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## Brief paper

On existence, optimality and asymptotic stability of the Kalman filter with partially observed inputs<sup>☆</sup>Jinya Su<sup>a</sup>, Baibing Li<sup>b,1</sup>, Wen-Hua Chen<sup>a</sup><sup>a</sup> Department of Aeronautical and Automotive Engineering, Loughborough University, Loughborough LE11 3TU, UK<sup>b</sup> School of Business and Economics, Loughborough University, Loughborough LE11 3TU, UK

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## ABSTRACT

For linear stochastic time-varying systems, we investigate the properties of the Kalman filter with partially observed inputs. We first establish the existence condition of a general linear filter when the unknown inputs are partially observed. Then we examine the optimality of the Kalman filter with partially observed inputs. Finally, on the basis of the established existence condition and optimality result, we investigate asymptotic stability of the filter for the corresponding time-invariant systems. It is shown that the results on existence and asymptotic stability obtained in this paper provide a unified approach to accommodating a variety of filtering scenarios as its special cases, including the classical Kalman filter and state estimation with unknown inputs.

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## 1. Introduction

State estimation plays an important role in state space modelling and control. It has been applied to a wide range of areas; see Li (2009) and Liang, Chen, and Pan (2010) for some recent applications in network control systems, transportation management, etc.

In the recent decades, state estimation for discrete-time linear stochastic systems with unknown inputs (also termed as unknown input filtering (UIF) problem) has received considerable attention since the original work of Kitanidis (1987) first appeared. Various filters were developed under different assumptions for the systems with unknown inputs; see, e.g., Cheng, Ye, Wang, and Zhou (2009), Darouach and Zasadzinski (1997), Darouach, Zasadzinski, and Xu (1994), Fang and Callafon (2012), Gillijns and De Moor (2007), Hsieh (2000, 2010) and Kitanidis (1987), among many others. Most of these researches used the technique of minimum variance unbiased estimation, hence leading to an unbiased minimum-variance

filter (UMVF). Another important research line is state estimation for descriptor systems (Hsieh, 2011, 2013). It has been recently shown in Hsieh (2013) that any linear descriptor systems can be transformed into a linear stochastic system with unknown inputs. This shows a close link between these two kinds of problem. In addition, various properties for these developed filters have been investigated, including the existence condition (Darouach & Zasadzinski, 1997), asymptotic stability (Fang & Callafon, 2012), and global optimality of the UMVF (Cheng et al., 2009 and Hsieh, 2010).

Recently, Li (2013) has developed a Kalman filter for linear systems with partially observed inputs, where the inputs are observed not at the level of interest but rather the input information is available at an aggregate level. It has been shown that the developed filter provides a unified approach to state estimation for linear systems with Gaussian noise. In particular, it includes two important extreme scenarios as its special cases: (a) the filter where all the inputs are completely available (i.e. the classical Kalman filter; see, e.g., Simon, 2006); and (b) the filter where all inputs are unknown (i.e. the filter investigated in Kitanidis, 1987 and many others for the UIF problem). Potentially the proposed filter can be applied to a variety of practical problems in many different areas such as population estimation and traffic control.

So far there is not any study discussing the existence and asymptotic stability issues of this newly proposed unified filter. In this paper we investigate the properties of the Kalman filter with partially observed inputs developed in Li (2013). For linear stochastic time-varying systems with partially observed inputs, we

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establish the existence condition for a general linear filter. Then we show that the developed filter is optimal in the sense of minimum error covariance matrix. Finally, we consider asymptotic stability of the filter for the corresponding time-invariant systems based on the established existence condition and optimality result.

This paper has provided a unified approach to accommodating existence and asymptotic stability conditions in a variety of filtering scenarios: it includes the results on existence and asymptotic stability for some important filters as its special cases, e.g., the filters developed for the problems where the inputs are completely available and where all the inputs are unknown. Note that the former is the classical Kalman filtering problem and the corresponding existence and asymptotic stability conditions are well established in the literature. For the latter case with unknown inputs, there has been a continuing research interest in existence and asymptotic stability conditions for various discrete-time systems (e.g., Cheng et al., 2009, Darouach & Zasadzinski, 1997, Fang & Callafon, 2012, Kitanidis, 1987) and continuous-time systems (e.g., Bejarano, Floquet, Perruquetti, & Zheng, 2013, Corless & Tu, 1998, Hou & Muller, 1992).

