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DISCONTINUOUS GALERKIN METHODS FOR FRIEDRICHS' SYSTEMS. PART II. SECOND-ORDER ELLIPTIC PDES*

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Abstract. This paper is the second part of a work attempting to give a unified analysis of discontinuous Galerkin methods. The setting under scrutiny is that of Friedrichs' systems endowed with a particular 2×2 structure in which one unknown can be eliminated to yield a system of second-order elliptic-like PDEs for the remaining unknown. A general discontinuous Galerkin method for approximating such systems is proposed and analyzed. The key feature is that the unknown that can be eliminated at the continuous level can also be eliminated at the discrete level by solving local problems. All the design constraints on the boundary operators that weakly enforce boundary conditions and on the interface operators that penalize interface jumps are fully stated. Examples are given for advection-diffusion-reaction, linear continuum mechanics, and a simplified version of the magnetohydrodynamics equations. Comparisons with well-known discontinuous Galerkin approximations for the Poisson equation are presented.

Key words. Friedrichs' systems, finite elements, partial differential equations, discontinuous Galerkin method

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1. Introduction.

Friedrichs' systems [10] are systems of first-order PDEs endowed with a symmetry and a positivity property. Such systems embrace both elliptic and hyperbolic PDEs; i.e., they include advection-reaction, advection-diffusion-reaction, linear continuum mechanics, and Maxwell's equations in the elliptic regime, to cite a few examples. The analysis of this class of problems and its approximation by means of discontinuous Galerkin (DG) methods has been initiated by Lesaint [13], Lesaint and Raviart [12], and Johnson, Nävert, and Pitkäranta [11]. A thorough systematic analysis generalizing [13, 12, 11] has been undertaken in the first part of this work [9].

In this second part, we specialize the setting to two-field Friedrichs' systems such that (i) the dependent variable z can be partitioned into the form $z = (z^\sigma, z^u)$, and (ii) the σ-component, z^σ, can be eliminated to yield a system of second-order PDEs for the u-component, z^u, which is of elliptic type. To efficiently approximate the above Friedrichs' systems using DG methods, it is desirable to reproduce at the discrete level the possibility of eliminating the σ-component of the discrete unknown *locally* on each mesh element. This feature induces a nontrivial modification of the analysis presented in [9] that constitutes the scope of the present work. In particular, the design of boundary and interface operators has to be revised. The analysis presented herein shows that to recover stability while allowing for the local elimination in question requires an enhanced penalty on the boundary conditions and on the interface jumps of the discrete u-component.

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This paper is organized as follows. Section 2 briefly restates the main theoretical results of [9] on the well-posedness of Friedrichs' systems and introduces the above-mentioned two-field structure. Section 3 presents three important examples of two-field Friedrichs' systems, namely advection-diffusion-reaction equations written in mixed form, linear continuum mechanics equations written in the stress-pressure-displacement form, and a simplified form of the magnetohydrodynamics (MHD) equations. Section 4 formulates a general DG method for two-field Friedrichs' systems and describes the technique to locally eliminate the σ-component of the discrete solution. The convergence analysis constitutes the scope of section 5. All the design assumptions on the boundary operators which weakly enforce boundary conditions and on the interface operators which penalize interface jumps are stated. The key results are Theorem 5.8, which contains the main estimate for the σ- and u-component of the approximation error, and Theorem 5.14, which contains an improved estimate for the u-component of the error in the L^2-norm obtained using a duality argument. Finally, section 6 applies the DG method to the PDE systems presented in section 3; in particular, the link with the unified analysis of Arnold et al. [1] for the Poisson equation is explicated to illustrate the fact that various DG methods presented in the literature, e.g., the local discontinuous Galerkin (LDG) method of Cockburn and Shu [7], the interior penalty (IP) method of Baker [3] and Arnold [2], the method of Brezzi et al. [6], and the methods of Bassi and Rebay [5] and Bassi et al. [4], fit into the present framework.

2. Two-field Friedrichs' systems. Section 2.1 is meant to recall well-posedness results proved in part I, [9]. The reader familiar with this material can jump to section 2.2, where the notion of two-field Friedrichs' systems is introduced.

2.1. Main results on one-field Friedrichs' systems. Let Ω be a bounded, open, connected, Lipschitz domain in $\mathbb{R}^d$. Let m be a positive integer and set $L = [L^2(\Omega)]^m$ equipped with the canonical L^2-induced inner product $(\cdot,\cdot)_L$. Let $\mathcal{K}$ and $\{\mathcal{A}^k\}_{1\le k\le d}$ be $(d+1)$ functions on Ω with values in $\mathbb{R}^{m,m}$ such that

$$\text{(A1)} \qquad \mathcal{K} \in [L^\infty(\Omega)]^{m,m},$$

$$\text{(A2)} \qquad \forall k \in \{1,\dots,d\}, \quad \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m} \quad \text{and} \quad \sum_{k=1}^{d} \partial_k \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m},$$

$$\text{(A3)} \qquad \forall k \in \{1,\dots,d\}, \quad \mathcal{A}^k = (\mathcal{A}^k)^t \ \text{ a.e. in } \Omega,$$

$$\text{(A4)} \qquad \exists \mu_0 > 0, \quad \mathcal{K} + \mathcal{K}^t - \sum_{k=1}^{d} \partial_k \mathcal{A}^k \ge 2\mu_0 \mathcal{I}_m \ \text{ a.e. on } \Omega,$$

where $\mathcal{I}_m$ is the identity matrix in $\mathbb{R}^{m,m}$. To alleviate notation we define the operator $K \in \mathcal{L}(L;L)$ by $K : L \ni z \longmapsto \mathcal{K}z \in L$ and it adjoint $K^* \in \mathcal{L}(L;L)$ by $K^* : L \ni z \longmapsto \mathcal{K}^t z \in L$.

Let $\mathfrak{D}(\Omega)$ be the space of $\mathfrak{C}^\infty$ functions that are compactly supported in Ω. A function z in L is said to have an A-weak derivative in L if the linear form

$$[\mathfrak{D}(\Omega)]^m \ni \phi \longmapsto -\int_\Omega \sum_{k=1}^{d} z^t \partial_k(\mathcal{A}^k \phi) \in \mathbb{R} \tag{2.1}$$

is bounded on L. In this case, the function in L that can be associated with the above linear form by means of the Riesz representation theorem is denoted by Az. Define the so-called graph space $W = \{z \in L; Az \in L\}$ equipped with the graph norm $\|z\|_W = \|Az\|_L + \|z\|_L$. The space W is endowed with a Hilbert structure when equipped with the scalar product $(z,y)_L + (Az, Ay)_L$. For $z \in W$, the function in L that can be associated with the linear form $[\mathfrak{D}(\Omega)]^m \ni \phi \longmapsto \int_\Omega \sum_{k=1}^d z^t \mathcal{A}^k \partial_k \phi \in \mathbb{R}$ is denoted by $\tilde{A}z$. Clearly, $A \in \mathcal{L}(W;L)$ and $\tilde{A} \in \mathcal{L}(W;L)$ and if z is smooth, e.g., $z \in [\mathfrak{C}^1(\overline{\Omega})]^m$,

$$Az = \sum_{k=1}^{d} \mathcal{A}^k \partial_k z, \qquad \tilde{A}z = -\sum_{k=1}^{d} \partial_k(\mathcal{A}^k z). \tag{2.2}$$

Furthermore, we set $T = K + A$, $\tilde{T} = K^* + \tilde{A}$. Note that $\tilde{A}$ and $\tilde{T}$ are the formal adjoints of A and T, respectively, owing to (A3). Assumption (A4) implies

$$\forall z \in W, \quad (Tz, z)_L + (z, \tilde{T}z)_L \geq 2\mu_0 \|z\|_L^2. \tag{2.3}$$

Let $D \in \mathcal{L}(W; W')$ be the operator defined by

$$\forall (z,y) \in W \times W, \quad \langle Dz, y \rangle_{W',W} = (Az, y)_L - (z, \tilde{A}y)_L. \tag{2.4}$$

Observe that D is self-adjoint by construction; moreover, it is a boundary operator in the sense that $\mathrm{Ker}(D)$ is the closure of $[\mathfrak{D}(\Omega)]^m$ in W; see [8] for further results.

Consider the following problem: For $f \in L$, seek $z \in W$ such that $Tz = f$. In general, boundary conditions must be enforced for this problem to be well-posed. In other words, one must find a closed subspace V of W such that the restricted operator $T : V \to L$ is an isomorphism. To achieve this goal, a simple approach inspired from Friedrichs' work [9, 10] consists of introducing an operator $M \in \mathcal{L}(W; W')$ such that

(M1) $\quad M$ is positive, i.e., $\langle Mz, z \rangle_{W',W} \geq 0 \quad \forall z$ in W,

(M2) $\quad W = \mathrm{Ker}(D - M) + \mathrm{Ker}(D + M)$.

Then by setting

$$V = \mathrm{Ker}(D - M) \qquad \text{and} \qquad V^* = \mathrm{Ker}(D + M^*), \tag{2.5}$$

where $M^* \in \mathcal{L}(W; W')$ is the adjoint of M and V and V^* are equipped with the graph norm, the following theorem can be proved (see [8, 9] for a proof).

THEOREM 2.1. *Assume* (A1)–(A4) *and* (M1)–(M2). *Then, the restricted operators* $T : V \to L$ *and* $\tilde{T} : V^* \to L$ *are isomorphisms.*

As a result, for f in L, the following two problems are well-posed:

$$\text{Seek } z \in V \text{ such that } Tz = f, \tag{2.6}$$

$$\text{Seek } z^* \in V^* \text{ such that } \tilde{T}z^* = f. \tag{2.7}$$

A key observation at this point is that the boundary conditions enforced in (2.6) and (2.7) are essential; i.e., they are enforced strongly by seeking the solutions in V and V^*, respectively. The key reason that led us to focus on the theory of Friedrichs' systems is that it yields a way to enforce boundary conditions naturally, thus leading to a suitable framework for developing a DG theory. To see this, we introduce the following bilinear forms on $W \times W$:

$$a(z,y) = (Tz, y)_L + \tfrac{1}{2}\langle (M - D)z, y \rangle_{W',W}, \tag{2.8}$$

$$a^*(z,y) = (\tilde{T}z, y)_L + \tfrac{1}{2}\langle (M^* + D)z, y \rangle_{W',W}. \tag{2.9}$$

It is clear that a and a^* are in $\mathcal{L}(W \times W; \mathbb{R})$. Equipped with these two new bilinear forms, we now consider the following problems: For $f \in L$,

$$\text{Seek } z \in W \text{ such that } a(z, y) = (f, y)_L \quad \forall y \in W, \tag{2.10}$$

$$\text{Seek } z^* \in W \text{ such that } a^*(z^*, y) = (f, y)_L \quad \forall y \in W. \tag{2.11}$$

The key result of this section is the following

THEOREM 2.2. *Assume* (A1)–(A4) *and* (M1)–(M2). *Then,*

(i) *there is a unique solution to* (2.10) *and this solution solves* (2.6)*;*

(ii) *there is a unique solution to* (2.11) *and this solution solves* (2.7).

Theorem 2.2 is proven in [9]. Contrary to (2.6) and (2.7), the boundary conditions in (2.10) and (2.11) are natural; i.e., they are weakly enforced. For this reason, problem (2.10) will constitute our working basis for designing DG methods; see section 4.

2.2. The two-field structure. We now particularize the above setting by assuming that the $(d+1)$ $\mathbb{R}^{m,m}$-valued fields $\mathcal{K}$ and $\{\mathcal{A}^k\}_{1\le k\le d}$ have a 2×2 block structure; i.e., there are two positive integers m_σ and m_u such that $m = m_\sigma + m_u$ and

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}^{\sigma\sigma} & \mathcal{K}^{\sigma u} \\ \mathcal{K}^{u\sigma} & \mathcal{K}^{uu} \end{bmatrix}, \qquad \mathcal{A}^k = \begin{bmatrix} 0 & \mathcal{B}^k \\ [\mathcal{B}^k]^t & \mathcal{C}^k \end{bmatrix}, \tag{2.12}$$

with obvious notation for the blocks of $\mathcal{K}$ and where for all $k \in \{1,\dots,d\}$, $\mathcal{B}^k$ is an $m_\sigma \times m_u$ matrix field and $\mathcal{C}^k$ is a symmetric $m_u \times m_u$ matrix field. To simplify the notation, define the operators $B = \sum_{k=1}^d \mathcal{B}^k\partial_k$, $B^\dagger = \sum_{k=1}^d [\mathcal{B}^k]^t\partial_k$, $\nabla\cdot B = \sum_{k=1}^d \partial_k\mathcal{B}^k$, $C = \sum_{k=1}^d \mathcal{C}^k\partial_k$, $C^\dagger = \sum_{k=1}^d [\mathcal{C}^k]^t\partial_k$, and $\nabla\cdot C = \sum_{k=1}^d \partial_k\mathcal{C}^k$. Set $L_\sigma = [L^2(\Omega)]^{m_\sigma}$ and $L_u = [L^2(\Omega)]^{m_u}$.

The two key hypotheses on which the present work is based are the following:

(A5) $\qquad \exists k_0 > 0 \quad \forall \xi \in \mathbb{R}^{m_\sigma}, \quad \xi^t\mathcal{K}^{\sigma\sigma}\xi \ge k_0\|\xi\|^2_{\mathbb{R}^{m_\sigma}} \quad$ a.e. on Ω,

(A6) $\qquad \forall k \in \{1,\dots,d\}, \quad$ the $m_\sigma \times m_\sigma$ upper-left block of $\mathcal{A}^k$ is zero.

Assumption (A5), which means that $\mathcal{K}^{\sigma\sigma}$ is uniformly positive definite, implies that the matrix $\mathcal{K}^{\sigma\sigma}$ is invertible.

Assumptions (A5) and (A6) allow for the elimination of z^σ from the PDE system $Tz = f$. With obvious notation, partition z and f into (z^σ, z^u) and (f^σ, f^u), respectively. Then, z^σ is given by

$$z^\sigma = [\mathcal{K}^{\sigma\sigma}]^{-1}\Big(f^\sigma - \mathcal{K}^{\sigma u}z^u - Bz^u\Big), \tag{2.13}$$

and z^u solves the following second-order PDE:

$$\begin{aligned} &-B^\dagger[\mathcal{K}^{\sigma\sigma}]^{-1}Bz^u + (C - B^\dagger[\mathcal{K}^{\sigma\sigma}]^{-1}\mathcal{K}^{\sigma u} - \mathcal{K}^{u\sigma}[\mathcal{K}^{\sigma\sigma}]^{-1}B)z^u \\ &\qquad + (\mathcal{K}^{uu} - \mathcal{K}^{u\sigma}[\mathcal{K}^{\sigma\sigma}]^{-1}\mathcal{K}^{\sigma u})z^u = f^u - (\mathcal{K}^{u\sigma} + B^\dagger)[\mathcal{K}^{\sigma\sigma}]^{-1}f^\sigma. \end{aligned} \tag{2.14}$$

The objective of the present work is to design DG methods for approximating (2.14). The strategy we are going to follow consists of constructing a DG approximation to (2.10), but at variance with what has been done in [9], the construction is now specialized to the above 2×2 block structure so that the approximate unknown corresponding to z^σ can be eliminated locally on each mesh element by solving local problems.

