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Geometric Transitions, Flops and Non-Kähler Manifolds: II

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Abstract

We continue our study of geometric transitions in type II and heterotic theories. In type IIB theory we discuss an F-theory setup which clarifies many of our earlier assumptions and allows us to study gravity duals of $\mathcal{N} = 1$ gauge theories with arbitrary global symmetry group G. We also point out the subtle differences between global and local metrics, and show that in many cases the global descriptions are far more complicated than discussed earlier. We determine the full global description in type I/heterotic theory.

In type IIA, our analysis gives rise to a local non-Kähler metric whose global description involves a particular orientifold action with gauge fluxes localised on branes. We are also able to identify the three form fields that allow for a smooth flop in the M-theory lift. We briefly discuss the issues of generalised complex structures in type IIB theory and possible half-twisted models in the heterotic duals of our type II models. In a companion paper we will present details on the topological aspects of these models.

Contents

1. Introduction and Summary

The study of geometric transitions has been an extensive field of research in the past five years. It started with the seminal papers [1][2][3] where the idea of connecting open string theory with branes to closed string theory with fluxes was spelled out. The upshot was a powerful recipe to derive effective field theory results from string theory compactification with branes and fluxes. Spectacular results have been obtained subsequently in many directions. For example, earlier development of the string flux compactifications [4][5] now applied to this scenario, showed a new "democratic" way to think about effective theories coming from string compactifications.

The direction opened in [3], based on the previous work on topological strings [6], had a powerful impact on stretching the skills to use string theory to describe explicitly large classes of effective field theories. By matching effective field theory quantities such as gluino condensate and meson vacuum expectation values with geometrical objects, one can use compactification details to get new insights into the strongly coupled field theories. There were several avenues to accomplish that. One was to use matrix models to obtain effective potentials and their extrema [7][8]. Another one was to translate geometrical results regarding period matrices into field theory language [9][10][11][12]. A dynamical way to undertand the above match is to use an M theory description where the branes and geometries have a description in terms of unique M5 branes [13][14][15][16].

More recently it has become clear that some aspects of geometric transitions in [3] have to be reconsidered to accomodate some "exotic" NS forms that are needed to have a precise match between field theory and geometric quantities. This was pointed out in [17] for compact manifolds and has been succesfully implemented for geometric transitions in [18][19]. This implementation requires a departure from the compactification with $SU(3)$ *holonomy*, i.e. standard Calabi-Yau manifolds, to a larger class of manifolds with $SU(3)$ *structure*, i.e. non-Kähler manifolds. The same happens in M theory where the G_2 holonomy condition should be dropped in favor of a more general G_2 structure condition.

However the above development of geometric transition [3], although spectacular in many respects, lacked the preciseness of the equivalent Klebanov-Strassler [1] or Maldacena-Nunez [2] models because of the absence of a complete supergravity description of the system. An attempt to get the full supergravity description was started in a series of papers [18], [19] by following a duality chain that used some aspects of T-duality and mirror symmetry advocated in [20]. The program of following this chain by a step-by-step application of known dualities turned out to be highly non-trivial because the duality rules were suited to some regime of parameter space that were opposite to the regime that we would be interested in for studying geometric transitions. Nevertheless, with some subtle manipulations we obtained the right metric in type II and M-theories. However, the metric that we got in [18] and [19] turned out to be only a *local* description of a much more involved *global* framework. The global framework involves non-trivial orientifold actions giving rise to other D-branes and orientifold planes in addition to the already present wrapped five branes. Our ideas can be summarized as follows.

The starting point of our program is a type IIB solution which is *locally* the one of [21]. Recall that the solution of [21] describes D5 branes wrapped on the resolution two-cycle of a resolved conifold, exactly what one wants for an IR description of a $\mathcal{N} = 1$ gauge theory. The full *global* solution of [21] is not supersymmetric as discussed in [22], [18]. Therefore, to get a supersymmetric background we embed this solution into an F-theory setup. In terms of type IIB language this is done by considering extra $D7/O7$ branes on a particular six dimensional manifold whose local metric (i.e. in the region where the effects of background $D7/O7$ branes are negligible) looks similar to the metric of [21], but now written in terms of *local coordinates*. From the F-theory point of view this is simply a compactification on a fourfold with G-fluxes [4][23]. We will present the details of the construction of this fourfold in sec. 2.1. In short, the fourfold is a T^2 fibration over a six-dimensional base that locally resembles a resolved conifold with fractional $D3$ branes[1]. The base is not a Calabi-Yau manifold because $c_1 \neq 0$ but will be Kähler[2]. This is similar to the embedding of the Klebanov-Strassler background [1] in F-theory [5]. In this case, the F-theory base is again approximately a conifold, and the G-fluxes appear in IIB as H_{NS} and H_{RR} fluxes.

In the light of these considerations, the field theory results should also be reinterpreted to accomodate the new branes and orientifolds. It is inherent to geometric transitions that the gluino condensate in the brane side is identified with geometrical structures in the flux side. For the $\mathcal{N} = 1$ theory on the D5 branes this means identification of Λ^3 with the size of S^3 cycles. In the setup presented here, additional fundamental matter is introduced. In type IIB this corresponds to the D7 branes situated on top of the O7-planes. Their distance from the resolution cycles is the mass of the corresponding fundamental quarks[3].

Consider now a scenario where the D5 branes wrapped on the resolution two-cycle are close to *one* orientifold plane. Clearly, then the other three orientifold O7 planes will be too far away from the wrapped D5 branes to be considered a part of the effective theory, so effectively there are only 4 relevant D7 branes on top of one O7 plane. This implies for the field theory that the global picture represents an $\mathcal{N} = 1$ $SU(N)$ gauge theory with matter in the fundamental representation and flavor group $SU(2)^4$, the quark masses of the flavors being m and the scale of the theory being $\Lambda_{\rm global}$. The global picture is obtained by

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[1] In [18] we constructed a fourfold with non–degenerating T^2 fiber. From our discussion, this would capture only the local behavior of the system as the degeneration points of the fiber (alternatively the positions of the seven branes) are not in the local neighborhood.

[2] The base could be Calabi-Yau only at one special point − the orientifold point [24]. Clearly we will not be in this region if we want to move the seven branes far away.

[3] There are *two* different matter multiplets (or flavor degrees of freedom) here in the UV. One of them are the matter multiplets coming from the strings that stretch between full and fractional D3 branes. These form the *bi-fundamental* matters, and are in general charged under both the gauge groups in the ultaviolet. The second one come from the strings stretching between full and fractional D3 branes, and the seven branes. They are the *fundamental* matters. At the far IR, when the D3 branes cascade away leaving only the fractional D3 branes, the theory will only have fundamental matter multiplets and no bi-fundamental matters. In this paper we will concentrate only in this region, leaving a more detailed discussion of the UV behavior for the sequel.

starting with $\mathcal{N} = 1$ $Sp(2N)$ gauge theory and giving an expectation value to the adjoint Higgs which breaks $Sp(2N)$ to $SU(N)$ and the global group from $SO(8)$ to $SU(2)^4$. The local picture therefore represents a pure $\mathcal{N} = 1$ $SU(N)$ theory which is the effective theory after integrating out the massive quarks with a scale

$$\Lambda^{3N}_{\rm local} = m^4 \Lambda^{3N-4}_{\rm global}. \tag{1.1}$$

Thus, to go from a local picture to a global picture one has to integrate in the massive quarks as in [25]. We will soon see that the local picture is forced upon us by the identifications of geometric transitions.

The local geometry versus global geometry issue is also discussed in [5] where the local Klebanov-Strassler solution [1] gets the extra flavor group G globally. This makes the Klebanov-Strassler solution an effective theory. The only difference is that in their case the solution of [1] is supersymmetric, whereas our starting point, the solution of [21] is not. This can be understood from the fact that [1] uses integer and fractional D3 branes, the latter being wrapped D5 on vanishing cycles. This is a susy configuration. When the vanishing cycle becomes finite (the conical singularity is resolved), the fractional D3 branes become genuine D5 wrapped on 2-cycles which[4], together with the integer D3 branes, are in general a non-susy system[5].

From the type IIB solution one can infer a mirror IIA solution [18]. The result is a family of non-Kähler manifolds, the non-Kählerity coming from the presence of type IIB NS flux. This means that the geometric transitions of [3] can actually be generalized to non–Kähler manifolds, as presented in [18].

The non-Kählerity in type IIA theory is proportional to the NS flux so the non-Kähler geometry can be considered as a function of the type IIB NS flux. However, in type IIB we know from [9] that the NS flux is proportional to the coupling constant of the dual gauge

[4] We will also refer to these branes as fractional D3 branes.

[5] As we will discuss in sec. 2.1, it doesn't really matter if there might exist such a solution that preserves supersymmetry (i.e allows primitive three form fluxes). Of course, to discuss the geometric transition of $\mathcal{N} = 1$ $SU(N)$ theory with fundamental flavor transforming under a group G, we *have* to introduce $D7/O7$ branes along with the fractional and whole $D3$ branes [26] (see also [27] where somewhat equivalent construction, but only with $D7$ branes, are made to study Klebanov-Strassler model [1] with fundamental flavors). Our local metric studied earlier in [18], [19] is much more robust and it *only* depends on the topology of the resolved conifold.

theory, and therefore the non-Kählerity of type IIA solutions are inversely proportional to the coupling constant of the field theory. Thus we see two possible limits:
(a) The NS flux is (almost) zero. Then $1/g^2$ in field theory is also near zero which means we are in the extreme IR. So the Kähler limit of our solution is dual to the extreme IR in the field theory.
(b) The NS flux is large so the deviation from Kählerity is big. Then $1/g^2$ in the field theory is large which means that we are in the far UV of the field theory.

Our solution maps naturally to the one discussed in [28], where an interpolating solution between the Klebanov-Strassler (KS) solution [1] and the Maldacena-Nunez (MN) solution [2] was considered, with the MN solution obtained in the extreme IR. Both these backgrounds are in type IIB. In the limit (a), our solution for wrapped D5 branes will only have RR flux. This is similar to the MN solution. In the limit (b) and any intermediate case between (a) and (b), our solution will have both NS and RR flux. This is a Klebanov-Strassler type solution[6]. The difference between our analysis and [28] is the fact that we have an interpolating IIA solution, which appears very naturally in the form of the non-Kähler deformation of the deformed conifold.

The distinction global/local picture translates onto the type IIA case. This will be discussed in detail in sec. 6. The wrapped $D5$ branes become wrapped $D6$ branes and the stacks of $D7/O7$ become stacks of $D6/O6$ which are now located at fixed points of a specific orientifold operation. The global group is $SO(4)^8$. In terms of field theory, the global picture represents an $\mathcal{N} = 1$ $SU(N)$ theory with matter in the fundamental representation and a *maximal* flavor group $SU(2)^{16}$. If in the global picture, the field theory has the same scale Λ_{global} and all the flavors have the same mass m, then the local scale is again given by the relation (1.1). This means that the mirror symmetry passes a field theory consistency condition, namely, the scales of the effective theories are the same in type IIA and type IIB.

We will discuss in sec. 7 how our IIA solutions fit into the class of known susy backgrounds with torsion. For example, one type of solutions are discussed in [29] with nontrivial RR two-form and nontrivial dilaton. Here the nonzero components of the torsion classes are W_2^+, W_4, W_5. The second type of solutions are the half-flat solutions which were

[6] A word of caution here. In [19] we showed that the gravity dual of $\mathcal{N} = 1$ $SU(N)$ gauge theory looks very similar to the Klebanov-Strassler [1] background with three form fluxes. However there were some small differences which we will elaborate on later.

first discussed in [17] and are obtained as mirrors of type IIB solutions with NS flux. For the half-flat solutions W_1^+, W_2^+, W_3 are nonzero. The half-flat solutions were lifted to G_2 manifolds by using Hitchin's Hamiltonian flow in [30].

Our solution will be neither of the type discussed in [29] nor half-flat. Due to the elaborate F–theory setup in IIB we will have fluxes turned on the lift to M–theory that destroy the G_2 holonomy. We therefore expect only a G_2 structure manifold[7]. Furthermore, we only have the local metrics available, so our torsion class analysis should not be taken too seriously. We cannot make statements about the global topology of our IIA backgrounds. The local solutions are not susy and will therefore not obey any classification for susy torsional backgrounds such as [29] or [17].

In contrast, in the heterotic theory we will propose, in sec. 3, a global background that reduces in the local limit to the one proposed in [19]. The global background will look very similar to the MN background [2], although it can in general differ from the MN background [2] by warp factors in the metric, but satisfies the torsional relation at least in the limit where it agrees with MN. The method to construct heterotic solutions is to start with a slightly different F–theory setup. We can trade the $D7/O7$ stacks for $D9/O9$. This way we reach a type I picture related to a heterotic picture by an S-duality. This mapping implies a possible geometric transition for heterotic strings, as observed in [19]. The global heterotic solution is not neccesarily dual to a global type IIB solution as this would imply going away from the orientifold limit. Since the type II transitions can be understood by studying the topological sector, we can use a half-twisted version to understand the transition for the (0,2) theories [31][32][33]. This will be addressed in sec. 3.1 and sec. 3.2.

We will also study heterotic vector bundles in sec. 4. They are associated with the global group G that we encounter in IIB. The global symmetries appear as gauge symmetries of the $SO(32)$ bundle in the heterotic side. In sec. 4.2 we will discuss possible ways to pull bundles across a conifold transition.

As we discussed above, the global type IIB picture leads us to the global type IIA model. This is basically the mirror of the type IIB background (once we take into account all the subtleties). In type IIA, however, we encounter some interesting scenarios. First,

[7] Thus the holonomy could even be $SU(2)$ in some of these models. However, susy will be determined *not* by the holonomy but by the structure. So G_2 structure but $SU(2)$ holonomy will preserve $\mathcal{N} = 1$ susy in 4 dimensions.

we find that the global picture is generated from a specific orientifold action. Due to the inherent orientifold action we find some gauge fluxes that are secretly localised on the D6 branes. Some part of these fluxes may even give rise to three-from fluxes in the internal space, although their effect in the local region of interest may be negligible. Globally they might have some effects. Secondly, the same orientifold action produces three-form potentials that are not pure gauge. We find this a novel aspect of our model. The three form potentials are responsible for a smooth flop operation in the M-theory lift of our model. This was anticipated in [34] and [35], and here we provide the first concrete realisation. These details will be presented in sec. 6; and in sec. 7 we will argue how these fluxes determine the torsion classes both before and after geometric transitions in type IIA theory.

Before we go into the detail discussions of the above aspects, we should point out a simple T-dual version in type IIA of the full type IIB global picture. First, one shouldn't confuse this T-dual model with the *mirror* type IIA model. The mirror is a different duality that captures the *full* content of our type IIB model. The brane dual that we are talking of are in the same spirit as [36], [37] and it captures only some aspects of our full type IIB global picture. Nevertheless, many of the UV expectations will be clear from the T-dual. First, the resolved conifold will T-dualise into two intersecting NS5 branes, and the fractional D3 branes will become half-D4 branes. This is same as our model in [13] and [14]. The D7/O7 system will T-dualise into a D6/O6 system that will be parallel to one of the NS5 branes and in fact will coincide with the NS5 branes. This is somewhat similar to the model advocated in [38] and in [27]. We differ from [38] in two respects: one, our brane construction is the T-dual of a resolved conifold, and two, we have both full D4 and half-D4 branes. The model of [38], that is T-dual to a conifold, takes only full D4 branes, and therefore could study gauge theories with vanishing Beta-function only. We also differ from [27] because [27] studies the T-dual of a Klebanov-Tseytlin model [39] with D7 branes as flavors. As we know [39] behaves badly at the IR because of naked-singularity there, and Klebanov-Strassler [1] will be the right model to study IR of a confining gauge theory. So the model advocated by [27] can say nothing about the IR behavior that we would be interested in. Plus the flavors are given by only D7 branes in [27], that in general should be viewed from an F-theory perspective which is lacking in [27][8].

[8] These issues were in fact already pointed out in [27], although no solutions were presented there.

In the rest of the paper we will solve all the issues that could in principle plague the models of [38] and [27]. As discussed above, we can show much more than merely giving the global story. We can discuss the full dynamics of geometric transitions in *all* string theories, including M-theory.

We start by revisiting the root of all of these issues: the resolved conifold.

2. Resolved conifold revisited

The original study of open-closed string duality in type II theory starts with D6 branes wrapping a three cycle of a non-compact deformed conifold. Naively one might expect the deformed conifold to be a complex Kähler manifold with a non-zero three cycle. However as discussed earlier in [18] this is not quite correct, and the manifold that actually would solve the string equations of motion is a non-Kähler deformation of the deformed conifold. It also turns out that the manifold has no integrable complex structure, but only has an almost complex structure. This is consistent with the prediction of [3]. The non-Kählerity of the underlying metric can be easily seen from its explicit form[9]

$$\begin{aligned} ds^2_{IIA} = g_1 \ & \Big[(dz - b_{z\mu}\ dx^\mu) + \Delta_1 \cot \widehat{\theta}_1\ (dx - b_{x\theta_i}\ d\theta_i) + \Delta_2 \cot \widehat{\theta}_2\ (dy - b_{y\theta_j}\ d\theta_j) + ..\Big]^2 \\ & + g_2\ [d\theta_1^2 + (dx - b_{x\theta_i}\ d\theta_i)^2] + g_3\ [d\theta_2^2 + (dy - b_{y\theta_j}\ d\theta_j)^2] + g_4 \sin\psi\ [(dx - b_{x\theta_i}\ d\theta_i)\ d\theta_2 \\ & + (dy - b_{y\theta_j}\ d\theta_j)\ d\theta_1] + \ g_4 \cos\psi\ [d\theta_1\ d\theta_2 - (dx - b_{x\theta_i}\ d\theta_i)(dy - b_{y\theta_j}\ d\theta_j)]. \end{aligned} \tag{2.1}$$

where the coefficients g_i and the coordinates $\theta_i, \widehat{\theta}_i$ etc. are defined in [18]. The background has non-trivial gauge fields (that form the sources of the wrapped D6 branes) and a non-zero string coupling (which could in principle be small).

Existence of such an exact supergravity background helps us to obtain the corresponding mirror type IIB background. One would expect that this can be easily achieved using

[9] In deriving the metric, we took a simpler model where all the spheres were replaced by tori with periodic coordinates (x, θ_1) and (y, θ_2). The coordinate z formed a non-trivial $U(1)$ fibration over the T^2 base. The replacement of spheres by two tori is directly motivated from the corresponding brane constructions of [36],[37], where non-compact NS5 branes required the existence of tori instead of spheres in the T-dual picture. In fact there is a deeper reason behind this choice. The metrics that we studied earlier in [18] and [19] are actually *local* descriptions of a much more elaborate global story. As we will show, the local description *only* allows tori. We will soon provide a derivation that will justify all the assumptions taken. In fact we will see that most of the assumptions are imposed on us by some stringent requirements on the type IIB metric.

the mirror rules of [20]. It turns out however that the mirror rules of [20], as discussed in [19] and [18], do not quite suffice[10]. A detailed analysis of this is presented in [19]. As discussed in [18] and [19], we have to be careful about various subtle issues while doing the mirror transformations:

(a) The mirror rules of [20] tells us that *any* Calabi-Yau manifold with a mirror admits, at least *locally*, a T^3 fibration over a three dimensional base. This seems to fail for the deformed conifold as it does not possess enough isometries to represent it as a T^3 fibration[11]. On the other hand, a resolved conifold does have a well defined T^3 torus over a 3-d base, which can be exploited to get the mirror (see also [40]). It also turns out that the T^3 torus is a lagrangian submanifold [41], so a mirror transformations will not break any supersymmetry.

(b) Viewing the mirror transformation naively as three T-dualities along the T^3 torus *does not* give the right mirror metric. There are various issues here. The rules of [20] tell us that the mirror transformation would only work when the three dimensional base is very large. The configuration that we have is exactly opposite of the case [20]. Recall that our configuration lies at the end of a much larger cascading theory. By UV/IR correspondences, this means that the base manifold is very small. Furthermore we are at the *tip* of the geometric transition and therefore we have to be in a situation with very small base (in fact very small fiber too). In [19] and in [18] we showed that we could still apply the rules of [20] if we impose a non-trivial large complex structure on the underlying T^3 torus. The complex structure can be integrable or non-integrable. Using an integrable complex structure, we showed in [19] and in [18] that we can come remarkably close to getting the right mirror metric. Our conjecture there was that if we use a non-integrable complex structure we can get the right mirror manifold.

It seems therefore natural to start with the manifold that exhibits three isometry directions — the resolved conifold. We can, however, not use the metric for D5 branes wrapping the S^2 of a resolved conifold as derived in [21], because it breaks all supersymmetry [22]. The metric that we proposed in [18] (where we kept the harmonic functions undetermined) is very close to the metric of [21] but differs in some subtle way:

(a) The type IIB resolved conifold metric that we proposed in [18] is a D5 wrapping a two cycle that *preserves* supersymmetry. We will discuss this issue in more detail below.

[10] The mirror transformations used in [18] was to take a specific type IIB background and get the metric (2.1).

[11] There is a subtlety about the local and global pictures that we will clarify soon.

(b) As explained in [18], [19], our IIB manifold also has seven branes (and possibly orientifold planes) along with the type IIB three-form fluxes. The metric constructed in [21] doesn't have seven branes but allows three-form fluxes.
The *local* behavior of the type IIB metric is expressed in terms of non-trivial complex structures τ_1 and τ_2 as $dz_1 = dx - \tau_1 d\theta_1$ and $dz_2 = dy - \tau_2 d\theta_2$. The local metric then reads

$$ds^2 = (dz + \Delta_1 \cot \theta_1 \; dx + \Delta_2 \cot \theta_2 \; dy)^2 + |dz_1|^2 + |dz_2|^2 \tag{2.2}$$

where all the warp factors can locally be absorbed in to the coordinate differentials. In this formalism the metric may naively look similar to the one studied in [21] but the global picture is completely different from the one proposed by [21].

2.1. The full global picture

To consider the full global picture, let us consider an F-theory compactification on a fourfold which is a non-trivial T^2 fibration over a resolved conifold base. For simplicity we will consider a compact base, although the non-compact base would be easy to generalize. To preserve charge conservation, therefore we will remove the D5 branes for the time being. Two results are immediately obvious:
(a) Since the fourfold is a Calabi-Yau, the base will no longer remain a Calabi-Yau resolved conifold, and will have non zero first Chern class. The non zero first Chern class comes precisely from the seven branes in the picture. Although not Calabi-Yau, the base is still Kähler[12].
(b) In the presence of fluxes the metric of the fourfold only changes by an overall conformal factor and therefore modulo this subtlety, we can consider a Calabi-Yau fourfold for our case.

In the following we will also be able to shed light simultaneously on the fourfold *after* the geometric transition in type IIB theory.

We start by considering F theory on a base B (whose local metric is given in (2.2)), so that $X \to B$ is an elliptically fibered Calabi-Yau fourfold. Suppose that B contains a smooth curve $E \simeq P^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and that there is a conifold

[12] Thus when the seven branes are kept at large distances, the torus fibration will look effectively trivial in local region away from the seven branes. Here the manifold will effectively simulate vanishing first Chern condition. This will be useful later when we will try to give a topological theory description for our background.

transition from B to B' obtained by contracting the P^1 to a conifold and then smoothing. This gives another elliptically fibered Calabi-Yau fourfold $X' \to B'$.

We compute the Euler characteristic of X by topology. Recall that the Weierstrass model for X is obtained as a Calabi-Yau hypersurface in the projective bundle

$$P = \mathcal{P}(\mathcal{O}_B \oplus \mathcal{O}_B(2K_B) \oplus \mathcal{O}_B(3K_B)). \tag{2.3}$$

with equation

$$y^2 = 4x^3 - g_2 x - g_3 \tag{2.4}$$

where y is a coordinate on the bundle $\mathcal{O}_B(3K_B)$ and x is a coordinate on the bundle $\mathcal{O}_B(2K_B)$ (hence they live in the dual bundles $\mathcal{O}_B(-3K_B)$ and $\mathcal{O}_B(-2K_B)$ respectively). The third homogeneous coordinate z in the P^2 fiber corresponding to the factor $\mathcal{O}_B$ has been suppressed. We also have that g_k is a section of $\mathcal{O}_B(-2kK_B)$ for $k = 2, 3$.

The elliptic fiber is generically smooth, but is a cuspidal cubic over points of B where $g_2 = g_3 = 0$, and is a nodal cubic over points of B where the discriminant[13] $\Delta = g_2^3 - 27g_3^2 = 0$ but g_2 and g_3 are not simultaneously zero. The zero locus of Δ is a complex surface S containing the curve D defined by $g_2 = g_3 = 0$.

Since the Euler characteristic of a smooth cubic curve, nodal cubic curve, resp. cuspidal cubic curve are respectively $0, 1, 2$ we obtain

$$\chi(X) = 0 \cdot \chi(B - S) + 1 \cdot \chi(S - D) + 2 \cdot \chi(D) = \chi(S) + \chi(D). \tag{2.5}$$

We now make the following claim: $\chi(X') = \chi(X)$, i.e the Euler characteristics do not change under geometric transition.

To see this, note first that by the adjunction formula, K_B is trivial after restriction to E. That means that g_2 and g_3 restricted to E are sections of the trivial bundle, i.e. constants. Choosing these constants to be nonzero and generic, we see that $S \cap E$ is empty.

For $X' \to B'$ we adopt corresponding notation $S',\ C',\ g_2',\ g_3'$. We can also contract E to a conifold B_0 and adopt similar notation $S_0,\ D_0,\ (g_2)_0,\ (g_3)_0$. Note that g_2 and g_3 uniquely fix $(g_2)_0$ and $(g_3)_0$ by assigning to the conifold point the (constant) value of g_2 resp. g_3 on E. Since $S \cap E$ is empty, we learn that S_0 does not contain the conifold, hence $S_0 \simeq S$ and $D_0 \simeq D$. Thus $\chi(X) = \chi(X_0)$. Now we see that S' and D' are deformations

[13] Not to be confused with Δ_i, the warp factors appearing in the metrics.

of S_0 and D_0 which avoid the conifold, so have the same Euler characteristic. It follows that $\chi(X') = \chi(X_0) = \chi(X)$.