This paper is structured as follows. First, Section 2 is devoted to problem statement. Then we establish the existence condition in Section 3. We focus on the properties of the filter proposed in Li (2013) in Section 4. In Section 5, we investigate asymptotic stability. Finally, this paper concludes in Section 6.

## 2. Problem statement

Consider a linear stochastic time-varying system:

$$\begin{aligned} x_{k+1} &= A_k x_k + G_k d_k + \omega_k \\ y_k &= C_k x_k + \nu_k, \end{aligned} \quad (1)$$

where  $x_k \in R^n$  is the state vector,  $d_k \in R^m$  is the input vector, and  $y_k \in R^p$  is the measurement vector at each time step  $k$  with  $p \geq m$  and  $n \geq m$ . The process noise  $\omega_k \in R^n$  and the measurement noise  $\nu_k \in R^p$  are assumed to be mutually uncorrelated with zero-mean and a known covariance matrix,  $Q_k = E[\omega_k \omega_k^T] \geq 0$  and  $R_k = E[\nu_k \nu_k^T] > 0$ , respectively.  $A_k$ ,  $G_k$  and  $C_k$  are known matrices. Without loss of generality, we follow Gillijns and De Moor (2007) and Kitanidis (1987), and assume that  $G_k$  has a full column-rank. The initial state  $x_0$  is independent of  $\omega_k$  and  $\nu_k$  with a known mean  $\hat{x}_0$  and covariance matrix  $P_0 > 0$ .

We consider the scenario where the input vector  $d_k$  is not fully observed at the level of interest but rather it is available only at an aggregate level. Specifically, let  $D_k$  be a  $q_k \times m$  known matrix with  $0 \leq q_k \leq m$  and  $F_{0k}$  an orthogonal complement of  $D_k^T$  such that  $D_k F_{0k} = O_{q_k \times (m-q_k)}$  and  $F_{0k}^T F_{0k} = I_{m-q_k}$ , where  $O$  and  $I$  represent the zero matrix and identity matrix of appropriate dimensions. We suppose that the input data is available only on some linear combinations:

$$r_k = D_k d_k, \quad (2)$$

where  $r_k$  is available at each time step  $k$ .  $D_k$  is assumed to have a full row-rank; otherwise the redundant rows can be removed.

As pointed out in Li (2013), the matrix  $D_k$  characterizes the availability of input information at each time step  $k$ . It includes two extreme scenarios that are usually considered: (a)  $q_k = m$  and  $D_k$  is an identity matrix, i.e. the complete input information is available; this is case that the classical Kalman filter can be applied; (b)  $q_k = 0$ , i.e. no information on the input variables is available; this is the problem investigated in Darouach and Zasadzinski (1997), Gillijns and De Moor (2007), Hsieh (2000) and Kitanidis (1987).

Throughout this paper, we use  $\lambda(B)$  to denote any eigenvalue of a square matrix  $B$ . For any two symmetric matrices  $A$  and  $B$  with suitable dimensions, the notation  $A \geq B$  is used if and only if

$A - B$  is non-negative definite. In addition, we use  $G_k^\perp$  to denote an orthogonal complement of  $G_k$  and  $\Omega_k = [G_k, G_k^\perp]$ . Define

$$\Pi_k = \begin{pmatrix} D_{k-1} \\ C_k G_{k-1} \end{pmatrix}. \quad (3)$$

## 3. Existence condition

To establish the existence condition of a general linear filter for system (1) and (2), we first consider an invertible linear transformation.

### 3.1. Transformation

Consider the following invertible matrix:

$$M_k = \begin{bmatrix} D_k & O_{q_k \times (n-m)} \\ O_{(n-m) \times m} & I_{n-m} \\ F_{0k}^T & O_{(m-q_k) \times (n-m)} \end{bmatrix} \Omega_k^{-1}.$$

It is straightforward to verify that  $M_k G_k d_k$  can be expressed as:

$$\begin{aligned} M_k G_k d_k &= [D_k^T, O_{m \times (n-m)}, F_{0k}]^T d_k \\ &= [(D_k d_k)^T, (O_{(n-m) \times m} d_k)^T, (F_{0k}^T d_k)^T]^T \\ &= [r_k^T, O_{1 \times (n-m)}, (F_{0k}^T d_k)^T]^T \\ &= \tilde{r}_k + \tilde{G}_k \delta_k, \end{aligned} \quad (4)$$

where  $\tilde{r}_k = [r_k^T, O_{1 \times (n-m)}, O_{1 \times (m-q_k)}]^T$ ,  $\delta_k = F_{0k}^T d_k$  and  $\tilde{G}_k = [O_{(m-q_k) \times q_k}, O_{(m-q_k) \times (n-m)}, I_{m-q_k}]^T$ . We note that  $\tilde{r}_k$  is completely available due to Eq. (2).

