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On varieties of Fano type and singularities in positive characteristic

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Declaration of originality

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Abstract

In this dissertation we explore the birational geometry of higher-dimensional algebraic varieties in positive characteristic, with a special emphasis on the study of varieties of Fano type and the singularities of the Minimal Model Program.

In the first two chapters we prove that many classical statements of the Minimal Model Program do not hold in characteristic $p > 0$ by exhibiting explicit counterexamples: we construct a klt del Pezzo surface violating the Kawamata-Viehweg vanishing theorem and Kawamata log terminal threefold singularities which are not rational in characteristic three, purely log terminal pairs with non-normal centres and terminal Fano varieties with non-vanishing intermediate cohomology in all positive characteristic.

Then, we discuss a joint work with H. Tanaka where we study the geometry of threefold del Pezzo fibrations in positive characteristic. This is done by carrying out a detailed analysis of surfaces of del Pezzo type over an imperfect field k: we bound the torsion index of numerically trivial line bundles and we show geometric integrality of such surfaces in characteristic at least seven. On the arithmetic side, we show that a surface of del Pezzo type over a C_1-field admits a closed point with purely inseparable residue field of bounded degree.

Finally, in the last chapter we prove a refinement of the Base point free Theorem for nef Cartier divisors of numerical dimension at least one on Kawamata log terminal threefolds in large characteristic.

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Con quella faccia un po' così
quell'espressione un po' così
che abbiamo noi
che abbiamo visto Genova.
PAOLO CONTE

Contents

1 Introduction

The discipline of Algebraic Geometry, whose origin can be traced back to the work on conics of Apollonius of Perga in the hellenistic period, studies the possible shapes of *algebraic varieties* - geometric objects defined by polynomial equations.

The ultimate goal of Algebraic Geometry lies in finding a classification theorem for algebraic varieties. As Hartshorne points out in the introduction of his influential book [Ha77, Section I.8] one can split the classification problem in two parts. First, we want to achieve a *birational* classification. This means to construct a 'minimal' representative in each birational equivalence class. The adjective minimal suggests that the internal geometry of the variety constructed should be as simple as possible. Once this is achieved, one can hope to construct a 'moduli space' parametrising minimal varieties.

In this thesis, we will mostly concentrate on the birational aspect of the classification problem, especially from the modern point of view of the Minimal Model Program. The first step in the birational classification of algebraic varieties were established by the Italian school of Algebraic Geometry of Castelnuovo, Enriques and Severi (whose period span the years 1885-1935), who completely settled the case of algebraic surfaces.

Despite the very early results on algebraic surfaces, it was not until the work of S. Mori in the eighties that a general framework to deal with the higher dimensional case was established. The basic idea is that most of the information on the birational geometry of an algebraic variety X is encoded in the positivity properties of the canonical line bundle $\omega_X := \mathcal{O}_X(K_X) = \wedge^{\dim X}\Omega^1_X$, where Ω^1_X is the cotangent bundle of X. The Minimal Model Program (in short, MMP) is an algorithm which uses K_X as a compass to construct the desired simple birational representative: starting from a smooth projective variety, the MMP performs birational transformation which increase the positivity of K_X. The following is the leading conjecture in modern birational geometry:

Conjecture 1.1 (Minimal Model Conjecture)**.** *Let W be a smooth projective variety over an algebraically closed field k. Then there exists a birational contraction map $\phi\colon W \dashrightarrow X$ to a projective variety with 'mild' (terminal) singularities such that one of the following alternative holds.*

1. *There exists a projective morphism $f\colon X \to Z$ with connected fibres such that $\dim(Z) < \dim(X)$, $-K_X$ is f-ample and the relative Picard rank $\rho(X/Z)$ is one. In this case we say f is a Mori fibre space.*
2. *The canonical divisor K_X is nef. Then one of the following alternatives hold:*
   - *(a) there exists a projective morphism $f\colon X \to Z$ with connected fibres such that $\dim(Z) < \dim(X)$ and $K_X \sim_{\mathbb{Q}} f^*A$ for an ample divisor A on Z. In this case we say that f is a Calabi-Yau fibre space;*
   - *(b) there exists a projective birational morphism $f\colon X \to X_{\mathrm{can}}$ such that X_{can} has canonical singularities and $K_{X_{\mathrm{can}}}$ is ample. We say that X_{can} is a canonically polarised variety.*

The conjecture can be thought as a uniformisation statement for higher dimensional algebraic varieties. It essentially states that that every algebraic variety can be birationally constructed starting from three type of building blocks with pure geometry: Fano varieties (for which K_X is negative), Calabi-Yau varieties (for which K_X is trivial) and canonically polarised varieties.

The minimal model conjecture for varieties of dimension three was solved in the eighties by Mori and later generalised to the logarithmic case (see [Mor88], [Sho93] and the book [Kol$^+$92]). However, Mori's proof relied on special properties of three-dimensional varieties and his approach revealed to be difficult to generalise to higher dimension. In 2006, in the seminal article [BCHM10] Birkar, Cascini, Hacon and M$^{\mathrm{c}}$Kernan proved the minimal model conjecture in the case of varieties of general type over any field of characteristic zero, building on earlier work of Shokurov and Hacon-M$^{\mathrm{c}}$Kernan (see the monograph [Cor07]).

All the results explained in the previous paragraph have been obtained for varieties over fields of characteristic zero. So, what happens for fields of positive characteristic? Answering this question is particularly interesting because, while later developments in the MMP in characteristic zero relied on the Kodaira vanishing theorem, the first results in Mori theory (such as the cone theorem for smooth varieties) were obtained using the bend-and-break technique, which ultimately depends on reduction to positive characteristic and the use of the Frobenius morphism.

In recent years, starting from the work of Keel [Kee99], it has been realised that the Frobenius morphism can sometimes act as a replacement to the failure of vanishing theorems. More precisely, techniques from the theory of F-splitting have merged in the field of birational geometry with tremendous success. In [Tan14, Tan15, Tan18a], Tanaka developed the full machinery of the logarithmic MMP for surfaces over excellent schemes. In [HX15], the authors proved the existence of the flips for threefolds in characteristic

$p > 5$ using tools from the theory of F-singularities. This was the starting point for the development of the threefold logarithmic MMP in characteristic at least five (see [CTX15, Bir16, BW17, HNT19]).

Motivated by these recent developments, my Ph.D. research focused on reaching a better understanding of the birational geometry of threefolds and higher-dimensional varieties in positive characteristic. I was particularly motivated by the following questions:

1. Which classical statements of the Minimal Model Program over the complex numbers are still valid in positive characteristic?

2. Which pathological phenomena can appear on threefolds in low characteristic?

3. Which properties of the singularities of the Minimal Model Program are still true in positive characteristic?

4. What can we say about the structure of the end-products of the MMP? For example, can we classify the bad behaviour of fibres of Mori fibre spaces and of Calabi-Yau fibre spaces for threefolds?

The above problems are all strictly connected to properties of varieties of Fano type in positive characteristic. In the following sections, we summarise our own original contribution to the theory of minimal models in positive characteristic.

1.1. Fano varieties and singularities of the MMP in characteristic $p > 0$

The Kodaira vanishing theorem, together with its vast generalisation known as Kawamata-Viehweg vanishing theorem, is ubiquitous in the modern approach to birational geometry over a field of characteristic zero. The most important results of the Minimal Model Program, such as the base point free theorem ([KM98, Theorem 3.3]), the finite generation of the canonical ring and the existence of flips ([BCHM10]) all rely on intensive applications of vanishing theorems.

Unfortunately, Raynaud showed in [Ray78] that vanishing theorems à la Kodaira do not hold in general for varieties over fields of positive characteristic. This failure is one of the main obstacle to the implementation of the MMP algorithm in positive characteristic. Since Raynaud's example, a lot of research in positive characteristic birational geometry focused on constructing examples of pathological varieties violating vanishing theorems and on studying their geography (see for example [Eke88, SB91, Muk13, dCF15]).

In this context, varieties of Fano type over fields of characteristic $p > 0$ violating Kodaira-type vanishing theorems revealed to be so elusive to construct that Kollár even speculated whether such examples exist or not (see [Kol13, Remark 3.5]). Finding Fano varieties violating the Kodaira vanishing theorem has wide-ranging consequences: it has been long understood that vanishing theorems for Fano varieties are strictly intertwined with the study of the singularities of the MMP and with extension theorems.

In Chapters 3 and 4, we investigate these ideas by constructing Fano varieties that violate vanishing theorems. Then, we use such examples to exhibit pathological behaviour of singularities and the failure of extension theorems in positive characteristic.

1.1.1. Varieties of Fano type and Kodaira vanishing in positive characteristic

In the recent article [Tot17], Totaro constructs a striking sequence of examples of smooth Fano varieties violating the Kodaira vanishing theorem in every positive characteristic. Before Totaro's examples, very few classes of varieties of Fano type violating Kawamata-Viehweg vanishing had been constructed and exclusively in small characteristic (see [LR97, CT, Ber17, Kov18]).

An important feature of Totaro's examples is that the dimension of the Fano varieties he constructs grows with the characteristic. Thus it is natural to ask the following

Question 1.2 ([Tot17, Section 6]). *Fix a natural number d. Does it exist a positive integer $p_0(d)$ such that varieties of Fano type of dimension d over a perfect field k of characteristic $p \geq p_0(d)$ satisfy Kawamata-Viehweg vanishing?*

The above question seems to be strictly related to the boundedness problem for Fano varieties over $\mathrm{Spec}(\mathbb{Z})$. While in higher dimension Question 1.2 is widely open, the case of surfaces has been settled by a recent work of Cascini-Tanaka-Witaszek.

Theorem 1.3 (cf. [CTW17, Theorem 1.2]). *There exists a positive integer $p_0(2)$ such that the following hold. Let X be a surface of del Pezzo type over a perfect field k of characteristic $p > p_0(2)$. Let B be a boundary divisor such that (X, B) is klt and let D be a Weil $\mathbb{Q}$-Cartier divisor such that $D - (K_X + B)$ is big and nef. Then*

$$H^1(X, \mathcal{O}_X(D)) = 0.$$

We actually know more in the case of surfaces: the Kawamata-Viehweg vanishing holds for smooth del Pezzo surfaces over any perfect field (see [CT18, Appendix A]) and even for regular del Pezzo surfaces over an imperfect field of characteristic $p > 3$ (see [Das, Theorem 1.1]).

Unfortunately, the costant $p_0(2)$ obtained in 1.3 is not explicit since it is constructed using boundedness results for surfaces of del Pezzo type in mixed characteristic and Noetherian induction.

In [CT, Theorem 4.2], the authors construct a family of klt del Pezzo surfaces in characteristic two violating Kawamata-Viehweg vanishing, thus showing $p_0(2) > 2$. In [Ber17], we produce a counterexample in characteristic three. It was the first example in literature of a variety of Fano type violating Kawamata-Viehweg in characteristic larger than two.

Theorem 1.4 (Theorem 3.1). *Over any field k of characteristic three, there exists a projective normal k-surface T such that*

1. *T has klt singularities and $-K_T$ is ample;*
2. *$\rho(T) = 1$;*
3. *there exists an ample $\mathbb{Q}$-Cartier Weil divisor A on T such that $H^1(T, \mathcal{O}_T(-A)) \neq 0$.*

Another well-known property of varieties of Fano type X in characteristic zero is the vanishing of $H^i(X, \mathcal{O}_X)$ for $i > 0$ (and thus $\chi(X, \mathcal{O}_X) = 1$). However, the vanishing of the coherent cohomology of the structure sheaf of varieties of Fano type in positive characteristic (especially of the first cohomology group) is still an open question.

The only known result in all dimensions is the beautiful theorem of Esnault [Esn03], where she proves that smooth Fano varieties in positive characteristic satisfy the vanishing of Witt-vector cohomology $H^i(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for $i > 0$. This vanishing was extended to the case of threefolds of Fano type in characteristic at least five by Gongyo-Nakamura-Tanaka (see [GNT]). As for coherent cohomology, it is well-known that surfaces of del Pezzo type have vanishing cohomology group and if X is a smooth Fano threefold, Shepherd-Barron shows in [SB97] that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

We prove that there exists mildly singular Fano varieties with non-vanishing intermediate cohomology:

Theorem 1.5 (Theorem 4.9). *Let k be a field of characteristic $p \geq 3$. Then there exists a Fano variety W with terminal singularities of dimension $2p + 2$ over k such that*

$$H^2(W, \mathcal{O}_W) \neq 0.$$

Let us remark that the existence of smooth Fano varieties with non-vanishing intermediate cohomology is still unknown.

1.1.2. Singularities of the MMP in positive characteristic

We shift our attention to the study of singularities of the Minimal Model Program in positive characteristic.

One of the most important applications of Kawamata-Viehweg vanishing in characteristic zero is the proof, originally due to Elkik [Elk81], that Kawamata log terminal singularities (in short, klt) are Cohen-Macaulay and rational (see also [KM98, Theorem 5.22]). In characteristic $p > 0$, due to the failure of vanishing theorems, general cohomological properties of klt singularities are still largely unknown. However, according to a local-global principle, they are expected to be strictly related to vanishing theorems for varieties of Fano type.

As an instance of this principle, Hacon and Witaszek show in [HW17] that Kawamata log terminal threefold singularities are Cohen-Macaulay and rational over perfect fields of characteristic $p > p_0(2)$ as a consequence of the Kawamata-Viehweg vanishing theorem for log del Pezzo surfaces.

As for low characteristic, in [CT, Theorem 1.3] the authors give an example of a klt not Cohen-Macaulay threefold in characteristic two. Building on Theorem 1.4 and using a cone construction (see Section 2.3), we find the first examples of threefold klt singularities which is not rational in characteristic larger than two:

Theorem 1.6 (Theorem 3.2). *Let k be a field of characteristic three. Then there exists a $\mathbb{Q}$-factorial klt threefold singularity X which is not Cohen-Macaulay.*

In [Tot17, Corollary 2.2], Totaro shows that there exists (not $\mathbb{Q}$-factorial) terminal singularities which are not Cohen-Macaulay in every positive characteristic by taking the cone over the Fano varieties he constructs. Recall that rational singularities implies Cohen-Macaulay, thus the above examples show also that klt singularities are not necessarily rational in positive characteristic.

Another important property of singularities in characteristic zero due to Kawamata-Viehweg vanishing is the normality of the centre S of a purely log terminal (plt) pair $(X, S + B)$ (see [KM98, Proposition 5.51]). The normality of plt centres is a crucial ingredient in the proof of the existence of pl-flips in characteristic zero (see [HM07] and [BCHM10]). It is also one of the key step for the MMP for threefolds: in [HX15, Theorem 3.1.1 and Proposition 4.1] the authors show the normality of plt centres for threefolds over an algebraically closed field of characteristic $p > 5$ using tools from the theory of F-singularities. This is the first step of their proof of the existence of pl-flips for threefolds ([HX15, Theorem 4.12]).

One might thus be led to conjecture that plt centres are normal also in positive characteristic. Unfortunately, this is not the case: in [CT, Theorem 1], Cascini and Tanaka construct an example of a plt threefold with non-normal centre in characteristic two. Inspired by their work, in [Ber18] we use Totaro's examples to construct non-normal plt centres for every prime $p \geq 3$

Theorem 1.7 (Theorem 4.4). *Let k be any field of characteristic $p \geq 3$. Then there exists a log pair (Z, S) such that*

1. *Z is an affine variety over k with terminal singularities of dimension $2p + 2$ and S is a prime divisor,*
2. *(Z, S) is a purely log terminal pair with $K_Z + S$ Cartier,*
3. *S is not normal.*

In Corollary 4.8 we show that the above theorem shows that the 'Main lifting lemma' of Hacon and M^{c}Kernan (see [HM07, Theorem 5.4.21]) may fail in positive characteristic.

In Section 4.4, we present a further pathology for singularities in positive characteristic. Over a field of characteristic zero, a well-known result of Ambro and Fujino asserts that the base of a Mori fibre space has klt singularities (see [Fu99, Corollary 4.6] and [Amb05, Theorem 0.2]). Their proof uses positivity theorems which are obtained via Hodge theory. One might thus wonder if a similar statement hold in positive characteristic. Recently Tanaka showed in [Tana, Theorem 1.1] that the answer is negative: he constructs examples of klt varieties admitting a Mori fibre spaces onto a basis with non-klt singularities in characteristic two and three. Building on work on wild quotient singularities of Yasuda ([Yas17]), we construct examples in larger characteristic:

Theorem 1.8 (Theorem 4.11)**.** *Let k be a field of characteristic $p \geq 5$. Then there exists a projective contraction $f\colon X \to Y$ of normal k-varieties such that*

1. *X is a $\mathbb{Q}$-factorial terminal quasi-projective variety of dimension $p+3$;*
2. *Y is a $\mathbb{Q}$-factorial affine variety of dimension three which is not log canonical;*
3. *$\rho(X/Y) = 1$ and $-K_X$ is f-ample, equivalently f is a Mori fibre space.*

1.2. Generic fibres of del Pezzo fibrations in positive characteristic

The minimal model conjecture predicts that an arbitrary variety is birational to either a Mori fibre space, a Calabi-Yau fibre space or a canonically polarised variety. Thus, from the point of view of the classification theory of varieties, it is particularly important to understand the properties of Mori fibre spaces and Calabi-Yau fibre space.

In positive characteristic, the study of fibre spaces has an additional difficulty: the failure of the generic smoothness theorem. Indeed, given a fibration $f\colon V \to B$ (i.e. a proper morphism between algebraic varieties such that $f_*\mathcal{O}_V = \mathcal{O}_B$ and $\dim(B) < \dim(V)$), the general fibre of f may be non-normal or even non-reduced. Such fibrations are often called in literature *wild fibrations*.

One can interpret the bad behaviour of the general fibres of fibrations f in terms the geometric properties of the generic fibre $V_{k(B)} := V \times_B \operatorname{Spec}(K(B))$, which is now a variety defined over an *imperfect field*. Let us recall here the difference between regular and smooth varieties over non-perfect fields and let us explain why it is important to consider regular varieties. We say that a variety V of dimension n is *regular* if for all points x in V the local rings $\mathcal{O}_{V,x}$ are regular, *i.e.* $\dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{V,x}$ where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{V,x}$. If V is a smooth variety over k, then V is regular but the converse is true only when k is a perfect field. However regularity is preserved by localisation and thus if V is a k-smooth variety admitting a fibration $f\colon V \to B$, the generic fibre $V_{k(B)}$ remains regular. The following shows exactly an example where $V_{k(B)}$ is not smooth.

Example 1.9. Let us show with a concrete example how general fibres of a fibration between smooth varieties can have wild behaviour in positive characteristic. Let k be an algebraically closed field of characteristic $p > 0$ and consider the p-Fermat hypersurface (see [MS03, Sch10]):

$$V := \{x_0 y_0^p + x_1 y_1^p + \dots x_n y_n^p = 0\} \subset \mathbb{P}^n_x \times \mathbb{P}^n_y.$$

The variety V is smooth over k and the generic fibre $V_{k(\mathbb{P}^n_x)}$ is a regular variety. However, all the fibres of the projection $\pi_x \colon V \to \mathbb{P}^n_x$ are non-reduced and the geometric generic fibre $V_{\overline{k(\mathbb{P}^n_x)}}$ is a non-reduced scheme.

One can hope to bound the possible pathological behaviour to small characteristic if further geometric conditions are imposed on the generic fibre. This is suggested by our experience with the classification theorem for algebraic surfaces over algebraically closed fields of positive characteristic (see [BM76]):

Theorem 1.10. *Let k be an algebraically closed field of characteristic $p > 0$. Let S be a projective surface over k and let $\pi \colon S \to C$ be a projective contraction onto a normal curve C. Then the following hold.*

1. *If $\pi \colon S \to C$ is a Mori fibre space, the general fibre of π is a smooth rational curve.*
2. *If $\pi \colon S \to C$ is a Calabi-Yau fibre space, then the general fibre is either a smooth elliptic curve or it is a cuspidal cubic. The latter case, known as quasi-elliptic fibrations, can happen only if $p \in \{2, 3\}$.*

The proof of Theorem 1.10 can be divided in two steps. First, one proves that the generic fibre is geometrically integral (see for example [Bad01, Corollary 7.3]). Then the Tate's genus-change formula for curves (see [Tat52, Sch09]) gives a bound on the wild behaviour of the generic fibre. For a more detailed, yet concise, discussion of the proof we refer to [Lie13, Section 5.1].

After the development of the MMP for threefolds, it is natural to look for a characterisation analogue to Theorem 1.10 on wild fibres of threefold Mori fibre spaces and Calabi-Yau fibre spaces.

Recently, Tate's formula has been generalised to the higher dimensional case as an adjunction formula for the canonical divisors under purely inseparable base change of the base fields (see [Tan18b, PW, Tanb]). One can thus hope to generalise the strategy of the proof of Theorem 1.10 to the higher dimensional case. The case of threefold Mori fibre spaces has especially received particular attention in recent work (see [PW, FS18, BT19]). Let us summarise the results in the literature and explain our own contribution.

In the case of a Mori fibre spaces of relative dimension one (the so-called conic bundles) the characterisation of wild fibres is classical: either the general fibre is a smooth rational curve or it is a double conic and the latter case can happen only for $p = 2$.

In the case of a contraction $\pi \colon V \to B$ of relative dimension two (the so-called del Pezzo fibrations), the situation is much more subtle. In [Kol91, Remark 1.2.1], Kollár asks whether the general fibre of a threefold del Pezzo fibration is normal. Since terminal singularities are regular in codimension two (see [Kol13, Corollary 2.30]), this is equivalent to show geometric normality of regular del Pezzo surfaces over imperfect fields.

As a first positive result on Kollár's question, Patakfalvi and Waldron show that no pathologies can appear on regular del Pezzo surfaces in sufficiently large characteristic.

Theorem 1.11 ([PW, Theorem 1.5]). *Let k be a field of positive characteristic $p \geq 5$. Let X be a regular projective del Pezzo surface over k such that $H^0(X, \mathcal{O}_X) = k$. Then X is geometrically normal.*

In a recent article, Fanelli and Schröer completely settle Kollár's question, proving that the general fibre of a Mori fibre space onto a curve is normal also in low characteristic.

Theorem 1.12 ([FS18, Theorem 14.1]). *Let k be a field of positive characteristic $p > 0$, let $\overline{k}$ be an algebraic closure of k. Suppose such that $[k : k^p] \leq p$, where $k^p = \left\{x \in \overline{k} \mid x^p \in k\right\}$ (e.g. k is the function field of a curve over an algebraically closed field). Let X be a regular projective del Pezzo surface over k such that $H^0(X, \mathcal{O}_X) = k$. If $\rho(X) = 1$, then X is geometrically normal, i.e. $X \times_k \overline{k}$ is normal.*

In both of the above results, the regularity (or at least the Gorenstein) assumption on X plays a crucial role. However, from the perspective of the log MMP, it is natural to investigate the properties of general fibres of $(K_X + \Delta)$-Mori fibre spaces, where (X, Δ) is a Kawamata log terminal threefold pair. In the log case, more pathologies on general fibres have been constructed in literature: in [Sch07] and [Tana] the authors exhibit surfaces of del Pezzo type with non-vanishing irregularity over any imperfect field of characteristic $p = 2, 3$.

Motivated by the previous results, in a joint work with H. Tanaka (see [BT19]), we carry a systematic study of the possible pathological behaviour on surfaces of del Pezzo type over imperfect fields. The following is the main result of Chapter 5. Let us stress that the most interesting case covered by the following result is when k is not perfect or X does not admit a smooth birational model.

Theorem 1.13 (Theorem 5.3, Theorem 5.4, Theorem 5.7). *Let k be a field of positive characteristic $p > 0$. Let X be a surface of del Pezzo type over k such that $k = H^0(X, \mathcal{O}_X)$. Then the following hold.*

1. *Let $p \geq 7$. Then X is geometrically integral and $H^1(X, \mathcal{O}_X) = 0$. If k is a C_1-field, then X has a k-rational point.*

2. *Let $p \in \{3, 5\}$. If L is a Cartier divisor on X such that $L \equiv 0$, then $pL \sim 0$. If k is a C_1-field, then X has a $k^{1/p}$-rational point.*

3. *Let $p = 2$. If L is a Cartier divisor on X such that $L \equiv 0$, then $4L \sim 0$. If k is a C_1-field, then X has a $k^{1/4}$-rational point.*

In Chapter 5, we also present various applications to the global geometry of del Pezzo fibrations. More precisely, we prove a bound on the torsion index of relatively numerical trivial Cartier divisors on del Pezzo threefold fibration and the existence of a purely inseparable section of bounded degree (see Theorem 5.1 and Theorem 5.2). Unfortunately, we still do not whether the general fibre of klt del Pezzo fibration is normal in characteristic at least seven.

1.3. On the base point free theorem for threefolds in large characteristic

Let X be a normal projective variety over k and let L be a nef Cartier divisor on X. One of the central questions in birational geometry is to find sufficient conditions on L to ensure it is semi-ample.

Over a field of characteristic zero, the most important semi-ampleness statement is the Base point free theorem developed by Kawamata, Kollár, Mori, Reid and Shokurov:

Theorem 1.14 ([KM98, Theorem 3.3])**.** *Let (X, Δ) be a projective Kawamata log terminal pair over a field of characteristic zero. Let L be a nef Cartier divisor such that $nL-(K_X+\Delta)$ is a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for some $n > 0$. Then the complete linear system $|mL|$ has no base points for $m \gg 0$.*

The proof uses heavily vanishing theorems that are not in general available over fields of positive characteristic. Since the base point free theorem is the first step needed in order to run the MMP, it is a natural question to ask whether it remains valid in positive characteristic.

In [Kee99], using special properties of the Frobenius morphism, Keel proves a base point free theorem for big and nef Cartier divisors on threefolds in positive characteristic, but working in the category of algebraic spaces (see [Kee99, Theorem 0.5]).

Using the Minimal Model Program for threefolds in characteristic $p > 5$, Birkar and Waldron extend Keel's result to nef (but not necessarily big) divisors and show that the contraction morphism is also projective:

Theorem 1.15 ([BW17, Theorem 1.2])**.** *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a projective Kawamata log terminal threefold pair over k. Let L be a nef Cartier divisor on X and assume that $nL-(K_X+\Delta)$ is a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for some $n > 0$. Then $|mL|$ has no base points for m sufficiently large and sufficiently divisible.*

Let us note that Theorem 2.5 is weaker than the classical base point free theorem: a priori we have to impose m to be sufficiently divisible to conclude that $|mL|$ is base point free. This divisibility condition is indeed necessary in general as shown by Tanaka in [Tana, Theorem 1.2].

However Tanaka's examples are in characteristic two and three and thus one can ask if the divisibility condition of Theorem 2.5 can be removed for threefolds defined over fields of large characteristic. This is especially interesting if one is interested in effectivity results.

In Chapter 6, we use the Kawamata-Viehweg vanishing theorem for log del Pezzo surfaces (see Theorem 1.3) and techniques from the Minimal Model Program to prove the stronger form of the base point free theorem for nef Cartier divisors of numerical dimension at least one in large characteristic.

Theorem 1.16 (Theorem 6.1)**.** *There exists an integer $p_0 \geq 5$ such that the following holds. Let k be a perfect field of characteristic $p > p_0$. Let (X, Δ) be a projective klt threefold log pair over k. Let L be a nef Cartier divisor on X such that $nL - (K_X + \Delta)$ is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for some $n > 0$. Suppose that the numerical dimension $\nu(L)$ is at least one. Then the linear system $|mL|$ is base point free for $m \gg 0$.*

2

Preliminaries

2.1. Notations and conventions

Throughout this thesis, k will denote an arbitrary field.

- For a field k, we denote $\overline{k}$ (resp. k^{sep}) an algebraic closure (resp. a separable closure) of k. If k is of characteristic $p > 0$, then we set $k^{1/p^\infty} := \bigcup_{e=0}^\infty k^{1/p^e} = \bigcup_{e=0}^\infty \{x \in \overline{k} \mid x^{p^e} \in k\}$.

- We say X is a k-variety, or simply a *variety* whenever the field k is clear from the context, if X is an integral scheme that is separated and of finite type over k. We say that X is a *curve* over k or a k-*curve* (resp. a *surface* over k or a k-*surface*, resp. a *threefold* over k) if X is a k-variety of dimension one (resp. two, resp. three). We denote by $K(X)$ its field of rational functions.

- Let $f\colon X \to Y$ be a projective morphism between normal varieties. We say that f is a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Y$.

- For a scheme X, its *reduced structure* X_{red} is the reduced closed subscheme of X such that the induced morphism $X_{\text{red}} \to X$ is surjective.

- If $k \subset k'$ is a field extension and X is a k-scheme, we denote $X \times_{\text{Spec } k} \text{Spec } k'$ by $X \times_k k'$ or $X_{k'}$.

- Let X be a scheme of finite type over a field k and let $k \subset k'$ be an algebraic field extension. We denote by $X(k')$ the subset of X consisting of the closed points x such that there is a k-algebra homomorphism $k(x) \to k'$.

- Let X be a normal variety over k. We denote by K_X the canonical divisor class of X (for a more detailed discussion of canonical class on singular varieties see [Kol13, pages 7-8]).

- We say (X, Δ) is a *log pair* if X is a normal variety, Δ is an effective $\mathbb{Q}$-divisor and $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

- We say that a normal variety X is $\mathbb{Q}$-factorial if for any Weil divisor D there exists an integer m such that mD is a Cartier divisor.

- Let (X, Δ) be a log pair. Let $f\colon Y \to X$ be a proper birational morphism, where Y is a normal variety. Then we have

$$K_Y + \Delta_Y = f^*(K_X + \Delta), \text{ where } \Delta_Y = f_*^{-1}\Delta - \sum_{E_i |\, f-\text{except.}} a(E_i, X, \Delta)E_i$$

for some $a(E_i, X, \Delta) \in \mathbb{Q}$. The coefficients $a(E_i, X, \Delta)$ are called the discrepancies of the pair (X, Δ) and they can be considered as a measure of the singularities of the pair (X, Δ). We say that (X, Δ) is *Kawamata log terminal* (for short *klt*) if for every birational morphism $f\colon Y \to X$ we have $\lfloor \Delta_Y \rfloor \leq 0$ (or equivalently $\lfloor \Delta \rfloor \leq 0$ and $a(E_i, X, \Delta) > -1$). We say that the pair is *terminal* if $a(E_i, X, \Delta) > 0$. For a more detailed discussion on the singularities of the MMP we refer to [Kol13, Section 2.1].

- We say that X is a *variety of Fano type* if there exists an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is a log Fano pair, i.e. (X, Δ) has Kawamata log terminal singularities and $-(K_X + \Delta)$ is ample. If X has dimension two, we say that it is a *surface of del Pezzo type* and the pair (X, Δ) is said to be a log del Pezzo pair.

- For an $\mathbb{F}_p$-scheme X we denote by $F_X\colon X \to X$ the *absolute Frobenius morphism.* The absolute Frobenius morphism acts on the scheme X as the identity on the topological space and as the p-th power map on the structure sheaf. For a positive integer e we denote by $F_X^e\colon X \to X$ the e-th iterated absolute Frobenius morphism.

- In a few occasions we will also need some notions from the theory of F-singularities. Let X be a normal scheme over a field k of characteristic $p > 0$ and suppose that the absolute Frobenius $F\colon X \to X$ is a finite morphism. Let Δ be an effective $\mathbb{Q}$-divisor on X. We say that (X, Δ) is *F-pure* if there exists $e > 0$ such that the morphism

$$\mathcal{O}_X \to F_*^e\mathcal{O}_X \hookrightarrow F_*^e\mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor)$$

locally splits in the category of $\mathcal{O}_X$-modules homomorphisms. We say that (X, Δ) is *strongly F-regular* if for any effective divisor E on X there exists $e > 0$ such that the morphism

$$\mathcal{O}_X \to F_*^e\mathcal{O}_X \hookrightarrow F_*^e\mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor + E)$$

locally splits in the category of $\mathcal{O}_X$-modules homomorphisms.

- Let X be a normal variety and let L be a reflexive sheaf on X. We denote by $L^{[m]} := (L^{\otimes m})^{\vee\vee}$. If L is a reflexive sheaf of rank one, we say that L is a Weil divisorial sheaf. In this case, there exists a Weil divisor D on X such that $L \simeq \mathcal{O}_X(D)$ and $L^{[m]} \simeq \mathcal{O}_X(mD)$. We say that L is $\mathbb{Q}$-Cartier if there exists an m such that $L^{[m]}$ is an invertible sheaf (or equivalently mD is a Cartier divisor).

- We recall some notions of positivity that appear frequently in birational geometry. Let $f\colon X \to Y$ be a proper morphism between integral schemes and let L be an invertible sheaf on X. We say tha L is *f-semi-ample* if there exists a positive integer $m > 0$ such that the canonical map

$$f^* f_* L^{\otimes m} \to L^{\otimes m} \tag{2.1}$$

is surjective. We say moreover it *f-ample* if the natural morphisms of schemes over Y

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}_Y(f_* L^{\otimes m}) \\ & \searrow \quad \swarrow & \\ & Y & \end{array} \quad . \tag{2.2}$$

is an embedding. We say that L is *f-big* if if there exists a positive integer $m > 0$ such that the canonical map of Equation 2.1 is generically surjective and the natural map of schemes over Y

$$\begin{array}{ccc} X & \dashrightarrow & \mathbb{P}_Y(f_* L^{\otimes m}) \\ & \searrow \quad \swarrow & \\ & Y & \end{array} \quad . \tag{2.3}$$

is generically an embedding. We say that L is *f-nef* if $L \cdot C \geq 0$ for any integral curve $C \in X$ such that $f(C)$ is zero dimensional. When $Y = \mathrm{Spec}(k)$ is the spectrum of a field, we say that the invertible sheaf L is semi-ample, ample, nef or big with no reference to the structure morphism $f\colon X \to \mathrm{Spec}(k)$. For more details and results on positivity in algebraic geometry we refer to the books [Laz04a] and [Laz04b].

- Let L be a Cartier divisor on a variety X over k. We define the *base locus* $\mathrm{Bs}(L)$ of L by

$$\mathrm{Bs}(L) := \bigcap_{s \in H^0(X,L)} \{x \in X \mid s|_x = 0\}.$$

In particular, $\mathrm{Bs}(L)$ is a closed subset of X.

- We recall the basic definition of the contractions that appear when running the Minimal Model Program. Let $f\colon X \to Y$ be a projective contraction between quasi-projective normal varieties. Let (X, Δ) be a klt pair. We say that f is a $(K_X + \Delta)$-*extremal negative contraction* if $\rho(X/Y) = 1$ and $-(K_X + \Delta)$ is f-ample. When the pair (X, Δ) is clear from the context, we will simply say it is an extremal negative contraction. If $\dim(Y) < \dim(X)$ we say that $f : X \to Y$ is a *Mori fibre space.* If $\dim(Y) = \dim(X)$ and the exceptional locus $\mathrm{Ex}(f)$ has pure codimension one, we say that f is a *divisorial contraction.* If the codimension of the exceptional locus $\mathrm{Ex}(f)$ is bigger than one we say that f is a *flipping contraction.* In the latter case, we say that $f^+ : X^+ \to Y$ is the *flip* if it is a projective birational morphism between normal varieties such that the exceptional locus $\mathrm{Ex}(f^+)$ has codimension at least two and the divisor $K_{X^+} + \Delta^+$ is an f^+-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, where Δ^+ is the strict transform of Δ. These notion can be extended to the dlt or log canonical setting. For more details on the MMP we refer to [KM98] and [BCHM10].

- Let k be an algebraically closed field. For a normal surface X over k and a canonical singularity $x \in X$ (i.e. a rational double point), we refer to the table at [Art77, pages 15-17] for the list of equations of type A_n, D_n^m and E_n^m. For example, we say that x is a canonical singularity of type A_n if the henselisation of $\mathcal{O}_{X,x}$ is isomorphic to $k\{x, y, z\}/(z^{n+1} + xy)$, where $k\{x, y, z\}$ denotes the henselisation of the local ring of $k[x, y, z]$ at the maximal ideal (x, y, z).

