *\mathrm{j}, \tag{2.34}$$

where $' = \frac{d}{d\mu}$ and the left-hand side is evaluated at $\mu = 0$. This is the usual statement that the first charge generated by the flat current $\mathcal{L}_\mu$ is the conserved charge of the $\mathfrak{psu}(2,2|4)$ algebra. This can be seen from the monodromy matrix[4]

$$\boldsymbol{M}(\mu) = \mathrm{P}\exp\left(\int_{-\infty}^{\infty} \mathcal{L}_\mu\right). \tag{2.35}$$

The charge of the $\mathfrak{psu}(2,2|4)$ algebra

$$\boldsymbol{Q}_{\mathfrak{psu}} = \int_{-\infty}^{\infty} *\mathrm{j} \tag{2.36}$$

is the coefficient of the first power of μ in the expansion of $\boldsymbol{M}(\mu)$.

3 Master symmetry

Following Klose, Loebbert, and Münkler [13], we can define a flat deformation of the Maurer-Cartan current by

$$\begin{aligned} L_\mu = J + g^{-1}\mathcal{L}_\mu g = &\left[A + e^{\mu}K_1 + e^{2\mu}K_2 + e^{3\mu}K_3 + (e^{4\mu} - 1)N\right] dz \\ &+ \left[\bar{A} + e^{-3\mu}\bar{K}_1 + \ e^{-2\mu}\bar{K}_2 + e^{-\mu}\bar{K}_3 + (e^{-4\mu} - 1)\bar{N}\right] d\bar{z}. \end{aligned} \tag{3.1}$$

Note that $L_0 = J$, since $\mathcal{L}_0 = 0$. Using the Σ operator defined in the previous section, L_μ can be written as

$$L_\mu = \left[e^{\mu\Sigma}(J + N) - N\right] dz + \left[e^{-\mu\Sigma}(\bar{J} + \bar{N}) - \bar{N}\right] d\bar{z}. \tag{3.2}$$

(It is actually more straightforward to verify that L_μ satifies a flatness condition.)

A deformation g_μ of the coset element g can be defined by the differential equation [30, 31]

$$dg_\mu(z,\bar{z}) = g_\mu(z,\bar{z})L_\mu \quad \text{with} \quad g_\mu(z_0,\bar{z}_0) = g(z_0,\bar{z}_0). \tag{3.3}$$

Here, $(z_0, \bar{z}_0)$ is any reference point on the worldsheet needed to fix an "initial condition". This equation is well-defined since L_μ is flat. Consequently, this deformation of g is only defined on-shell. An ansatz to solve it is

$$g_\mu(z,\bar{z}) = \chi_\mu(z,\bar{z})g(z,\bar{z}), \tag{3.4}$$

[4]For simplicity and to avoid global subtleties, we will assume the worldsheet is infinite and open. This is not essential to any of the results discussed below which are classical. It would be interesting to check that finite size effects do not spoil our conclusions, but we expect they do not as Berkovits has shown that the Yangian symmetries are preserved even when such quantum corrections are taken into account [16].

where χ_μ satisfies

$$d\chi_\mu = \chi_\mu \mathcal{L}_\mu \quad \text{with} \quad \chi_\mu(z_0, \bar{z}_0) = \mathbf{1}. \tag{3.5}$$

Again, this equation is well defined since $\mathcal{L}_\mu$ is flat. We can expand this differential equation in a power series about 0. The flat current vanishes for $\mu = 0$, so the first two equations are

$$d\chi^{(0)} = 0, \quad d\chi^{(1)} = *\mathrm{j} = j_z dz - j_{\bar{z}} d\bar{z}, \tag{3.6}$$

where we used (2.34). Written in this way, it is clear that $\chi^{(1)}$ only exists if the equations of motion are satisfied. The solution is given by

$$\chi^{(0)}(z, \bar{z}) = \mathbf{1}, \quad \chi^{(1)}(z, \bar{z}) = \int_{(z_0, \bar{z}_0)}^{(z, \bar{z})} (dz j_z - d\bar{z} j_{\bar{z}}). \tag{3.7}$$

With this, we are finally in the position to define the "master symmetry" [13]

$$\hat{\delta} g(z, \bar{z}) := \chi^{(1)}(z, \bar{z}) g(z, \bar{z}). \tag{3.8}$$