To illustrate this a bit more, let us consider an example. Consider a singular quadric hypersurface in P^4 with coordinates $(x_0 \ldots, x_4)$ defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0. \tag{2.6}$$

This is B_0. The point $(1,0,0,0,0)$ is the conifold. It has two Kähler small resolutions, related by flops. One of them, call it B, can be described by blowing up the plane defined by $x_1 = x_2 = 0$. This also leads to a toric description if desired. The conifold transition is completed by taking the equation of B' to be:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = tx_0^2 \tag{2.7}$$

for any nonzero value of t. The Euler characteristics can be computed by *schubert* [42], obtaining 19728 for $\chi(X)$ and $\chi(X')$. This is perfectly consistent with the result derived in [14] (see also [13]) where the F-theory background for a fourfold after geometric transition was derived. The resulting Euler characteristics evaluated there were precisely 19728, confirming the generic derivation presented here.

The above picture is not complete unless we devise a way to add five branes in the setup. This would mean that we take a non-compact base. The analysis of adding five branes to the picture has been given in [18]. The novelty of this derivation is that we are no longer required to consider a local picture where the fourfold is viewed as a product manifold (as in [18]). The full global picture implies the existence of seven branes on a six dimensional manifold that has a blown up P^1 with wrapped D5 branes on it. This therefore raises the question: can we show now that the correct local metric for our case is (2.2) as we have the full global picture at hand?

Knowing the local type IIB metric before the geometric transition will clarify many of the assumptions that we used in [18]. The metric that has been proposed in [21] is not the full global picture because it does not contain information about seven branes etc. in the metric (not to mention the issue of supersymmetry). It could at best be valid locally, but *not* in the global coordinates the metric has been formulated in. We will therefore rewrite the metric of [21] using only local coordinates. The proposed metric of D5 wrapped on a $\mathbf{P}^1$ of a resolved conifold is given by [21]:

$$
\begin{aligned}
ds^2 = h^{-1/2} ds^2_{0123} + h^{1/2}\Big[\gamma' d\tilde{r}^2 + \frac{1}{4}\gamma' \tilde{r}^2 (d\tilde{\psi} + \cos\,\tilde{\theta}_1 d\tilde{\phi}_1 + \cos\,\tilde{\theta}_2 d\tilde{\phi}_2)^2 + \\
+ \frac{1}{4}\gamma (d\tilde{\theta}_1^2 + \sin^2\,\tilde{\theta}_1 d\tilde{\phi}_1^2) + \frac{1}{4}(\gamma + 4a^2)(d\tilde{\theta}_2^2 + \sin^2\,\tilde{\theta}_2 d\tilde{\phi}_2^2)\Big],
\end{aligned}
\tag{2.8}
$$

where the coefficients etc. are defined in [21], [18]. Globally this metric breaks supersymmetry, and the harmonic function h doesn't contain the information about the seven branes (and orientifold seven planes). To rewrite the metric, let us define the coordinates in the following way:

$$
\begin{aligned}
&\tilde{\psi} = \langle\psi\rangle + \frac{2z}{\sqrt{\gamma_0'\sqrt{h_0}}}, && \tilde{\phi}_2 = \langle\phi_2\rangle + \frac{2y}{\sqrt{(\gamma_0 + 4a^2)\sqrt{h_0}}\ \sin\,\langle\theta_2\rangle} \\
&\tilde{\phi}_1 = \langle\phi_1\rangle + \frac{2x}{\sqrt{\gamma_0\sqrt{h_0}}\ \sin\,\langle\theta_1\rangle}, && \tilde{r} = r_0 + \frac{r}{\sqrt{\gamma_0'\sqrt{h_0}}} \\
&\tilde{\theta}_1 = \langle\theta_1\rangle + \frac{2\theta_1}{\sqrt{\gamma_0\sqrt{h_0}}}, && \tilde{\theta}_2 = \langle\theta_2\rangle + \frac{2\theta_2}{\sqrt{(\gamma_0 + 4a^2)\sqrt{h_0}}}
\end{aligned}
\tag{2.9}
$$

where $h_0 = h(r_0), \gamma_0 = \gamma(r_0)$ are constants measured at $r = r_0$. The coordinates $(x, y, \theta_1, \theta_2, z, r)$ are small deviations from their respective expectation values. Therefore writing the metric using these coordinates will only give information about a small local region. The metric after coordinate redefinitions (2.9) will take the following form:

$$
ds^2 = dr^2 + (dz + \Delta_1^0 \cot\,\langle\theta_1\rangle\ dx + \Delta_2^0 \cot\,\langle\theta_2\rangle\ dy)^2 + (d\theta_1^2 + dx^2) + (d\theta_2^2 + dy^2) + ... \tag{2.10}
$$

where we have absorbed $\gamma_0'\sqrt{h_0}$ in the defination of dr, and the dotted terms are higher orders in $(x, y, \theta_1, \theta_2, z, r)$. Furthermore the coefficients Δ_1^0, Δ_2^0 appearing in (2.10) are defined as:

$$
\Delta_1^0 \equiv \Delta_1(r_0) = \sqrt{\frac{\gamma_0'}{\gamma_0}}\ r_0\ , \qquad \Delta_2^0 \equiv \Delta_2(r_0) = \sqrt{\frac{\gamma_0'}{(\gamma_0 + 4a^2)}}\ r_0 \tag{2.11}
$$

Comparing the metric (2.10) with the one from [18], i.e (2.2), we see that (2.10) is the limit of (2.2) when $\theta_1 \to \langle\theta_1\rangle$ and $\theta_2 \to \langle\theta_2\rangle$ (along with $r = r_0$) in (2.2)! Thus (2.2) is the correct local description of D5 branes wrapped on $\mathbf{P}^1$ of a resolved conifold that would preserve supersymmetry[14]. Not only that, now we see why we expect two tori instead of two spheres in our metric (2.2). Locally a sphere is similar to a degenerating torus, and therefore expanding the sphere metric with coordinates (ϕ_i, θ_i) $(i = 1, 2)$ about a point $(\langle\phi_i\rangle, \langle\theta_i\rangle)$ we get the metric of a torus. In fact the tori in (2.10) are with complex structures $\tau_{1,2} = i$, which we boosted to obtain the appropriate large complex structure limit [18]. Finally, as we have argued earlier, (2.2) is also consistent with the T-dual brane picture [37], [36] where it is natural to obtain flat tori from brane configurations. Thus all of these fit perfectly now.

Before moving ahead, let us entertain another possible scenario. The metric (2.8) is non-supersymmetric, and therefore susceptible to quantum corrections. A generic quantum correction may change all the warp factors, and could even make the two spheres asymmetric (or even squashed). However if we remove the wrapped D5 branes, then the metric (2.8) is the metric of a resolved conifold that could become supersymmetric in the absence of fluxes. Question now is how robust is our local description (2.2)? Our local description along with the seven branes should preserve susy even in the presence of fluxes, as we argued above. To check this, let us assume that instead of (2.8) we start with another more generic metric whose explicit form can be written as

$$
\begin{aligned}
ds^2 = F_0(\tilde{r})\ ds^2_{0123} + F_1(\tilde{r})\ d\tilde{r}^2 + F_2(\tilde{r})\ (d\tilde{\psi} + \cos\ \tilde{\theta}_1 d\tilde{\phi}_1 + \cos\ \tilde{\theta}_2 d\tilde{\phi}_2)^2 + \\
+ \left[F_3(\tilde{r})\ d\tilde{\theta}_1^2 + F_4(\tilde{r})\ \sin^2\ \tilde{\theta}_1 d\tilde{\phi}_1^2\right] + \left[F_5(\tilde{r})\ d\tilde{\theta}_2^2 + F_6(\tilde{r})\ \sin^2\ \tilde{\theta}_2 d\tilde{\phi}_2^2\right]
\end{aligned}
\tag{2.12}
$$

where $F_i(\tilde{r})$ are the warp factors, and the two spheres are now asymmetric and squashed. The above metric (2.12) could either be interpreted as the deformation of (2.8) after quantum corrections, or as a generic type IIB metric on a resolved conifold with wrapped D5 branes. Whatever be our interpretation, far away from the wrapped D5 branes we should simply see the resolved conifold part, as the effect of wrapped D5 branes would be negligible. Furthermore primitivity is also restored at large distances, and this would imply that the warp factors should approach the values set by (2.8) with $h(\tilde{r}) \rightarrow 1$. We are of course not so concerned about the global behavior as we want to study the local

[14] Thus the replacement $\Delta_i^0(r_0)\ \cot\ \langle\theta_i\rangle \rightarrow \Delta_i(r_0)\ \cot\ \theta_i$ in (2.10) is the local back-reactions of the added seven branes to finally convert (2.10) to (2.2) and preserve supersymmetry. When the F-theory torus do not degenerate in local neighborhood, like what we took in [18], then the metric (2.2) has very small θ_i dependences which means we are in the limit (2.10). Thus $\langle\theta_i\rangle \rightarrow \theta_i$ keeping $r = r_0$, is exactly what we meant by *delocalization* in [18].

metric so that we can put in D7 branes and O7 planes in the system. The local metric can be easily extracted from (2.12) using the small expansion trick that we had devised earlier. In the present case, the small expansion will be

$$\begin{aligned} \tilde{\psi} &= \langle\psi\rangle + \frac{z}{\sqrt{F_2(r_0)}}, & \tilde{\phi}_2 &= \langle\phi_2\rangle + \frac{y}{\sqrt{F_6(r_0)}\,\sin\langle\theta_2\rangle} \\ \tilde{\phi}_1 &= \langle\phi_1\rangle + \frac{x}{\sqrt{F_4(r_0)}\,\sin\langle\theta_1\rangle}, & \tilde{r} &= r_0 + \frac{r}{\sqrt{F_1(r_0)}} \\ \tilde{\theta}_1 &= \langle\theta_1\rangle + \frac{\theta_1}{\sqrt{F_3(r_0)}}, & \tilde{\theta}_2 &= \langle\theta_2\rangle + \frac{\theta_2}{\sqrt{F_5(r_0)}} \end{aligned} \tag{2.13}$$

where it is assumed that none of the warp factors $F_i(\tilde{r})$ would vanish near $\tilde{r} \rightarrow r_0$. Furthermore our small expansion for the warp factor

$$F_i(\tilde{r}) \;=\; F_i(r_0) \;+\; \frac{r}{\sqrt{F_1(r_0)}}\,\frac{\partial F_i}{\partial \tilde{r}}\Big|_{\tilde{r}=r_0} \approx F_i(r_0) \;\; \text{for} \;\; r \rightarrow 0 \tag{2.14}$$

is thus justifiable. Therefore plugging (2.13) into (2.12), we can easily show that the local metric is exactly of the form (2.10) with the only difference being that the Δ_i^0 are now defined as

$$\Delta_1^0 \;=\; \frac{F_2(r_0)}{\sqrt{F_4(r_0)}}, \qquad \Delta_2^0 \;=\; \frac{F_2(r_0)}{\sqrt{F_6(r_0)}} \tag{2.15}$$

which reduce to (2.11) once we remove the quantum corrections.

Our small calculation discussed above actually has some profound consequences. It tells us that (2.2) is not only the right local metric for our case, but also that (2.2) is not affected by the details of the quantum corrections envisioned on (2.8). All the possible corrections simply change the warp factors Δ_1 and Δ_2. But there is an even more intriguing possibilty related to footnote 5 that we mentioned earlier. Imagine we do manage to find a supersymmetric wrapped D5 brane system. The new metric should look like (2.12) precisely because (2.12) captures all the essentials of a metric of D5 wrapped on two cycles of a resolved conifold. Thus the quantum corrected metric for (2.8) and the expected susy solution will bear strong resemblance. However in both cases the local metric is exactly (2.2) so our local description with the metric (2.2) will not change and we can consider seven branes in the system to study $\mathcal{N} = 1$ gauge theory with flavors. Integrating out the flavors (i.e seven branes are far away or non-degenerating F-theory torus of [18]) is equivalent to considering the metric (2.10) locally and this is what was taken in [18][15].

[15] To make this precise, the $(r_0, \langle\theta_i\rangle)$ in eq. 5.2 of [18] is the $(r_0, \langle\theta_i\rangle)$ of (2.9) here also.

2.2. Field Theory Interpretation

In geometric transitions, gluino condensates are identified with sizes of geometrical cycles. In the global picture discussed in the previous section, this identification will be between the gluino condensate describing the IR of the $\mathcal{N}=1$ $SU(N)$ theory and the size of the S^3 cycle in the deformed geometry.

To understand what happens in the local picture let us start from the discussion of type I in [43]. On one wrapped D5 brane the field theory is $\mathcal{N}=1$ $Sp(2)$ with some twisting of the normal bundle in order to account for the wrapping on a 2–cycle. It also contains 32 $Sp(2)$ "half-hypermultiplets" which will become massive due to Wilson lines. After a T-duality to type IIB, we get a D5 wrapped on a P^1 cycle. If one concentrates on the vicinity of a single fixed point and the D7 branes which are located around this fixed point, this gives rise to an $Sp(2)$ or $SU(2)$ theory with 4 massive flavors and a superpotential

$$W = \sum_i m_i Q_i \bar{Q}_i \ + \ Q_i A \bar{Q}_i, \qquad\qquad i = 1, \cdots, 4 \tag{2.16}$$

Because of the Wilson lines in type I, the gauge group $Sp(2)$ breaks to $U(1)$ and the masses of the fundamental flavors will change as $m_i \pm a$, where a is the expectation value of an adjoint Higgs field.

The difference between global and local picture then amounts to whether or not we consider the fundamental flavors to be part of the gauge theory[16]. If we integrate out the flavors, then the IR gauge coupling of the effective theory will be constant.

If we do not integrate out the flavors, the gauge coupling constant will be given by

$$\tau(z) \ = \ \tau_0 + \frac{1}{2\pi i}\left(\sum_{i=1}^{4} \log\,(z - m_i^2) - 4\log\, z\right) \tag{2.17}$$

where z is the complex direction orthogonal to the D7 branes. As τ depends on the size of cycles in the metric (2.1), functions like g_2 are part of τ and will depend nontrivially on z, too. But the D7 branes are extended in directions both parallel and orthogonal to the P^1 cycle in the resolved conifold, therefore they will also be both parallel and orthogonal to the S^3 cycle in the deformed conifold. The coupling τ will then depend on both the

[16] The other flavors corresponding to the other three fixed points are already integrated out from an even richer theory. We will consider an intermediate scale to accomodate only one flavor from a specific fixed point.

integral over A-cycle and B-cycle. The identification between τ and $\int (\chi + ie^{-2\phi}) \ H_{NS}$ will change if τ is given by (2.17). This is to be compared with the case in [16] where the flavors were introduced as infinite D5 branes wrapped on 2-cycles which do not intersect the P^1 cycles and which become part of the S^3 after the transition.

This modifies the approach of [3] if the flavors introduced by D7 branes are still part of the theory, but the original geometric transition (on the level of the effective field theory) is obtained if the flavors are integrated out.

In the case of N D5 branes in type I (and several D5 branes in IIB), we find gauge group $Sp(2N)$, which is then broken by the adjoint Higgs field to $SU(N)$. Again, the usual geometric transition arises if all massive flavors are integrated out.

The local picture constructed in [18] and in [19] captures almost all the details of geometric transitions envisioned by [3], as in this limit the effect of D7/O7 branes is invisible. If the flavors corresponding to D7 branes are integrated in, then the picture of [3] has to be extended accordingly to accomodate the flavor degrees of freedom. In this paper, unless mentioned otherwise, we will stick with the simplest model where the D7 branes and orientifold seven planes are far from the resolution two-cycle of our manifold.

2.3. Topological aspects and excursions into generalized geometries

The analysis of supersymmetry in type IIB theory that we gave above therefore relies heavily on the correct choices of NS and RR fluxes (including the RR fluxes from seven branes). On the other hand, the topological sector of this model, which would be — say after geometric transition — a B-model on a deformed conifold has *no* dependence on the RR fields. Does that mean that we are allowed to choose any arbitrary background H_{RR} fields? We will at least show that the RR fields are quantized without leaning on the assumption that they are governed by the number of branes before geometric transition (because that would already assume correctness of the geometric transition).

An alternative verification of the quantization of RR fields can be done by considering which correlation function we expect to measure. Taking into account the H_{NS} fields, the fermionic part of a Green-Schwarz superstring can be written as

$$S^g_{\text{fermionic}} = 4i\psi^p \Delta_+ \psi^p + 4i\psi^{\dot{q}} \Delta_- \psi^{\dot{q}} + R_{ijkl}\sigma^{ij}_{\dot{p}\dot{q}}\sigma^{kl}_{rs}\psi^{\dot{p}}\psi^{\dot{q}}\psi^r\psi^s \tag{2.18}$$

where $(\psi^p, \psi^{\dot{q}})$ are the two inequivalent spinor representations of the transverse D_4 and the sigma matrices are defines as $\sigma^{ij}_{\dot{p}\dot{q}} \equiv \Gamma^{[i}_{r[\dot{p}}\Gamma^{j]}_{\dot{q}]r}$ with a similar definition for the other components. The Gamma matrix has 8×8 blocks given as $\Gamma^i \equiv \begin{pmatrix} 0 & \Gamma^i_{p\dot{q}} \\ \Gamma^i_{\dot{r}s} & 0 \end{pmatrix}$ which are used to define the σ's above. In this notation we have to specify what we mean by the covariant derivative $\Delta_\pm$ and R_{ijkl}. The most generic definition of the covariant derivative will be given as [44],[45]:

$$\Delta_\pm \psi^{q(\dot{q})} = \partial_\pm \psi^{q(\dot{q})} + \frac{1}{2}\left(\omega - \frac{1}{2}H\right)^{ab} \sigma_{ab}^{pq(\dot{p}\dot{q})} \psi^{p(\dot{p})} \tag{2.19}$$

where $\frac{1}{2}H$ forms the torsion and we have chosen the torsional connection $\omega_+ \equiv \omega - \frac{1}{2}H$, and not the other one i.e ω_- (see [45] for details on this). Observe that in the absence of H_{NS} there is no such ambiguity and the GS superstring can be defined unambiguously.

The curvature R_{ijkl} that we defined above is actually measured w.r.t. the connection ω_+. We could also measure the curvature w.r.t. the other connection ω_-. If we define $R^+_{ijkl} \equiv R_{ijkl}$ and R^-_{ijkl} as the other curvature[17], then the following identity can be easily verified:

$$R^+_{ijkl} = R^-_{ijkl} - 2H_{[ijk,l]} \tag{2.20}$$

which also serves as the definition of the torsion [44], [45], [46]. We want to concentrate on the topological nature of these models.

An efficient way to study the topological theory is to convert the GS fermions into NS forms, i.e measure the fermions as worldsheet fermions with (p,q) $U(1)$ charges. This will elucidate the (2,2) nature of the lagrangian. Thus we will have

$$(\psi^p, \psi^{\dot{q}}) \rightarrow (\psi^i_\pm, \bar{\psi}^{\bar{j}}_\pm) \in \left(\left(K_\Sigma^{1/2}, \bar{K}_\Sigma^{1/2}\right) \otimes \phi^*\left(T^{(1,0)}\mathcal{M}\right), \left(K_\Sigma^{1/2}, \bar{K}_\Sigma^{1/2}\right) \otimes \phi^*\left(T^{(0,1)}\mathcal{M}\right)\right) \tag{2.21}$$

where Σ is the two dimensional world-sheet and the background manifold is given by $\mathcal{M}$ with a particular choice of almost complex structure. Therefore the 2d local complex coordinates would be $(z, \bar{z})$ and the corresponding 6d ones would be $(T^{(1,0)}\mathcal{M}, T^{(0,1)}\mathcal{M})$. The manifold $\mathcal{M}$ therefore admits an almost complex structure which may or may not be integrable. In type IIB $\mathcal{M}$ will be a *conformally* Kähler manifold with an integrable complex structure [19], whereas in type IIA, $\mathcal{M}$ will in general be a non-Kähler manifold

[17] The definition of these curvatures are $R^{(\pm)i}_{jkl} = \partial_k \Gamma^{(\pm)i}_{lj} + \Gamma^{(\pm)i}_{km}\Gamma^{(\pm)m}_{lj} - (k \leftrightarrow l)$ where $\Gamma^\pm$ is the torsional connection.

with a non-integrable complex structure [18]. The sigma model action can now be re-written from (2.18) to

$$S^n_{\rm fermionic} = 2i(g_{i\bar{j}} + B_{i\bar{j}})\left(\bar{\psi}^{\bar{j}}_- \Delta_+ \psi^i_- + \bar{\psi}^{\bar{j}}_+ \Delta_- \psi^i_+\right) + R^+_{i\bar{j}k\bar{l}} \psi^i_+ \psi^k_- \bar{\psi}^{\bar{j}}_+ \bar{\psi}^{\bar{l}}_- \tag{2.22}$$

where the curvature etc. are measured w.r.t. the right connection. Observe the appearance of $\mathbf{g} + \mathbf{B}$ in the above action. This takes care of the torsion.

The topological twist in the absence of torsion acts on the bundle in the following way: it converts one of the fermion to a world-sheet one-form and the other fermion to a world-sheet zero form [47]. In particular its action is:

$$(\psi^i_+, \psi^i_-) \quad \to \quad (\psi^i_z, \chi^i) \in \left(K_\Sigma \otimes \phi^*(T^{(1,0)}\mathcal{M}), \phi^*(T^{(1,0)}\mathcal{M})\right) \tag{2.23}$$

with a similar kind of action for the other pair. This implies that the correlation function of operators $< \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 .... \mathcal{O}_n >$ is given by the integral of the top form on the manifold $\mathcal{M}$. Generically, operators in the twisted sigma model are of the form[18]

$$\mathcal{O}_f = f_{i_1...i_p\bar{j}_1...\bar{j}_q} \chi^{i_1}...\chi^{i_p} \chi^{\bar{j}_1}...\chi^{\bar{j}_q} = f_{i_1...i_p\bar{j}_1...\bar{j}_q} dZ^{i_1}...dZ^{i_p} d\bar{Z}^{j_1}...d\bar{Z}^{j_q} \tag{2.24}$$

which is basically a (p,q) form on the manifold $\mathcal{M}$ because $(Z^i, \bar{Z}^j)$ are the complex coordinates on the manifold determined by an almost complex structure.

In the presence of torsion the issue of a topological twist is much more complicated. (2,2) nonlinear sigma models (before twist) with $H \equiv H_{NS} = dB \neq 0$ were first discussed in [49]. In fact the earlier mentioned two different spin-connections $\omega_\pm$ will become useful now[19]. Due to the existence of $\omega_\pm$, there are two different complex structures compatible w.r.t. to the choices of torsional spin-connections [49]. These complex structures can then be used to decompose the fermionic components of the (2,2) sigma model [50]. The zero mode distribution will be similar to the case without torsion:

$$k_\mp \equiv \text{dim ker } \Delta_\pm - \text{dim ker } \Delta^\dagger_\pm \tag{2.25}$$

[18] For a review of topological correlation function the readers may want to refer to the original work of Witten [47], or some of the recent review papers [48].

[19] The heterotic side of the story is completed in [45],[46].

implying vanishing first Chern-classes[20]. Once such constraints are observed, the twisting proceeds in the standard way (see [50] for recent discussion where the techniques of [51] is used to do the twisting). A generic operator in such a twisted model looks similar to (2.24), and is therefore related to the cohomological states of the underlying manifold.

The above form of the correlation function and the operators solve the RR–flux question we raised. Since the correlators involve (p, q) forms, they are related to $H^{p,q}$ of the manifold $\mathcal{M}$. Its elements are quantized and normalizable forms of the manifold, and therefore the RR forms will be given by quantized normalizable three forms. And indeed, these forms are related to the *number* of wrapped D5 branes in the dual picture[21].

The analysis presented so far only skims the surface of a much more rigorous structure. Not only the twisting can be elaborated for sigma models with $(2, 2)$ supersymmetry, but they also have a deeper connection to generalized complex structures developed by Hitchin [52] and Gualtieri [53]. These will be studied in the sequel to this paper. We will also try to connect the type II solutions that we presented in [18] and [19] with the Hitchin–Gualtieri system. A more recent paper analyzing some aspect of this connection is [50].

Before we proceed with the analysis of topological aspects of $(2, 2)$ models, let us make a few observations. These will be useful to relate the analysis to heterotic $(0, 2)$ models. First, the $(2, 2)$ action can be written in a concise way using superfield notation as [49]:

$$S = \int d^2\sigma d^2\theta \Big[g_{ij} D^\alpha \mathbf{\Phi}^i \ D_\alpha \mathbf{\Phi}^j + B_{ij} D^\alpha \mathbf{\Phi}^i \ (\sigma^3 D)_\alpha \mathbf{\Phi}^j\Big] + ... \tag{2.26}$$

where σ^3 is the third Pauli matrix and the dotted terms involve the four–fermi terms. As is clear from the above action, the superfields $\mathbf{\Phi}^i$, $i = 1, .., d$ realize only the $(1, 1)$ part of the action. Additional supersymmetry can be implemented w.r.t. the two allowed choices of the complex structure $J^i_{\pm j}$ as[22]

$$\delta_\epsilon \mathbf{\Phi}^i = -i(\epsilon_+ D_- \mathbf{\Phi}^j) J^i_{+j} + i(\epsilon_- D_+ \mathbf{\Phi}^j) J^i_{-j} \tag{2.27}$$

[20] This doesn't mean that the target manifold has to be Kähler. In [45], [46] numerous non-Kähler manifolds have been constructed with vanishing first Chern class.

[21] The dual picture that we are talking about here is only the far IR story *a-la* [1]. We can also build a setup with spatially varying NS and RR fluxes (or infinite sequences of flop transitions in the dual) like [12] but will not do so here.

[22] We are using both $D_\pm$ and $\Delta_\pm$ to write the derivatives in a sigma-model. The distinction between them should be clear from the context.

(2.27) is a combination of two different variations: one for chiral superfields in a (2,2) sigma model, and the other one for a twisted[23] chiral superfield (see sec. 8 of [49]). This means that the very existence of B_{ij} fields in the background implies a (2,2) sigma model action *coupled* to twisted chiral fields[24]. This is basically the content of the action (2.26) as twisted chiral fields (say η^p) couple with chiral fields (say Φ^q) as [49]:

$$S_{\text{coupling}} = \int d^2\sigma d^2\theta \Big(K_{p\bar{q}} (D^a \Phi^p)(\sigma^3 D)_a \bar{\eta}^{\bar{q}} + K_{\bar{p}q} (D^a \bar{\Phi}^{\bar{p}})(\sigma^3 D)_a \eta^q \Big) \tag{2.28}$$

which is precisely the B field coupling in (2.26) if we interpret $\mathbf{\Phi}^i$ as a superfield with both chiral and twisted chiral components.

The above observation has in fact interesting consequences for the connection of this model with generalized complex geometry developed recently by Hitchin [52] and Gualtieri [53]. This connection is explored in a series of papers [55],[56]. We only give a brief account here and leave the detailed discussion for future work. The starting point is the observation repeatedly mentioned above, namely: the action (2.26) as constructed naively shows (1,1) world–sheet supersymmetry. The additional supersymmetry is implemented via the transformation (2.27) which eventually enhances it to (2,2).