2.2. The Minimal Model Program in positive characteristic

In this section, we recall the results we will need from the Minimal Model Program for surfaces and threefolds. These will be used in Chapters 5 and 6.

2.2.1. Minimal model program for excellent surfaces

In [Tan18a], Tanaka proves the minimal model conjecture for surfaces over excellent base schemes. Since the proofs in minimal model theory often rely on induction on the dimension, this can be thought as a first step towards establishing both the MMP for arithmetic threefolds and the MMP for threefolds over imperfect fields.

For a reference to excellent schemes we refer to [Liu02, Section 8.2.3]. For us we will need the case where the excellent scheme B is either the spectrum of a local ring obtained by localising an algebraic variety over a possibly non-closed point or the spectrum of a (possibly imperfect) field.

Let us summarise some of Tanaka's results we will use in the following.

Theorem 2.1 (MMP for excellent surfaces, cf. [Tan18a, Theorem 1.1])**.** *Let B be a regular excellent scheme of finite type. Let $\pi : X \to S$ be projective B-morphism from a integral normal quasi-projective B-scheme X of dimension two to a quasi-projective B-scheme S. Let Δ be an effective $\mathbb{R}$-divisor. Assume that the pair (X, Δ) is log canonical. Then there exists a sequence of birational projective S-morphisms:*

$$X := X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{n-1}} X_n =: Y,$$

and divisors Δ_i and Δ_Y such that

$$\Delta_0 := \Delta, \Delta_i := (\varphi_{i-1})_*\Delta_{i-1}, \Delta_Y := \Delta_n$$

for which the following hold:

1. *each X_i is an integral normal scheme,*
2. *$\rho(X_i/X_{i+1}) = 1$ and $-(K_{X_i/B} + \Delta_i)$ is φ_i-ample.*
3. *(Y, Δ_Y) satisfies one of the following:*
   (i) $K_{Y/B} + \Delta_Y$ is nef over S,
   (ii) There is a projective S-contraction $\mu\colon Y \to Z$ such that $\dim(Z) < \dim(Y)$, $\rho(Y/Z) = 1$ and $-(K_{Y/B} + \Delta)$ is μ-ample.

The main ingredient in developing the minimal model program is the following base point free theorem on excellent surfaces.

Theorem 2.2 (Base point free theorem for excellent surface, cf. [Tan18a, Theorem 4.2])**.** *Let B be a regular excellent separated scheme of finite dimension. Let $\pi\colon X \to S$ be a projective B-morphism from a quasi-projective normal B-surface to a quasi-projective B-scheme. Assume that there are a effective $\mathbb{Q}$-divisor Δ on X and a nef Cartier divisor D on X which satisfy the following properties:*

1. *(X, Δ) is a Kawamata log terminal pair,*
2. *$D - (K_{X/B} + \Delta)$ is π-nef and π-big, and*
3. *if T is the affine spectrum of a field of positive characteristic k for the Stein factorisation $\pi : X \to T \to S$ of π, then either k is a perfect fied or $D \not\equiv 0$.*

Then there exists a positive integer b_0 such that $|bD|$ is π-free for any integer b with $b \geq b_0$.

Remark 2.3. The remaining case where k is an imperfect field and $D \equiv 0$ will be settled in Theorem 5.3.

Comment 2.4. Let us explain why, even if one is exclusively interested in birational geometry over algebraically closed fields of positive characteristic, it is natural to discuss birational geometry over more general base schemes. The reasons for this degree of generality are mainly the following two.

1. Singularity theory: it is often convenient to localise at non-closed point of a variety to prove statements in singularity theory. For example, the fact that terminal (resp. canonical) singularities are regular (resp. Gorenstein) in codimension two is due to the statement on terminal (resp. canonical) singularities of excellent surfaces (see [Kol13, Theorem 2.29 and Theorem 2.30]). Another example is the proof that general hyperplane sections of klt threefolds singularities are still klt in characteristic $p > 5$ (see [ST17, Main Theorem]). We will use this principle in the proof of Theorem 6.22.

2. Fibrations in positive characteristic: if one is interested in properties of the generic fibres of fibre spaces (and deduce some properties of general fibres), one has to study varieties defined over imperfect fields.

2.2.2. Minimal model program for threefolds in characteristic $p > 5$

In recent years, starting from the ground-breaking result of Hacon and Xu ([HX15]) on the existence of dlt flips for threefolds log pairs (X, Δ) with standard coefficients over an algebraically closed field k of characteristic $p > 5$, the Minimal Model Program for threefolds over fields of characteristic at least five has undergone a major development (see [CTX15, Bir16, BW17, GNT, HNT19])

Here we summarise the results of the Minimal Model Program for threefolds in characteristic $p > 5$ that are known in the literature. We start from the base point free theorem for klt pairs.

Theorem 2.5 ([BW17, Theorem 1.2])**.** *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a klt threefold pair over k. Let L be a nef Cartier divisor on X and assume that $L - (K_X + \Delta)$ is big and nef. Then L is semi-ample.*

Next, we recall the existence of flips:

Theorem 2.6 ([HX15, Theorem 1.1], [Bir16, Theorem 1.1], [HNT19, Theorem 3.2])**.** *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a log canonical threefold pair over k. Let $f : X \to Z$ be a $(K_X + \Delta)$-negative flipping contraction. Then its flip exists.*

We now discuss the problem of termination of sequence of flips. The most general results are the following two:

Theorem 2.7 (Special termination, cf. [GNT, Theorem 2.6], [Bir16, Proposition 5.5])**.** *Let k be a perfect field. Let Z be a quasi-projective variety over k. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt threefold pair, projective over Z. Then every sequence of $(K_X + \Delta)$-flips over Z terminates around $\Delta^{=1}$.*

The following theorem shows the termination for a special type of MMP, called *MMP with scaling*. This is an MMP where divisorial and flipping contraction are chosen with the aid of an additional effective divisor H, which act as a compass in the choice of the contractions. For an explanation of the algorithm of the MMP with scaling, we refer to [BCHM10, Section 3.10].

Theorem 2.8 (Termination of MMP with scaling, cf. [BW17, Theorem 1.6], [HNT19, Theorem 6.12])**.** *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a threefold log pair over k. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then*

1. *If (X, Δ) is klt and H is an ample divisor, then any $(K_X + \Delta)$-*MMP *over Z with scaling of H terminates.*

2. *If (X, Δ) is log canonical, then there exists a $(K_X + \Delta)$-*MMP *which terminates.*

2.3. Cone constructions

Taking cones over normal projective varieties will be the main technical tool used in Chapters 3 and 4 to construct pathological examples of singularities of the MMP in positive characteristic. The main idea to construct such pathologies consists in considering a log Fano variety X for which some form of Kodaira vanishing fails and then taking the cone over X.

2.3.1. Cones of $\mathbb{Q}$-Cartier Weil divisors

For the theory of cones on polarised algebraic varieties (X, L) where X is a projective normal variety and L is an ample Cartier divisor, we refer to [Kol13, Chapter 3]. In Chapter 3, we need to deal with a generalisation, originally due to Demazure [Dem88], of the cone constructions to the case of ample $\mathbb{Q}$-Cartier Weil divisors. We thus extend some of the results explained by Kollár to this setting.

Let (X, Δ) be a log pair of dimension n and let L be an ample $\mathbb{Q}$-Cartier Weil divisorial sheaf on X. We denote by

$$C_a(X, L) := \mathrm{Spec}_k \sum_{m \geq 0} H^0(X, L^{[m]})$$

the cone over X induced by L. The point defined by the ideal $\sum_{m\geq 1} H^0(X, L^{[m]})$ is called the *vertex* of the cone and we denote it by v. Over X we consider the affine morphism:

$$\pi\colon BC_a(X, L) := \mathrm{Spec}_X \sum_{m \geq 0} L^{[m]} \to X.$$

The morphism π comes with a natural section X^- defined by the vanishing of the ideal sheaf $\sum_{m\geq 1} L^{[m]}$. The open subset of $BC_a(X, L)$

$$\mathrm{Spec}_X \sum_{m \in \mathbb{Z}} L^{[m]} = BC_a(X, L) \setminus X^-,$$

is isomorphic to $C_a^*(X, L) := C_a(X, L) \setminus v$. We have the following diagram:

$$\begin{array}{ccc} BC_a(X, L) & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f} & & \\ C_a(X, L) & & . \end{array} \tag{2.4}$$

The birational morphism f contracts exactly the section X^- of π with anti-ample $\mathbb{Q}$-Cartier divisorial sheaf $\mathcal{O}_{X^-}(X^-) \simeq L^\vee$. Given a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X, we can construct a $\mathbb{Q}$-divisor $D_{C_a(X,L)} = f_*\pi^*D$ on $C_a(X, L)$.

The following result (originally due to [Wat81]) describes the divisor class group of the cone and the condition under which the log canonical class is $\mathbb{Q}$-Cartier.

Proposition 2.9 (cf. [Kol13, Proposition 3.14])**.** *With the same notation as in diagram (2.4), we have*

1. $\mathrm{Pic}(C_a(X,L)) = 0$,
2. $\mathrm{Cl}(C_a(X,L)) = \mathrm{Cl}(X)/\langle L\rangle$,
3. $m(K_{C_a(X,L)} + \Delta_{C_a(X,L)})$ *is Cartier if and only if there exists* $r \in \mathbb{Q}$ *such that* $\mathcal{O}_X(m(K_X+\Delta)) \simeq L^{[rm]}$. *Moreover,*

$$K_{BC_a(X,L)} + \pi^*\Delta + (1+r)X^- = f^*(K_{C_a(X,L)} + \Delta_{C_a(X,L)}). \tag{2.5}$$

Proof. Let Z be the locus where L is not a Cartier divisor and denote the open set $V := X \setminus Z$. Since X is normal, Z has codimension at least two and thus $\mathrm{Cl}(V) \simeq \mathrm{Cl}(X)$ and $\mathrm{Pic}(V) \simeq \mathrm{Pic}(X)$. By construction, $\pi\colon \pi^{-1}(V) \to V$ is an $\mathbb{A}^1$-bundle and thus $\mathrm{Cl}(\pi^{-1}(V)) \simeq \mathrm{Cl}(V)$ and $\mathrm{Pic}(\pi^{-1}(V)) \simeq \mathrm{Pic}(V)$. Since π is equi-dimensional, we conclude that the codimension of $\pi^{-1}(Z)$ is at least two and thus

$$\mathrm{Cl}(BC_a(X,L)) \simeq \mathrm{Cl}(\pi^{-1}(V)) \simeq \mathrm{Cl}(V) \simeq \mathrm{Cl}(X),$$

and analogously we have $\mathrm{Pic}(BC_a(X,L)) \simeq \mathrm{Pic}(X)$.

We prove (1). Let M be a line bundle on $C_a(X,L)$. Then f^*M is trivial on X^-, thus concluding that M is trivial on $C_a(X,L)$.

We prove (2). Since v has codimension at least two in $C_a(X,L)$, we have $\mathrm{Cl}(C_a(X,L)) \simeq \mathrm{Cl}(C_a^*(X,L))$. Thus we have

$$\mathbb{Z}[X^-] \to \mathrm{Cl}(BC_a(X,L)) \to \mathrm{Cl}(C_a^*(X,L)) \to 0.$$

Since $\mathcal{O}_{X^-}(X^-) \simeq L^\vee$, we conclude (2).

We prove (3). We have $\pi^{-1}(V)\backslash X^- \to V$ is a $\mathbb{G}_m$-bundle. Thus there is a natural linear equivalence $K_{\pi^{-1}(V)} + X^-|_{\pi^{-1}(V)} \sim \pi^*K_V$ which extends to a natural linear equivalence $K_{BC_a(X,L)} + X^- \sim \pi^*K_X$. Thus $K_{BC_a(X,L)} + \pi^*\Delta + X^- \sim \pi^*(K_X+\Delta)$. By assertion (2), the divisor $mK_{C_a(X,L)} + m\Delta_{C_a(X,L)}$ is Cartier if and only if $mK_{C_a^*(X,L)} + m\Delta_{C_a^*(X,L)}$ is linearly equivalent to zero. This is equivalent to $m(K_X+\Delta) \sim rmL$ for some $r \in \mathbb{Q}$. As for the last equality, we have

$$K_{BC_a(X,L)} + \pi^*\Delta + (1+a)X^- = f^*(K_{C_a(X,L)} + \Delta_{C_a(X,L)}),$$

for some a. By restricting to X^- we have

$$0 \sim_{\mathbb{Q}} (K_{BC_a(X,L)} + \pi^*\Delta + (1+a)X^-)|_{X^-} \sim K_X + \Delta + aX^-|_{X^-} \sim_{\mathbb{Q}} rL - aL,$$

thus concluding. □

From the point of view of the singularities of the MMP we have the following

Proposition 2.10 (cf. [Kol13, Lemma 3.1])**.** *With the same notation as in diagram (2.4), let us assume that* $K_X + \Delta \sim_{\mathbb{Q}} rL$.

1. *If* L *is Cartier, then the pair* $(C_a(X, L), \Delta_{C_a(X,L)})$ *is terminal (resp. klt) if and only if the pair* (X, Δ) *is terminal (resp. klt) and* $r < -1$ *(resp.* $r < 0$*).*

2. *If* L *is Cartier, then the pair* $(C_a(X, L), \Delta_{C_a(X,L)})$ *is dlt if the pair* (X, Δ) *is dlt and* $r < 0$.

3. *If* X *is* $\mathbb{Q}$*-factorial and* $\mathrm{char}(k) = 0$*, then* $(C_a(X, L), \Delta_{C_a(X,L)})$ *is klt if and only if* (X, Δ) *is klt and* $r < 0$.

Proof. Assertions (1) and (2) are proven in [Kol13, Lemma 3.1]. We prove assertion (3). Since (X, Δ) is klt, we conclude by inversion of adjunction (see [KM98, Theorem 5.50]) that the pair $(BC_a(X, L), \pi^*\Delta + X^-)$ is plt and the unique plt centre is X^-. Since $r < 0$, we have $(BC_a(X, L), \pi^*\Delta + (1 + r)X^-)$ is klt and thus by Proposition 2.9 we conclude the proof. □

Comment 2.11. Proposition 2.10 is probably the most evident instance of the general principle that klt singularities should correspond to a local analogue of varieties of Fano type.

We will be interested in understanding whether the singularity at the vertex of the cone is Cohen-Macaulay or not.

For this reason, we show that the local cohomology at the vertex of the cone is controlled by the cohomology groups of L:

Proposition 2.12. *For* $i \geq 2$,

$$H^i_v(C_a(X, L), \mathcal{O}_{C_a(X,L)}) \simeq \bigoplus_{m \in \mathbb{Z}} H^{i-1}(X, L^{[m]}).$$

Proof. Since $C_a(X, L)$ is affine, the cohomology groups $H^i(C_a(X, L), \mathcal{O}_{C_a(X,L)})$ vanish for $i \geq 1$. Thus, by the long exact sequence in local cohomology (see [Ha77, Chapter III, ex. 2.3]), we have

$$H^i_v(C_a(X, L), \mathcal{O}_{C_a(X,L)}) \simeq H^{i-1}(U, \mathcal{O}_U) \text{ for } i \geq 2.$$

Since f is affine, we have

$$H^{i-1}(U, \mathcal{O}_U) = H^{i-1}(X, f_*\mathcal{O}_U) = \bigoplus_{m \in \mathbb{Z}} H^{i-1}(X, L^{[m]}),$$

thus concluding. □

2.3.2. Projective cones

Let X be a normal variety over k and let L be an ample $\mathbb{Q}$-Cartier Weil divisor on X. We define the projective cone of X with respect to L as in [Kol13, Section 3.1, page 97]:

$$C_p(X,L) = \mathrm{Proj}_k \sum_{m\geq 0} (\sum_{r=0}^{m} H^0(X, L^{[r]}) x_{n+1}^{m-r}).$$

It contains as an open dense subset $C_a(X,L)$. The following result, which shows how to compute the cohomology of the structure sheaf of $C_p(X,L)$, is probably well-known to experts. We include a proof for sake of completeness.

Proposition 2.13. *For $i \geq 2$,*

$$H^i(C_p(X,L), \mathcal{O}_{C_p(X,L)}) \simeq \bigoplus_{m>0} H^{i-1}(X, L^{[m]}).$$

Proof. We denote by v the vertex of $C_a(X,L) \subset C_p(X,L)$ and we consider the natural inclusion

$$V := C_a(X,L) \setminus v = \mathrm{Spec}_X \sum_{m\in\mathbb{Z}} L^{[m]} \subset \mathrm{Spec}_X \sum_{m\leq 0} L^{[m]} = C_p(X,L) \setminus v =: U.$$

Considering the long exact sequences in local cohomology (see [Ha77, Chapter III, Ex 2.3]) we have the following natural commutative diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H^{i-1}(U, \mathcal{O}_U) & \xrightarrow{\delta_i} & H^i_v(C_p(X,L), \mathcal{O}_{C_p(X,L)}) & \longrightarrow & H^i(C_p(X,L), \mathcal{O}_{C_p(X,L)}) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle i^*} & \\
\ldots \longrightarrow & H^{i-1}(V, \mathcal{O}_V) & \xrightarrow{\eta_i} & H^i_v(C_a(X,L), \mathcal{O}_{C_a(X,L)}) & \longrightarrow & H^i(C_a(X,L), \mathcal{O}_{C_a(X,L)}) & \longrightarrow \ldots,
\end{array}$$

where by i we mean the natural inclusion maps. It is easy to see that the diagram

$$\begin{array}{ccc}
H^i(U, \mathcal{O}_U) & \xrightarrow{i^*} & H^i(V, \mathcal{O}_V) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
\bigoplus_{m\leq 0} H^{i-1}(X, L^{[m]}) & \longrightarrow & \bigoplus_{m\in\mathbb{Z}} H^{i-1}(X, L^{[m]})
\end{array}$$

commutes where the bottom arrow is the natural inclusion, thus showing that i^* is injective. Since $H^{i-1}(C_a(X,L), \mathcal{O}_{C_a(X,L)}) = 0$ for $i \geq 2$, we have that the maps η_i are isomorphisms and thus we conclude that δ_i are injective. Therefore we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^{i-1}(U, \mathcal{O}_U) & \xrightarrow{\delta_i} & H^i_v(C_p(X,L), \mathcal{O}_{C_p(X,L)}) & \longrightarrow & H^i(C_p(X,L), \mathcal{O}_{C_p(X,L)}) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & \\
0 \longrightarrow & \bigoplus_{m\leq 0} H^{i-1}(X, L^{[m]}) & \longrightarrow & \bigoplus_{m\in\mathbb{Z}} H^{i-1}(X, L^{[m]}) & \longrightarrow & H^i(C_p(X,L), \mathcal{O}_{C_p(X,L)}) & \longrightarrow 0,
\end{array}$$

which concludes the proof. □

2.4. Vanishing theorems in positive characteristic

We collect the two main tools that we use to discuss vanishing theorems on log del Pezzo surfaces. The first one comes from a blending of Ekedahl's cyclic construction and Bend and Break and it has already been successfully applied to the study of varieties in positive characteristic.

Theorem 2.14 ([Kol96, Theorem II.6.2, Remark II.6.7.2]). *Let k be a perfect field. Let X be a normal variety over k. Let L be an ample Weil $\mathbb{Q}$-Cartier divisor on X such that $H^1(X, L^\vee) \neq 0$. Assume that X is covered by a family of curves $\{D_t\}$ such that X is smooth along the general curve D_t and such that*

$$((p-1)L - K_X) \cdot D_t > 0.$$

Then for every point $x \in X$ there exists a rational curve C_x passing through x such that

$$L \cdot C_x \leq 2 \dim X \frac{L \cdot D_t}{((p-1)L - K_X) \cdot D_t}.$$

Remark 2.15. If $\dim(X) = 2$, the same statement holds if L is assumed to be big and nef.

The second tool we use is a Kodaira vanishing type theorem for varieties in positive characteristic without irregularity.

Proposition 2.16 ([CT, Lemma 3.2]). *Let k be a perfect field. Let X be a normal variety over k such that $H^1(X, \mathcal{O}_X) = 0$ and let A be an effective Weil divisor. Then the natural map*

$$H^1(X, \mathcal{O}_X(-A)) \to H^1(X, F_* \mathcal{O}_X(-pA))$$

is injective.

As a direct consequence we have

Theorem 2.17. *Let k be a perfect field. Let X be a Cohen-Macaulay normal variety over k of dimension $n \geq 2$ such that $H^1(X, \mathcal{O}_X) = 0$ and let A be an ample (if $n = 2$ it is sufficient big and nef) and effective Cartier divisor. Then*

$$H^1(X, \mathcal{O}_X(-A)) = 0.$$

Proof. By Proposition 2.16 we have that $h^1(\mathcal{O}_X(-A)) \leq h^1(\mathcal{O}_X(-p^k A))$ for every $k \geq 1$. By Serre duality we have $h^1(\mathcal{O}_X(-p^k A)) = h^{n-1}(\mathcal{O}_X(K_X + p^k A))$, which vanishes for k sufficiently large by Serre vanishing (resp. see [Kol96, Chapter II, Remark 6.2.4]). □

2.5. Geometry over imperfect fields

In this section, we summarise some results on varieties defined over imperfect fields. For us, the main motivation to consider such varieties lies in understanding the geometry of fibrations in positive characteristic.

2.5.1. Behaviour of the canonical class under base change

Let k be an imperfect field and let X be a normal proper variety over k. Assume that k is integrally closed in $K(X)$, equivalently $k = H^0(X, \mathcal{O}_X)$. Let $k \subset k'$ a non-separable field extension. Then the base change $X_{k'}$ could be non-reduced or non-normal. We consider the natural induced morphism

$$f \colon Y := (X_{k'})^N_{\mathrm{red}} \to X,$$

where $(X_{k'})^N_{\mathrm{red}}$ is the normalisation of $(X_{k'})_{\mathrm{red}}$. For applications to birational geometry it is particularly important to understand the relation between K_Y and K_X. The following is the higher-dimensional generalisation of Tate's genus change formula for curves:

Theorem 2.18 ([Tan18b, Theorem 4.2], [PW, Theorem 1.1], [Tanb, Theorem 3.16])**.** *Let k be an imperfect field and let X be a normal proper variety over k such that $k = H^0(X, \mathcal{O}_X)$. Let $k \subset k'$ be a field extension. Let Y be the normalisation of the reduced scheme $(X_{k'})_{\mathrm{red}}$ and let $f \colon Y \to X$ be the natural morphism. Then there exists an effective Weil divisor C on Y such that*

$$K_Y + (p-1)C \sim f^*K_X.$$

Moreover C can be chosen to be non-zero if and only if $X_{k'}$ is not normal.

The above theorem have been successfully used to show global properties of wild fibrations: rational chain connectedness of fibres of Mori fibre spaces ([Tan18b, Theorem 1.4]), uniruledness of Calabi-Yau fibre spaces with non-normal general fibres ([Tan18b, Theorem 1.3]), smoothness of the normalisation of the general fibres ([PW, Theorem 1.4]).

2.5.2. Frobenius length of geometric non-normality

In [Tanb], H. Tanaka introduces four invariants to measure the possible bad behaviour of varieties over imperfect fields when passing to some inseparable extension. In Chapter 5, we will study geometric non-normality of varieties over imperfect fields. Recall that a variety X over a field k is said to be *geometrically normal over k* if $X \times_k \overline{k}$ is normal. We start by recalling the definition of Frobenius length of geometric non-normality (Definition 2.19) and some fundamental properties (Remark 2.20).

Definition 2.19. Let k be a field of characteristic $p > 0$. Let X be a proper normal variety over k such that $k = H^0(X, \mathcal{O}_X)$. The *Frobenius length of geometric non-normality*

$\ell_F(X/k)$ of X/k is defined by

$$\ell_F(X/k) := \min\{\ell \in \mathbb{Z}_{\geq 0} \mid (X \times_k k^{1/p^\ell})^N_{\rm red} \text{ is geometrically normal over } k^{1/p^\ell}\}.$$

Properties 2.20. Let k and X be as in Definition 2.19. Set $\ell := \ell_F(X/k)$. Let (k', Y) be one of $(k^{1/p^\infty}, (X \times_k k^{1/p^\infty})^N_{\rm red})$ and $(\overline{k}, (X \times_k \overline{k})^N_{\rm red})$. Let $f\colon Y \to X$ be the natural morphism. We summarise some results from [Tanb, Section 5].

1. The existence of the right hand side of Definition 2.19 is assured by [Tanb, Remark 5.2].
2. If X is not geometrically normal, then ℓ is a positive integer [Tanb, Remark 5.3] and there exist nonzero effective Weil divisors $D_1, ..., D_\ell$ such that
$$K_Y + (p-1)\sum_{i=1}^{\ell} D_i \sim f^*K_X,$$
where $f : Y \to X$ denotes the induced morphism [Tanb, Proposition 5.11].
3. The ℓ-th iterated absolute Frobenius morphism $F^\ell_{X \times_k k'}$ factors through the induced morphism $Y \to X \times_k k'$ [Tanb, Proposition 5.4 and Theorem 5.9]:
$$F^\ell_{X\times_k k'} : X \times_k k' \to Y \to X \times_k k'.$$

2.5.3. Geometrically klt singularities

The purpose of this subsection is to introduce the notion of geometrically klt singularities and its variants.

Definition 2.21. Let (X, Δ) be a log pair over a field k such that k is algebraically closed in $K(X)$. We say that (X, Δ) is *geometrically klt* (resp. terminal, canonical, lc) if $(X \times_k \overline{k}, \Delta \times_k \overline{k})$ is klt (resp. terminal, canonical, lc).

Lemma 2.22. *Let k be a field. Let X and Y be varieties over k which are birational to each other. Then X is geometrically reduced over k if and only if Y is geometrically reduced over k.*

Proof. Recall that for a k-scheme, being geometrically reduced is equivalent to satisfy Serre's condition S_1 (i.e. for any local ring of the variety there exists a regular element) and geometrically R_0 (i.e. $X_{\overline{k}}$ is regular at its generic point). Since both X and Y are S_1, the assertion follows from the fact that being geometrically R_0 is a condition on the generic point. □

We prove a descent result for such singularities.

Proposition 2.23. *Let (X, Δ) be a geometrically klt (resp. terminal, canonical, lc) pair such that k is algebraically closed in $K(X)$. Then (X, Δ) is klt (resp. terminal, canonical, lc).*

Proof. We only treat the klt case, as the others are analogous. Let $\pi\colon Y \to X$ be a birational k-morphism, where Y is a normal variety and we write $K_Y+\Delta_Y = \pi^*(K_X+\Delta)$. It suffices to prove that $\lfloor \Delta_Y \rfloor \leq 0$. Thanks to Lemma 2.22, Y is geometrically integral. Let $\nu\colon W \to Y \times_k \overline{k}$ be the normalisation morphism and let us consider the following commutative diagram:

$$\begin{array}{ccc} W & & \\ \nu\downarrow & & \\ Y \times_k \overline{k} & \xrightarrow{g} & Y \\ \pi_{\overline{k}}\downarrow & & \pi\downarrow \\ X \times_k \overline{k} & \xrightarrow{f} & X. \end{array}$$

Denote by $\psi := \pi_{\overline{k}} \circ \nu$ and $h := g \circ \nu$ the composite morphisms. We have

$$K_W + \Delta_W := \psi^*(K_{X_{\overline{k}}} + \Delta_{\overline{k}}) = h^*\pi^*(K_X + \Delta) = h^*(K_Y + \Delta_Y).$$

By [Tan18b, Theorem 4.2], there exists an effective $\mathbb{Z}$-divisor D such that

$$h^*(K_Y + \Delta_Y) = K_W + D + h^*\Delta_Y,$$

and thus $\Delta_W = D + h^*\Delta_Y \geq h^*\Delta_Y$. Since $(X_{\overline{k}}, \Delta_{\overline{k}})$ is klt, any coefficient of Δ_W is < 1. Then any coefficient of Δ_Y is < 1, thus (X, Δ) is klt. □

If k is a perfect field, being klt is equivalent to being geometrically klt by [Kol13, Proposition 2.15]. However, over imperfect fields, being geometrically klt is a strictly stronger condition. Let us show with the following examples how bad singularities can appear after base change, even if the variety we are considering is geometrically normal.

Example 2.24. We fix k to be an imperfect field of characteristic $p > 0$.

1. Consider the log pair $(\mathbb{A}^1_k, \frac{2}{3}P)$, where P is a closed point whose residue field $k(P)$ is a purely inseparable extension of k of degree p. This pair is klt over k, but it is not geometrically log canonical.

2. Let $t \in k \setminus k^p$. Let $q := p^e$ for some $e > 0$ and consider the following surface

$$X := \operatorname{Spec} k[x, y, z]/(t + x^q + y^{q+1} + z^{q+1})$$

The surface X is regular, and thus klt. Let us consider the base-change to the algebraic closure: $X_{\overline{k}}$. A simple substitution $w := x + t^{1/q}$ defines an isomorphism

$$X_{\overline{k}} \simeq \operatorname{Spec} \overline{k}[w, y, z]/(w^q + y^q + z^{q+1}).$$

The surface $X_{\overline{k}}$ is normal with a unique singular point of multiplicity q at the origin. Thus X is geometrically normal but not geometrically log canonical if $q > 3$.

3

KV vanishing fails for log del Pezzo surfaces in characteristic three

3.1. Introduction

In this chapter we show counterexamples to the Kodaira vanishing theorem on surfaces of del Pezzo type and to the rationality of klt threefold singularities in characteristic three. These examples answers questions of Hacon-Witaszek ([HW17, Question 5.4]) and Kovács ([Kov18]).

Inspired by an example of Keel and McKernan (see [KM99, Section 9] and [CT, Section 4]), we construct a log del Pezzo surface violating the Kawamata-Viehweg vanishing in characteristic three:

Theorem 3.1 (See Theorem 3.14)**.** *Over any field k of characteristic three, there exists a projective normal k-surface T such that*

1. *T has klt singularities and $-K_T$ is ample;*
2. *$\rho(T) = 1$;*
3. *there exists an ample $\mathbb{Q}$-Cartier Weil divisor A on T such that $H^1(T, \mathcal{O}_T(-A)) \neq 0$.*

Using the cone construction of Section 2.3 and Theorem 3.1 we show:

Theorem 3.2 (See Theorem 3.17)**.** *Let k be a field of characteristic three. Then there exists a $\mathbb{Q}$-factorial Kawamata log terminal threefold singularity X which is not Cohen-Macaulay.*

In the last paragraph of the section, we present a Kodaira-type vanishing theorem for big and nef Cartier divisors on log del Pezzo surfaces of characteristic $p \geq 5$, which partially answers a question of Cascini and Tanaka (see [CT18, Remark 3.2]).

Theorem 3.3 (See Theorem 3.22)**.** *Let X be a log del Pezzo surface defined over an algebraically closed field k of characteristic $p \geq 5$ and let A be a big and nef Cartier divisor. Then,*

$$H^1(X, \mathcal{O}_X(A)) = 0.$$

3.1.1. Frobenius splitting and liftability

We fix a perfect field k of characteristic $p > 0$. For the convenience of the reader we recall the definition of F-splitting.

Definition 3.4. Let X be a projective variety over k. We say that X is *globally F-split* if for some $e > 0$ the natural map

$$\mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X$$

splits as a homomorphism of $\mathcal{O}_X$-modules.

Remark 3.5. In the definition of F-splitting for a variety X one can equivalently ask that for *all* $e > 0$ the $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ splits.

Being globally F-split implies powerful vanishing results for the cohomology of ample divisors on X. We will need the following result, which is a mild generalization of [BK05, Theorem 1.2.9] to $\mathbb{Q}$-Cartier Weil divisors on surfaces:

Proposition 3.6. *Let X be a normal projective surface over k. If X is globally F-split, then for any ample $\mathbb{Q}$-Cartier Weil divisor A*

$$H^1(X, \mathcal{O}_X(-A)) = 0.$$

Proof. By Remark 3.5 we know that for any large $g \gg 0$ there exists a splitting:

$$\mathcal{O}_X \to F^g_* \mathcal{O}_X \to \mathcal{O}_X.$$

Restricting to the regular locus U and tensoring by $\mathcal{O}_U(-A)$ we have the following splitting:

$$\mathcal{O}_U(-A) \to F^g_* \mathcal{O}_U(-p^g A) \to \mathcal{O}_U(-A).$$

Since X is a normal variety and each sheaf in the sequence is reflexive we deduce that the splitting holds on the whole X:

$$\mathcal{O}_X(-A) \to F^g_* \mathcal{O}_X(-p^g A) \to \mathcal{O}_X(-A).$$

Passing to cohomology, we have an injection:

$$H^1(X, \mathcal{O}_X(-A)) \hookrightarrow H^1(X, \mathcal{O}_X(-p^g A)).$$

Let m be the Cartier index of A and let us write $m = p^f h$ where $\gcd(p, h) = 1$. Then for large enough and sufficiently divisible e we have that m divides $p^f(p^e-1)$ and thus

$$p^f(p^e-1)A \text{ is Cartier.}$$

Now consider $g = f + e$. We have

$$-p^g A = -(p^{e+f} - p^f)A - p^f A.$$

Since X is a normal surface, any divisorial sheaf is Cohen-Macaulay and thus we can apply Serre duality (see [KM98, Theorem 5.71]) to deduce

$$H^1(X, \mathcal{O}_X(-p^g A)) \simeq H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-p^g A), \mathcal{O}_X(K_X)))^*.$$

Being X normal, we have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-p^g A), \mathcal{O}_X(K_X)) \simeq \mathcal{O}_X(K_X + p^g A)$ since both sheaves are reflexive and they are isomorphic on U. Therefore

$$H^1(X, \mathcal{O}_X(-p^g A)) \simeq H^1(X, \mathcal{O}_X(K_X + p^g A))^* = H^1(X, \mathcal{O}_X(K_X + p^f A) \otimes \mathcal{O}_X((p^{e+f} - p^f)A))^*.$$

By choosing e sufficiently large and divisible we conclude that the last cohomology group vanishes by Serre vanishing criterion for ample line bundles. □

We recall the definition of liftability for log pairs to $W_2(k)$.

Definition 3.7 (cf. [EV92, Definition 8.11])**.** Let X be a smooth variety over a perfect field k of characteristic $p > 0$. Let $D = \sum_i D_i$ be a snc divisor on X. We say that the log pair (X, D) lifts to $W_2(k)$ if there exists a flat family $\mathcal{X} \to \operatorname{Spec}(W_2(k))$ and divisors $\mathcal{D}_i \subset \mathcal{X}$ such that

1. the induced morphism $\mathcal{D}_i \to \operatorname{Spec}(W_2(k))$ is flat,
2. $(X, \sum_i D_i) \simeq (\mathcal{X} \times_{W_2(k)} k, \sum_i \mathcal{D}_i \times_{W_2(k)} k)$.

Log pairs in positive characteristic admitting a resolution satisfy strong vanishing theorems:

Proposition 3.8 ([CTW17, Lemma 6.1])**.** *Let k be a perfect field of characteristic $p > 2$. Let (X, Δ) be a klt log pair of dimension two. Suppose there exists a log resolution $\pi\colon V \to X$ such that $(V, \operatorname{Ex}(\pi) + \lceil \mu_*^{-1}\Delta \rceil))$ lifts to $W_2(k)$. Let D be a Weil divisor such that $D - (K_X + \Delta)$ is ample. Then $H^1(X, \mathcal{O}_X(D)) = 0$.*

3.2. Keel-McKernan surface in characteristic three

In this section and in the following we fix k to be a field of characteristic three. We prove Theorem 3.1 by constructing a log del Pezzo surface of Picard rank one not satisfying the Kawamata-Viehweg vanishing theorem.