Left-multiplying both sides of Eq. (4) by  $M_k^{-1}$ ,  $G_k d_k$  can be decoupled into two parts:

$$G_k d_k = M_k^{-1} \tilde{r}_k + M_k^{-1} \tilde{G}_k \delta_k. \quad (5)$$

From Eq. (5), the dynamics of  $x_{k+1}$  can be rewritten as:

$$\begin{aligned} x_{k+1} &= A_k x_k + M_k^{-1} \tilde{r}_k + M_k^{-1} \tilde{G}_k \delta_k + \omega_k \\ &= A_k x_k + u_k + F_k \delta_k + \omega_k, \end{aligned}$$

where  $u_k = M_k^{-1} \tilde{r}_k$  is a known term, and  $F_k$  is given by

$$F_k = M_k^{-1} \tilde{G}_k = [G_k, G_k^\perp] \begin{bmatrix} F_{0k} \\ O \end{bmatrix} = G_k F_{0k}. \quad (6)$$

Consequently, linear system (1) with the partially observed inputs  $r_k = D_k d_k$  can be equivalently represented by the following system:

$$\begin{aligned} x_{k+1} &= A_k x_k + u_k + F_k \delta_k + \omega_k \\ y_k &= C_k x_k + \nu_k. \end{aligned} \quad (7)$$

The above manipulation shows that a linear stochastic system with partially observed inputs (2) is equivalent to a linear system with unknown inputs; similar property is also found for linear descriptor systems (Hsieh, 2013).

### 3.2. Existence condition

In this subsection, we will establish the existence condition of a general, asymptotically stable and unbiased linear filter for system (7) and hence for its equivalent system, Eqs. (1) and (2).

Motivated by the linear filter structure in the literature (e.g. Darouach et al., 1994), we consider a general linear filter for discrete-time linear system (7) of the form

$$\hat{x}_{k+1} = E_k \hat{x}_k + J_k u_k + K_{k+1} y_{k+1}, \quad (8)$$

where the gain matrices  $E_k, J_k$  and  $K_{k+1}$  are to be designed. Based on (7) and (8), one can obtain the error dynamics  $e_{k+1} = x_{k+1} - \hat{x}_{k+1}$ :

$$\begin{aligned} e_{k+1} &= (A_k x_k + u_k + F_k \delta_k + \omega_k) - (E_k \hat{x}_k + J_k u_k + K_{k+1} y_{k+1}) \\ &= E_k e_k - (J_k - I + K_{k+1} C_{k+1}) u_k \\ &\quad + (A_k - K_{k+1} C_{k+1} A_k - E_k) x_k - (K_{k+1} C_{k+1} F_k - F_k) \delta_k \\ &\quad + (I - K_{k+1} C_{k+1}) \omega_k - K_{k+1} v_{k+1}. \end{aligned}$$

To ensure the filter is unbiased, it is required that the filtering error is independent of  $u_k, x_k$  and  $\delta_k$ . In addition, it is expected that the error approaches to zero as time  $k$  increases. Hence the existence condition for filter (8) is given by:

- (i)  $E_k$  is stable (i.e., any eigenvalue of  $E_k$  satisfies  $|\lambda(E_k)| < 1$ );
- (ii)  $E_k = A_k - K_{k+1} C_{k+1} A_k$ ;
- (iii)  $K_{k+1} C_{k+1} F_k = F_k$ ;
- (iv)  $J_k = I - K_{k+1} C_{k+1}$ .

For system (1)–(2), however, the existence condition for system (7) should be expressed in terms of matrices  $A_k, G_k, C_k$  and  $D_k$ . For this end, we first state a lemma.

**Lemma 1.** For system (1)–(2), we have

$$\begin{aligned} \text{rank} \left( \begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) \\ = \text{rank} \left( \begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} & O \end{bmatrix} \right) + \text{rank}(D_k D_k^T). \end{aligned}$$

See the Appendix for proof. We now provide a condition for the existence of a general linear filter for a dynamic system with partially observed inputs.