Where does the mathematical structure of generalized complex geometry fit into this? Note that once we have the (2,2) action in this form we can easily relate it to the fermionic term in (2.18) or (2.22). Defining the bosonic part of $\mathbf{\Phi}^i$ as X^i, the action (2.26) takes the following form:

$$S = \frac{1}{8\pi\alpha'} \int d^2\sigma \Big[(g_{ij} + B_{ij}) \partial_+ X^i \partial_- X^j + \frac{1}{4} S^g_{\text{fermionic}} \Big]. \tag{2.29}$$

---

[23] Not to be confused with the topological twist that we perform on this model.

[24] As is obvious from the above analysis, in the absence of B-field the target manifold has to be Kähler for (2.27) to exist [54]. The presence of B fields allows two different complex structures $J^i_{\pm j}$ that are covariantly constant w.r.t. to the two connections $\omega_\mp$, respectively. The two complex structures appearing in the susy variation (2.27) are perfectly consistent with the fact that chiral and twisted chiral superfield susy transformations require two *different* complex structures. These two choices of complex structure differ by the Pauli matrix σ^3, and this is the reason why σ^3 appears in the sigma-model action (2.26). All these details fit nicely with the expected theory of chiral and twisted-chiral superfields, as shown in [49]. Furthermore, as noted in [50], this bi–Hermitian structure satisfies the definition for a twisted generalized Kähler target [53].

Written this way, the action will require no other corrections[25] and consequently be anomaly free. Observe that the right moving sector has eight fermions denoted as ψ^p in (2.18). On the other hand, the left moving sector also has eight fermions denoted as $\psi^{\dot{q}}$. Together they give rise to the (2,2) world sheet action. The bosonic part (i.e. the X^i part) of the above action can be re–written in the following way:

$$S_{\text{bosonic}} = \int p_\mu \wedge dX^\mu + \frac{1}{2}\theta^{\mu\nu}\eta_\mu \wedge \eta_\nu + \frac{1}{2}G^{\mu\nu}\eta_\mu \wedge *\eta_\nu + \frac{1}{2}B \tag{2.30}$$

where η_μ is a one–form on the world–sheet and $G_{\mu\nu}, \theta_{\mu\nu}$ are related to the *open–string* data of a Seiberg–Witten non–commutative theory [57]. The above form of the action first appeared in [58] and is subsequently used by [55].

Supersymmetrizing (2.30) is not difficult. The original action (2.29) is supersymmetrized using $S^g_{\text{fermionic}}$ terms. Here we see that susy requires the following substitutions

$$\partial_\pm X^i \;\; \rightarrow \;\; D_\pm \mathbf{\Phi}^i, \qquad \eta_\mu \;\; \rightarrow \;\; \Psi_{\pm\mu}, \qquad (G_{\mu\nu}, \theta_{\mu\nu}) \;\; \rightarrow \;\; \Big(E_{(\mu\nu)}, E_{[\mu\nu]}\Big). \tag{2.31}$$

We can now write the most generic action using the ingredients of (2.31) and (2.30). This will typically look like:

$$S_{\text{gen}} = \int d^2\sigma d^2\theta \Big(a_1\; D_+\mathbf{\Phi}^\mu \Psi_{-\mu} + a_2\; D_-\mathbf{\Phi}^\mu \Psi_{+\mu} + a_3\; \Psi_{+\mu}\Psi_{-\nu}E^{\mu\nu} + a_4\; D_+\mathbf{\Phi}^\mu D_-\mathbf{\Phi}^\nu B_{\mu\nu}\Big) \tag{2.32}$$

where a_i are some unknown coefficients. In writing the above action we strictly followed (2.30). This would explain the non–existence of terms like $D_+\mathbf{\Phi}^\mu D_-\mathbf{\Phi}^\nu E_{\mu\nu}$ in the action. In [55] it was pointed out that that to achieve (1,1) supersymmetry, the coefficients in (2.32) have to be precisely

$$a_1 \; = \; 1, \qquad a_2 \; = \; -1, \qquad a_3 \; = \; -1, \qquad a_4 \; = \; 1. \tag{2.33}$$

Above technique therefore demonstrates a simple way to obtain a supersymmetric action from the bosonic action of [58]. The next question will is: under what conditions does (2.32) have (2,2) supersymmetry? Since we have already fixed the unknown coefficients a_i, the only other way is to look for target space geometry. This is exactly where the mathematical construction of generalized complex geometries fits in! It turns out, as was discussed by Lindstrom in [55] (see the first paper in the list), the target manifold has to

[25] For example Chern-Simons corrections.

be a bi–Hermitian or, equivalently, twisted generalized Kähler manifold [53] to attain the full (2,2) world–sheet supersymmetry.

The discussion how the manifolds constructed in [18] and [19] fit into this (twisted) generalized complex framework will be left for the sequel to this paper. Our aim in the next section is to study the heterotic side of the story, i.e. (0,2) models. We will be able to provide a full global description of the particular setup initiated in [19]. But before doing so, let us conclude this section by pointing out one obvious result: integrating out $\Psi_{\pm\mu}$ in (2.32) reproduces back (2.29).

3. Global Heterotic models

In the type II analysis in [18], [19] and above we showed that the global descriptions for the geometric transition backgrounds are particularly involved. In type IIB we gave a local metric, and the global background is a Kähler manifold with additional D7 branes and O7 planes. On the other hand, the type IIA background is locally and globally non–Kähler. We now want to discuss the type I/Heterotic models whose local descriptions were presented in [19]. These backgrounds turned out to be, not surprisingly, non–Kähler (but complex) manifolds. In the following we present global descriptions of these models.

3.1. Heterotic (0,2) models: brief discussion

To develop (0,2) models in the context of complex structures and not generalized complex structures, we have to go back to (2.29). In this action we have the freedom to add non–interacting fields. This ruins the carefully balanced (2,2) supersymmetry of this model. We can use this to our advantage by adding non–interacting fields *only* in the left–moving sector. This breaks the left moving supersymmetry, and one might therefore hope to obtain an action for (0,2) models from (2.29), at least *classically*. On the other hand, a possible (0,2) action is also restricted because this will be the action for heterotic string. It turns out, there are few allowed changes one can do to find the classical (0,2) action from a given (2,2) action:

- Keep the right moving sector unchanged, i.e. ψ^p remain as before.
- In the left moving sector, replace $\psi^{\dot{q}}$ by eight fermions Ψ^a, $a = 1, ...8$. Also add 24 additional non–interacting fermions Ψ^b, $b = 9, ...32$.
- Replace ω_+ by gauge fields A, i.e. embed the *torsional* spin connection into the gauge connection.

The above set of transformations will convert the classical (2,2) action given in (2.29) to a classical (0,2) one. One might, however, wonder about the Bianchi identity in the heterotic theory. The type IIB three–form fields are closed, whereas heterotic three–form fields satisfy the Bianchi identity. One immediate reconciliation would be that because of the *embedding* $\omega_+ = A$, the heterotic three–form should be closed. This may seem like an admissible solution to the problem, but because of subtleties mentioned in [45], [59], [60] the above embedding will not allow any compact non–Kähler manifolds to appear in the heterotic theory. Therefore, as a first approximation, we will assume an embedding of the form

$$\omega_+ \ = \ A \ + \ \mathcal{O}(\alpha'). \tag{3.1}$$

Using this the new action with (0,2) supersymmetry becomes:

$$\begin{aligned} S = \frac{1}{8\pi\alpha'}\int d^2\sigma \Big[&(g_{ij}+B_{ij})\partial_+ X^i \partial_- X^j + i\psi^p(\Delta_+\psi)^p + i\Psi^A(\Delta_-\Psi)^A + \\ &+ \ \frac{1}{2}F_{ij(AB)}\sigma^{ij}_{pq} \ \psi^p\psi^q\Psi^A\Psi^B + \mathcal{O}(\alpha')\Big] \end{aligned} \tag{3.2}$$

where due to the embedding (3.1), F^a_{ij} forms the Yang–Mills field strength measured w.r.t. Lie algebra matrices T^a_{AB}. The fermion indices are $A = 1, ..., 32$, which means there are 32 fermions, and hence T^a form tensors of rank 16. The reader can easily identify the above action as an action for a heterotic sigma model with torsion [44], [45], [46], [61], [62], [63], [64]. The action of the Laplacian (2.19) changes accordingly to

$$\begin{aligned} &\Delta_-\Psi^A = \partial_-\Psi^A + A^{AB}_i(\partial_- X^i) \ \Psi^B \\ &\Delta_+\psi^p = \partial_+\psi^p + \frac{1}{2}(\omega_+)^{ab}\sigma^{pq}_{ab}\psi^q \\ &H_{ijk} = \frac{1}{2}\left(B_{ij,k} + B_{jk,i} + B_{kj,i}\right) \ . \end{aligned} \tag{3.3}$$

As expected, this set of actions (3.3) determines the (0,1) supersymmetric heterotic sigma model [62]. This is similar to the (1,1) action for the type II case. The full (0,2) susy will be determined by additional actions on the fields (exactly as for the (1,1) case before).

The above set of manipulations that convert a classical (2,2) action to a classical (0,2) one will help us to understand various things about the heterotic theory using data of type II theories. In particular we would like to ask the following questions:

• Can we use this to find new torsional background in heterotic theory? In [19] we provided a possible background in heterotic theory which *might* show geometric transition. However, this background was duality chased from the orientifold corner of type IIB theory.

This means, in particular, that quantum corrections would modify both type II as well as the heterotic background once we shift the type IIB background from the orientifold point. Although it is possible to infer the quantum corrections to the type IIB background (i.e. from F-theory), the corresponding correction in the heterotic picture is not easy to determine. Therefore, using above manipulations we might be able to infer a possible background in heterotic theory that would show geometric transition.

• From above manipulations we also observe that both metric and B_{ij} fields can be taken to the heterotic side. However, this is only possible if the original (2,2) background does not have any H_{RR} fields. In the presence of H_{RR} the simple manipulations that we performed cannot give a (0,2) or a (0,1) model. We are therefore particularly interested in type II models that allow only for an NS three–form, like MN [2]. When both NS and RR backgrounds are present, it might still be possible to perform above manipulations if one can find an equivalent U–dual background. This U–dual background will in general not be Kähler (not even conformally Kähler). Observe that these U-dualities *do not* require the original background to be at the orientifold point. This is therefore different from the analysis performed in [4], [23][26].

• Once a particular heterotic background is found one has to address the issue of vector bundles[27]. Two sets of questions arise:

(a) What are the allowed vector bundles on the manifold? How do we study the stability of these bundles?

(b) How do we pull bundles through a geometric transition (or conifold transition)?

The first issue, of the existence of the bundle, can be inferred from the detailed analysis given (at least for the $U(1)$ case) in [46], [66]. However, the situation here may become a

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[26] Another interesting question would be to allow for both NS and RR background in the S–dual type I picture. This is a highly restrictive scenario as the allowed values of NS fluxes in the S–dual type I picture are only two discrete choices [65].

[27] The vector bundles that we are interested in are a subgroup of the $SO(32)$ bundle of the heterotic theory. In fact, there are two possible bundles here. If the proposed heterotic background is dual to an $SU(N)$ gauge theory, where N is the number of wrapped NS5 branes, then the $SU(N)$ gauge group may also appear in the geometry. This would be a much more involved case. If a dual theory exists, we have to determine the full UV behavior of this theory. We can assume (as in the type II cases) that the geometric dual just describes the far IR of a much more detailed theory, and − if we also assume that the theory will eventually confine − then $SU(N)$ will be broken. This means we only have to consider $SO(32)$ (or possibly its subgroup because of the embedding (3.1)). More details will follow.

little simpler than the one in [46], because of the embedding (3.1). This embedding implies that for a given complex structure J

$$i\partial\bar{\partial}J \;=\; 0+\mathcal{O}(\alpha')\,. \tag{3.4}$$

This means we can study the stability of bundles using the recent analysis of Li and Yau [67] and Fu and Yau [68]. The second issue, of pulling the bundles through a conifold transition, is important if we want to present a heterotic model that does show geometric transition. In [46] it was shown how bundles can be pulled through flops. Here we would like to study a more subtle phenomena: the analysis of the bundles after geometric transition.

• Our manipulations can now be used to understand half–twisted sigma models from type II twisted models. The half–twist is performed only to the right moving sector of the sigma model (3.2). This was presented first in [51]. Our aim is to understand in more detail the Chiral de Rham complex CDR [69]. Because of the embedding (3.1), we expect a connection between the chiral differential operators CDO [69], [70] and the chiral de Rham complex. This issue has also recently been addressed by Witten [33]. CDO's are relevant for the study of conformal field theories for models that are *twisted* or *half-twisted*. Therefore, all the relevant operators in the theory are written as BRST exact operators w.r.t. the BRST charge Q. In this way the CFT (which is also a TFT) is different from the usual CFT for string sigma models. As discussed in [33], under the embedding (3.1), the CDO is then a half–twisted version of a sigma model with (2,2) supersymmetry called the CDR.

3.2. Heterotic (0,2) models: detailed discussion

To use the sigma-model analysis that we presented in the above section, we have to determine a particular heterotic background and show that such an equivalent background can also appear in the type IIB set-up. In our earlier paper [19] we provided a heterotic background that has non-trivial dilaton, metric and torsion. The explicit vielbeins for this background *after* geometric transition were given in [19] as:

$$\begin{aligned}
&e^1 = (h_2+a_2^2h_1)^{-\frac{1}{2}}(C'D')^{-\frac{1}{4}}(dy-b_{yj}\;d\zeta^j), \qquad e^2 = (h_4+a_1^2h_1)^{-\frac{1}{2}}(C'D')^{-\frac{1}{4}}(dx-b_{xi}\;d\zeta^i)\\
&e^3 = \frac{1}{\sqrt{2}}(h_3h_4)^{\frac{1}{2}}(C'D')^{-\frac{1}{4}}(d\theta_1+\gamma_4 d\theta_2), \qquad e^4 = \frac{1}{\sqrt{2}}(h_3h_4)^{\frac{1}{2}}(C'D')^{-\frac{1}{4}}(d\theta_1+\gamma_5 d\theta_2)\\
&e^5 = \gamma'^{\frac{1}{2}}H^{\frac{1}{4}}(C'D')^{-\frac{1}{4}}dr, \qquad e^6 = h_1^{\frac{1}{2}}(C'D')^{-\frac{1}{4}}dz
\end{aligned} \tag{3.5}$$

where, as one might recall from [19], the vielbeins e^1 and e^2 are simple because Re $\tau_1 = 0$, whereas the vielbeins e^3 and e^4 contain mixed components because Re $\tau_2 \neq 0$, where

$$d\chi_1 \equiv dx + \tau_1 dy, \qquad d\chi_2 \equiv d\theta_1 + \tau_2 d\theta_2. \tag{3.6}$$

It was pointed out that for the background after transition the complex structure is specified by Re $\tau_1 = 0$, Re $\tau_2 \neq 0$, whereas the opposite holds true for the background before transition. The variables γ_4 and γ_5 appearing in (3.5) are defined as:

$$\gamma_4 = \text{Re } \tau_2 \pm \text{Im } \tau_2, \quad \gamma_5 = \text{Re } \tau_2 \mp \text{Im } \tau_2 \tag{3.7}$$

with τ_2 defined above in (3.6). Similarly the other variables were given in [19] as:

$$\begin{aligned}
& h_1 = \frac{e^{-2\phi}}{\alpha_0 CD}, \quad h_2 = \alpha_0(C + e^{2\phi}E^2), \quad h_4 = \alpha_0(D + e^{2\phi}F^2) \\
& h_3 = \frac{C - \beta_1^2 E^2}{\alpha_0(D + e^{2\phi}F^2)}, \ a_1 = -\alpha_0 e^{2\phi} ED, \ a_2 = -\alpha_0 e^{2\phi} FC, \ \alpha = \frac{1}{1 + E^2 + F^2} \\
& \alpha_0 = \frac{1}{CD + (CF^2 + DE^2)e^{2\phi}}, \quad \beta_1 = \frac{\sqrt{\alpha_0} e^{2\phi}}{\sqrt{e^{2\phi}CD - (C + e^{2\phi}E^2)(1 - D^2)F^{-2}}} \\
& C_\pm \ = \ \frac{\alpha}{2}\left(1 + F^2 \pm \frac{EF}{\sqrt{(1+E^2)/(1+F^2)}}\right), \quad C \ = \ C_-, \quad D \ = \ C_+ \\
& E = \Delta_1 \ \cot\theta_1, \quad F = \Delta_2 \ \cot\theta_2, \quad r \ = \ r_0, \quad \theta_i \ = \ \text{arbitrary}
\end{aligned} \tag{3.8}$$

where h_i, C, D and the dilaton ϕ are all evaluated at fixed radial coordinates $r = r_0$ and C', D' are defined for fixed $\theta_i = \langle\theta_i\rangle$ also. $\gamma(r^2)$ satisfies $3\gamma'\gamma(\gamma + 4a^2) = 2r^2$ where $\gamma' = \partial\gamma/\partial r^2$ (again both measured at $r = r_0$). H is a warp factor due to D5 and D7-branes as well as O7–planes, but it was not further specified in [19]. Using the above definitions the metric after geometric transition is simply $ds^2 = \eta_{ab}e^a e^b$ with $e^a = e^a_\mu dx^\mu$. This is equivalent to:

$$ds^2 = \mathcal{A}_1 \ dz^2 + \mathcal{A}_2 \ (dy - b_{yj} \ d\zeta^j)^2 + \mathcal{A}_3 \ (dx - b_{xi} \ d\zeta^i)^2 + \mathcal{A}_4 \ |d\chi_2|^2 + \mathcal{A}_5 \ dr^2 \tag{3.9}$$

with the coefficients $\mathcal{A}_i$ are written in terms of h_i, a_i etc. as

$$\begin{aligned}
& \mathcal{A}_2 = \frac{1}{(h_2 + a_2^2 h_1)\sqrt{C'D'}}, \quad \mathcal{A}_3 = \frac{1}{(h_4 + a_1^2 h_1)\sqrt{C'D'}} \\
& \mathcal{A}_1 = \frac{h_1}{\sqrt{C'D'}}, \quad \mathcal{A}_4 = \frac{h_3 h_4}{\sqrt{C'D'}}, \quad \mathcal{A}_5 = \frac{\gamma'\sqrt{H}}{\sqrt{C'D'}}
\end{aligned} \tag{3.10}$$

The metric (3.9) is somewhat similar to the type I picture developed earlier in [19]. The torsion and the coupling are

$$\begin{aligned} H_{\text{het}} \equiv \tilde{H} &= \tilde{\mathcal{H}}^b_{xz\theta_1}\, dy \wedge dz \wedge d\theta_2 - \tilde{\mathcal{H}}^b_{yz\theta_2}\, dx \wedge dz \wedge d\theta_1 + \\ &+ \tilde{\mathcal{H}}^b_{xzr}\, dy \wedge dz \wedge dr - \tilde{\mathcal{H}}^b_{yzr}\, dx \wedge dz \wedge dr \\ g^{\text{het}} &= \frac{1}{\sqrt{C'D'}} \end{aligned} \tag{3.11}$$

The above background is of course *not* the complete background. We need stable vector bundles satisfying torsional DUY equations. This will be dealt a little later. Here we make the following observations:

• In this background the dilaton is proportional to the warp factors appearing in the metric. This is consistent with the expectations for a torsional background [44],[61],[4], [23], [45], [71].

• As emphasized in [19], it could be possible that the dual background before geometric transition is related to $NS5$ branes wrapped on two cycles of the non-Kähler manifold. These stem from D5–branes wrapped on a two cycle of a resolved conifold (which survives the orientifold operation) in IIB. After the geometric transition we get a heterotic background which has only fluxes but no branes. The question here is whether we can shrink the two cycle on which we have wrapped $NS5$ branes and get another background with fluxes. At a more fundamental level, can the closed string background with fluxes compute anything of the world volume dynamics on the wrapped $NS5$ branes? To get an answer to this question, we have to carefully study the heterotic background.

• We would also like to give a heterotic background that is *away* from the orientifold point (in IIB). This is a tricky question, because the U-dualities by which we obtained our heterotic background are only defined for type IIB theory at the orientifold point. To find a more generic heterotic background we could rely on the sigma–model derivation. It states that a specific torsional background in type IIB theory may be lifted to heterotic theory as long as there are no RR background fluxes. One such RR–flux free background is known in type IIB theory: it is the Maldacena-Nunez torsional background [2]. However, this cannot be lifted naively because the three–form is closed and the background has no vector bundles. We will show that a similar background can be found in heterotic theory for some specific choices of b_{ij} in (3.9).

The issue of vector bundles is important. It can be easily shown that the action (3.2) is invariant under supersymmetry transformation by a spinor σ provided

$$F^a_{ij}\ \Gamma^{[i}\ \Gamma^{j]}\ =\ 0, \qquad \partial\sigma = \frac{1}{2}\omega_-^{ab}\ \Gamma^{[b}\ \Gamma^{a]}\sigma. \tag{3.12}$$

This ensures $\mathcal{N} = 1$ supersymmetry for the background (3.9). Observe also the appearance of the other spin-connection ω_- in the susy transformations. This stems from the relation of the (0,2) sigma model with the (2,2) sigma model. The reader will also recognize that the first condition in (3.12) is $J^{ij}F^a_{ij} = 0$ for a fundamental form J_{ij} that is covariantly constant with respect to a connection with torsion.

At this point one can also entertain the following puzzle: if the metric (3.9) is related to some wrapped NS5 branes in the dual theory, then two different theories on the NS5 branes seem possible: a (2,0) theory with self-dual $B^+_{\mu\nu}$ field[28] and a (1,1) gauge theory. It turns out that U-dualities from type IIB background always lead to the (1,1) theory and not the (2,0) theory[29].

We now turn to the issue of finding a heterotic metric away from the type IIB orientifold limit. First, we see that the metric (3.9) can be re–written in the following way:

$$\begin{aligned} ds^2\ =&\mathcal{A}_1\ dz^2 + \mathcal{A}_2\ dy^2 + \mathcal{A}_3\ dx^2 + \mathcal{A}_4\ |d\chi_2|^2 + \mathcal{A}_2\ b^2_{yj}(d\zeta^j)^2 + \\ &+ \mathcal{A}_3\ b^2_{xi}(d\zeta^i)^2 - 2\left[\mathcal{A}_2\ b_{yj}\ dy\ d\zeta^j + \mathcal{A}_3\ b_{xi}\ dx\ d\zeta^i\right] \end{aligned} \tag{3.13}$$

where $\zeta^i = \theta_i$ are the coordinates along angular directions. We allow for a generic choice of type IIB B_{NS} fields at the orientifold point given by:

$$b\ =\ b_{x\theta_1}\ dx\wedge d\theta_1 + b_{x\theta_2}\ dx\wedge d\theta_2 + b_{y\theta_1}\ dy\wedge d\theta_1 + b_{y\theta_2}\ dy\wedge d\theta_2. \tag{3.14}$$

Apparently, this choice of B_{NS} field is more generic than the one for the type IIB background in [19]. Does the type IIB theory allow such a choice of B_{NS} field? For an arbitrary choice of type IIB metric the answer is of course no. However, using the construction in [19], we see that the type IIB geometry at the orientifold point is $T^2\times T^2\times S^1\times R$, with R being the direction parameterized by dr^2. It was shown in [19] that this kind of type IIB

[28] This is the world volume B^+ field, and shouldn't be confused with heterotic torsion.

[29] The (2,0) theory on the other hand can be derived directly from M-theory on an interval *a-la* Horava–Witten [72]. In fact SO(32) heterotic theory yields a (1,1) five–brane theory and $E_8\times E_8$ gives the (2,0) theory [73],[74].

geometry and a B_{NS} field with choice $(b_{x\theta_1}, b_{y\theta_2})$ only, requires the two T^2 tori (χ_1, χ_2) in (3.6) to have the following complex structures (after geometric transition):

$$\tau_1 \;=\; i\sqrt{\frac{h_2 + a_2^2 h_1}{h_4 + a_1^2 h_1}}, \qquad \tau_2 = \frac{1}{h_3 h_4}\left[a_1 a_2 h_1 + i\sqrt{h_2 h_3 h_4 - a_1^2 a_2^2 h_1^2}\right]. \tag{3.15}$$

We see that one of the tori is a square torus, whereas the other is not. Such a compactification is a non–compact version of T^6 flux compactifications studied in [4], [23][30]. Therefore it allows for all possible fluxes that are *invariant* under orbifold and orientifold actions. The choice of B_{NS} field (3.14) is clearly invariant under such actions, and is therefore a valid choice for the IIB background.

We now focus on relating the heterotic background (3.13) from [19] to a background similar to Maldacena–Nunez with the metric of a deformed conifold and B_{NS} field, but no RR–flux. A first step would be to achieve a metric with two square tori, accompanied by an appropriate change in fluxes. Here we will assume that we can consistently employ the other choice of IIB B_{NS} in (3.14), i.e

$$b \;=\; b_{x\theta_2}\; dx \wedge d\theta_2 + b_{y\theta_1}\; dy \wedge d\theta_1 \tag{3.16}$$

along with the χ_2 torus in the heterotic metric (3.13) converted to a square one. The supersymmetry of the background will be restored by a modified type IIB B_{RR} field, so that both (χ_1, χ_2) tori are now square.

Once we change B_{RR} in type IIB, the heterotic torsion will change completely from the one in (3.11). Similarly the coefficients in the metric (3.13) will have to be re–evaluated. The question is if we can determine a valid torsional background now[31].

In the following we will show that the metric and the torsion can be determined to satisfy the torsional relation from the superpotential [75], [76], [77]

$$H \;=\; e^{2\phi}\; * d\left(e^{-2\phi}\; J\right) \tag{3.17}$$

with the dilaton ϕ, the torsion H and the fundamental 2-form J. Alternatively, once we know a particular metric — and the dilaton from the warp factors — the torsion can be

[30] Any orientifold action will not be visible in the metric as each of these terms are invariant under any such actions.

[31] The dilaton will be determined from the warp factors in the metric.

evaluated via (3.17). We will also be able to infer possible deviation of the metric that would lead to $\mathcal{N}=2$ spacetime supersymmetry.