Remark 2.1. The present study does not cover the DG approximation of the whole realm of second-order PDEs. Indeed, it is clear from (2.14) that the leading-order term in the PDE, namely $B^\dagger[\mathcal{K}^{\sigma\sigma}]^{-1}Bz^u$ (up to first-order terms), has a very particular structure since the matrices $(\mathcal{B}^k)^t[\mathcal{K}^{\sigma\sigma}]^{-1}\mathcal{B}^k$ are positive semidefinite. Hence, the PDEs covered by this work are elliptic-like; see section 3 for various examples.

Remark 2.2. In some applications, K has no local representation; i.e., there is no local field $\mathcal{K}$ to represent K. This is indeed the case for the neutron transport equation, where K is a scattering operator. Everything that is said hereafter is also valid in this case, provided the matrix block representation of $\mathcal{K}$ is replaced by the operator block representation of K and provided $K^{\sigma\sigma}$ has a local representation, i.e., $(K^{\sigma\sigma}z^\sigma, y^\sigma)_{L_\sigma} = \int_\Omega (y^\sigma)^t \mathcal{K}^{\sigma\sigma} z^\sigma$.

2.3. Integral representation of boundary operators. Let $n = (n_1, \ldots, n_d)^t$ be the unit outward normal to $\partial\Omega$. Henceforth, we assume that the fields $\{\mathcal{A}^k\}_{1\le k\le d}$ are sufficiently smooth for the matrix $\mathcal{D} = \sum_{k=1}^d n_k \mathcal{A}^k$ to be meaningful at the boundary. Hence, the following representation holds:

$$\langle Dz, y\rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{D} z \tag{2.15}$$

whenever z and y are smooth functions. Owing to (2.12), $\mathcal{D}$ has a 2×2 block structure with $\mathcal{D}^{\sigma u} = \sum_{k=1}^d n_k \mathcal{B}^k$, $\mathcal{D}^{u\sigma} = [\mathcal{D}^{\sigma u}]^t$, $\mathcal{D}^{uu} = \sum_{k=1}^d n_k \mathcal{C}^k$, and

$$\mathcal{D}^{\sigma\sigma} = 0. \tag{2.16}$$

Likewise, we assume that the boundary operator M has an integral representation; i.e., there exists a matrix-valued field $\mathcal{M} : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$ such that

$$\langle Mz, y\rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{M} z \tag{2.17}$$

whenever z and y and smooth functions. We denote by $\mathcal{M}^{\sigma u}$, $\mathcal{M}^{u\sigma}$, and $\mathcal{M}^{uu}$ the top-right, bottom-left, and bottom-right blocks of $\mathcal{M}$, respectively. Henceforth, we assume that

$$\mathcal{M}^{\sigma\sigma} = 0. \tag{2.18}$$

This assumption holds for all the two-field Friedrichs' systems presented in section 3. For instance, the Dirichlet-like boundary condition $\mathcal{D}^{\sigma u}z^u = 0$ can be enforced by taking

$$\mathcal{M} = \left[\begin{array}{c|c} 0 & -\mathcal{D}^{\sigma u} \\ \hline \mathcal{D}^{u\sigma} & \mathcal{M}^{uu} \end{array}\right], \tag{2.19}$$

where $\mathcal{M}^{uu}$ is a positive matrix in $\mathbb{R}^{m_u,m_u}$ (this means that for all $\zeta \in \mathbb{R}^{m_u}$, $\zeta^t\mathcal{M}^{uu}\zeta \ge 0$) and is constructed so that $\mathrm{Ker}(\mathcal{D}^{\sigma u}) \subset \mathrm{Ker}(\mathcal{M}^{uu} - \mathcal{D}^{uu})$ (for instance take $\mathcal{M}^{uu} = \mathcal{D}^{uu} + c\,(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}}$ with c large enough for $\mathcal{M}^{uu}$ to be positive). Similarly, taking

$$\mathcal{M} = \left[\begin{array}{c|c} 0 & \mathcal{D}^{\sigma u} \\ \hline -\mathcal{D}^{u\sigma} & \mathcal{M}^{uu} \end{array}\right], \tag{2.20}$$

where $\mathcal{M}^{uu}$ is a positive matrix in $\mathbb{R}^{m_u,m_u}$, yields the Robin boundary condition $2\mathcal{D}^{u\sigma}z^\sigma + (\mathcal{D}^{uu} - \mathcal{M}^{uu})z^u = 0$. The homogeneous Neumann boundary condition is obtained by setting $\mathcal{M}^{uu} = \mathcal{D}^{uu}$ whenever $\mathcal{D}^{uu}$ is positive. See (3.7) and (6.3) for examples.

3. Examples. This section presents three examples of Friedrichs' systems endowed with the 2×2 block structure introduced in section 2.2.

3.1. Advection-diffusion-reaction. Consider the PDE

$$-\nabla{\cdot}(\kappa\nabla u) + \beta{\cdot}\nabla u + \mu u = f, \tag{3.1}$$

with $\beta \in [L^\infty(\Omega)]^d$, $\nabla{\cdot}\beta \in L^\infty(\Omega)$, $\mu \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and where $\kappa = (\kappa_{kl})_{1\le k,l\le d}$ is a symmetric positive definite tensor-valued field defined on Ω whose lowest eigenvalue is uniformly bounded away from zero. Assume also that

$$\mu - \tfrac{1}{2}\nabla{\cdot}\beta \ge \mu_0 > 0 \quad \text{a.e. in } \Omega. \tag{3.2}$$

The PDE (3.1) can be written as a system of first-order PDEs in the form

$$\begin{cases} \kappa^{-1}\sigma + \nabla u = 0, \\ \mu u + \nabla{\cdot}\sigma + \beta{\cdot}\nabla u = f. \end{cases} \tag{3.3}$$

Set $m = d+1$, $m_\sigma = d$, and $m_u = 1$. Then, the mixed formulation (3.3) can be cast into the form of a two-field Friedrichs' system by introducing $(d+1)$ functions with values in $\mathbb{R}^{m,m}$, namely $\mathcal{K}$ and $\{\mathcal{A}^k\}_{1\le k\le d}$ such that

$$\mathcal{K} = \left[\begin{array}{c|c} \kappa^{-1} & 0 \\ \hline 0 & \mu \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c|c} 0 & e^k \\ \hline (e^k)^t & \beta^k \end{array}\right], \tag{3.4}$$

where e^k is the kth vector in the canonical basis of $\mathbb{R}^d$ and β^k is the kth component of β in this basis. It is clear that hypotheses (A1)–(A6) hold. The graph space is $W = H(\text{div};\Omega) \times H^1(\Omega)$ and for all $(\sigma,u),(\tau,v) \in W$,

$$\langle D(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma{\cdot}n, v\rangle_{-\frac12,\frac12} + \langle \tau{\cdot}n, u\rangle_{-\frac12,\frac12} + \int_{\partial\Omega}(\beta{\cdot}n)uv, \tag{3.5}$$

where $\langle,\rangle_{-\frac12,\frac12}$ denotes the duality pairing between $H^{-\frac12}(\partial\Omega)$ and $H^{\frac12}(\partial\Omega)$. Note that (3.5) makes sense since functions in $H^1(\Omega)$ have traces in $H^{\frac12}(\partial\Omega)$ and vector fields in $H(\text{div};\Omega)$ have normal traces in $H^{-\frac12}(\partial\Omega)$.

Homogeneous Dirichlet boundary conditions can be enforced by setting

$$\langle M(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma{\cdot}n, v\rangle_{-\frac12,\frac12} - \langle \tau{\cdot}n, u\rangle_{-\frac12,\frac12}. \tag{3.6}$$

With this choice $V = V^* = H(\text{div};\Omega)\times H^1_0(\Omega)$. Let $\varrho \in L^\infty(\partial\Omega)$ be such that $2\varrho + \beta{\cdot}n \ge 0$ a.e. in $\partial\Omega$. Then, setting

$$\langle M(\sigma,u),(\tau,v)\rangle_{W',W} = -\langle \sigma{\cdot}n, v\rangle_{-\frac12,\frac12} + \langle \tau{\cdot}n, u\rangle_{-\frac12,\frac12} + \int_{\partial\Omega}(2\varrho + \beta{\cdot}n)uv, \tag{3.7}$$

the spaces V and V^* are defined by $V = \{(\sigma,u) \in W; (-\sigma{\cdot}n + \varrho u)|_{\partial\Omega} = 0\}$ and $V^* = \{(\sigma,u) \in W; (\sigma{\cdot}n + (\varrho + \beta{\cdot}n)u)|_{\partial\Omega} = 0\}$; i.e., a Robin boundary condition is enforced. A Neumann condition corresponds to $\varrho = 0$. We refer the reader to [9] for more details.

Remark 3.1. When κ is not invertible, Friedrichs' formalism can be extended as detailed in [8].

3.2. Linear continuum mechanics. Let α and γ be two positive functions in $L^\infty(\Omega)$ uniformly bounded away from zero by α_0 and γ_0, respectively. Consider the following set of PDEs:

$$\begin{cases} \sigma + p\mathcal{I}_d - \frac{1}{2}(\nabla u + (\nabla u)^t) = 0, \\ \mathrm{tr}(\sigma) + (d+\gamma)p = 0, \\ -\frac{1}{2}\nabla\cdot(\sigma + \sigma^t) + \alpha u = f, \end{cases} \tag{3.8}$$

where σ is $\mathbb{R}^{d,d}$-valued, p is scalar-valued, u is $\mathbb{R}^d$-valued, and $f \in [L^2(\Omega)]^d$. The first and second equations in (3.8) imply $p = -\gamma^{-1}\nabla\cdot u$ and $\sigma = \frac{1}{2}(\nabla u + (\nabla u)^t) + \gamma^{-1}(\nabla\cdot u)\mathcal{I}_d$; γ is a compressibility coefficient, σ is the stress tensor, $\frac{1}{2}(\nabla u + (\nabla u)^t)$ is the strain tensor, and u represents the displacement field in solid mechanics and the velocity field in fluid mechanics. In the usual solid mechanics equations, the function α vanishes identically. The function α has been introduced in (3.8) to ensure that the positivity property (A4) holds; see (3.10). In a forthcoming work, it will be shown that provided mild additional assumptions are made, the positivity property (A4) can be replaced by the weaker assumption (7.1), thus allowing α to vanish identically.

Set $m = d^2 + 1 + d$. The tensor σ in $\mathbb{R}^{d,d}$ is identified with the vector $\overline{\sigma} \in \mathbb{R}^{d^2}$ by setting $\overline{\sigma}_{[ij]} = \sigma_{ij}$ with $1 \le i,j \le d$ and $[ij] = d(j-1)+i$. Then, the mixed formulation (3.8) can be cast into the form of a Friedrichs' system by introducing the $(d+1)$ $\mathbb{R}^{m,m}$-valued fields with the following 3×3 block structure

$$\mathcal{K} = \left[\begin{array}{c:c:c} \mathcal{I}_{d^2} & \mathcal{Z} & 0 \\ \hdashline (\mathcal{Z})^t & (d+\gamma) & 0 \\ \hdashline 0 & 0 & \alpha\mathcal{I}_d \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c:c:c} 0 & 0 & \mathcal{E}^k \\ \hdashline 0 & 0 & 0 \\ \hdashline (\mathcal{E}^k)^t & 0 & 0 \end{array}\right], \tag{3.9}$$

where $\mathcal{Z} \in \mathbb{R}^{d^2}$ has components given by $\mathcal{Z}_{[ij]} = \delta_{ij}$ with $1 \le i,j \le d$, and for all $k \in \{1,\dots,d\}$, $\mathcal{E}^k \in \mathbb{R}^{d^2,d}$ has components given by $\mathcal{E}^k_{[ij],l} = -\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $1 \le i,j,l \le d$; here, the δ's denote Kronecker symbols.

To recover the 2×2 structure introduced in section 2.2, set $m_\sigma = d^2+1$ and $m_u = d$; i.e., the σ-component corresponds to the pair $(\overline{\sigma}, p)$. Then, hypotheses (A1)–(A6) hold. In particular, (A4)–(A5) result from the fact that for all $z = (\overline{\sigma}, p, u) \in \mathbb{R}^m$,

$$z^t\mathcal{K}z \ge \left(1 - \frac{d}{d+\frac{\gamma_0}{2}}\right)\overline{\sigma}^2 + \frac{\gamma_0}{2}p^2 + \frac{d}{d+\frac{\gamma_0}{2}}\left(\overline{\sigma} + \frac{d+\frac{\gamma_0}{2}}{d}p\mathcal{Z}\right)^2 + \alpha_0 u^2 \ge c(\overline{\sigma}^2 + p^2 + u^2), \tag{3.10}$$

where c depends only on d, α_0, and γ_0. Using the second Korn inequality for the variable u, it is readily seen that the graph space is $W = H_{\overline{\sigma}} \times L^2(\Omega) \times [H^1(\Omega)]^d$ with $H_{\overline{\sigma}} = \{\overline{\sigma} \in [L^2(\Omega)]^{d^2};\ \nabla\cdot(\sigma+\sigma^t) \in [L^2(\Omega)]^d\}$. The boundary operator D takes the following form: For all $(\overline{\sigma}, p, u), (\overline{\tau}, q, v) \in W$,

$$\langle D(\overline{\sigma}, p, u), (\overline{\tau}, q, v)\rangle_{W',W} = -\langle \tfrac{1}{2}(\tau+\tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tfrac{1}{2}(\sigma+\sigma^t)\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}}, \tag{3.11}$$

where $\langle,\rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing between $[H^{-\frac{1}{2}}(\partial\Omega)]^d$ and $[H^{\frac{1}{2}}(\partial\Omega)]^d$.

To enforce boundary conditions for (3.8), one possibility consists of setting for all $(\overline{\sigma}, p, u), (\overline{\tau}, q, v) \in W$,

$$\langle M(\overline{\sigma}, p, u), (\overline{\tau}, q, v)\rangle_{W',W} = \langle \tfrac{1}{2}(\tau+\tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tfrac{1}{2}(\sigma+\sigma^t)\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}}. \tag{3.12}$$

With this choice, the u-component is set to zero at $\partial\Omega$ (i.e., a homogeneous Dirichlet boundary condition on the displacement (in solid mechanics) or on the velocity (in fluid mechanics) is enforced) as shown in the following

LEMMA 3.1. *Let M be given by* (3.12). *Then,* $V = V^* = H_{\overline{\sigma}} \times L^2(\Omega) \times [H_0^1(\Omega)]^d$.