3.2.1. Construction of Keel-M^{c}Kernan surface in characteristic three

In [KM99, Section 9], the authors construct a family of log del Pezzo surfaces in characteristic 2 violating the Bogomolov bound on the number of singular points. In [CT] it was noted that their example gives various counterexamples to the Kawamata-Viehweg vanishing theorem. We adapt their construction to the characteristic 3 case.

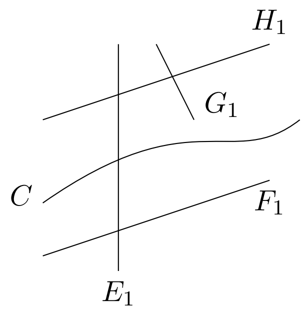
Let us consider the smooth rational curve C inside $\mathbb{P}^1_x \times \mathbb{P}^1_y$ defined by the equation:

$$C := \left\{([x_1 : x_2], [y_1 : y_2]) \mid x_2 y_1^3 - x_1 y_2^3 = 0\right\}.$$

We denote by $\pi_x \colon \mathbb{P}^1_x \times \mathbb{P}^1_y \to \mathbb{P}^1_x$ the natural projection onto the first coordinate and we say $F_p = \pi_x^{-1}(p)$ for $p \in \mathbb{P}^1_x$ is the vertical fibre over p.

The main property of C is that the morphism $\pi_x|_C \colon C \to \mathbb{P}^1_x$ is the relative Frobenius morphism. Geometrically, the curve C has the following "funny" property: every vertical fibre F_p is a triple tangent to C.

Fix a closed point p_1 on C and consider the vertical fiber F_1 passing through this point. Since such a fiber is a triple tangent to C at the point p_1 we perform three successive blow-ups starting from $\mathbb{P}^1_x \times \mathbb{P}^1_y$ to separate C from F_1. The order of the blow-ups is as follows: at each step we blow-up the intersection point of the strict transform of F_1 and the strict transform of C. After these birational modifications the strict transforms of C and F_1 (which, by abuse of notation, are denoted by the same letter) and the exceptional divisors E_1, G_1, H_1 are in the following configuration:



where all the curves are smooth and rational with the following intersection numbers:

$$H_1^2 = -2, G_1^2 = -2, F_1^2 = -3, E_1^2 = -1, C \cdot E_1 = 1, E_1 \cdot F_1 = 1, E_1 \cdot H_1 = 1, H_1 \cdot G_1 = 1.$$

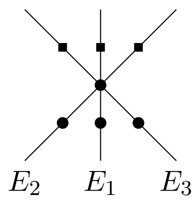
Note that the self-intersection of C has dropped by 3. Performing the same operation with other two points p_2, p_3 on the curve C we construct a birational morphism $f \colon S \to \mathbb{P}^1_x \times \mathbb{P}^1_y$ where the strict transform of C has become a (-3)-curve. Over each point p_i we have the exceptional curves H_i, G_i, E_i and the strict transform of the fibre F_i in the same configuration as the one described above for p_1.

On S there are the (-3)-curves F_1, F_2, F_3 and three cycles of type A_2 of (-2)-curves formed by H_i and G_i for $i = 1, \ldots, 3$. Let $\psi \colon S \to T$ be the birational contraction of the curves F_i, H_i, G_i for $i \in \{1, 2, 3\}$ and C. We can construct ψ by running a suitable MMP (see [Tan14]) with respect to the pair

$$(S, \Delta := \sum_{i=1}^{3} \frac{2}{3} F_i + \sum_{i=1}^{3} \frac{1}{2}(H_i + G_i) + \frac{2}{3} C),$$

which at each step of the MMP we contract exactly one of the curves appearing in Supp(Δ). We denote, with a slight abuse of notation, the pushforward of a divisor D via ψ with the same letter D.

On T we have the following configuration of curves and singular points:



▪ = A_2-type singular point

• = A_1-type singular point

Remark 3.9. The singularity at the points of type A_2 (resp. A_1) is formally isomorphic to the quotient of $\mathbb{A}^2_k$ by the action of the group scheme μ_3 with weights $(1, 2)$ (resp. $(1, 1)$).

The following proposition justifies why this surface is a generalization of Keel-M^{c}Kernan construction in characteristic three:

Proposition 3.10. *The surface T is a klt del Pezzo surface of Picard rank one. Moreover, $-K_T$ is numerically equivalent to E_1.*

Proof. It is straightforward to see that $\rho(T) = 1$. Indeed, $\rho(S) = 10$ and the morphism ψ contracts nine exceptional curves. Since we contract only cycle of (-2)-curves and (-3)-curves, T has klt singularities. We are only left to show that $-K_T$ is an ample divisor. By an explicit computation we have

$$\psi^* K_T \sim_{\mathbb{Q}} K_S + \sum_{i=1}^{3} \frac{1}{3} F_i + \frac{1}{3} C. \tag{3.1}$$

Since $\rho(T) = 1$ it is enough to prove that the anticanonical divisor has the same intersection with an effective curve as E_1.

Let F_p be the fibre of the the map $\pi_x \circ f \colon S \to \mathbb{P}^1_x$ over a general point $p \in \mathbb{P}^1_x$. By the projection formula we have:

$$-K_T \cdot \psi_* F_p = -\psi^* K_T \cdot F_p = -K_S \cdot F_p - \frac{1}{3} C \cdot F_p = 1. \qquad \square$$

We also have

$$E_1 \cdot \psi_* F_p = \psi^* E_1 \cdot F_p = \frac{1}{3} C \cdot F_p = 1.$$

Remark 3.11. It is possible to perform a similar construction for higher characteristic, but the resulting surface has ample canonical divisor class.

Remark 3.12. In [KM99, Section 9] the authors prove the Bogomolov bound: a log del Pezzo surface of Picard rank one over an algebraically closed field of characteristic zero has at most six singular points. The bound was later improved to four singular points in characteristic zero in [Bel09]. The surface T has seven singular points and thus shows that the Bogomolov bound cannot hold in characteristic 3. It is an open question whether the Bogomolov bound holds for large characteristic.

We show that there are no anticanonical sections on T:

Proposition 3.13. $H^0(T, \mathcal{O}_T(-K_T)) = 0$.

Proof. By formula (3.1) we have

$$H^0(T, \mathcal{O}_T(-K_T)) = H^0(S, \mathcal{O}_S(-K_S - \sum_{i=1}^{3} F_i - C)).$$

A direct computation shows

$$-K_S - \sum_{i=1}^{3} F_i - C \sim f^*(-K_{\mathbb{P}^1_x \times \mathbb{P}^1_y} - \sum_{i=1}^{3} F_i - C) + \sum_{i=1}^{3}(G_i + 2H_i + 3E_i).$$

Therefore

$$H^0(T, \mathcal{O}_T(-K_T)) = H^0(\mathbb{P}^1_x \times \mathbb{P}^1_y, \mathcal{O}(-K_{\mathbb{P}^1_x \times \mathbb{P}^1_y} - \sum_{i=1}^{3} F_i - C)) = H^0(\mathbb{P}^1_x \times \mathbb{P}^1_y, \mathcal{O}(-2, -1)) = 0.$$

□

3.2.2. Failure of Kawamata-Viehweg vanishing theorem on T

We show that various Kawamata-Viehweg vanishing theorem fails on the surface T.

We consider the following ample $\mathbb{Q}$-Cartier Weil divisor

$$A := E_2 + E_3 - E_1.$$

Theorem 3.14. *The Kawamata-Viehweg vanishing theorem fails for the Weil divisor A;* i.e.

$$H^1(T, \mathcal{O}_T(-A)) \neq 0.$$

Proof. The strategy is to pull-back the divisor to the minimal resolution S and compute there the cohomology groups. Let us consider the pull-back of A to S as a $\mathbb{Q}$-divisor:

$$-\psi^* A = E_1 + \frac{1}{3}F_1 + \frac{2}{3}H_1 + \frac{1}{3}G_1 + \frac{1}{3}C - E_2 - \frac{1}{3}F_2 - \frac{2}{3}H_2 - \frac{1}{3}G_2 - \frac{1}{3}C - E_3 - \frac{1}{3}F_3 - \frac{2}{3}H_3 - \frac{1}{3}G_3 - \frac{1}{3}C;$$

thus

$$\lfloor -\psi^* A \rfloor = E_1 - E_2 - F_2 - H_2 - G_2 - E_3 - F_3 - H_3 - G_3 - C.$$

We have

$$\psi_* \mathcal{O}_S(\lfloor -\psi^* A \rfloor) = \mathcal{O}_T(-A).$$

Indeed, it is immediate to see that $\psi_*\mathcal{O}_S(\lfloor -\psi^*A \rfloor) \subset \mathcal{O}_T(-A)$ and we only need to check equality. Let U be an open set of T and $f \in k(T)$ be a rational function such that $\mathrm{div}_T(f) \geq E_2 + E_3 - E_1$ over the open set U. Therefore it is straightforward to check that $\mathrm{div}_S(\psi^* f) \geq E_2 + F_2 + H_2 + G_2 + E_3 + H_3 + G_3 + E_3 + F_3 + C - E_1$, thus showing that $f \in \psi_*\mathcal{O}_S(\lfloor -\psi^*A \rfloor)(U)$.

We now compute the cohomology group $H^1(\mathcal{O}_T(-A))$ using the Leray spectral sequence

$$E_2^{i,j} = H^j(T, R^i\psi_*\mathcal{O}_S(\lfloor -\psi^*A \rfloor)) \Rightarrow H^{i+j}(S, \mathcal{O}_S(\lfloor -\psi^*A \rfloor)). \tag{3.2}$$

We show that $R^i\psi_*\mathcal{O}_S(\lfloor -\psi^*A \rfloor) = 0$ for $i > 0$. By the Kawamata-Viehweg vanishing theorem for birational morphism between surfaces (see [Kol13, Theorem 10.4]) we just need to check that $\lfloor -\psi^*A \rfloor$ is $\psi-$nef:

$$\begin{aligned}
&\lfloor -\psi^*A \rfloor \cdot C = 2,\\
&\lfloor -\psi^*A \rfloor \cdot F_1 = 1, \lfloor -\psi^*A \rfloor \cdot H_1 = 1, \lfloor -\psi^*A \rfloor \cdot G_1 = 0,\\
&\lfloor -\psi^*A \rfloor \cdot F_i = 2, \lfloor -\psi^*A \rfloor \cdot H_i = 0, \lfloor -\psi^*A \rfloor \cdot G_i = 1 \text{ for } i = 2, 3.
\end{aligned}$$

Therefore the Leray spectral sequence (3.2) degenerates at the E_2 page and we have for all $i \geq 0$:

$$H^i(T, \mathcal{O}_T(-A)) \simeq H^i(S, \mathcal{O}_S(\lfloor -\psi^*A \rfloor)).$$

By a direct computation we have

$$K_S \cdot \lfloor -\psi^*A \rfloor = -2 \text{ and } \lfloor -\psi^*A \rfloor^2 = -6.$$

Therefore, by the Riemann-Roch theorem on S, we deduce

$$\chi(T, \mathcal{O}_T(-A)) = \chi(S, \mathcal{O}_S(\lfloor -\psi^*A \rfloor)) = -1,$$

which implies $h^1(T, \mathcal{O}_T(-A)) \neq 0$. □

We now conclude that the surface T gives a generalization to [CTW17, Theorem 1.3] to characteristic three.

Corollary 3.15. *Over any algebraically closed field k of characteristic 3 there exists a log del Pezzo surface T which is not globally F-split and such that for any log resolution $\mu\colon S \to T$ the log smooth pair $(S, \mathrm{Exc}(\mu))$ does not lift to $W_2(k)$.*

Proof. The surface T constructed above is not globally F-split by Proposition 3.6 and Theorem 3.14.

By Proposition 3.14 and Serre duality (see [KM98, Theorem 5.71, Proposition 5.75]) we have

$$H^1(T, \mathcal{O}_T(K_T + A)) \neq 0.$$

If the pair $(S, \mathrm{Exc}(\mu))$ lifted to $W_2(k)$, we could apply Proposition 3.8 to the $\mathbb{Z}$-divisor $D := K_T + A$, thus getting a contradiction with the non-vanishing above. □

3.3. A klt not Cohen-Macaulay threefold singularity in characteristic three

In this section we prove there exists klt not CM threefold singularities in characteristic three. With the same notation as in Section 2.3, let us consider the cone over the log del Pezzo surface T:

$$X := C_a(T, \mathcal{O}_T(A)),$$

and denote the vertex by v.

We prove that X has klt singularities. If we were working over a field of characteristic zero we would conclude immediately by Proposition 2.10. However, since we are working in positive characteristic, we cannot apply Inversion of Adjunction and thus we need to further study the singularities of X to conclude it is klt. We start by studying the singularities of its partial resolution:

$$Y := \mathrm{Spec}_T \textstyle\sum_{m \geq 0} \mathcal{O}_T(mA) \xrightarrow{f} X.$$

The exceptional locus of the birational morphism π is the prime divisor E, which is isomorphic to T. We denote by f the natural affine map $\pi \colon Y \to T$. We thus have the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & T \\ {\scriptstyle f}\downarrow & & \\ X & & . \end{array}$$

Proposition 3.16. *The variety Y is a $\mathbb{Q}$-factorial threefold with isolated singularities and the pair (Y, E) is toroidal (hence log canonical).*

Proof. To check that Y is $\mathbb{Q}$-factorial it is sufficent to work in an analytic neighbourhood of the singular locus by [Mat80, (24.E)]. The same is true to compute the discrepancies. Thus we can reduce to study the preimage $\pi^{-1}(U) \subset Y$ of an analytic neighbourhood U of the singular points of T because outside the preimage of those points the pair (Y, E) is log smooth.

As explained in Remark 3.9, there are two different types of singular points in T. We show the result is true for the A_2-type singular points; for the A_1-type singular points the computation is similar.

Let us consider a singular point $p \in T$, which is formally isomorphic to the quotient of $\mathbb{A}^2_{u,v}$ by the group μ_3 with weight $(1, 2)$. In local coordinates,

$$V := \mathbb{A}^2_{u,v} /\!/ \mu_3 = \mathrm{Spec}_k\, k[u^3, v^3, uv] \simeq \mathrm{Spec}_k\, k[x, y, z]/(z^3 - xy),$$

and the Weil divisorial sheaf $\mathcal{O}_T(A)$ is isomorphic to the Weil divisorial ideal $\mathcal{O}_V(-D) :=$

(x, z). In this case we have to compute the local equations for the variety:

$$\mathrm{Spec}_V \sum_{k \geq 0} \mathcal{O}_V(-mD).$$

We have

$$\mathcal{O}_V(-D) = (x, z), \quad \mathcal{O}_V(-2D) = (x, z^2), \quad \mathcal{O}_V(-3D) = (x)$$

and these generate the $\mathcal{O}_V$-algebra $\sum_{k \geq 0} \mathcal{O}_V(-mD)$. If we set

$$\mathcal{O}_V(-D) = (x, z) = (a, b) \quad \mathcal{O}_V(-2D) = (x, z^2) = (c, b^2) \quad \mathcal{O}_V(-3D) = (x) = (d),$$

one can write the relations between the generators and show that

$$\mathrm{Spec}_V \sum_{k \geq 0} \mathcal{O}_V(-mD) \simeq$$

$$\simeq \mathrm{Spec}_k \frac{k[x, y, z, a, b, c, d]}{(z^3 - xy, a^2 - cx, ab - cz, a^3 - dx^2, ac - dx, b^3 - dy, bc - dz)}.$$

The fibration f corresponds to the natural morphism

$$\mathrm{Spec}_k \frac{k[x, y, z, a, b, c, d]}{(z^3 - xy, a^2 - cx, ab - cz, a^3 - dx^2, ac - dx, b^3 - dy, bc - dz)} \to \mathrm{Spec}_k \frac{k[x, y, z]}{(z^3 - xy)},$$

and the section E is the subvariety defined by the ideal (a, b, c, d).

A more conceptual way to understand the $\mathbb{A}^1$-fibration $Y \to T$ and its singularities is to see it locally as a quotient of the trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^2$. Let us consider the line bundle

$$\mathbb{L} := \mathrm{Spec}_{\mathbb{A}^2} \sum_{k \geq 0} (u)^k \simeq \mathrm{Spec}_k \, k[u, v, s]$$

together with the section $S = (s = 0)$. We have a natural action of μ_3 on $\mathbb{L}$ of weight $(1, 2, 1)$ and we can construct the quotient

$$p \colon \mathbb{L} \to \mathbb{L} /\!/ \mu_3.$$

A direct computation shows that the quotient pair $(\mathbb{L} /\!/ \mu_3, p(S))$ is isomorphic to (Y, E). With this description, we deduce that Y is a $\mathbb{Q}$-factorial variety by [KM98, Lemma 5.16] and that the singularities of Y are isolated.

Moreover we have shown that, near the preimage via f of the singular points of T, the pair (Y, E) is toroidal and thus by [CLS11, Proposition 11.4.24] we conclude it has log canonical singularities. □

Theorem 3.17. *The variety X has klt singularities and it is not Cohen-Macaulay.*

Proof. By Proposition 2.12 and Theorem 3.14 we deduce

$$H^2_v(X, \mathcal{O}_X) \simeq \sum_{m\in\mathbb{Z}} H^1(T, \mathcal{O}_T(mA)) \neq 0,$$

thus proving X is not Cohen-Macaulay. We are left to check it is klt. We have $-K_T \sim_{\mathbb{Q}} A$ by Proposition 3.10. Thus by Proposition 2.9, K_X is $\mathbb{Q}$-Cartier and $K_Y \sim_{\mathbb{Q}} f^*K_X$.

By Proposition 3.16, Y is a $\mathbb{Q}$-factorial variety, the pair (Y, E) is toroidal and thus Y is klt. This concludes that X has klt singularities. □

3.4. Kodaira-type vanishing for log del Pezzo surfaces in characteristic $p > 0$

Throughout this section, k is an algebraically closed field of characteristic $p > 0$. The aim of this section is to collect some Kodaira-type vanishing results for big and nef line bundles on klt del Pezzo surfaces for arbitrary $p > 0$ and to prove Theorem 3.3.

Lemma 3.18. *Let X be a surface of del Pezzo type over k. Then X is a rational surface and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

Proof. The surface X is rational by [Tan15, Fact 3.4 and Theorem 3.5]. Let Y be the minimal resolution. By the Kawamata-Viehweg vanishing theorem for birational morphism between surfaces, we have $H^i(X, \mathcal{O}_X) \simeq H^i(Y, \mathcal{O}_Y)$. Since Y is a smooth rational surface we conclude. □

We start by discussing the case of log del Pezzo surfaces with at worst canonical singularities.

Proposition 3.19. *Let A be a big and nef Cartier divisor on a klt del Pezzo surface X such that $|-K_X| \neq \emptyset$. Then*

$$H^1(X, \mathcal{O}_X(A)) = 0.$$

Proof. Since A is effective, the divisor $A - K_X$ is effective and ample by hypothesis and thus by an application of Serre duality we have

$$h^1(X, \mathcal{O}_X(A)) = h^1(X, \mathcal{O}_X(K_X - A)),$$

which is zero by Theorem 2.17. □

Proposition 3.20. *Let X be a klt del Pezzo surface over k with at most three singular non du Val singular points. Suppose that all of them are formally isomorphic to the quotient of $\mathbb{A}^2_k$ by the action of a group scheme μ_m for some $m > 0$. Then for any big and nef Cartier divisor A*

$$H^1(X, \mathcal{O}_X(A)) = 0.$$

Proof. Let us recall that given a Weil divisor D on a surface X with only μ_m-quotient singularities we have

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X) + \sum_{P \in \mathrm{NotCart(D)}} c_P(D),$$

where $c_P(D)$ is a rational number depending on the type of singularity of the pair (X, D) near P (for more details see [Rei87]). In [PV07, Corollary 4.1], the authors prove that $c_P(-K_X) > -1$ (let us note that the assumption on the characteristic of the base field is unnecessary). Applying the Riemann-Roch formula we have

$$\begin{aligned} h^0(\mathcal{O}_X(A - K_X)) &\geq 1 + \frac{1}{2}(A^2 - 3A \cdot K_X + 2K_X^2) + \sum_{P \in \mathrm{NotCart}(K_X)} c_P(-K_X) \\ &\geq 3 + K_X^2 - 3 > 0. \end{aligned}$$

Thus by Proposition 2.17 we conclude $h^1(X, \mathcal{O}_X(K_X - A)) = 0$ and thus, by Serre duality, $h^1(\mathcal{O}_X(A)) = 0$. □

Remark 3.21. By [PV07, Corollary 4.1], we have that $c_P(-K_X) \geq -\frac{1}{3}$ if the singularity P is formally a quotient of $\mathbb{A}_k^2$ by μ_3. Then the same proof of the previous Proposition shows that on a log del Pezzo surface X with at most four non du Val singular points which are formally quotients by the group scheme μ_3, then any big and nef Cartier divisor A satisfies $H^1(X, \mathcal{O}_X(A)) = 0$. In particular vanishing theorems for Cartier divisors hold on the surface T we constructed in Section 3.2, which explains why we had to look for a $\mathbb{Q}$-Cartier Weil divisor violating the vanishing theorem.

We now deduce an effective vanishing for the H^1 of a positive line bundle on a log del Pezzo surface, thus answering a question of Cascini and Tanaka (see [CT18, Remark 3.2]).

Theorem 3.22. *Let X be a klt del Pezzo surface over k and let A be a big and nef Cartier divisor. Then*

1. $H^1(X, \mathcal{O}_X(-A)) = 0$*;*
2. *If* $p \geq 5$*, then* $H^1(X, \mathcal{O}_X(A)) = 0$*;*
3. *If* $p = 3$*, then* $H^1(X, \mathcal{O}_X(2A)) = 0$*;*
4. *If* $p = 2$*, then* $H^1(X, \mathcal{O}_X(4A)) = 0$*.*

Proof. To prove (1), it is enough to show that $H^0(X, A) \neq 0$ because we can thus consider an effective divisor D linearly equivalent to A and apply Theorem 2.17. So denoting by $f : Y \to X$ the minimal resolution, we have

$$H^0(X, \mathcal{O}_X(A)) = H^0(Y, \mathcal{O}_Y(f^*A)).$$

Since Y is a rational surface we have $h^2(Y, \mathcal{O}_Y(f^*A)) = h^0(Y, \mathcal{O}_Y(K_Y - f^*A)) = 0$ and therefore

$$h^0(Y, \mathcal{O}_Y(f^*A)) \geq 1 + \frac{1}{2}(f^*A)(f^*A - K_Y) = 1 + \frac{1}{2}(A^2 - K_X \cdot A) > 0.$$

To prove (2), let us note that if $H^1(X, \mathcal{O}_X(A)) \neq 0$ we have $H^1(X, \mathcal{O}_X(K_X - A)) \neq 0$ by Serre duality. Let us define a Weil $\mathbb{Q}$-Cartier ample divisor

$$L := A - K_X.$$

Consider a covering family $\{D_t\}$ of curves for X belonging to a very ample linear system. Since L and $-K_X$ are ample we have that

$$(p-1)(L \cdot D_t) - K_X \cdot D_t > 0.$$

Therefore we can apply Theorem 2.14 for every point $x \in X$ we can find a curve C_x passing through x such that

$$L \cdot C_x \leq 4 \frac{L \cdot D_t}{(p-1)L \cdot D_t - K_X \cdot D_t} < \frac{4}{p-1}.$$

Moreover, if $x \in X$ is chosen to be generic we have that $A \cdot C_x \geq 1$ since A is big Cartier divisor and therefore

$$L \cdot C_x = A \cdot C_x - K_X \cdot C_x > 1.$$

Thus concluding that $p < 5$.

In the case where $p = 3$, we apply the same proof to $L = 2A - K_X$ with the same notation to the curves D_t. In this case by Theorem 2.14 we can find that for any point x there exists a rational curve C_x passing through x such that

$$L \cdot C_x < \frac{4}{p-1} = 2.$$

However choosing x generic enough we have

$$L \cdot C_x = 2A \cdot C_x - K_X \cdot C_x > 2,$$

thus getting a contradiction. The proof for the case $p = 2$ is analogous. □

Remark 3.23. In the proof of assertions (2)-(4) the assumption that the singularities of X are klt is superfluous. Indeed the same proof works for projective normal surfaces with ample $\mathbb{Q}$-Cartier anti-canonical divisor.

We conclude by discussing the special case where the linear system induced by A is birational.

Proposition 3.24. *Let (X, Δ) be a log del Pezzo pair over k. Let A be a big and nef Cartier* *divisor such that the linear system $|A|$ is base point free and the morphism associated is birational onto the image. Then*

$$H^1(X, \mathcal{O}_X(A)) = 0.$$

Proof. Let $f\colon Y \to X$ be the minimal resolution and consider the effective divisor Δ_Y such that:

$$K_Y + \Delta_Y = f^*(K_X + \Delta).$$

Since klt surface singularities are rational, we have $R^1 f_* \mathcal{O}_Y(f^*A) = R^1 f_* \mathcal{O}_Y \otimes \mathcal{O}_X(A) = 0$. Thus we deduce

$$H^i(Y, \mathcal{O}_Y(f^*A)) = H^i(X, \mathcal{O}_X(A)).$$

By hypothesis, there exists an irreducible section $C \in |f^*A|$. The curve C is Gorenstein with dualizing sheaf:

$$\omega_C = \mathcal{O}_Y(K_Y + C) \otimes \mathcal{O}_C.$$

Consider the following short exact sequence:

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(C) \to \mathcal{O}_C(C) \to 0.$$

By Lemma 3.18 we have $H^i(Y, \mathcal{O}_Y) = 0$ for $i = 1, 2$. Thus taking the long exact sequence in cohomology we have

$$H^1(Y, \mathcal{O}_Y(f^*A)) = H^1(Y, \mathcal{O}_Y(C)) \simeq H^1(C, \mathcal{O}_C(C)).$$

Now, using Serre duality on C we have

$$H^1(C, \mathcal{O}_C(C)) \simeq H^1(C, \omega_C \otimes \mathcal{O}_C(-K_Y|_C))) \simeq H^0(C, \mathcal{O}_C(K_Y|_C))^*.$$

It is easy to see that $K_Y|_C$ is an anti-ample divisor because

$$K_Y \cdot C = (f^*(K_X + \Delta) - \Delta_Y) \cdot f^*A = (K_X + \Delta) \cdot A - \Delta_Y \cdot f^*A < 0.$$

Therefore $H^0(C, \mathcal{O}_C(K_Y|_C)) = 0$, thus concluding the proof. □

4

Pathologies in positive characteristic birational geometry

4.1. Introduction

Singularities play a crucial role in the recent development of the Minimal Model Program. Most of the results in the study of singularities in characteristic zero, such as normality of plt centres and klt-ness of the base of Mori fibre spaces, rely on subtle applications of the Kawamata-Viehweg vanishing theorem and semi-positivity theorems, which originate from Hodge theoretic methods. One may thus ask whether such results still hold true in positive characteristic.

In this chapter we show various counterexamples in positive characteristic to some of the by now classical results on singularity theory for varieties in characteristic zero, extending previous work of Cascini and Tanaka ([CT] and [Tana]). Let us summarise the properties of our pathological examples:

Theorem 4.1. *Let k be a perfect field of characteristic $p \geq 3$. Then*

1. *There exists a purely log terminal pair (Z, S) of dimension $2p+2$ such that $S = \lfloor S \rfloor$ is not normal (see Theorem 4.4);*
2. *there exists a terminal Fano variety W of dimension $2p+2$ such that $H^2(W, \mathcal{O}_W) \neq 0$ (see Theorem 4.9);*
3. *if $p \geq 5$, there exists a Mori fibre space $f \colon X \to Y$, where X is a terminal variety of dimension $p+3$ and Y is a threefold which is not log canonical (see Theorem 4.11).*

4.2. Non-normal purely log terminal centres

In this section, we construct examples of non-normal purely log terminal centres in all positive characteristic $p \geq 3$.

The main ingredient is the following

Theorem 4.2 ([Tot17, Theorem 2.1]). *Let k be a field characteristic $p \geq 3$. Then there exists a smooth Fano variety X over k of dimension $2p+1$ with a very ample Cartier divisor A such that*

1. $\rho(X) = 2$,
2. $-K_X = 2A$ *and*
3. $H^1(X, \mathcal{O}_X(A)) \neq 0$.

Remark 4.3. The Fano varieties constructed above are homogeneous spaces under the action of SL_n with non-reduced stabilizers. They have already been a prolific source of examples of 'pathologies' in positive characteristic: by taking cones over X, Totaro shows that for every $p \geq 3$ there exists a terminal not Cohen-Macaulay singularity in dimension $2p+2$ ([Tot17, Corollary 2.2]). In [AZ17] the authors construct smooth Calabi-Yau varieties in positive characteristic which are not liftable to characteristic zero by considering general anticanonical sections of X and double covers along a general member of the linear system $|-2K_X|$.

To prove the main result we consider a cone over the Fano variety X constructed by Totaro and we show that the prime divisor induced on the cone by a smooth section $E \in |A|$ is not normal.

Theorem 4.4. *Let k be any field of characteristic $p \geq 3$. Then there exists a log pair (Z, S) such that*

1. *Z is an affine variety over k with terminal singularities of dimension $2p+2$ and S is a prime divisor,*
2. *(Z, S) is a purely log terminal pair with $K_Z + S$ Cartier,*
3. *S is not normal.*

Proof. Let us fix a field k of characteristic $p \geq 3$. Let us consider the smooth Fano variety X with the ample divisor A of Theorem 4.2. Since A is very ample, by Bertini theorem there exists a smooth divisor $E \in |A|$. We define the pair

$$(Z, S) := (C_a(X, \mathcal{O}_X(A)), E_{C_a(X, \mathcal{O}_X(A))}).$$

Since (X, E) is log smooth and $K_X + E \sim -A$ we conclude by Proposition 2.9 that $K_Z + S$ is Cartier and by Proposition 2.10 that the pair (Z, S) is dlt. Since S is the only irreducible component in the boundary, the pair (Z, S) is actually plt. We denote by (Y, S^Y) the pair given by $(BC_a(X, \mathcal{O}_X(A)), f_*^{-1}S)$.

We check that S is not normal. The subvariety S can be written as

$$S = \operatorname{Spec}_k \sum_{m \geq 0} \operatorname{Im}(H^0(X, \mathcal{O}_X(mA)) \to H^0(E, \mathcal{O}_E(mA)).$$

Let us consider the short exact sequence

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0,$$

and tensor it with $\mathcal{O}_X(nA)$, where n is a positive integer. Taking the long exact sequence in cohomology we have

$$H^0(X, \mathcal{O}_X(nA)) \to H^0(E, \mathcal{O}_E(nA)) \to H^1(X, \mathcal{O}_X((n-1)A)) \to H^1(X, \mathcal{O}_X(nA)).$$

Since A is ample and $H^1(X, \mathcal{O}(A)) \neq 0$, we can consider, by Serre vanishing, the largest $n \geq 2$ such that $H^1(X, \mathcal{O}_X((n-1)A)) \neq 0$ and $H^1(X, \mathcal{O}_X(nA)) = 0$. Thus the morphism

$$H^0(X, \mathcal{O}_X(nA)) \to H^0(E, \mathcal{O}(nA))$$

is not surjective and therefore the morphism

$$\nu \colon S^\nu := C_a(E, \mathcal{O}_X(A)|_E) \to S,$$

induced by the natural injection of k-algebras

$$\sum_{m \geq 0} \operatorname{Im}(H^0(X, \mathcal{O}_X(mA)) \to H^0(E, \mathcal{O}_E(mA)) \subset \sum_{m \geq 0} H^0(E, \mathcal{O}_E(mA))$$

is not an isomorphism. The morphism ν is finite and it is birational since for $m > n$ we have $H^0(E, \mathcal{O}_E(mA)) = \operatorname{Im}(H^0(X, \mathcal{O}_X(mA)) \to H^0(E, \mathcal{O}_E(mA))$. Thus we conclude that the variety S is not normal and that ν is the normalisation morphism, since S^ν is normal. □

Remark 4.5. We note that, since $\rho(X) = 2$, the affine variety Z is not $\mathbb{Q}$-factorial by Proposition 2.9.

Remark 4.6. We point out that S is regular in codimension one and thus by Serre's criterion we deduce that S does not satisfy the S_2 condition. In general, plt centres satisfy the R_1 condition also in positive characteristic as explained in [GNT, Lemma 2.5].

Remark 4.7. The normalisation morphism $\nu \colon S^\nu \to S$ is a universal homeomorphism since the morphism $S^Y \to S$ has connected fibers. In [GNT, Theorem 3.15], the authors prove, assuming the existence of pl-flips, that this is always the case for plt centres on threefolds in any characteristic.

We show that the lifting lemma of Hacon-M^cKernan (see [HM07, Theorem 5.4.21]) fails over fields of positive characteristic:

Corollary 4.8. *With the same notation as in Theorem 4.4, there exists a projective birational morphism* $f\colon (Y, S^Y) \to (Z, S)$ *such that*

1. (Y, S^Y) *is log smooth and* S^Y *is a prime divisor,*
2. $-\operatorname{Exc}(f)$ *is an* f*-ample divisor,*
3. $K_Y + S^Y$ *is semiample and big, and*
4. *for every* $m \geq 0$*, the restriction map*

$$H^0(Y, \mathcal{O}_Y(m(K_Y + S^Y))) \to H^0(S^Y, \mathcal{O}_Y(mK_{S^Y})),$$

is not surjective.

Proof. By construction, the pair (Y, S^Y) is an $\mathbb{A}^1$-bundle over a log smooth pair. Thus property (1) is immediate. To show property (2) we note that the divisor $-X^-$ is f-ample over the affine variety Z, thus it is ample. We have, by formula (2.5) of Proposition 2.9, that

$$K_Y + S^Y = f^*(K_Z + S),$$

and since $K_Z + S$ is ample on Z we conclude property (3) holds.

We now show property (4). Since S is not normal we have that the morphism

$$f_*\mathcal{O}_Y \to f_*\mathcal{O}_{S^Y} \tag{4.1}$$

is not surjective. Indeed, since $S^Y \to S^\nu$ is a surjective birational projective morphism between normal varieties, we have by Zariski's main theorem the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{O}_Z & \twoheadrightarrow & \mathcal{O}_S & & \\ \downarrow \simeq & & \downarrow & \searrow & \\ & & & & \nu_*\mathcal{O}_{S^\nu} \\ & & & \swarrow \simeq & \\ f_*\mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_{S^Y} & & \end{array}.$$

Since $\mathcal{O}_S \to \nu_*\mathcal{O}_{S^\nu}$ is not surjective, we conclude that the bottom arrow is not surjective.

By Proposition 2.9 we have that $K_Z + S \sim 0$ and thus, using the projection formula, we have

$$f_*\mathcal{O}_Y(m(K_Y + S^Y)) = f_*(f^*\mathcal{O}_Z(m(K_Z + S))) = f_*\mathcal{O}_Y,$$

and in the same way

$$f_*\mathcal{O}_{S^Y}(mK_{S^Y}) = f_*\mathcal{O}_{S^Y}.$$

Since Z is affine, we have therefore that the morphism in (4.1) is not surjective if and only if

$$H^0(Y, \mathcal{O}_Y(m(K_Y + S^Y))) \to H^0(S^Y, \mathcal{O}_{S^Y}(mK_{S^Y})),$$

is not surjective for any integer $m > 0$, thus concluding the proof. □

4.3. Terminal Fano varieties with $H^2(\mathcal{O}_Z) \neq 0$

In this section we construct Fano varieties with terminal singularities and non-vanishing intermediate cohomology.

We recall the Euler sequence on a projective vector bundle. Let E be a vector bundle of rank $r+1$ on a smooth variety X and let $\pi\colon \mathbb{P}_X(E) \to X$ be the associated projective bundle, then we have the following short exact sequence:

$$0 \to \Omega^1_{\mathbb{P}(E)/X} \to \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^* E \to \mathcal{O}_{\mathbb{P}(E)} \to 0,$$

which shows $K_{\mathbb{P}(E)} = \mathcal{O}_{\mathbb{P}(E)}(-r-1) \otimes \pi^*(K_X \otimes \det E)$.