This is a non-local transformation acting on the currents as

$$\begin{aligned} \hat{\delta} J = g^{-1} d\chi^{(1)} g = \hat{\delta} K_2 + \hat{\delta} K_1 + \hat{\delta} K_3 + \hat{\delta} A = g^{-1}(j_z dz - j_{\bar{z}} d\bar{z}) g = \\ = (K_1 + 2K_2 + 3K_3 + 4N)\, dz - \left(\bar{K}_3 + 2\bar{K}_2 + 3\bar{K}_1 + 4\bar{N}\right) d\bar{z} \end{aligned} \tag{3.9}$$

Up until this point, the master symmetry has been discussed at the level of geometry. The pure spinor string also has ghosts, and we should consider that $\hat{\delta}$ also acts on them. Since the structure under discussion is on-shell, we will consider N and $\bar{N}$ as fundamental fields with defining equations of motion $\nabla \bar{N} + \bar{\nabla} N = 0$ and $\nabla \bar{N} - \bar{\nabla} N - 2[N, \bar{N}] = 0$. We will define the extension of the master symmetry to act on them as

$$\hat{\delta} N = 4N, \quad \hat{\delta} \bar{N} = -4\bar{N}. \tag{3.10}$$

With these transformations, it is immediate to verify that (3.9) is a symmetry of the equations of motion (2.22), turning it into a Maurer-Cartan identity together with $\nabla \bar{N} - \bar{\nabla} N - 2[N, \bar{N}] = 0$. As argued in [30], χ_μ generates an infinite tower of non-local symmetries of a $\mathbb{Z}_2$ coset. The analogous statement for the pure spinor string for any value of μ will be proved at the end of this paper. The main difference here, apart from the presence of fermionic terms, is that the gauge field A transforms into the ghost current N, thereby mixing matter and ghosts.

Since this symmetry is only defined on-shell, discussing it at the level of the action is potentially meaningless. Nevertheless, we follow reference [13] and try to use Noether procedure to calculate the current associated to the master symmetry anyway. We include a local parameter $\epsilon(z, \bar{z})$ in the transformation of g

$$\hat{\delta} g = \epsilon(z, \bar{z}) \chi^{(1)} g. \tag{3.11}$$

The currents transform as

$$\hat{\delta} J = d\epsilon g^{-1} \chi^{(1)} g + \epsilon \Sigma\, (K + N)\, dz - \epsilon \Sigma \left(\bar{K} + \bar{N}\right) d\bar{z}. \tag{3.12}$$

Inserting this transformation into the action and only collecting terms depending on $d\epsilon$, we find

$$\hat{\delta} S = \frac{1}{4}\int d^2 z \,\mathrm{Str}\Big(\partial\epsilon\chi^{(1)} j_{\bar{z}} + \bar{\partial}\epsilon\chi^{(1)} j_z\Big). \tag{3.13}$$

Using the same normalization as (2.21), we read off the conserved current

$$\boldsymbol{J}^{(0)} = \mathrm{Str}\Big(\chi^{(1)} j_z\Big) dz + \mathrm{Str}\Big(\chi^{(1)} j_{\bar{z}}\Big) d\bar{z}. \tag{3.14}$$

Note that by (3.6),

$$*\boldsymbol{J}^{(0)} = \frac{1}{2} d\big(\mathrm{Str}(\chi^{(1)}\chi^{(1)})\big). \tag{3.15}$$

Using this, we can perform the integral to find the conserved charge

$$\mathfrak{C}^{(0)} = \int *\boldsymbol{J}^{(0)} = \frac{1}{2}\mathrm{Str}(\chi^{(1)}\chi^{(1)})\Big|_{-\infty}^{\infty}. \tag{3.16}$$

If we choose the point z_0 in the initial condition (3.5) of $\chi_\mu(z,\bar{z})$ to be at spacial $-\infty$ and use that the $\mathfrak{psu}(2,2|4)$ charge is given by

$$\boldsymbol{Q}_{\mathfrak{psu}} = \chi^{(1)}(\infty), \tag{3.17}$$

we see that

$$\mathfrak{C}^{(0)} = \frac{1}{2}\mathrm{Str}\big(\boldsymbol{Q}_{\mathfrak{psu}}\boldsymbol{Q}_{\mathfrak{psu}}\big) \tag{3.18}$$

is the Casimir of the $\mathfrak{psu}(2,2|4)$ algebra, as in [13].