Proof. It is clear that $V = V^*$ since $M + M^* = 0$. Observe that

$$\langle (D-M)(\overline{\sigma}, p, u), (\overline{\tau}, q, v)\rangle_{W',W} = -\langle (\tau + \tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}}. \tag{3.13}$$

Hence, it is clear that $H_{\overline{\sigma}} \times L^2(\Omega) \times [H_0^1(\Omega)]^d \subset \mathrm{Ker}(D-M) = V$. Conversely, let $(\overline{\sigma}, p, u) \in \mathrm{Ker}(D-M)$. Let $\theta \in [H^{-\frac{1}{2}}(\partial\Omega)]^d$. Consider the following problem: Seek $v_\theta \in [H^1(\Omega)]^d$ such that for all $w \in [H^1(\Omega)]^d$,

$$(v_\theta, w)_{[L^2(\Omega)]^d} + (\nabla v_\theta + (\nabla v_\theta)^t, \nabla w + (\nabla w)^t)_{[L^2(\Omega)]^{d,d}} = \langle \theta, w\rangle_{-\frac{1}{2},\frac{1}{2}}.$$

This problem is well-posed owing to the second Korn inequality and the Lax–Milgram lemma. Set $\tau_\theta = \nabla v_\theta + (\nabla v_\theta)^t$. Since $\overline{\tau}_\theta \in H_{\overline{\sigma}}$, one can take $(\overline{\tau}, q, v) = (\overline{\tau}_\theta, 0, 0)$ in (3.13) yielding $\langle \theta, u\rangle_{-\frac{1}{2},\frac{1}{2}} = 0$. Since θ is arbitrary in $[H^{-\frac{1}{2}}(\partial\Omega)]^d$, it is inferred that $u \in [H_0^1(\Omega)]^d$. ◻

3.3. Simplified MHD. For the sake of simplicity we assume that the space dimension is three, i.e., $d = 3$. Let ν, μ, and σ be three functions in $L^\infty(\Omega)$, and let $\beta \in [L^\infty(\Omega)]^3$ be a vector field. A simplified (time-discretized) version of the MHD equations consists of seeking the electric field E and the magnetic field H such that

$$\begin{cases} \nu H + \nabla\times E = 0, \\ \sigma(E + \beta\times(\mu H)) - \nabla\times H = j, \end{cases} \tag{3.14}$$

where $j \in [L^2(\Omega)]^3$ is a given source term. The separation of the electromagnetic field (H, E) into magnetic and electric fields induces a natural partitioning of $[L^2(\Omega)]^6$ into $[L^2(\Omega)]^3 \times [L^2(\Omega)]^3$. The PDEs (3.14) are recast into the form of a Friedrichs' system by introducing the following block structured matrices in $\mathbb{R}^{6,6}$:

$$\mathcal{K} = \left[\begin{array}{c:c} \nu\mathcal{I}_3 & 0 \\ \hdashline \sigma\mu\mathcal{V} & \sigma\mathcal{I}_3 \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & \mathcal{R}^k \\ \hdashline (\mathcal{R}^k)^t & 0 \end{array}\right], \tag{3.15}$$

where $\mathcal{R}^k_{ij} = \epsilon_{ikj}$ is the Levi-Civita permutation tensor, $1 \le i, j, k \le 3$, and $\mathcal{V}_{ij} = \sum_{k=1}^d \epsilon_{ikj}\beta^k$. Assume that ν and σ are positive functions on Ω uniformly bounded away from zero and that there is $\alpha_0 > 0$ such that a.e. in Ω, $2\left(\frac{\nu}{\sigma}\right)^{\frac{1}{2}} - \mu\|\beta\|_{[L^\infty(\Omega)]^d} \ge \alpha_0$. In the above framework, one readily verifies that (A1)–(A6) hold with $m = 6$, $m_\sigma = 3$, and $m_u = 3$. In the full MHD equations, the off-diagonal term induced by β is compensated by a term originating from the conservation of momentum in the Navier–Stokes equations so that the condition for (A4) to hold is simply that ν and σ be uniformly bounded away from zero.

The graph space is $W = H(\mathrm{curl};\Omega) \times H(\mathrm{curl};\Omega)$ and for all $(H, E), (h, e) \in W$,

$$\begin{aligned} \langle D(H,E), (h,e)\rangle_{W',W} = {}& (\nabla\times E, h)_{[L^2(\Omega)]^3} - (E, \nabla\times h)_{[L^2(\Omega)]^3} \\ & + (H, \nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H, e)_{[L^2(\Omega)]^3}. \end{aligned} \tag{3.16}$$

When (H, E) and (h, e) are smooth, the above duality product can be interpreted as the boundary integral $\int_{\partial\Omega}[(n\times E)\cdot h + (n\times e)\cdot H]$.

An admissible boundary condition for (3.14) consists of setting

$$\langle M(H,E),(h,e)\rangle_{W',W} = -(\nabla\times E,h)_{[L^2(\Omega)]^3} + (E,\nabla\times h)_{[L^2(\Omega)]^3} + (H,\nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H,e)_{[L^2(\Omega)]^3} \tag{3.17}$$

for all $(H,E),(h,e)\in W$. Assuming $[H^1(\Omega)]^3$ is dense in $H(\text{curl};\Omega)$, this choice yields $V=V^*=H(\text{curl};\Omega)\times H_0(\text{curl};\Omega)$; i.e., the tangential component of the electric field is set to zero; see [8] for the analysis.

4. Two-field DG approximation.

In this section we design a DG method to approximate the two-field Friedrichs' systems introduced in section 2.2. The key feature is that the discrete σ-component can be eliminated locally.

4.1. The discrete setting.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of meshes of Ω. The meshes are assumed to be affine to avoid unnecessary technicalities; i.e., Ω is assumed to be a polyhedron. For $K\in\mathcal{T}_h$, h_K denotes its diameter and we set $h=\max_{K\in\mathcal{T}_h} h_K$. Henceforth, the notation $\xi\lesssim\zeta$ means that there is a positive c, *independent of* h, such that $\xi\le c\zeta$. For any measurable subset E of Ω, we denote by $(\cdot,\cdot)_{L,E}$ the usual scalar product in $[L^2(E)]^m$. We define similarly $(\cdot,\cdot)_{L_u,E}$ and $(\cdot,\cdot)_{L_\sigma,E}$.

We denote by $\mathcal{F}_h^{\mathrm{i}}$ the set of interfaces; i.e., $F\in\mathcal{F}_h^{\mathrm{i}}$ if F is a (d-1)-dimensional manifold and there are $K_1(F)$ and $K_2(F)\in\mathcal{T}_h$ such that $F=K_1(F)\cap K_2(F)$. For $F\in\mathcal{F}_h^{\mathrm{i}}$, we set $\mathcal{T}(F)=K_1(F)\cup K_2(F)$. We denote by $\mathcal{F}_h^{\partial}$ the set of the faces that separate the mesh from the exterior of Ω; i.e., $F\in\mathcal{F}_h^{\partial}$ if F is a (d-1)-dimensional manifold and there is $K(F)\in\mathcal{T}_h$ such that $F=K(F)\cap\partial\Omega$. For $F\in\mathcal{F}_h^{\partial}$, we set $\mathcal{T}(F)=K(F)$. For all $F\in\mathcal{F}_h^{\mathrm{i}}$, we denote by n_F the unit normal vector on F pointing from $K_1(F)$ to $K_2(F)$. For all $F\in\mathcal{F}_h^{\partial}$, we denote by n_F the unit normal vector on F pointing outside Ω. Finally, we set $\mathcal{F}_h=\mathcal{F}_h^{\mathrm{i}}\cup\mathcal{F}_h^{\partial}$. For all $F\in\mathcal{F}_h$, it is assumed that

$$h_{\mathcal{T}(F)}\lesssim h_F, \tag{4.1}$$

where $h_{\mathcal{T}(F)}$ denotes the diameter of $\mathcal{T}(F)$ and h_F that of F. No other assumption than (4.1) is made on the matching of element faces.

For a nonnegative integer p, consider the finite element space of scalar-valued functions

$$P_{h,p}=\{v_h\in L^2(\Omega);\,\forall K\in\mathcal{T}_h,\ v_h|_K\in\mathbb{P}_p\}, \tag{4.2}$$

where $\mathbb{P}_p$ denotes the vector space of polynomials with real coefficients and with total degree less than or equal to p. The mesh family $\{\mathcal{T}_h\}_{h>0}$ is assumed to be regular enough for the following inverse and trace inverse inequalities to hold: For all $v_h\in P_{h,p}$,

$$\forall K\in\mathcal{T}_h,\quad \|\nabla v_h\|_{[L^2(K)]^d}\lesssim h_K^{-1}\|v_h\|_{L^2(K)}, \tag{4.3}$$

$$\forall F\in\mathcal{F}_h,\quad \|v_h\|_{L^2(F)}\lesssim h_F^{-\frac12}\|v_h\|_{L^2(\mathcal{T}(F))}. \tag{4.4}$$

Let p_u and p_σ be two integers such that

$$1\le p_u \qquad\text{and}\qquad p_u-1\le p_\sigma. \tag{4.5}$$

Define the following vector spaces:

$$U_h=[P_{h,p_u}]^{m_u},\qquad \Sigma_h=[P_{h,p_\sigma}]^{m_\sigma},\qquad W_h=U_h\times\Sigma_h, \tag{4.6}$$

and set $U(h) = [H^1(\Omega)]^{m_u} + U_h$, $\Sigma(h) = [H^1(\Omega)]^{m_\sigma} + \Sigma_h$, and $W(h) = [H^1(\Omega)]^m + W_h$. Obviously, inequalities (4.3) and (4.4) can be applied componentwise to all functions in U_h and in Σ_h. Moreover, since every function v in $U(h)$ has a (possibly two-valued) trace a.e. on $F \in \mathcal{F}_h^{\mathrm{i}}$, we set

$$[\![v]\!] = v^1 - v^2, \qquad \{v\} = \tfrac{1}{2}(v^1 + v^2), \tag{4.7}$$

where for a.e. $x \in F$, $v^\nu(x) = \lim_{y \to x} v(y)|_{K_\nu(F)}$, $\nu \in \{1, 2\}$. We define τ^1, τ^2, and $[\![\tau]\!]$ similarly for all τ in $\Sigma(h)$. The arbitrariness in the choice of $K_1(F)$ and $K_2(F)$ could be avoided by choosing intrinsic notations that would, however, unnecessarily complicate the presentation; nothing that is said hereafter depends on this choice. The above mean and jump operators are extended to boundary faces $F \in \mathcal{F}_h^\partial$ by taking the value of the function on that face.

4.2. Boundary and interface operators. For all $F \in \mathcal{F}_h$, we define the matrix-valued field $\mathcal{D}_F : F \to \mathbb{R}^{m,m}$ by

$$\mathcal{D}_F(x) = \sum_{k=1}^{d} n_{F,k} \mathcal{A}^k(x) \quad \text{a.e. on } F, \tag{4.8}$$

where $n_F = (n_{F,1}, \dots, n_{F,d})^t$. Owing to (2.12), $\mathcal{D}_F$ has a 2×2 block structure with $\mathcal{D}_F^{\sigma u} = \sum_{k=1}^d n_{F,k} \mathcal{B}^k$, $\mathcal{D}_F^{u\sigma} = [\mathcal{D}_F^{\sigma u}]^t$, $\mathcal{D}_F^{uu} = (\mathcal{D}_F^{uu})^t = \sum_{k=1}^d n_{F,k} \mathcal{C}^k$, and

$$\mathcal{D}_F^{\sigma\sigma} = 0. \tag{4.9}$$

The definition (4.8) is clearly compatible with that of $\mathcal{D}$; i.e., if $F \in \mathcal{F}_h^\partial$, $\mathcal{D}_F = \mathcal{D}$. Moreover, observe that for all z, y in $W(h)$ and for all $K \in \mathcal{T}_h$,

$$\sum_{F \subset \partial K} n_F \cdot n_K (\mathcal{D}_F z, y)_{L,F} = (Az, y)_{L,K} - (z, \tilde{A} y)_{L,K}. \tag{4.10}$$

We now extend the matrix-valued field $\mathcal{D}$ to interfaces as follows. For all $F \in \mathcal{F}_h^{\mathrm{i}}$, $\mathcal{D}|_F$ is two-valued, the two values being $n_F \cdot n_{K_1(F)} \mathcal{D}_F$ and $n_F \cdot n_{K_2(F)} \mathcal{D}_F$. Note that $\{\mathcal{D}\} = 0$ a.e. on $\mathcal{F}_h^{\mathrm{i}}$ since $\sum_{k=1}^d \partial_k \mathcal{A}^k$ is bounded owing to (A2).

To weakly enforce boundary conditions, we introduce for all $F \in \mathcal{F}_h^\partial$ a linear operator

$$M_F = \left[\begin{array}{c:c} M_F^{\sigma\sigma} & M_F^{\sigma u} \\ \hdashline M_F^{u\sigma} & M_F^{uu} \end{array}\right] \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m). \tag{4.11}$$

Note that M_F is not necessarily the restriction of M to functions defined on F; see Remark 5.2 below. Similarly, to penalize interface jumps, we introduce for all $F \in \mathcal{F}_h^{\mathrm{i}}$ a linear operator

$$S_F = \left[\begin{array}{c:c} S_F^{\sigma\sigma} & S_F^{\sigma u} \\ \hdashline S_F^{u\sigma} & S_F^{uu} \end{array}\right] \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m). \tag{4.12}$$

Star superscripts denote the L^2-adjoint of M_F, S_F, or any block thereof. For instance, $(M_F^{u\sigma})^* \in \mathcal{L}([L^2(F)]^{m_u}; [L^2(F)]^{m_\sigma})$ is defined such that $((M_F^{u\sigma})^*(v), \tau)_{L_\sigma,F} = (M_F^{u\sigma}(\tau), v)_{L_u,F}$ for all $v \in [L^2(F)]^{m_u}$ and for all $\tau \in [L^2(F)]^{m_\sigma}$. Finally, we introduce for all $F \in \mathcal{F}_h$ a linear operator

$$R_F \in \mathcal{L}([L^2(\mathcal{F}_h)]^{m_u}; [L^2(F)]^{m_u}). \tag{4.13}$$

The purpose of this operator is to reduce computational costs when solving the discrete problem for the u-component once the discrete σ-component has been eliminated locally; see section 4.4 and, in particular, (4.31). A simple choice consists of setting $R_F \equiv 0$ for all $F \in \mathcal{F}_h$; an example with nonzero R_F's is the IP method discussed in section 6.1.2.