Our starting metric is (3.13), where we account for the changes made to B_{NS} (3.16) and the complex structure of the χ_2 torus by letting the coefficients $\mathcal{A}_i$ be arbitrary. Let $\tau_2 = i|\tau|$ in (3.6). The choice (3.16) leads to $dy\ d\theta_1$ and $dx\ d\theta_2$ cross–terms in the metric. Requiring those to have the same prefactor can be achieved by

$$\mathcal{A}_3 \ = \ \mathcal{A}_2\ b_{y\theta_1}\ b^{-1}_{x\theta_2}. \tag{3.18}$$

Furthermore imposing equal prefactors on dy^2 and $d\theta_2^2$ leads to

$$\mathcal{A}_4 \ = \ |\tau|^{-2}\mathcal{A}_2\,(1 - b_{y\theta_1}\ b_{x\theta_2})\,. \tag{3.19}$$

After that the metric becomes:

$$\begin{aligned} ds^2 \ &= \mathcal{A}_1\ dz^2 + \mathcal{A}_2\ \Big[(dy^2 + d\theta_2^2) + b_{y\theta_1}\cdot b^{-1}_{x\theta_2}\ dx^2 + \left(|\tau|^{-2} - |\tau|^{-2} b_{y\theta_1}\cdot b_{x\theta_2} + b^2_{y\theta_1}\right)\ d\theta_1^2 \\ &\qquad - 2\ b_{y\theta_1}\,(dy\ d\theta_1 + dx\ d\theta_2)\Big] + \mathcal{A}_5\ dr^2 + ds^2_{0123}. \end{aligned} \tag{3.20}$$

If we furthermore restrict also dx^2 and $d\theta_1^2$ to have equal coefficients, the b field becomes constrained, too:

$$\mathcal{A}_3 \ = \ \frac{\mathcal{A}_2}{|\tau|^2}, \qquad \mathcal{A}_4 \ = \ \mathcal{A}_2\left(\frac{1}{|\tau|^2} - b^2_{y\theta_1}\right), \qquad b_{x\theta_2} \ = \ |\tau|^2 b_{y\theta_1} \tag{3.21}$$

where $\mathcal{A}_1$ is still arbitrary. The final metric, after we do all these substitutions, becomes

$$ds^2 = \mathcal{A}_5\ dr^2 + \mathcal{A}_1\ dz^2 + \mathcal{A}_2\ \Big[(dy^2 + d\theta_2^2) + \frac{1}{|\tau|^2}(dx^2 + d\theta_1^2) - 2\ b_{y\theta_1}\,(dy\ d\theta_1 + dx\ d\theta_2)\Big] \tag{3.22}$$

along with flat Minkowski spacetime. To bring (3.21) into some familiar form, we perform the following set of transformations:

$$\begin{aligned} y \quad &\rightarrow \quad \sin\,\langle\psi\rangle\ y + \cos\,\langle\psi\rangle\ \theta_2 \\ \theta_2 \quad &\rightarrow \quad -\cos\,\langle\psi\rangle\ y + \sin\,\langle\psi\rangle\ \theta_2 \\ z \quad &\rightarrow \quad z + a_1\ \cot\,\langle\theta_1\rangle\ x + b_1\ \cot\,\langle\theta_2\rangle\ y \end{aligned} \tag{3.23}$$

with (θ_1, x, r) remaining unchanged. We have also denoted the expectation values of the angles as $\langle\Theta\rangle$, with $\Theta = (\theta_2, \psi, \theta_1)$ and a_1, b_1 are constants. Furthermore, in our notation $d\psi$ is related to dz (see sec. 5 of [18] for notations).

The metric (3.22), after inserting in the set of transformations (3.23), takes the following form:

$$
\begin{aligned}
ds^2 \ &= \mathcal{A}_1 \left(dz + a_1 \cot \langle\theta_1\rangle \ dx + b_1 \cot \langle\theta_2\rangle \ dy\right)^2 + \mathcal{A}_2 \left[(dy^2 + d\theta_2^2) + \frac{1}{|\tau|^2}(dx^2 + d\theta_1^2)\right] + \\
&\quad - 2 \ \mathcal{A}_2 \ b_{y\theta_1} \left[\sin \langle\psi\rangle(dy \ d\theta_1 + dx \ d\theta_2) + \cos \langle\psi\rangle(d\theta_1 \ d\theta_2 - dx \ dy)\right] + \mathcal{A}_5 \ dr^2.
\end{aligned}
\tag{3.24}
$$

One observes that for non–constant Θ, the metric will be exactly a *warped* deformed conifold[32]. In the following we will therefore try to answer the following questions:

- When can we have non-constant Θ?
- What is the background dilaton?
- What will be the torsion for this background?
- What are the stable vector bundles allowed for such a torsional background?

Another related question would be to ask for the allowed deformations of the above metric (3.24) that would preserve $\mathcal{N} = 2$ supersymmetry. Of course once we make Θ non-constant, both dilaton and torsion will have to change so that the background still preserves $\mathcal{N} = 1$ supersymmetry. Therefore deforming *away* from $\mathcal{N} = 1$ supersymmetry should be non–trivial.

Coming back to the metric (3.24), we see that (3.24) has a close relation with Maldacena–Nunez (MN) $\mathcal{N} = 1$ supergravity solution. The MN solution [2] is a warped deformed conifold, with the metric taking the following form[33]:

$$
\begin{aligned}
ds^2_{\rm MN} \ &= \ N \ dr^2 + \frac{N}{4}\left(d\psi + \cos \tilde{\theta}_1 \ d\phi_1 + \cos \tilde{\theta}_2 \ d\phi_2\right)^2 + \\
&\quad + \frac{N}{4}\left(e^{2g} + a^2\right)\left(d\tilde{\theta}_2^2 + \sin^2\tilde{\theta}_2 \ d\phi_2^2\right) + \frac{N}{4}\left(d\tilde{\theta}_1^2 + \sin^2\tilde{\theta}_1 \ d\phi_1^2\right) + \\
&\quad - \frac{Na}{2}\left[\cos \psi(d\tilde{\theta}_1 d\tilde{\theta}_2 - \sin \tilde{\theta}_1 \sin \tilde{\theta}_2 \ d\phi_1 d\phi_2) + \sin \psi(\sin \tilde{\theta}_1 \ d\phi_1 \ d\tilde{\theta}_2 + \sin \tilde{\theta}_2 \ d\phi_2 \ d\tilde{\theta}_1)\right]
\end{aligned}
\tag{3.25}
$$

[32] This doesn't mean that the metric will be conformally Ricci-flat. By *warped* deformed conifold we mean generic warping and not just conformally warped. We will soon distinguish between the various possible warpings allowed in these set-ups.

[33] The metric written in [2] uses left-invariant one-forms. One can easily show the following result by writing the metric in components.

where we have used $(\tilde{\theta}_1, \tilde{\theta}_2)$ in (3.25) to distinguish it from (θ_1, θ_2) used in (3.24). In fact to see the relation between (3.25) and (3.24) we have to use *local* coordinates. Let us therefore use the following substitutions:

$$\begin{aligned}
&\psi = \langle\psi\rangle + z, \quad \tilde{\theta}_1 = \langle\theta_1\rangle + \theta_1, \quad \tilde{\theta}_2 = \langle\theta_2\rangle + \theta_2 \\
&\phi_1 = \langle\phi_1\rangle + \frac{x}{\sin\langle\theta_1\rangle}, \qquad \phi_2 = \langle\phi_2\rangle + \frac{y}{\sin\langle\theta_2\rangle}
\end{aligned} \tag{3.26}$$

along with the following definitions for a and g (we follow the notation of [28]):

$$a(r) \; = \; -\frac{2r}{\sinh 2r}, \qquad e^{2g} \; = \; 4r \coth 2r - \frac{4r^2}{\sinh^2 2r} - 1 \tag{3.27}$$

which are in fact necessary to make (3.25) a solution to the string equation of motion. Recall however that the MN solution (3.25) is a metric in type IIB theory with closed three–form field H_{NS} and non–trivial dilaton. This solution has vanishing H_{RR}, axion and five–form. To establish the connection between (3.25) and (3.24), let us substitute (3.26) in (3.25) letting $(\theta_1, \theta_2, x, y) \to 0$. Under these assumptions (3.25) takes the following form (ignoring numerical factors of $\frac{N}{4}$ for simplicity):

$$\begin{aligned}
ds^2_{\rm MN} \; = \; & \Big(dz + \cot\langle\theta_1\rangle\, dx + \cot\langle\theta_2\rangle\, dy\Big)^2 + \left(e^{2g} + a^2\right)(dy^2 + d\theta_2^2) + (dx^2 + d\theta_1^2) + \\
& - 2\, a\Big[\sin\langle\psi\rangle(dy\, d\theta_1 + dx\, d\theta_2) + \cos\langle\psi\rangle(d\theta_1\, d\theta_2 - dx\, dy)\Big] + 4\, dr^2 + ...
\end{aligned} \tag{3.28}$$

where the dotted terms are corrections that are higher orders in $(\theta_1, \theta_2, x, y)$.

Comparing (3.28) with our metric (3.24), we see that they are of the same form up to higher order corrections on (3.28). Therefore one might be tempted to conjecture that deforming away from the point $(\langle\theta_{1,2}\rangle, \langle\phi_{1,2}\rangle)$ should give the global description of the heterotic metric (3.24). One possible solution for the coefficients $\mathcal{A}_i$ in (3.24) (or in (3.13)) that allows for non-constant Θ would then be:

$$\mathcal{A}_1 = \mathcal{A}_3 = \frac{\mathcal{A}_5}{4} = \frac{N}{4}, \quad \mathcal{A}_2 = \frac{N(e^{2g} + a^2)}{4}, \quad \mathcal{A}_4 = \frac{Ne^{2g}}{e^{2g} + a^2} \tag{3.29}$$

where we have re–introduced the numerical factor of $\frac{N}{4}$. Finally, the type IIB B_{NS} field and the (χ_1, χ_2) tori will have the following form:

$$\begin{aligned}
&b = a\; dx \wedge d\theta_2 + \frac{a}{e^{2g} + a^2}\; dy \wedge d\theta_1 \\
&d\chi_1 = dx + i\; dy, \quad d\chi_2 = d\theta_1 + i\sqrt{e^{2g} + a^2}\; d\theta_2.
\end{aligned} \tag{3.30}$$

Under the above choices of coefficients, our metric (3.24) agrees exactly with the MN metric (3.25). Of course we should remind the readers that we have not yet *derived* the coefficients of (3.24) from our duality chain. We haven't identified a dual background before geometric transition that would allow us to do so. So (3.29) is simply a possibility at this point (albeit a strong one).

It is also worth mentioning that the MN background was derived for the IR regime (for small r). The same is true for our local IIB metric. However, once we infer the local heterotic metric and then lift it to a global background, there is no a–priori restriction to be in the IR. Our global solution should be valid for UV as well, but since MN fails in this regime, we will not specify the coefficients in the global metric. The UV limit for the MN background has been found in [28] and we will come back to this issue later.

In the case where the coefficients $\mathcal{A}_i$ in (3.24) are *different* from the one considered in (3.29), our ansatz would be the global form of the metric (3.24) with non–constant Θ. The heterotic metric is finally given by:

$$\begin{aligned} ds^2_{\rm het} = \ & ds^2_{0123} + \mathcal{A}_1 \ (d\psi + a_1 \cos\theta_1 \ d\phi_1 + b_1 \cos\theta_2 \ d\phi_2)^2 + \mathcal{A}_3 \ \left(d\theta_1^2 + \sin^2\theta_1 \ d\phi_1^2\right) + \\ & + \mathcal{A}_2 \ \left(d\theta_2^2 + \sin^2\theta_2 \ d\phi_2^2\right) - 2 \ \mathcal{A}_2 \ b_{y\theta_1} \Big[\cos\psi \ (d\theta_1 \ d\theta_2 - \sin\theta_1 \sin\theta_2 \ d\phi_1 \ d\phi_2) + \\ & + \sin\psi \ (\sin\theta_1 \ d\phi_1 \ d\theta_2 + \sin\theta_2 \ d\phi_2 \ d\theta_1)\Big] + \mathcal{A}_5 \ dr^2 \end{aligned} \tag{3.31}$$

where $(\psi, \theta_{1,2}, \phi_{1,2})$ are the global coordinates; and we have kept the warp factors $\mathcal{A}_i$ arbitrary. Notice that the above metric written in terms of global coordinates has a lesser number of isometries than (3.24) written in terms of local coordinates. In fact (3.24) is a T^3 fibration over a three–dimensional base, i.e in SYZ [20] form, whereas (3.31) cannot be brought into SYZ–form at all. This observation may also resolve one puzzle regarding mirror transformation of a resolved or deformed conifold. The resolved conifold has a natural T^3 torus built inside it irrespective of whether we use global or local coordinates. On the other hand a deformed conifold is *not* a T^3 fibration over a three dimensional base. But when we use local coordinates to write the metric of a deformed conifold, we regain some of the isometries, and then a SYZ–form for the deformed conifold can be written. This is precisely the reason for the success of the duality chain proposed in [18] and [19]. Since this duality chain involves T–dualities, it seems impossible that one could *loose* isometries along the way. Thus, the metrics of [18], [19] all have three isometries in terms of local coordinates used therein. This is perfectly consistent with the arguments in this article.

The above metric (3.31) is another example of a warped deformed conifold. When the warp factors $\mathcal{A}_i$ are different from (3.29) this metric does *not* coincide with the MN metric (3.25). We should now compare this with other available warped–deformed–conifold solutions:

- The Klebanov-Strassler (KS) metric of [1], and
- The Geometric transition dual metric of [19], [18].

Both of these are in type IIB theory with background NS and RR fluxes. Furthermore whereas the KS solution [1] is written in terms of global coordinates $(\psi, \theta_{1,2}, \phi_{1,2}, r)$, the solution presented in [19], [18] are written only in terms of *local* coordinates $(z, x, y, \theta_{1,2})$. The full global picture involves seven branes and orientifold planes, as discussed earlier. To compare all the different pictures, let us cite the metric of a simple (i.e. Ricci–flat) deformed conifold [78], [79], [80]:

$$\begin{aligned} ds^2_{\rm DC} \;=\; & \frac{2}{3K^2}\;(d\psi + \cos\,\theta_1\;d\phi_1 + \cos\,\theta_2\;d\phi_2)^2 + K\;\cosh\,\rho\;\left(d\theta_1^2 + \sin^2\theta_1\;d\phi_1^2\right) + \\ & + K\;\cosh\,\rho\;\left(d\theta_2^2 + \sin^2\theta_2\;d\phi_2^2\right) - 2K\;\Big[\cos\,\psi\;(d\theta_1\;d\theta_2 - \sin\,\theta_1\;\sin\,\theta_2\;d\phi_1\;d\phi_2) + \\ & + \sin\,\psi\;(\sin\,\theta_1\;d\phi_1\;d\theta_2 + \sin\,\theta_2\;d\phi_2\;d\theta_1)\Big] + \frac{2}{3K^2}\;d\rho^2 \end{aligned} \tag{3.32}$$

where ρ is the "radial" coordinate, and $K = K(\rho) = \frac{(\sinh\,2\rho - 2\rho)^{1/3}}{2^{1/3}\sinh\,\rho}$ (see [78], [1], [79], [80] for more details). The metric (3.32) shows the same warp factors for the two spheres parameterized by (θ_1, ϕ_1) and (θ_2, ϕ_2). So does the KS solution [1]. On the other hand, all the other solutions, Maldacena-Nunez (3.25), the geometric transition solutions of [19], [18], and the heterotic solution (3.31) have different warp factors for the two spheres (or the two tori in case of [19], [18]). For example, the IIB solution for gravity dual presented in [19] is given (in local coordinates) by:

$$\begin{aligned} ds^2_{IIB} \;=\; & h_1(dz + a_1\;dx + a_2\;dy)^2 + h_2(dy^2 + d\theta_2^2) + h_4(dx^2 + h_3\;d\theta_1^2) + \\ & + h_5\big[\sin\,\psi\;(dx\;d\theta_2 + dy\;d\theta_1) + \cos\,\psi\;(d\theta_1\;d\theta_2 - dx\;dy)\big] + \gamma'\sqrt{H}\;dr^2 \end{aligned} \tag{3.33}$$

where h_i and a_i were defined in (3.8). The global picture for this solution has not been worked out yet. But comparing locally shows that the solution of [19], i.e. (3.33), forms a new class of supergravity dual with markedly different warping behavior of a deformed conifold metric. This results from the different, more elaborate set-up of [19] and [18] compared to [1] and [2].

We still have to determine the torsion for our background (3.31), where we will use (3.29) since we know it results in a valid background in terms of global coordinates[34]. We need the fundamental 2–form J in terms of vielbeins e_i and the dilaton. Since the background has changed, the vielbeins are not those in (3.5). Instead we choose:

$$\begin{aligned}
e^1 &= \sqrt{N}\ dr, \quad e^5 = \frac{\sqrt{N}}{2}\ e^g\ d\theta_2, \quad e^2 = \frac{\sqrt{N}}{2}\ (d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2) \\
e^3 &= \frac{\sqrt{N}}{2}\ (\sin\psi\sin\theta_1\ d\phi_1 + \cos\psi\ d\theta_1 - a\ d\theta_2) \\
e^4 &= -\frac{\sqrt{N}}{2}\ \Big[\mathcal{B}\ e^g\sin\theta_2\ d\phi_2 + \mathcal{A}(\cos\psi\sin\theta_1\ d\phi_1 - \sin\psi\ d\theta_1 + a\sin\theta_2\ d\phi_2)\Big] \\
e^6 &= -\frac{\sqrt{N}}{2}\ \Big[\mathcal{A}\ e^g\sin\theta_2\ d\phi_2 - \mathcal{B}(\cos\psi\sin\theta_1\ d\phi_1 - \sin\psi\ d\theta_1 + a\sin\theta_2\ d\phi_2)\Big]
\end{aligned} \tag{3.34}$$

which gives rise to the metric (3.31) with $\mathcal{A}_i$ defined as in (3.29). It was pointed out in [28] that these are the correct vielbeins for observing the $SU(3)$ structure of this background and they were first given in [81]. They are not quite those of the deformed conifold since our background (3.31), as noted earlier, is neither a Ricci–flat deformed nor a conformally deformed conifold[35]. The $\mathcal{A},\mathcal{B}$ used in (3.34) satisfy $\mathcal{A}^2+\mathcal{B}^2=1$, with $\mathcal{A},\mathcal{B}$ given by

$$\mathcal{A} = \coth 2r - 2r\mathrm{csch}^2 2r\ , \qquad \mathcal{B} = \mathrm{csch} 2r\ \sqrt{-1+4r\coth 2r - 4r^2\mathrm{csch}^2 2r}\ . \tag{3.35}$$

The fundamental two–form is evaluated with a choice of complex structure such that $J = (e^1\wedge e^2 + e^3\wedge e^4 + e^5\wedge e^6)$. This amounts to

$$\begin{aligned}
J &= \frac{N}{2}\ dr\wedge(d\psi + \cos\theta_1\ d\phi_1 + \cos\theta_2\ d\phi_2) \\
&- \frac{N}{4}\ \mathcal{A}\sin\theta_1\ d\theta_1\wedge d\phi_1 - \frac{N}{4}\ \left(-\mathcal{A}^2 a + \mathcal{A}e^{2g} - 2\mathcal{B}ae^g\right)\sin\theta_2\ d\theta_2\wedge d\phi_2 \\
&+ \frac{N}{4}\ (\mathcal{A}a + \mathcal{B}e^g)\ [\sin\psi(d\theta_1\wedge d\theta_2 - \sin\theta_1\sin\theta_2\ d\phi_1\wedge d\phi_2) \\
&\qquad + \cos\psi(\sin\theta_1\ d\theta_2\wedge d\phi_1 - \sin\theta_2\ d\theta_1\wedge d\phi_2)]\ .
\end{aligned} \tag{3.36}$$

---

34 In other words, we neglect the UV regime for the time being. The derivation would be equivalent, the appropriate coefficients for the vielbeins can be found in [28].

35 Although not a conformally deformed conifold, our model will have many of the characteristic features of a deformed conifold as can be seen from (3.31). For example there will be a three-cycle that can undergo a geometric transition. The three cycle will also have a similar Hopf-fibration structure as a deformed conifold. This is clear from the $d\psi$ fibration structure of (3.31). These similarities will be exploited in the next section to study the vector bundles across conifold transitions.

From the value of J one can easily see that the manifold is non-Kähler. This is of course expected because the local metric is also non-Kähler with torsion. The background dilaton can be extracted from the warped metric (3.31) or from [2], [81], and is given by

$$e^{2\Phi} \;=\; \frac{e^{g+2\Phi_0}}{\sinh\, 2r} \tag{3.37}$$

where Φ_0 is some constant value that could be fixed from the U–dual type IIB background. With the dilaton (3.37) one computes

$$d(e^{-2\Phi}J) \;=\; e^{-2\Phi}\left(-2\frac{\partial\Phi}{\partial r}\; dr\wedge J + dJ\right)\,. \tag{3.38}$$

The Hodge dual of this expression is most easily found in terms of vielbeins, since in a non–coordinate basis simply:

$$*\,(e^{\alpha_1}\wedge e^{\alpha_2}\wedge e^{\alpha_3}) \;=\; \frac{1}{3!}\;\epsilon^{\alpha_1\alpha_2\alpha_3}{}_{\mu_1\mu_2\mu_3}\; e^{\mu_1}\wedge e^{\mu_2}\wedge e^{\mu_3}\,. \tag{3.39}$$

We choose the orientation so that $\epsilon^{123456}=1$. Inverting (3.34) and replacing the coordinate differentials by vielbeins one finds

$$\begin{aligned} e^{2\Phi}\; * \; d(e^{-2\Phi}J) \;=\; & \frac{1}{\sqrt{N}\;F_2(r)}\;\Bigg[\frac{F_2(r)\;(1+8r^2-\cosh 4r)\;(4r-\sinh 4r)}{F_1(r)\,\sinh^2 2r}\; e^1\wedge e^2\wedge e^6 \\ & +\frac{2\;(-1+2r\coth 2r)}{\sinh 2r}\; e^1\wedge e^3\wedge e^5 + \frac{(1+8r^2-\cosh 4r)}{\sinh^3 2r}\; e^1\wedge e^4\wedge e^6 \\ & +\frac{F_2^2(r)}{\sinh r\;\cosh r}\; e^2\wedge e^3\wedge e^6 + \left(-\frac{r}{\sinh^2 r}+\frac{1}{\sinh r\;\cosh r}-\frac{r}{\cosh^2 r}\right)\; e^2\wedge e^4\wedge e^5 \\ & +\frac{(-4r+\sinh 4r)}{\sinh^2 2r}\; e^3\wedge e^4\wedge e^6\Bigg] \end{aligned} \tag{3.40}$$

with $F_1(r)$ and $F_2(r)$ defined by[36]

$$\begin{aligned} F_1(r) \;&=\; -1+8r^2+\cosh 4r - 4r\;\sinh 4r \\ F_2(r) \;&=\; \sqrt{-1+4r(\coth 2r - r\;\mathrm{csch}^2\;2r)} \end{aligned} \tag{3.41}$$

[36] Again these relations are motivated from the similarity of (3.31) with the MN background [2]. We may not consider the values taken by [2], and represent all our results using the unknown warp-factors $\mathcal{A}_i$. This way we can allow a non-trivial vector bundle for our background. More details on this will be presented elsewhere.

The 3–form (3.40) is the torsion for our background (3.31) with dilaton (3.37). In terms of global coordinates $(r, \theta_i, \phi_i, \psi)$ the torsion H is given as

$$
\begin{aligned}
H &= e^{2\Phi} \ * \ d(e^{-2\Phi} J) \\
&= -\frac{Na'}{4} \cos\psi \ dr \wedge (d\theta_1 \wedge d\theta_2 - \sin\theta_1 \sin\theta_2 \ d\phi_1 \wedge d\phi_2) + \\
&\quad - \frac{Na'}{4} \sin\psi \ dr \wedge (\sin\theta_2 \ d\theta_1 \wedge d\phi_2 - \sin\theta_1 \ d\theta_2 \wedge d\phi_1) + \\
&\quad + \frac{Na}{4} \sin\psi \ d\theta_1 \wedge d\theta_2 \wedge (d\psi + \cos\theta_1 \ d\phi_1 + \cos\theta_2 \ d\phi_2) + \\
&\quad - \frac{N}{4} (\sin\theta_1 \cos\theta_2 - a \cos\psi \cos\theta_1 \sin\theta_2) \ d\theta_1 \wedge d\phi_1 \wedge d\phi_2 + \\
&\quad - \frac{N}{4} (\sin\theta_2 \cos\theta_1 - a \cos\psi \cos\theta_2 \sin\theta_1) \ d\theta_2 \wedge d\phi_1 \wedge d\phi_2 + \\
&\quad - \frac{N}{4} \sin\theta_1 \ d\theta_1 \wedge d\phi_1 \wedge d\psi + \frac{N}{4} \sin\theta_2 \ d\theta_2 \wedge d\phi_2 \wedge d\psi + \\
&\quad - \frac{Na}{4} \cos\psi \ (\sin\theta_2 \ d\theta_1 \wedge d\phi_2 \wedge d\psi - \sin\theta_1 \ d\theta_2 \wedge d\phi_1 \wedge d\psi) + \\
&\quad - \frac{Na}{4} \sin\psi \sin\theta_1 \sin\theta_2 \ d\phi_1 \wedge d\phi_2 \wedge d\psi
\end{aligned}
\tag{3.42}
$$

with $a' = \partial a / \partial r$. At this point the attentive reader might question the existence of torsion and warped solution from the point of view of [59], [60], [45]. We seem to have used an embedding where $\omega_+ = A$, i.e. the gauge connection A is embedded in the torsional spin connection, and not the one of (3.1). We hasten to point out that this is justified as long as the underlying manifold is *non–compact*. For compact non–Kähler manifolds the situation is much more subtle and delicate as was pointed out in the series of papers [45], [75], [46]. To complete the analysis of the proposed heterotic background (3.31) with torsion (3.42) and dilaton (3.37) we need to consider two more issues:

- The existence of vector bundles, and
- The global behavior of $b_{y\theta_1}, b_{x\theta_2}$ and $|\tau|$.

The vector bundles will be studied in the next section. Here we concentrate on the behavior of $b_{y\theta_1}$ and $b_{x\theta_2}$. In the process we will also be able to study the complex structure $|\tau|$ in detail.

The global behavior of type IIB B–fields $b_{y\theta_1}$ and $b_{x\theta_2}$ can only be determined after we solve the background equation of motions. However, the situation is complicated because of the additional seven branes and orientifold planes. That was one of the motivations for using local coordinates in the first place, because we lack the global metric in IIB.