**Theorem 1.** Suppose that both matrices  $D_k^T$  and  $G_k$  have a full column-rank. Then a sufficient condition for the existence of a general linear filter (8) for system (1)–(2) is given by:

$$\text{rank}(\Pi_{k+1}) = m \tag{9}$$

and for all  $z \in \mathcal{C}$  ( $\mathcal{C}$  is the field of complex numbers) such that  $|z| \geq 1$ :

$$\text{rank} \left( \begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) = n + m. \tag{10}$$

**Proof.** We note that we can select matrices  $E_k = A_k - K_{k+1} C_{k+1} A_k$  and  $J_k = I - K_{k+1} C_{k+1}$  to ensure that condition parts (ii) and (iv) are satisfied. Hence, we will focus on condition parts (i) and (iii). We first show that Eq. (9) guarantees there exists a matrix  $K_{k+1}$  such that condition part (iii) holds. We note

$$\begin{aligned} \begin{bmatrix} D_k \\ C_{k+1} G_k \end{bmatrix} \begin{bmatrix} F_{0k} & D_k^T \end{bmatrix} \\ = \begin{bmatrix} O_{q_k \times (m-q_k)} & D_k D_k^T \\ C_{k+1} G_k F_{0k} & C_{k+1} G_k D_k^T \end{bmatrix}. \end{aligned} \tag{11}$$

Since  $[F_{0k}, D_k^T]$  is invertible and  $\Pi_{k+1}$  has a full column-rank, we obtain that  $C_{k+1} G_k F_{0k} = C_{k+1} F_k$  (see Eq. (6)) is also of full column-rank, i.e.

$$\text{rank}(C_{k+1} F_k) = m - q_k. \tag{12}$$

Eq. (12) guarantees there exists a matrix  $K_{k+1}$  such that condition part (iii) holds.

Next, since  $C_{k+1} F_k$  has a full column-rank, there exists an invertible matrix  $N_k \in \mathbb{R}^{p \times p}$  such that

$$N_k C_{k+1} F_k = \begin{bmatrix} O_{(p-m+q_k) \times (m-q_k)} \\ I_{m-q_k} \end{bmatrix}.$$

The general solution  $K_{k+1}$  of  $K_{k+1} C_{k+1} F_k = F_k$  is given by  $K_{k+1} = [\Gamma_k, F_k] N_k$ , where  $\Gamma_k$  can be any matrix of suitable dimensions and is to be designed for the gain matrix  $K_{k+1}$ .

Now define  $S_{1k}$  and  $S_{2k}$  such that

$$\begin{bmatrix} S_{1k} \\ S_{2k} \end{bmatrix} = N_k C_{k+1} A_k. \tag{13}$$

Then from condition part (ii), we can obtain

$$\begin{aligned} E_k &= A_k - K_{k+1} C_{k+1} A_k \\ &= A_k - [\Gamma_k, F_k] N_k C_{k+1} A_k = A_k - [\Gamma_k, F_k] \begin{bmatrix} S_{1k} \\ S_{2k} \end{bmatrix} \\ &= A_k - F_k S_{2k} - \Gamma_k S_{1k}. \end{aligned} \tag{14}$$

According to Anderson and Moore (1979, p. 342), the existence condition part (i) holds if and only if the following equivalent conditions hold:

- (a)  $A_k - F_k S_{2k} - \Gamma_k S_{1k}$  is stable for a matrix  $\Gamma_k$ ;
- (b)  $S_{1k} \eta = 0$  and  $(A_k - F_k S_{2k}) \eta = \lambda \eta$  for some constant  $\lambda$  and vector  $\eta$  implies  $|\lambda| < 1$  or  $\eta = 0$ .

The condition (b) can be expressed in the following equivalent form for all  $z \in \mathcal{C}$  and  $|z| \geq 1$ :

$$\text{rank} \left( \begin{bmatrix} zI_n - A_k + F_k S_{2k} \\ S_{1k} \end{bmatrix} \right) = n. \tag{15}$$

The following identity, in conjunction with Lemma 1, shows that Eq. (15) is satisfied:

$$\begin{aligned} \text{rank} \left( \begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} & O \end{bmatrix} \right) \\ = \text{rank} \left( \begin{bmatrix} I_n & O \\ -C_{k+1} & zI \end{bmatrix} \begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} & O \end{bmatrix} \right) \\ = \text{rank} \left( \begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} A_k & C_{k+1} F_k \end{bmatrix} \right) \\ = \text{rank} \left( \begin{bmatrix} I_n & O \\ O & N_k \end{bmatrix} \begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} A_k & C_{k+1} F_k \end{bmatrix} \right) \\ = \text{rank} \left( \begin{bmatrix} zI_n - A_k & -F_k \\ S_{1k} & O \\ S_{2k} & I_{m-q_k} \end{bmatrix} \right) \\ = \text{rank} \left( \begin{bmatrix} zI_n - A_k + F_k S_{2k} & -F_k \\ S_{1k} & O \\ O & I_{m-q_k} \end{bmatrix} \right) \\ = \text{rank} \left( \begin{bmatrix} zI_n - A_k + F_k S_{2k} & O \\ S_{1k} & O \\ O & I_{m-q_k} \end{bmatrix} \right) \\ = \text{rank} \left( \begin{bmatrix} zI_n - A_k + F_k S_{2k} \\ S_{1k} \end{bmatrix} \right) + m - q_k. \end{aligned}$$