Theorem 4.9. *Let k be a field of characteristic $p \geq 3$. Then there exists a Fano variety W with terminal singularities of dimension $2p+2$ over k such that*

$$H^2(W, \mathcal{O}_W) \neq 0.$$

Proof. Let us fix a field k of characteristic $p \geq 3$ and let us consider the Fano variety X with the ample divisor A of Theorem 4.2. We define the projective variety $W := C_p(X, \mathcal{O}_X(A))$. Since $H^1(X, \mathcal{O}_X(A)) \neq 0$, we conclude $H^2(W, \mathcal{O}_W) \neq 0$ by Proposition 2.13. The variety W has terminal singularities since the only singular point is the vertex of the cone $C_a(X, \mathcal{O}_X(A))$.

We are only left to prove that W is a Fano variety. For this it is sufficient to check that the projective bundle $\pi\colon Y := \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(-A)) \to X$ is Fano. On Y there is a unique negative section X^- such that $\mathcal{O}_Y(X^-)|_{X^-} \simeq \mathcal{O}_X(-A)$ and there is also a positive section X^+ such that $\mathcal{O}_Y(X^+)|_{X^+} \simeq \mathcal{O}_X(A)$ (which shows X^+ is a big and nef divisor on Y). A simple computation shows $X^+ \sim X^- + \pi^* A$. We note that

$$\mathcal{O}_Y(1) = \mathcal{O}_Y(X^-),$$

and by the relative Euler sequence we have

$$K_Y = \mathcal{O}_Y(-2) \otimes \pi^*(K_X - A).$$

Thus we have that the anticanonical class

$$-K_Y = 2X^- + \pi^*(3A) = 2X^+ + \pi^* A, \tag{4.2}$$

is a big and nef divisor. To conclude that $-K_Y$ is ample, we show that the null locus $\mathrm{Null}(-K_Y)$ (see [Laz04b, Definition 10.3.4]) is empty. By equation (4.2), the null locus must be contained in X^-, but since $-K_Y|_{X^-} = A$ is ample we conclude it must be empty. □

4.4. Singularities of the base of Mori fibre spaces in positive characteristic

The aim of this section is to construct further examples of klt varieties admitting a Mori fibre space over a base with non-klt singularities in higher characteristic. Since our construction is based on the work of Yasuda on wild quotient singularities (see [Yas14, Yas17]) we start by recalling some of his results.

We fix a field k of characteristic $p > 0$ and the cyclic group $G := \mathbb{Z}/p\mathbb{Z}$. Let us recall that for every integer $1 \leq i \leq p$ we have a unique indecomposable representation of G over k on a k-vector space of dimension i denoted by V_i, which is given by the following matrix $(i \times i)$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & \dots & 1 & 1 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

Let V be a G-representation and consider the quotient $V \to X := V/G$. The representation V decomposes into sum of indecomposable ones

$$V = \bigoplus_{\lambda=1}^{l} V_{d_\lambda}$$

with $1 \leq d_\lambda \leq p$ and we introduce the following invariant for X:

$$D_V = \sum_{i=1}^{l} \frac{d_\lambda(d_\lambda - 1)}{2}.$$

Theorem 4.10 ([Yas17]). *Suppose $D_V \geq 2$. Then the quotient X is terminal (resp. canonical, log canonical) if and only if $D_V > p$ (resp. $D_V \geq p$, $D_V \geq p - 1$).*

We can now construct new examples of Mori fibre space with bad singularities on the base.

Theorem 4.11. *Let k be a field of characteristic $p \geq 5$. Then there exists a projective contraction $f\colon X \to Y$ of normal k-varieties such that*

1. *X is a $\mathbb{Q}$-factorial terminal quasi-projective variety of dimension $p + 3$;*

2. Y is a $\mathbb{Q}$-factorial affine variety of dimension three which is not log canonical;

3. $\rho(X/Y) = 1$ and $-K_X$ is f-ample, equivalently f is a Mori fibre space.

Proof. Consider the indecomposable representation of G on the three-dimensional space V_3. Since $D_{V_3} = 3 < p - 1$ we have by Theorem 4.10 that the quotient $Y := V_3/G$ is not log canonical.

Now we consider the space $(\mathbb{P}^1)^p$ with the following G action:

$$T(([x_1 : y_1], [x_2 : y_2], \ldots, [x_p : y_p])) = ([x_p : y_p], [x_1 : y_1], \ldots, [x_{p-1} : y_{p-1}]).$$

Now we consider the space $(\mathbb{P}^1)^p \times V_3$ and we let G act diagonally.

Claim. The quotient $X := ((\mathbb{P}^1)^p \times V_3)/G$ has terminal singularities.

Proof. Consider the affine charts $U_x := \{x_i \neq 0\} \subset (\mathbb{P}^1)^p$ and $U_y := \{y_i \neq 0\} \subset (\mathbb{P}^1)^p$. These affine charts are isomorphic to $\mathbb{A}^p_k$. The fixed locus of T is contained in $U_x \cup U_y$ and $T(U_x) \subset U_x$ (resp. $T(U_y) \subset U_y$). So we only need to understand the action of G on $U_x \times V_3$ and $U_y \times V_3$. Since the situation is symmetric it is sufficient to understand the action on $U_x \times V_3$. Since the action of G on $U_x \times V_3$ is the sum of the irreducible representations $V_p \oplus V_3$ and since $D_{V_p \oplus V_3} = \frac{p(p-1)}{2} + 3 \geq p + 1$ we conclude it is terminal by Theorem 4.10. □

We are only lef to prove that $X \to V_3/G$ is a Mori fibre space. Consider now the following diagram

$$\begin{array}{ccc} (\mathbb{P}^1)^p \times V_3 & \longrightarrow & X \\ \downarrow & & \downarrow \\ V_3 & \longrightarrow & Y \end{array}$$

We are only left to check that $X \to Y$ is a Mori fibre space. It is sufficient to check that $\rho(X/Y) = 1$ and this is immediate since $\mathrm{Pic}(X) \hookrightarrow (\mathrm{Pic}((\mathbb{P}^1)^p \times V_3))^G = \mathbb{Z}$. □

Let us note that in Theorem 4.11, the relative dimension of the fibration increases with the characteristic p. Thus we are led to consider the following

Conjecture 4.12. *Let n and d be positive integers. Does there exists $p_0(d)$ such that if $f\colon X \to Y$ is a Mori fibre space where X is a klt $\mathbb{Q}$-factorial variety of dimension n and $\dim(X) - \dim(Y) = d$, then Y has klt singularities? In particular, does there exist $p_0(1)$?*

Even in the case of threefold conic bundles, we have very little evidence for the above conjecture. Let us note that Kollár proves that the base is smooth if the total space is smooth (see [Kol91, Complement 4.11.2]). In [NT, Theorem 3.8] the authors prove that the base of a threefold conic bundle has $W\mathcal{O}$-rational singularities for $p > 5$ (recall that klt surface singularities are rational and in particular $W\mathcal{O}$-rational).

In the remaining part of the chapter, we present a partial positive result on the descent of singularities of a Mori fibre space onto the base if the total space has Gorenstein canonical singularities. To achieve it, we will use some notions from the theory of F-singularities of which we now recall the basic definitions.

Lemma 4.13. *Let k be a field of characteristic $p > 0$. Let $f\colon X \to Y$ be a finite morphism of normal k-schemes of degree n such that $p \nmid n$. Then if X is F-pure (respectively, it is strongly F-regular), then Y is F-pure (respectively, it is strongly F-regular).*

Proof. The statement is local, so we can suppose that X is an affine globally F-split (resp. globally F-regular) scheme. Choose a splitting ψ of the Frobenius morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ in the category of $\mathcal{O}_X$-modules. We have the following commutative square:

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{F_Y} & F_*\mathcal{O}_Y \\ {\scriptstyle f^\#}\downarrow & & \downarrow{\scriptstyle F_*f^\#} \\ f_*\mathcal{O}_X & \xrightarrow{f_*F_X} & f_*F_*\mathcal{O}_X \end{array}$$

Since $p \nmid n$, we have that $\frac{1}{n}\mathrm{Trace}_{X/Y}\colon f_*\mathcal{O}_X \to \mathcal{O}_Y$ is a splitting for $f^\#$. Thus a splitting of F_Y is given by $\frac{1}{n}\mathrm{Trace}_{X/Y} \circ f_*\psi \circ F_*f^\#$.

For the case of strongly F-regular singularities, we fix D an effective Cartier divisor on Y and we consider the natural commutative diagram for any integer $e > 0$:

$$\begin{array}{ccc} \mathcal{O}_Y & \longrightarrow & F^e_*\mathcal{O}_Y(D) \\ \downarrow & & \downarrow \\ f_*\mathcal{O}_X & \longrightarrow & f_*F^e_*\mathcal{O}_X(f^*D) \end{array}$$

The same argument as before shows that a splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X(f^*D)$ in the category of $\mathcal{O}_X$-modules induces a splitting of $\mathcal{O}_Y \to F^e_*\mathcal{O}_Y(D)$ in the category of $\mathcal{O}_Y$-modules. $\square$

Proposition 4.14. *Let k be an algebraically closed field of characteristic $p > 5$. Let X be a $\mathbb{Q}$-factorial normal quasi-projective threefold over k with klt singularities and let $f\colon X \to S$ be a Mori fibre space of relative dimension one. If p does not divide the Cartier index of K_X (e.g. X has canonical Gorenstein singularities), then the surface S is $\mathbb{Q}$-factorial and has klt singularities.*

Proof. The $\mathbb{Q}$-factoriality of the base S of a Mori fibre space in characteristic $p > 5$ is proven in [HNT19, Theorem 5.4].

We are left to prove that S is klt. Since a general fibre F is $\mathbb{P}^1_k$, we have by adjunction that $-K_X \cdot F = 2$. Let us denote by i the Cartier index of K_X. Consider L a sufficiently ample Cartier divisor on S such that $A := -iK_X + f^*L$ is an ample Cartier divisor on X. This implies that there exists n_0 such that for any $n \geq n_0$, the linear system $|nA|$ is very ample. Let us choose n such that $p \nmid n$. By [ST17, Main Theorem], we deduce that a general section H of $|nA|$ has klt singularities. Since $H \cdot F = 2in$, the morphism $H \to S$ is finite of degree $2in$. Since $p > 5$, the singularities of H are strongly F-regular by [Ha98]. Since the degree of the finite morphism $H \to S$ is not divisible by p, we conclude by Lemma 4.13 that the singularities of S are strongly F-regular. Since S is $\mathbb{Q}$-factorial, we conclude it is klt. □

Let us note however that Conjecture 4.12 cannot be extended to the case of lc-trivial (or log Calabi-Yau) fibrations in positive characteristic. Indeed, over a field of characteristic zero, we have the following stronger statement on singularities due to Ambro:

Theorem 4.15 ([Amb05, Theorem 0.2])**.** *Let k be a field of characteristic zero. Let (X, Δ) be a normal quasi-projective klt pair over k. Let $f\colon X \to Y$ be a contraction between normal varieties such that $K_X + \Delta \sim_{f,\mathbb{Q}} 0$. Then there exists a boundary Δ_Y on Y such that (Y, Δ_Y) is a klt pair.*

Unfortunately in characteristic $p > 0$, one cannot hope to bound the singularities of the base of a log-canonical trivial fibration even in large characteristic as we show in the following example, based on a generalisation due to Lorenzini and Schröer ([LS18]) to large characteristic of the $\mathbb{Z}/2\mathbb{Z}$-wild quotients in characteristic two due to Artin (see [Art75])

Proposition 4.16. *Let k be a perfect field of characteristic $p \geq 5$. Then there exists a projective contraction $f\colon X \to S$ of normal k-varieties with the following properties.*

1. *X is a smooth threefold,*
2. *S is a surface that is not log canonical.*
3. *$K_X \sim_{f,\mathbb{Q}} 0$.*

Proof. Let us consider the ring

$$A = k[x, y][u, v]/(u^p - x^{p-1}u - x, v^p - y^{p-1}v - y).$$

Let us consider the $(\mathbb{Z}/p\mathbb{Z})$-action

$$\sigma : (x, y, u, v) \mapsto (x, y, u + a, v + b)$$

on A. By the Jacobi's criterion, the ring A is regular. As explained in [LS18, Theorem 7.5], the invariant ring under the action of σ is

$$A^{\mathbb{Z}/p\mathbb{Z}} = k[x, y, z]/(z^p - x^{p-1}y^{p-1}z - x^p y + y^p x),$$

where $z = yu - vx$. For $p \geq 5$, the multiplicity at the origin $\mathrm{mult}_O(z^p - x^{p-1}y^{p-1}z - x^p y + y^p x) \geq 5$ and thus the singularity is not log canonical.

Let E be an ordinary elliptic curve and consider a non-zero p-torsion point $a \in E[p]$. We have a natural $(\mathbb{Z}/p\mathbb{Z})$-action on E given by the translation τ_a. So we have the equivariant $(\mathbb{Z}/p\mathbb{Z})$-morphism $p_1\colon \mathrm{Spec}(A) \times E \to \mathrm{Spec}(A)$. Let us consider the induced morphism $f\colon X := (\mathrm{Spec}(A) \times E)/(\mathbb{Z}/p\mathbb{Z}) \to \mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A) \times E & \xrightarrow{\pi} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}}) \end{array}$$

Since the action of $(\mathbb{Z}/p\mathbb{Z})$ has no fixed points on $\mathrm{Spec}(A) \times E$ we conclude that X is a regular scheme. Moreover $\pi^*(K_{X/\mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})}) \sim 0$, implies $K_{X/\mathrm{Spec}(A^{\mathbb{Z}/p\mathbb{Z}})} \sim_{\mathbb{Q}} 0$, thus concluding. $\square$

5

On del Pezzo fibrations in positive characteristic

Based on joint work with H. Tanaka.

5.1. Introduction

The minimal model conjecture predicts that an arbitrary algebraic variety is birational to either a minimal model or a Mori fibre space $\pi\colon V \to B$. A distinguished property of Mori fibre spaces in characteristic zero is that any relative numerically trivial line bundles is automatically trivial (cf. [KMM87, Lemma 3.2.5]). Let us recall that given a morphism $\pi\colon V \to B$ a line bundle L on V is said to be π-torsion if there exists a positive integer $m > 0$ such that $mL \sim_\pi 0$. The smallest integer m for which mL is trivial is called the torsion index of L. In [Tana, Theorem 1.4], Tanaka constructs counterexamples to the same statement in positive characteristic. More specifically, if the characteristic is two or three, then there exists a Mori fibre space $\pi\colon V \to B$ and a line bundle L on V such that $\dim V = 3, \dim B = 1, L \equiv_\pi 0$, and $L \not\sim_\pi 0$. Then it is tempting to ask how bad the torsion indices can be.

One of the main results of this chapter is to give such an explicit upper bound of torsion indices for three-dimensional del Pezzo fibrations.

Theorem 5.1 (Theorem 5.70)**.** *Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi\colon V \to B$ be a projective k-morphism such that $\pi_*\mathcal{O}_V = \mathcal{O}_B$, where V is a three-dimensional $\mathbb{Q}$-factorial normal quasi-projective variety over k and B is a smooth curve over k. Assume there exists an effective $\mathbb{Q}$-divisor Δ such that (V, Δ) is klt and $\pi\colon V \to B$ is a $(K_V + \Delta)$-Mori fibre space. Let L be a π-numerically trivial Cartier divisor on V.*

Then the following hold.

1. *If* $p \geq 7$, *then* $L \sim_{\pi} 0$.
2. *If* $p \in \{3, 5\}$, *then* $p^2 L \sim_{\pi} 0$.
3. *If* $p = 2$, *then* $16L \sim_{\pi} 0$.

We also prove a theorem of Graber–Harris–Starr type for del Pezzo fibrations in positive characteristic.

Theorem 5.2 (Theorem 5.69). *Let* k *be an algebraically closed field of characteristic* $p > 0$. *Let* $\pi : V \to B$ *be a projective* k*-morphism such that* $\pi_* \mathcal{O}_V = \mathcal{O}_B$, V *is a normal three-dimensional variety over* k, *and* B *is a smooth curve over* k. *Assume that there exists an effective* $\mathbb{Q}$*-divisor* Δ *such that* (V, Δ) *is klt and* $-(K_V + \Delta)$ *is* π*-nef and* π*-big. Then the following hold.*

1. *There exists a curve* C *on* V *such that* $C \to B$ *is surjective and the following properties hold.*
   - (a) *If* $p \geq 7$, *then* $C \to B$ *is an isomorphism.*
   - (b) *If* $p \in \{3, 5\}$, *then* $K(C)/K(B)$ *is a purely inseparable extension of degree* $\leq p$.
   - (c) *If* $p = 2$, *then* $K(C)/K(B)$ *is a purely inseparable extension of degree* ≤ 4.
2. *If* B *is a rational curve, then* V *is rationally chain connected.*

Theorem 5.2 can be considered as a generalisation of classical Tsen's theorem, i.e. the existence of sections on ruled surfaces. Tsen's theorem was used to establish the log minimal model program in characteristic $p > 5$ [BW17, Section 3.4]. Also, Tsen's theorem was used to show that $H^i(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for threefolds X of Fano type in characteristic $p > 5$ when $i > 0$ (cf. [GNT, Theorem 1.3]).

The proofs of Theorem 5.1 and Theorem 5.2 are carried out by studying the generic fibre $X := V \times_B \operatorname{Spec} K(B)$ of π, which is a surface of del Pezzo type defined over an imperfect field. Roughly speaking, Theorem 5.1 and Theorem 5.2 hold by the following two theorems.

Theorem 5.3 (Theorem 5.45). *Let* k *be a field of characteristic* $p > 0$. *Let* X *be a* k*-surface of del Pezzo type. Let* L *be a numerically trivial Cartier divisor on* X. *Then the following hold.*

1. *If* $p \geq 7$, *then* $L \sim 0$.
2. *If* $p \in \{3, 5\}$, *then* $pL \sim 0$.
3. *If* $p = 2$, *then* $4L \sim 0$.

Theorem 5.4 (Theorem 5.65). *Let* k *be a* C_1*-field of characteristic* $p > 0$. *Let* X *be a* k*-surface of del Pezzo type such that* $k = H^0(X, \mathcal{O}_X)$. *Then*

1. *If* $p \geq 7$, *then* $X(k) \neq \emptyset$;

2. *If* $p \in \{3, 5\}$, *then* $X(k^{1/p}) \neq \emptyset$;

3. *If* $p = 2$, *then* $X(k^{1/4}) \neq \emptyset$.

5.1.1. Sketch of the proof of Theorem 5.3

Let us overview some of the ideas used in the proof of Theorem 5.3. By considering the minimal resolution and running a minimal model program, the problem can be reduced by [Tan18a, Theorem 4.4(3)] to the case when X is a regular surface of del Pezzo type which has a K_X-Mori fibre space structure $X \to B$. In particular, it holds that $\dim B = 0$ or $\dim B = 1$.

The case when $\dim B = 0$

Assume that $\dim B = 0$. In this case, X is a regular del Pezzo surface. We first classify $Y := (X \times_k \overline{k})^N_{\mathrm{red}}$ (Theorem 5.5). We then compare $X \times_k \overline{k}$ with $Y = (X \times_k \overline{k})^N_{\mathrm{red}}$ (Theorem 5.6).

Theorem 5.5 (Theorem 5.33)**.** *Let* k *be a field of characteristic* $p > 0$. *Let* X *be a projective normal surface over* k *with canonical singularities such that* $k = H^0(X, \mathcal{O}_X)$ *and* $-K_X$ *is ample. Then the normalisation* Y *of* $(X \times_k \overline{k})_{\mathrm{red}}$ *satisfies one of the following properties.*

1. $X \times_k \overline{k}$ *is geometrically normal over* k. *Moreover,* $X \times_k \overline{k}$ *has at worst canonical singularities. In particular,* $Y \simeq X \times_k \overline{k}$ *and* $-K_Y$ *is ample.*

2. Y *is isomorphic to a Hirzebruch surface, i.e. a* $\mathbb{P}^1$*-bundle over* $\mathbb{P}^1$.

3. Y *is isomorphic to a weighted projective surface* $\mathbb{P}(1, 1, m)$ *for some positive integer* m.

Theorem 5.6 (cf. Theorem 5.35)**.** *Let* k *be a field of characteristic* $p > 0$. *Let* X *be a projective normal surface over* k *with canonical singularities such that* $k = H^0(X, \mathcal{O}_X)$ *and* $-K_X$ *is ample. Let* Y *be the normalisation of* $(X \times_k \overline{k})_{\mathrm{red}}$ *and let*

$$\mu : Y \to X \times_k \overline{k}$$

be the induced morphism.

1. *If* $p \geq 5$, *then* μ *is an isomorphism and* Y *has at worst canonical singularities.*

2. *If* $p = 3$, *then the absolute Frobenius morphism* $F_{X \times_k \overline{k}}$ *of* $X \times_k \overline{k}$ *factors through* μ*:*

$$F_{X \times_k \overline{k}} : X \times_k \overline{k} \to Y \xrightarrow{\mu} X \times_k \overline{k}.$$

3. *If $p = 2$, then the second iterated absolute Frobenius morphism $F^2_{X\times_k\overline{k}}$ of $X \times_k \overline{k}$ factors through μ:*
$$F^2_{X\times_k\overline{k}} : X \times_k \overline{k} \to Y \xrightarrow{\mu} X \times_k \overline{k}.$$

Note that Theorem 5.5 shows that $Y = (X \times_k \overline{k})^N_{\mathrm{red}}$ is a rational surface. In particular, any numerically trivial line bundle on Y is trivial. By Theorem 5.6, if L' denotes the pullback of L to $X \times_k \overline{k}$, then it holds that $L'^4 \simeq \mathcal{O}_{X\times_k\overline{k}}$ in the case (3). Then the flat base change theorem implies that also L^4 is trivial.

We now discuss the proofs of Theorem 5.5 and Theorem 5.6. Roughly speaking, we apply Reid's idea ([Rei94, cf. the proof of Theorem 1.1]) to prove Theorem 5.5 by combining with a rationality criterion (Lemma 5.32). As for Theorem 5.6, we use the notion of Frobenius length of geometric normality $\ell_F(X/k)$ introduced in [Tanb] (cf. Definition 2.19, Remark 2.20). Roughly speaking, if $p = 2$, then we can prove that $\ell_F(X/k) \leq 2$ by computing certain intersection numbers (cf. the proof of Proposition 5.34). Then general result on $\ell_F(X/k)$ (Remark 2.20) implies (3) of Theorem 5.6.

The case when $\dim B = 1$

Assume that $\dim B = 1$, i.e. $\pi : X \to B$ is a K_X-Mori fibre space to a curve B. Since X is of del Pezzo type, we have that the extremal ray R of $\overline{\mathrm{NE}}(X)$ that is not corresponding to $\pi : X \to B$ is spanned by an integral curve Γ, i.e. $R = \mathbb{R}_{\geq 0}[\Gamma]$. In particular, $\Gamma \to B$ is a finite surjective morphism of curves. If $K_X \cdot \Gamma < 0$, then the problem is reduced to the above case (5.1.1) by contracting Γ. Even if $K_X \cdot \Gamma = 0$, then we may contract Γ and apply the same strategy. Hence, it is enough to treat the case when $K_X \cdot \Gamma > 0$. Note that the numerically trivial Cartier divisor L on X descends to B, i.e. we have $L \sim \pi^* L_B$ for some Cartier divisor L_B on B. Then, a key observation is that the extension degree $[K(\Gamma) : K(B)]$ is at most five (Proposition 5.42). For example, if $p > 5$, then $\Gamma \to B$ is separable. Then the Hurwitz formula implies that $-K_B$ is ample, hence $L_B \sim 0$. If $K(\Gamma)/K(B)$ is purely inseparable of degree p^e, then it hold that $L_B^{p^e} \sim 0$, since $-K_{\Gamma^N}$ is ample. For the remaining case, i.e. $p = 2$, $[K(\Gamma) : K(B)] = 4$, and $K(\Gamma)/K(B)$ is inseparable but not purely inseparable, we prove that $H^0(B, L_B^4) \neq 0$ by applying Galois descent for the separable closure of $K(\Gamma)/K(B)$ (cf. the proof of Proposition 5.44).

5.1.2. Sketch of the proof of Theorem 5.4

Let us overview some of the ideas used in the proof of Theorem 5.4. The first step is the same as Subsection 5.1.1, i.e. considering the minimal resolution and running a minimal model program, we reduce the problem to the case when X is a regular surface of del Pezzo type which has a K_X-Mori fibre space structure $X \to B$.

The case when $\dim B = 0$

Assume that $\dim B = 0$. In this case, X is a regular del Pezzo surface with $\rho(X) = 1$. Since the p-degree of a C_1-field is at most one (Lemma 5.54), it follows from [FS18, Theorem 14.1] that X is geometrically normal. Then Theorem 5.5 implies that the base change $X \times_k \overline{k}$ is a canonical del Pezzo surface, i.e. $X \times_k \overline{k}$ has at worst canonical singularities and $-K_{X\times_k\overline{k}}$ is ample. In particular, we have that $1 \leq K_X^2 \leq 9$. Note that if X is smooth, then it is known that X has a k-rational point (cf. [Kol96, Theorem IV.6.8]). Following the same strategy as in [Kol96, Theorem IV.6.8], we can show that $X(k) \neq \emptyset$ if $K_X^2 \leq 4$ (Lemma 5.56). For the remaining cases $5 \leq K_X^2 \leq 9$, we use results established in [Sch08], which restrict the possibilities for the type of singularities on $X \times_k \overline{k}$. For instance, if $p \geq 11$, then [Sch08, Theorem 6.1] shows that the singularities on $X \times_k \overline{k}$ are of type A_{p^e-1}. However, such singularities cannot appear, because the minimal resolution V of $X \times_k \overline{k}$ satisfies $\rho(V) \leq 9$. Hence, X is actually smooth if $p \geq 11$ (Proposition 5.47). For the remaining cases $p \leq 7$, we study the possibilities one by one, so that we are able to deduce what we desire. For more details, see Subsection 5.6.1.

The case when $\dim B = 1$

Assume that $\dim B = 1$, i.e. $\pi : X \to B$ is a K_X-Mori fibre space to a curve B. Then the outline is similar to the one in (5.1.1). Let us use the same notation as in (5.1.1). The typical case is that $-K_B$ is ample. In this case, B has a rational point. Then also the fibre of π over a rational point, which is a conic curve, has a rational point. Although we need to overcome some technical difficulties, we may apply this strategy up to suitable purely inseparable covers for almost all the cases (cf. the proof of Proposition 5.63). There is one case we can not apply this strategy: $p = 2$, $K_X \cdot \Gamma > 0$, and $K(\Gamma)/K(B)$ is inseparable and not purely inseparable. In this case, we can prove that $-K_B$ is actually ample (Proposition 5.62).

5.1.3. Large characteristic

Using the techniques developed in this paper, we also prove the following theorem, which shows that some a priori possible pathologies of log del Pezzo surfaces over imperfect fields can appear exclusively in small characteristic.

Theorem 5.7 (cf. Corollary 5.50 and Theorem 5.52)**.** *Let k be a field of characteristic $p \geq 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then X is geometrically integral over k and $H^i(X, \mathcal{O}_X) = 0$ for any $i > 0$.*

As a consequence, we deduce the following result on del Pezzo fibrations in large characteristic:

Corollary 5.8. *Let k be an algebraically closed field of characteristic $p \geq 7$. Let $\pi\colon V \to B$ be a projective k-morphism of normal k-varieties such that $\pi_*\mathcal{O}_V = \mathcal{O}_B$ and* $\dim V -$

$\dim B = 2$. *Assume that there exists an effective* $\mathbb{Q}$*-divisor* Δ *on* V *such that* (V, Δ) *is klt and* $-(K_V + \Delta)$ *is* π*-nef and* π*-big. Then general fibres of* π *are integral schemes and there is a non-empty open subset* B' *of* B *such that the equation* $(R^i\pi_*\mathcal{O}_V)|_{B'} = 0$ *holds for any* $i > 0$.

The authors do not know whether surfaces of del Pezzo type are geometrically normal if the characteristic is sufficiently large. On the other hand, even if p is sufficiently large, regular surfaces of del Pezzo type can be non-smooth. More specifically, for an arbitrary imperfect field k of characteristic $p > 0$, we construct a regular surface of del Pezzo type which is not smooth (Proposition 5.67).

5.2. Preliminaries

5.2.1. Surfaces of del Pezzo type

In this subsection, we summarise some basic properties of surfaces of del Pezzo type over arbitrary fields. For later use, we introduce some terminology. Note that del Pezzo surfaces in our notation allow singularities.

Definition 5.9. Let k be a field. A k-surface X is *del Pezzo* if X is a projective normal surface such that $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. A k-surface X is *weak del Pezzo* if X is a projective normal surface such that $-K_X$ is a nef and big $\mathbb{Q}$-Cartier divisor.

We study how the property of being of del Pezzo type behaves under birational transformations.

Lemma 5.10. *Let* k *be a field. Let* X *be a* k*-surface of del Pezzo type. Let* $f : Y \to X$ *be the minimal resolution of* X*. Then* Y *is a* k*-surface of del Pezzo type.*

Proof. Let Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a log del Pezzo pair. We define a $\mathbb{Q}$-divisor Δ_Y by $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Since $f : Y \to X$ is the minimal resolution of X, we have that Δ_Y is an effective $\mathbb{Q}$-divisor. The pair (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is nef and big. Thus there exists E effective divisor such that for all $0 < \varepsilon \ll 1$ the divisor $-(K_Y + \Delta_Y) - \varepsilon E$ is ample. If ε is sufficienly small, then $(Y, \Gamma := \Delta_Y + \varepsilon E)$ is klt by the existence of log resolution for excellent schemes (see [Lip78]). Therefore the pair (Y, Γ) is klt and $-(K_Y + \Gamma)$ is ample, i.e. it is a log del Pezzo pair. □

Lemma 5.11. *Let* k *be a field. Let* (X, Δ) *be a two-dimensional projective klt pair over* k*. Let* H *be a nef and big* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor. Then there exists an effective* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor* A *such that* $A \sim_{\mathbb{Q}} H$ *and* $(X, \Delta + A)$ *is klt.*

Proof. Thanks to the existence of log resolutions for excellent surfaces [Lip78], the same proof of [GNT, Lemma 2.8] works in our setting. □

Lemma 5.12. *Let k be a field. Let X be a k-surface of del Pezzo type. Let $f : X \to Y$ be a birational k-morphism to a projective normal k-surface Y. Then Y is a k-surface of del Pezzo type.*

Proof. Let Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a log del Pezzo pair. Set $H := -(K_X + \Delta)$, which is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. By Lemma 5.11, there exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A such that $A \sim_{\mathbb{Q}} H$ and $(X, \Delta + A)$ is klt. Then the pair $(Y, f_*\Delta + f_*A)$ is klt and $K_X + \Delta + A \sim_{\mathbb{Q}} f^*(K_Y + f_*\Delta + f_*A) \sim_{\mathbb{Q}} 0$. It follows from [Tan18a, Corollary 4.11] that Y is $\mathbb{Q}$-factorial. By Nakai's criterion, the $\mathbb{Q}$-divisor f_*A is ample. In particular $(Y, f_*\Delta)$ is a log del Pezzo pair. □

5.2.2. Geometrically canonical del Pezzo surfaces

In this subsection we collect results on the anti-canonical systems of geometrically canonical del Pezzo surfaces we will need later.

Canonical del Pezzo surfaces over algebraically closed fields

We verify that the results in [Kol96, Chapter III, Section 3] hold for del Pezzo surfaces with canonical singularities over algebraically closed fields.

Proposition 5.13. *Let X be a canonical weak del Pezzo surface over an algebraically closed field k. Then the following hold.*

1. $H^2(X, \mathcal{O}_X(-mK_X)) = 0$ *for any non-negative integer m.*
2. $H^i(X, \mathcal{O}_X) = 0$ *for any $i > 0$.*
3. $H^0(X, \mathcal{O}_X(-K_X)) \neq 0$.
4. $H^1(X, \mathcal{O}_X(mK_X)) = 0$ *for any integer m.*
5. $h^0(X, \mathcal{O}_X(-mK_X)) = 1 + \frac{m(m+1)}{2}K_X^2$ *for any non-negative integer m.*

Proof. The assertion (1) follows from Serre duality. We now show (2). It follows from [Tan14, Theorem 5.4 and Remark 5.5] that X has at worst rational singularities. Then the assertion (2) follows from the fact that X is a rational surface [Tan15, Theorem 3.5].

We now show (3). By $H^2(X, \mathcal{O}_X(-K_X)) = 0$ and the Riemann–Roch theorem, we have $h^0(X, \mathcal{O}_X(-K_X)) \geq 1 + K_X^2 > 0$. Thus (3) holds.

We now show (4). By (3), there exists an effective Cartier divisor D such that $D \sim -K_X$. In particular, D is effective, nef, and big. It follows from [CT, Proposition 3.3] that

$$H^1(X, \mathcal{O}_X(-nD)) = H^1(X, \mathcal{O}_X(K_X + nD)) = 0$$

for any $n \in \mathbb{Z}_{>0}$. Replacing D by $-K_X$, the assertion (4) holds. Thanks to (1) and (4), assertion (5) follows from the Riemann–Roch theorem. □

Lemma 5.14. *Let Y be a canonical weak del Pezzo surface over an algebraically closed field k. If a divisor $\sum_{i=1}^{r} a_iC_i \in |-K_Y|$ is not irreducible or not reduced, then every C_i is a smooth rational curve.*

Proof. Taking the minimal resolution of Y, we may assume that Y is smooth. Fix an index $1 \leq i_0 \leq r$. By adjunction, we have

$$2p_a(C_{i_0}) - 2 = -C_{i_0} \cdot \left(\sum_{i \neq i_0} \frac{a_i}{a_{i_0}} C_i \right) - \frac{a_{i_0} - 1}{a_{i_0}} C_{i_0} \cdot (-K_Y). \tag{5.1}$$

Note that both the terms on the right hand side are non-positive.

Since Y is smooth and $\sum_i a_iC_i$ is nef and big, it follows from [Tan15, Theorem 2.6] that $H^1(X, -n\sum_i a_iC_i) = 0$ for $n \gg 0$. Hence, $\sum_i a_iC_i$ is connected. Therefore, if $\sum_i a_iC_i$ is reducible, the first term in the right hand side of (5.1) is strictly negative, hence $p_a(C_{i_0}) < 0$.

If $a_{i_0} \geq 2$ and $C_{i_0} \cdot K_Y < 0$, then the second term in the right hand side of (5.1) is strictly negative, hence $p_a(C_{i_0}) < 0$. If $C_{i_0} \cdot K_Y = 0$, then C_i is a smooth rational curve with $C_i^2 = -2$. □

Proposition 5.15. *Let Y be a canonical weak del Pezzo surface over an algebraically closed field k. Let $\mathrm{Bs}(-K_Y)$ be the base locus of $-K_Y$, which is a closed subset of Y. Then the following hold.*

1. $\mathrm{Bs}(-K_Y)$ *is empty or* $\dim(\mathrm{Bs}(-K_Y)) = 0$.

2. *A general member of the linear system $|-K_Y|$ is irreducible and reduced.*

Proof. Taking the minimal resolution of Y, we may assume that Y is smooth. Using Proposition 5.13, the same proof of [Dol12, Theorem 8.3.2.i] works in our setting, so that (1) holds and general members of $|-K_Y|$ are irreducible.

It is enough to show that a general member of $|-K_Y|$ is reduced. Suppose it is not. Then there exist $a > 1$ such that a general member is of the form $aC \in |-K_Y|$ for some curve C. In particular, C is a smooth rational curve by Lemma 5.14. Recall that we have the short exact sequence

$$0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y(C)) \to H^0(C, \mathcal{O}_C(C)) \to 0.$$

Since $H^1(Y, \mathcal{O}_Y) = 0$ (Proposition 5.13), we have that $h^0(Y, \mathcal{O}_Y(C)) = 1 + h^0(C, \mathcal{O}_C(C))$. As C is a smooth rational curve, we conclude by the Riemann–Roch theorem that $h^0(Y, \mathcal{O}_Y(C)) = 2 + C^2$.