Nevertheless, the global behavior of the IIB B–fields can be extracted from the relation between type IIB and heterotic picture. Recall that the global heterotic metric (3.31) contains the global IIB B–fields, and was obtained by connecting the local pictures in both theories and then using the similarity of the heterotic metric with Maldacena-Nunez [2] to obtain the global picture. In our case of interest, a background with only NS flux, we know MN to be a valid solution in the IR. Comparing (3.31) with the MN metric determines B[37]. For small r:

$$\begin{aligned} b_{x\theta_2} &= -1+\frac{2}{3}r^2-\frac{14}{45}r^4+\mathcal{O}(r^6) \\ b_{y\theta_1} &= -1+\frac{10}{3}r^2-\frac{446}{45}r^4+\mathcal{O}(r^6) \end{aligned} \tag{3.43}$$

Near $r \to 0$ both B–field components are constant as one might have expected. Having determined the B field we can also fix the χ_2-torus in (3.6). The complex structure is given as

$$|\tau|^2 = 1+\frac{8}{3}r^2-\frac{32}{45}r^4+\mathcal{O}(r^6) \tag{3.44}$$

which tells us how the (θ_1, θ_2) torus varies as we move along the radial direction. In fact, near $r \to 0$: $\tau_2 \equiv i|\tau| = i$ which, along with $\tau_1 = i$, completely specifies the IR behavior in IIB.

The discussion in [2] does not extend to the UV regime. Here we can rely on the analysis of [28] which embeds the MN background in a class of interpolating solutions between MN and KS. Using their results we can obtain the large r behavior of the B–fields:

$$\begin{aligned} b_{x\theta_2} &= -2\ e^{-2r}+a_{\rm uv}\ (2r-1)\ e^{-\frac{10r}{3}}-\frac{1}{2}\ a_{\rm uv}^2\ (2r-1)^2\ e^{-\frac{14r}{3}}+\mathcal{O}(e^{-6r}) \\ b_{y\theta_1} &= -2\ e^{-2r}-a_{\rm uv}\ (2r-1)\ e^{-\frac{10r}{3}}-\frac{1}{2}\ a_{\rm uv}^2\ (2r-1)^2\ e^{-\frac{14r}{3}}+\mathcal{O}(e^{-6r}) \end{aligned} \tag{3.45}$$

where $a_{\rm uv} = -\infty$ corresponds to MN in the interpolating scenario. The complex structure then results in

$$|\tau|^2 = 1+2\ e^{-4r}-a_{\rm uv}\ (2r-1)\ e^{-\frac{4r}{3}}+\frac{1}{2}\ a_{\rm uv}^2\ (2r-1)^2\ e^{-\frac{8r}{3}}+\mathcal{O}(e^{-\frac{16r}{3}}). \tag{3.46}$$

Notice that for $r \to \infty$ the B–fields vanish and the complex structure approaches again $\tau_2 = i$ and $\tau_1 = i$.

[37] One might as well solve the heterotic equations of motion (see also [28]).

This finalizes the study of the heterotic background (3.31). Before we move on to study vector bundles we would like to comment on possible deformations to $\mathcal{N}=2$ susy. Let us go back and consider (3.20), written in terms of local coordinates. We have already observed that in this metric the relative coefficients of dy^2 and $d\theta_2^2$ are the same, whereas the relative coefficients of dx^2 and $d\theta_1^2$ are different. Instead of requiring the latter also to be equal let us keep them inhomogeneous. We furthermore assume that the B–fields go to zero in the following way:

$$b_{y\theta_1} \quad \rightarrow \quad g_1\epsilon, \qquad\quad b_{x\theta_2} \quad \rightarrow \quad g_2\epsilon \tag{3.47}$$

with ϵ approaching zero. Under this requirement the ratio of $b_{y\theta_1}$ and $b_{x\theta_2}$ approaches $\frac{b_{y\theta_1}}{b_{x\theta_2}} = \frac{g_1}{g_2}$ although individually they are very small everywhere. Using (3.47) and the coordinate transformation (3.23) we obtain the following local form of the metric in the limit $\epsilon \rightarrow 0$:

$$\begin{aligned} ds^2 \; &= \mathcal{A}_1 \left(dz + a_1 \cot\langle\theta_1\rangle\, dx + b_1 \cot\langle\theta_2\rangle\, dy\right)^2 + \mathcal{A}_2\,(dy^2 + d\theta_2^2) + \\ &+ \mathcal{A}_2 \left(\frac{g_1}{g_2}\, dx^2 + \frac{1}{|\tau|^2}\, d\theta_1^2\right) + \mathcal{A}_5\, dr^2 + ds^2_{0123} \end{aligned} \tag{3.48}$$

where $\mathcal{A}_i$ can be generic functions of the local coordinates, and a_1, b_1 are constants[38]. If we set $a_1 = 0,\ b_1 = 1$, we arrive at a metric that is the local form of the $\mathcal{N}=2$ metric studied by [82]. We can therefore conjecture our metric (3.48) to have the global form (recall (3.26) for the relation between local and global coordinates):

$$\begin{aligned} ds^2 \; &= \mathcal{A}_1 \left(d\psi + \cos\theta_2\, d\phi_2\right)^2 + \mathcal{A}_2\,(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2) + \\ &+ \mathcal{A}_2 \left(\frac{1}{|\tau|^2}\, d\theta_1^2 + \frac{g_1}{g_2}\sin^2\theta_1\, d\phi_1^2\right) + \mathcal{A}_5\, dr^2 + ds^2_{0123}. \end{aligned} \tag{3.49}$$

To determine the precise values of the various coefficients we can compare (3.49) to the $\mathcal{N}=2$ metric of [82] (see also [83]):

$$\begin{aligned} ds^2 \; &= ds^2_{0123} + \frac{e^{-x}}{g_c^2\,\Omega}\cos^2\theta_1 \left(d\psi + \cos\theta_2\, d\phi_2\right)^2 + F(r)\,(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2) + \\ &+ F(r)\, \left(\frac{1}{g_c^2\, F(r)}\, d\theta_1^2 + \frac{e^x}{\Omega\, g_c^2\, F(r)}\sin^2\theta_1\, d\phi_1^2\right) + g_c^2\, e^{2x}\left(\frac{\partial F}{\partial r}\right)^2 dr^2 \end{aligned} \tag{3.50}$$

[38] For $\mathcal{N}=1$ case studied earlier, $a_1 = b_1 = 1$.

where g_c is the *coupling* constant and $F(r)$ is a function of the radial coordinate r [82]. The other two variables, Ω and e^x are defined in terms of the coordinates, g_c, $F(r)$ and an integration constant κ, as (see [82] for derivations):

$$\Omega = e^x \cos^2\theta_1 + e^{-x} \sin^2\theta_1, \quad e^{-2x} = 1 - \frac{1 + \kappa\, e^{-2g_c^2 F}}{2g_c^2 F.} \tag{3.51}$$

Comparing (3.49) and (3.50), it is obvious that with the correct identifications of the coefficients $\mathcal{A}_i$ and $g_1, g_2, |\tau|$ we can obtain an $\mathcal{N} = 2$ deformation of our background (3.31). The appropriate dilaton Φ and torsion H for this case have already been considered in [82] and therefore we will not re–derive the torsion for this background. The interested reader may find all the details in [82].

One final comment on the metrics (3.49) and (3.50): Notice that the relative coefficients of $d\theta_1^2$ and $\sin^2\theta_1 d\phi_1^2$ are different. This is reminiscent of a similar behavior that we encountered for the type IIB $\mathcal{N} = 1$ metric (3.33) (see also [19] for derivation). In (3.33) the local metric showed equal coefficients for $(dy, d\theta_2)$ but different relative coefficients for $(dx, d\theta_1)$. Since we encounter the same behavior for the $\mathcal{N} = 2$ case herein, this might be an indication that the metric (3.33) actually shows $\mathcal{N} = 2$ supersymmetry, and only the NS and RR fluxes break the supersymmetry to $\mathcal{N} = 1$. If the above statement is true (we have not verified this yet) this would be perfectly consistent with the predictions for geometric transitions considered in [84], [9], [12], [11], [14], [13], [15], [10] where it was clearly stated that the underlying manifolds preserve $\mathcal{N} = 2$ supersymmetry and the fluxes break susy from $\mathcal{N} = 2$ to $\mathcal{N} = 1$.

4. Pulling Rank 2 vector bundles through conifold transitions

In the previous subsection we gave a detailed construction of a heterotic background on a manifold that resembles a warped deformed conifold. To complete the story we need to study vector bundles on the manifold, and also discuss how the bundles are pulled through conifold transitions. In the following therefore we will address these questions in some detail.

Since the analysis will involve some mathematics that might not be too well-known to physicists, we will start by discussing some standard facts about rank 2 vector bundles. For simplicity of exposition, we specialize to vector bundles on Calabi-Yau threefolds.

4.1. The Serre construction

Consider a holomorphic rank 2 vector bundle E on a Calabi-Yau threefold X satisfying anomaly cancellation. Pick a holomorphic section $s \in H^0(X, E)$, and suppose that the set of zeros of s is a smooth complex curve $C \subset X$.[39] Multiplication and wedging by s gives an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{O}_X(\det E).$$

Here and in the sequel, $\mathcal{O}_X(E)$ as usual denotes the sheaf of holomorphic sections of the vector bundle E. The notion of a sheaf gives a framework for simultaneously discussing sections over arbitrary open sets.

Note that $\det E$ is the trivial bundle by the condition $c_1(E) = 0$, so we can rewrite the above as

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{O}_X.$$

The sections in the image of the rightmost map clearly vanish on C. Denoting the sheaf of functions vanishing on C by $\mathcal{I}_C$,[40] the above exact sequence induces a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{I}_C \to 0. \tag{4.1}$$

The *Serre construction* allows us to reverse this process to give a construction of E from C and some extra data. The Serre construction is explained in [85].

The exact sequence (4.1) describes $\mathcal{O}_X(E)$ as an extension of $\mathcal{I}_C$ by $\mathcal{O}_X$. Such extensions are classified by the Ext group $\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X)$. There are the same groups as have appeared in the physics literature in describing open string states at large radius.

Ext groups enjoy a number of properties, explained in [85].

- If $\mathcal{E}$ and $\mathcal{F}$ are any sheaves, then $\mathrm{Ext}^0(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{F})$.
- If E is a vector bundle and $\mathcal{F}$ is any sheaf, then $\mathrm{Ext}^i(\mathcal{O}(E), \mathcal{F}) \simeq H^i(X, O(E^*) \otimes \mathcal{F})$.
- If $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is a short exact sequence of sheaves and $\mathcal{F}$ is any sheaf, then there are long exact sequences

$$\cdots \to \mathrm{Ext}^i(\mathcal{E}_2, \mathcal{F}) \to \mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \to \mathrm{Ext}^i(\mathcal{E}_1, \mathcal{F}) \to \mathrm{Ext}^{i+1}(\mathcal{E}_2, \mathcal{F}) \to \cdots$$

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[39] The existence of such a section is in a sense not really a restrictive assumption. If $\mathcal{O}(1)$ as usual denotes the hyperplane bundle on X given by the realization of X in some $\mathbb{P}^n$, then $E \otimes \mathcal{O}(N)$ admits such a section for N sufficiently large.

[40] *Warning:* $\mathcal{I}_C$ is not a line bundle.

and

$$\cdots \to \mathrm{Ext}^i(\mathcal{F},\mathcal{E}_1) \to \mathrm{Ext}^i(\mathcal{F},\mathcal{E}) \to \mathrm{Ext}^i(\mathcal{F},\mathcal{E}_2) \to \mathrm{Ext}^{i+1}(\mathcal{F},\mathcal{E}_1) \to \cdots$$

The ideal sheaf $\mathcal{I}_C$ fits into the short exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

where the last nontrivial map is the restriction map. It follows that the desired Ext group $\mathrm{Ext}^1(\mathcal{I}_C,\mathcal{O}_X)$ fits into the short exact sequence

$$\mathrm{Ext}^1(\mathcal{O}_X,\mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{I}_C,\mathcal{O}_X) \to \mathrm{Ext}^2(\mathcal{O}_C,\mathcal{O}_X) \to \mathrm{Ext}^2(\mathcal{O}_X,\mathcal{O}_X).$$

By the properties of Ext, we have that $\mathrm{Ext}^i(\mathcal{O}_X,\mathcal{O}_X) \simeq H^i(X,\mathcal{O}_X)$. But $H^1(X,\mathcal{O}_X) = H^1(X,\mathcal{O}_X) = 0$ for Calabi-Yau threefolds. We conclude that

$$\mathrm{Ext}^1(\mathcal{I}_C,\mathcal{O}_X) \simeq \mathrm{Ext}^2(\mathcal{O}_C,\mathcal{O}_X). \tag{4.2}$$

We now need some mathematical results which generalize Serre duality and the adjunction formula. For this, we first need to introduce local Exts. As a preliminary, we discuss the case of Ext^0 first, i.e. Hom. Given bundles E and F, we have the group $\mathrm{Hom}(E,F)$, but we also have a local version, namely the bundle $E^* \otimes F$. In sheaf language, we would write the sheaf $\underline{\mathrm{Hom}}(\mathcal{O}(E),\mathcal{O}(F))$, which is nothing but the sheaf $\mathcal{O}(E^* \otimes F)$. This is the same thing as $\underline{\mathrm{Ext}}^0(\mathcal{O}(E),\mathcal{O}(F))$ by definition. The connection between local and global Hom (or equivalently Ext^0) is

$$H^0(X,\underline{\mathrm{Hom}}(\mathcal{O}(E),\mathcal{O}(F))) \simeq \mathrm{Hom}(\mathcal{O}(E),\mathcal{O}(F)).$$

An analogous equality is true for arbitrary sheaves which are not necessarily the sheaves of sections of a holomorphic vector bundle.

Given sheaves $\mathcal{E},\mathcal{F}$, we have the local Ext sheaves $\underline{\mathrm{Ext}}^i(\mathcal{E},\mathcal{F})$. These enjoy several properties, some analogous to the properties of *Ext* groups. These properties are all explained in [85].

• If E and F are vector bundles, then $\underline{\mathrm{Ext}}^0(\mathcal{O}(E),\mathcal{O}(F)) \simeq \mathcal{O}(E^* \otimes F)$.

• If E is a vector bundle and $\mathcal{F}$ is any sheaf, then $\mathrm{Ext}^i(\mathcal{O}(E),\mathcal{F}) = 0$ for $i > 0$. This is a local version of a corresponding property for Ext groups, since higher cohomology vanishes on disks.

• If $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is a short exact sequence of sheaves and $\mathcal{F}$ is any sheaf, then there are long exact sequences of sheaves

$$\cdots \to \underline{\mathrm{Ext}}^i(\mathcal{E}_2, \mathcal{F}) \to \underline{\mathrm{Ext}}^i(\mathcal{E}, \mathcal{F}) \to \underline{\mathrm{Ext}}^i(\mathcal{E}_1, \mathcal{F}) \to \underline{\mathrm{Ext}}^{i+1}(\mathcal{E}_2, \mathcal{F}) \to \cdots$$

and

$$\cdots \to \underline{\mathrm{Ext}}^i(\mathcal{F}, \mathcal{E}_1) \to \underline{\mathrm{Ext}}^i(\mathcal{F}, \mathcal{E}) \to \underline{\mathrm{Ext}}^i(\mathcal{F}, \mathcal{E}_2) \to \underline{\mathrm{Ext}}^{i+1}(\mathcal{F}, \mathcal{E}_1) \to \cdots$$

But it is not the case that $H^0(X, \underline{\mathrm{Ext}}^i(\mathcal{O}(E), \mathcal{O}(F))) \simeq \mathrm{Ext}^i(\mathcal{O}(E), \mathcal{O}(F))$ for $i > 0$. Instead there is a more complicated relation between local and global Ext groups described by a spectral sequence. Rather than beginning a lengthy digression into the formalism of spectral sequences, we content ourselves with a simple consequence which suffices for our purposes.

• If $\mathcal{E}, \mathcal{F}$ are sheaves and $\underline{\mathrm{Ext}}^j(\mathcal{E}, \mathcal{F}) = 0$ for $j < i$, then $H^0(X, \underline{\mathrm{Ext}}^i(\mathcal{O}(E), \mathcal{O}(F))) \simeq \mathrm{Ext}^i(\mathcal{O}(E), \mathcal{O}(F))$.

We now apply this to duality. We have, as explained in [86],

$$\underline{\mathrm{Ext}}^i(\mathcal{O}_C, \mathcal{O}_X) \simeq \begin{cases} \Omega^1_C & i = 2 \\ 0 & i < 2 \end{cases}$$

So the final bullet above applies, and we conclude that

$$\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X) \simeq \mathrm{Ext}^2(\mathcal{O}_C, \mathcal{O}_X) \simeq H^0(C, \Omega^1_C).$$

Putting this all together, E can be recovered from the curve C and a holomorphic 1-form on C.

Now let C be an arbitrary smooth curve and ω a holomorphic 1-form on C. Reversing the above construction, we get an extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_C \to 0 \tag{4.3}$$

whose extension class in $\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_X)$ corresponds to ω under the above isomorphisms. We cannot conclude that $\mathcal{E}$ is a vector bundle, only that it is a sheaf. Furthermore, a local calculation shows that $\mathcal{E}$ fails to be (the sheaf of sections of) a vector bundle precisely at the points of C where ω vanishes. If C is a smooth curve of genus g and ω is not identically zero, then there are $2g - 2$ such points (including multiplicity).

We conclude that we get a vector bundle only for $g = 1$. But as elliptic curves are abundant on Fano or Calabi-Yau threefolds B, we learn that rank 2 bundles are straightforward to construct on these threefolds.

We now have to satisfy anomaly cancellation, which we presume to be of the form $c_i(\mathcal{E}) = c_i(B)$ for $i = 1, 2$. From (4.3), we get $c_1(\mathcal{E}) = 0$ and we compute that $c_2(\mathcal{E})$ is represented by the homology class of the curve C. So we clearly cannot use the above form of the Serre construction if $c_1(B) \neq 0$. But we can easily cancel the anomaly by tensoring $\mathcal{E}$ by an appropriate line bundle in many cases. And since most bundles cannot be produced by the Serre construction, there is no reason to doubt that the anomaly can be cancelled for most or even all B by using more general bundles. The above comments notwithstanding, we will stick with the Serre construction for now so that we can be more precise.

To cancel the c_1 anomaly, we have to replace $\mathcal{E}$ by $\mathcal{E} \otimes L$, where L is a line bundle with $2c_1(L) = c_1(B)$. Note that not every element of integral cohomology is divisible by 2, so L need not exist. But in a rough sense, "about half" of the B's have this property. For example, for a hypersurface $B \subset \mathbf{P}^4$ of odd degree $2n + 1$, we have $c_1(B) = \mathcal{O}(4 - 2n)$ so we can take $L = \mathcal{O}(2 - n)$.

Now

$$c_2(\mathcal{E} \otimes L) = c_2(\mathcal{E}) + c_1(\mathcal{E}) \cdot c_1(L) + c_1(L)^2 = c_2(\mathcal{E}) + c_1(L)^2,$$

so we just have to take any elliptic curve C representing the cohomology class $c_2(B) - c_1(L)^2$, and anomaly cancellation is satisfied. It is very easy to find such curves, at least as long as $c_2(B) - c_1(B)^2/4$ is non-negative, since elliptic curves abound on Fano and Calabi-Yau threefolds B.

4.2. Conifold Transitions

It is now a simple matter to use the results of the previous section to pull the bundle $\mathcal{E} \otimes L$ through the conifold transition, at least as far as the complex structure is concerned. The metric will be left for future work.

We first note how the Chern classes of B change under a conifold transition [87]:

$$c_1(B) = c_1(B'), \qquad c_2(B) = c_2(B') - [E], \tag{4.4}$$

where we are using the notation introduced in sec. 2.1, i.e. $E \subset B$ is the $\mathbf{P}^1$ that contracts to the conifold B_0, which then smooths to B'. If we let $C \subset B$ and $C' \subset B'$ be the elliptic curves used in the Serre construction to construct bundles $\mathcal{E}$, $\mathcal{E}'$, it follows from (4.4) and the calculation of the Chern classes in the previous section that we must require $C = C' - E$, and then we will be done.

This situation is easy to arrange. We let $C_0 \subset B_0$ be an elliptic curve containing the conifold with the required cohomology class. Then C_0 deforms to the desired elliptic curve $C' \subset B'$. On the other end of the transition, the preimage of C_0 under the contraction map $B \to B_0$ is a reducible curve of the form $C + E$, where C projects isomorphically to C_0, hence is elliptic. So $C = C' - E$ in the required sense, and we get bundles $\mathcal{E} \otimes L$ and $\mathcal{E}' \otimes L$ which satisfy the required properties. Note that we have used (4.4) to identify the appropriate bundle L on B with another bundle on B' which has also been denoted by L.

5. M-theory and non-Kähler manifolds

So far we have discussed the issues of vector bundles for the kind of heterotic compactifications related to geometric transitions. In this section we want to address a more generic question: Under what condition does a F-theory (or an equivalent M-theory) compactification) with fluxes allow Kähler or non-Kähler manifolds with vector bundles? In other words, how does the back–reaction of vector bundles affect the underlying heterotic geometries?

A brief discussion of this issue appeared in [88], where it was discussed that localized and non–localized fluxes from M-theory play an important role in determing the precise back–reaction effects on the underlying geometry. The conclusion was that the heterotic manifold is always non–Kähler when there are only non–localized M–theory fluxes. On the other hand, in the presence of only localized fluxes, the manifold may or may not be Kähler. The generic formula for the non–Kählerity is given by the relation:

$$dJ \;=\; \alpha' * [\Omega_3(\omega_+) - \Omega_3(A)] + e^{-2\phi} d\phi \wedge J \tag{5.1}$$

where A is the one-form gauge bundle and ϕ is the heterotic dilaton. Apparently, even in the absence of a background three–form, the manifold can become non–Kähler due to vector bundles and non–trivial dilaton. Thus, Kählerity is restored only when

$$\omega_+ \;=\; A, \qquad \phi \;=\; \text{constant} \tag{5.2}$$

which is precisely the condition studied in [89]! Here we have derived the condition by demanding a Kähler compactification from the generic equation for dJ.

Let us now imagine that we do not turn on any non–localized gauge fluxes and at the same time do not allow the standard embedding. Then naively we would expect to get a non–Kähler manifold with the non–Kählerity coming precisely from the difference $\Omega_3(\omega_+) - \Omega_3(A)$ (and the dilaton). In fact, the three–form fluxes will typically look like [75], [46]

$$H = f + \frac{\alpha'}{2} \,\mathrm{Tr}\left(\omega_0 \wedge \tilde{f} \wedge \tilde{f} + \tilde{f} \wedge \mathcal{R}_{\omega_0} + \frac{1}{2}\tilde{f} \wedge d\tilde{f} - \frac{1}{6}\tilde{f} \wedge \tilde{f} \wedge \tilde{f}\right) + \mathcal{O}(\alpha'^2) \qquad (5.3)$$

where $f = \alpha'[\Omega_3(\omega_0) - \Omega_3(A)]$ and ω_0 is the *gravitational* spin-connection at zeroth order in α' (see [46] for more details). We have also defined $\tilde{f}$ as a one–form created from f using the vielbeins, and

$$\mathcal{R}_{\omega_0} = d\omega_0 + \frac{2}{3}\omega_0 \wedge \omega_0. \qquad (5.4)$$

The above three–form back-reacts on the geometry to make the space non–Kähler. Thus, vector bundles without the standard embedding seem to be allowed only on non–Kähler manifolds. This would seem to contradict the result of [90] where it was found that a fractional gauge Chern–Simons term can appear in an ordinary Kähler compactification. However, this apparent puzzle can be resolved by taking the background *gaugino condensate* into account[41]. This can be explained as follows:

The existence of torsion in the heterotic theory is a direct consequence of G–fluxes in M–theory [4], [91], [23]. As mentioned above, there are two kinds of fluxes: localized and non–localized ones [45], [46]. M–theory anomaly cancellation (i.e. non–zero Euler number) requires fluxes for consistency. Considering only non–localised fluxes gives rise to heterotic torsion. But this picture is in general not complete. This is because localized fluxes at the orbifold singularities are inevitable consequences of the existence of *normalizable* harmonic forms near the singularities [92]. The situation becomes more involved because of the presence of non–localized fluxes that back–react on the harmonic forms by reverse–backreacting on the geometry. In a generic situation, the harmonic forms may not be easy to evaluate. However for our case this could be done [92]. A more standard analysis of how both kinds of fluxes conspire to give the right Bianchi identity in the heterotic theory has been presented in [45]. We will not go into details here, and conclude that flux

[41] The gaugino condensate contributes to the (3,0) and the (0,3) part of the three–form, and can break susy. But we will consider a condensate that preserves susy.

compactifications in M–theory generically lead to vector bundles in the heterotic side, and in special cases, also to torsion[42].

To conclude, the manifold can still become non–Kähler in the absence of flux via the relation (5.1). In [90] the $\Omega_3(\omega)$ term was cancelled by one of the Chern-Simons terms of the gauge fields. Therefore, the non–Kählerity in this model arose from vector bundles of one of the E_8 gauge groups. To obtain a Calabi–Yau space we need $dJ = 0$. This can only be true with an additional contribution to the superpotential of [75], [76]. This additional contribution has been worked out in [77] following the work of [93] (see also [94] for some recent works). The equation for non–Kählerity is now given by

$$dJ \;=\; \alpha' * [\Omega_3(\omega_+) - \Omega_3(A)] - \alpha' * \langle \bar{\chi}^A \Gamma \chi^A \rangle + e^{-2\phi} d\phi \wedge J \qquad (5.5)$$

where χ^A are the gaugino fields of heterotic theory.

The above equation can in principle have solutions with $dJ = 0$ if the dilaton is chosen to be constant. In fact, both the gaugino condensate term and the anomaly term are of the same order in α' if we ignore the $\mathcal{O}(\alpha'^2)$ terms from (5.3). If we cancel the Ω_3 term with one of the E_8 terms $\Omega_3(A_1)$ in (5.5), then dJ vanishes (for a constant dilaton) iff we choose a negative sign for the condensate. This could in principle resolve the puzzle raised for [90]. Furthermore, the above identification restricts the possible values for the gaugino condensate.