Hence, Eqs. (9) and (10) guarantee there exists a gain  $K_{k+1}$  such that: (a)  $K_{k+1} C_{k+1} F_k = F_k$ ; and (b)  $E_k = A_k - K_{k+1} C_{k+1} A_k$  is stable. This completes the proof.

**Remarks.** (i) Eq. (9) is the estimability condition for the filter developed in Li (2013) for system (1) with partially observed inputs (2). From the proof of Theorem 1, it also guarantees the unbiasedness of a general linear filter. In addition, Theorem 1 shows that to ensure the estimation error of a general linear filter is stable as time  $k$  increases, a detectability condition (10) needs to be met.

(ii) When condition parts (i)–(iv) are satisfied, the general linear filter (8) is given by

$$\hat{x}_{k+1} = (A_k - K_{k+1}C_{k+1}A_k)\hat{x}_k + (I - K_{k+1}C_{k+1})u_k + K_{k+1}y_{k+1}. \quad (16)$$

(iii) The error dynamics of the above filter (16) that satisfy condition (i)–(iv) become

$$e_{k+1} = (A_k - K_{k+1}C_{k+1}A_k)e_k + [I - K_{k+1}C_{k+1}, -K_{k+1}][\omega_k, v_{k+1}]^T. \quad (17)$$

### 3.3. Relationships with the existing filters

As mentioned earlier, system (1) with partially observed inputs (2) includes two important scenarios as its special cases: (a) the complete input information is available; and (b) no information on the input variables is available. In this subsection, we compare the developed existence condition in the previous subsection for partially observed inputs to the condition derived for the classical Kalman filter with complete information on the inputs, and to that of the filter with unknown inputs.

**Theorem 2.** *The proposed existence condition for filter (8) in Theorem 1 reduces to: (a) the existence condition of the classical Kalman filter when the complete information on the inputs is available, i.e.,  $D_k$  is invertible; and (b) the existence condition of the filter with unknown inputs, i.e.  $D_k$  is an empty matrix.*

**Proof.** First, we consider the case that matrix  $D_k$  is invertible. It is clear that Eq. (9) is satisfied due to the non-singularity of  $D_k$ . In addition, we have

$$\begin{aligned} \text{rank} \left( \begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) &= \text{rank} \left( \begin{bmatrix} zI_n - A_k & O \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) \\ &= \text{rank} \left( \begin{bmatrix} zI_n - A_k \\ C_{k+1} \end{bmatrix} \right) + \text{rank}(D_k D_k^T). \end{aligned}$$

Since  $\text{rank}(D_k D_k^T) = m$ , the existence condition (10) reduces to

$$\text{rank} \left( \begin{bmatrix} zI_n - A_k \\ C_{k+1} \end{bmatrix} \right) = n, \quad \forall z \in \mathcal{C}, |z| \geq 1$$

which is the detectability condition of the classical Kalman filter (see, e.g. Anderson & Moore, 1979; Simon, 2006).

Next, we turn to the scenario where no information on the inputs  $d_k$  is available. Since matrix  $D_k$  reduces to a zero-by-zero empty matrix in this case, Eq. (9) becomes

$$\text{rank}(C_{k+1}G_k) = m. \quad (18)$$

In addition, Eq. (10) reduces to

$$\text{rank} \left( \begin{bmatrix} zI_n - A_k & G_k \\ C_{k+1} & O \end{bmatrix} \right) = n + m, \quad \forall z \in \mathcal{C}, |z| \geq 1. \quad (19)$$

Eqs. (18)–(19) are identical to the results for the filter with unknown inputs (Darouach & Zasadzinski, 1997). This completes the proof.

Theorem 2 shows that the obtained existence condition in this paper is a more generic condition. In addition, comparing the existence condition (9) and (10) of the general linear filter (8) for systems with partially available inputs to the existence condition (18)–(19), it can be seen that partial information on the unknown inputs has relaxed the existence condition of a general linear filter. In other words, with the information on the unknown inputs at an

aggregate level (2), it is more likely that the general linear filter (8) exists.