We now consider the induced map

$$\begin{array}{ccc} H^0(Y, \mathcal{O}_Y(C)) & \to & H^0(Y, \mathcal{O}_Y(aC)) \simeq H^0(Y, \mathcal{O}_Y(-K_Y)) \\ \varphi & \mapsto & \varphi^a \end{array}$$

Since a general member of $|-K_Y|$ is of the form aD for some $D \geq 0$, θ is a dominant morphism if we consider θ as a morphism of affine spaces. Therefore, it holds that

$$h^0(Y, \mathcal{O}_Y(-K_Y)) \leq h^0(Y, \mathcal{O}_Y(C)) = 2 + C^2 = -K_Y \cdot C \leq K_Y^2,$$

which contradicts Proposition 5.13. □

Anti-canonical systems on geometrically canonical del Pezzo surfaces

In this section, we study anticanonical systems on geometrically canonical del Pezzo surfaces over an arbitrary field k and we describe their anti-canonical model when the anti-canonical degree is small.

We need the following results on geometrically integral curves of genus one.

Lemma 5.16. *Let k be a field. Let C be a geometrically integral Gorenstein projective curve over k of arithmetic genus one with $k = H^0(C, \mathcal{O}_C)$. Let L be a Cartier divisor on C and let $R(C, L) := \bigoplus_{m \geq 0} H^0(C, mL)$ be the graded k-algebra. Then the following hold.*

(i) If $\deg_k(L) = 1$, then $\mathrm{Bs}(L) = \{P\}$ for some k-rational point P and $R(C, L)$ is generated by $\bigoplus_{1 \leq j \leq 3} H^0(C, jL)$ as a k-algebra.

(ii) If $\deg_k(L) \geq 2$, then L is globally generated and $R(C, L)$ is generated by $H^0(C, L) \oplus H^0(C, 2L)$ as a k-algebra.

(iii) If $\deg_k L \geq 3$, then L is very ample and $R(C, L)$ is generated by $H^0(C, L)$ as a k-algebra.

Proof. See [Tanb, Lemma 11.10 and Proposition 11.11]. □

Proposition 5.17. *Let k be a field. Let X be a geometrically canonical weak del Pezzo surface over k such that $k = H^0(X, \mathcal{O}_X)$. Let $R(X, -K_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(-mK_X))$ be the graded k-algebra. Then the following hold.*

1. *If m is a positive integer such that $mK_X^2 \geq 2$, then $|-mK_X|$ is base point free.*
2. *If $K_X^2 = 1$, then $\mathrm{Bs}(-K_X) = \{P\}$ for some k-rational point P.*
3. *If $K_X^2 = 1$, then $R(X, -K_X)$ is generated by $\bigoplus_{1 \leq j \leq 3} H^0(X, -jK_X)$ as a k-algebra.*
4. *If $K_X^2 = 2$, then $R(X, -K_X)$ is generated by $H^0(X, -K_X) \oplus H^0(X, -2K_X)$ as a k-algebra.*
5. *If $K_X^2 \geq 3$, then $R(X, -K_X)$ is generated by $H^0(X, -K_X)$ as a k-algebra.*

In particular, if $-K_X$ is ample, then $|-6K_X|$ is very ample.

Proof. Consider the following condition.

(2)' If $K_X^2 = 1$, then $\mathrm{Bs}(-K_X)$ is not empty and of dimension zero.

Since $K_X^2 = 1$, (2) and (2)' are equivalent. Note that to show that (1), (2)', and (3)–(5), we may assume that k is algebraically closed.

From now on, let us prove (1)–(5) under the condition that k is algebraically closed. It follows from Proposition 5.15 that a general member C of $|-K_X|$ is a prime divisor.

Since C is a Cartier divisor and X is Gorenstein, then C is a Gorenstein curve. By adjunction, C is a Gorenstein curve of arithmetic genus $p_a(C) = 1$. By Proposition 5.13, we have the following exact sequence for every integer m:

$$0 \to H^0(X, -(m-1)K_X) \to H^0(X, -mK_X) \to H^0(C, -mK_X|_C) \to 0.$$

By the above exact sequence, the assertions (1) and (2) follow from (3) and (2) of Lemma 5.16, respectively.

We prove the assertions (3), (4) and (5). By the above short exact sequence, it is sufficient to prove the same statement for the k-algebra $R(C, \mathcal{O}_C(-K_X))$, which is the content of Lemma 5.16. □

Theorem 5.18. *Let k be a field. Let X be a geometrically canonical del Pezzo surface over k such that $H^0(X, \mathcal{O}_X) = k$. Then the following hold.*

1. *If $K_X^2 = 1$, then X is isomorphic to a weighted hypersurface in $\mathbb{P}_k(1,1,2,3)$ of degree six.*

2. *If $K_X^2 = 2$, then X is isomorphic to a weighted hypersurface in $\mathbb{P}_k(1,1,1,2)$ of degree four.*

3. *If $K_X^2 = 3$, then X is isomorphic to a hypersurface in $\mathbb{P}_k^3$ of degree three.*

4. *If $K_X^2 = 4$, then X is isomorphic to a complete intersection of two quadric hypersurfaces in $\mathbb{P}_k^4$.*

Proof. Using Proposition 5.17, the proof is the same as in [Kol96, Theorem III.3.5]. □

5.2.3. Mori fibre spaces to curves

In this subsection, we summarise properties of regular curves with anti-ample canonical divisor and of Mori fibre space of dimension two over arbitrary fields.

Lemma 5.19. *Let k be a field. Let C be a projective Gorenstein integral curve over k. Then the following are equivalent.*

1. *ω_C^{-1} is ample.*

2. *$H^1(C, \mathcal{O}_C) = 0$.*

3. *C is a conic curve of $\mathbb{P}_K^2$, where $K := H^0(C, \mathcal{O}_C)$.*

4. *$\deg_k \omega_C = -2\dim_k(H^0(C, \mathcal{O}_C))$.*

Proof. It follows from [Tan18a, Corollary 2.8] that (1), (2), and (4) are equivalent. Clearly, (3) implies (1). By [Kol13, Lemma 10.6], (1) implies (3). □

Lemma 5.20. *Let k be a field and let C be a projective Gorenstein integral curve over k such that $k = H^0(C, \mathcal{O}_C)$ and ω_C^{-1} is ample. Then the following hold.*

1. *If C is geometrically integral over k, then C is smooth over k.*
2. *If the characteristic of k is not two, then C is geometrically reduced over k.*
3. *If the characteristic of k is not two and C is regular, then C is smooth over k.*

Proof. By Lemma 5.19, C is a conic curve in $\mathbb{P}^2_k$. Thus, the assertion (1) follows from the fact that an integral conic curve over an algebraically closed field is smooth.

Let us show (2) and (3). Since the characteristic of k is not two and C is a conic curve in $\mathbb{P}^2_k$, we can write

$$C = \mathrm{Proj}\ k[x, y, z]/(ax^2 + by^2 + cz^2)$$

for some $a, b, c \in k$. Since C is an integral scheme, two of a, b, c are not zero. Hence, C is reduced. Thus (2) holds. If C is regular, then each of a, b, c is nonzero, hence C is smooth over k. □

Proposition 5.21. *Let k be a field. Let $\pi : X \to B$ be a K_X-Mori fibre space from a projective regular k-surface X to a projective regular k-curve with $k = H^0(B, \mathcal{O}_B)$. Let b be a (not necessarily closed) point. Then the following hold.*

1. *The fibre X_b is irreducible.*
2. *The equation $k(b) = H^0(X_b, \mathcal{O}_{X_b})$ holds.*
3. *The fibre X_b is reduced.*
4. *The fibre X_b is a conic in $\mathbb{P}^2_{k(b)}$.*
5. *If $\mathrm{char}\, k \neq 2$, then any fibre of π is geometrically reduced.*
6. *If $\mathrm{char}\, k \neq 2$ and k is separably closed, then π is a smooth morphism.*

Proof. If X_b is not irreducible, it contradicts the hypothesis $\rho(X/B) = 1$. Thus (1) holds.

Let us show (2). Since π is flat, the integer

$$\chi := \dim_{k(b)} H^0(X_b, \mathcal{O}_{X_b}) - \dim_{k(b)} H^1(X_b, \mathcal{O}_{X_b}) \in \mathbb{Z}$$

is independent of $b \in B$. Since $H^1(X_b, \mathcal{O}_{X_b}) = 0$ for any $b \in B$, it suffices to show that $\dim_{k(b)} H^0(X_b, \mathcal{O}_{X_b}) = 1$ for some $b \in B$. This holds for the case when b is the generic point of B. Hence, (2) holds.

Let us prove (3). It is clear that the generic fibre is reduced. We may assume that $b \in B$ is a closed point. Assume that X_b is not reduced. By (1), we have $X_b = mC$ for some prime divisor C and $m \in \mathbb{Z}_{\geq 2}$. Since $-K_X \cdot_{k(b)} X_b = 2$, we have that $m = 2$. Then we obtain an exact sequence:

$$0 \to \mathcal{O}_X(-C)|_C \to \mathcal{O}_{X_b} \to \mathcal{O}_C \to 0.$$

Since $C^2 = 0$ and ω_C^{-1} is ample, we have that $\mathcal{O}_X(-C)|_C \simeq \mathcal{O}_C$. Since $H^1(C, \mathcal{O}_C) = 0$, we get an exact sequence:

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(X_b, \mathcal{O}_{X_b}) \to H^0(C, \mathcal{O}_C) \to 0.$$

Then we obtain $\dim_{k(b)} H^0(X_b, \mathcal{O}_{X_b}) \geq 2$, which contradicts (2). Hence (3) holds.

We now show (4). By [Tan18a, Corollary 2.9], $\deg_{k(b)} \omega_{X_b} = (K_X + X_b) \cdot_{k(b)} X_b < 0$. Hence (4) follows from (2) and Lemma 5.19.

The assertions (5) and (6) follow from Proposition 5.20. □

5.2.4. Twisted forms of canonical singularities

The aim of this subsection is to prove Proposition 5.30.

The main idea is to bound the purely inseparable degree of regular non smooth points on geometrically normal surfaces according to the type of singularities. For this, the notion of Jacobian number plays a crucial role.

Definition 5.22. Let k be a field of characteristic $p > 0$. Let R be an equi-dimensional k-algebra essentially of finite type over k. Let $J_{R/k}$ be its Jacobian ideal of R over k (cf. [HS06, Definition 4.4.1 and Proposition 4.4.4]). We define the *Jacobian number* of R/k as $\nu(R) := \nu(R/k) := \dim_k(R/J_{R/k})$. Note that $\nu(R/k) < \infty$ if $R/J_{R/k}$ is an artinian ring and its residue fields are finite extensions of k.

Remark 5.23. Let $k \subset k'$ be a field extension of characteristic $p > 0$ and let R be an equi-dimensional k-algebra essentially of finite type over k. Then the following hold.

1. By [HS06, Definition 4.4.1], we get

$$J_{R/k} \cdot (R \otimes_k k') = J_{R \otimes_k k'/k'}.$$

   In particular, if $R/J_{R/k}$ is an artinian ring and its residue fields are finite extensions of k, then we have $\nu(R/k) = \nu(R \otimes_k k'/k')$.

2. Assume that k is a perfect field. By [HS06, Definition 4.4.9], $\mathrm{Spec}\,(R/J_{R/k})$ set-theoretically coincides with the the non-regular locus of Spec R.

3. Assume that R is of finite type over k. Then (1) and (2) imply that $\mathrm{Spec}\,(R/J_{R/k})$ set-theoretically coincides with the the non-smooth locus of Spec $R \to$ Spec k.

Remark 5.24. In our application, R will be assumed to be a local ring $\mathcal{O}_{X,x}$ at a closed point x of a geometrically normal surface X over k. In this case, (3) of Remark 5.23 implies that $R/J_{R/k}$ is an artinian local ring whose residue field is a finite extension of k. Hence, $\nu(R/k) = \dim_k(R/J_{R/k})$ is well-defined as in Definition 5.22.

To treat local situations, let us recall the notion of essentially étale ring homomorphisms. For its fundamental properties, we refer to [Fu15, Subsection 2.8].

Definition 5.25. Let $f\colon R \to S$ be a local homomorphism of local rings. We say that f is *essentially étale* if there exists an étale R-algebra $\overline{S}$ and a prime ideal $\mathfrak{p}$ of $\overline{S}$ such that $\mathfrak{p}$ lies over the maximal ideal of R and S is R-isomorphic to $\overline{S}_{\mathfrak{p}}$.

In the following we will use some basic properties of Jacobian ideals, that rely on the fact that they are Fitting ideals. Let us recall the definition of Fitting ideals for a finitely presented module M over a ring R. Let us choose a presentation

$$R^{\oplus m} \xrightarrow{A} R^{\oplus n} \to M \to 0,$$

and define the k-th fitting ideals $\mathrm{Fit}_k(M)$ as the ideal generated by that $(n-k)\times(n-k)$-minors of the matrix A. By [SP, Tag 07Z8], this ideal $\mathrm{Fit}_k(M)$ is indipendent on the choice of the presentation.

If R is an equi-dimensional k-algebra essentially of finite type over k of dimension n, then as explained in [HS06, Discussion 4.4.7] we have the equality

$$J_{R/k} = \mathrm{Fit}_n(\Omega^1_{R/k}).$$

Lemma 5.26. *Let k be a field. Let $f\colon R \to S$ be an essentially étale local k-algebra homomorphism of local rings which are essentially of finite type over k. Let $\mathfrak{m}_R$ and $\mathfrak{m}_S$ be the maximal ideals of R and S, respectively. Set $k(R) := R/\mathfrak{m}_R$ and $k(S) := S/\mathfrak{m}_S$. Then the following hold.*

1. *If M is an R-module of finite length whose support is contained the maximal ideal $\mathfrak{m}_R$, then the equation*

$$\dim_k(M \otimes_R S) = [k(S) : k(R)] \dim_k M$$

holds.

2. *Suppose that R is an integral domain, $R/J_{R/k}$ is an artinian ring, and $k(R)$ is a finite extension of k. Then the equation*

$$\nu(S/k) = [k(R) : k(S)]\nu(R/k)$$

holds.

Proof. Let us show (1). Since M is a finitely generated R-module, there exists a sequence of R-submodules $M =: M_0 \supset M_1 \supset \cdots \supset M_n = 0$ such that $M_i/M_{i+1} \simeq R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ by [Mat89, Theorem 6.4] . Since the support of M is $\mathfrak{m}_R$, we have $\mathfrak{p} = \mathfrak{m}_R$.

As $R \to S$ is flat, the problem is reduced to the case when $M = R/\mathfrak{m}_R = k(R)$. In this case, we have

$$k(R) \otimes_R S = (R/\mathfrak{m}_R) \otimes_R S \simeq S/\mathfrak{m}_R S = S/\mathfrak{m}_S = k(S),$$

where the equality $S/m_R S = S/m_S$ follows from the assumption that f is a localisation of an unramified homomorphism. Hence, (1) holds.

Let us show (2). Set $n := \dim R$. We use the description of the Jacobian of R via Fitting ideals : $J_{R/k} = \mathrm{Fit}_n(\Omega^1_{R/k})$ and $J_{S/k} = \mathrm{Fit}_n(\Omega^1_{S/k})$. We have

$$J_{S/k} = \mathrm{Fit}_n(\Omega^1_{S/k}) = \mathrm{Fit}_n(\Omega^1_{R/k} \otimes_R S) = \mathrm{Fit}_n(\Omega^1_{R/k})S = J_{R/k}S,$$

where the third equality follows from (3) of [SP, Tag 07ZA]. As $f : R \to S$ is flat, we obtain $S/J_{S/k} \simeq (R/J_{R/k}) \otimes_R S$. By (1) and Definition 5.22, the assertion (2) holds. □

Example 5.27. Let k be a field of characteristic $p > 0$. Let $X = \mathrm{Spec}\, R$ be a surface over k such that

(i) $X \times_k \overline{k} = \mathrm{Spec}\,(R \otimes_k \overline{k})$ is a normal surface,

(ii) $X \times_k \overline{k}$ has a unique singular point x, and x is a canonical singularity of type A_{p^n-1}.

We prove that $\nu(R/k) = p^n$. By Remark 5.23, we have $\nu(R/k) = \nu(R \otimes_k \overline{k}/\overline{k})$. In order to compute $\nu(R \otimes_k \overline{k}/\overline{k})$, it is sufficient to localise at the singular point by [HS06, Corollary 4.4.5]. Thus we can suppose that k is algebraically closed and R is a local k-algebra.

By [Art77, pages 16-17] (cf. Section 2.1), the henselisation R^h of R is isomorphic to

$$k\{x, y, z\}/(z^{p^n} + xy).$$

In particular there exist essentially étale local k-algebra homomorphisms $R \to S$ and $k[x, y, z]/(z^{p^n} - xy) \to S$. A direct computation shows $\nu(k[x, y, z]/(z^{p^n} - xy)) = p^n$. Thus by Lemma 5.26, we have

$$\nu(R) = \nu(S) = \nu(k[x, y, z]/(z^{p^n} - xy)) = p^n.$$

The following is a generalisation of [FS18, Lemma 14.2].

Lemma 5.28. *Let k be a field of characteristic $p > 0$. Let $X = \mathrm{Spec}\, R$, where R is an equi-dimensional local k-algebra of essentially finite type over k. Let x be the closed point of X. Suppose that $R/J_{R/k}$ is a local artinian ring and its residue field $k(x)$ is a finite extension of k. Then $[k(x) : k]$ is a divisor of $\nu(R/k)$.*

Proof. Let $R/J_{R/k} =: M_0 \supset M_1 \supset \cdots \supset M_n = 0$ be a composition sequence of $R/J_{R/k}$-submodules (cf. [Mat89, Theorem 6.4]). Since $R/J_{R/k}$ is an artinian local ring, it holds that $M_i/M_{i+1} \simeq k(x)$ for any i. We have

$$\nu(R/k) = \dim_k(R/J_{R/k}) = \sum_{i=0}^{n-1} \dim_k(M_i/M_{i+1}) = n \dim_k k(x) = n[k(x) : k].$$

We thus conclude that $[k(x) : k]$ is a divisor of $\nu(R/k)$. □

Lemma 5.29. *Let X be a regular variety over a separably closed field k. Suppose that $X_{\overline{k}} = X \times_k \overline{k}$ is a normal variety with a unique singular point y. Let x be the image of y by the induced morphism $X_{\overline{k}} \to X$. Then the following hold.*

1. *$[k(x) : k]$ is a divisor of $\nu(\mathcal{O}_{X,x})$.*
2. *$X \times_k k(x)$ is not regular.*

Proof. Since k is separably closed, the induced morphism $X_{\overline{k}} \to X$ is a universal homeomorphism. Note that the local ring $\mathcal{O}_{X,x}$ is not geometrically regular over k. Applying Lemma 5.28 to the local ring $\mathcal{O}_{X,x}$, we deduce that $[k(x) : k]$ is a divisor of $\nu(\mathcal{O}_{X,x})$. Thus (1) holds. Consider the base change $\pi \colon X \times_k k(x) \to X$. Let x' be the point on $X \times_k k(x)$ lying over x. Note that x' is a $k(x)$-rational point of $X \times_k k(x)$ whose base change by $(-) \times_{k(x)} \overline{k}$ is not regular. By [FS18, Corollary 2.6], we conclude that $X \times_k k(x)$ is not regular at x'. □

We now explain how the previous results can be used to construct closed points with purely inseparable residue field on a regular surface. This will be used in Section 5.6 to find purely inseparable points on regular del Pezzo surfaces.

Proposition 5.30. *Let X be a regular surface over k. Suppose that $X_{\overline{k}} = X \times_k \overline{k}$ is a normal surface over $\overline{k}$ with a unique singular point y. Assume that y is a canonical singularity of type A_{p^n-1}. Let z be the image of y by the induced morphism $X_{\overline{k}} \to X_{k^{1/p^n}} = X \times_k k^{1/p^n}$. Then z is a k^{1/p^n}-rational point on $X_{k^{1/p^n}}$.*

Proof. Set $R := \mathcal{O}_{X,x}$, where x is the unique closed point along which X is not smooth. Let k^{sep} be the separable closure of k. For $R_{k^{\mathrm{sep}}} := R \otimes_k k^{\mathrm{sep}}$, it follows from Example 5.27 that $\nu(R_{k^{\mathrm{sep}}}) = p^n$. Lemma 5.29 implies that $k^{\mathrm{sep}} \subset k(z)$ is purely inseparable and $[k(z) : k^{\mathrm{sep}}]$ is a divisor of p^n. In particular, $k(z) \subset (k^{\mathrm{sep}})^{1/p^n}$.

Consider the Galois extension $k^{1/p^n} \subset (k^{\mathrm{sep}})^{1/p^n}$ and denote by G its Galois group. For $X_{(k^{\mathrm{sep}})^{1/p^n}} := X \times_k (k^{\mathrm{sep}})^{1/p^n}$, G acts on the set $X_{(k^{\mathrm{sep}})^{1/p^n}}((k^{\mathrm{sep}})^{1/p^n})$. The unique singular $(k^{\mathrm{sep}})^{1/p^n}$-rational point on $X_{(k^{\mathrm{sep}})^{1/p^n}}$ is fixed under the G-action. Thus it descends to a k^{1/p^n}-rational point on $X_{k^{1/p^n}}$. □

5.3. Behaviour of del Pezzo surfaces under base changes

In this section, we study the behaviour of canonical del Pezzo surfaces over an imperfect field k under the base changes to the algebraic closure $\overline{k}$.

5.3.1. Classification of base changes of del Pezzo surfaces

In this subsection, we give classification of base changes of del Pezzo surfaces with canonical singularities over imperfect fields (Theorem 5.33). To this end, we need two auxiliary lemmas: Lemma 5.31 and Lemma 5.32. The former one classify $\mathbb{Q}$-factorial surfaces over algebraically closed fields whose anti-canonical bundles are sufficiently positive. Its proof is based on a simple but smart idea by Reid (cf. the proof of [Rei94, Theorem 1.1]). A similar idea has also been used by I. Cheltsov in [Che96]. The latter one, i.e. Lemma 5.32, gives a rationality criterion for the base changes of log del Pezzo surfaces.

Lemma 5.31. *Let k be an algebraically closed field. Let Y be a projective normal $\mathbb{Q}$-factorial surface over k such that $-K_Y \equiv A + D$ for an ample Cartier divisor A and a pseudo-effective $\mathbb{Q}$-divisor D. Let $\mu : Z \to Y$ be the minimal resolution of Y. Then one of the following assertions holds.*

1. *$D \equiv 0$ and Y has at worst canonical singularities.*
2. *Z is isomorphic to a $\mathbb{P}^1$-bundle over a smooth projective curve.*
3. *$Z \simeq \mathbb{P}^2$.*

Proof. Assuming that (1) does not hold, let us prove that either (2) or (3) holds. We have

$$K_Z + E = \mu^* K_Y$$

for some effective μ-exceptional $\mathbb{Q}$-divisor E on Z. In particular, it holds that

$$K_Z + E + \mu^*(D) = \mu^*(K_Y + D) \equiv -\mu^* A.$$

Since (1) does not hold, we have that $D \not\equiv 0$ or $E \neq 0$. Then we get

$$K_Z + \mu^* A \equiv -E - \mu^*(D) \not\equiv 0,$$

hence $K_Z + \mu^* A$ is not nef. By the cone theorem for a smooth projective surface [KM98, Theorem 1.24], there is a curve C that spans a $(K_Z + \mu^* A)$-negative extremal ray R of $\overline{\mathrm{NE}}(Z)$. Note that C is not a (-1)-curve. Indeed, otherwise $\mu(C)$ is a curve and we obtain $\mu^* A \cdot C > 0$, which induces a contradiction:

$$(K_Z + \mu^* A) \cdot C \geq -1 + 1 = 0.$$

It follows from the classification of the K_Z-negative extremal rays [KM98, Theorem 1.28] that either $Z \simeq \mathbb{P}^2$ or Z is a $\mathbb{P}^1$-bundle over a smooth projective curve. In any case, one of (2) and (3) holds. □

Lemma 5.32. *Let (X, Δ) be a projective two-dimensional klt pair over a field of characteristic $p > 0$ such that $-(K_X + \Delta)$ is nef and big. Assume that $k = H^0(X, \mathcal{O}_X)$. Then $(X \times_k \overline{k})_{\mathrm{red}}$ is a rational surface.*

Proof. See [NT, Proposition 2.20]. □

We now give a classification of the base changes of del Pezzo surfaces with canonical singularities.

Theorem 5.33. *Let k be a field of characteristic $p > 0$. Let X be a canonical del Pezzo surface over k with $k = H^0(X, \mathcal{O}_X)$. Then the normalisation Y of $(X \times_k \overline{k})_{\mathrm{red}}$ satisfies one of the following properties.*

1. *$X \times_k \overline{k}$ is geometrically canonical over k. In particular, $Y \simeq X \times_k \overline{k}$ and $-K_Y$ is ample.*

2. *X is not geometrically normal over k and Y is isomorphic to a Hirzebruch surface, i.e. a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.*

3. *X is not geometrically normal over k and Y is isomorphic to a weighted projective surface $\mathbb{P}(1, 1, m)$ for some positive integer m.*

Proof. Replacing k by its separable closure, we may assume that k is separably closed. Let $f : Y \to X$ be the induced morphism and let $\mu : Z \to Y$ be the minimal resolution of Y. By [Tan18b, Theorem 4.2], there is an effective $\mathbb{Z}$-divisor D on Y such that

- $K_Y + D = f^*K_X$, and
- if $X \times_k \overline{k}$ is not normal, then $D \neq 0$.

Since $-K_X$ is an ample Cartier divisor, so is $-f^*K_X$. Moreover, it follows from [Tan18b, Lemma 2.2 and Lemma 2.5] that Y is $\mathbb{Q}$-factorial. Hence, we may apply Lemma 5.31 to $-K_Y = -f^*K_X + D$.

By Lemma 5.32, Y is a rational surface. Thus, if (2) or (3) of Lemma 5.31 holds, then one of (1)–(3) of Theorem 5.33 holds, as desired. Therefore, let us treat the case when (1) of Lemma 5.31 holds. Then it holds that $D = 0$ and Y has at worst canonical singularities. In this case, we have that $Y = X \times_k \overline{k}$ and X is geometrically canonical. Hence, (1) of Theorem 5.33 holds, as desired. □

5.3.2. Bounds on Frobenius length of geometric non-normality

In this subsection, we give an upper bound for the Frobenius length of geometric non-normality for canonical del Pezzo surfaces (Proposition 5.34).

Proposition 5.34. *Let k be a field of characteristic $p > 0$. Let X be a canonical del Pezzo surface over k with $k = H^0(X, \mathcal{O}_X)$. Let Y be the normalisation of $(X \times_k \overline{k})_{\mathrm{red}}$ and let $f : Y \to X$ be the induced morphism. Assume that the linear equivalence*

$$K_Y + \sum_{i=1}^{r} C_i \sim f^*K_X$$

holds for some prime divisors $C_1, ..., C_r$ (not necessarily $C_i \neq C_j$ for $i \neq j$). Then it holds that $r \leq 2$.

Proof. Set $C := \sum_{i=1}^r C_i$. We have $K_Y + C \sim f^*K_X$. If $C = 0$, then there is nothing to show. Hence, we may assume that $C \neq 0$. In particular, X is not geometrically normal. In this case, it follows from Theorem 5.33 that Y is isomorphic to either a Hirzebruch surface or $\mathbb{P}(1,1,m)$ for some $m > 0$.

We first treat the case when $Y \simeq \mathbb{P}(1,1,m)$. If $m = 1$, then the assertion is obvious. Hence, we may assume that $m \geq 2$. In this case, for the minimal resolution $g : Z \to Y$, we have that

$$K_Z + \frac{m-2}{m}\Gamma = g^*K_Y$$

where Γ is the negative section of the fibration $Z \to \mathbb{P}^1$ such that $\Gamma^2 = -m$. Note that m is the $\mathbb{Q}$-factorial index of Y, i.e. mD is Cartier for any $\mathbb{Z}$-divisor D on Y. We have that

$$-K_Z = \frac{m-2}{m}\Gamma - g^*K_Y \equiv \frac{m-2}{m}\Gamma + g^*C - g^*f^*K_X$$

Consider the intersection number with a fibre F_Z of $Z \to \mathbb{P}^1$:

$$2 = \left(\frac{m-2}{m}\Gamma + g^*C - g^*f^*K_X\right) \cdot F_Z \geq \frac{m-2}{m} + C \cdot g_*(F_Z) + 1.$$

Thus we obtain

$$2 \geq C \cdot (mg_*(F_Z)) \geq r,$$

where the last inequality holds since $mg_*(F_Z)$ is an ample Cartier divisor. Therefore, we obtain $r \leq 2$, as desired.

It is enough to treat the case when Y is a Hirzebruch surface. For a fibre F of $\pi : Y \to \mathbb{P}^1$, we have that

$$-2 + C \cdot F = (K_Y + C) \cdot F = f^*K_X \cdot F \leq -1,$$

hence $C \cdot F \leq 1$. There are two possibilities: $C \cdot F = 1$ or $C \cdot F = 0$.

Assume that $C \cdot F = 1$. Then there is a section Γ of π and a π-vertical $\mathbb{Z}$-divisor C' such that $C = \Gamma + C'$. Consider the intersection number with Γ:

$$-2 + \Gamma \cdot C' = (K_Y + \Gamma + C') \cdot \Gamma = (K_Y + C) \cdot \Gamma = f^*K_X \cdot \Gamma \leq -1.$$

Therefore, we have $\Gamma \cdot C' \leq 1$. This implies that either $C' = 0$ or C' is a prime divisor. In any case, we get $r \leq 2$, as desired.

We may assume that $C \cdot F = 0$, i.e. C is a π-vertical divisor. Let Γ be a section of π such that $\Gamma^2 \leq 0$. We have that

$$-2 + C \cdot \Gamma = (K_Y + \Gamma + C) \cdot \Gamma \leq (K_Y + C) \cdot \Gamma = f^*K_X \cdot \Gamma \leq -1.$$

Hence, we obtain $C \cdot \Gamma \leq 1$, which implies $r \leq 1$. □

Theorem 5.35. *Let k be a field of characteristic $p > 0$. Let X be a canonical del Pezzo surface over k such that $k = H^0(X, \mathcal{O}_X)$. Let Y be the normalisation of $(X \times_k \overline{k})_{\mathrm{red}}$ and let*

$$\mu : Y \to X \times_k \overline{k}$$

be the induced morphism.

1. *If $p \geq 5$, then X is geometrically canonical, i.e. μ is an isomorphism and Y has at worst canonical singularities.*
2. *If $p = 3$, then $\ell_F(X/k) \leq 1$ and the absolute Frobenius morphism $F_{X \times_k \overline{k}}$ of $X \times_k \overline{k}$ factors through μ:*
$$F_{X \times_k \overline{k}} : X \times_k \overline{k} \to Y \xrightarrow{\mu} X \times_k \overline{k}.$$
3. *If $p = 2$, then $\ell_F(X/k) \leq 2$ and the second iterated absolute Frobenius morphism $F^2_{X \times_k \overline{k}}$ of $X \times_k \overline{k}$ factors through μ:*
$$F^2_{X \times_k \overline{k}} : X \times_k \overline{k} \to Y \xrightarrow{\mu} X \times_k \overline{k}.$$

Proof. The assertion follows from Remark 2.20 and Proposition 5.34. □

5.4. Numerically trivial line bundles on log del Pezzo surfaces

The purpose of this section is to give an explicit upper bound on the torsion index of numerically trivial line bundles on log del Pezzo surfaces over imperfect fields (Theorem 5.45). To achieve this result, we use the minimal model program to reduce the problem to the case when our log del Pezzo surface admits a Mori fibre space structure $\pi : X \to B$. The cases $\dim B = 0$ and $\dim B = 1$ will be settled in Theorem 5.36 and Proposition 5.44, respectively.

5.4.1. Canonical case

In this subsection, we study numerically trivial Cartier divisor on del Pezzo surfaces with canonical singularities.

Theorem 5.36. *Let k be a field of characteristic $p > 0$. Let X be a canonical weak del Pezzo surface over k such that $k = H^0(X, \mathcal{O}_X)$. Let L be a numerically trivial Cartier divisor on X. Then the following hold.*

1. *If $p \geq 5$, then $L \sim 0$.*
2. *If $p = 3$, then $3L \sim 0$.*
3. *If $p = 2$, then $4L \sim 0$.*

Proof. We first reduce the problem to the case when $-K_X$ is ample. It follows from [Tan18a, Theorem 4.2] that $-K_X$ is semi-ample. As $-K_X$ is also big, $|-mK_X|$ induces a birational morphism $f : X \to Y$ to a projective normal surface Y. Then it holds that K_Y is $\mathbb{Q}$-Cartier and $K_X = f^*K_Y$. In particular, Y has at worst canonical singularities. Then [Tan18a, Theorem 4.4] enables us to find a numerically trivial Cartier divisor L_Y on Y such that $f^*L_Y \sim L$. Hence the problem is reduced to the case when $-K_X$ is ample.

Let us discuss the case where $p \geq 5$. In this case by Theorem 5.35, $X_{\overline{k}}$ is a del Pezzo surface with canonical singularities. Set $\mathcal{L} := \mathcal{O}_X(L)$ and let $\mathcal{L}_{\overline{k}}$ be the pull-back to $X_{\overline{k}}$. Since $X_{\overline{k}}$ is a rational surface, we have $H^0(X_{\overline{k}}, \mathcal{L}_{\overline{k}}) \neq 0$. By base change we thus have

$$H^0(X, \mathcal{L}) \otimes_k \overline{k} \simeq H^0(X_{\overline{k}}, \mathcal{L}_{\overline{k}}) \neq 0,$$

thus concluding.

We only treat the case when $p = 2$, as the case $p = 3$ is analogous. By Theorem 5.35, the second iterated absolute Frobenius morphism

$$F^2_{X \times_k \overline{k}} : X \times_k \overline{k} \to X \times_k \overline{k}$$

factors through the normalisation $(X \times_k \overline{k})^N_{\mathrm{red}}$ of $(X \times_k \overline{k})_{\mathrm{red}}$:

$$F^2_{X \times_k \overline{k}} : X \times_k \overline{k} \to (X \times_k \overline{k})^N_{\mathrm{red}} \xrightarrow{\mu} X \times_k \overline{k},$$

where μ denotes the induced morphism. Set $\mathcal{L} := \mathcal{O}_X(L)$ and let $\mathcal{L}_{\overline{k}}$ be the pullback of $\mathcal{L}$ to $X \times_k \overline{k}$. Since $(X \times_k \overline{k})^N_{\mathrm{red}}$ is a normal rational surface by Lemma 5.32, any numerically trivial invertible sheaf is trivial: $\mu^*\mathcal{L}_{\overline{k}} \simeq \mathcal{O}_{(X \times_k \overline{k})^N_{\mathrm{red}}}$. As $F^2_{X \times_k \overline{k}}$ factors through μ, we have that

$$\mathcal{L}^4_{\overline{k}} = (F^2_{X \times_k \overline{k}})^*\mathcal{L}_{\overline{k}} \simeq \mathcal{O}_{X \times_k \overline{k}}.$$

Then it holds that

$$H^0(X, \mathcal{L}^4) \otimes_k \overline{k} \simeq H^0(X \times_k \overline{k}, \mathcal{L}^4_{\overline{k}}) \simeq H^0(X \times_k \overline{k}, \mathcal{O}_{X \times_k \overline{k}}) \neq 0.$$

Hence we obtain $H^0(X, \mathcal{L}^4) \neq 0$, i.e. $4L \sim 0$. □

5.4.2. Essential step for the log case

In this subsection, we study the torsion index of numerically trivial line bundles on log del Pezzo surfaces admitting the following special Mori fibre space structure onto a curve.