The operators M_F, S_F, and R_F satisfy various design criteria which are collected in section 5.1. For the time being, we solely mention the important assumption

$$M_F^{\sigma\sigma} = 0 \qquad \text{and} \qquad S_F^{\sigma\sigma} = 0. \tag{4.14}$$

Hence, the jumps across interfaces of the σ-component of the unknown are not controlled. This is the key property that allows for the local elimination of the σ-component of the discrete solution z_h; see section 4.4. This is the most important difference with respect to the DG method analyzed in [9].

4.3. The discrete problem and the notion of fluxes. Drawing inspiration from (2.10), we introduce the bilinear form a_h such that for all z, y in $W(h)$,

$$\begin{aligned} a_h(z,y) = &\sum_{K\in\mathcal{T}_h} (Tz,y)_{L,K} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(z) - \mathcal{D}z, y)_{L,F} - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}z\},\{y\})_{L,F} \\ &+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F([\![z]\!]),[\![y]\!])_{L,F} + \sum_{F\in\mathcal{F}_h} (R_F([\![z^u]\!]),[\![y^u]\!])_{L_u,F}. \end{aligned} \tag{4.15}$$

The first and second term in the right-hand side come directly from (2.8). The third term is meant to ensure that a_h satisfies a coercivity property on W_h (see Lemma 5.4) in a manner consistent with the continuous setting (this term is zero whenever z is smooth). The fourth term is used to control the jump of the discrete solution across interfaces. The last term is a perturbation (possibly $R_F \equiv 0$) which allows for some modifications of the second and third terms to alleviate computational costs; see the end of section 4.4 and the IP method discussed in section 6.1.2.

The discrete counterpart of (2.10) is the following: For $f = (f^\sigma, f^u) \in L$,

$$\begin{cases} \text{Seek } z_h = (z_h^\sigma, z_h^u) \in W_h \text{ such that} \\ a_h(z_h, y_h) = (f, y_h)_L \quad \forall y_h = (y_h^\sigma, y_h^u) \in W_h. \end{cases} \tag{4.16}$$

As in [9], the discrete problem (4.16) can be localized by using the notion of flux. Let K be a mesh element in $\mathcal{T}_h$ and let $z \in W(h)$. The element flux of z on ∂K, say $\phi_{\partial K}(z) \in [L^2(\partial K)]^m$, is defined by its restriction to the faces F of ∂K as follows:

$$\phi_{\partial K}(z)|_F = \begin{cases} \frac{1}{2}(\mathcal{D}z + M_F(z) + 2R'_F(z^u)) & \text{if } F \in \mathcal{F}_h^\partial, \\ n_F{\cdot}n_K(\mathcal{D}_F\{z\} + S_F([\![z]\!]) + R'_F([\![z^u]\!])) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}, \end{cases} \tag{4.17}$$

where $R'_F(z^u) = (0, R_F(z^u)) \in [L^2(F)]^m$.

The discrete problem (4.16) is equivalently reformulated in terms of the following local problems posed for all $K \in \mathcal{T}_h$:

(4.18)
$$\begin{cases} \text{Seek } z_h \in W_h \text{ such that } \forall q = (q^\sigma, q^u) \in [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma} \times [\mathbb{P}_{p_u}(K)]^{m_u}, \\ (Kz_h, q)_{L,K} + (Az_h, q)_{L,K} + (\phi_{\partial K}(z_h) - n_F{\cdot}n_K\mathcal{D}_F z_h|_K, q)_{L,\partial K} = (f, q)_{L,K}, \end{cases}$$

or equivalently using the local integration by parts formula (4.10),

$$\text{(4.19)}\qquad \begin{cases} \text{Seek } z_h \in W_h \text{ such that } \forall q = (q^\sigma, q^u) \in [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma} \times [\mathbb{P}_{p_u}(K)]^{m_u}, \\ (Kz_h, q)_{L,K} + (z_h, \tilde{A}q)_{L,K} + (\phi_{\partial K}(z_h), q)_{L,\partial K} = (f, q)_{L,K}. \end{cases}$$

4.4. Eliminating the σ-component. We now rewrite (4.18) using the 2×2 block structure, and we show how the unknown z_h^σ can be locally eliminated. To simplify, we assume that $f^\sigma \equiv 0$ (this is a natural assumption to define z^σ in physical models). Recall that the σ-component of the element flux is

$$\text{(4.20)}\qquad \phi^\sigma_{\partial K}(z^u)|_F = \begin{cases} \frac{1}{2}(\mathcal{D}^{\sigma u} + M_F^{\sigma u})z^u & \text{if } F \in \mathcal{F}_h^\partial, \\ n_F{\cdot}n_K(\mathcal{D}_F^{\sigma u}\{z^u\} + S_F^{\sigma u}([\![z^u]\!])) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}, \end{cases}$$

where we stress that $\phi^\sigma_{\partial K}$ solely depends on z^u owing to (4.14). Then, (4.18) implies that z_h^σ solves the following local problems: For all $q^\sigma \in \mathbb{P}_\sigma(K) := [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma}$,

$$\text{(4.21)}\qquad (\mathcal{K}^{\sigma\sigma} z_h^\sigma + \mathcal{K}^{\sigma u} z_h^u + Bz_h^u, q^\sigma)_{L_\sigma,K} + (\phi^\sigma_{\partial K}(z_h^u) - \mathcal{D}^{\sigma u}_{\partial K} z_h^u|_K, q^\sigma)_{L_\sigma,\partial K} = 0.$$

For all $K \in \mathcal{T}_h$, let θ_K^1 be the L^2-orthogonal projection from $[L^2(K)]^{m_\sigma}$ onto $\mathbb{P}_\sigma(K)$ and let $\theta_K^2 : \mathbb{P}_\sigma(K) \to \mathbb{P}_\sigma(K)$ be the mapping such that for all $q^\sigma \in \mathbb{P}_\sigma(K)$, $(\theta_K^2(q^\sigma), r^\sigma)_{L_\sigma,K} = (\mathcal{K}^{\sigma\sigma} q^\sigma, r^\sigma)_{L_\sigma,K}$ for all $r^\sigma \in \mathbb{P}_\sigma(K)$ (note that θ_K^2 is the identity whenever $\mathcal{K}^{\sigma\sigma}$ is the identity matrix in $\mathbb{R}^{m_\sigma,m_\sigma}$). Let $F \in \mathcal{F}_h$. Define the mapping $r_F : [L^2(F)]^{m_\sigma} \longrightarrow \Sigma_h$ so that for all $z^\sigma \in [L^2(F)]^{m_\sigma}$, $r_F(z^\sigma)$ solves

$$\text{(4.22)}\qquad (r_F(z^\sigma), y_h^\sigma)_{L_\sigma} = (z^\sigma, \{y_h^\sigma\})_{L_\sigma,F} \quad \forall y_h^\sigma \in \Sigma_h.$$

Observe that the support of $r_F(z^\sigma)$ is contained in $\mathcal{T}(F)$. Then, (4.21) yields the local reconstruction formula for the discrete σ-component in the form

$$\text{(4.23)}\qquad \forall K \in \mathcal{T}_h, \quad z_h^\sigma|_K = \mathfrak{R}_K(z_h^u) + \mathfrak{R}_{\Delta_K}([\![z_h^u]\!]),$$

where

$$\text{(4.24)}\qquad \mathfrak{R}_K(z_h^u) = -(\theta_K^2)^{-1}\theta_K^1(\mathcal{K}^{\sigma u} z_h^u + Bz_h^u|_K)$$

is supported on K, and where

$$\text{(4.25)}\qquad \mathfrak{R}_{\Delta_K}([\![z_h^u]\!]) = -(\theta_K^2)^{-1} \sum_{F \subset \partial K} r_F(\psi_{F,K}([\![z_h^u]\!]))$$

is supported on $\Delta_K = \{K' \in \mathcal{T}_h;\ \exists F \in \mathcal{F}_h^{\mathrm{i}};\ F = K \cap K'\}$. Here,

$$\text{(4.26)}\qquad \psi_{F,K}(v) = \begin{cases} \frac{1}{2}(M_F^{\sigma u} - \mathcal{D}^{\sigma u})v & \text{if } F \in \mathcal{F}_h^\partial, \\ (2n_F{\cdot}n_K S_F^{\sigma u} - \mathcal{D}_F^{\sigma u})v & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}. \end{cases}$$

Then, using (4.23) in (4.19) shows that z_h^u solves the following problems: For all $K \in \mathcal{T}_h$ and for all $q^u \in \mathbb{P}_u(K) := [\mathbb{P}_{p_u}(K)]^{m_u}$,

(4.27)

$$((\mathcal{K}^{u\sigma} - (\nabla{\cdot}B)^*)(\mathfrak{R}_K(z_h^u) + \mathfrak{R}_{\Delta_K}([\![z_h^u]\!])) + (\mathcal{K}^{uu} - \nabla{\cdot}C)z_h^u - f^u, q^u)_{L_u,K}$$
$$- (z_h^u, C^\dagger q^u)_{L_u,K} - (\mathfrak{R}_K(z_h^u) + \mathfrak{R}_{\Delta_K}([\![z_h^u]\!]), B^\dagger q^u)_{L_u,K} + (\phi^u_{\partial K}(z_h^u), q^u)_{L_u,\partial K} = 0,$$

where for $F \in \mathcal{F}_h^\partial$,

$$\phi^u_{\partial K}(z^u_h)|_F = \tfrac{1}{2}(M^{u\sigma}_F + \mathcal{D}^{u\sigma})(\mathfrak{R}_K(z^u_h) + \mathfrak{R}_{\Delta_K}([\![z^u_h]\!])) + \tfrac{1}{2}(M^{uu}_F + \mathcal{D}^{uu})z^u_h + R_F([\![z^u_h]\!]), \tag{4.28}$$

and for $F \in \mathcal{F}^{\mathrm{i}}_h$,

$$\phi^u_{\partial K}(z^u_h)|_F = n_F{\cdot}n_K(\mathcal{D}^{u\sigma}_F\,\{\mathfrak{R}_K(z^u_h) + \mathfrak{R}_{\Delta_K}([\![z^u_h]\!])\} + \mathcal{D}^{uu}_F\,\{z^u_h\} + S^{u\sigma}_F([\![\mathfrak{R}_K(z^u_h) + \mathfrak{R}_{\Delta_K}([\![z^u_h]\!])]\!]) + S^{uu}_F([\![z^u_h]\!]) + R_F([\![z^u_h]\!])). \tag{4.29}$$

This readily yields the following.

PROPOSITION 4.1. *If the pair* (z^σ_h, z^u_h) *solves* (4.16)*, then* (4.23) *holds and* z^u_h *solves* (4.27)*. Conversely, if* z^u_h *solves* (4.27) *and if* z^σ_h *is defined by* (4.23)*, then the pair* (z^σ_h, z^u_h) *solves* (4.16).

At this point, it is important to observe that owing to the presence of the nonlocal term $\mathfrak{R}_{\Delta_K}$ in the flux $\phi^u_{\partial K}$, the problem (4.27) couples the degrees of freedom for z^u_h in a given element to those in the neighboring elements and also to those in the neighbors of the neighbors. Let us assume that $S^{u\sigma}_F \equiv 0$ and, for simplicity, that Dirichlet boundary conditions are enforced so that $M^{\sigma u}_F = -\mathcal{D}^{\sigma u}$ and $M^{u\sigma}_F = \mathcal{D}^{u\sigma}$ (Neumann/Robin boundary conditions can be treated as well). Then, if R_F is defined so that for all $F \subset \partial K$,

$$R_F([\![z^u_h]\!]) + \mathcal{D}^{u\sigma}_F\,\{\mathfrak{R}_{\Delta_K}([\![z^u_h]\!])\} = 0, \tag{4.30}$$

the terms involving $\mathfrak{R}_{\Delta_K}([\![z^u_h]\!])$ are eliminated from (4.28)–(4.29). Owing to this elimination, problem (4.27) couples the degrees of freedom for z^u_h in a given element only to those in the neighboring elements. Using (4.25), it is readily verified that (4.30) holds if R_F is designed such that

$$R_F([\![z^u_h]\!]) = \tfrac{1}{2}\mathcal{D}^{u\sigma}_F \sum_{i=1}^{2} (\theta^2_{K_i(F)})^{-1} \sum_{F' \in \partial K_i(F)} r_{F'}(\psi_{F',K_i(F)}([\![z^u_h]\!]))|_F. \tag{4.31}$$

Finally, a further simplification occurs whenever $\mathcal{K}^{u\sigma} - (\nabla{\cdot}B)^* \equiv 0$ since, in this case, the term $\mathfrak{R}_{\Delta_K}([\![z^u_h]\!])$ needs not be evaluated to solve (4.27) for z^u_h; i.e., the reconstruction of z^σ_h from (4.23) can be performed as a postprocessing step.

5. Convergence analysis. In this section, we present the design criteria for the above DG method and perform the error analysis. The main results are Theorem 5.8, which estimates the error in the norm (5.10), and Theorem 5.14, which improves the L_u-estimate of the u-component of the error by means of a duality argument. Throughout this section, we assume the following:

- For all $k \in \{1, \dots, d\}$ and for all $K \in \mathcal{T}_h$, $\mathcal{B}^k \in [\mathcal{C}^{0,1}(K)]^{m_\sigma, m_u}$.
- The mesh family $\{\mathcal{T}_h\}_{h>0}$ is such that (4.1), (4.3), and (4.4) hold.
- The approximation spaces are defined according to (4.2), (4.5), and (4.6).