## 4. The filter with partially observed inputs

Now we focus on the filter proposed in Li (2013) for linear stochastic systems when the inputs are partially observed. Note that this filter was derived under the Bayesian framework with the assumption that  $\omega_k$  and  $v_k$  follow a Gaussian distribution, and  $\delta_k$  has a noninformative prior distribution. We summarize the results of the filter below. Define

$$\tilde{D}_k = \begin{pmatrix} D_k & O_{q_k \times (n-m)} \\ O_{(n-m) \times m} & I_{n-m} \end{pmatrix}.$$

Let  $\tilde{M}_k = \tilde{D}_k \Omega_k^{-1}$ . It is shown in Li (2013) that for system (1) with the input data available at an aggregate level (2), if matrix  $\Pi_k$  has a full column-rank, then the posterior distribution for  $x_k$  at any time step  $k$  is a Gaussian distribution with posterior mean  $\hat{x}_{k|k}$  and posterior covariance matrix  $P_{k|k}$  given by:

$$\begin{aligned} \hat{x}_{k|k} &= A_{k-1}\hat{x}_{k-1|k-1} + P_{k|k}\tilde{M}_{k-1}^T(\tilde{M}_{k-1}P_{k-1|k-1}M_{k-1}^T)^{-1} \\ &\quad \times \bar{r}_{k-1} + K_k(y_k - C_k A_{k-1}\hat{x}_{k-1|k-1}), \end{aligned} \quad (20)$$

and

$$\begin{aligned} P_{k|k} &= P_{k|k-1} - P_{k|k-1}C_k^T H_k^{-1} C_k P_{k|k-1} + [F_{k-1} \\ &\quad - P_{k|k-1}C_k^T H_k^{-1} C_k F_{k-1}][F_{k-1}^T C_k^T H_k^{-1} C_k F_{k-1}]^{-1} \\ &\quad \times [F_{k-1} - P_{k|k-1}C_k^T H_k^{-1} C_k F_{k-1}]^T, \end{aligned} \quad (21)$$

with

$$\begin{aligned} K_k &= P_{k|k-1}C_k^T H_k^{-1} + [F_{k-1} - P_{k|k-1}C_k^T H_k^{-1} C_k F_{k-1}] \\ &\quad \times [F_{k-1}^T C_k^T H_k^{-1} C_k F_{k-1}]^{-1} F_{k-1}^T C_k^T H_k^{-1}, \end{aligned} \quad (22)$$

$\bar{r}_k = [r_k^T, O^T]^T$ ,  $P_{k|k-1} = A_{k-1}P_{k-1|k-1}A_{k-1}^T + Q_{k-1}$  and  $H_k = C_k P_{k|k-1}C_k^T + R_k > 0$ . Note that Eq. (9) guarantees Eq. (12) holds, and hence  $F_{k-1}^T C_k^T H_k^{-1} C_k F_{k-1}$  is invertible in the above equations.

Under the Bayesian framework,  $\hat{x}_{k|k}$  was shown to be a minimum mean square error (MMSE) estimate in Li (2013). However, no further properties of the filter were explored.

We now derive the dynamics of the state estimation error  $e_k = x_k - \hat{x}_{k|k}$ .

**Lemma 2.** *The estimation error  $e_k = x_k - \hat{x}_{k|k}$  of the filter (20)–(22) follows the recursive equation*

$$e_k = (A_{k-1} - K_k C_k A_{k-1})e_{k-1} + [I - K_k C_k, -K_k][\omega_{k-1}, v_k]^T, \quad (23)$$

where  $K_k$  is given by Eqs. (21)–(22).

**Proof.** Let  $W_{k-1} = P_{k|k}\tilde{M}_{k-1}^T(\tilde{M}_{k-1}P_{k-1|k-1}\tilde{M}_{k-1}^T)^{-1}$ . The error dynamics of the filter (20)–(22) are given by

$$\begin{aligned} e_k &= A_{k-1}x_{k-1} + G_{k-1}d_{k-1} + \omega_{k-1} - A_{k-1}\hat{x}_{k-1|k-1} \\ &\quad - W_{k-1}\bar{r}_{k-1} - K_k(y_k - C_k A_{k-1}\hat{x}_{k-1|k-1}) \\ &= (A_{k-1} - K_k C_k A_{k-1})e_{k-1} + (G_{k-1} - K_k C_k G_{k-1})d_{k-1} \\ &\quad - W_{k-1}\bar{r}_{k-1} + (I - K_k C_k)\omega_{k-1} - K_k v_k. \end{aligned}$$