To summarize, a dynamical way to study a flux background is to take our proposed superpotential and solve for dJ using the various contributions (tree level, perturbative

[42] Recall that the complex three–form that appears in heterotic theory in the presence of torsion has to be imaginary self–dual (ISD) to preserve supersymmetry in four dimensions. This implies the background equation (5.1). This equation is more general than the constraint derived in [61],[44], [59] and it reduces to the known form when the manifold is complex [75]. This equation makes the non–Kähler nature of the manifold manifest. Observe, that if we scale the metric then the three–form H scales linearly, too. On the other hand, from the Bianchi identity we observe that the three–form does not scale (at least to the lowest order in α'). This implies that the radial modulus should get stabilized [45], [75]. This argument, although correct, is rather naive at this point. The fact that the Bianchi identity does not scale is only true for the Kähler case. In the non–Kähler case the three form appears on both sides of the identity (as we saw in the derivation of the correct background three–form). Therefore the correct way to study the potential for the radial modulus would be to evaluate the three–form flux order by order in α' and use the kinetic term to calculate the potential. This was done in [75], [46].

and non-perturbative). If $dJ = 0$ with integrable complex structure we obtain Kähler CY compactifications. All other cases, i.e. when $dJ \neq 0$ with integrable or non–integrable complex structure, will correspond to non–Kähler compactifications. In this case the radius is stabilized at tree level. In the Kähler case, the radius is fixed non–perturbatively [95],[96],[90]. In both cases the σ–model conformal invariance is restored at *that* particular radius.

Thus, along with the mathematical analysis of vector bundles in sec. 4, we have a nearly complete picture of the gravity dual in heterotic theory. Recently, a detailed study of vector bundles on *compact* non–Kähler manifolds that are some T^2 bundle over a $K3$ base (the examples studied in [4], [23], [91], [97], [45], [75], [46]) was presented in [68] (see also [67] for earlier works on the subject from a mathematical point of view and [66], [46] for a physical point of view).

6. Global type IIA background

We have now sufficiently elaborated on the heterotic and the type IIB setup. In type IIB the *local* metric is the metric of D5 branes wrapped on a two– cycle of a resolved conifold *a-la* [21]. The *global* picture is obtained from an F–theory background that we discussed in sec. 2.1, and has F–theory seven–branes distributed in some particular way determined by an underlying Weierstrass equation. The whole configuration preserves $\mathcal{N} = 1$ supersymmetry.

We will now discuss the global type IIA background, which will turn out to have intersecting D6–branes and O6–planes on a geometry that looks locally like the non–Kähler deformed conifold constructed in [18]. However, the F–theory setup will change the allowed fluxes, which have to be invariant under a certain orientifold action. This will affect the metric only in a minor way, but we will show that we find fluxes which cannot be completely gauged away. This is in contrast to the $\mathcal{M}$–theory lift advocated in [18], where our non–Kähler manifold lifted to a purely geometrical background in 11 dimensions. The background constructed here will lift to an 11–dimensional background with G–fluxes. We comment on the implications for G_2 structure versus G_2 holonomy in the next section.

6.1. Type IIA background revisited

The full *global* type IIA is now determined as a mirror of the global IIB background. The local metric was already discussed in [18] and [19]. The mirror D6 branes in the local picture now wrap the three cycle of a non–Kähler deformed conifold. Globally there are additional D–branes and O–planes. They are the mirror of the type IIB D7/O7 system. To specify the precise coordinates of the branes in type IIA picture, let us consider carefully the orientations of the branes in type IIB case. In the local type IIB background, the resolved conifold has two two–tori oriented along (x, θ_1) and (y, θ_2). The $U(1)$ fibration is given by the coordinate z, and the radial direction is r. In this local framework, let us assume that the D5 branes wrap the torus (y, θ_2) and are stretched along spacetime coordinates $x^{0,1,2,3}$. A mirror transformation results in D6 branes oriented along (z, x, θ_2) and stretched along the usual spacetime directions $x^{0,1,2,3}$. The local non-Kähler metric of a *deformed* conifold is given by (2.1). The three–cycle is a $U(1)$ fibration over a compact two–dimensional space specified by (α, β) such that

$$ds_2^2 = \tilde{g}_2\ d\theta_1(d\theta_1 - 2\alpha) + g_3\ dy(dy - 2\beta) \tag{6.1}$$

forms *another* two dimensional subspace inside the non–Kähler deformed conifold[43]. If we denote the type IIB B_{NS} as $(b_{x\theta_1},\ b_{y\theta_2}) \equiv (b_1,\ b_2)$, then the differential forms (α, β) and the function $\tilde{g}_2$ are

$$\alpha\ =\ \sin(2\ \tan^{-1} b_1)\ dx, \qquad \beta\ =\ b_2\ dy, \qquad \tilde{g}_2\ =\ g_2(1 + b_1^2) \tag{6.2}$$

with g_2, g_3 defined in [18], [19] and in (2.1). The metric of the three–cycle can now be written as $ds_3^2 = G_{mn}\ dy^m dy^n$ with $y^m = (z, x, \theta_2)$, and

$$G = \begin{pmatrix} G_{zz} & G_{zx} & G_{z\theta_2} \\ G_{xz} & G_{xx} & G_{x\theta_2} \\ G_{\theta_2 z} & G_{\theta_2 x} & G_{\theta_2\theta_2} \end{pmatrix}$$

$$= \begin{pmatrix} g_1 & g_1\ \Delta_1\ \cot\widehat{\theta}_1 & -g_1\ b_2\ \Delta_2\ \cot\widehat{\theta}_2 \\ g_1\ \Delta_1\ \cot\widehat{\theta}_1 & g_2 + g_1\ \Delta_1^2\ \cot^2\widehat{\theta}_1 & -g_1\ b_2\ \Delta_1\ \Delta_2\ \cot\widehat{\theta}_1\ \cot\widehat{\theta}_2 \\ -g_1\ b_2\ \Delta_2\ \cot\widehat{\theta}_2 & -g_1\ b_2\ \Delta_1\ \Delta_2\ \cot\widehat{\theta}_1\ \cot\widehat{\theta}_2 & \tilde{g}_3 + g_1\ b_2^2\ \Delta_2^2\ \cot^2\widehat{\theta}_2 \end{pmatrix} \tag{6.3}$$

[43] In the language of the non–Kähler metric, the $U(1)$ fibration is over the *full* four–dimensional subspace and is not restricted to the 2d base (α, β).

with $\tilde{g}_3$ defined as $\tilde{g}_3 = g_3(1 + b_2^2)$. (6.3) is the precise metric of the three–cycle on which we have wrapped D6 branes in the local geometry.

To study the full global picture, we have to contemplate a few possible scenarios in type IIB theory, all resulting from different torus fibrations of the F–theory fourfold:

(1) The F–theory torus is fibered non–trivially over the two–dimensional torus parametrized by (x, y). This would mean that the seven branes wrap directions (z, θ_1, θ_2) and are stretched along the non–compact directions $(r, x^{0,1,2,3})$.

(2) The F–theory torus is fibered non–trivially over a different two dimensional torus parametrised by (x, θ_1). Here the seven branes would wrap directions (y, θ_2, z), and are stretched along the same non–compact directions as above.

(3) The F–theory torus is fibered non–trivially over a compact two dimensional surface parametrised by (θ_1, θ_2). The seven branes in this scenario will wrap directions (x, y, z) and are stretched along the other non–compact directions as above.

Three T–dualities along (x, y, z) will act non–trivially in all these cases. Let us consider case (1) first. Two T–dualities along (x, y) will take us to type I $SO(32)$ theory. This is precisely the background studied in [19]. An S–duality to this background will give us the required heterotic manifold, whose global geometry we studied in section 3.

A further T–duality along z direction to the type I geometry will take us to type IIA. In fact, the type IIA background is an orientifold background. But since an orientifold operation is *not* a symmetry of type IIA theory, we should also accompany the orientifold action Ω with a space reversal along the z direction, $\mathcal{I}_z : z \rightarrow -z$. Thus the duality chain for this case is

$$\text{Type I on } \frac{S^1_z}{\Omega} \quad \xrightarrow{T_z} \quad \text{Type IIA on } \frac{S^1_z}{\Omega \cdot \mathcal{I}_z \cdot (-1)^{F_L}} \tag{6.4}$$

where T_z denote T–duality along the z direction and $(-1)^{F_L}$ changes the sign of the left–moving Ramond sector. The two fixed points of $\mathcal{I}_z$ are the two $O8$ planes, their charge being cancelled by D8–branes [72], [98].

The global picture has therefore wrapped D6–branes on three–cycle of a non–Kähler manifold, along with a $D8 - O8$ system at the two fixed points of z (or ψ in global coordinates). The $D8 - O8$ system intersects the D6 branes along a compact two dimensional torus (x, θ_2) and the 3+1 dimensional non–compact spacetime $x^{0,1,2,3}$.

In case (2), a mirror transformation gives rise to another type IIA background modded out by an orientifold action. The precise duality chain is

$$\text{Type IIB on} \quad \frac{T^5}{\Omega\cdot\mathcal{I}_{x\theta_1}\cdot(-1)^{F_L}} \xrightarrow{T_{xyz}} \text{Type IIA on} \quad \frac{T^5}{\Omega\cdot\mathcal{I}_{yz\theta_1}\cdot(-1)^{F_L}} \tag{6.5}$$

where the T^5 torus is given by $T^5 = T^2_{(x\theta_1)} \times T^2_{(y\theta_2)} \times S^1_z$, with S^1_z forming the $U(1)$ fibration discussed above. The fixed points of this action are the $O6$ planes whose charges are cancelled by D6–branes. This $D6-O6$ system intersects with the other wrapped D6 branes along the same compact two–torus (x,θ_2) and the 3+1 dimensional non–compact spacetime.

Finally, in case (3), we expect the type IIA orientifold operation to be $\Omega\cdot\mathcal{I}_{xyz\theta_1\theta_2}\cdot(-1)^{F_L}$. The fixed points of this action are the $O4$ planes which need $D4$ branes to cancel the total charge locally. These four–branes are oriented along the non–compact directions $(r, x^{0,1,2,3})$, and therefore overlap with the wrapped D6 branes only along the spacetime directions.

Let us now study the metric. Case (1) with intersecting $D6-D8-O8$ system cannot be the global completion of our type IIA background. Recall, that the original type IIB (local) metric is *not* invariant under the orientifold operation of case (1) [19]. There are cross terms in the metric that are not invariant under orientation reversal. The invariant part of the metric can eventually give us a valid heterotic background, but it lacks the cross terms characteristic for the deformed conifold. This will not lead to the non–Kähler deformed conifold we require for the geometric transtition in IIA [44].

One can also easily rule out case (3), where the F–theory fibration is along the compact direction (θ_1,θ_2) in the local picture. A coordinate reversal transformation $\mathcal{I}_{\theta_1\theta_2} : \theta_i \to -\theta_i$ does not appear to be a symmetry of the local metric (2.2). So the metric cannot provide the type IIA global picture.

This leaves us with case (2) where the F–theory torus is fibered over the (x,θ_1) torus. Not only is the IIB metric (2.2) invariant under $(x,\theta_1) \;\to\; (-x,-\theta_1)$, but also the two tori with non–trivial complex structure (boosted by $f_{1,2}$). This means that after a mirror

[44] Note, that even an attempt to obtain a IIA background in this scenario is futile, because the global metric in type I/heterotic lost its isometry along z (or ψ). Therefore we cannot perform the last T–duality that would lead to a IIA background. But, as explained above, even if it was possible to T–dualize along non–isometry directions (see e.g. [99]) the resulting background would not be the one we require in IIA.

transformation we will have a global intersecting $D6 - O6 - D6$ system in type IIA with a local configuration of wrapped D6 branes on the three cycle of a non–Kähler deformed conifold. Since no part of the metric is projected out under orientifold operation, this will serve as the full global picture in type IIA.

Having obtained a complete global picture in terms of $D7/O7$ in type IIB or $D6/O6$ in type IIA, we now turn to the question what the global picture means in terms of the underlying $\mathcal{N} = 1$ gauge theory. The answer, as anticipated in section 2.2., is simple because in type IIB the existence of $D7/O7$ implies *global* symmetries in the gauge theory [26]. In our case the global symmetry will be $SU(2)^{16}$ instead of the expected $SO(8)^4$ because of Wilson lines. This means that we are actually studying $\mathcal{N} = 1$ $SU(N)$ gauge theory with fundamental flavors transforming under the global symmetry $SU(2)^{16}$.

6.2. More subtleties

The above conclusion, although correct, is unfortunately premature. The underlying orientifolding/orbifolding operation makes the situation rather tricky.

First, observe that the type IIB orientifold operation has a non–trivial effect on the B_{NS} and B_{RR} fields. The original choice of B_{NS} fields $(b_{x\theta_1}, b_{y\theta_2})$ are projected out. This would mean that the fibration structures in the type IIA local metric would also have to change. In fact the three cases studied earlier give rise to the following choices of type IIB B_{NS} fields:

• Case (1): Ω and $\mathcal{I}_{xy}$ allow $b_{x\theta_1}, b_{x\theta_2}, b_{y\theta_1}, b_{y\theta_2}$. We used these choices to determine the heterotic manifold after geometric transition in earlier sections.

• Case (2): Ω and $\mathcal{I}_{x\theta_1}$ only allow $b_{x\theta_2}$ and $b_{y\theta_1}$ [45]. This implies that the metric in IIA will have a different fibration structure compared to the one that gave rise to the heterotic bckground.

• Case (3): Ω and $\mathcal{I}_{\theta_1\theta_2}$ also allow $b_{x\theta_1}, b_{x\theta_2}, b_{y\theta_1}, b_{y\theta_2}$. However this choice of orientifolding is not useful enough to get a global IIA metric, as we discussed above.

The type IIA *local* metric will change accordingly to accomodate the actions of $O6$ planes. The metric still remains a non–Kähler deformation of a Calabi-Yau deformed

[45] The orientifold action also allows for other components of B_{NS} fields like $b_{xz}, b_{xr}, b_{z\theta_1}, b_{r\theta_1}$ but we can set them to zero and study the theory with only two components consistently.

conifold, but is now given as

$$
\begin{aligned}
ds^2_{IIA} = \;\; & g_1 \left[dz + \Delta_1 \cot \widehat{\theta}_1 \; (dx - b_{x\theta_2} \; d\theta_2) + \Delta_2 \cot \widehat{\theta}_2 \; (dy - b_{y\theta_1} \; d\theta_1) + ...\right]^2 + \\
& + g_2 \; [d\theta_1^2 + (dx - b_{x\theta_2} \; d\theta_2)^2] + g_3 \; [d\theta_2^2 + (dy - b_{y\theta_1} \; d\theta_1)^2] + g_4 \sin \psi \; [(dx - b_{x\theta_2} \; d\theta_2) \; d\theta_2 \\
& + (dy - b_{y\theta_1} \; d\theta_1) \; d\theta_1] + \; g_4 \cos \psi \; [d\theta_1 \; d\theta_2 - (dx - b_{x\theta_2} \; d\theta_2)(dy - b_{y\theta_1} \; d\theta_1)].
\end{aligned}
\tag{6.6}
$$

One can also check that the metric remains invariant under the orientifold operation $\Omega \cdot \mathcal{I}_{yz\theta_1}$ when rotating by ψ as in [18]. This implies that we should set $\psi = \langle\psi\rangle$ in (2.9). Allowing for non–constant ψ (i.e. extending ψ to z) would result in a non–trivial constraint on the warp factor g_4 in (6.6):

$$
g_4 \;\; = \;\; g_4(r, \theta_1, \theta_2)|_{r=r_0} \;\; = \;\; \sum_n a_n \; \theta_1^{2n+1} \; \tilde{g}_4(r_0, \theta_2) \tag{6.7}
$$

where a_n are some non–zero numbers and $\tilde{g}_4$ is another function of $(r = r_0, \theta_2)$.

One might also contemplate a more generic relation between ψ and z as $\psi \;\; = \;\; f(z)$. For the metric (6.6) to remain invariant under orientifold action, the function $f(z)$ has to satisfy

$$
f(z) \;\; + \;\; f(-z) \;\; = \;\; \pi. \tag{6.8}
$$

Although such a choice would put no constraints on the warp factors g_i in (6.6), the functional dependence would not be consistent with the simple choice of the small coordinate shifts in type IIB (2.9). Therefore, (6.7) seems to be the only reasonable constraint we can impose on the warp factor g_4.

The next subtleties are related to the choice of B fields in type IIB. The new choice of B_{NS} fields ($b_{x\theta_2}$ and $b_{y\theta_1}$) will induce the following B_{NS} fields in the mirror type IIA background:

$$
\begin{aligned}
B_{\rm IIA} \;\; = & \sqrt{\langle\alpha\rangle_1} \; (dx - b_{x\theta_2} d\theta_2) \wedge d\theta_1 - \sqrt{\langle\alpha\rangle_2} \; (dy - b_{y\theta_1} d\theta_1) \wedge d\theta_2 + A\sqrt{\langle\alpha\rangle_1} \; d\theta_1 \wedge dz + \\
& + B\sqrt{\langle\alpha\rangle_2} \; (dy \sin \psi - \cos \psi \; d\theta_2 - b_{y\theta_1} \; \sin \psi \; d\theta_1) \wedge dz
\end{aligned}
\tag{6.9}
$$

where $\langle\alpha\rangle_i$ and (A, B) were defined in [18], [19]. We wrote the B_{NS} field using $\langle\alpha\rangle_i$ to emphasize the fact that some of the terms are pure gauge (see [19] for more details on this). Does that means that we can gauge them away? It turns out that because of the presence of D6 branes, we *cannot* gauge away all components of B_{NS}. The ungauged part of the B_{NS} field will appear as gauge fluxes on the D6 branes.

Since this issue is a little subtle, we will tread carefully in the following. We can perform a ψ rotation on (y, θ_2) as in (3.23) by identifying $\langle\psi\rangle = \psi$ (see also [18]). This will change the B_{NS} field into

$$\begin{aligned} B_{\rm IIA} &= (dx - b_{x\theta_2} d\theta_2) \wedge d\tilde{\theta}_1 - (dy - b_{y\theta_1} d\theta_1) \wedge d\tilde{\theta}_2 + (d\widehat{\theta}_1 + d\widehat{\theta}_2) \wedge dz \\ &= b_{x\theta_2}\ d\tilde{\theta}_1 \wedge d\theta_2 + b_{y\theta_1}\ d\theta_1 \wedge d\tilde{\theta}_2 + dx \wedge d\tilde{\theta}_1 - dy \wedge d\tilde{\theta}_2 + (d\widehat{\theta}_1 + d\widehat{\theta}_2) \wedge dz \end{aligned} \tag{6.10}$$

where we have isolated the B–field part that depends on the type IIB B–fields. The quantities appearing here are defined as:

$$\begin{aligned} \tilde{\theta}_1 &= -\frac{1}{k\sqrt{b_1}} \left[\ln(k\ \cos\ \theta_1 + \widehat{\Delta}_1)\right] \\ \widehat{\theta}_1 &= \frac{1}{k\sqrt{b_1}} \left[\sqrt{b_1}\ \arctan\ \frac{k\ \sin\ \theta_1}{\widehat{\Delta}_1}\right] \\ k &= \sqrt{\frac{b_1 - a_1}{a_1}}, \qquad \widehat{\Delta}_1 = \sqrt{1 - k^2\ \sin^2\ \theta_1} \\ \langle\alpha\rangle_1 &= \frac{1}{1 + (\Delta_1^0)^2\ \cot^2\ \theta_1 + (\Delta_2^0)^2\ \cot^2\ \langle\theta_2\rangle} \equiv \frac{1}{a_1 + b_1\ \cot^2\ \theta_1} \end{aligned} \tag{6.11}$$

with an equivalent description for $\tilde{\theta}_2$ and $\widehat{\theta}_2$. Using this one can show that the b–independent parts of (6.10) are pure gauge.

How many components of the b–independent part of (6.10) can be gauged away? In the absence of $D6$ branes, all the components can be gauged to zero. Ignoring the spacetime orientations of the six–branes and planes, we see that the b–independent part of (6.10) that would appear as gauge flux F on the wrapped $D6$ branes would be

$$F_{z\theta_2} = \lim_{\epsilon\to 0} \left[\epsilon^{-1}\ B\sqrt{\langle\alpha\rangle_2}\right]\ d\theta_2 \wedge dz \tag{6.12}$$

on a non–trivial two–cycle inside the three–cycle of the non–Kähler deformed conifold. This gauge flux is very large and will provide a non–commutativity parameter to the gauge theory on wrapped $D6$ branes[46]. All other b–independent components of (6.10) are gauge equivalent to zero[47].

Let us now consider the b–dependent part of (6.10). The type IIA B–field depends on the type IIB components $(b_{x\theta_2}, b_{y\theta_1})$ which could have non–zero field strength. Because of

[46] Observe that the $3+1$ dimensional $\mathcal{N} = 1$ $SU(N)$ gauge theory is still commutative.

[47] Observe that there are no possibilities of any *pinning* effect [100] or any *dipole* behavior [101] here. For topologically trivial cycles, we would have encountered more involved scenarios.

the underlying mirror transformation the components of (6.10) should be independent of (x, y, z). Furthermore, the b–dependent terms in (6.10) have wedge structures of the form $d\theta_1 \wedge d\tilde{\theta}_2$ and $d\tilde{\theta}_1 \wedge d\theta_2$. If the components $(b_{x\theta_2}, b_{y\theta_1})$ were functions of (θ_1, θ_2) *only* and not of r, the radial coordinate, we would find $dB_{\rm IIB} \neq 0, dB_{\rm IIA} = 0$, i.e. the type IIA B–field (6.10) will have no field strengths, whereas the type IIB B–field will have a non–zero field strengths[48]. Such B–field components are projected out *at* the orientifold point, but away from the orientifold point[49] we can have $b_{x\theta_2} = b_{x\theta_2}(\theta_1, \theta_2)$ and $b_{y\theta_1} = b_{y\theta_1}(\theta_1, \theta_2)$. Therefore, if we we want to gauge away as many components of $H_{\rm IIA}$ as possible, our first ansätze for the type IIB B–field components would be

$$b_{x\theta_2} = \sum_{m,n} b_{mn} \cot^m \theta_1 \cot^{2n} \theta_2, \qquad b_{y\theta_1} = \sum_{k,l} c_{kl} \cot^k \theta_1 \cot^{2l} \theta_2 \tag{6.13}$$

where b_{mn} and c_{kl} are integers independent of (x, y, z, θ_i, r). The precise numbers will be determined by the background equations of motion. Plugging this ansatz (6.13) into (6.10), one of the b–dependent component takes the following form:

$$b_{y\theta_1}\, d\theta_1 \wedge d\tilde{\theta}_2 = \sum_{k,l} c_{kl}\, d\Theta_1 \wedge d\Theta_2 \tag{6.14}$$

which is again a pure gauge. Similarly, the other b–dependent component can also be expressed as a pure gauge. The variables Θ_i are defined in terms of θ_i as

$$\Theta_1 = \int \cot^k \theta_1, \qquad \Theta_2 = \int \frac{\cot^{2l} \theta_2}{a_2 + b_2 \cot^2 \theta_2}\ . \tag{6.15}$$

Expressing (6.10) in terms of Θ_i, none of these components are parallel to the six–branes. Thus, they would not contribute to fluxes in type IIA theory and could be completely gauged to zero. Then the only non–trivial effect from type IIA B–field will be the appearence of the gauge flux (6.12) on an internal two–cycle of the non–Kähler deformed conifold[50].

[48] Type IIB NS threeform field $H_{\rm IIB} \equiv \mathcal{H}$ components will be $\mathcal{H}_{x\theta_1\theta_2} = \partial_{[\theta_1} b_{|x|\theta_2]}$ and $\mathcal{H}_{y\theta_1\theta_2} = \partial_{[\theta_2} b_{|y|\theta_1]}$.

[49] In fact away from the orientifold point, all components of the B–field can be allowed.

[50] The reader might wonder about switching on a type IIB component $b_{z\theta_2}$ which in type IIA would be mirror to a component parallel to the six–branes. However, such a component would be projected out in IIB by the orientifold operation. Away from the orientifold point, such a component will be allowed.

Above analysis assumed the seven branes of type IIB being far away so that we are no longer at the orientifold point. This is consistent with the picture of integrating out all the flavors to study $\mathcal{N} = 1$ pure $SU(N)$ Yang–Mills theory. The choice of type IIB B_{NS}–field determines the allowed components of the type IIB B_{RR}–field from the linearized equation of motion

$$H_{RR} = *_6 H_{NS} \tag{6.16}$$

where the Hodge–$*_6$ is w.r.t. the six dimensional internal space. (6.16) allows for the following components of $H_{RR} \equiv h$: h_{yzr} and h_{xzr}. This is equivalent to choosing $\partial B_{RR} \equiv \partial B$ as $\partial_r B_{xz}$ and $\partial_r B_{yz}$. This means that one particular consistent choice of B–fields in type IIB *away* from the orientifold point is

$$B_{NS}: \ [b_{x\theta_2}(\theta_1, \theta_2), b_{y\theta_1}(\theta_1, \theta_2)] \qquad B_{RR}: \ [B_{xz}(r), B_{yz}(r)] \ . \tag{6.17}$$

So now the natural question would be: which B–field components in type IIB and type IIA are allowed *at* the orientifold point? Observe that the choice of B–fields away from the orientifold point (6.17) is not very symmetric. It would seem more natural to switch on *similar* components for both B_{NS} and B_{RR} (see the solutions in [1] and [21]). A possible choice at the orientifold point, that also satisfies (6.16), would be to define the B_{NS}–field as $B_{NS}: \ [b_{x\theta_2}(r), b_{y\theta_1}(r)]$ and the field strengths as:

$$\begin{aligned} H_{NS} &: [\mathcal{H}_{x\theta_2 r} = \partial_r b_{x\theta_2}, \qquad \mathcal{H}_{y\theta_1 r} = \partial_r b_{y\theta_1}] \\ H_{RR} &: [h_{x\theta_2 z} = *_6 \mathcal{H}_{y\theta_1 r}, \qquad h_{y\theta_1 z} = *_6 \mathcal{H}_{x\theta_2 r}] \, . \end{aligned} \tag{6.18}$$

Are these choices of B–fields away from the orientifold point (6.17) and at the orientifold point (6.18) consistent? We propose the following scenario:

• Locally, the radial dependence in (6.18) will not be visible because of (2.9). Thus, H_{NS} and H_{RR} are very small in our region of interest.

• Once we go away from the orientifold point, then non–perturbative effects will change the background [43]. In particular, $b_{x\theta_2}, b_{y\theta_1}$ will now also depend on θ_i.

Thus, given a background (6.18) we can deform it into (6.17) with non–perturbative corrections. The deformation happens when we want to move the seven branes away from the wrapped D5 branes. This is the case when one of the orientifold points coincides with the resolution two–cycle of the F–theory base discussed in sec. 2.1. So we can still remain *at* the orientifold point if the resolution two–cycle *does not* coincide with *any* of the orientifold points. This is again similar to the situation encountered in the F–theory construction with Klebanov–Strassler base [5]. The seven branes and orientifold points do not coincide with the conifold point (where we have wrapped D5 branes).