5.1. The design criteria for the boundary and interface operators. For all $F \in \mathcal{F}_h^\partial$, for all $v, w \in [L^2(F)]^{m_u}$, and for all $\tau \in [L^2(F)]^{m_\sigma}$, we assume that

(DG1) $M_F^{\sigma\sigma} = 0,$

(DG2) $M_F^{\sigma u} + (M_F^{u\sigma})^* = 0,$

(DG3) $(M_F^{uu}(v), v)_{L_u,F} \geq 0,$

(DG4) $|(M_F^{\sigma u}(v) - \mathcal{D}^{\sigma u}v, \tau)_{L_\sigma,F}| \lesssim h_F^{\frac{1}{2}} |v|_{M,F} \|\tau\|_{L_\sigma,F},$

(DG5) $|(M_F^{uu}(v) + \mathcal{D}^{uu}v, w)_{L_u,F}| \lesssim h_F^{-\frac{1}{2}} \|v\|_{L_u,F} |w|_{M,F},$

(DG6) $|(M_F^{uu}(v) - \mathcal{D}^{uu}v, w)_{L_u,F}| \lesssim h_F^{-\frac{1}{2}} |v|_{M,F} \|w\|_{L_u,F},$

(DG7) $\mathrm{Ker}(\mathcal{M} - \mathcal{D}) \subset \mathrm{Ker}(M_F - \mathcal{D}),$

(DG8) $\mathrm{Ker}(\mathcal{M}^t + \mathcal{D}) \subset \mathrm{Ker}(M_F^* + \mathcal{D}),$

where we have introduced the following seminorms:

$$(5.1) \quad \forall v \in U(h), \qquad |v|_M^2 = \sum_{F \in \mathcal{F}_h^\partial} |v|_{M,F}^2 \qquad \text{with} \qquad |v|_{M,F}^2 = (M_F^{uu}(v), v)_{L_u,F}.$$

For all $F \in \mathcal{F}_h^{\mathrm{i}}$, for all $v, w \in [L^2(F)]^{m_u}$, and for all $\tau \in [L^2(F)]^{m_\sigma}$, we assume that

(DG9) $S_F^{\sigma\sigma} = 0,$

(DG10) $S_F^{\sigma u} + (S_F^{u\sigma})^* = 0,$

(DG11) $(S_F^{uu}(v), v)_{L_u,F} \geq 0,$

(DG12) $|(S_F^{uu}(v), w)_{L_u,F}| \lesssim h_F^{-\frac{1}{2}} \|v\|_{L_u,F} |w|_{S,F},$

(DG13) $|(S_F^{uu}(v), w)_{L_u,F}| \lesssim h_F^{-\frac{1}{2}} |v|_{S,F} \|w\|_{L_u,F},$

(DG14) $|(S_F^{\sigma u}(v), \tau)_{L_\sigma,F}| \lesssim h_F^{\frac{1}{2}} |v|_{S,F} \|\tau\|_{L_\sigma,F},$

(DG15) $|(\mathcal{D}^{\sigma u}v, \tau)_{L_\sigma,F}| \lesssim h_F^{\frac{1}{2}} |v|_{S,F} \|\tau\|_{L_\sigma,F},$

(DG16) $|(\mathcal{D}^{uu}v, w)_{L_u,F}| \lesssim h_F^{-\frac{1}{2}} |v|_{S,F} \|w\|_{L_u,F},$

where we have introduced the following seminorms:

$$(5.2) \quad \forall v \in U(h), \qquad |v|_S^2 = \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} |v|_{S,F}^2 \qquad \text{with} \qquad |v|_{S,F}^2 = (S_F^{uu}(v), v)_{L_u,F}.$$

Finally, the design of the operators R_F is based on the following assumptions:

(DG17) $\forall z_h \in W_h, \quad \rho_h([\![z_h^u]\!], [\![z_h^u]\!]) \geq -\frac{1}{4}(|z_h^u|_J^2 + |z_h^u|_M^2),$

(DG18) $\forall (z, y_h) \in W(h) \times W_h, \quad \rho_h([\![z^u]\!], [\![y_h^u]\!]) \lesssim (|z^u|_J + |z^u|_M)(|y_h^u|_J + |y_h^u|_M),$

where $\rho_h([\![z^u]\!], [\![y^u]\!]) := \sum_{F \in \mathcal{F}_h} (R_F([\![z^u]\!]), [\![y^u]\!])_{L_u,F}$ and where for all $z^u \in U(h)$,

$$(5.3) \qquad |z^u|_J^2 = \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} |z^u|_{J,F}^2 \qquad \text{with} \qquad |z^u|_{J,F} = |[\![z^u]\!]|_{S,F}.$$

Theorem 5.8 relies only on assumptions (DG1)–(DG5), (DG7), (DG9)–(DG12), (DG14)–(DG15), and (DG17)–(DG18), collectively referred to as (DG$^\flat$). The additional assumptions (DG6), (DG8), (DG13), and (DG16) are needed to prove Theorem 5.14. Assumptions (DG1)–(DG18) are collectively referred to as (DG$^\sharp$).

Remark 5.1. Assumptions (DG1)–(DG6) imply that for all $(\tau, v) \in [L^2(F)]^m$,

$$|v|_{M,F} \lesssim h_F^{-\frac{1}{2}} \|v\|_{L_u,F}, \tag{5.4}$$

$$|(M_F^{\sigma u}(v), \tau)_{L_\sigma,F}| \lesssim \|v\|_{L_u,F} \|\tau\|_{L_\sigma,F}, \tag{5.5}$$

$$|(M_F^{u\sigma}(\tau) + \mathcal{D}^{u\sigma}\tau, v)_{L_u,F}| \lesssim h_F^{\frac{1}{2}} |v|_{M,F} \|\tau\|_{L_\sigma,F}. \tag{5.6}$$

For instance, taking $v = w$ in (DG6) and using the fact that $\mathcal{D}^{uu}$ is bounded yields $|v|_{M,F}^2 \lesssim \|v\|_{L_u,F}^2 + h_F^{-\frac{1}{2}} |v|_{M,F} \|v\|_{L_u,F}$, whence (5.4) readily follows. Properties (5.4)–(5.6) will be used in what follows.

Remark 5.2. Assumptions (DG7) and (DG8) are consistency hypotheses which trivially hold if $M_F(z) = \mathcal{M}z|_F$. However, it is not always possible to make this simple choice because it is sometimes necessary to penalize the boundary values of the u-component of the unknown. For instance, when Dirichlet-like boundary conditions are enforced, i.e., $\mathcal{M}^{\sigma u} = -\mathcal{D}^{\sigma u}$, it may happen that $\mathcal{M}^{uu} = 0$ (see the examples discussed in section 3). In this circumstance, assumptions (DG4)–(DG6) cannot be satisfied if we set $M_F^{uu}(v) = \mathcal{M}^{uu}v|_F = 0$, since $|v|_{M,F} = 0$ for all $v \in [L^2(F)]^{m_u}$. Instead, it is necessary that M_F^{uu} scale like h_F^{-1}. The consistency hypotheses (DG7) and (DG8) then mean that the extra control required by (DG4)–(DG6) is compatible with the way boundary conditions are enforced (see also Remark 6.2 and section 6.1.1, section 6.2, and section 6.3 for examples).

While assumptions (DG$^\sharp$) are just what it takes to prove Theorems 5.8 and 5.14, it is simpler in practice to work with a simplified set of assumptions. These are summarized in the following lemmas. Lemma 5.1 is tailored for the case when Dirichlet-like boundary conditions are enforced, while Lemma 5.2 is tailored for the case when Neumann or Robin boundary conditions are enforced. For brevity, only the proof of Lemma 5.1 is detailed, the other two proofs being similar.

LEMMA 5.1 (Dirichlet-like BCs). *Assume $M_F^{\sigma\sigma} = 0$, $M_F^{\sigma u}(v) = -\mathcal{D}^{\sigma u}v$ for all $v \in [L^2(F)]^{m_u}$, $M_F^{u\sigma} = -(M_F^{\sigma u})^*$, M_F^{uu} is self-adjoint, and*

$$h_F|\mathcal{D}^{uu}| + h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} \lesssim M_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}, \tag{5.7}$$

where $\mathcal{I}_{m_u}$ is the identity matrix in $\mathbb{R}^{m_u,m_u}$. Then, (DG1)–(DG6) *hold.*

Proof. Assumptions (DG1)–(DG3) are evident. To prove (DG4), observe that for every positive semidefinite matrix $\mathcal{Z} \in \mathbb{R}^{m_u,m_u}$ and for all $x \in \mathbb{R}^{m_u}$, $(\mathcal{Z}x, x) \leq \|\mathcal{Z}^{1/2}\|(\mathcal{Z}^{1/2}x, x)$. Let $v \in [L^2(F)]^{m_u}$; upon observing that $\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u}$ is positive semidefinite, we apply the above result to derive

$$\begin{aligned}\|\mathcal{D}^{\sigma u}v\|_{L_\sigma,F} &= (\mathcal{D}^{\sigma u}v, \mathcal{D}^{\sigma u}v)_{L_\sigma,F}^{\frac{1}{2}} = (\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u}v, v)_{L_u,F}^{\frac{1}{2}} \\ &\lesssim ((\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}}v, v)_{L_u,F}^{\frac{1}{2}} \lesssim h_F^{\frac{1}{2}}|v|_{M,F},\end{aligned}$$

whence (DG4) is readily inferred. To prove (DG5)–(DG6), let $v, w \in [L^2(F)]^{m_u}$. Then, $|(M_F^{uu}(v), w)_{L_u,F}| \lesssim |v|_{M,F}|w|_{M,F}$ and since $(\mathcal{D}^{uu})^2$ is positive semidefinite,

$$\|\mathcal{D}^{uu}v\|_{L_u,F} \lesssim (|\mathcal{D}^{uu}|v, v)_{L_u,F}^{\frac{1}{2}} \lesssim h_F^{-\frac{1}{2}}|v|_{M,F},$$

whence (DG5)–(DG6) are readily deduced. ◻

LEMMA 5.2 (Neumann–Robin BCs). *Assume $M_F^{\sigma\sigma} = 0$, $M_F^{\sigma u}(v) = \mathcal{D}^{\sigma u}v$ for all $v \in [L^2(F)]^{m_u}$, $M_F^{u\sigma} = -(M_F^{\sigma u})^*$, M_F^{uu} is self-adjoint, and*

$$h_F|\mathcal{D}^{uu}| \lesssim M_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}. \tag{5.8}$$

Then, (DG1)–(DG6) *hold.*

LEMMA 5.3 (interface operator). *Assume $S_F^{\sigma\sigma}=0$, $S_F^{u\sigma}=0$, $S_F^{\sigma u}=0$, S_F^{uu} is self-adjoint, and*

$$h_F|\mathcal{D}^{uu}|+h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}}\lesssim S_F^{uu}\lesssim h_F^{-1}\mathcal{I}_{m_u}. \tag{5.9}$$

Then, (DG9)–(DG16) *hold.*

Remark 5.3. Conditions (5.7) and (5.9) generally imply that S_F^{uu} and M_F^{uu} are of order h_F^{-1}; this differs from the condition derived in [9], where S_F and M_F are of order 1. Roughly speaking, to be able to eliminate the discrete σ-component, it is necessary to have a stronger control of the interface jumps and of the boundary values of the discrete u-component.

5.2. The direct argument. To perform the error analysis we introduce the following two discrete norms on $W(h)$:

$$\|z\|_{h,A}^2=\|z^\sigma\|_{L_\sigma}^2+\|z^u\|_{L_u}^2+|z^u|_J^2+|z^u|_M^2+\sum_{K\in\mathcal{T}_h}\|Bz^u\|_{L_\sigma,K}^2, \tag{5.10}$$

$$\|z\|_{h,1}^2=\|z\|_{h,A}^2+\sum_{K\in\mathcal{T}_h}[h_K^{-2}\|z^u\|_{L_u,K}^2+h_K^{-1}\|z^u\|_{L_u,\partial K}^2+h_K\|z^\sigma\|_{L_\sigma,\partial K}^2]. \tag{5.11}$$

The norm $\|\cdot\|_{h,A}$ is used to measure the approximation error, and the norm $\|\cdot\|_{h,1}$ serves to measure the interpolation properties of the discrete space W_h. In this section, it is implicitly assumed that (DG$^\flat$) holds.

LEMMA 5.4 (L-coercivity). *For all h and for all $z_h=(z_h^\sigma,z_h^u)$ in W_h,*

$$\|z_h^\sigma\|_{L_\sigma}^2+\|z_h^u\|_{L_u}^2+|z_h^u|_J^2+|z_h^u|_M^2\lesssim a_h(z_h,z_h). \tag{5.12}$$

Proof. Proceeding as in the proof of Lemma 4.1 in [9] and using the skew-symmetry assumptions (DG2) and (DG10) yields for all $z_h\in W_h$,

$$\|z_h^\sigma\|_{L_\sigma}^2+\|z_h^u\|_{L_u}^2+|z_h^u|_J^2+\tfrac{1}{2}|z_h^u|_M^2+\rho_h([\![z_h^u]\!],[\![z_h^u]\!])\lesssim a_h(z_h,z_h).$$

Then, the desired result follows from (DG17). ∎

LEMMA 5.5 (stability). *The following holds:*

$$\forall z_h\in W_h,\quad \|z_h\|_{h,A}\lesssim\sup_{y_h\in W_h\backslash\{0\}}\frac{a_h(z_h,y_h)}{\|y_h\|_{h,A}}. \tag{5.13}$$

Proof. Let $z_h=(z_h^\sigma,z_h^u)\in W_h\backslash\{0\}$ and set $\mathbb{S}=\sup_{y_h\in W_h\backslash\{0\}}\frac{a_h(z_h,y_h)}{\|y_h\|_{h,A}}$.

(1) Owing to Lemma 5.4, it is inferred that

$$\|z^\sigma\|_{L_\sigma}^2+\|z^u\|_{L_u}^2+|z^u|_J^2+|z^u|_M^2\lesssim a_h(z_h,z_h)\le\mathbb{S}\,\|z_h\|_{h,A}.$$

(2) Control of Bz_h^u. Let $K\in\mathcal{T}_h$. Denote by $\overline{\mathcal{B}_K^k}$ the mean-value of $\mathcal{B}^k$ over K; then,

$$\|\mathcal{B}^k-\overline{\mathcal{B}_K^k}\|_{[L^\infty(K)]^{m_\sigma,m_u}}\le h_K\|\mathcal{B}^k\|_{[\mathcal{C}^{0,1}(K)]^{m_\sigma,m_u}}. \tag{5.14}$$

Define the field π_h such that $\pi_h|_K=\sum_{k=1}^d\overline{\mathcal{B}_K^k}\partial_k z_h^u$. Set $\varpi_h=(\pi_h,0)$. It is clear that $\pi_h\in\Sigma_h$ since $p_u-1\le p_\sigma$; hence, $\varpi_h\in W_h$. Using (5.14), together with the inverse inequalities (4.3) and (4.4), leads, for all $F\subset\partial K$, to

$$\begin{cases}\|\pi_h\|_{L_\sigma,F}\lesssim h_F^{-\frac{1}{2}}\|\pi_h\|_{L_\sigma,\mathcal{T}(F)}, & \text{if } F\in\mathcal{F}_h^\partial,\\ \|\{\pi_h\}\|_{L_\sigma,F}+\|[\![\pi_h]\!]\|_{L_\sigma,F}\lesssim h_F^{-\frac{1}{2}}\|\pi_h\|_{L_\sigma,\mathcal{T}(F)} & \text{if } F\in\mathcal{F}_h^{\mathrm{i}},\end{cases} \tag{5.15}$$

$$\|\pi_h\|_{L_\sigma,K}\lesssim\|Bz_h^u\|_{L_\sigma,K}+\|z_h^u\|_{L_u,K}, \tag{5.16}$$

whence it is readily inferred that

$$\|\varpi_h\|_{h,A} = \|\pi_h\|_{L_\sigma} \lesssim \|z_h\|_{h,A}.$$

Furthermore, from the definition of a_h it follows that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2 &= a_h(z_h,\varpi_h) + \sum_{K\in\mathcal{T}_h}(Bz_h^u, Bz_h^u - \pi_h)_{L_\sigma,K} \\ &\quad - (\mathcal{K}^{\sigma\sigma} z_h^\sigma + \mathcal{K}^{\sigma u} z_h^u, \pi_h)_{L_\sigma} - \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F^{\sigma u}(z_h^u) - \mathcal{D}^{\sigma u} z_h^u, \pi_h)_{L_\sigma,F} \\ &\quad + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{\sigma u} z_h^u\}, \{\pi_h\})_{L_\sigma,F} - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F^{\sigma u}([\![z_h^u]\!]), [\![\pi_h]\!])_{L_\sigma,F} \\ &= a_h(z_h,\varpi_h) + R_1 + R_2 + R_3 + R_4 + R_5,\end{aligned}$$

where R_1 to R_5 denote the second to sixth terms in the right-hand side. Proceeding as in the proof of Lemma 4.3 in [9] and using (DG4), (DG14), (DG15), the terms R_1–R_5 are bounded from above as follows:

$$\sum_{i=1}^{5} |R_i| \lesssim (\|z_h^\sigma\|_{L_\sigma}^2 + \|z_h^u\|_{L_u}^2 + |z_h^u|_M^2 + |z_h^u|_J^2) + \gamma \sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2,$$

where $\gamma > 0$ can be chosen as small as needed. Hence,

$$\sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2 \lesssim a_h(z_h,\varpi_h) + a_h(z_h,z_h) \lesssim \mathbb{S}\,\|z_h\|_{h,A}.$$