Notation 5.37. We use the following notation.

1. k is a field of characteristic $p > 0$.
2. X is a regular k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$ and $\rho(X) = 2$.
3. B is a regular projective curve over k such that $k = H^0(B, \mathcal{O}_B)$.

4. $\pi : X \to B$ is a K_X-Mori fibre space.

5. Let $R = \mathbb{R}_{\geq 0}[\Gamma]$ be the extremal ray which does not correspond to π, where Γ denotes a curve on X. Note that $\pi(\Gamma) = B$. Set $d_\Gamma := \dim_k H^0(\Gamma, \mathcal{O}_\Gamma) \in \mathbb{Z}_{>0}$ and $m_\Gamma := [K(\Gamma) : K(B)] \in \mathbb{Z}_{>0}$. We denote by $\pi_\Gamma : \Gamma \to B$ the induced morphism.

6. Assume that $K_X \cdot \Gamma > 0$.

Lemma 5.38. *We use Notation 5.37. Then the following hold.*

(7) $\Gamma^2 \leq 0$.

(8) There exists a rational number α *such that* $0 \leq \alpha < 1$ *and* $(X, \alpha\Gamma)$ *a log del Pezzo pair.*

Proof. The assertion (7) follows from Lemma 5.39 below. Let us prove (8). By Notation 5.37(2), there is an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is a log del Pezzo pair. We write $\Delta = \alpha\Gamma + \Delta'$ for some rational number $0 \leq \alpha < 1$ and an effective $\mathbb{Q}$-divisor Δ' with $\Gamma \not\subset \mathrm{Supp}(\Delta')$. Since $\overline{\mathrm{NE}}(X)$ is generated by Γ and a fibre F of the morphism $\pi : X \to B$, we conclude that any prime divisor C such that $C \neq \Gamma$ is nef. In particular, Δ' is nef. Hence, $(X, \alpha\Gamma)$ is a log del Pezzo pair. Thus, (8) holds. ☐

Lemma 5.39. *Let* k *be a field. Let* X *be a projective* $\mathbb{Q}$*-factorial normal surface over* k *Let* $R = \mathbb{R}_{\geq 0}[\Gamma]$ *is an extremal ray of* $\overline{\mathrm{NE}}(X)$*, where* Γ *is a curve on* X*. If* $\Gamma^2 > 0$*, then* $\rho(X) = 1$*.*

Proof. We may apply the same argument as in [Tan14, Theorem 3.21, Proof of the case where $C^2 > 0$ in page 20]. ☐

The first step is to prove that $m_\Gamma \leq 5$ (Proposition 5.42). To this end, we find an upper bound and a lower bound for α (Lemma 5.40, Lemma 5.41).

Lemma 5.40. *We use Notation 5.37. Take a closed point* b *of* B *and set* $F_b := \pi^*(b)$*. Let* $k(b)$ *be the residue field at* b *and set* $d(b) := [k(b) : k]$*. Then the following hold.*

1. $K_X \cdot_k F_b = -2d(b)$.

2. $\Gamma \cdot_k F_b = m_\Gamma d(b)$.

3. If α *is a rational number such that* $-(K_X + \alpha\Gamma)$ *is ample, then* $\alpha m_\Gamma < 2$*.*

Proof. Let us show (1). We have that

$$\deg_k \omega_{F_b} = (K_X + F_b) \cdot_k F_b = K_X \cdot_k F_b < 0.$$

Hence, Lemma 5.19 implies that

$$K_X \cdot_k F_b = \deg_k \omega_{F_b} = -2d(b).$$

Thus (1) holds. Clearly, (2) holds.

Let us show (3). Since $-(K_X + \alpha\Gamma)$ is ample, (1) and (2) imply that

$$0 > (K_X + \alpha\Gamma) \cdot_k F_b = -2d(b) + \alpha m_\Gamma d(b).$$

Thus (3) holds. □

Lemma 5.41. *We use Notation 5.37. Then the following hold.*

1. $(K_X + \Gamma) \cdot_k \Gamma = -2d_\Gamma < 0$.
2. *For a rational number β with $0 \leq \beta \leq 1$, it holds that*
$$(K_X + \beta\Gamma) \cdot_k \Gamma \geq d_\Gamma(1 - 3\beta).$$
3. *If α is a rational number such that $0 \leq \alpha < 1$ and $-(K_X + \alpha\Gamma)$ is ample, then it holds that $1/3 < \alpha$.*

Proof. We fix a rational number α such that $0 \leq \alpha < 1$ and $-(K_X + \alpha\Gamma)$ is ample, whose existence is guaranteed by Lemma 5.38.

Let us show (1). It holds that

$$(K_X + \Gamma) \cdot_k \Gamma \leq (K_X + \alpha\Gamma) \cdot_k \Gamma < 0,$$

where the first inequality follows from $\Gamma^2 \leq 0$ and $0 \leq \alpha < 1$, whilst the second one holds since $-(K_X + \alpha\Gamma)$ is ample. Therefore, by adjunction and Lemma 5.19, we deduce $(K_X + \Gamma) \cdot_k \Gamma = \deg_k \omega_\Gamma = -2d_\Gamma$. Thus (1) holds.

Let us show (2). For $k_\Gamma := H^0(\Gamma, \mathcal{O}_\Gamma)$, the equation $d_\Gamma = [k_\Gamma : k]$ (Notation 5.37(5)) implies that

$$K_X \cdot_k \Gamma = \deg_k(\omega_X|_\Gamma) = d_\Gamma \cdot \deg_{k_\Gamma}(\omega_X|_\Gamma) \in d_\Gamma\mathbb{Z}.$$

Combining with $K_X \cdot_k \Gamma > 0$ (Notation 5.37(6)), we obtain $K_X \cdot_k \Gamma \geq d_\Gamma$. Hence, it holds that

$$(K_X + \beta\Gamma) \cdot_k \Gamma = (1 - \beta)K_X \cdot_k \Gamma + \beta(K_X + \Gamma) \cdot_k \Gamma$$
$$= (1 - \beta)K_X \cdot_k \Gamma + \beta(-2d_\Gamma) \geq (1 - \beta)d_\Gamma + \beta(-2d_\Gamma) = d_\Gamma(1 - 3\beta).$$

Thus (2) holds. The assertion (3) follows from (2). □

Proposition 5.42. *We use Notation 5.37. It holds that $m_\Gamma \leq 5$.*

Proof. We fix a rational number α such that $0 \leq \alpha < 1$ and $-(K_X + \alpha\Gamma)$ is ample, whose existence is guaranteed by Lemma 5.38. Then the inequality $m_\Gamma < 6$ holds by

$$\frac{2}{m_\Gamma} > \alpha > \frac{1}{3},$$

where the first and second inequalities follow from Lemma 5.40 and Lemma 5.41, respectively. ☐

To prove the main result of this subsection (Proposition 5.44), we first treat the case when $K(\Gamma)/K(B)$ is separable or purely inseparable.

Lemma 5.43. *We use Notation 5.37. Let L_B be a numerically trivial Cartier divisor on B. Then the following hold.*

1. *If $K(\Gamma)/K(B)$ is a separable extension, then ω_B^{-1} is ample and $L_B \sim 0$.*
2. *If $K(\Gamma)/K(B)$ is a purely inseparable morphism of degree p^e for some $e \in \mathbb{Z}_{>0}$, then $p^e L_B \sim 0$.*

Proof. We first prove (1). Assume that $K(\Gamma)/K(B)$ is a separable extension. Let $\Gamma^N \to \Gamma$ be the normalisation of Γ. Set $\pi_{\Gamma^N} : \Gamma^N \to B$ to be the induced morphism. Since ω_Γ^{-1} is ample, so is $\omega_{\Gamma^N}^{-1}$. Hence we obtain $H^1(\Gamma^N, \mathcal{O}_{\Gamma^N}) = 0$ (Lemma 5.19). Thanks to the Hurwitz formula (cf. [Liu02, Theorem 4.16 in Section 7]), we have that $H^1(B, \mathcal{O}_B) = 0$, thus ω_B^{-1} is ample (Lemma 5.19). In particular, the numerically trivial Cartier divisor L_B is trivial, i.e. $L_B \sim 0$. Thus (1) holds.

We now show (2). Since $K(\Gamma)/K(B)$ is a purely inseparable morphism of degree p^e, the e-th iterated absolute Frobenius morphism $F_B^e : B \to B$ factors through the induced morphism $\pi_{\Gamma^N} : \Gamma^N \to B$:

$$F_B^e : B \to \Gamma^N \xrightarrow{\pi_{\Gamma^N}} B.$$

It holds that $\pi_{\Gamma^N}^* L_B \sim 0$, hence $p^e L_B = (F_B^e)^* L_B \sim 0$. Thus (2) holds. ☐

Proposition 5.44. *We use Notation 5.37. Let L be a numerically trivial Cartier divisor on X. Then the following hold.*

1. *If $p \geq 7$, then $L \sim 0$.*
2. *If $p \in \{3, 5\}$, then $pL \sim 0$.*
3. *If $p = 2$, then $4L \sim 0$.*

Proof. By [Tan18a, Theorem 4.4], there exists a numerically trivial Cartier divisor L_B on B such that $\pi^* L_B \sim L$. If $K(\Gamma)/K(B)$ is separable, then Lemma 5.43(1) implies that $L \sim 0$. Therefore, we may assume that $K(\Gamma)/K(B)$ is not a separable extension. Thanks to Proposition 5.42, we have

$$[K(\Gamma) : K(B)] = m_\Gamma \leq 5.$$

Let us show (1). Assume $p \geq 7$. In this case, there does not exist an inseparable extension $K(\Gamma)/K(B)$ with $[K(\Gamma) : K(B)] \leq 5$. Thus (1) holds.

Let us show (2). Assume $p \in \{3, 5\}$. Since $K(\Gamma)/K(B)$ is not a separable extension and $[K(\Gamma) : K(B)] \leq 5$, it holds that $K(\Gamma)/K(B)$ is a purely inseparable extension of degree p. Hence, Lemma 5.43(2) implies that $pL \sim 0$. Thus (2) holds.

Let us show (3). Assume $p = 2$. Since $K(\Gamma)/K(B)$ is not a separable extension and $[K(\Gamma) : K(B)] \leq 5$, there are the following three possibilities (i)–(iii).

(i) $K(\Gamma)/K(B)$ is a purely inseparable extension of degree 2.

(ii) $K(\Gamma)/K(B)$ is a purely inseparable extension of degree 4.

(iii) $K(\Gamma)/K(B)$ is an inseparable extension of degree 4 which is not purely inseparable.

If (i) or (ii) holds, then Lemma 5.43(2) implies that $4L \sim 0$. Hence we may assume that (iii) holds. Let $\Gamma^N \to \Gamma$ be the normalisation of Γ. Corresponding to the separable closure of $K(B)$ in $K(\Gamma) = K(\Gamma^N)$, we obtain the following factorisation

$$\Gamma^N \to B_1 \to B$$

where $K(\Gamma^N)/K(B_1)$ is a purely inseparable extension of degree two and $K(B_1)/K(B)$ is a separable extension of degree two. In particular, $K(B_1)/K(B)$ is a Galois extension. Set $G := \mathrm{Gal}(K(B_1)/K(B)) = \{\mathrm{id}, \sigma\}$. Since $L_B|_{\Gamma^N} \sim L|_{\Gamma^N} \sim 0$ and the absolute Frobenius morphism $F_{B_1} : B_1 \to B_1$ factors through $\Gamma^N \to B_1$, it holds that $2L_B|_{B_1} \sim 0$. In particular, we have that $H^0(B_1, 2L_B|_{B_1}) \neq 0$. Fix $0 \neq s \in H^0(B_1, 2L_B|_{B_1})$. We obtain

$$0 \neq s\sigma(s) \in H^0(B_1, 4L_B|_{B_1})^G.$$

As $s\sigma(s)$ is G-invariant, $s\sigma(s)$ descends to B, i.e. there is an element

$$t \in H^0(B, 4L_B)$$

such that $t|_{B_1} = s\sigma(s)$. In particular, we obtain $t \neq 0$, hence $4L_B \sim 0$. Therefore, we have $4L \sim 0$. □

5.4.3. General case

We are ready to prove the main theorem of this section.

Theorem 5.45. *Let k be a field of characteristic $p > 0$. Let X be a k-surface of del Pezzo type. Let L be a numerically trivial Cartier divisor on X. Then the following hold.*

1. *If $p \geq 7$, then $L \sim 0$.*
2. *If $p \in \{3, 5\}$, then $pL \sim 0$.*
3. *If $p = 2$, then $4L \sim 0$.*

Proof. Let us consider the Stein factorisation $X \to \mathrm{Spec}(H^0(X, \mathcal{O}_X)) \to \mathrm{Spec}(k)$. Since we only need to check that for a given m the space of global sections $H^0(X, mL)$ does not vanish we can replace k with $H^0(X, \mathcal{O}_X)$ without any loss of generality.

Furthermore, replacing X by its minimal resolution, we may assume that X is regular by Lemma 5.10. We run a K_X-MMP:

$$\varphi : X =: X_0 \to X_1 \to \cdots \to X_n.$$

Since $-K_X$ is big, the end result X_n is a K_{X_n}-Mori fibre space. It follows from [Tan18a, Theorem 4.4(3)] that there exists a Cartier divisor L_n with $\varphi^* L_n \sim L$. Since also X_n is of del Pezzo type by Lemma 5.12, we may replace X by X_n. Let $\pi : X \to B$ be the induced K_X-Mori fibre space.

If $\dim B = 0$, then we conclude by Theorem 5.36. Hence we may assume that $\dim B = 1$. Since X is a surface of del Pezzo type, there is an effective $\mathbb{Q}$-divisor such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample. Hence any extremal ray of $\overline{\mathrm{NE}}(X)$ is spanned by a curve. Note that $\rho(X) = 2$ and a fibre of $\pi : X \to B$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$. Let $R = \mathbb{R}_{\geq 0}[\Gamma]$ be the other extremal ray, where Γ is a curve on X. To summarise, (1)–(5) of Notation 5.37 hold. There are the following three possibilities:

(i) $\Gamma^2 \geq 0$.

(ii) $\Gamma^2 < 0$ and $K_X \cdot \Gamma \leq 0$.

(iii) $\Gamma^2 < 0$ and $K_X \cdot \Gamma > 0$.

Assume (i). In this case, any curve C on X is nef. Since $-(K_X + \Delta)$ is ample, also $-K_X$ is ample. Therefore, we conclude by Theorem 5.36.

Assume (ii). In this case, $-K_X$ is nef and big. Again, Theorem 5.36 implies the assertion of Theorem 5.45.

Assume (iii). In this case, all the conditions (1)–(6) of Notation 5.37 hold. Hence the assertion of Theorem 5.45 follows from Proposition 5.44. $\square$

5.5. Results in large characteristic

In this section, we prove the existence of geometrically normal birational models of log del Pezzo surfaces over imperfect fields of characteristic at least seven (Theorem 5.49). As consequences, we prove geometric integrality (Corollary 5.50) and vanishing of irregularity for such surfaces (Theorem 5.52).

5.5.1. Analysis up to birational modification

The purpose of this subsection is to prove Theorem 5.49. To this end, we establish auxiliary results on Mori fibre spaces (Proposition 5.47, Proposition 5.48) We start by recalling the following well-known relation between the Picard rank and the anti-canonical volume of del Pezzo surfaces.

Lemma 5.46. *Let Y be a smooth weak del Pezzo surface over an algebraically closed field k. Then $\rho(Y) = 10 - K_Y^2$. In particular, it holds that $\rho(Y) \leq 9$.*

Proof. Let $Y =: Y_1 \to Y_2 \to \cdots \to Y_n = Z$ be a K_Y-MMP, where Z is a weak del Pezzo surface endowed with a K_Z-Mori fibre space $Z \to B$. It is sufficient to prove the relation $\rho(Z) = 10 - K_Z^2$, which is well known (cf. [KM98, Theorem 1.28]). □

Proposition 5.47. *Let k be field of characteristic $p \geq 11$. Let X be a regular del Pezzo k-surface such that $k = H^0(X, \mathcal{O}_X)$. Then X is smooth over k.*

Proof. By Theorem 5.35, $X \times_k \overline{k}$ has at most canonical singularities. By [Sch08, Theorem 6.1] such singularities are of type A_{p^e-1}. Since $X \times_k \overline{k}$ is a canonical del Pezzo surface, its minimal resolution $\pi\colon Y \to X \times_k \overline{k}$ is a smooth weak del Pezzo surface and we have

$$9 \geq \rho(Y) \geq \rho(X \times_k \overline{k}) + \sum_{x \in \mathrm{Sing}(X \times_k \overline{k})} (p-1) \geq \sum_{x \in \mathrm{Sing}(X \times_k \overline{k})} 10,$$

where the first inequality follows from Lemma 5.46 and the last inequality holds by $p \geq 11$. Thus, we obtain $\mathrm{Sing}(X \times_k \overline{k}) = \emptyset$, as desired. □

Proposition 5.48. *Let k be field of characteristic $p > 0$. Let X be a regular k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Assume that there is a K_X-Mori fibre space $\pi : X \to B$ to a projective regular k-curve B. Let Γ be a curve which spans the extremal ray of $\overline{\mathrm{NE}}(X)$ not corresponding to π. Then the following hold.*

1. *If $K_X \cdot \Gamma < 0$ (resp. ≤ 0), then $-K_X$ is ample (resp. nef and big). If $p \geq 5$, then ω_B^{-1} is ample and B is smooth over k.*
2. *If $K_X \cdot \Gamma > 0$ and $p \geq 7$, then ω_B^{-1} is ample and B is smooth over k.*
3. *If $K_X \cdot \Gamma > 0$, $p \geq 7$, and k is separably closed, then Γ is a section of π and π is smooth. In particular, X is smooth over k.*

Proof. The first part of assertion (1) follows immediately from Kleimann's criterion for ampleness (resp. [Laz04a, Theorem 2.2.16]). Assume $p \geq 5$. The anti-canonical model Z of X is geometrically normal by Theorem 5.35 and thus $H^1(Z, \mathcal{O}_Z) = 0$. This implies that $H^1(X, \mathcal{O}_X) = 0$ and $H^1(B, \mathcal{O}_B) = 0$. Hence, the assertion (1) holds by Lemma 5.19 and Lemma 5.20.

Let us show (2). The field extension $K(\Gamma)/K(B)$ corresponding to the induced morphism $\pi_\Gamma : \Gamma \to B$ is separable (Proposition 5.42). Thus B is a curve such that ω_B^{-1} is ample (Lemma 5.43). Since $p > 2$, B is a k-smooth curve by Lemma 5.20. Thus (2) holds.

Let us show (3). It follows from Proposition 5.21(6) that π is a smooth morphism. Hence it suffices to show that $\pi_\Gamma : \Gamma \to B$ is a section of π. Since $K(\Gamma)$ is separable over $K(B)$ and B is smooth over k, $K(\Gamma)$ is separable over k, i.e. $K(\Gamma)$ is geometrically reduced over k. Hence also Γ is geometrically reduced over k. Since $X_{\overline{k}}$ is a smooth projective rational surface with $\rho(X_{\overline{k}}) = 2$, $X_{\overline{k}}$ is a Hirzebruch surface and $\pi_{\overline{k}} : X_{\overline{k}} \to B_{\overline{k}}$ is a projection. Since the pullback $\Gamma_{\overline{k}}$ of Γ is a curve with $\Gamma_{\overline{k}}^2 < 0$ by Lemma 5.39, $\Gamma_{\overline{k}}$ is a section of $\pi_{\overline{k}} : X_{\overline{k}} \to B_{\overline{k}}$. The base change $\Gamma_{\overline{k}} \to B_{\overline{k}}$ is an isomorphism, hence so is the original one $\pi_\Gamma : \Gamma \to B$. Thus (3) holds. □

Theorem 5.49. *Let k be a separably closed field of characteristic $p \geq 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then there exists a birational map $X \dashrightarrow Y$ to a projective normal k-surface Y such that one of the following properties holds.*

1. *Y is a regular del Pezzo surface such that $k = H^0(Y, \mathcal{O}_Y)$ and $\rho(Y) = 1$. In particular, Y is geometrically canonical over k. Moreover, if $p \geq 11$, then Y is smooth over k.*
2. *There is a smooth projective morphism $\pi : Y \to B$ such that $B \simeq \mathbb{P}^1_k$ and the fibre $\pi^{-1}(b)$ is isomorphic to $\mathbb{P}^1_{k(b)}$ for any closed point b of B, where $k(b)$ denotes the residue field of b. In particular, Y is smooth over k and $Y \times_k \overline{k}$ is a Hirzebruch surface.*

Proof. Let $f : Z \to X$ be the minimal resolution of X. By Lemma 5.10, Z is a k-surface of del Pezzo type. We run a K_Z-MMP:

$$Z =: Z_0 \to Z_1 \to \cdots \to Z_n =: Y.$$

By Lemma 5.12, the surfaces Z_i are of del Pezzo type. The end result Y is a K_Y-Mori fibre space $\pi : Y \to B$. If $\dim B = 0$, then Y is a regular del Pezzo surface, hence (1) holds by Theorem 5.35 and Proposition 5.47. If $\dim B = 1$, then Proposition 5.48 implies that (2) holds. □

Corollary 5.50. *Let k be a field of characteristic $p \geq 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then X is geometrically integral over k.*

Proof. We may assume k is separably closed. It is enough to show that X is geometrically reduced [Tan18b, Lemma 2.2]. By Lemma 2.22, we may replace X by a surface birational to X. Then the assertion follows from Theorem 5.49. □

5.5.2. Vanishing of $H^1(X, \mathcal{O}_X)$

In this subsection, we prove that surfaces of del Pezzo type over an imperfect field of characteristic $p \geq 7$ have vanishing irregularity.

Lemma 5.51. *Let k be a field of characteristic $p > 0$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. If X is geometrically normal over k, then it holds that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

Proof. The assertion immediately follows from Lemma 5.32. □

Theorem 5.52. *Let k be a field of characteristic $p \geq 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

Proof. We may assume that k is separably closed. Let $X \dashrightarrow Y$ be the birational morphism as in the statement of Theorem 5.49. Lemma 5.51 implies that $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$.

Let $\varphi : W \to X$ and $\psi : W \to Y$ be birational morphisms from a regular projective surface W. Since both Y and W are regular, we have that $H^i(W, \mathcal{O}_W) = 0$ for $i > 0$. Then the Leray spectral sequence implies that $H^1(X, \mathcal{O}_X) = 0$. It is clear that $H^j(X, \mathcal{O}_X) = 0$ for $j \geq 2$. □

In characteristic zero, it is known that the image of a variety of Fano type under a surjective morphism remains of Fano type (cf. [FG12, Theorem 5.12]). The same result is false over imperfect fields of low characteristic as shown in [Tana, Theorem 1.4]. We now prove that this phenomenon can appear exclusively in low characteristic for surfaces.

Corollary 5.53. *Let k be a field of characteristic $p \geq 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$ and let $\pi \colon X \to Y$ be a projective k-morphism such that $\pi_* \mathcal{O}_X = \mathcal{O}_Y$. Then Y is a k-variety of Fano type. Furthermore, if $\dim Y = 1$, then Y is smooth over k.*

Proof. We distinguish two cases according to $\dim Y$. If $\dim Y = 2$, then π is birational and we conclude by Lemma 5.12. If $\dim Y = 1$, then thanks to the Leray spectral sequence, we have an injection:

$$H^1(Y, \mathcal{O}_Y) \hookrightarrow H^1(X, \mathcal{O}_X),$$

where $H^1(X, \mathcal{O}_X) = 0$ by Theorem 5.52. Therefore ω_Y^{-1} is ample by Lemma 5.19 and Y is smooth over k by Lemma 5.20. □

5.6. Purely inseparable points on log del Pezzo surfaces

The aim of this section is to construct purely inseparable points of bounded degree on log del Pezzo surfaces X over C_1-fields of positive characteristic (Theorem 5.65). Since we may take birational model changes, the problem is reduced to the case when X has a Mori fibre space structure $X \to B$. The case when $\dim B = 0$ and $\dim B = 1$ are treated in Subsection 5.6.1 and Subsection 5.6.2, respectively. In Subsection 5.6.3, we prove the main result of this section (Theorem 5.65).

5.6.1. Purely inseparable points on regular del Pezzo surfaces

In this subsection we prove the existence of purely inseparable points with bounded degree on geometrically normal regular del Pezzo surfaces over C_1-fields. If $K_X^2 \leq 4$, then we apply the strategy as in [Kol96, Theorem IV.6.8] (Lemma 5.56). We analyse the remaining cases by using a classification result given by [Sch08, Section 6] and Proposition 5.30. We first relate the C_r-condition (for definition of C_r-field, see [Kol96, Definition IV.6.4.1]) for a field of positive characteristic to its p-degree.

Lemma 5.54. *Let k be a field of characteristic $p > 0$. If r is a positive integer and k is a C_r-field, then* $\text{p-deg}(k) \leq r$, *where* $\text{p-deg}(k) := \log_p[k : k^p]$. *In particular, if k is a C_1-field, then* $\text{p-deg}(k) \leq 1$.

Proof. Suppose by contradiction that $[k : k^p] \geq p^{r+1}$. Let $s_1, ..., s_{p^r+1}$ be elements of k which are linearly independent over k^p. Let us consider the following homogeneous polynomial of degree p:

$$P := \sum_{k=1}^{p^r+1} s_i x_i^p = s_1 x_1^p + \cdots + s_{p^r+1} x_{p^r+1}^p \in k[x_1, \ldots, x_{p^r+1}].$$

Since $s_1, ..., s_{p^r+1}$ are linearly independent over k^p, the polynomial P has only the trivial solution in k. In particular k is not a C_r-field. □

We then study rational points on geometrically normal del Pezzo surfaces of degree ≤ 4 (compare with [Kol96, Exercise IV.6.8.3]). We need the following result.

Lemma 5.55 (cf. Exercise IV.6.8.3.2 of [Kol96])**.** *Let k be a C_1-field. Let S be a weighted hypersurface of degree 4 in $\mathbb{P}_k(1, 1, 1, 2)$. Then $S(k) \neq \emptyset$.*

Proof. Let us recall the definition of normic forms ([Kol96, Definition IV.6.4.2]). A homogeneous polynomial $h \in k[y_1, ..., y_m]$ of degree m is called a normic form if $h = 0$ has only the trivial solution in k. If k has a normic form of degree two, then the same argument as in the proof of [Kol96, Theorem IV.6.7] works.

Suppose now that k does not have a normic form of degree two. We can write $\mathbb{P}_k(1, 1, 1, 2) = \text{Proj } k[x_0, x_1, x_2, x_3]$, where $\deg x_0 = \deg x_1 = \deg x_2 = 1$ and $\deg x_3 = 2$. Let

$$F(x_0, x_1, x_2, x_3) := cx_3^2 + f(x_0, x_1, x_2)x_3 + g(x_0, x_1, x_2) \in k[x_0, x_1, x_2, x_3]$$

be the defining polynomial of S, where $c \in k$ and $f(x_0, x_1, x_2), g(x_0, x_1, x_2) \in k[x_1, x_2, x_3]$. If $c = 0$, then $F(0, 0, 0, 1) = 0$. Thus, we may assume that $c \neq 0$. Fix $(a_0, a_1, a_2) \in k^3 \setminus \{(0, 0, 0)\}$. Set $\alpha := f(a_0, a_1, a_2) \in k$ and $\beta := g(a_0, a_1, a_2) \in k$. Since $h(X, Y) := cX^2 + \alpha XY + \beta Y^2$ is not a normic form, there is $(u, v) \in k^2 \setminus \{(0, 0)\}$ such that $h(u, v) = cu^2 + \alpha uv + \beta v^2 = 0$. Since $c \neq 0$, we obtain $v \neq 0$. Therefore, it holds that $F(a_0, a_1, a_2, u/v) = c(u/v)^2 + \alpha(u/v) + \beta = 0$, as desired. □

Lemma 5.56. *Let X be a geometrically normal regular del Pezzo surface over a C_1-field k of characteristic $p > 0$ such that $k = H^0(X, \mathcal{O}_X)$. If $K_X^2 \leq 4$, then $X(k) \neq \emptyset$.*

Proof. Since X is geometrically normal, then it is geometrically canonical by Theorem 5.33. Thus we can apply Theorem 5.18 and we distinguish the cases according to the degree of K_X.

If $K_X^2=1$, then X has a k-rational point by Proposition 5.17(2). If $K_X^2=2$, then X can be embedded as a weighted hypersurface of degree 4 in $\mathbb{P}_k(1,1,1,2)$ and we apply Lemma 5.55 to conclude it has a k-rational point. If $K_X^2=3$, then X is a cubic hypersurface in $\mathbb{P}^3_k$ and thus it has a k-rational point by definition of C_1-field. If $K_X^2=4$, then X is a complete intersection of two quadrics in $\mathbb{P}^4$ and thus it has a k-rational point by [Lan52, Corollary in page 376]. □

We now discuss the existence of purely inseparable points on geometrically normal regular del Pezzo surfaces over C_1-fields.

Proposition 5.57. *Let X be a regular del Pezzo surface over a C_1-field k of characteristic $p \geq 7$ such that $k=H^0(X,\mathcal{O}_X)$. Then $X(k)\neq\emptyset$.*

Proof. If X is a smooth del Pezzo surface, we conclude that there exists a k-rational point by [Kol96, Theorem IV.6.8]. If $p\geq 11$, then X is smooth by Proposition 5.47 and we conclude.

It suffices to treat the case when $p=7$ and X is not smooth. By Theorem 5.35(2), X is geometrically canonical. By [Sch08, Theorem 6.1], any singular point of the base change $X_{\overline{k}}=X\times_k\overline{k}$ is of type A_{p^n-1}. It follows from Lemma 5.46 that $X_{\overline{k}}$ has a unique A_6 singular point. Thus by Lemma 5.46 we have $K_X^2\leq 3$, hence Lemma 5.56 implies $X(k)\neq\emptyset$. □

Proposition 5.58. *Let X be a regular del Pezzo surface over a C_1-field k of characteristic $p\in\{3,5\}$ such that $k=H^0(X,\mathcal{O}_X)$. If X is geometrically normal over k, then $X(k^{1/p})\neq\emptyset$.*

Proof. It is sufficient to consider the case when X is not smooth by [Kol96, Theorem IV.6.8]. By Theorem 5.33, $X_{\overline{k}}$ has canonical singularities.

If $p=5$ and X is not smooth, then the singularities of $X_{\overline{k}}$ must be of type A_4 or E_8^0 according to [Sch08, Theorem 6.1 and Theorem 6.4]. If $X_{\overline{k}}$ has one singular point of type E_8^0 or two singular points of type A_4, then $K_X^2=1$ by Lemma 5.46. Thus we conclude that X has a k-rational point by Lemma 5.56. If $X_{\overline{k}}$ has a unique singular point of type A_4, it follows from Proposition 5.30 that $X(k^{1/p})\neq\emptyset$.

If $p=3$ and X is not smooth, then the singularities of $X_{\overline{k}}$ must be of type A_2, A_8, E_6^0 or E_8^0 according to [Sch08, Theorem 6.1 and Theorem 6.4]. If one of the singular points is of the type A_8, E_6^0 and E_8^0, then $K_X^2\leq 3$ by Lemma 5.46 and we conclude $X(k)\neq\emptyset$ by Lemma 5.56. Thus, we may assume that all the singularities of $X_{\overline{k}}$ are of type A_2. If there is a unique singularity of type A_2 on $X_{\overline{k}}$, then it follows from Proposition 5.30 that $X(k^{1/3})\neq\emptyset$. Therefore, we may assume that there are at least two singularities of type A_2 on $X_{\overline{k}}$. Then it holds that $K_X^2\leq 5$. By [Dol12, Table 8.5 in page 431], we have that $K_X^2\neq 5$, hence $K_X^2\leq 4$. Thus Lemma 5.56 implies $X(k)\neq\emptyset$. □

Proposition 5.59. *Let X be a regular del Pezzo surface over a C_1-field k of characteristic $p=2$ such that $k=H^0(X,\mathcal{O}_X)$. If X is geometrically normal, then $X(k^{1/4})\neq\emptyset$.*

Proof. It is sufficient to consider the case when X is not smooth by [Kol96, Theorem IV.6.8]. The singularities of $X_{\overline{k}}$ are canonical by Theorem 5.33. Hence, by [Sch08, Theorem in page 57], they must be of type A_1, A_3 A_7, D_n^0 with $4 \leq n \leq 8$ or E_n^0 for $n = 6, 7, 8$. We distinguish five cases for the singularities appearing on $X_{\overline{k}}$.

1. There exists at least a singular point of type A_7, D_n^0 with $n \geq 5$ or E_n^0 for $n = 6, 7, 8$.
2. There are at least two singular points with one being of type A_3.
3. There exists at least one singular point of type D_4^0.
4. There is a unique singular point of type A_3.
5. All the singular points are of type A_1.

In case (1), it holds that $K_X^2 \leq 4$. Hence, we obtain $X(k) \neq \emptyset$ by Lemma 5.56. In case (2), if $K_X^2 \leq 4$, then Lemma 5.56 again implies $X(k) \neq \emptyset$. Hence, we may assume that $K_X^2 = 5$. Then there exist exactly two singular points P and Q on $X_{\overline{k}}$ such that P is of type A_3 and Q is of type A_1. However, this cannot occur by [Dol12, Table 8.5 at page 431].

In case (3) we have that $K_X^2 \leq 5$. However a D_4^0 singularity cannot appear on a del Pezzo of degree five according to [Dol12, Table 8.5 at page 431]. Thus $K_X^2 \leq 4$ and Lemma 5.56 implies $X(k) \neq \emptyset$. In case (4), we apply Proposition 5.30 to conclude that $X(k^{1/4}) \neq \emptyset$.

In case (5), consider $X_{(k^{\mathrm{sep}})^{1/2}}$. By Proposition 5.30, on $X_{(k^{\mathrm{sep}})^{1/2}}$ there are singular points $\{P_i\}_{i=1}^m$ of type A_1 such that $k(P_i) = (k^{\mathrm{sep}})^{1/2}$ and their union $\coprod_i P_i$ is the non-smooth locus of $X_{(k^{\mathrm{sep}})^{1/2}}$. Let $Y = \mathrm{Bl}_{\coprod_i P_i} X_{(k^{\mathrm{sep}})^{1/2}}$ be the blowup of $X_{(k^{\mathrm{sep}})^{1/2}}$ along $\coprod_i P_i$. Since each P_i is a $(k^{\mathrm{sep}})^{1/2}$-rational point whose base change to the algebraic closure is a canonical singularity of type A_1, the surface Y is smooth. Since the closed subscheme $\coprod_i P_i$ is invariant under the action of the Galois group $\mathrm{Gal}((k^{\mathrm{sep}})^{1/2}/k^{1/2})$, the birational $(k^{\mathrm{sep}})^{1/2}$-morphism $Y \to X_{(k^{\mathrm{sep}})^{1/2}}$ descends to a birational $k^{1/2}$-morphism $Z \to X^{k^{1/2}}$, where Z is a smooth projective surface over $k^{1/2}$ whose base change to the algebraic closure is a rational surface. It holds that $Z(k^{1/2}) \neq \emptyset$ by [Kol96, Theorem IV.6.8], which implies $X(k^{1/2}) \neq \emptyset$. □

5.6.2. Purely inseparable points on Mori fibre spaces

In this subsection, we discuss the existence of purely inseparable points on log del Pezzo surfaces over C_1-fields admitting Mori fibre space structures onto curves. We start by recalling auxiliary results.