In principle we could find the higher scalar charges associated with higher powers of the μ expansion of χ_μ in a similar way, but we can guess the result as follows. The form in (3.14) suggests that the scalar current containing all higher master symmetry charges is $*\mathrm{Str}\big(\chi_\mu \mathcal{L}'_\mu\big)$. However, there are a few problems with this first attempt: χ_μ is not an element of the algebra, it is not conserved, and its μ^2 coefficient does not match the Casimir we obtain from acting with $\hat{\delta}$ on $\mathfrak{C}^{(0)}$. A better guess is

$$\boldsymbol{J}_\mu = *\mathrm{Str}\big(\chi_\mu^{-1}\chi'_\mu\mathcal{L}'_\mu\big). \tag{3.19}$$

This is conserved $d{*}\boldsymbol{J}_\mu = 0$ using that $d(\chi_\mu^{-1}\chi'_\mu) = [\chi_\mu^{-1}\chi'_\mu, \mathcal{L}_\mu] + \mathcal{L}'_\mu$ and the first derivative with respect to μ of the flatness condition (2.33). Then, the complete tower of nonlocal charges can be defined by

$$\mathfrak{C}_\mu = \int_{-\infty}^{\infty} *\boldsymbol{J}_\mu = \int_{-\infty}^{\infty}\mathrm{Str}\big(\chi_\mu^{-1}\chi'_\mu\mathcal{L}'_\mu\big) = \frac{1}{2}\mathrm{Str}\Big((\chi_\mu^{-1}\chi'_\mu(\infty))(\chi_\mu^{-1}\chi'_\mu(\infty))\Big). \tag{3.20}$$

The zeroth power of μ in the expansion gives the Casimir $\mathfrak{C}^{(0)}$, and it is easy to show that the coefficient of μ is the result of calculating $\hat{\delta}\mathfrak{C}^{(0)}$.

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4 Yangians

Having defined $\hat{\delta}$, we can see how it affects other symmetries of the string. If we act with $\hat{\delta}$ on (2.21) we obtain

$$\begin{aligned}\hat{\delta}\mathrm{j} = \mathrm{j}^{(1)} &= g(K_1 + 4K_2 + 9K_3 + 16N)g^{-1}dz \\ &\quad - g(9\bar{K}_1 + 4\bar{K}_2 + \bar{K}_3 + 16\bar{N})g^{-1}d\bar{z} + [\chi^{(1)}, \mathrm{j}],\end{aligned} \tag{4.1}$$

which is the first non-local current given by the monodromy matrix (2.35) constructed from the flat current (2.30). Schematically, this current generates a transformation on the coset element of the form $\delta g \sim [\eta, \chi^{(1)}]g$, where $\eta \in \mathfrak{psu}(2,2|4)$ is constant. We can try to use the Noether method again to see if we can obtain the non-local current as the Noether current associated with this transformation. However, carrying this out, we obtain only the last term in (4.1). As mentioned previously, it is not surprising that this time we could not obtain the desired result since this is an on-shell symmetry. In the case of the principal chiral model it is possible to interpret these non-local currents as Noether currents [32, 33], but it is not clear that we can use the same method here.

We could obtain the higher non-local currents by successive applications of the master symmetry generator $\hat{\delta}$, but as before there is a faster way to obtain these currents as we now show.

4.1 Non-local current

The non-local current associated with the global symmetry and all higher Yangian charges is calculated by replacing $g \to g_\mu$ (3.3), (3.4) in the definition of the Noether current (2.21) and defining a finite μ deformation of ghost currents as

$$N \to N_\mu = e^{4\mu}N, \quad \bar{N} \to \bar{N}_\mu = e^{-4\mu}\bar{N}, \tag{4.2}$$

from which we can see that the master symmetry (3.10) corresponds to the first power of the μ deformation. With these definitions we have

$$\begin{aligned}\mathbb{J}_\mu &= g_\mu((K_\mu)_1 + 2(K_\mu)_2 + 3(K_\mu)_3 + 4N_\mu)g_\mu^{-1}dz \\ &\quad + g_\mu(3(\bar{K}_\mu)_1 + 2(\bar{K}_\mu)_2 + (\bar{K}_\mu)_3 + 4\bar{N}_\mu)g_\mu^{-1}d\bar{z} \\ &= \chi_\mu g(e^{\mu}K_1 + 2e^{2\mu}K_2 + 3e^{3\mu}K_3 + 4e^{4\mu}N)g^{-1}\chi_u^{-1}dz \\ &\quad + \chi_\mu g(3e^{-3\mu}\bar{K}_1 + 2e^{-2\mu}\bar{K}_2 + e^{-\mu}\bar{K}_3 + 4e^{-4\mu}\bar{N})g^{-1}\chi_u^{-1}d\bar{z},\end{aligned} \tag{4.3}$$