(3) Collecting the above bounds yields $\|z_h\|_{h,A}^2 \lesssim \mathbb{S}\,\|z_h\|_{h,A}$, thereby completing the proof. ◻

LEMMA 5.6 (continuity). *The following holds:*

$$\forall (z,y_h) \in W(h)\times W_h, \quad a_h(z,y_h) \lesssim \|z\|_{h,1}\|y_h\|_{h,A}. \tag{5.17}$$

Proof. The main idea is to integrate by parts $a_h(z,y_h)$ by using the formal adjoint $\tilde{A}$. Proceeding as in the proof of Lemma 4.4 in [9] leads to

$$\begin{aligned}a_h(z,y_h) = &\sum_{K\in\mathcal{T}_h} [(Kz,z)_{L,K} + (z,\tilde{A}y_h)_{L,K}] + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(z) + \mathcal{D}z, y_h)_{L,F} \\ &+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \tfrac{1}{2}([\![\mathcal{D}z]\!], [\![y_h]\!])_{L,F} + \rho_h([\![z^u]\!], [\![y_h^u]\!]) + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F([\![z]\!]), [\![y_h]\!])_{L,F}.\end{aligned} \tag{5.18}$$

Let R_1 to R_5 be the five terms in the right-hand side.

(1) Using the Cauchy–Schwarz inequality and inverse inequalities, we obtain

$$|R_1| \lesssim \sum_{K\in\mathcal{T}_h} \|z\|_{L,K}\|y_h\|_{L,K} + \|z^\sigma\|_{L_\sigma,K}\|By_h^u\|_{L_\sigma,K} + h_K^{-1}\|z^u\|_{L_u,K}\|y_h\|_{L,K}.$$

Hence, $|R_1| \lesssim \|z\|_{h,1}\|y_h\|_{h,A}$.

(2) For the second term, we have

$$\begin{aligned}|R_2| \le \tfrac{1}{2} \sum_{F\in\mathcal{F}_h^\partial} |&(M_F^{\sigma u}(z^u) + \mathcal{D}^{\sigma u} z^u, y_h^\sigma)_{L_\sigma,F} + (M_F^{uu}(z^u) + \mathcal{D}^{uu} z^u, y_h^u)_{L_u,F} \\ &+ (M_F^{u\sigma}(z^\sigma) + \mathcal{D}^{u\sigma} z^\sigma, y_h^u)_{L_u,F}|.\end{aligned}$$

Using (5.5), (DG5), the boundedness of $\mathcal{D}$, (5.6), and the inverse inequality (4.4), each term in the above equality is bounded as follows:

$$
\begin{aligned}
&|(M_F^{\sigma u}(z^u) + \mathcal{D}^{\sigma u} z^u, y_h^\sigma)_{L_\sigma,F}| \lesssim \|z^u\|_{L_u,F} \|y_h^\sigma\|_{L_\sigma,F} \lesssim h_F^{-\frac{1}{2}} \|z^u\|_{L_u,F} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)},\\
&|(M_F^{uu}(z^u) + \mathcal{D}^{uu} z^u, y_h^u)_{L_u,F}| \lesssim h_F^{-\frac{1}{2}} \|z^u\|_{L_u,F} |y_h^u|_{M,F},\\
&|(M_F^{u\sigma}(z^\sigma) + \mathcal{D}^{u\sigma} z^\sigma, y_h^u)_{L_u,F}| \lesssim h_F^{\frac{1}{2}} \|z^\sigma\|_{L_\sigma,F} |y_h^u|_{M,F}.
\end{aligned}
$$

As a result, $|R_2| \lesssim \|z\|_{h,1} \|y_h\|_{h,A}$.

(3) For the third term, we have

$$
|R_3| \le \tfrac{1}{2} \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} |([\![\mathcal{D}^{\sigma u} z^u]\!], [\![y_h^\sigma]\!])_{L_\sigma,F} + ([\![\mathcal{D}^{uu} z^u]\!], [\![y_h^u]\!])_{L_u,F} + ([\![\mathcal{D}^{u\sigma} z^\sigma]\!], [\![y_h^u]\!])_{L_u,F}|.
$$

Using the boundedness of $\mathcal{D}$, the inverse inequality (4.4), and (DG15), each term in the above equality is bounded as follows:

$$
\begin{aligned}
&|([\![\mathcal{D}^{\sigma u} z^u]\!], [\![y_h^\sigma]\!])_{L_\sigma,F}| \lesssim \|\{z^u\}\|_{L_u,F} \|[\![y_h^\sigma]\!]\|_{L_\sigma,F} \lesssim h_F^{-\frac{1}{2}} \|\{z^u\}\|_{L_u,F} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)},\\
&|([\![\mathcal{D}^{uu} z^u]\!], [\![y_h^u]\!])_{L_u,F}| \lesssim \|\{z^u\}\|_{L_u,F} \|[\![y_h^u]\!]\|_{L_u,F} \lesssim h_F^{-\frac{1}{2}} \|\{z^u\}\|_{L_u,F} \|y_h^u\|_{L_u,\mathcal{T}(F)},\\
&|([\![\mathcal{D}^{u\sigma} z^\sigma]\!], [\![y_h^u]\!])_{L_u,F}| = |(\{z^\sigma\}, \mathcal{D}_F^{\sigma u} [\![y_h^u]\!])_{L_\sigma,F}| \lesssim h_F^{\frac{1}{2}} \|\{z^\sigma\}\|_{L_\sigma,F} |y_h^u|_{J,F}.
\end{aligned}
$$

As a result, $|R_3| \lesssim \|z\|_{h,1} \|y_h\|_{h,A}$.

(4) The fourth term is controlled using (DG18).

(5) For the fifth term, we have

$$
|R_5| \le \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} |(S_F^{\sigma u}([\![z^u]\!]), [\![y_h^\sigma]\!])_{L_\sigma,F} + (S_F^{uu}([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F} + (S_F^{u\sigma}([\![z^\sigma]\!]), [\![y_h^u]\!])_{L_u,F}|.
$$

Using (DG12) and (DG14), together with the inverse inequality (4.4), each term in the above equality is bounded as follows:

$$
\begin{aligned}
&|(S_F^{\sigma u}([\![z^u]\!]), [\![y_h^\sigma]\!])_{L_\sigma,F}| \lesssim h_F^{\frac{1}{2}} |z^u|_{J,F} \|[\![y_h^\sigma]\!]\|_{L_\sigma,F} \lesssim |z^u|_{J,F} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)},\\
&|(S_F^{uu}([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F}| \lesssim h_F^{-\frac{1}{2}} \|[\![z^u]\!]\|_{L_u,F} |y_h^u|_{J,F},\\
&|(S_F^{u\sigma}([\![z^\sigma]\!]), [\![y_h^u]\!])_{L_u,F}| \lesssim h_F^{\frac{1}{2}} \|[\![z^\sigma]\!]\|_{L_\sigma,F} |y_h^u|_{J,F}.
\end{aligned}
$$

As a result, $|R_5| \lesssim \|z\|_{h,1} \|y_h\|_{h,A}$. The proof is complete. □

LEMMA 5.7 (consistency). *Let* $z \in V \cap [H^1(\Omega)]^m$ *solve* (2.6) *and let* z_h *solve* (4.16). *Then,*

$$
\forall y_h \in W_h, \quad a_h(z - z_h, y_h) = 0. \tag{5.19}
$$

Proof. Let $y_h \in W_h$ and use (4.15) to evaluate $a_h(z, y_h)$. Since z solves (2.6), the first term in the right-hand side of (4.15) is equal to $(f, y_h)_L$. Owing to the consistency assumption (DG7), the second term in the right-hand side of (4.15) vanishes. Furthermore, since for all $F \in \mathcal{F}_h^{\mathrm{i}}$, $\{\mathcal{D}z\} = \mathcal{D}_F[\![z]\!] = 0$ and $[\![z]\!] = 0$ because $z \in [H^1(\Omega)]^m$, the third, fourth, and fifth terms in (4.15) are also zero. As a result, $a_h(z, y_h) = (f, y_h)_L = a_h(z_h, y_h)$, completing the proof. □

THEOREM 5.8 (convergence). *Let $z \in V \cap [H^1(\Omega)]^m$ solve* (2.6) *and let z_h solve* (4.16). *Then,*

$$\|z - z_h\|_{h,A} \lesssim \inf_{y_h \in W_h} \|z - y_h\|_{h,1}. \tag{5.20}$$

Proof. The proof follows from the second Strang lemma. ∎

Owing to the regularity of the mesh family $\{\mathcal{T}_h\}_{h>0}$, the following interpolation property holds: For all $z \in [H^{p_\sigma+1}(\Omega)]^{m_\sigma} \times [H^{p_u+1}(\Omega)]^{m_u}$, there is $y_h \in W_h$ satisfying

$$\|z - y_h\|_{h,1} \lesssim (h^{p_\sigma+1} + h^{p_u})\big(\|z^\sigma\|_{[H^{p_\sigma+1}(\Omega)]^{m_\sigma}} + \|z^u\|_{[H^{p_u+1}(\Omega)]^{m_u}}\big). \tag{5.21}$$

Since $p_u - 1 \le p_\sigma$, the above interpolation error is of order h^{p_u}.

COROLLARY 5.9. *Let $z \in [H^{p_\sigma+1}(\Omega)]^{m_\sigma} \times [H^{p_u+1}(\Omega)]^{m_u}$ solve* (2.6) *and let z_h solve* (4.16). *Then,*

$$\|z - z_h\|_{h,A} \lesssim h^{p_u}\big(\|z^\sigma\|_{[H^{p_\sigma+1}(\Omega)]^{m_\sigma}} + \|z^u\|_{[H^{p_u+1}(\Omega)]^{m_u}}\big). \tag{5.22}$$

Remark 5.4. For both the σ- and the u-component of the solution, the error estimate in the L^2-norm is $\mathcal{O}(h^{p_u})$. If $p_\sigma = p_u := p$, this result is suboptimal when compared with that obtained using the DG method analyzed in [9], which yields $\mathcal{O}(h^{p+\frac{1}{2}})$ error estimates. The reason for this slight optimality loss is that in the present method the interface jumps of the σ-component are not controlled to allow for this component to be locally eliminated, the consequence being that the jumps on the u-component must be penalized with an $\mathcal{O}(h^{-1})$ weight. If $p_\sigma = p_u - 1$, (5.22) is still suboptimal for the u-component but is optimal in the L^2-norm for the σ-component.

Finally, when the exact solution z is only in the graph space W, i.e., when z is not in $[H^1(\Omega)]^m$ so that $a_h(z, \cdot)$ may not be meaningful, we use a density argument to infer the convergence of the DG approximation. For $z \in W + W_h$, define the norm

$$\|z\|_{W^-} = \|z\|_L + \left(\sum_{K \in \mathcal{T}_h} \|Bz^u\|^2_{L_\sigma,K}\right)^{\frac{1}{2}}. \tag{5.23}$$

Observe that $\|z\|_{W^-} \le \|z\|_{h,A}$.

COROLLARY 5.10. *Assume that there is $\gamma > 0$ such that $[H^{\gamma+1}(\Omega)]^m \cap V$ is dense in V. Let z solve* (2.6) *and let z_h solve* (4.16). *Then,*

$$\lim_{h \to 0} \|z - z_h\|_{W^-} = 0. \tag{5.24}$$

Proof. Let $\epsilon > 0$. There is $z_\epsilon \in [H^{\gamma+1}(\Omega)]^m \cap V$ such that $\|z - z_\epsilon\|_W \le \frac{\epsilon}{2}$. Let $z_{\epsilon h}$ be the unique solution in W_h such that $a_h(z_{\epsilon h}, y_h) = (Tz_\epsilon, y_h)_L$ for all $y_h \in W_h$. From the regularity of z_ϵ together with Theorem 5.8 and Corollary 5.9, it is inferred that $\lim_{h\to 0} \|z_{\epsilon h} - z_\epsilon\|_{h,A} = 0$. Furthermore, using the discrete inf-sup condition (5.13) yields

$$\begin{aligned}\|z_{\epsilon h} - z_h\|_{W^-} &\lesssim \sup_{y_h \in W_h \setminus \{0\}} \frac{a_h(z_{\epsilon h}, y_h) - a_h(z_h, y_h)}{\|y_h\|_{h,A}} = \sup_{y_h \in W_h \setminus \{0\}} \frac{(T(z_\epsilon - z), y_h)_L}{\|y_h\|_{h,A}} \\ &\le \|T(z_\epsilon - z)\|_L \sup_{y_h \in W_h \setminus \{0\}} \frac{\|y_h\|_L}{\|y_h\|_{h,A}} \le \|z - z_\epsilon\|_W \le \frac{\epsilon}{2},\end{aligned}$$

where we have used the fact that for all $y_h \in W_h$, $a_h(z_h, y_h) = (Tz, y_h)_L$. Finally, using the triangle inequality $\|z - z_h\|_{W^-} \le \|z - z_\epsilon\|_{W^-} + \|z_\epsilon - z_{\epsilon h}\|_{W^-} + \|z_{\epsilon h} - z_h\|_{W^-}$, we deduce that $\limsup_{h\to 0} \|z - z_h\|_{W^-} \le \epsilon$. ∎

5.3. The duality argument. We now improve the error estimate on the L^2-norm of the u-component of the solution by using a duality argument. In this section, it is implicitly assumed that (DG$^\sharp$) holds.