Lemma 5.60. *Let k be a C_1-field and let C be a regular projective curve such that $k = H^0(C, \mathcal{O}_C)$ and $-K_C$ is ample. Then it holds that $C \simeq \mathbb{P}_k^1$. In particular, $C(k) \neq \emptyset$.*

Proof. Since C is a geometrically integral conic curve in $\mathbb{P}_k^2$ (Lemma 5.19), the assertion follows from definition of C_1-field. □

Lemma 5.61. *Let X be a regular projective surface over a C_1-field k of characteristic $p > 0$ such that $k = H^0(X, \mathcal{O}_X)$. Let $\pi\colon X \to B$ be a K_X-Mori fibre space to a regular projective curve B. Then the following hold.*

1. *Let $k \subset k'$ be an algebraic field extension. If $B(k') \neq \emptyset$, then $X(k') \neq \emptyset$.*
2. *If $-K_B$ is ample, then $X(k) \neq \emptyset$.*

Proof. Let us show (1). Let b be a closed point in B such that $k \subset k(b) \subset k'$. By Proposition 5.21, the fibre X_b is a conic in $\mathbb{P}^2_{k(b)}$. By [Lan52, Corollary in page 377], $k(b)$ is a C_1-field, hence we deduce $X_{k(b)}(k(b)) \neq \emptyset$. Thus, (1) holds. The assertion (2) follows from Lemma 5.60 and (1) for the case when $k' = k$. □

To discuss the case when $p = 2$, we first handle a complicated case in characteristic two.

Proposition 5.62. *Let k be a field of characteristic two such that $[k : k^2] \leq 2$. Let X be a regular k-surface of del Pezzo type and let $\pi : X \to B$ be a K_X-Mori fibre space to a curve B. Let Γ be a curve which spans the K_X-negative extremal ray which does not correspond to π. Assume that*

1. *$K_X \cdot \Gamma > 0$, and*
2. *$K(\Gamma)/K(B)$ is an inseparable extension of degree four which is not purely inseparable.*

Then $-K_B$ is ample.

Proof. We divide the proof in several steps. Let us note that since $\pi\colon X \to B$ is a Mori fibre space we have that $\rho(X) = 2$.

Step 1. *In order to show the assertion of Proposition 5.62, we may assume that*

3. *B is not smooth over k,*
4. p-deg$(k) = 1$, *i.e.* $[k : k^2] = 2$, *and*
5. *the generic fibre of π is not geometrically reduced.*

Proof. If (3) does not hold, then B is a smooth curve over k. Since $(X_{\overline{k}})_{\mathrm{red}}$ is a rational surface by Lemma 5.32, $B_{\overline{k}}$ is a smooth rational curve. Then $-K_B$ is ample, as desired. Thus, we may assume (3). From now on, we assume (3).

If (4) does not hold, then k is a perfect field. In this case, B is smooth over k, which contradicts (3). Thus, we may assume (4).

Let us prove the assertion of Proposition 5.62 if (5) does not hold. In this case, the generic fibre $X_{K(B)}$ of $\pi : X \to B$ is a geometrically integral regular conic over $K(B)$. Thus it is smooth over $K(B)$ by Lemma 5.20. We use notation as in Notation 5.37. Lemma 5.38(8) enables us to find a rational number α such that $0 \leq \alpha < 1$ and $(X, \alpha\Gamma)$ is a log del Pezzo pair. Then Lemma 5.40(3) implies that $\alpha m_\Gamma < 2$. Since our assumption (2) implies

$m_\Gamma = [K(\Gamma) : K(B)] = 4$, we have that $\alpha < 1/2$. By the assumption (2) and $\alpha < 1/2$, the induced pair $(X_{\overline{K(B)}}, \alpha\Gamma|_{X_{\overline{K(B)}}})$ on the geometric generic fibre is F-pure. It follows from [Eji, Corollary 4.10] that $-K_B$ is ample. Hence, we may assume that (5) holds. This completes the proof of Step 1. □

From now on, we assume that (3)–(5) of Step 1 hold.

Step 2. *X and B are geometrically integral over k. X is not geometrically normal over k.*

Proof. Since $[k : k^2] = 2$, it follows from [Sch10, Theorem 2.3] that X and B are geometrically integral over k (note that $\log_2[k : k^2]$ is called the degree of imperfection for k in [Sch10, Theorem 2.3]). If X is geometrically normal over k, then also B is geometrically normal over k, i.e. B is smooth over k. This contradicts (3) of Step 1. This completes the proof of Step 2. □

We now introduce some notation. Set $k_1 := k^{1/2}$. By Step 2, $X \times_k k_1$ is integral and non-normal (cf. [Tanb, Proposition 2.10(3)]). Let $\nu : X_1 := (X \times_k k_1)^N \to X \times_k k_1$ be its normalisation. Let $X_1 \to B_1$ be the Stein factorisation of the induced morphism $X_1 \to X \to B$. To summarise, we have a commutative diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{\nu} & X \times_k k_1 & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B \times_k k_1 & \longrightarrow & B. \end{array}$$

Let $C \subset X \times_k k_1$ and $D \subset X_1$ be the closed subschemes defined by the conductors for ν. For $K := K(B)$, we apply the base change $(-) \times_B \operatorname{Spec} K$ to the above diagram:

$$\begin{array}{ccccc} V_1 & \longrightarrow & V \times_K L & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} K_1 & \longrightarrow & \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} K, \end{array}$$

where $V := X \times_B K$, $L := K(B \times_k k_1) = K(B) \otimes_k k_1$, and $K_1 = K(B_1)$. Since taking Stein factorisations commute with flat base changes, the morphism $V_1 \to \operatorname{Spec} K_1$ coincides the Stein factorisation of the induced morphism $V_1 \to \operatorname{Spec} K$.

Step 3. *C dominates B.*

Proof. Assuming that C does not dominate B, let us derive a contradiction. Since B is geometrically integral over k (Step 2), we can find a non-empty open subset B' of B such that B' is smooth over k and the image of C on B is disjoint from B'. Let B'_1, X', and X'_1 be the inverse images of B' to B_1, X, and X_1, respectively. Then the resulting diagram is as follows

$$\begin{array}{ccccc} X'_1 & \xrightarrow{\simeq} & X' \times_k k_1 & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \pi' \\ B'_1 & \xrightarrow{\simeq} & B' \times_k k_1 & \longrightarrow & B'. \end{array}$$

Since $X'_1 \simeq X' \times_k k_1 = X' \times_k k^{1/2}$ is normal, it holds that X' is geometrically normal over k.

Let $\pi'_{\overline{k}} : X'_{\overline{k}} \to B'_{\overline{k}}$ be the base change of π' to the algebraic closure $\overline{k}$. Since X' is geometrically normal over k, $X'_{\overline{k}}$ is a normal surface. Note that $B'_{\overline{k}}$ is a smooth curve. Since general fibres of $\pi'_{\overline{k}} : X'_{\overline{k}} \to B'_{\overline{k}}$ are $K_{X'_{\overline{k}}}$-negative and $(\pi'_{\overline{k}})_*\mathcal{O}_{X'_{\overline{k}}} = \mathcal{O}_{B'_{\overline{k}}}$, general fibres of $\pi'_{\overline{k}}$ are isomorphic to $\mathbb{P}^1_{\overline{k}}$. Then the generic fibre of $\pi'_{\overline{k}} : X'_{\overline{k}} \to B'_{\overline{k}}$ is smooth, hence so is the generic fibre of $\pi : X \to B$. This contradicts (5) of Step 1. This completes the proof of Step 3. □

Step 4. *The following hold.*

(i) *L/K is a purely inseparable extension of degree two.*

(ii) *V is a regular conic curve on $\mathbb{P}^2_K$ which is not geometrically reduced over K.*

(iii) *$V_1 \to V \times_K L$ is the normalisation of $V \times_K L$.*

(iv) *$V \times_K L$ is an integral scheme which is not regular.*

(v) *The restriction $D|_{V_1}$ of the conductor D to V_1 satisfies $D_{V_1} = Q$, where Q is a K_1-rational point.*

(vi) *V_1 is isomorphic to $\mathbb{P}^1_{K_1}$.*

(vii) *K_1/L is a purely inseparable extension of degree two, and $K_1 = K^{1/2}$.*

Proof. The assertions (i)–(iii) follows from the construction. Step 3 implies (iv). Let us show (v). For the induced morphism $\varphi : V_1 \to V$, we have that

$$K_{V_1} + D|_{V_1} \sim \varphi^* K_V.$$

Since $-K_V$ is ample, it holds that

$$0 > \deg_{K_1}(K_{V_1} + D|_{V_1}) \geq -2 + \deg_{K_1}(D|_{V_1}),$$

which implies $\deg_{K_1}(D|_{V_1}) \leq 1$. Step 3 implies that $D|_{V_1} \neq 0$, hence $D|_{V_1}$ consists of a single rational point. Thus, (v) holds.

Let us show (vi). Since V_1 has a K_1-rational point around which V_1 is regular, V_1 is smooth around this point. In particular, Lemma 2.22 implies that V_1 is geometrically reduced. Then V_1 is a geometrically integral conic curve in $\mathbb{P}^2_{K_1}$. Therefore, V_1 is smooth over K_1. Since V_1 has a K_1-rational point, V_1 is isomorphic to $\mathbb{P}^1_{K_1}$. Thus, (vi) holds.

Let us show (vii). The inclusion $K_1 \subset K^{1/2}$, which is equivalent to $K_1^2 \subset K$, follows from the fact that K is algebraically closed in $K(V)$ and the following:

$$K_1^2 \subset K(V_1)^2 = K(V \times_K L)^2 = (K(V) \otimes_K L)^2 \subset K(V).$$

It follows from [BM40, Theorem 3] that the p-degree p-deg(K) is two, i.e. $[K^{1/2} : K] = 4$ (note that the p-degree is called the degree of imperfection in [BM40]). Hence, it is enough to show that $K_1 \neq L$. Assume that $K_1 = L$. Then V_1 is smooth over L by (vi). Hence, $V \times_K L$ is geometrically integral over L. Therefore, V is geometrically integral over K, which contradicts (5) of Step 1. This completes the proof of Step 4. □

Step 5. *Set-theoretically, C does not contain $\Gamma \times_k k_1$.*

Proof. Assuming that C contains $\Gamma \times_k k_1$, let us derive a contradiction. In this case, the set-theoretic inclusion

$$f^{-1}(\Gamma) \subset \nu^{-1}(C) = D$$

holds, where $f : X_1 \to X$ is the induced morphism. Since $B_1 \to B$ is a universal homeomorphism and the geometric generic fibre $\Gamma \times_B \operatorname{Spec} \overline{K}$ of $\Gamma \to B$ consists of two points, the geometric generic fibre of $D \to B_1$ contains two distinct points. In particular, it holds that $\deg_{K_1}(D|_{V_1}) \geq 2$. However, this contradicts (v) of Step 4. This completes the proof of Step 5. □

Step 6. *$-K_{B_1}$ is ample.*

Proof. It follows from Lemma 5.38(8) that there is a rational number α such that $0 \leq \alpha < 1$ and $(X, \alpha\Gamma)$ is a log del Pezzo pair. Consider the pullback:

$$K_{X_1} + D + \alpha f^*\Gamma = f^*(K_X + \alpha\Gamma).$$

Take the geometric generic fibre W of $\pi_1 : X_1 \to B_1$, i.e. $W = V_1 \times_{K_1} \operatorname{Spec} \overline{K_1} \simeq \mathbb{P}^1_{\overline{K_1}}$ (Step 4(vi)). It is clear that $-(K_W + (D + \alpha f^*\Gamma)|_W)$ is ample. Since $D|_{V_1} = Q$ is a rational point (Step 4(v)), its pullback $D|_W =: Q_W$ to W is a closed point on W. As $-(K_W + (D + \alpha f^*\Gamma)|_W)$ is ample, all the coefficients of $B := (\alpha f^*\Gamma)|_W$ must be less than one. Therefore, Step 5 implies that $(W, (D + \alpha f^*\Gamma)|_W)$ is F-pure. It follows from [Eji, Corollary 4.10] that $-K_{B_1}$ is ample. This completes the proof of Step 6. □

Step 7. *$-K_B$ is ample.*

Proof. As $-K_{B_1}$ is ample (Step 6), Lemma 5.19 implies that $H^1(B_1, \mathcal{O}_{B_1}) = 0$. Since $K(B_1) = K(B)^{1/2}$ (Step 4(vii)), the morphism $B_1 \to B$ coincides with the absolute Frobenius morphism of B. Hence, B_1 and B are isomorphic as schemes. Thus, the vanishing $H^1(B_1, \mathcal{O}_{B_1}) = 0$ implies $H^1(B, \mathcal{O}_B) = 0$. Then $-K_B$ is ample by Lemma 5.19. This completes the proof of Step 7. □

Step 7 completes the proof of Proposition 5.62. □

Proposition 5.63. *Let X be a regular k-surface of del Pezzo type over a C_1-field k of characteristic $p > 0$ such that $k = H^0(X, \mathcal{O}_X)$. Let $\pi\colon X \to B$ be a K_X-Mori fibre space to a regular projective curve. Then the following hold.*

1. *If $p \geq 7$, then $X(k) \neq \emptyset$.*
2. *If $p = \{3, 5\}$, then $X(k^{1/p}) \neq \emptyset$.*
3. *If $p = 2$, then $X(k^{1/4}) \neq \emptyset$.*

Proof. Since X is a surface of del Pezzo type, the cone theorem tells that $\overline{\mathrm{NE}}(X)$ is rational polyhedral and the extremal rays are generated by curves. Since $\rho(X) = 2$, there are only two extremal rays and let $R = \mathbb{R}_{\geq 0}[\Gamma]$ be the extremal ray of $\overline{\mathrm{NE}}(X)$ not corresponding to $\pi : X \to B$. In particular, we have $\pi(\Gamma) = B$. We distinguish two cases:

(I) $K_X \cdot \Gamma \leq 0$;

(II) $K_X \cdot \Gamma > 0$.

Suppose that (I) holds. In this case, $-K_X$ is nef and big. If $p > 2$, then the generic fibre $X_{K(B)}$ is a smooth conic. In particular, the base change $X_{\overline{K(B)}}$ is strongly F-regular. By [Eji, Corollary 4.10], $-K_B$ is ample. Hence, Proposition 5.63 implies $X(k) \neq \emptyset$.

We now treat the case when (I) holds and $p = 2$. Then $-K_X$ is semi-ample and big. Let Z be its anti-canonical model. In particular, Z is a canonical del Pezzo surface. By Theorem 5.35, we have $\ell_F(Z/k) \leq 2$. Therefore, for $k_W := k^{1/4}$ and $W := (Z \times_k k_W)^N_{\mathrm{red}}$, W is geometrically normal over k_W. In particular, $H^0(W, \mathcal{O}_W) = k_W = k^{1/4}$. We have the following commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\nu} & X \times k^{1/4} & \longrightarrow & X \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ W & \xrightarrow{\mu} & Z \times_k k^{1/4} & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\, k^{1/4} & = & \mathrm{Spec}\, k^{1/4} & \longrightarrow & \mathrm{Spec}\, k, \end{array}$$

where μ and ν are the normalisations. It follows from Theorem 5.33 that W is geometrically klt and $H^1(W, \mathcal{O}_W) = 0$. Since the morphism $Y \to W$ is birational and W is klt by Proposition 2.23, it holds that $H^1(Y, \mathcal{O}_Y) = 0$.

Consider the Stein factorisation $\pi_1\colon Y \to B_1$ of the induced morphism $Y \to X \xrightarrow{\pi} B$. Since $H^1(Y, \mathcal{O}_Y) = 0$, we conclude that $H^1(B_1, \mathcal{O}_{B_1}) = 0$. In particular, since k_W is a C_1-field, it holds that $B_1 \simeq \mathbb{P}^1_{k_W}$ (Lemma 5.60). Thanks to [Tan18b, Theorem 4.2], we can find an effective divisor D on Y such that $K_Y + D = f^*K_X$. Since $-K_X$ is big, also $-K_Y$ is big. Fix a general k_W-rational point $c \in B_1$ and let F_c be its π_1-fibre. Since we take c to be general, F_c avoids the non-regular points of Y. By adjunction, $\omega_{F_c}^{-1}$ is ample. This implies that F is a conic on $\mathbb{P}^2_{k_W}$. Hence, $Y(k^{1/4}) = Y(k_W) \neq \emptyset$. Therefore, we deduce $X(k^{1/4}) \neq \emptyset$.

We suppose (II) holds. We have $[K(\Gamma) : K(B)] \leq 5$ by Proposition 5.42. If $K(\Gamma)/K(B)$ is separable, then $-K_B$ is ample (Lemma 5.43). Then Proposition 5.63 implies $X(k) \neq \emptyset$. Hence, we may assume that $K(\Gamma)/K(B)$ is inseparable. If $K(\Gamma)/K(B)$ is not purely inseparable, then $-K_B$ is ample by Proposition 5.62. Again, Proposition 5.63 implies $X(k) \neq \emptyset$. Hence, it is enough to treat the case when $K(\Gamma)/K(B)$ is purely inseparable. Since $[K(\Gamma) : K(B)] \leq 5$, it suffices to prove that $X(k^{1/p^e}) \neq \emptyset$ for the positive integer e defined by $[K(\Gamma) : K(B)] = p^e$. Set $C := \Gamma^N$. Since ω_Γ^{-1} is ample, also $-K_C$ is ample. Hence Proposition 5.63 implies $C(k') \neq \emptyset$, where $k' := H^0(C, \mathcal{O}_C)$. Since

$$k'^{p^e} \subset K(\Gamma)^{p^e} \subset K(B),$$

it holds that $k'^{p^e} \subset k$. Therefore, we obtain $X(k^{1/p^e}) \neq \emptyset$, as desired. □

5.6.3. General case

In this subsection, using the results proven above, we prove the main result in this section (Theorem 5.65) We present a generalisation of the Lang–Nishimura theorem on rational points. Although the argument is similar to the one in [RY00, Proposition A.6], we include the proof for the sake of completeness.

Lemma 5.64 (Lang-Nishimura). *Let k be a field. Let $f : X \dashrightarrow Y$ be a a rational map between k-varieties. Suppose that X is regular and Y is proper over k. Fix a closed point P on X. Then there exists a closed point Q on Y such that $k \subset k(Q) \subset k(P)$, where $k(P)$ and $k(Q)$ denote the residue fields.*

Proof. The proof is by induction on $n := \dim X$. If $n = 0$, then there is nothing to show. Suppose $n > 0$. Consider the blowup $\pi : \mathrm{Bl}_P X \to X$ at the closed point P. Since X is regular, the π-exceptional divisor E is isomorphic to $\mathbb{P}^{n-1}_{k(P)}$ by [Liu02, Section 8, Theorem 1.19]. Consider now the induced map $f : \mathrm{Bl}_P X \dashrightarrow Y$. By the valuative criterion of properness, the map f induces a rational map $E = \mathbb{P}^{n-1}_{k(P)} \dashrightarrow Y$ from the π-exceptional divisor E. Then by the induction hypothesis Y has a closed point Q whose residue field is contained in $k(P)$. □

Theorem 5.65. *Let k be a C_1-field of characteristic $p > 0$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then the following hold.*

1. *If $p \geq 7$, then $X(k) \neq \emptyset$;*

2. *If $p \in \{3, 5\}$, then $X(k^{1/p}) \neq \emptyset$;*

3. *If $p = 2$, then $X(k^{1/4}) \neq \emptyset$.*

Proof. Let $Y \to X$ be the minimal resolution of X. We run a K_Y-MMP $Y =: Y_0 \to Y_1 \to \cdots \to Y_n =: Z$. Note that the end result is a Mori fibre space. Thanks to Lemma 5.64, we may replace X by Z. Hence it is enough to treat the following two cases.

(i) X is a regular del Pezzo surface with $\rho(X) = 1$.

(ii) There exists a Mori fibre space structure $\pi\colon X \to B$ to a curve B.

Assume (i). By Lemma 5.54, we have p-deg$(k) \leq 1$. Therefore X is geometrically normal by [FS18, Theorem 14.1]. Thus we conclude by Propositions 5.57, Proposition 5.58, and Proposition 5.59. If (ii) holds, then the assertion follows from Propositions 5.63. □

5.7. Pathological examples

In this section, we collect pathological feature appearing on surfaces of del Pezzo type over imperfect fields.

5.7.1. Summary of known results

We first summarise previously known examples of pathologies appearing on del Pezzo surfaces over imperfect fields.

Geometric properties

We have shown that if $p \geq 7$ and X is a surface of del Pezzo type, then X is geometrically integral (Corollary 5.50). We have established a partial result on geometric normality (Theorem 5.49). Let us summarise known examples in small characteristic related to these properties.

1. Let $\mathbb{F}$ be a perfect field of characteristic $p > 0$ and let $k := \mathbb{F}(t_1, t_2, t_3)$. Then

$$X := \mathrm{Proj}\ k[x_0, x_1, x_2, x_3]/(x_0^p + t_1 x_1^p + t_2 x_2^p + t_3 x_3^p)$$

   is a regular projective surface which is not geometrically reduced over k. Since X is an hypersurface in $\mathbb{P}^3_k$ of degree p we have

$$H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}) \to H^0(X, \mathcal{O}_X) \to H^1(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-p)) = 0,$$

   thus showing that $H^0(X, \mathcal{O}_X) = k$. If the characteristic of k is two or three, then $-K_X$ is ample, hence X is a regular del Pezzo surface.

2. There exist a field of characteristic 2 and a regular del Pezzo surface X over k such that $H^0(X, \mathcal{O}_X) = k$, X is geometrically reduced over k, and X is not geometrically normal over k (see [Mad16, Main Theorem]).

3. If k is an imperfect field of characteristic $p = 2, 3$ there exists a geometrically normal regular del Pezzo surface X of Picard rank one which is not smooth (see [FS18, Section 14, Equation 27]). In [FS18, Theorem 14.8], an example of a regular geometrically integral but geometrically non-normal del Pezzo surface of Picard rank two is constructed when $p = 2$.

4. If k is an imperfect field of characteristic $p \in \{2, 3\}$, then there exists a k-surface X of del Pezzo type such that $H^0(X, \mathcal{O}_X) = k$, X is geometrically reduced over k, and X is not geometrically normal over k ([Tana]).

Vanishing of $H^1(X, \mathcal{O}_X)$

We have shown that if X is a surface of del Pezzo type over a field of characteristic $p \geq 7$, then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Let us summarise known examples in small characteristic which violate the vanishing of $H^1(X, \mathcal{O}_X)$.

1. If k is an imperfect field of characteristic 2, then there exists a regular weak del Pezzo surface X such that $H^1(X, \mathcal{O}_X) \neq 0$ (see [Sch07]).

2. There exist an imperfect field of characteristic 2 and a regular del Pezzo surface X such that $H^1(X, \mathcal{O}_X) \neq 0$ (see [Mad16, Main theorem]).

3. If k is an imperfect field of characteristic $p \in \{2, 3\}$, then there exists a surface X of del Pezzo type such that $H^1(X, \mathcal{O}_X) \neq 0$ (see [Tana]).

Remark 5.66. Since $h^1(X, \mathcal{O}_X)$ is a birational invariant for surfaces with klt singularities, the previous examples do not admit regular k-birational models which are geometrically normal. This shows that Theorem 5.49 cannot be extended to characteristic two and three.

5.7.2. Non-smooth regular log del Pezzo surfaces

In this subsection, we construct examples of regular k-surfaces of del Pezzo type which are not smooth (cf. Theorem 5.49).

Proposition 5.67. *Let k be an imperfect field of characteristic $p > 0$. Then there exists a k-regular surface X of del Pezzo type which is not smooth over k.*

Proof. Fix a k-line L on $\mathbb{P}^2_k$. Let $Q \in L$ be a closed point such that $k(Q)/k$ is a purely inseparable extension of degree p whose existence is guaranteed by the assumption that k is imperfect. Consider the blow-up $\pi : X \to \mathbb{P}^2_k$ at the point Q. We have

$$K_X = \pi^* K_{\mathbb{P}^2_k} + E \quad \text{and} \quad \widetilde{L} + E = \pi^* L,$$

where E denotes the π-exceptional divisor and $\widetilde{L}$ is the proper transform of L. Since $\widetilde{L} \cup E$ is simple normal crossing and the $\mathbb{Q}$-divisor

$$-(K_X + \widetilde{L} + \epsilon E) = \pi^*(K_X + L) - \epsilon E$$

is ample for any $0 < \epsilon \ll 1$, the pair $(X, \widetilde{L} + \epsilon E)$ is log del Pezzo. Hence, X is of del Pezzo type.

It is enough to show that X is not smooth. There exists an affine open subset $\operatorname{Spec} k[x, y] = \mathbb{A}^2_k$ of $\mathbb{P}^2_k$ such that $Q \in \operatorname{Spec} k[x, y]$ and the maximal ideal corresponding to Q can be written as $(x^p - \alpha, y)$ for some $\alpha \in k \setminus k^p$. Let X' be the inverse image of $\operatorname{Spec} k[x, y]$ by π. Since blowups commute with flat base changes, the base change $X'_{\overline{k}}$ is isomorphic to the blowup of $\operatorname{Spec} \overline{k}[x, y]$ along the non-reduced ideal $((x - \beta)^p, y)$, where $\beta \in \overline{k}$ with $\beta^p = \alpha$.

Let us make the following change of coordinates over $\overline{k}$:

$$x' := x - \beta; y' := y.$$

In this new coordinates, $X'_{\overline{k}}$ is the blowup of $\mathbb{A}^2_{\overline{k}} = \operatorname{Spec} \overline{k}[x', y']$ along the ideal (x'^p, y'). We can directly check that $X'_{\overline{k}}$ contains an affine open subset of the form $\operatorname{Spec} \overline{k}[s, y, u]/(st - u^p)$, which is not smooth. □

Remark 5.68. The surface X constructed in Proposition 5.67 is del Pezzo (resp. weak del Pezzo) if and only if $p = 2$ (resp. $p \leq 3$). Indeed, $-E^2 = [k(Q) : k] = p$ implies $K_X \cdot_k E = (K_X + E) \cdot_k E - E^2 = -2p + p = -p$. Thus the desired conclusion follows from

$$K_X \cdot_k \widetilde{L} = K_X \cdot_k \pi^* L - K_X \cdot_k E = -3 + p.$$

5.8. Applications to del Pezzo fibrations

In this section, we give applications of Theorem 5.45 and Theorem 5.65 on log del Pezzo surfaces over imperfect fields to the birational geometry of threefold fibrations. The first application is to rational chain connectedness.

Theorem 5.69. *Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi : V \to B$ be a projective k-morphism such that $\pi_* \mathcal{O}_V = \mathcal{O}_B$, V is a normal threefold over k, and B is a smooth curve over k. Assume that there exists an effective $\mathbb{Q}$-divisor Δ such that (V, Δ) is klt and $-(K_V + \Delta)$ is π-nef and π-big. Then the following hold.*

1. *There exists a curve C on V such that $C \to B$ is surjective and the following properties hold.*
   - *(a) If $p \geq 7$, then $C \to B$ is an isomorphism.*
   - *(b) If $p \in \{3, 5\}$, then $K(C)/K(B)$ is a purely inseparable extension of degree $\leq p$.*
   - *(c) If $p = 2$, then $K(C)/K(B)$ is a purely inseparable extension of degree ≤ 4.*
2. *If B is a rational curve, then V is rationally chain connected.*

Proof. Let us show (1). Thanks to [Kol96, Ch. IV, Theorem 6.5], $K(B)$ is a C_1-field. Then Theorem 5.65 implies the assertion (1). The assertion (2) follows from (1) and the fact that general fibres are rationally connected (see Lemma 5.32). □

The second application is to Cartier divisors on Mori fibre spaces which are numerically trivial over the bases.

Theorem 5.70. *Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi\colon V \to B$ be a projective k-morphism such that $\pi_*\mathcal{O}_V = \mathcal{O}_B$, where X is a $\mathbb{Q}$-factorial normal quasi-projective threefold and B is a smooth curve. Assume there exists an effective $\mathbb{Q}$-divisor Δ such that (V, Δ) is klt and $\pi\colon V \to B$ is a $(K_V + \Delta)$-Mori fibre space. Let L be a π-numerically trivial Cartier divisor on V. Then the following hold.*

1. *If $p \geq 7$, then $L \sim_\pi 0$.*
2. *If $p \in \{3, 5\}$, then $p^2 L \sim_\pi 0$.*
3. *If $p = 2$, then $16L \sim_\pi 0$.*

Proof. We only prove the theorem in the case when $p = 2$, since the other cases are similar and easier. Since the generic fibre $V_{K(B)}$ is a $K(B)$-surface of del Pezzo type, we have by Theorem 5.45 that $4L|_{V_{K(B)}} \sim 0$. Therefore, $4L$ is linearly equivalent to a vertical divisor, i.e. we have

$$4L \sim \sum_{i=1}^{r} \ell_i D_i,$$

where $\ell_i \in \mathbb{Z}$ and D_i is a prime divisor such that $\pi(D_i)$ is a closed point b_i.

Since $\rho(V/B) = 1$ and V is $\mathbb{Q}$-factorial, all the fibres of π are irreducible. Hence, we can write $\pi^*(b_i) = n_i D_i$ for some $n_i \in \mathbb{Z}_{>0}$. Let m_i be the Cartier index of D_i, i.e. the minimum positive integer m such that mD_i is Cartier. Since the divisor $\pi^*(b_i) = n_i D_i$ is Cartier, then there exists $r_i \in \mathbb{Z}_{>0}$ such that $n_i = r_i m_i$.

We now prove that r_i is a divisor of 4. Since $K(B)$ is a C_1-field and the generic fibre is a surface of del Pezzo type, we conclude by Theorem 5.65 that there exists a curve Γ on V such that the degree d of the morphism $\Gamma \to B$ is a divisor of 4. By the equation

$$r_i \cdot (m_i D_i) \cdot \Gamma = n_i D_i \cdot \Gamma = \pi^*(b_i) \cdot \Gamma = d,$$

r_i is a divisor of 4.

Therefore, it holds that $4m_i D_i \sim_\pi 0$. On the other hand, the divisor $4L = \sum_{i=1}^{r} \ell_i D_i$ is Cartier, hence we have that $\ell_i = s_i m_i$ for some $s_i \in \mathbb{Z}$. Therefore it holds that

$$16L \sim \sum_{i=1}^{r} 4\ell_i D_i \sim \sum_{i=1}^{r} s_i (4m_i D_i) \sim_\pi 0,$$

as desired. □

6

On the base point free theorem for klt threefolds in large characteristic

6.1. Introduction

The base point free theorem is one of the cornerstones of the Minimal Model Program (MMP for short) over a field of characteristic zero. Due to the failure of the Kodaira vanishing theorem and its generalisation, it is not known whether it still holds for varieties over fields of positive characteristic.

However, in the case of threefolds the base point free theorem has been established with increasing generality in recent years. In the seminal article [Kee99], Keel proved that if L is assumed to be big and nef, then it is endowed with a map (EWM) without any restriction on the characteristic. After the development of the MMP for threefolds ([HX15]), in [Xu15, Theorem 1.1], [Bir16, Theorem 1.5] and [BW17, Theorem 1.2] the authors prove with increasing generality that the linear system $|mL|$ is base point free for sufficiently large and *sufficiently divisible* $m > 0$ over perfect fields of characteristic $p > 5$.

Let us recall that over a field of characteristic zero, the divisibility condition on m is indeed superfluous (see [KM98, Theorem 3.3]). Thus one may wonder whether we can remove it also in characteristic $p > 0$. Unfortunately, this is not possible in low characteristic: in [Tana, Theorem 1.2], Tanaka showed that the divisibility assumption is indeed necessary over fields of characteristic two and three when the numerical dimension of L is one.

The aim of this section is to present a refinement of the base point free theorem for threefolds by proving that we can remove the divisibility assumption if the characteristic is sufficiently large.

Theorem 6.1 (Theorem 6.24)**.** *There exists a constant $p_0 \geq 5$ such that the following holds. Let k be a perfect field of characteristic $p > p_0$. Let (X, Δ) be a projective klt threefold log pair over k. Let L be a nef Cartier divisor on X such that*

1. *the numerical dimension $\nu(L)1$ is at least one;*

2. *$nL - (K_X + \Delta)$ is a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for some $n > 0$.*

Then there exists an integer $m_0 > 0$ such that the linear system $|mL|$ is base point free for all $m \geq m_0$.

Let us recall that the numerical dimension of a nef Cartier divisor on a normal projective variety is defined as follows:

$$\nu(L) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \mid L^k \not\equiv 0 \right\}.$$

Remark 6.2. The author does not know whether Theorem 6.1 can be extended to the case where $\nu(L) = 0$. This is related to understanding the torsion of the Picard group of threefolds of Fano type in positive characteristic.

Apart from the intrinsic interest of understanding the differences between characteristic zero and characteristic p birational geometry, Theorem 6.1 is important if one desires to obtain effective statements in positive characteristic. Indeed, if the divisibility required on m is arbitrarily large, there is no hope that the Effective base point free theorem of Kollár (see [Kol93]) could hold in positive characteristic.

Theorem 6.1 is a consequence of the following descent result for numerically trivial Cartier divisors under $(K_X + \Delta)$-negative birational contractions, which is the main technical result of this paper.

Theorem 6.3 (Theorem 6.19)**.** *There exists a constant $p_0 \geq 5$ with the following property. Let k be a perfect field of characteristic $p > p_0$. Let $\pi \colon X \to Z$ be a projective contraction between quasi-projective normal varieties. Suppose that there exists an effective $\mathbb{Q}$-divisor $\Delta \geq 0$ such that*

1. *(X, Δ) is a klt threefold log pair;*

2. *$-(K_X + \Delta)$ is π-big and π-nef;*

3. $\dim(Z) \geq 1$.

Let L be a Cartier divisor on X such that $L \equiv_\pi 0$. Then there exists a Cartier divisor M on Z such that $L \sim \pi^* M$.

Remark 6.4. The constant p_0 in Theorem 6.1 and Theorem 6.3 comes from the Kawamata-Viehweg vanishing theorem for log del Pezzo surfaces in large characteristic (see [CTW17, Theorem 1.2]).

6.1.1. Sketch of the proof

The proof of Theorem 6.3 is divided in two steps: first we discuss Cartier divisors which are numerically trivial for pl-contractions over surfaces and threefolds (see Theorem 6.17). Then we prove the general case (see Theorem 6.19).

Let us overview the case of pl-contractions treated in Section 6.3. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt threefold pair and let S be a prime divisor in $\lfloor \Delta \rfloor$. Let $\pi\colon (X, \Delta) \to Y$ be a $(K_X + \Delta)$-negative contraction where S is a prime divisor which is π-anti-nef and $\dim(Y) \geq 2$. Let L be a π-numerically trivial Cartier divisor. We aim to prove that L is π-trivial over a neighbourhood of $\pi(S)$. Since $L|_S$ is $\pi|_S$-trivial (Proposition 6.5), it is sufficient to lift sections from S to prove that L is π-trivial. In order to do so, we show that the higher direct image $R^1\pi_*\mathcal{O}_X(L - S)$ vanishes. First we prove the vanishing in the case where the fibres of π are at most one dimensional (Proposition 6.13), for which we generalise a vanishing theorem of Das and Hacon (see Proposition 6.12). To prove the general case of pl-contractions in Theorem 6.17, we use some techniques developed by Hacon and Witaszek to prove the rationality of klt threefold singularities (see [HW17]). The main ingredient for proving the vanishing of $R^1\pi_*\mathcal{O}_X(L-S)$ is the Kawamata-Viehweg vanishing theorem for surfaces of del Pezzo type in large characteristic (see [CTW17]) and the aforementioned case of pl-contractions whose maximum dimension of the fibres is one.

In Section 6.4, we prove Theorem 6.3. First in Subsections 6.4.1, 6.4.2 we discuss with the case where $\dim(Z) \geq 2$. The idea is to use the MMP and by replacing a fibre of $\pi\colon X \to Z$ with a surface of del Pezzo type by Proposition 6.10, we can apply Theorem 6.17 to conclude the descent. For the case where $\dim(Z) = 1$, we blend the previous cases with some results on del Pezzo fibrations proven in [BT19].