where we defined $K_\mu = (g_\mu^{-1}dg_\mu)|_{\mathfrak{m}} = (g^{-1}\mathcal{L}_\mu g + g^{-1}dg)|_{\mathfrak{m}}$. From the last two lines we can identify this current as

$$\mathbb{J}_\mu = \chi_\mu(*\mathcal{L}'_\mu)\chi_\mu^{-1}. \tag{4.4}$$

Computing the ∂ and $\bar{\partial}$ derivatives of its components, we see that

$$\begin{aligned}\bar{\partial}\mathbb{J}_\mu &= \chi_\mu\Big(g[\bar{\nabla}+\bar{K},2e^{2\mu}K_2+e^{\mu}K_1+3e^{3\mu}K_3+4e^{4\mu}N]g^{-1}\\&\qquad+[\bar{l}_\mu,g(2e^{2\mu}K_2+e^{\mu}K_1+3e^{3\mu}K_3+4e^{4\mu}N)g^{-1}]\Big)\chi_\mu^{-1}\\&=\chi_\mu\Big(\bar{\partial}l'_\mu+[\bar{l}_\mu,l'_\mu]\Big)\chi_\mu^{-1}. \qquad (4.5)\\ \partial\bar{\mathbb{J}}_\mu &= \chi_\mu\Big(g[\nabla+K,2e^{-2\mu}\bar{K}_2+3e^{-3\mu}\bar{K}_1+e^{-\mu}\bar{K}_3+4e^{-4\mu}\bar{N}]g^{-1}\\&\qquad+[l_\mu,g(2e^{-2\mu}\bar{K}_2+3e^{-3\mu}\bar{K}_1+e^{-\mu}\bar{K}_3+4e^{-4\mu}\bar{N})g^{-1}]\Big)\chi_\mu^{-1}\\&=\chi_\mu\Big(-\partial\bar{l}'_\mu+[l_\mu,-\bar{l}'_\mu]\Big)\chi_\mu^{-1}. \qquad (4.6)\end{aligned}$$

So, the conservation of $\mathbb{J}_\mu$ is simply the first derivative with respect to μ of the flatness condition (2.33) of $\mathcal{L}_\mu$:

$$\bar{\partial}\mathbb{J}_\mu+\partial\bar{\mathbb{J}}_\mu=\chi_\mu\Big(\bar{\partial}l'_\mu-\partial\bar{l}'_\mu+[\bar{l}_\mu,l'_\mu]+[\bar{l}'_\mu,l_\mu]\Big)\chi_\mu^{-1}=0. \qquad (4.7)$$

This relation proves that if g is a solution, then the deformation g_μ is also a solution. The current $\mathbb{J}_\mu$ contains a whole tower of non-local conserved currents of the model, starting with the global $\mathfrak{psu}(2,2|4)$ current (2.4). It is easily checked that

$$\mathbb{J}_\mu=\mathrm{j}+\mu\mathrm{j}^{(1)}+\cdots, \qquad (4.8)$$

where $\mathrm{j}^{(1)}$ is the first Yangian current. To prove the higher μ powers are all higher Yangian currents and that $\hat{\delta}$ acts as a raising operator, we proceed as follows. First we note that

$$\hat{\delta}\mathcal{L}'_\mu=\mathcal{L}''_\mu+[\chi^{(1)},\mathcal{L}'_\mu]. \qquad (4.9)$$

After some manipulations, one can show that

$$\chi_\mu^{-1}\hat{\delta}\mathbb{J}_\mu\chi_\mu=*\mathcal{L}''_\mu+[\chi_\mu^{-1}\hat{\delta}\chi_\mu,*\mathcal{L}'_\mu]+[\chi^{(1)},\mathcal{L}'_\mu]. \qquad (4.10)$$