Before moving ahead let us clarify one more point. The B_{RR} fields for the two cases that we discussed above i.e (6.17) and (6.18), would seem to imply that we have changed the orientation of the wrapped D5 branes. Recall that our initial wrapped D5 branes were on a two-torus along (y, θ_2) directions. However the choice (6.18) would seem *not* to allow such wrapped D5 branes. This is an illusion, because the choice (6.18) will change completely as soon as we go away from the orientifold point. Once we are away from the orientifold point, the sources for the wrapped D5 can be easily shown to be present there. However, there is another much deeper reason why we can allow all kind of configurations of wrapped D5 branes in our scenario. This has to do with the fact that we are defining the background only locally. In the local version, the type IIB metric can be rewritten as

$$ds^2 = dr^2 + (dz + A\ dx + B\ dy)^2 + (d\theta_1^2 + dx^2) + (d\theta_2^2 + dy^2) + ... \tag{6.19}$$

where we can extract the values of A and B from (2.10) as:

$$A \ = \ \Delta_1^0 \cot\langle\theta_1\rangle + \mathcal{O}(\theta_1), \qquad B \ = \ \Delta_2^0 \cot\langle\theta_2\rangle + \mathcal{O}(\theta_2) \tag{6.20}$$

which tells us that they are basically constants. This would imply that the fibration structure can be written as $d\tilde{z} \ \approx \ dz + A\ dx + B\ dy$. Therefore from the above local metric, D5 branes wrapping directions (y, θ_2), can be easily traded with (r, y) directions and vice-versa. So the precise way by which we can go from the global type IIB model to the local metric (2.2) is to put non-trivial complex structure on the (x, θ_1) and (y, r) tori, trade the local r coordinate with local θ_2 and then redefine the $d\tilde{z}$ fibration structure accordingly. This way configuration *at* the orientifold point, and configuration *away* from the orientifold point will be identical.

Thus at the orientifold point, we can stick with our configuration (6.18). Now assuming such a scenario simplifies the ensuing analysis, because we do not have to worry about non–perturbative corrections from the splitting of orientifold–seven planes in type IIB, the resulting O6 planes in type IIA will not split either. We then employ the following generic choice of type IIB RR field:

$$\begin{aligned} B_{RR} \ &= \ c_1\ dx \wedge dz + c_2\ dx \wedge dy + c_3\ dx \wedge d\theta_2 \ + \\ &+ c_4\ dy\ \wedge d\theta_1 + c_5\ dz \wedge d\theta_1 + c_6\ d\theta_1 \wedge d\theta_2 \end{aligned} \tag{6.21}$$

where we have allowed for two new components c_2 and c_6 that are naively allowed under orientifold action. The coefficients c_i are in general functions or $(r,\ \theta_1,\ \theta_2)$. The B_{NS} components are still $(b_{x\theta_2}, b_{y\theta_1})$. With this choice of B_{RR}, we obtain the following one–form field $\mathcal{A}$ in the mirror type IIA:

$$\mathcal{A} = \frac{2AB}{1+A^2}(c_1 - \alpha Bc_2)\ (dx - b_{x\theta_2}d\theta_2) + (c_1 - 2\alpha Bc_2)\ (dy - b_{y\theta_1}d\theta_1) - c_2\ dz \tag{6.22}$$

where we used the freedom of choosing the orientation of B_{NS} similar to [18]. The fibration structure of the metric (6.6) remains as before. The one–form (6.22) will be sourced by the wrapped D6 branes.

Apart from the one form, there will also be three–forms in the mirror IIA. They arise precisely because of our choices of B_{NS} components in IIB. They are given by:

$$\begin{aligned}\mathcal{C} \ = \ \ & C_{xy1}\ (dx - b_{x\theta_2}d\theta_2)\wedge(dy - b_{y\theta_1}d\theta_1)\wedge d\theta_1\\ & + C_{xy2}\ (dx - b_{x\theta_2}d\theta_2)\wedge(dy - b_{y\theta_1}d\theta_1)\wedge d\theta_2\\ & + C_{xz1}\ (dx - b_{x\theta_2}d\theta_2)\wedge dz\wedge d\theta_1 + C_{yz2}\ (dy - b_{y\theta_1}d\theta_1)\wedge dz\wedge d\theta_2\\ & + C_{xz2}\ (dx - b_{x\theta_2}d\theta_2)\wedge dz\wedge d\theta_2 + C_{yz1}\ (dy - b_{y\theta_1}d\theta_1)\wedge dz\wedge d\theta_1\end{aligned} \tag{6.23}$$

where the coefficients are defined in the following way:

$$\begin{aligned} C_{xy1} &= -c_5 + f_1c_1 + \frac{2\alpha f_1c_1A^2B^2}{1+A^2} + 2\alpha B(c_4 - f_1c_2)\\ C_{xy2} &= \frac{2\alpha AB^2}{1+A^2}\Big[-c_3 + f_2(Bc_1 - c_2)\Big]\\ C_{xz1} &= c_4 - f_1c_2 + \frac{2f_1c_1A^2B}{1+A^2}, \qquad C_{yz2} = -c_3 + f_2(Bc_1 - c_2)\\ C_{xz2} &= \frac{2AB}{1+A^2}\Big[-c_3 + f_2(Bc_1 - c_2)\Big], \qquad C_{yz1} \ = \ A\ f_1c_1\end{aligned} \tag{6.24}$$

We can make the following interesting observations:

• All individual coefficients can be separated into f_i–dependent and f_i–independent parts.

• The terms containing f_1 are always accompanied by $d\theta_1$. The same statement is true for the f_2–dependent terms. These terms (like $f_1d\theta_1$ and $f_2d\theta_2$) are exact functions, and can therefore be written in terms of $d\tilde{\theta}_i$ of (6.11).

• The f_i–dependent terms are also accompanied by either c_1 or c_2 or both.

This raises the suspicion that maybe all RR–fluxes are pure gauge artifacts, too. We will show in the following that this is not the case, but that we obtain G–fluxes which will contribute to $\mathcal{M}$–theory fluxes. These, in turn, allow for a smooth flop transition, avoiding the singular point in the conifold transition.

It suffices first to consider a simple case where some of the coefficients c_i in (6.21) are allowed to vanish. What is the minimum number of c_i that have to be non–zero? Obviously, the one–form (6.22) depends only on c_1 and c_2. Since this has to be non–zero because of the wrapped D6–branes, we will make the simplifying assumption that all other c_i vanish. This doesn't quite work because if c_2 is non-zero, then from (6.21) we see that this switches on a component $dx \wedge dy$. But from (6.18) we know that such a component doesn't exist, at least for the simplfied case that we wanted to analyse. Thus we take:

$$\textbf{Case 1}: \quad c_1 = c_1(\theta_2), \quad c_2 = c_3 = c_4 = c_5 = c_6 = 0 \tag{6.25}$$

where the θ_2 dependence of c_1 is motivated from the choice of background H_{RR} in (6.18)[51]. This implies for the type IIB H_{RR} fields:

$$h_{xz\theta_2} = \frac{\partial c_1}{\partial \theta_2}, \qquad h_{yz\theta_1} = 0 \tag{6.26}$$

which is consistent with the expected components of H_{RR} in type IIB from (6.18) if $b_{x\theta_2}$ also vanishes, or is independent of the radial coordinate. Thus the choice (6.25) implies that we choose in type IIB the local metric (2.2) and the B-fields:

$$B_{NS} = b_{y\theta_1}(r)\ dy \wedge d\theta_1, \quad B_{RR} = c_1(\theta_2)\ dx \wedge dz \tag{6.27}$$

and in mirror type IIA the following background

$$\begin{aligned} ds^2_{IIA} = &\ g_1 \left[dz + \Delta_1 \cot \widehat{\theta}_1\ dx + \Delta_2 \cot \widehat{\theta}_2\ (dy - b_{y\theta_1}\ d\theta_1) + ...\right]^2 + \\ & + g_2\ [d\theta_1^2 + dx^2] + g_3\ [d\theta_2^2 + (dy - b_{y\theta_1}\ d\theta_1)^2] + g_4 \sin\psi\ [dx\ d\theta_2 + \\ & + (dy - b_{y\theta_1}\ d\theta_1)\ d\theta_1] + \ g_4 \cos\psi\ [d\theta_1\ d\theta_2 - dx\ (dy - b_{y\theta_1}\ d\theta_1)] \\ \mathcal{A} = &\ \frac{2ABc_1}{1+A^2}\ dx + c_1\ (dy - b_{y\theta_1} d\theta_1) \\ B_{\rm IIA} = &\ b_{y\theta_1}\ d\theta_1 \wedge d\tilde{\theta}_2 \end{aligned} \tag{6.28}$$

[51] There is a little more to it. The choice of H_{RR} as in (6.25) also means that we have chosen the *orientation* of the wrapped D5 branes in our type IIB model. As we saw earlier, it doesn't really matter how the D5 branes are wrapped as long as we are taking consistent background NS and RR fields satisfying equations of motion and correct orientifold projections. Clearly (6.21) satisfies the second constraint, and we can make sure that (6.25) will satisfy both the constraints. Furthermore as we will see below, many more cases can be entertained in our global picture.

We now see that the choice (6.25) simplifies the coefficients (6.23) quite a bit. Infact, combining (6.25), (6.11) in (6.23) we can re-express (6.23) as $\mathcal{C} = \sum_{1=1}^{6} C_i$ with

$$
\begin{aligned}
C_1 &\equiv C_{[xy1]} = \left(c_1 + \frac{2\alpha c_1 A^2 B^2}{1+A^2}\right) dx \wedge dy \wedge d\tilde{\theta}_1 \\
C_2 &\equiv C_{[xy2]} = \frac{2\alpha A B^2 c_1}{1+A^2}\, d\widehat{\theta}_2 \wedge dx \wedge (dy - b_{y\theta_1} d\theta_1) \\
C_3 &\equiv C_{[xz1]} = \frac{2ABc_1}{1+A^2}\, d\widehat{\theta}_1 \wedge dx \wedge dz \\
C_4 &\equiv C_{[yz2]} = c_1\, d\widehat{\theta}_2 \wedge (dy - b_{y\theta_1} d\theta_1) \wedge dz \\
C_5 &\equiv C_{[xz2]} = \frac{2ABc_1}{1+A^2}\, d\widehat{\theta}_2 \wedge dx \wedge dz, \quad C_6 \equiv C_{[yz1]} = c_1\, dy \wedge dz \wedge d\widehat{\theta}_1
\end{aligned}
\tag{6.29}
$$

where we have separated the terms with $d\tilde{\theta}_i$ and $d\widehat{\theta}_i$ as defined in (6.11).

The existence of the $\mathcal{C}$ field component C_{xz2} implies that we can define a combination

$$\mathcal{V} = \sqrt{\det G} + i|C_5| \tag{6.30}$$

where G is the metric of the three–cycle along (x, z, θ_2) directions (6.3) and $|C_5|$ from (6.29) is the coefficient along $d\zeta \equiv dx \wedge dz \wedge d\theta_2$ but now evaluated for $c_1 = c_1(\theta_1, \theta_2)$ by deforming away from the orientifold point. Therefore, the effective volume of the three cycle along (x, z, θ_2) is $\mathcal{V}$ and not just $\sqrt{\det G}$ from (6.3). This implies that in the limit

$$\sqrt{\det G} \rightarrow 0, \qquad |\mathcal{V}| \equiv \sqrt{\det G + |C_5|^2} \rightarrow |C_5| \tag{6.31}$$

which is non-zero. Thus, when we lift the type IIA background to $\mathcal{M}$–theory as in [18] to a seven dimensional manifold with G_2 structure, a flop operation is a completely *non-singular* operation. This is perfectly consistent with the predictions of [34] and [35] and justifies the analysis of [18].

Recall that the type IIA B_{NS} fields were almost all pure gauge[52]. Let us now show that this is not the case for B_{RR}. We can simply evaluate the field strenths of the fluxes. The G–fluxes are found to be

$$G_{xy\theta_1\theta_2} = \left[\frac{A^2}{1+A^2}\frac{\partial}{\partial\theta_2}(c_1 + 2\alpha c_1 B^2)\, d\tilde{\theta}_1 \wedge d\theta_2 - 2B^2 c_1 \frac{\partial}{\partial\theta_1}\left(\frac{\alpha A^2}{1+A^2}\right) d\theta_1 \wedge d\widehat{\theta}_2\right] dx \wedge dy \tag{6.32}$$

[52] Of course this does not mean that they can all be gauged away. As discussed above, gauge flux along the brane world volume cannot be gauged away.

which is still non–zero even in the local limit where A, B and α are just constants. The most important G–flux component is

$$G_{xz\theta_1\theta_2} = \left[\frac{2A^2}{1+A^2}\frac{\partial}{\partial\theta_2}(Bc_1)\, d\tilde{\theta}_1 \wedge d\theta_2 - 2Bc_1\frac{\partial}{\partial\theta_1}\left(\frac{A}{1+A^2}\right)\, d\theta_1 \wedge d\widehat{\theta}_2\right] dx \wedge dz \quad (6.33)$$

which shows that $|C_5|$ cannot be gauged away. Finally, there is one more component that is similar to the type IIB RR three form (6.26). It is given by

$$G_{yz\theta_1\theta_2} = \frac{\partial c_2}{\partial\theta_1}\, d\theta_1 \wedge d\tilde{\theta}_2 \wedge dy \wedge dz \quad (6.34)$$

with all other components vanishing. The existence of these G–fluxes is precisely the reason why our $\mathcal{M}$–theory manifold does not have a G_2 holonomy, but only a G_2 structure. In the next section we discuss the implications for $SU(3)$ and G_2 torsion classes in IIA and $\mathcal{M}$–theory, respectively.

For completeness let us also mention that our background may also have non–zero five–form

$$C_{xyz\theta_1\theta_2} = c_6 + f_1c_3 - f_2(1+f_1)c_4 + B(c_5 - f_1c_1)f_2\,. \quad (6.35)$$

For the simple background (6.25) this is a pure gauge, and will therefore not affect type IIA dynamics.

Our next simple case would be when we allow all the components in (6.18). This would imply that both the B_{NS} components $(b_{x\theta_2}, b_{y\theta_1})$ are non-zero. The c_i in (6.21) will be

$$\textbf{Case 2}: \quad c_1 = c_1(\theta_2), \quad c_4 = c_4(z), \quad c_2 = c_3 = c_5 = c_6 = 0 \quad (6.36)$$

and the type IIA metric will become (6.6) instead of (6.28). The type IIB threeform will have both the components, with $h_{yz\theta_1}$ now given by $h_{yz\theta_1} = \frac{\partial c_4}{\partial z}$ and $h_{xz\theta_2}$ as before (6.26). It is also easy to see that the $\mathcal{C}$ components $C_{xy2}, C_{yz2}, C_{yz1}$ and C_{xz2} will also remain as before (6.29). The only changes will be to the following components: C_{xy1} and C_{xz1}, and they will be given by

$$\delta C_{[xy1]} = 2\alpha Bc_4\, dx \wedge dy \wedge d\theta_1, \quad \delta C_{[xz1]} = c_4\, dx \wedge dz \wedge d\theta_1 \quad (6.37)$$

These changes will be reflected on the G-fluxes also. As one would have expected, the changes in the G-fluxes will be precisely from the additional c_4 terms. In fact we will have one new component of G-flux in addition to the ones that we already evaluated in (6.32), (6.33), (6.34)[53]. This is given by

$$G_{[xyz\theta_1]} \;=\; -2\alpha B \frac{\partial c_4}{\partial z}\; dx \wedge dy \wedge dz \wedge d\theta_1 \tag{6.38}$$

The five form, which is now given by

$$C_{[xyz\theta_1\theta_2]} \;=\; -\Big[c_4\; d\bar{\theta}_1 \wedge d\tilde{\theta}_2 - c_1\; d\tilde{\theta}_1 \wedge d\widehat{\theta}_2\Big] \wedge dx \wedge dy \wedge dz \tag{6.39}$$

still remains a pure gauge. We have also defined $\bar{\theta}_1 \;=\; \theta_1 + \tilde{\theta}_1$. From above we see that the case of flop in the M-theory lift of our background still remains non-singular. The flux configurations are now much more involved, but the basic physics have not changed. One little concern here would be the $c_4(z)$ part. This is explicitly a function of z and therefore might create problems for our T-duality transformations. One could overcome this problem by making a T-duality transformation on the H_{RR} directly following the rules given in [102]. All we now require is to make $h_{yz\theta_1}$ independent of z direction. For our case, this doesn't seem to be of any concern.

The above discussion then leads us to describe yet another case, that could in principle occur if we make the field strengths in type IIB independent of the duality directions. This would be when in (6.21) we allow

$$\textbf{Case 3}: \quad c_1 \;=\; c_1(\theta_2), \;\; c_3 \;=\; c_3(z), \;\; c_4 \;=\; c_4(z), \;\; c_5 \;=\; c_5(y), \;\; c_2 \;=\; c_6 \;=\; 0 \tag{6.40}$$

For this case, the H_{RR} would be more complicated than the ones that we had earlier. The components of H_{RR} would become

$$h_{xz\theta_2} \;=\; \frac{\partial c_1}{\partial \theta_2} - \frac{\partial c_3}{\partial z}, \qquad h_{yz\theta_1} \;=\; \frac{\partial c_5}{\partial y} - \frac{\partial c_4}{\partial z} \tag{6.41}$$

Both these components are to be arranged so that they satisfy the type IIB equation of motion. Since the B_{NS} components have remained unchanged we can easily satisfy the imaginary self-duality condition of the type IIB three form fluxes. Rest of the analysis follows a straightforward route, using the mirror rules. The type IIA metric and the one-forms remain same as before, but the three forms and their corresponding G-fluxes change.

[53] These components of G-fluxes will not change although the corresponding C-fields have changed.

Before we go into the analysis of torsion classes, consider two other hypothetical scenarios. First one is a charge zero configuration. Due to the F-theory construction behind our type II models, the global charges all cancel when we consider compact fourfold in sec. 2.1. Imagine we continue the charge zero configuration even after we allow non-compact fourfolds. Of course such a charge zero state will allow only a finite number of fractional D3 branes. Next, let us also allow local charge cancellation. This way in (6.21) we can allow:

$$\textbf{Case 4}: \quad c_3 = c_3(z), \;\; c_4 = c_4(z), \;\; c_1 \to 0, \;\; c_5 = c_2 = c_6 = 0 \tag{6.42}$$

For this case the H_{RR} can be easily extracted from (6.41) now. The three-forms are very interesting now. They will be completely independent of f_1 or f_2. The non-zero components will be given by

$$C_{xy2} = -\frac{2\alpha c_3 AB^2}{1+A^2}, \quad C_{xz1} = c_4, \quad C_{xz2} = -\frac{2c_3 AB}{1+A^2}, \quad C_{yz2} = -c_3 \tag{6.43}$$

with all other components vanishing. The G-fluxes for such a configuration are simpler than the earlier examples. This is because of the coordinate structure of c_3 and c_4. One can easily show that the G-fluxes will be independent of c_4, and will be given entirely by c_3 as

$$\begin{aligned} G_{xyz\theta_2} &= \frac{2\alpha AB^2}{1+A^2}\frac{\partial c_3}{\partial z}, \qquad G_{xy\theta_1\theta_2} = 2c_3B^2\frac{\partial}{\partial\theta_1}\left(\frac{\alpha A}{1+A^2}\right) \\ G_{xz\theta_1\theta_2} &= 2Bc_3\frac{\partial}{\partial\theta_1}\left(\frac{A}{1+A^2}\right) \end{aligned} \tag{6.44}$$

where the relevant component of G-flux $G_{xz\theta_1\theta_2}$ vanishes in the local region implying that the corresponding potential $C_{xz\theta_2}$ can be gauged away. The operation of flop in M-theory may however still remain non-singular because globally the G-flux component $G_{xz\theta_1\theta_2}$ is not expected to vanish.

The second one, although not very important for our main line of thought followed in this paper, is another generalisation that could be done to (6.21). What we mean is to add two more terms to (6.21) that are invariant under our global orientifold operation in type IIB theory. The only allowed ones are

$$\textbf{Case 5}: \quad c_1 = c_1(\theta_2), \; c_i = 0, \quad \delta B_{RR} = c_7 \, dr \wedge dx + c_8 \, dr \wedge d\theta_1 \tag{6.45}$$

where the total type IIB background RR field will be $B_{RR} + \delta B_{RR}$ and $i = 2, ..., 6$. We are also assuming that c_7, c_8 would in general be functions of (θ_i, z).

This change in type IIB B_{RR} field will effect the mirror picture. It turns out that the type IIA one form will not change at all from the value derived earlier in (6.22). However, the mirror type IIA three form will now change from the one derived earlier in (6.23) to

$$\begin{aligned}\delta\mathcal{C} \;=\; & C_{ryz}\; dr\wedge(dy - b_{y\theta_1}d\theta_1)\wedge dz + C_{rxz}\; dr\wedge(dx - b_{x\theta_2}d\theta_2)\wedge dz \\ & + C_{rxy}\; dr\wedge(dx - b_{x\theta_2}d\theta_2)\wedge(dy - b_{y\theta_1}d\theta_1)\end{aligned} \tag{6.46}$$

where as before, the total three form field will be $\mathcal{C}+\delta\mathcal{C}$, and the coefficients in (6.46) are now given by

$$C_{ryz} \;=\; c_7, \quad C_{rxz} \;=\; \frac{2c_7AB}{1+A^2}, \quad C_{rxy} \;=\; \frac{2\alpha c_7AB^2}{1+A^2} \tag{6.47}$$

Looking carefully at the additional components of the three form we observe that the three form component C_{rxz} is exactly equal to the part of $C_{xz\theta_2}$ if we replace $c_7 \leftrightarrow -c_3$. In fact all the above components are exactly the same under the substitution. What does this mean? We already gave an answer to this question when we discussed the local equivalence between the r and θ_2 directions. Under this equivalence we clearly see why c_7 and c_8 terms of (6.45) do not contribute to the one form (6.22). Furthermore it is also clear why c_8 term do not contribute to the three form above.

The equivalences between (r,θ_2) and $(c_7,-c_3)$ are also consistent with the wrapped D5 brane scenario. Recall that because of the orientifold action the D5 branes can be thought to be along the (r,y) directions in the internal space. This would imply that we should look for the combination

$$\mathcal{V}_2 \;=\; \sqrt{g_1g_2g_r} \;+\; \frac{2i\; c_7AB}{1+A^2} \tag{6.48}$$

as our complexified volume, where g_1, g_2, g_r are the warp factors in the type IIA metric (6.6). This combination goes over to the original combination (6.30) once we go away from the orientifold limit and consider c_1 in (6.21) as $c_1(\theta_1,\theta_2)$. In the same scenario we can define another effective volume along (x,z,θ_2) direction as

$$\mathcal{V}_3 \;=\; \sqrt{g_1g_2g_3 - \frac{g_1g_4^2\sin^2\,\langle\psi\rangle}{4}} \;-\; \frac{2i\; c_3AB}{1+A^2} \tag{6.49}$$

which has somewhat similar behavior as (6.48) and would match precisely with (6.48) if $g_3 \;=\; g_r$ and $\langle\psi\rangle$ small. The connection to (6.30) will now be the following. Away from the orientifold point $b_{x\theta_2}$ and $b_{y\theta_1}$ will be functions of (θ_1, θ_2) also. This would mean that we have the other two components $b_{x\theta_1}$ and $b_{y\theta_2}$. The D5 branes in type IIB can be assumed to warp directions (y, θ_2) and then the volume element will be (6.30). Our picture at the orientifold point with volume element (6.48) should be considered at this stage as a toy model that is helpful in clarifying all the expected details that one would expect from a model away from the orientifold point. Thus (6.49) will serve as an *intermediate* scenario between (6.48) and (6.30).

Now from the earlier discussion we should expect that the c_8 term in (6.45) only contributes to the five form, exactly the way c_6 term contributes to the five form. This is easily verified by computing the mirror five form

$$\delta\mathcal{C}_5 \;=\; -(c_8 + c_7 f_1)\; dr \wedge (dx - b_{x\theta_2} d\theta_2) \wedge (dy - b_{y\theta_1} d\theta_1) \wedge dz \wedge d\theta_1 \tag{6.50}$$

where one can show that the other expected components of the five form $C_{rxyz\theta_2}$ and $C_{rxz\theta_1\theta_2}$ are proportional to ϵf_1 and since f_1 goes like $f_1 \sim \frac{1}{\sqrt{\epsilon}}$ these components go to zero when $\epsilon \to 0$. Finally, if we also take

$$c_8 \;=\; 0, \qquad\qquad h_{xzr} \;\equiv\; \frac{\partial c_7}{\partial z} \tag{6.51}$$

where h_{xzr} is the type IIB three form, then one can show that the additional contribution to the five form (6.50) is also a pure gauge. With this the analysis of type IIA will be complete.

7. Torsion classes

After all the detailed discussion about the global type IIA manifolds, we will now attempt to classify the IIA non–Kähler manifolds that we constructed. As already pointed out in [18] we do not find a half–flat manifold after performing three T–dualities with fluxes. This might appear to contradict current literature on this subject, but we will show the contrary. First, it has been shown that lifting a 10dim manifold on a twisted circle (i.e. with gauge field as in our case) can still give a supersymmetric M–theory background, even if the 10dim manifold was not half–flat. But we can only obtain torsion classes for our local background, which does not show supersymmetry. We therefore neither expect a half–flat manifold nor the one discussed in [29] on the grounds of supersymmetry. Furthermore, we actually do not expect a 11dim manifold with G_2 holonomy, since our M–theory background has flux turned on.

7.1. $SU(3)$ and G_2 structure manifolds

It is by now widely known that string theory backgrounds with fluxes do not require a Calabi–Yau manifold (with $SU(3)$ holonomy). The 4dim $\mathcal{N}=1$ supersymmetry condition of a covariantly constant spinor is relaxed to the existence of a globally defined spinor that is constant with respect to a *torsional* connection. This reduces the structure group of the 6dim manifold from $SO(6)$ to $SU(3)$ and the intrinsic torsion decomposes under representations of $SU(3)$. In particular, the torsion lies in 5 classes [66], [103]: $\tau_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$.