6.2. Preliminaries

6.2.1. Numerically trivial Cartier divisors on excellent surfaces

We will need descent results for numerically trivial Cartier divisors on excellent surfaces. Thus we recall the the base point free theorem for excellent surfaces proven by Tanaka (see [Tan18a]).

Proposition 6.5. *Let B be a regular excellent separated scheme of finite dimension. Let $\pi\colon X \to S$ be a projective B-morphism of normal quasi-projective B-schemes. Suppose (X, Δ) is a $\mathbb{Q}$-factorial surface log pair where Δ is a boundary. Let L be a π-nef Cartier divisor on X and suppose that $L - (K_X + \Delta)$ is π-ample. Then the following hold.*

1. *Suppose that $L \not\equiv_\pi 0$. Then there exists b_0 such that for all $b \geq b_0$, bL is π-free. In particular, there exists a factorisation $\pi\colon X \xrightarrow{\pi_L} T \xrightarrow{g} S$ such that $L \sim \pi_L^* H$ for a g-ample Cartier divisor H on T.*

2. *Suppose that (X, Δ) is dlt and $L \equiv_\pi 0$ and the Stein factorisation of $X \to S$ is $X \to \mathrm{Spec}(k) \to S$, where k is a perfect field. Then $L \sim 0$.*

Proof. Since X is $\mathbb{Q}$-factorial, we can perturb the boundary Δ and assume that $\lfloor \Delta \rfloor = 0$. We show (1). The first part of the statement is [Tan18a, Theorem 4.2]. As for the remaining part, there exist H_{b_0} and H_{b_0+1} Cartier divisors on T such that $b_0 L = \pi_L^* H_{b_0}$ and $(b_0+1)L = \pi_L^* H_{b_0+1}$. In particular, $L = \pi_L^*(H_{b_0+1} - H_{b_0})$, thus concluding.

We show (2). Since $\nu(L) = 0$, then (X, Δ) is a log del Pezzo pair over a perfect field. By the Riemann-Roch formula, we have $h^0(L) \geq 1$. Since $\nu(L) = 0$, we conclude $L \sim 0$. □

6.2.2. Pl-contractions

We introduce the notion of (weak) pl-contractions, which is a natural generalisation of the notions of pl-divisorial contractions and pl-flipping contractions (see [GNT, Definition 3.2]) to the case of contractions of fibre type.

Definition 6.6. Let k be a field. Let (X, Δ) be a dlt pair over k and let S be a prime divisor contained in $\lfloor \Delta \rfloor$. Let $\pi \colon X \to Y$ be a projective k-morphism between quasi-projective normal varieties. We say that π is a $(K_X + \Delta, S)$-*pl-contraction* (resp. a *weak* $(K_X + \Delta, S)$-*pl-contraction*) if

1. $-(K_X + \Delta)$ is π-ample,
2. $-S$ is π-ample (resp. π-nef).

We collect some properties of weak pl-contractions for later use.

Lemma 6.7. *Let k be a field. Let X be a normal variety over k and let S be a $\mathbb{Q}$-Cartier prime divisor. Let $\pi \colon X \to Y$ be a proper contraction between normal varieties such that $-S$ is π-nef. Then for all closed points $x \in \pi(S)$, we have $\pi^{-1}(x) \subset S$.*

Proof. Immediate since $-S$ is π-nef. □

Lemma 6.8. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a $\mathbb{Q}$-factorial threefold dlt pair over k and let S be a prime divisor contained in $\lfloor \Delta \rfloor$. Let $\pi \colon X \to Y$ be a weak $(K_X + \Delta, S)$-pl-contraction. Then $-S$ is π-semi-ample.*

Proof. We write $-S = K_X + \Delta - (K_X + \Delta + S)$. Thus we conclude $-S$ is π-semi-ample by the relative base point free theorem (see [GNT, Theorem 2.9]). □

Lemma 6.9. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a $\mathbb{Q}$-factorial threefold dlt pair over k and let S be a prime divisor contained in $\lfloor \Delta \rfloor$. Let $\pi \colon X \to Y$ be a weak $(K_X + \Delta, S)$-pl-contraction. Let $\pi \colon X \xrightarrow{g} Z \xrightarrow{h} Y$ be the relative semi-ample fibration associated to $-S$ given by Lemma 6.8. Assume*

1. *$-S$ is not π-ample over any neighbourhood of $\pi(S)$,*
2. *$\dim(Y) \geq 2$.*

Then the dimension of the fibres of g is at most one in a neighbourhood of $g(S)$.

Proof. The dimension of the image of S is either a surface, a curve or a point. In the first two cases, by a dimension argument the fibres of the morphism $S \to \pi(S)$ must be at most one dimensional. Thus we can assume that $\pi(S)$ is a closed point. Since $\dim(Y) \geq 2$, there is a neighbourhood U of y such that the dimension of the fibres of π over $U \setminus y$ is at most one. If S gets contracted to a point by g, we have that h is an isomorphism over an open neighbourhood of y, thus contradicting assumption (1). $\square$

The following extraction result, which motivated the definition of weak pl-contraction, will be used repeatedly in the following. It is essentially stating that given a $(K_X + \Delta)$-negative contraction from a threefold onto a variety of positive dimension we can replace, after some birational modification, a fibre with a surface of del Pezzo type.

Proposition 6.10 (cf. [GNT, Proposition 2.15])**.** *Let k be a perfect field of characteristic $p > 5$. Let $\pi\colon X \to Z$ be a projective contraction of normal quasi-projective varieties over k with the following properties:*

i) (X, Δ) *is a klt threefold log pair,*

ii) $-(K_X + \Delta)$ *is π-big and π-nef.*

iii) $0 < \dim(Z) \leq 3$.

Fix a closed point $z \in Z$. Then there exists a commutative diagram of quasi-projective normal varieties

$$\begin{array}{ccc} W & \xrightarrow{\psi} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\pi} & Z, \end{array}$$

and an effective $\mathbb{Q}$-divisor Δ_Y on Y such that

1. (Y, Δ_Y) *is a $\mathbb{Q}$-factorial plt pair;*

2. $S = (g^{-1}(z))_{\text{red}}$ *is an irreducible component of $\lfloor \Delta_Y \rfloor$ and g is a weak $(K_Y + \Delta_Y, S)$-pl-contraction;*

3. *W is a smooth threefold and φ and ψ are projective birational morphisms.*

In particular, S is a surface of del Pezzo type by adjunction.

6.2.3. Vanishing for pl-contractions with one dimensional fibres

In this subsection, we present a generalisation of a relative vanishing theorem of Kodaira type in positive characteristic due to Das and Hacon (see [DH16]). This will be used in Proposition 6.13 to prove a descent result for numerically trivial Cartier divisors on pl-contractions with one-dimensional fibres. We refer to [Sch14] for a thorough treatment of the trace map of the Frobenius morphism on log pairs.

Proposition 6.11 (cf. [DH16, Proposition 3.2])**.** *Let k be a perfect field of characteristic $p > 5$. Let $(X, \Delta = S + \Gamma)$ be a $\mathbb{Q}$-factorial threefold plt pair where S is a prime Weil divisor and $\lfloor \Gamma \rfloor = 0$. Suppose that there exists an integer $f > 0$ such that $(p^f - 1)(K_X + \Delta)$ is an integral divisor. Then there exists $e > 0$ such that the morphism induced by the trace map of Frobenius*

$$\psi_{ne} \colon F_*^{ne}\mathcal{O}_X((1 - p^{ne})(K_X + \Gamma) - p^{ne}mS) \to \mathcal{O}_X(-mS).$$

is surjective for all $n \geq 1$ and for all $m \geq 1$ at all codimension two points of X contained in S.

Proof. By the proof of [DH16, Proposition 3.2], we obtain that there exists an integer $e > 0$ such that the natural morphism

$$F_*^{ne}\mathcal{O}_X((1 - p^{ne})(K_X + \Gamma)) \to \mathcal{O}_X$$

admits a splitting in the category of $\mathcal{O}_X$-modules at all codimension two points of X contained in S for all $n \geq 1$ (cf. [DH16, Equations 3.4, 3.5]). By tensoring with $\mathcal{O}_X(-mS)$ and considering reflexive hulls, we thus deduce that

$$F_*^{ne}\mathcal{O}_X((1 - p^{ne})(K_X + \Gamma) - p^{ne}mS) \to \mathcal{O}_X(-mS)$$

is surjective at all codimension two points of X contained in S. □

The following vanishing theorem is an easy generalisation of [DH16, Theorem 3.5].

Proposition 6.12. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a $\mathbb{Q}$-factorial threefold dlt pair and let S be a prime divisor contained in $\lfloor \Delta \rfloor$. Assume there exists $e > 0$ such that $(p^e - 1)(K_X + \Delta)$ is integral. Let $\pi \colon X \to Y$ be a projective contraction between normal quasi-projective varieties such that*

1. *the maximum dimension of the fibres of π is one;*
2. *π is a weak $(K_X + \Delta, S)$-pl-contraction.*

Let L be a Cartier divisor on X such that L is π-nef. Then for all $m > 0$ we have

$$R^1\pi_*\mathcal{O}_X(L - mS) = 0$$

in a neighbourhood of $\pi(S)$.

Proof. We follow the proof of [DH16, Theorem 3.5] and we show how to adapt their arguments in order to prove the desired vanishing theorem.

Let us write $\Delta = S + \Delta'$. Since X is $\mathbb{Q}$-factorial we can slightly perturb the boundary Δ' in order to find a boundary Γ such that $(X, S + \Gamma)$ is plt and $(1 - p^f)(K_X + S + \Gamma)$ is Cartier and π-ample for some $f > 0$.

If π is birational, then by Lemma 6.7 we can suppose $\mathrm{Ex}(\pi) \subset S$. If $\dim(Y) = 2$, then by Lemma 6.7 S is a vertical divisor for π (i.e. $\pi(S)$ is an irreducible curve in Y). Moreover, by localising at codimension one points of Y and applying the relative Kawamata-Viehweg vanishing theorem for excellent surfaces (see [Tan18a, Theorem 3.3]), we can suppose that $R^1\pi_*\mathcal{O}_X(L - mS)$ is supported on a finite number of closed points of $\pi(S)$.

Consider now $e > 0$ given by Proposition 6.11. Using the trace map of the Frobenius morphism for all $n \geq 1$ we have the exact sequence

$$0 \to \ker(\psi_{ne}) \to F_*^{ne}\mathcal{O}_X((1-p^{ne})(K_X + \Gamma) + p^{ne}L - p^{ne}mS) \xrightarrow{\psi_{ne}} \mathcal{O}_X(L - mS).$$

Now let us split the sequence in two short exact sequences:

$$0 \to \ker(\psi_{ne}) \to F_*^{ne}\mathcal{O}_X((1-p^{ne})(K_X + \Gamma) + p^{ne}L - p^{ne}mS) \to \mathrm{im}(\psi_{ne}) \to 0; \quad (6.1)$$

and

$$0 \to \mathrm{im}(\psi_{ne}) \to \mathcal{O}_X(L - mS) \to \mathcal{G}_{ne} \to 0. \quad (6.2)$$

Consider the long exact sequence in cohomology obtained by applying the push-forward π_* to the short exact sequence (6.1). Since the fibres of π are at most one dimensional, we have $R^2\pi_* \ker(\psi_e) = 0$. Let us denote $H = -(K_X + mS - L + \Gamma) = -(K_X + S + \Gamma) + (L - (m-1)S)$ and let r be the Cartier index of H. Since L and $-S$ are π-nef, we have H is π-ample for all $m \geq 0$. Write $(p^{ne} - 1) = r \cdot a + b$ for some positive integer a, b such that $0 \leq b < r$. Thus we have

$$\begin{aligned} &R^1\pi_*(F_*^{ne}\mathcal{O}_X((1-p^{ne})(K_X + \Gamma) + p^{ne}L - p^{ne}mS)) = \\ &\qquad = F_*^{ne}(R^1\pi_*\mathcal{O}_X(raH - b(K_X + \Gamma + mS - L) + L - mS)), \end{aligned}$$

which vanishes for sufficiently large $n > 0$ by Serre vanishing. Therefore we conclude $R^1\pi_*\mathrm{im}(\psi_{ne}) = 0$.

Since ψ_{ne} is surjective at codimension two points contained in S by Proposition 6.11, we have $\mathcal{G}_{ne}$ is supported on a finite number of points. Therefore $R^1\pi_*\mathcal{G}_{ne} = 0$ and thus we conclude $R^1\pi_*\mathcal{O}_X(L - mS) = 0$. □

6.3. Numerically trivial Cartier divisors on pl-contractions

In this section, we prove a descent result for numerically trivial Cartier divisors on threefolds under weak pl-contractions over surfaces and threefolds (see Theorem 6.17).

First, in Subsection 6.3.1 we discuss the case of a weak pl-negative contraction with fibres whose maximum dimension is one. In this case the main tool we use is the vanishing theorem proven in Proposition 6.12.

In Subsection 6.3.2 we discuss the general case. Our strategy is based on techniques developed by Hacon and Witaszek in [HW17] to prove that klt threefold singularities are rational in large characteristic. The main ingredient in the proof of Theorem 6.17 are

Proposition 6.13 and the Kawamata-Viehweg vanishing theorem for log del Pezzo surfaces in large characteristic ([CTW17, Theorem 1.2]).

6.3.1. Descent for pl-contractions with one dimensional fibres

We apply the vanishing theorem of Proposition 6.12 to discuss descent of numerically trivial Cartier divisors in the case of pl-contractions with one-dimensional fibres.

Proposition 6.13. *Let k be a perfect field of characteritic $p > 5$. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt threefold pair over k and let S be a prime divisor contained in $\lfloor \Delta \rfloor$. Let $\pi \colon X \to Y$ be a projective k-morphism between quasi-projective normal varieties such that*

1. *the maximum dimension of the fibres of π is one;*

2. *π is a weak $(K_X + \Delta, S)$-pl-contraction.*

Let L be a Cartier divisor on X such that $L \equiv_\pi 0$. Then $L \sim_\pi 0$ over a neighbourhood of $\pi(S)$.

Proof. Since the question is local over the base, we can assume Y to be affine. By [GNT, Theorem 2.11] S is a normal variety. Moreover by adjunction the pair $(S, \mathrm{Diff}_S(\Delta'))$ is dlt where

$$(K_X + \Delta)|_S = K_S + \mathrm{Diff}_S(\Delta'),$$

where $\Delta' = \Delta - S$. Let us denote by $\pi|_S \colon S \to T$ the restricted morphism. Since $-(K_S + \mathrm{Diff}_S(\Delta'))$ is $\pi|_S$-ample, Proposition 6.5 implies $L|_S \sim_{\pi|_S} 0$.

By Lemma 6.7 for any $y \in T := \pi(S)$ we have $\pi^{-1}(y) \subset S$. Since $L|_S \sim_{\pi|_S} 0$, it is sufficient to prove that the morphism

$$\pi_* \mathcal{O}_X(L) \to (\pi|_S)_* \mathcal{O}_S(L)$$

is surjective to prove that $L \sim_\pi 0$ over a neighbourhood of T. By Proposition 6.12 we have the vanishing $R^1\pi_*\mathcal{O}_X(L - S) = 0$ and thus we conclude. □

6.3.2. Descent for pl-contractions over threefolds and surfaces

We begin by recalling the Kawamata-Viehweg vanishing theorem for log del Pezzo surfaces in large characteristic (see [CTW17, Theorem 1.2]), which plays a key role in the proof of Theorem 6.17.

Theorem 6.14. *There exists a constant $p_0 > 0$ with the following property. Let k be a perfect field of characteristic $p > p_0$. Let (S, Δ) be a dlt surface log pair over k such that $-(K_S + \Delta)$ is ample. Let B be an effective $\mathbb{Q}$-divisor such that (S, B) is dlt and let L be a Weil divisor such that $L - (K_S + B)$ is ample. Then $H^1(S, L) = 0$.*

Proof. Fix p_0 as in [CTW17, Theorem 1.2]. Let us decompose $\Delta = \Delta^{=1} + \Delta^{<1}$. Since S is $\mathbb{Q}$-factorial, we have that for $0 < \varepsilon \ll 1$ the pair $(X, \Delta' := (1-\varepsilon)\Delta^{=1} + \Delta^{<1})$ is klt and $-(K_X + \Delta')$ remains amples. Similarly we can perturb B to B' such that (S, B) is klt and $L - (K_S + B')$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-ample divisor. We thus conclude by Theorem [CTW17, Theorem 1.2]. □

Remark 6.15. We do not know an explicit bound on p_0. However the examples constructed in [CT, Theorem 4.2] and [Ber17, Theorem 1.1] show that $p_0 > 3$.

We recall the restriction short exact sequence constructed in [HW17].

Proposition 6.16. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt threefold log pair defined over k. Let S be an irreducible component of $\lfloor \Delta \rfloor$ and let $\Delta = \Delta' + S$. Let D be a Weil divisor on X. Then for all $n \in \mathbb{Z}_{\geq 0}$ there exists a short exact sequence*

$$0 \to \mathcal{O}_X(-(n+1)S + D) \to \mathcal{O}_X(-nS + D) \to \mathcal{O}_S(G_n) \to 0,$$

where $G_n \sim_{\mathbb{Q}} -nS|_S + D|_S - \Delta_n$ and $\Delta_n \leq \mathrm{Diff}_S(\Delta')$.

Proof. Since X is $\mathbb{Q}$-factorial, the pair (X, S) is plt and we can apply [HW17, Corollary 3.7]. Since $\Delta_n \leq \mathrm{Diff}_S(0) \leq \mathrm{Diff}_S(\Delta')$ we conclude. □

We are now ready to prove the main result of this section.

Theorem 6.17. *There exists an integer $p_0 \geq 5$ such that the following hold. Let k be a perfect field of characteritic $p > p_0$. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt threefold log pair over k, and let S be a prime Weil divisor contained in $\lfloor \Delta \rfloor$. Let $\pi \colon X \to Y$ be a projective contraction between quasi-projective normal varieties such that*

1. *π is a weak $(K_X + \Delta, S)$-pl-contraction;*
2. *$\dim(Y) \geq 2$.*

Let L be a Cartier divisor on X such that $L \equiv_\pi 0$. Then $L \sim_\pi 0$ over a neighbourhood of $\pi(S)$.

Proof. Consider $p_0 > 5$ for which the statement of Theorem 6.14 holds. As in the proof of Proposition 6.13, we write adjunction

$$(K_X + \Delta)|_S = K_S + \mathrm{Diff}_S(\Delta')$$

where $\Delta' = \Delta - S$ and we have $L|_S \sim_{\pi|_S} 0$ by Proposition 6.5. Since by Lemma 6.7 for any $y \in T := \pi(S)$ we have $\pi^{-1}(y) \subset S$, it is sufficient to prove that the morphism

$$\pi_* \mathcal{O}_X(L) \to (\pi|_S)_* \mathcal{O}_S(L)$$

is surjective. To prove the surjectivity, we show the vanishing of $R^1\pi_*\mathcal{O}_X(L-S)$.

Let $n \geq 0$ and let us consider the following exact sequence given by Proposition 6.16:

$$0 \to \mathcal{O}_X(-(n+1)S) \to \mathcal{O}_X(-nS) \to \mathcal{O}_S(G_n) \to 0,$$

where $G_n \sim_{\mathbb{Q}} -nS|_S - \Delta_n$ for some $\Delta_n \leq \mathrm{Diff}_S(\Delta')$. We tensor with L and we consider the following exact sequence of $\mathcal{O}_Y$-modules:

$$\begin{aligned} 0 &\to \pi_*\mathcal{O}_X(L-(n+1)S)) \to \pi_*\mathcal{O}_X(L-nS) \to (\pi|_S)_*\mathcal{O}_S(L+G_n) \to \\ &\to R^1\pi_*\mathcal{O}_X(L-(n+1)S)) \to R^1\pi_*\mathcal{O}_X(L-nS)) \to R^1(\pi|_S)_*\mathcal{O}_S(L+G_n). \end{aligned}$$

Thus in order to prove $R^1\pi_*\mathcal{O}_X(L-(n+1)S)=0$ for all $n \geq 0$ it is sufficient to prove the following two vanishing results in cohomology:

(i) $R^1(\pi|_S)_*\mathcal{O}_S(L+G_n)=0$ for every $n \geq 0$.

(ii) $R^1\pi_*\mathcal{O}_X(L-mS)=0$ for $m \gg 0$ and sufficiently divisible.

To prove (i) let us note that

$$L|_S + G_n \sim_{\mathbb{Q}} (K_S + \mathrm{Diff}_S(\Delta') - \Delta_n) + L|_S - nS|_S \sim_{\mathbb{Q}} (K_S + B) + A,$$

where $(S, B := \mathrm{Diff}_S(\Delta') - \Delta_n)$ is klt and A is $(\pi_S)_*$-ample. If $\dim(T) \geq 1$, we conclude by the relative Kawamata-Viehweg vanishing (see [Tan18a, Proposition 3.2]). If $\dim(T)=0$, we have

$$R^1(\pi|_S)_*\mathcal{O}_S(L+G_n) = H^1(S, \mathcal{O}_S(L+G_n)).$$

Since $(S, \mathrm{Diff}_S(\Delta'))$ is a dlt pair such that $-(K_S + \mathrm{Diff}_S(\Delta'))$ is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, we conclude $H^1(S, \mathcal{O}_S(L+G_n))=0$ by Theorem 6.14.

We now prove (ii). If $-S$ is π-ample over a neighbourhood of T we conclude by the relative Serre vanishing theorem. Thus we can suppose $-S$ is not π-ample over any neighbourhood of T. By Lemma 6.8 we can consider the semi-ample fibration over Y associated to $-S$:

$$\pi \colon X \xrightarrow{g} Z \xrightarrow{h} Y,$$

and let us consider an integer $k > 0$ such that $-kS = g^*H$ for a h-ample Cartier divisor on Z.

Since by Lemma 6.9 the fibres of g are at most one dimensional, we can apply Proposition 6.12 to deduce that $R^ig_*\mathcal{O}_X(L-mS)=0$ for $i>0$ and $m>0$. By Proposition 6.13 we have $L=g^*M$ for some Cartier divisor M which is h-trivial. By the Grothendieck spectral sequence, we thus deduce $R^1\pi_*\mathcal{O}_X(L-mkS) = R^1h_*\mathcal{O}_Z(M+mH)$. Since H is g-ample, we apply relative Serre vanishing to $R^1h_*\mathcal{O}_Z(M+mH)$ to conclude that for all m sufficiently large we have $R^1\pi_*\mathcal{O}_X(L-mkS)=0$. □

6.4. The base point free theorem in large characteristic

The aim of this section is to prove Theorem 6.23. For this we discuss first descent of numerically trivial Cartier divisors in the birational case (see Subsection 6.4.1) and in the case of conic bundles (see Subsection 6.4.2). In these cases, the main techniques used are the MMP for threefolds and Proposition 6.10 to reduce to the case of pl-contractions proven in Theorem 6.17.

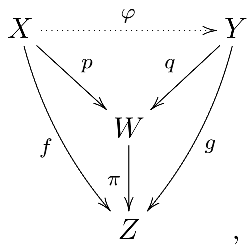
In subsection 6.4.3, we combine these results together with results on del Pezzo fibrations from [BT19] to prove Theorem 6.23.

6.4.1. Birational case

In this subsection, we prove descent of numerically Cartier divisors under $(K_X + \Delta)$-negative birational contraction of threefolds in large characteristic.

We need the following easy lemma on birational maps which are isomorphisms in codimension one.

Lemma 6.18. *Let k be a field. Let $f\colon X \to Z$ be a proper contraction of normal varieties over k. Let us consider the following commutative diagram*



,

where $p\colon X \to W$ and $q\colon Y \to W$ are small proper birational contractions between normal varieties. Let L be a Cartier divisor on X and suppose that there exists a Cartier divisor M on W such that $L \sim p^*M$. Then φ_*L is a Cartier divisor. Moreover, the following are equivalent:

(i) there exists a Cartier divisor H on Z such that $L \sim f^*H$,

(ii) there exists a Cartier divisor H on Z such that $M \sim \pi^*H$,

(iii) there exists a Cartier divisor H on Z such that $\varphi_*L \sim g^*H$.

Proof. Since φ_*L and q^*M are both Weil divisors on a normal variety and they coincide outside a codimension two subset, we conclude that $\varphi_*L \sim q^*M$. In particular, φ_*L is Cartier.

We now prove that (i) is equivalent to (ii). Obviously, (ii) implies (i). If (i) holds, then $L \sim f^*H \sim p^*\pi^*H \sim p^*M$ implies $M \sim \pi^*H$. We can repeat the same proof using the equality $\varphi_*L \sim q^*M$ to conclude (ii) is equivalent to (iii). □

Theorem 6.19. *There exists a constant $p_0 \geq 5$ with the following property. Let k be a perfect field of characteristic $p > p_0$. Let $\pi\colon X \to Z$ be a projective birational morphism between quasi-projective normal varieties. Suppose that there exists an effective $\mathbb{Q}$-divisor Δ such that*

1. *(X, Δ) is a klt threefold log pair,*
2. *$-(K_X + \Delta)$ is π-big and π-nef.*

Let L be a Cartier divisor such that $L \equiv_\pi 0$. Then $L \sim_\pi 0$.

Proof. Fix $p_0 > 5$ for which Theorem 6.17 holds. Since the question is local over the base, we can assume Z to be affine and we fix $z \in Z$ a closed point. By Proposition 6.10, there exists a birational morphism $g\colon Y \to Z$ such that there exists an effective $\mathbb{Q}$-divisor Δ_Y such that

(i) (Y, Δ_Y) is a $\mathbb{Q}$-factorial plt pair,

(ii) $S := (g^{-1}(z))_{\mathrm{red}}$ is an irreducible component of $\lfloor \Delta_Y \rfloor$ and g is a weak $(K_X + \Delta, S)$-pl-contraction.

Let us consider the following diagram

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & Y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\pi} & Z, \end{array}$$

where φ and ψ are log resolution. Denote by $f := \pi \circ \psi$.

In order to prove the theorem, it is sufficient to prove $\psi^* L \sim_f 0$. We first prove that $\psi^* L$ descends to a Cartier divisor M on Y. To accomplish this, we apply Theorem 6.17 inductively.

Claim 6.20. *There exists a Cartier divisor M on Y such that $\psi^* L \sim \varphi^* M$.*

Proof. Since Y is $\mathbb{Q}$-factorial variety with klt singularities, we have

$$K_W + \sum_{i \in I} E_i \sim_{\mathbb{Q}} \varphi^* K_Y + \sum_{i \in I} (1 + a_i) E_i,$$

where $\bigcup_{i \in I} E_i = \mathrm{Supp}(\mathrm{Ex}(\varphi))$ and $1 + a_i > 0$ for all $i \in I$. In particular,

$$K_W + \sum_{i \in I} E_i \sim_{\mathbb{Q}, Y} \sum_{i \in I} (1 + a_i) E_i.$$

By [HNT19, Theorem 1.1] we can run a $(K_W + \sum_i E_i)$-MMP over Y

$$h\colon W \dashrightarrow W_2 \dashrightarrow \cdots \dashrightarrow W_n =: T,$$

which terminates with a morphism $p\colon T \to Y$ such that $\sum_i (1+a_i)h_*E_i$ is p-nef. By negativity lemma, this implies that all the divisors E_i are contracted by h. Thus $p\colon T \to Y$ is a small morphism. Since Y is $\mathbb{Q}$-factorial, we conclude p is an isomorphism. Since we run a $(\sum_i (1+a_i)E_i)$-MMP, each step is a pl-divisorial contraction or a pl-flip. Thus we can apply Theorem 6.17 and Lemma 6.18 inductively to conclude that there exists a Cartier divisor M on Y such that $\psi^*L \sim \varphi^*M$. □

We can now apply Proposition 6.17 once more to shows that there exists a Cartier divisor N on Z such that $M \sim g^*N$, thus concluding the proof. □

As a corollary, we obtain a descent result for numerically trivial Cartier divisors on threefolds admitting a birational morphism over a klt pair.

Corollary 6.21. *There exists a constant $p_0 \geq 5$ with the following property. Let k be a perfect field of characteristic $p > p_0$. Let $\pi\colon X \to Z$ be a projective birational morphism between quasi-projective normal varieties. Suppose that there exists a $\mathbb{Q}$-divisor $B \geq 0$ such that (Z, B) is a klt threefold. Let L be a Cartier divisor on X such that $L \equiv_\pi 0$. Then $L \sim_\pi 0$.*

Proof. Without any loss of generality, we can assume that $\pi\colon X \to Z$ is a log resolution for the pair (Z, B). Thus we can write

$$K_X + \pi_*^{-1}B + \sum_{i \in I} E_i = \pi^*(K_Z + B) + \sum_{i \in I}(1+a_i)E_i.$$

Consider $0 < \varepsilon \ll 1$ such that $1 + a_i - \varepsilon > 0$ for any $i \in I$. By [GNT, Theorem 2.13] we run a $(K_X + \pi_*^{-1}B + \sum_{i \in I}(1-\varepsilon)E_i)$-MMP over Z:

$$f\colon X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n =: Y,$$

which ends with a relative minimal model $g\colon Y \to Z$. By applying Lemma 6.18 and Theorem 6.19 at each step of the MMP inductively, it is sufficient to prove that any Cartier divisor N on Y which is numerically g-trivial then it is g-trivial. By negativity lemma g is a small birational morphism and thus we have $(Y, g_*^{-1}B)$ is klt and $(K_Y + g_*^{-1}B)$ is g-trivial. In particular, $-(K_Y + g_*^{-1}B)$ is g-big and g-nef and thus we can apply Theorem 6.19 once more to conclude. □

6.4.2. Conic bundles

In this section we prove descent of numerically trivial Cartier divisors under $(K_X + \Delta)$-negative contraction of relative dimension one (also known as conic bundles).

Theorem 6.22. *There exists a constant $p_0 \geq 5$ with the following property. Let k be a perfect field of characteristic $p > p_0$. Let $\pi\colon X \to Z$ be a projective contraction between quasi-projective normal varieties. Suppose that there exists an effective $\mathbb{Q}$-divisor Δ such that*

1. *(X, Δ) is a klt threefold log pair,*

2. *$-(K_X + \Delta)$ is π-big and π-nef,*

3. *$\dim(Z) = 2$.*

Let L be a Cartier divisor on X such that $L \equiv_\pi 0$. Then $L \sim_\pi 0$.

Proof. By Proposition 6.5, there exists open subset $U \subset Z$ such that $L|_{\pi^{-1}(U)} \sim_\pi 0$ and $Z \setminus U$ has codimension two. Let z be a closed point in $Z \setminus U$. By Proposition 6.10, there exists a birational morphism $g\colon Y \to Z$ and an effective $\mathbb{Q}$-divisor Δ_Y on Y such that

(i) (Y, Δ_Y) is a $\mathbb{Q}$-factorial plt pair (in particular, Y is klt),

(ii) $S := (g^{-1}(z))_{\text{red}}$ is an irreducible component of $\lfloor \Delta_Y \rfloor$ and g is a weak $(K_X + \Delta, S)$-pl-contraction.

Let us consider the following diagram

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & Y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\pi} & Z, \end{array}$$

where φ and ψ are log resolution. Since L is π-trivial outside z, to prove the statement it is sufficient to prove $\varphi^* L$ is $(\pi \circ \psi)$-trivial over a neighbourhood of z. By Corollary 6.21 there exists a Cartier divisor M on Y such that $\psi^* L = \varphi^* M$. It is thus sufficient to prove that the Cartier divisor M on Y is g-trivial over a neighbourhood of z. This is a consequence of Theorem 6.17. $\square$

6.4.3. General case

We prove now the main theorem of this chapter. To deal with the remaining case of $(K_X + \Delta)$-negative contractions of relative dimension two, we combine Theorem 6.19 and Theorem 6.22 with a result on relatively numerically trivial Cartier divisors on del Pezzo fibrations (see [BT19, Theorem 1.1]), on the image of surfaces of del Pezzo type over imperfect fields of characteristic at least seven (see [BT19, Corollary 5.8]) and the MMP for $\mathbb{Q}$-factorial surfaces (see [Tan18a]).

Theorem 6.23. *There exists a constant $p_0 \geq 5$ with the following property. Let k be a perfect field of characteristic $p > p_0$. Let (X, Δ) be a klt threefold log pair and let $\pi\colon X \to Z$ be a projective contraction between quasi-projective normal varieties. Suppose that*

1. *$-(K_X + \Delta)$ is π-big and π-nef,*

2. *$\dim(Z) \geq 1$.*

Let L be a Cartier divisor such that $L \equiv_\pi 0$. Then $L \sim_\pi 0$.

Proof. We can assume k is algebraically closed by a base change to an algebraic closure. By taking a $\mathbb{Q}$-factoralization ([Bir16, Theorem 1.6]) we can further assume X is $\mathbb{Q}$-factorial. If $\dim(Z) \geq 2$, we conclude by Theorem 6.19 and Theorem 6.22.

If $\dim(Z) = 1$, by [GNT, Theorem 2.12] we can run a $(K_X + \Delta)$-MMP over Z:

$$X_0 := X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n =: Y,$$

which terminates with a Mori fibre space $g\colon Y \to T$ over Z and let us denote by $f\colon Y \to Z$ the natural morphism. Let us note that since $X_{k(Z)}$ is a surface of del Pezzo type over $k(Z)$, we deduce that also the generic fibre $Y_{k(Z)}$ is a surface of del Pezzo type over $k(Z)$ by [BT19, Lemma 2.9].

By Corollary 6.21 it is now sufficient to prove that given M a Cartier divisor on Y such that $M \equiv_f 0$, then $M \sim_f 0$. We subdivide the proof according to the dimension of T. If $\dim(T) = 1$, then $T = Z$ and thus we conclude by [BT19, Theorem 1.1].

If $\dim(T) = 2$, we have a factorisation $Y \xrightarrow{g} T \xrightarrow{h} Z$ and by Theorem 6.22 there exists a Cartier divisor N on T such that $N \equiv_h 0$. Note that T is a $\mathbb{Q}$-factorial surface by [HNT19, Theorem 5.4]. Since $Y_{k(Z)}$ is a surface of del Pezzo type, we deduce that the generic fibre $T_{k(Z)}$ is a Fano curve over $k(Z)$ by [BT19, Corollary 5.8]. In particular K_T is not pseudoeffective over Z. Since T is $\mathbb{Q}$-factorial, we can run a K_T-MMP over Z by [Tan18a, Theorem 1.1]:

$$\psi\colon T_0 := T \to T_1 \to T_2 \to \cdots \to T_n =: V$$

which terminates with a Mori fibre space $p\colon V \to Z$. By Proposition 6.5 we show that there exists a Cartier divisor D on V such that $\psi^* D = N$. Again by Proposition 6.5 there exists a Cartier divisor E on Z such that $D = p^* E$, thus concluding that $M = f^* E$. □

We now show our improvement of the base point free theorem in large characteristic.

Theorem 6.24. *There exists a constant $p_0 \geq 5$ such that the following holds. Let k be a perfect field of characteristic $p > p_0$. Let (X, Δ) be a quasi-projective klt threefold log pair over k and let $\pi\colon X \to Z$ be a projective k-morphism of quasi-projective normal varieties. Let L be a π-nef Cartier divisor on X such that*

1. $\dim(Z) \geq 1$ *or* $\dim(Z) = 0$ *and* $\nu(L) \geq 1$*;*
2. $nL - (K_X + \Delta)$ *is a π-big and π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for some $n > 0$.*

Then there exists $m_0 > 0$ such that mL is π-free for all $m \geq m_0$.

Proof. By the relative base point free theorem for threefolds ([GNT, Theorem 2.9]), L is π-semi-ample. Let us denote by $\pi\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ be the semi-ample fibration over Z given by L. Since $L \equiv_f 0$, we deduce by Theorem 6.19 that there exists a g-ample Cartier divisor H on Y such that $L \sim f^* H$. Thus we conclude. □

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