Noting that  $\bar{r}_{k-1} = \tilde{M}_{k-1}G_{k-1}d_{k-1}$ , we obtain

$$\begin{aligned} (G_{k-1} - K_k C_k G_{k-1})d_{k-1} - W_{k-1}\bar{r}_{k-1} \\ = [I - K_k C_k - W_{k-1}\tilde{M}_{k-1}]G_{k-1}d_{k-1}. \end{aligned} \quad (24)$$

Inserting (21) and (22) into (24), we can obtain (23) by noting that  $I - K_k C_k - W_{k-1} \tilde{M}_{k-1} = O$ . This completes the proof.

Lemma 2 shows that, for the gain  $K_k$  given in Eqs. (21)–(22), if  $A_{k-1} - K_k C_k A_{k-1}$  is stable, the error of the developed filter in Li (2013) will be stable as time  $k$  increases. In addition, the estimation error Eq. (23) shares the same structure as that of Eq. (17), upon which we can conclude that the filter (20)–(22) falls into the filter family with the generic linear structure Eq. (8).

We now consider the error covariance matrix  $P_{k|k}$ .

**Theorem 3.** Let  $\tilde{P}_{k|k}$  denote the error covariance matrix of any filter  $\hat{x}_k(Y_k)$  based on the sequence of measurements  $Y_k = \{y_0, y_1, \dots, y_k\}$ . Then for linear system (1) with partially observed inputs (2), we have  $\tilde{P}_{k|k} \geq P_{k|k}$ , where  $P_{k|k}$  is given by Eq. (21).

**Proof.** By definition, the conditional covariance matrix of the estimate  $\hat{x}_k(Y_k)$  for given  $Y_k$  is

$$\tilde{P}_{k|k} = E\{[x_k - \hat{x}_k(Y_k)][x_k - \hat{x}_k(Y_k)]^T | Y_k\}.$$

It is easy to verify the following identity:

$$\begin{aligned} \tilde{P}_{k|k} &= E\{[x_k - \hat{x}_{k|k} + \hat{x}_{k|k} - \hat{x}_k(Y_k)] \\ &\quad \times [x_k - \hat{x}_{k|k} + \hat{x}_{k|k} - \hat{x}_k(Y_k)]^T | Y_k\} \\ &= P_{k|k} + E\{[\hat{x}_{k|k} - \hat{x}_k(Y_k)][\hat{x}_{k|k} - \hat{x}_k(Y_k)]^T | Y_k\} \\ &\quad + E\{[x_k - \hat{x}_{k|k}][\hat{x}_{k|k} - \hat{x}_k(Y_k)]^T | Y_k\} \\ &\quad + E\{[\hat{x}_{k|k} - \hat{x}_k(Y_k)][x_k - \hat{x}_{k|k}]^T | Y_k\}. \end{aligned}$$

Li (2013) shows that the estimated state vector  $\hat{x}_{k|k}$  in Eq. (20) is the posterior mean conditional on the sequence of measurements  $Y_k = \{y_0, y_1, \dots, y_k\}$ . Hence, we have  $E\{x_k | Y_k\} = \hat{x}_{k|k}$  and the last two terms on the right-hand side of the above equation vanish, i.e.

$$\tilde{P}_{k|k} = P_{k|k} + E\{[\hat{x}_{k|k} - \hat{x}_k(Y_k)][\hat{x}_{k|k} - \hat{x}_k(Y_k)]^T | Y_k\}.$$

We thus conclude that  $\tilde{P}_{k|k}$  attains the minimum if and only if the second term of the right-hand side is equal to zero, i.e.  $\hat{x}_{k|k} = \hat{x}_k(Y_k)$ . This completes the proof.

Theorem 3 shows that the filter given by Eqs. (20)–(22) is optimal in the sense of both MMSE and minimum covariance matrix. This result is not only important in its own right but also useful in the subsequent asymptotic stability analysis.

### 5. Asymptotic stability

In this section, we discuss the asymptotic stability of the filter developed in Li (2013) for time-invariant system (1) and (2). We hence suppress the subscript  $k$  of matrices  $A_k, G_k, C_k, D_k, Q_k$  and  $R_k$ .