To get to final result let us now compute the derivative with respect to μ of $\mathbb{J}_\mu$

$$\chi_\mu^{-1}\mathbb{J}'_\mu\chi_\mu=*\mathcal{L}''_\mu+[\chi_\mu^{-1}\chi'_\mu,*\mathcal{L}'_\mu]. \qquad (4.11)$$

If we subtract both equations we have

$$\chi_\mu^{-1}(\hat{\delta}\mathbb{J}_\mu-\mathbb{J}'_\mu)\chi_\mu=[\chi_\mu^{-1}\hat{\delta}\chi_\mu-\chi_\mu^{-1}\chi'_\mu,*\mathcal{L}'_\mu]+[\chi^{(1)},*\mathcal{L}'_\mu]. \qquad (4.12)$$

Let us call $\phi_\mu:=\chi_\mu^{-1}\hat{\delta}\chi_\mu-\chi_\mu^{-1}\chi'_\mu$ and note that $\phi_0=-\chi^{(1)}$. Using that

$$\hat{\delta}\mathcal{L}_\mu=\mathcal{L}'_\mu+[\chi^{(1)},\mathcal{L}_\mu]-*\mathrm{j}, \qquad (4.13)$$

we can calculate that the differential equation satisfied by ϕ_μ is

$$d\phi_\mu+*\mathrm{j}=[\phi_\mu+\chi^{(1)},\mathcal{L}_\mu]. \qquad (4.14)$$

Since $*\mathrm{j} = d\chi^{(1)}$, we can change variables $\phi_\mu \to \psi_\mu := \phi_\mu + \chi^{(1)}$ and arrive at

$$d\psi_\mu = [\psi_\mu, \mathcal{L}_\mu], \tag{4.15}$$

where now the initial condition is $\psi_0 = 0$. Since $\mathcal{L}_0 = 0$, all higher powers of μ will vanish. So we conclude that $\phi_\mu = -\chi^{(1)}$ for any value of μ.[5] Thus, we have finally proven that

$$\hat{\delta}\mathbb{J}_\mu = \frac{d}{d\mu}\mathbb{J}_\mu, \tag{4.16}$$

so that $\hat{\delta}$ acts as a raising operator for non-local symmetires, exactly as in [13]. Similarly, it can be shown that the non-local Casimir (3.20) satisfies

$$\hat{\delta}\mathfrak{C}_\mu = \frac{d}{d\mu}\mathfrak{C}_\mu. \tag{4.17}$$

5 Conclusions and prospects

We have shown that the classical pure spinor string in the $AdS_5 \times S^5$ background has the full set of classical non-local symmetries extending those recently studied by Klose, Loebbert and Münkler in the context of $\mathbb{Z}_2$ cosets [13]. We find that the inclusion of ghosts in a sense makes the $\mathbb{Z}_4$ symmetry manifest, and all non-local symmetries can be lifted to the super-coset $\mathrm{PSU}(2,2|4)/\mathrm{SO}(1,4) \times \mathrm{SO}(5)$.

An immediate extension of the results of this paper is to derive the analog for the Green-Schwarz superstring. That can be done by erasing the ghosts and imposing an appropriate gauge choice. Classical solutions of the pure spinor string should preserve BRST symmetry which means the BRST charge should vanish when evaluated on the solution. If we do not set the ghosts to zero, this means that the currents K_3 and $\bar{K}_1$ should vanish. In reference [12], it was shown that the pure spinor flat current is equivalent to the one in the Green-Schwarz formalism [10] in this gauge. We expect that the GS string enjoys all of the symmetries discussed in the present work.

It would be interesting to apply the results of this paper to supersymmetric Wilson loops in AdS_5 as in reference [13]. However, it is as yet not known how to study such classical solutions in the pure spinor formalism. There is hope such a task can be done, since it was shown by explicit computations in references [20, 21] that the semi-classical quantization of the pure spinor string is equivalent to the Green-Schwarz string in a certain class of solutions. In [34] it was argued that the equivalence holds for any physical solution. With these results in mind, it is likely that one can extend the results of, for example, references [35, 36] to the pure spinor string.

A more speculative line of research is the relevance of the master symmetry in the quantum theory. Since the Yangian currents are still conserved at quantum level [16] it is possible that there is some quantum version of the master symmetry. However we cannot say if it will provide any additional help in achieving an exact solution of the model.

[5] This result can be used as a practical way to calculate $\hat{\delta}\chi_\mu = \chi'_\mu - \chi_\mu\chi^{(1)}$ for any power of μ.

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