Let z solve (2.6) and let z_h solve (4.16). Let $\psi := (\psi^\sigma, \psi^u) \in V^*$ solve

$$\tilde{T}\psi = (0, z^u - z_h^u). \tag{5.25}$$

We assume that the above problem yields (elliptic) regularity; i.e., ψ^u is in $[H^2(\Omega)]^{m_u}$, ψ^σ is in $[H^1(\Omega)]^{m_\sigma}$, and the following uniform bound holds:

$$\|\psi^u\|_{[H^2(\Omega)]^{m_u}} + \|\psi^\sigma\|_{[H^1(\Omega)]^{m_\sigma}} \lesssim \|z^u - z_h^u\|_{L_u}. \tag{5.26}$$

LEMMA 5.11. *Under the above hypotheses, the following holds:*

$$a_h(y, \psi) = (y^u, z^u - z_h^u)_{L_u} \quad \forall y \in W(h). \tag{5.27}$$

Proof. Let $y \in W(h)$. By integrating by parts (i.e., using (5.18)) and using the fact that ψ is continuous across interfaces, we obtain

$$a_h(y, \psi) = \sum_{K \in \mathcal{T}_h} (y, \tilde{T}\psi)_{L,K} + \sum_{F \in \mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(y) + \mathcal{D}y, \psi)_{L,F}.$$

Since $\psi \in V^* \cap [H^1(\Omega)]^m$, (DG8) implies $(M_F(y) + \mathcal{D}y, \psi)_{L,F} = 0$ for all $F \in \mathcal{F}_h^\partial$. The conclusion is straightforward since ψ solves (5.25). □

To avoid lengthy technicalities, we introduce the following norms:

$$\|y^\sigma\|_{h,\tilde{1}} = \left(\sum_{K \in \mathcal{T}_h} [h_K^2 \|y^\sigma\|^2_{[H^1(K)]^{m_\sigma}} + h_K \|y^\sigma\|^2_{L_\sigma, \partial K}] \right)^{\frac{1}{2}}, \tag{5.28}$$

$$\|y\|_{h,A^+} = \|y\|_{h,A} + \|y^\sigma\|_{h,\tilde{1}}, \tag{5.29}$$

$$\|y\|_{h,1^+} = \|y\|_{h,1} + \|y^\sigma\|_{h,\tilde{1}}. \tag{5.30}$$

The DG method converges optimally in the $\|\cdot\|_{h,A^+}$-norm as stated in the following.

COROLLARY 5.12. *Let $z \in V \cap [H^1(\Omega)]^m$ solve* (2.6) *and let z_h solve* (4.16). *Then,*

$$\|z - z_h\|_{h,A^+} \lesssim \inf_{y_h \in W_h} \|z - y_h\|_{h,1^+}. \tag{5.31}$$

Proof. Let y_h be an arbitrary element in W_h. Using inverse inequalities yields

$$\begin{aligned}
\|z^\sigma - z_h^\sigma\|_{h,\tilde{1}} &\le \|z^\sigma - y_h^\sigma\|_{h,\tilde{1}} + \|y_h^\sigma - z_h^\sigma\|_{h,\tilde{1}} \lesssim \|z^\sigma - y_h^\sigma\|_{h,\tilde{1}} + \|y_h^\sigma - z_h^\sigma\|_{L_\sigma} \\
&\le \|z^\sigma - y_h^\sigma\|_{h,\tilde{1}} + \|y_h^\sigma - z^\sigma\|_{L_\sigma} + \|z^\sigma - z_h^\sigma\|_{L_\sigma} \\
&\le \|z^\sigma - y_h^\sigma\|_{h,\tilde{1}} + \|z - y_h\|_{h,A} + \|z - z_h\|_{h,A} \\
&\lesssim \|z - y_h\|_{h,A^+} + \|z - z_h\|_{h,A}.
\end{aligned}$$

Hence, using the above inequality along with (5.20) leads to

$$\|z - z_h\|_{h,A^+} \lesssim \|z - y_h\|_{h,A^+} + \|z - y_h\|_{h,1} \lesssim \|z - y_h\|_{h,1^+}.$$

That concludes the proof since y_h is arbitrary in W_h. □

LEMMA 5.13 (continuity). *Assume that for all $K \in \mathcal{T}_h$ and for all $y \in W(h)$,*

$$\|Cy^u\|_{L_u,K} \lesssim \|By^u\|_{L_\sigma,K} + \|y^u\|_{L_u,K}. \tag{5.32}$$

Then, the following holds:

$$\forall (r,y)\in W(h)\times W(h),\quad a_h(r,y)\lesssim \|r\|_{h,A^+}\|y\|_{h,1}. \tag{5.33}$$

Proof. Let us bound all the terms in the right-hand side of (4.15).

(1) For the first term, say R_1, we proceed as follows:

$$\begin{aligned}|(Tr,y)_{L,K}| &\le |(Kr,y)_{L,K}|+|(Br^u,y^\sigma)_{L_\sigma,K}|+|(B^\dagger r^\sigma+Cr^u,y^u)_{L_u,K}|\\ &\lesssim \|r\|_{L,K}\|y\|_{L,K}+\|Br^u\|_{L_\sigma,K}\|y\|_{L,K}+\|r^\sigma\|_{[H^1(K)]^{m_\sigma}}\|y^u\|_{L_u,K}\\ &\lesssim (\|r\|^2_{L,K}+\|Br^u\|^2_{L_\sigma,K}+h_K^2\|r^\sigma\|^2_{[H^1(K)]^{m_\sigma}})^{\frac12}(\|y\|^2_{L,K}+h_K^{-2}\|y^u\|^2_{L_u,K})^{\frac12},\end{aligned}$$

where (5.32) has been used to bound $\|Cr^u\|$. Hence, $|R_1|\lesssim \|r\|_{h,A^+}\|y\|_{h,1}$.

(2) To bound the second term, say R_2, use (DG4), (DG6), (5.5), and the boundedness of $\mathcal{D}$ to infer

$$\begin{aligned}&|(M_F^{\sigma u}(r^u)-\mathcal{D}^{\sigma u}r^u,y^\sigma)_{L_\sigma,F}|\lesssim |r^u|_{M,F}\,h_F^{\frac12}\|y^\sigma\|_{L_\sigma,F},\\ &|(M_F^{uu}(r^u)-\mathcal{D}^{uu}r^u,y^u)_{L_u,F}|\lesssim |r^u|_{M,F}\,h_F^{-\frac12}\|y^u\|_{L_u,F},\\ &|(M_F^{u\sigma}(r^\sigma)-\mathcal{D}^{u\sigma}r^\sigma,y^u)_{L_u,F}|\lesssim \|r^\sigma\|_{L_\sigma,F}\|y^u\|_{L_u,F}\lesssim h_F^{\frac12}\|r^\sigma\|_{L_\sigma,F}\,h_F^{-\frac12}\|y^u\|_{L_u,F}.\end{aligned}$$

As a result, $|R_2|\lesssim \|r\|_{h,A^+}\|y\|_{h,1}$.

(3) To bound the third term, say R_3, use (DG15), (DG16), and the boundedness of $\mathcal{D}$ to infer

$$\begin{aligned}&|(\{\mathcal{D}^{\sigma u}r^u\},\{y^\sigma\})_{L_\sigma,F}|=|2(\mathcal{D}^{\sigma u}_{\partial K_1(F)}[\![r^u]\!],\{y^\sigma\})_{L_\sigma,F}|\lesssim |r^u|_{J,F}\,h_F^{\frac12}\|\{y^\sigma\}\|_{L_\sigma,F},\\ &|(\{\mathcal{D}^{uu}r^u\},\{y^u\})_{L_u,F}|=|2(\mathcal{D}^{uu}_{\partial K_1(F)}[\![r^u]\!],\{y^u\})_{L_u,F}|\lesssim |r^u|_{J,F}\,h_F^{-\frac12}\|\{y^u\}\|_{L_u,F},\\ &|(\{\mathcal{D}^{u\sigma}r^\sigma\},\{y^u\})_{L_u,F}|\lesssim \|[\![r^\sigma]\!]\|_{L_\sigma,F}\|\{y^u\}\|_{L_u,F}\lesssim h_F^{\frac12}\|[\![r^\sigma]\!]\|_{L_\sigma,F}\,h_F^{-\frac12}\|\{y^u\}\|_{L_u,F}.\end{aligned}$$

These bounds yield $|R_3|\lesssim \|r\|_{h,A^+}\|y\|_{h,1}$.

(4) To bound the fourth term, use (DG18).

(5) To bound the fifth term, say R_5, use (DG10), (DG13), and (DG14) to infer

$$\begin{aligned}&|(S_F^{\sigma u}([\![r^u]\!]),[\![y^\sigma]\!])_{L_\sigma,F}|\lesssim |r^u|_{J,F}\,h_F^{\frac12}\|[\![y^\sigma]\!]\|_{L_\sigma,F},\\ &|(S_F^{uu}([\![r^u]\!]),[\![y^u]\!])_{L_u,F}|\lesssim |r^u|_{J,F}\,h_F^{-\frac12}\|[\![y^u]\!]\|_{L_u,F},\\ &|(S_F^{u\sigma}([\![r^\sigma]\!]),[\![y^u]\!])_{L_u,F}|\lesssim h_F^{\frac12}\|[\![r^\sigma]\!]\|_{L_\sigma,F}\,|y^u|_{J,F}.\end{aligned}$$

Hence, $|R_5|\lesssim \|r\|_{h,A^+}\|y\|_{h,1}$. The proof is complete. □

THEOREM 5.14 (convergence). *Let $z\in V\cap[H^1(\Omega)]^m$ solve* (2.6) *and let z_h solve* (4.16)*. Assume elliptic regularity, i.e.,* (5.26)*, and that* (5.32) *holds. Then,*

$$\|z^u-z_h^u\|_{L_u}\lesssim h\inf_{y_h\in W_h}\|z-y_h\|_{h,1^+}. \tag{5.34}$$

Proof. Using $z-z_h$ as test function in (5.27) we infer $a_h(z-z_h,\psi)=\|z^u-z_h^u\|^2_{L_u}$. Then, using the consistency property stated in Lemma 5.7, this yields for all $\psi_h\in W_h$, $a_h(z-z_h,\psi-\psi_h)=\|z^u-z_h^u\|^2_{L_u}$. Lemma 5.13 in turn implies

$$\|z^u-z_h^u\|^2_{L_u}\lesssim \|z-z_h\|_{h,A^+}\|\psi-\psi_h\|_{h,1}\quad \forall\psi_h\in W_h.$$

Then, using the elliptic regularity (5.26) and the fact that $p_u \geq 1$ leads to

$$\begin{aligned}\|z^u - z_h^u\|_{L_u}^2 &\lesssim \|z - z_h\|_{h,A^+} \inf_{\psi_h \in W_h} \|\psi - \psi_h\|_{h,1} \\ &\lesssim h\|z - z_h\|_{h,A^+}(\|\psi^u\|_{[H^2(\Omega)]^{m_u}} + \|\psi^\sigma\|_{[H^1(\Omega)]^{m_\sigma}}) \\ &\lesssim h\|z - z_h\|_{h,A^+}\|z^u - z_h^u\|_{L_u}.\end{aligned}$$

The conclusion follows readily using Corollary 5.12. ∎

Remark 5.5. Stability and convergence in the $\|\cdot\|_{h,A^+}$-norm could have been proved directly by adding the quantity $(\sum_{K\in\mathcal{T}_h} h_K^2 \|B^\dagger y^\sigma + Cy^u\|_{L_u,K}^2)^{\frac{1}{2}}$ in the definition of the $\|\cdot\|_{h,A}$-norm, but this significantly lengthens the proof of Lemma 5.5. With this modification of the $\|\cdot\|_{h,A}$-norm, hypothesis (5.32) can be removed. However, this appears to be a minor issue since (5.32) holds for all the two-field Friedrichs' systems presented in section 3.

6. Applications. In this section we apply the DG method designed in section 4 and analyzed in section 5 to the Friedrichs' systems presented in section 3.

6.1. Advection-diffusion-reaction. We describe various DG methods that can be used to approximate the advection-diffusion-reaction equation introduced in section 3.1 and in which the σ-component of the unknown can be eliminated locally. Comparisons with the unified approached developed by Arnold et al. [1] are presented to illustrate the fact that the present DG method generalizes some of the DG methods that have been previously developed in the literature for the Poisson equation.

6.1.1. A first example: The LDG method. Consider first Dirichlet boundary conditions. Owing to (3.5) and (3.6), the integral representations (2.15) and (2.17) hold with the $\mathbb{R}^{d+1,d+1}$-valued boundary fields

$$\mathcal{D} = \left[\begin{array}{c|c} 0 & n \\ \hline n^t & \beta\cdot n \end{array}\right] \quad \text{and} \quad \mathcal{M} = \left[\begin{array}{c|c} 0 & -n \\ \hline n^t & 0 \end{array}\right], \tag{6.1}$$

where n is the unit outward normal to $\partial\Omega$. Let $\varsigma > 0$ and $\eta > 0$ (these design parameters can vary from face to face). For all $F \in \mathcal{F}_h$, set $R_F \equiv 0$ and

$$\mathcal{M}_F = \left[\begin{array}{c|c} 0 & -n_F \\ \hline n_F^t & \varsigma h_F^{-1} \end{array}\right], \qquad \mathcal{S}_F = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \eta h_F^{-1} \end{array}\right], \tag{6.2}$$

and define for all $y \in [L^2(F)]^{d+1}$, $M_F(y) = \mathcal{M}_F y$ and $S_F(y) = \mathcal{S}_F y$.

LEMMA 6.1. *Let M_F, S_F, and R_F be defined as above. Then, properties* (DG$^\sharp$) *hold.*

Proof. The consistency properties (DG7) and (DG8) are readily verified. Properties (DG17)–(DG18) are evident. The remaining properties are direct consequences of Lemmata 5.1 and 5.3. ∎

Remark 6.1. Let $\delta \in \mathbb{R}^d$. A slightly more general choice for the interface operator consists of setting for all $F \in \mathcal{F}_h^{\rm i}$, $\mathcal{S}_F^{\sigma u} = (\delta\cdot n_F)n_F$, where n_F is any of the two unit normal vectors to F. This choice leads to the so-called LDG method of Cockburn and Shu [7] as considered in the unified approach of [1] for the Poisson equation.