We note from Lemma 2 that the covariance matrix in Eq. (21) can be re-written as

$$\begin{aligned} P_{k|k} &= (A - K_k CA)P_{k-1|k-1}(A - K_k CA)^T \\ &\quad + (I - K_k C)Q(I - K_k C)^T + K_k R K_k^T. \end{aligned} \quad (25)$$

Under the condition given in Theorem 1 and in conjunction with Theorem 3 that the covariance matrix of the filter given by Eqs. (20)–(22) is optimal, it can be shown that the covariance matrix  $P_{k|k}$  in Eq. (25) is bounded for all  $k$  and for an arbitrary bounded initial covariance  $P_{0|0}$ . On the basis of boundedness of  $P_{k|k}$  and inspired by the approaches in Anderson and Moore (1979) and Fang and Callafon (2012), we can further show the following result. The proof is omitted here for lack of space and is available upon request.

**Theorem 4.** If the condition in Theorem 1 is satisfied and  $(A, Q^{\frac{1}{2}})$  is stabilizable, then the covariance matrix  $P_{k|k}$  of the filter (20)–(22) will converge to a unique fixed positive semi-definite matrix  $\bar{P}$  for any given initial condition  $P_{0|0}$ . Moreover, with the associated limiting gain matrices  $\bar{K}$ , the time-invariant filter is also stable, i.e. all the eigenvalues of  $A - \bar{K}CA$  satisfy  $|\lambda(A - \bar{K}CA)| < 1$ .

It is of interest to compare the asymptotic stability condition obtained with partially observed inputs to the asymptotic stability conditions when the complete information on the inputs is available and when the inputs are completely unknown. This is investigated in the following theorem. It shows that Theorem 4 provides a unified approach to accommodating asymptotic stability conditions in a variety of filtering scenarios.

**Theorem 5.** The asymptotic stability condition for the filter (20)–(22) in Theorem 4 reduces to: (a) the asymptotic stability condition of the classical Kalman filter when the complete information on the inputs is available, i.e.  $D$  is invertible; and (b) the asymptotic stability condition of the filter with unknown inputs, i.e.  $D$  is an empty matrix.

**Proof.** First, we note that when matrix  $D$  is invertible, the asymptotic stability condition reduces to: (a)  $(A, C)$  is detectable; and (b)  $(A, Q^{\frac{1}{2}})$  is stabilizable. These are the asymptotic stability conditions of the classical Kalman filter (see, e.g. Anderson & Moore, 1979).

Next, when no information on the inputs is available, we know from Theorem 2 that Eq. (10) in Theorem 1 reduces to Eq. (19). In addition, condition  $(A, Q^{\frac{1}{2}})$  along with  $R > 0$  (and hence  $R^{\frac{1}{2}} > 0$ ) can guarantee that the matrix below has a full row-rank, i.e.,

$$\text{rank} \left( \begin{bmatrix} A - e^{jw} I & G & Q^{\frac{1}{2}} & O \\ e^{jw} C & O & O & R^{\frac{1}{2}} \end{bmatrix} \right) = n + p, \quad \forall w \in [0, 2\pi]. \quad (26)$$

Eqs. (19) and (26) are identical to the asymptotic stability condition for the filter with unknown inputs (Darouach & Zasadzinski, 1997). This completes the proof.

### 6. Conclusions

This paper has established existence and asymptotic stability conditions for the recently developed filter with partially observed inputs in Li (2013). The obtained existence and asymptotic stability conditions provide a unified approach to accommodating a variety of filtering scenarios as its special cases, including the important Kalman filtering and the unknown input filtering problems. In practice, information on inputs and/or outputs may sometimes be only partially available in applications. This work takes a further step towards the development of more generic filtering techniques where different levels of information are exploited.

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### Appendix. Proof of Lemma 1

$$\begin{aligned} &\text{rank} \left( \begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \right) \\ &= \text{rank} \left( \begin{bmatrix} zI_n - A_k & -G_k \\ C_{k+1} & O \\ O & D_k \end{bmatrix} \begin{bmatrix} I_n & O \\ O & [F_{0k} D_k^T] \end{bmatrix} \right) \end{aligned}$$

$$\begin{aligned}
&= \text{rank} \left( \begin{bmatrix} zI_n - A_k & -G_k F_{0k} & -G_k D_k^T \\ C_{k+1} & O & O \\ O & D_k F_{0k} & D_k D_k^T \end{bmatrix} \right) \\
&= \text{rank} \left( \begin{bmatrix} zI_n - A_k & -F_k \\ C_{k+1} & O \end{bmatrix} \right) + \text{rank}(D_k D_k^T).
\end{aligned}$$

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