The failure of the torsional connection to be the Levi–Civita connection is measured in the failure of fundamental 2–form and holomorphic 3–form to be closed. Defining a set of real vielbeins $\{e_i\}$ one can define an almost complex structure as

$$\begin{aligned} E_1 &= e_1 + i\ e_2 \\ E_2 &= e_3 + i\ e_4 \\ E_3 &= e_5 + i\ e_6 \end{aligned} \tag{7.1}$$

which gives rise to a (1,1)–form w.r.t. this almost complex structure

$$J = e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6\,. \tag{7.2}$$

Similarly, one defines a holomorphic 3–form w.r.t. this almost complex structure

$$\Omega = \Omega_+ + i\ \Omega_- = (e_1 + i\ e_2) \wedge (e_3 + i\ e_4) \wedge (e_5 + i\ e_5)\,, \tag{7.3}$$

where $\Omega_\pm$ are the real adn imaginary part of Ω, repectively. J and Ω fulfill the compatibility relations

$$\begin{aligned} J \wedge \Omega_+ &= J \wedge \Omega_- = 0 \\ \Omega_+ \wedge \Omega_- &= \frac{2}{3} J \wedge J \wedge J\,. \end{aligned} \tag{7.4}$$

The torsion classes are then determined by the following forms:

$$\begin{aligned} \mathcal{W}_1 &\quad \leftrightarrow \quad dJ^{(3,0)} \\ \mathcal{W}_2 &\quad \leftrightarrow \quad (d\Omega)_0^{(2,2)} \\ \mathcal{W}_3 &\quad \leftrightarrow \quad (dJ)_0^{(2,1)} \\ \mathcal{W}_4 &\quad \leftrightarrow \quad J \wedge dJ \\ \mathcal{W}_5 &\quad \leftrightarrow \quad d\Omega^{(3,1)}\,, \end{aligned} \tag{7.5}$$

where the subscript 0 refers to the primitive part, i.e. in the cases in question $\beta \in \bigwedge_0^{p,q}$ if $\beta \wedge J = 0$. It is immediately obvious that complex manifolds have to have vanishing $\mathcal{W}_1$ and $\mathcal{W}_2$ and Kähler manifolds are determined by $\tau_1 \in \mathcal{W}_5$. Decomposing $\Omega = \Omega_+ + i\ \Omega_-$ we can write more precisely

$$\begin{aligned} d\Omega_\pm \wedge J = \Omega_\pm \wedge dJ &= \mathcal{W}_1^\pm\ J \wedge J \wedge J \\ d\Omega_\pm^{(2,2)} &= \mathcal{W}_1^\pm\ J \wedge J + \mathcal{W}_2^\pm \wedge J \\ dJ^{(2,1)} &= (J \wedge \mathcal{W}_4)^{(2,1)} + \mathcal{W}_3\,, \end{aligned} \tag{7.6}$$

so $\mathcal{W}_1$ is given by two real numbers, $\mathcal{W}_1 = \mathcal{W}_1^+ + \mathcal{W}_1^-$, $\mathcal{W}_2$ is a (1,1) form and $\mathcal{W}_3$ is a (2,1) form. With the definition of the contraction

$$\rightharpoonup\ :\ \bigwedge^k T^* \otimes \bigwedge^n T^* \longrightarrow \bigwedge^{n-k} T^* \tag{7.7}$$

and the convention $e_1 \wedge e_2 \ \rightharpoonup\ e_1 \wedge e_2 \wedge e_3 \wedge e_4 \ =\ e_3 \wedge e_4$ we define

$$\begin{aligned} \mathcal{W}_4 &= \frac{1}{2}\ J \rightharpoonup dJ \\ \mathcal{W}_5 &= \frac{1}{2}\ \Omega_+ \rightharpoonup d\Omega_+\,. \end{aligned} \tag{7.8}$$

A half–flat manifold is specified by $\tau_1 \in \mathcal{W}_1^+ \oplus \mathcal{W}_2^+ \oplus \mathcal{W}_3$, which follows from $J \wedge dJ = 0$ and $d\Omega_- = 0$, but $d\Omega_+ \neq 0$ (this lead to the terminology "half–flat"). Note that the assignment of Ω_- and Ω_+ may be switched.

Similar statements hold true for M–theory on 7–manifolds, which would require G_2 holonomy to preserve 4dim $\mathcal{N} = 1$ supersymmetry in the absence of flux. The fundamental object now is a nowhere vanishing 3–form Φ and its failure to be closed and/or co–closed determines the torsion. The relevant structure group is G_2 and the intrinsic torsion decomposes under this group. This results in 4 torsion classes for the 7–manifold: $\tau_2 \in \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$. There are given by

$$\begin{aligned} d\Phi &= \mathcal{X}_1\ (*\Phi) + \mathcal{X}_4 \wedge \Phi + \mathcal{X}_3 \\ d(*\Phi) &= \frac{4}{3}\ \mathcal{X}_4 \wedge (*\Phi) + \mathcal{X}_2 \wedge \Phi\,. \end{aligned} \tag{7.9}$$

In [52],[66] it was demonstrated how a manifold with $SU(3)$ structure can be lifted on a trivial circle or interval to a G_2 holonomy. One defines the G_2 invariant 3–form as

$$\Phi\ =\ \Omega_+ + J \wedge e_7 \tag{7.10}$$

where e_7 parametrizes the 7th direction, such that the resulting 7–manifold is $M \times I$ with $I \subset \mathbb{R}$. This produces Hitchin's flow equations if the 6–manifold is half–flat.

In contrast, we are more interested in the case discussed in [29]. Starting with an $SU(3)$–structure manifold X they constructed a G_2–structure manifold Y as a lift over a twisted circle with dilaton and gauge field:

$$ds_Y^2 \;=\; e^{-2\alpha\phi}\, ds_X^2 \;+\; e^{2\beta\phi}\, (dz + A)^2\,. \tag{7.11}$$

We will adopt the string frame in which $\alpha = 1/3$ and $\beta = 2/3$. In this case one should define the 3–form on the 7–manifold rather as

$$\Phi \;=\; e^{-\phi}\, \Omega_+ \;+\; e^{-\frac{2}{3}\phi}\, J \wedge e_7\,. \tag{7.12}$$

This gives straightforward relations between the torsion classes $\mathcal{W}_i$ and $\mathcal{X}_j$ that generally involve the field strength $F = dA$ and the dilaton ϕ. It was shown in [29] that requiring G_2 holonomy (i.e. $d\Phi = d(*\Phi) = 0$ or equivalently $\mathcal{X}_i = 0$) leads to the following constraints on the $SU(3)$ torsion classes:

$$\begin{aligned} \mathcal{W}_1^{\pm} \;&=\; \mathcal{W}_2^- \;=\; \mathcal{W}_3 \;=\; \mathcal{W}_4 \;=0 \\ \mathcal{W}_2^+ \;&=\; -e^{\phi}\, F_0^{(1,1)}\,, \qquad \mathcal{W}_5 \;=\; \frac{1}{3}\, d\phi\,. \end{aligned} \tag{7.13}$$

Note, that only in the string frame $\mathcal{W}_4 = 0$, otherwise it is also proportional to $d\phi$. This shows that the 6–manifold does not need to be Kähler (if $F_0^{(1,1)} \neq 0$), but it does not need to be half-flat either (it still could be if $d\phi = 0$).

This short discussion was intended to clarify that half–flat manifolds are not the only manifolds that can be lifted to a G_2 holonomy. One has to be specific about which type of lift is chosen. It is immediately clear that our scenario requires the 7-th direction to be a twisted circle, since the IIA background has a gauge field A. But since we have also other background fluxes turned on, we obtain a torsional M–theory background after the lift. Therefore, the manifold we propose in IIA is neither half–flat nor has it torsion restricted to $\mathcal{W}_2^+ \oplus \mathcal{W}_5$.

We now turn to the question what type of manifold we have constructed in IIA. Unfortunately, we had to take a local limit of the metric to be able to perform T–dualities from IIB to IIA. Therefore, the global information about our manifold is lost. It does strictly speaking not make sense to discuss the metric (2.1) when one is looking for topological properties, such as torsion classes. In the local background supersymmetry is not visible, but for the global supersymmetric background we do not know the metric. At this point there is no way out of this dilemma and we have to postpone the topological analysis of the supersymmetric background to future work.

However, we find it instructive to discuss the torsion of the local metric found in [18] as an example. This is not the background obtained from the F–theory construction in IIB, but shows a different fibration structure. This does not have major consequences for the torsion classes, we therefore restrict ourselves to this case.

7.2. Torsion classes before geometric transition

In this section we will only discuss the IIA case. It is shown that with a quite generic ansatz for the almost complex structure we can find a symplectic structure on the local metric, but no half–flat structure[54]. One possible choice for vielbeins in IIA is

$$\begin{aligned}
e_1 &= dr\,, \qquad\qquad e_2 = \sqrt{g_1}\,\Big(dz + A(dx - b_{x\theta_1}d\theta_1) + B(dy - b_{y\theta_2}d\theta_2)\Big) \\
e_3 &= \frac{1}{2}\sqrt{\frac{4g_2g_3 - g_4^2}{g_2}}\;d\theta_2\,, \qquad e_4 = \frac{1}{2}\sqrt{\frac{4g_2g_3 - g_4^2}{g_2}}\;(dy - b_{y\theta_2}d\theta_2) \\
e_5 &= \sqrt{g_2}\left(\sin\psi_0(dx - b_{x\theta_1}d\theta_1) + \cos\psi_0\;d\theta_1 + \frac{g_4}{2g_2}\;d\theta_2\right) \\
e_6 &= \sqrt{g_2}\left(\cos\psi_0(dx - b_{x\theta_1}d\theta_1) - \sin\psi_0\;d\theta_1 - \frac{g_4}{2g_2}\;(dy - b_{y\theta_2}d\theta_2)\right)
\end{aligned} \tag{7.14}$$

Following the discussion in [81] we write down the most generic candidate for a fundamental 2–form

$$J \;=\; \sum_{i<j} a_{ij}\; e^i \wedge e^j\,, \tag{7.15}$$

where the coefficients a_{ij} could in principle depend on all coordinates. This has to be compatible with an almost complex structure $\mathcal{J}_B^A = \delta^{AC} J_{CB}$, i.e. we require $\mathcal{J}^2 = -1$. If we furthermore make the assumption that $a_{13} = a_{14} = 0$,[55] the almost complex structure

[54] This statement holds true if one considers the IIA background abtained from IIB with orientifold action, but the precise torsion classes differ.

[55] This seems sensible because it holds for all known conifold geometries [81], and basically only implies that the complex vielbein containing dr does not contain e_3 and e_4. It can still contain dy and $d\theta_2$, since those also appear in e_5 and e_6.

takes a particularly simple form [81]. The complex vielbeins can be written as

$$
\begin{aligned}
E_1 &= e_1 + i\, e_2 \\
E_2 &= e_3 + i\, (X\, e_4 - P\, e_6) \\
E_3 &= e_5 + i\, (X\, e_6 + P\, e_4)\,.
\end{aligned}
\tag{7.16}
$$

with the restriction $P^2 + X^2 = 1$. In the following we will make the simplifying assumption that P (and therefore X) is a function of r only.

With this setup, J and Ω are defined and one can calculate dJ and $d\Omega$. One immediately notices that $\mathcal{W}_4 = 0$, because

$$
J \wedge dJ \;=\; -\frac{4g_2g_3 - g_4^2}{2}\Big(P(r)P'(r) + X(r)X'(r)\Big)\, dr \wedge dx \wedge dy \wedge d\theta_1 \wedge d\theta_2\,, \tag{7.17}
$$

which is identically zero because of $P(r)^2 + X(r)^2 = 1$. It is nevertheless not a half-flat manifold, because there is no choice of $P(r)$ that makes either $d\Omega_+ = 0$ or $d\Omega_- = 0$.

We can, however, choose $P(r)$ to give a symplectic structure. Consider $\mathcal{W}_1^\pm$ given by $d\Omega_\pm \wedge J = \mathcal{W}_1^\pm\; J \wedge J \wedge J$:

$$
\begin{aligned}
\mathcal{W}_1^+ &= \frac{\sqrt{g_1}\,(4g_2g_3 - g_4^2)}{2\sqrt{1-P(r)^2}}\; P'(r)\; dr \wedge dx \wedge dy \wedge dz \wedge d\theta_1 \wedge d\theta_2 \\
\mathcal{W}_1^- &= -\frac{1}{4}\,\sqrt{g_1}\,\left(P(r)(g_4^2 - 4g_2g_3) + g_4\sqrt{4g_2g_3 - g_4^2}\;\sqrt{1-P(r)^2}\right) \times \\
&\qquad \Big(b'_{x\theta_1}(r) + b'_{y\theta_2}(r)\Big)\; dr \wedge dx \wedge dy \wedge dz \wedge d\theta_1 \wedge d\theta_2\,.
\end{aligned}
\tag{7.18}
$$

Note that in our local background the function g_i are simply constants and we have assumed the IIB $B_{\rm NS}$–field components to have r–dependence only. Obviously, $\mathcal{W}_1^+$ vanises if $P(r)$ is constant (the prime denotes derivative w.r.t. r). $\mathcal{W}_1^-$ vanishes if

$$
P \;=\; \frac{g_4}{2\sqrt{g_2g_3}} \;=\; \text{constant}. \tag{7.19}
$$

It turns out, that this is also the only value for which $\mathcal{W}_3$ vanishes, and in this case $dJ = 0$. Let us stress again, that there is no choice for $P(r)$ that would give $\mathcal{W}_5 = 0$ or $\mathcal{W}_2^\pm = 0$. The remaining torsion classes could only vanish if the IIB $B_{\rm NS}$ field was constant. In that case we would trivially recover a CY, since then all metric components would be constant.

For completeness, let us also give $\mathcal{W}_5$ and $\mathcal{W}_2^{\pm}$ with the choice (7.19) for P:

$$\begin{aligned}
\mathcal{W}_2^+ \;&=\; -\frac{i}{8\sqrt{g_2 g_3 g_5}}\Big[2A\sqrt{g_1 g_3 g_5}\; b'_{x\theta_1}\; e_1\wedge(\cos\psi_0\; e_3+\sin\psi_0\; e_4)\\
&\quad + 2A\sqrt{g_1 g_3}\; g_4\; b'_{x\theta_1} e_1\wedge(\cos\psi_0\; e_5-\sin\psi_0\; e_6)\\
&\quad + 4A\sqrt{g_1 g_2}\; g_3\; b'_{x\theta_1} e_2\wedge(\cos\psi_0\; e_4-\sin\psi_0\; e_3)\\
&\quad + g_4\sqrt{g_5}\sin 2\psi_0\; b'_{x\theta_1}\;(e_3\wedge e_4-e_5\wedge e_6)+g_5\sin 2\psi_0\; b'_{x\theta_1}\;(e_3\wedge e_6-e_4\wedge e_5)\\
&\quad - 4g_2 g_3\;(\cos 2\psi_0\; b'_{x\theta_1}+B\; b'_{y\theta_2})\;(e_3\wedge e_5+e_4\wedge e_6)\\
&\quad - 2B\sqrt{g_1}\; b'_{y\theta_2}\;\big(\sqrt{g_2 g_5}\; e_2\wedge e_6+g_2\sqrt{g_3}\; e_1\wedge e_5+g_4\sqrt{g_2}\; e_2\wedge e_4\big)\Big]
\end{aligned}\tag{7.20}$$

$$\begin{aligned}
\mathcal{W}_2^- \;&=\; \frac{i}{16}\Big[4A\sqrt{\frac{g_1}{g_2}}\; b'_{x\theta_1}\;(\cos\psi_0\; e_2\wedge e_3+\sin\psi_0\; e_2\wedge e_4)+8A\sqrt{\frac{g_1 g_3}{g_5}}\;\sin\psi_0\; b'_{x\theta_1}\; e_1\wedge e_3\\
&\quad + 4A\frac{\sqrt{g_1}\; g_4}{\sqrt{g_2 g_5}}\;\sin\psi_0\; b'_{x\theta_1}\; e_2\wedge e_6-4\sin 2\psi_0\; b'_{x\theta_1}\;(e_3\wedge e_5+e_4\wedge e_6)\\
&\quad + 4B\sqrt{\frac{g_1}{g_3}}\; b'_{y\theta_2}\; e_1\wedge e_6+4\sqrt{\frac{g_1}{g_2 g_5}}\big(Ag_4\cos\psi_0\; b'_{x\theta_1}-2Bg_2\; b'_{y\theta_2}\big)\; e_2\wedge e_5\\
&\quad + \frac{(-g_5\cos\psi_0+g_4^2)\; b'_{x\theta_1}+4g_2 g_3\; b'_{y\theta_2}}{g_2 g_3}\;\left(e_3\wedge e_6-e_4\wedge e_5+\frac{g_4}{\sqrt{g_5}}\;(e_3\wedge e_4-e_5\wedge e_6)\right)\\
&\quad + 4\sqrt{\frac{g_1}{g_3}}\;(-2Ag_3\cos\psi_0\; b'_{x\theta_1}+Bg_4\; b'_{y\theta_2})\; e_1\wedge e_4\Big]
\end{aligned}\tag{7.21}$$

$$\begin{aligned}
\mathcal{W}_5 \;&=\; \sqrt{\frac{g_2 g_3}{g_5}}\;(b'_{x\theta_1}+b'_{y\theta_2})\; e_2+2A\sqrt{\frac{g_1 g_3}{g_5}}\sin\psi_0\; b'_{x\theta_1}\; e_5\\
&\quad + \frac{\sqrt{g_1}}{4\sqrt{g_2 g_5}}\;(2Ag_4\cos\psi_0\; b'_{x\theta_1}-4Bg_2\; b'_{y\theta_2})\; e_3+A\sqrt{\frac{g_1}{g_2}}\cos\psi_0\; b'_{x\theta_1}\; e_6
\end{aligned}\tag{7.22}$$

where we have defined $g_5 = 4g_2 g_3 - g_4^2$. The vielbeins $\{e_i\}$ are defined in (7.14).

7.3. Torsion classes after geometric transition

Very similar remarks hold true for the local metric after transition. We can find a symplectic structure, but $\mathcal{W}_2^{\pm}$ and $\mathcal{W}_5$ are nonzero. The metric after transition was obtained in [18] to be:

$$\begin{aligned}
ds^2 \;&=\; dr^2+e^{2\phi}\;\big(dz+A(dx-b_{x\theta_1}d\theta_1)+B(dy-b_{y\theta_2}d\theta_2)\big)^2\\
&\quad + \left(\frac{g_2}{2}-\sqrt{\frac{g_2}{g_3}}\;\frac{g_4}{4}\right)\left(d\theta_1^2+\big(dx-b_{x\theta_1}d\theta_1\big)^2\right)\\
&\quad + \left(\frac{g_2}{2}+\sqrt{\frac{g_2}{g_3}}\;\frac{g_4}{4}\right)\left(d\theta_2^2+\big(dy-b_{y\theta_2}d\theta_2\big)^2\right).
\end{aligned}\tag{7.23}$$

Taking the ansatz (7.16) but now with real vielbeins

$$\begin{aligned} e_1 &= dr\,, & e_2 &= e^{\phi}\,\big(dz + A(dx - b_{x\theta_1}d\theta_1) + B(dy - b_{y\theta_2}d\theta_2)\big) \\ e_3 &= \frac{1}{2}\sqrt{g_+}\;d\theta_2\,, & e_4 &= \frac{1}{2}\sqrt{g_+}\;(dy - b_{y\theta_2}d\theta_2) \\ e_5 &= \frac{1}{2}\sqrt{g_-}\,(\sin\psi_0(dx - b_{x\theta_1}d\theta_1) + \cos\psi_0\;d\theta_1) \\ e_6 &= \frac{1}{2}\sqrt{g_-}\,(-\cos\psi_0(dx - b_{x\theta_1}d\theta_1) + \sin\psi_0\;d\theta_1) \end{aligned} \tag{7.24}$$

also gives $\mathcal{W}_4 = 0$ automatically. Again, $\mathcal{W}_1^+$ can only vanish if $P(r)$ is constant and solving $\mathcal{W}_1^- = 0$ gives $P(r) = 0$. There is no choice of $P(r)$ that would allow for $\mathcal{W}_5 = 0$ or $\mathcal{W}_2^\pm = 0$. With the choice $P = 0$ the remaining torsion classes are

$$\begin{aligned} \mathcal{W}_2^+ &= -\frac{i}{4}\,\bigg[\frac{2Ae^{\phi}}{\sqrt{g_-}}\;b'_{x\theta_1}\,\big(\sin\psi_0\;(e_1\wedge e_3 + e_2\wedge e_4) + \cos\psi_0\;(e_1\wedge e_4 - e_2\wedge e_3)\big) \\ &\quad + \sin 2\psi_0\;b'_{x\theta_1}\;(e_3\wedge e_5 + e_4\wedge e_6) + \cos 2\psi_0\;b'_{x\theta_1}\;(e_4\wedge e_5 - e_3\wedge e_6) \\ &\quad - \frac{2Be^{\phi}}{\sqrt{g_+}}\;b'_{y\theta_2}\;(e_1\wedge e_6 - e_2\wedge e_5) + b_{y\theta_2}\;(e_4\wedge e_5 - e_3\wedge e_6)\bigg] \\ \mathcal{W}_2^- &= -\frac{i}{4}\,\bigg[\frac{2Ae^{\phi}}{\sqrt{g_-}}\;b'_{x\theta_1}\,\big(\cos\psi_0\;(e_1\wedge e_3 + e_2\wedge e_4) - \sin\psi_0\;(e_1\wedge e_4 - e_2\wedge e_3)\big) \\ &\quad + \cos 2\psi_0\;b'_{x\theta_1}\;(e_3\wedge e_5 + e_4\wedge e_6) - \sin 2\psi_0\;b'_{x\theta_1}\;(e_4\wedge e_5 - e_3\wedge e_6) \\ &\quad - \frac{2Be^{\phi}}{\sqrt{g_+}}\;b'_{y\theta_2}\;(e_1\wedge e_5 + e_2\wedge e_6) - b'_{y\theta_2}\;(e_3\wedge e_5 + e_4\wedge e_6)\bigg] \\ \mathcal{W}_5 &= \frac{2Ae^{\phi}}{\sqrt{g_-}}\;b'_{x\theta_1}\;\big(\cos\psi_0\;e_6 - \sin\psi_0\;e_5\big) - \frac{Be^{\phi}}{\sqrt{g_+}}\;b'_{y\theta_2}\;e_3 - \frac{1}{2}\;(b'_{x\theta_1} - b'_{y\theta_2})\;e_2 \end{aligned} \tag{7.25}$$

where we have defined

$$g_\pm \;=\; 2g_2 \pm \sqrt{\frac{g_2}{g_3}}\;g_4 \tag{7.26}$$

and ϕ is the IIA dilaton from before transition. In the limit that we took when performing the T–dualities to go from IIB to IIA, this is exactly the same as the IIB dilaton and constant (at least in the local limit). We see that the geometric transition maps the torsion classes $W_2^\pm$ and W_5 into themselves. By using the Hitchin flow equations, this means that the corresponding G_2 torsion classes are mapped into themselves. But we know that the flop just replaces the usual x^{11} lifting direction with the fiber direction inside the Hopf fibration $S^3/\mathbb{Z}_N$ over S^2. These two circles are used to lift $SU(3)$ torsion classes to G_2 torsion classes and this tells us that the G_2 torsion classes are not changed during the flop.

8. Conclusions and Future Directions

Geometric transitions are a powerful method to connect string theory and field theory. They have been used extensively to check and predict effective results in field theory from string theory. In this project, which is a continuation of our previous work [18], [19], we addressed several questions related to the geometric transition framework and reached conclusions which point out some subtle modification of the original framework of [3].

First, we showed that, in terms of SUGRA solutions, an acceptable way to to construct a SUSY solution is to consider a global picture with additional branes and planes. Away from the wrapped D5 branes on resolution two cycles of our manifold, there are additional seven branes and orientifold seven planes during the geometric transition. Such a configuration has two immediate advantages: on the one hand their presence allows a consistent lift to a SUSY F–theory picture and on the other hand they do not directly influence the effective theory as they can be viewed as massive flavors that are integrated out. If we consider them as part of the effective theory (by raising the energy scale of the theory), the field theory would be an $SU(N)$ gauge theory with fundamental flavors. This is a somewhat different configuration from the original framework of [3] (see also [16]) and would change the geometric transition identifications. These further complications will translate into a currently unknown metric for D5–D7 system.

Second, the local picture that we proposed in this paper captures the dynamics of $\mathcal{N} = 1$ $SU(N)$ gauge theory without any flavors. The local picture also explains the fact that in the mirror IIA picture our non–Kähler manifold is not expected to resemble any of the examples proposed in the literature [104]. Globally, in type IIA there are extra D6 branes which correspond to massive flavors which are integrated out to obtain the local model. In both type IIA and type IIB, the global framework involves subtle orientifold actions that have roots in our proposed F–theory setup[56].

Another important topic in our work was to deepen the evidence of a heterotic transition, even though the usual topological string methods are still missing for the (0,2)

[56] It would be interesting to see whether the global M-theory lift of our type IIA background is connected to any of the solutions presented in [105]. A way to search for such a solution is to consider the solutions of [105] and look for four dimensional slices of the seven dimensional manifolds that would resemble Atiyah-Hitchin spaces. In addition to that, if the local form of the metric resembles the local M-theory metric presented in [18], then this solution can be a candidate for our global M-theory picture. More details on this will be presented in the sequel.

theories. We managed to show that there is a surprising analogy between two hitherto different solutions, namely, the Maldacena–Nunez solution [2] in type IIB and the heterotic dual to the type IIB solution proposed in [19]. In fact, an interesting generalization of the Maldacena–Nunez solution with several warp factors was mapped to a global heterotic solution which then takes us away from the orientifold point. The existence of such a mapping indicates the possibility that our global heterotic model may indeed be dual to the theory on wrapped heterotic NS5 branes. However, to make this more concrete we need to provide more evidence, and the simple identification between two metrics is probably not too rigorous.

Our discussion is just the tip of an iceberg. Clearly, much work remains in order to clarify above points. It would be interesting to find information about the global picture by integrating in the massive flavors. This would imply changes in H_{NS} and H_{RR} due to the presence of D7 branes which consequently change the dual gauge theory[57]. This method should remove the non–SUSY problems of [21] and offer us consistent global geometry.

Our work should also be viewed as connecting gauge theories with the dynamics of non–Kähler manifolds. This would open up an extensive direction of research. An immediate next step would be to further connect our results with the generalized complex geometries of Hitchin [52] and Gualtieri [53], which unify metric and NS field. It should also be possible to consider the heterotic solution in the context of Chiral de Rham complex and chiral differential operators. This would provide a method for performing genuine heterotic topological computations with important application in describing moduli spaces of $\mathcal{N} = 1$ theories.

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[57] Similar results were obtained for the case of the solution of [1] in [27].

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