When Neumann and Robin boundary conditions are enforced, the integral representation (2.17) holds for the $\mathbb{R}^{d+1,d+1}$-valued boundary field

$$\mathcal{M} = \left[\begin{array}{c|c} 0 & n \\ \hline -n^t & 2\varrho + \beta\cdot n \end{array}\right]. \tag{6.3}$$

Assume that $\varrho \geq (\beta{\cdot}n)^-$, the negative part of $\beta{\cdot}n$ (this is not restrictive in practice since the usual Robin condition at an inflow boundary uses $\varrho = -\beta{\cdot}n \geq 0$). For all $F \in \mathcal{F}_h$, set $R_F \equiv 0$ and

$$\mathcal{M}_F = \left[\begin{array}{c|c} 0 & n_F \\ \hline -n_F^t & 2\varrho + \beta{\cdot}n_F \end{array}\right], \qquad \mathcal{S}_F = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \eta h_F^{-1} \end{array}\right], \tag{6.4}$$

and for all $y \in [L^2(F)]^{d+1}$, define $M_F(y) = \mathcal{M}_F y$ and $S_F(y) = \mathcal{S}_F y$. Then, it is easily verified that (5.8) holds. Hence, Lemma 5.2 implies that assumptions (DG1)–(DG6) hold. Moreover, the consistency assumptions (DG7) and (DG8) trivially hold. Of course, (DG9)–(DG16) hold since the definition of $\mathcal{S}_F$ is independent of the type of boundary condition. Finally, (DG17)–(DG18) are evident since $R_F \equiv 0$.

Remark 6.2. Observe that the scalings of the block $\mathcal{M}_F^{uu}$ are radically different whether Dirichlet or Robin/Neumann boundary conditions are enforced.

6.1.2. Comparison with other methods. In this section we restrict the setting to the equation $u - \Delta u = f$ and to homogeneous Dirichlet boundary conditions so as to make comparisons with the unified approach developed in [1], where it is shown that most of the DG methods amount to solving the following problem:

$$\begin{cases} \text{Seek } z_h = (\sigma_h, u_h) \in W_h \text{ such that } \forall y_h \in [\mathbb{P}_{p_\sigma}(K)]^d \times \mathbb{P}_{p_u}(K), \\ (z_h, \tilde{T} y_h)_{L,K} + (\widehat{\phi}_{\partial K}(z_h), y_h)_{L,\partial K} = (f, y_h)_{L,K}, \end{cases} \tag{6.5}$$

where the so-called numerical fluxes $\widehat{\phi}_{\partial K}(z_h)$ depend on the method under consideration. In view of (4.17) and (4.19), the link between the present formalism and that of [1] is based on the identification $\widehat{\phi}_{\partial K}(z_h)|_F = \phi_{\partial K}(z_h)|_F$. For the purpose of comparison, we restrict ourselves to boundary and interface operators such that for all $F \in \mathcal{F}_h$, for all $v \in L^2(F)$, and for all $\tau \in [L^2(F)]^d$,

$$M_F^{\sigma u}(v) = -n_F v, \qquad M_F^{u\sigma}(\tau) = \tau{\cdot}n_F, \tag{6.6}$$

$$S_F^{\sigma u}(v) = 0, \qquad S_F^{u\sigma}(\tau) = 0. \tag{6.7}$$

Therefore, the methods that can be constructed from this set of assumptions differ only in the design of M_F^{uu}, S_F^{uu}, and R_F. We set $\widehat{\phi}_{\partial K}(z_h) = (\widehat{u}_K n_K, \widehat{\sigma}_K{\cdot}n_K)$ (note that $\widehat{u}_K$ is $\mathbb{R}$-valued, $\widehat{\sigma}_K$ is $\mathbb{R}^d$-valued, and the sign convention we use herein for σ_h and $\widehat{\sigma}_K$ is opposite to that in [1]). Then, the above identification of the fluxes is possible if the DG method under consideration is such that

$$\widehat{\phi}_{\partial K}(z_h) = \begin{cases} (0, \sigma_h{\cdot}n_F + \frac{1}{2} M_F^{uu}(u_h) + R_F(u_h)) & \text{if } F \in \mathcal{F}_h^\partial, \\ (\{u_h\}\, n_K, \{\sigma_h\}{\cdot}n_K + n_F{\cdot}n_K (S_F^{uu}([\![u_h]\!]) + R_F([\![u_h]\!]))) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}. \end{cases} \tag{6.8}$$

The DG methods that belong to this class are those from [3, 5, 4, 6] together with that of [7] already discussed above. Observe that in this setting, the local flux reconstruction formula (4.23) takes the form

$$\forall K \in \mathcal{T}_h, \quad z_h^\sigma|_K = -\nabla z_h^u|_K + \sum_{F \subset \partial K} r_F([\![z_h^u]\!] n_F). \tag{6.9}$$

Comparison with the method of Brezzi et al. The method described by Brezzi et al. [6] (see also [1]) is such that

$$\widehat{\phi}_{\partial K}(z_h) = \begin{cases} (0, \sigma_h{\cdot}n_F + \frac{1}{2} \varsigma r_F(u_h n_F){\cdot}n_F) & \text{if } F \in \mathcal{F}_h^\partial, \\ (\{u_h\}\, n_K, \{\sigma_h\}{\cdot}n_K + \eta\, \{r_F([\![u_h]\!] n_F)\}{\cdot}n_K) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}, \end{cases} \tag{6.10}$$

where ς and η are positive constants. This amounts to specifying M_F^{uu}, S_F^{uu}, and R_F such that for all $v \in L^2(F)$,

$$M_F^{uu}(v) = \varsigma r_F(vn_F)\cdot n_F, \qquad S_F^{uu}(v) = \eta \{r_F(vn_F)\} \cdot n_F, \qquad R_F(v) \equiv 0. \tag{6.11}$$

The operator r_F is endowed with the following property.

LEMMA 6.2. *For all $F \in \mathcal{F}_h$ and for all $\tau_h \in [\mathbb{P}_{p_\sigma}(F)]^d$,*

$$h_F^{-\frac{1}{2}} \|\tau_h\|_{L_\sigma,F} \lesssim \|r_F(\tau_h)\|_{L_\sigma,\mathcal{T}(F)} \lesssim h_F^{-\frac{1}{2}} \|\tau_h\|_{L_\sigma,F}. \tag{6.12}$$

This lemma and the definition of r_F imply that for all $F \in \mathcal{F}_h$ and for all $v_h \in \mathbb{P}_{p_u}(F)$,

$$h_F^{-1} \|v_h\|_{L_u,F}^2 \lesssim (\{r_F(v_h n_F)\} \cdot n_F, v_h)_{L_u,F} \lesssim h_F^{-1} \|v_h\|_{L_u,F}^2. \tag{6.13}$$

These inequalities are just what is takes to prove that if the boundary and interface operators are defined using (6.6), (6.7), and (6.11), properties (DG$^\sharp$) hold. Therefore, the conclusions of Theorems 5.8 and 5.14 hold.

Comparison with the IP method. Let ς and η be two positive constants. The IP method of Baker [3] (see also Arnold [2]) is such that the flux is defined by

$$\widehat{\phi}_{\partial K}(z_h) = \begin{cases} (0, \sigma_h \cdot n_F + \frac{1}{2} \frac{\varsigma}{h_F} u_h + \rho_F([\![u_h]\!]) \cdot n_F) & \text{if } F \in \mathcal{F}_h^\partial, \\ (\{u_h\} n_K, \{\sigma_h\} \cdot n_K + \frac{\eta}{h_F} [\![u_h]\!] n_F \cdot n_K + \rho_F([\![u_h]\!]) \cdot n_K) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}, \end{cases} \tag{6.14}$$

where the operator $\rho_F : L^2(\Delta_F) \longrightarrow L^2(F)$ is defined by

$$\rho_F(v) = - \sum_{F' \in \Delta_F} \{r_{F'}(vn_{F'})\}, \tag{6.15}$$

and $\Delta_F = \{F' \in \mathcal{F}_h;\ \exists K' \in \mathcal{T}_h,\ F \cup F' \subset \partial K'\}$. This method fits the present framework if we set

$$M_F^{uu}(v) = \varsigma h_F^{-1} v, \qquad S_F^{uu}(v) = \eta h_F^{-1} v, \qquad R_F(v) = \rho_F(v) \cdot n_F. \tag{6.16}$$

Using Lemma 6.2, it is readily seen that (DG18) holds and that (DG17) holds if the design parameters ς and η are large enough. Therefore, the conclusions of Theorems 5.8 and 5.14 hold for the IP method. Note that the expression (4.31) derived for R_F in the general setting of two-field Friedrichs' systems reduces to (6.16) for the Poisson problem with Dirichlet boundary conditions.

Comparison with the methods of Bassi et al. The method proposed by Bassi and Rebay [5] corresponds to the choice of $M_F^{uu} \equiv 0$, $S_F^{uu} \equiv 0$, and $R_F \equiv 0$. Our analysis needs to be revised to account for this situation. Obviously, the L^2-coercivity still holds in the form $\|y\|_L^2 \lesssim a_h(y, y)$ for all $y \in W(h)$. Moreover, one easily derives the following continuity estimate: For all $(y, y_h) \in W(h) \times W_h$,

$$|a_h(y, y_h)| \lesssim \left(\sum_{K \in \mathcal{T}_h} [\|Ty\|_{L,K}^2 + h_K^{-1} \|y\|_{L,\partial K}^2] \right)^{\frac{1}{2}} \|y_h\|_L. \tag{6.17}$$

Then, provided $p_\sigma = p_u := p$, the second Strang lemma implies $\|z - z_h\|_L \lesssim h^p \|z\|_{[H^{p+1}(\Omega)]^m}$. Although this estimate is not optimal, it shows that the method of Bassi and Rebay is (possibly nonoptimally) convergent. Finally, the method proposed by Bassi et al. [4] fits the present framework by defining the operators

$$M_F^{uu}(v) = \varsigma r_F(vn_F) \cdot n_F, \qquad S_F^{uu}(v) = \eta \{r_F(vn_F)\} \cdot n_F, \tag{6.18}$$

and the operator R_F as in the IP method, i.e., (6.16). By using what has been shown above for the method of Brezzi et al. and the IP method, it is clear that the conclusions of Theorems 5.8 and 5.14 hold in this case also, provided ς and η are large enough.

6.2. Linear continuum mechanics. Consider the linear continuum mechanics equations introduced in section 3.2 and let us describe a DG method where the $(\overline{\sigma}, p)$-component of the unknown can be eliminated locally. Owing to (3.11) and (3.12), the integral representations (2.15) and (2.17) hold with the $\mathbb{R}^{m,m}$-valued boundary fields (recall that $m = d^2 + 1 + d$)

$$\mathcal{D} = \left[\begin{array}{c:c} 0 & \mathcal{H} \\ \hdashline \mathcal{H}^t & 0 \end{array}\right] \qquad \text{and} \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -\mathcal{H} \\ \hdashline \mathcal{H}^t & 0 \end{array}\right], \tag{6.19}$$

where $\mathcal{H} = \sum_{k=1}^{d} n_k(\mathcal{E}^k, 0)^t \in \mathbb{R}^{d^2+1,d}$. Observe that for all $\xi \in \mathbb{R}^d$, $\mathcal{H}\xi = (-\frac{1}{2}(n\otimes\xi + \xi\otimes n), 0)$. Let $\varsigma > 0$ and $\eta > 0$ (these design parameters can vary from face to face). For all $F \in \mathcal{F}_h$, set $R_F \equiv 0$ and

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0 & -\mathcal{H}_F \\ \hdashline \mathcal{H}_F^t & \varsigma h_F^{-1}\mathcal{I}_d \end{array}\right], \qquad \mathcal{S}_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \eta h_F^{-1}\mathcal{I}_d \end{array}\right], \tag{6.20}$$

where $\mathcal{H}_F$ is defined as $\mathcal{H}$ with n_F substituting for n. Define, for all $y \in [L^2(F)]^m$, $M_F(y) = \mathcal{M}_F y$ and $S_F(y) = \mathcal{S}_F y$. Then, using Lemmata 5.1 and 5.3, one readily verifies that properties (DG$^\sharp$) hold. An IP-like method can be derived as well.

6.3. Simplified MHD. Consider the simplified MHD equations introduced in section 3.3 and let us describe a DG method where the H-component of the unknown can be eliminated locally (the derivation of a DG method where the E-component of the unknown can be eliminated locally is similar). To recover the notation of section 5, set $\sigma \equiv H$ and $u \equiv E$. Owing to (3.16) and (3.17), the integral representations (2.15) and (2.17) hold with the $\mathbb{R}^{6,6}$-valued boundary fields

$$\mathcal{D} = \left[\begin{array}{c:c} 0 & \mathcal{N} \\ \hdashline \mathcal{N}^t & 0 \end{array}\right] \qquad \text{and} \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -\mathcal{N} \\ \hdashline \mathcal{N}^t & 0 \end{array}\right], \tag{6.21}$$

where $\mathcal{N} = \sum_{k=1}^{3} n_k \mathcal{R}^k$, and the $\mathbb{R}^{3,3}$-valued fields $\mathcal{R}^1$, $\mathcal{R}^2$, and $\mathcal{R}^3$ are defined in section 3.3. Observe that for all $\xi \in \mathbb{R}^3$, $\mathcal{N}\xi = n\times\xi$. Let $\varsigma > 0$ and $\eta > 0$ (these design parameters can vary from face to face). For all $F \in \mathcal{F}_h$, set $R_F \equiv 0$ and

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0 & -\mathcal{N}_F \\ \hdashline \mathcal{N}_F^t & \varsigma h_F^{-1}\mathcal{N}_F^t\mathcal{N}_F \end{array}\right], \qquad \mathcal{S}_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \eta h_F^{-1}\mathcal{N}_F^t\mathcal{N}_F \end{array}\right], \tag{6.22}$$

where $\mathcal{N}_F$ is defined as $\mathcal{N}$ by using n_F instead of n. For all $y \in [L^2(F)]^6$, let $M_F(y) = \mathcal{M}_F y$ and $S_F(y) = \mathcal{S}_F y$. Then, using Lemmata 5.1 and 5.3, one readily verifies that properties (DG$^\sharp$) hold. An IP-like method can be derived as well.

Remark 6.3. As opposed to advection-diffusion-reaction equations, the upper bounds in (5.7) and (5.9) are not sharp for the simplified MHD equations since the operators M_F and S_F do not need to control the whole L^2-norm of the electric field.

7. Conclusions. It happens sometimes that (A4) does not hold; instead, the following weaker inequality holds:

$$\exists \mu_0 > 0 \quad \forall z \in W, \qquad (Tz, z)_L + (z, \tilde{T}z)_L \geq 2\mu_0 \|\pi z^\sigma\|_{L_\sigma}^2, \tag{7.1}$$

where $\pi \in \mathcal{L}(L_\sigma; L_\sigma)$ may not be injective. In other words, coercivity no longer holds on the u-component of the unknown but holds only on a piece of the σ-component, namely πz^σ. The equation $-\Delta u = f$ corresponds to this situation with π equal to the identity. The linear continuum mechanics equations in the incompressible limit, e.g., the Stokes equations, also fall in this framework with a nontrivial noninjective operator π. It will be shown in a forthcoming third part that, provided additional mild assumptions are made on the differential operators and on the DG setting, all that has been said herein in the fully L-coercive case remains valid in the situation with partial coercivity.

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