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Abstract

A complete c-colouring of a graph is a proper colouring in which every pair of distinct colours from $[c] = \{1, 2, \ldots, c\}$ appears as the colours of endvertices of some edge. We consider the following generalisation of this concept to uniform hypergraphs. A complete c-colouring for a k-uniform hypergraph H is a mapping from the vertex set of H to $[c]$, such that (i) the colour set used on each edge has exactly k elements, and (ii) every k-element subset of $[c]$ appears as the colour set of some edge. In this paper we exhibit some differences between complete colourings of graphs and hypergraphs. First, it is known that every graph has a complete c-colouring for some c.

In contrast, we show an infinite family of hypergraphs H that do not admit a complete c-colouring for any c. We also extend this construction to λ-complete colourings (for $0 < \lambda \leq 1$), where condition (ii) is substituted with: at least $\lambda\binom{c}{k}$ different colour sets appear on edges.

We establish upper and lower bounds on a maximum degree, which guarantees the existence of a complete colouring of any hypergraph. Moreover, we prove that it is NP-complete to determine if a given hypergraph has a λ-complete colouring. Next, we show that, unlike graphs, hypergraphs do not have the so-called interpolation property, i.e., we construct hypergraphs that have a complete r-colouring and a complete s-colouring, but no complete t-colouring for some t such that $r < t < s$.

Finally, we investigate the notion of λ-complete colourings of graphs (i.e., 2-uniform hypergraphs). We show that λ-complete colourings have the same interpolation property as complete colourings. Moreover, we prove that it is NP-complete to decide whether a tree with $\binom{c}{2}$ edges has a λ-complete c-colouring, which strengthens the result by Cairnie and Edwards [JGT, 1997].

Keywords: complete colouring, achromatic number, hypergraph, strong colouring
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1. Introduction

For an integer c, a c-colouring of a graph G is *complete* if it is proper, i.e., no two adjacent vertices have the same colour, and for every pair of colours in $[c] := \{1, 2, \ldots, c\}$ there exists an edge of G, whose endvertices receive exactly these colours. The concept of complete colourings was introduced in 1967 by Harary, Hedetniemi, and Prins [17] in the language of (complete) homomorphisms to a complete graph. It is well known that every

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graph has a complete colouring (i.e., a complete c-colouring for some c). Indeed, every colouring of G using the minimum possible number of colours (i.e., the chromatic number $\chi(G)$ of G) is necessarily complete. Since any complete colouring is proper, clearly we cannot obtain any complete colouring with less than $\chi(G)$ colours. Thus it makes sense to ask what is the *maximum* possible number of colours in a complete colouring. This number is called the *achromatic* number of G, and denoted by $\psi(G)$.

Complete colourings and achromatic number have been extensively studied by many authors, both from a purely combinatorial, and from a computational perspective. For example Cairnie and Edwards [4] investigated the achromatic number of trees T with bounded degrees. They provided some tight bounds for $\psi(T)$ and a polynomial algorithm computing it. These results were later extended by Edwards [8] to the *fragmentable* classes of graphs with bounded vertex degrees. Intuitively, a class of graphs is fragmentable if every member of it can be split into disjoint components of constant size by removal of a small fraction of vertices. All graph classes admitting a separator theorem (e.g. planar graphs) are fragmentable.

On the other hand, determining the achromatic number is known to be NP-complete for cographs and interval graphs (see Bodlaender [2]), and even for trees (see Cairnie and Edwards [3]). We refer the reader to the survey by Edwards [9] for more information about the topic. We also point out the regularly updated bibliography maintained by Edwards [7].

A natural step is to generalise the concept of complete colourings to wider families of combinatorial objects. For example, Edwards [10] considered complete colourings of directed graphs, and investigated similarities and differences between the cases of graphs and digraphs.

Another natural way is to generalise the concept to hypergraphs. There are two natural ways of doing this, following from the fact that there are two common ways of defining proper colourings of hypergraphs. The first (and more common) variant was introduced by Erdős and Hajnal [14] and requires that no edge is monochromatic (see also Toft [19]). Such a colouring is sometimes called *weak colouring*. In the other variant, called *strong colouring* or *rainbow colouring*, we require that no two vertices belonging to a single edge share the same colour (see e.g. Agnarsson, M. Halldórsson [1]).

Nešetřil, Phelps, and Rödl [18] considered complete colourings of hypergraphs, which are proper in the weak sense, and for every pair of colours a, b there exists an edge, whose colour set is exactly $\{a, b\}$. A different concept was recently introduced by Dębski, Lonc, and Rzążewski [6]. Here a *complete colouring* of a k-uniform hypergraph H (alternatively called its *achromatic colouring*) is a proper strong colouring of H, with the additional property that every k-element set of colours appears on some edge of H. The authors of [6] showed that if the input hypergraph belongs to a fragmentable class and has bounded degree, then it has a complete colouring with many colours (in fact, asymptotically matching the trivial upper bound), similarly to the case of graphs (see Edwards [11], Cairnie and Edwards [3]).

It is perhaps interesting to mention so-called *mixed hypergraphs*, which are triples (V, E, F), where V is the set of vertices, and E and F are families of subsets of V (called *edges* and *co-edges*, respectively). Voloshin [21, 22], and Tuza and Voloshin [20] investigated colourings of mixed hypergraphs, such that no edge is monochromatic and every co-edge contains two vertices with the same colour. This problem has a similar flavour to complete colourings, forbidding the number of colours from growing too much (in particular excluding colouring each vertex with a unique colour).

In this paper we investigate the differences between complete colourings (in the strong sense) of graphs and hypergraphs. We show that some basic properties of complete colourings of graphs do not carry over to the case of hypergraphs.

First, as mentioned before, every graph has a complete colouring. This is, however, not the case with complete colourings of uniform hypergraphs. As already noted by Dębski *et al.* [6], there are uniform hypergraphs which have no complete colouring. In Section 3 we show that for any $k \geq 3$ and arbitrarily large n there exists a connected k-uniform hypergraph with n vertices and maximum degree $\Theta(n^{1/k})$, which does not have any complete colouring. Moreover, for any $0 < \lambda \leq 1$, they do not even have any strong c-colouring, where at least $\lambda\binom{c}{k}$ k-subsets of colours appear on edges (we call such colourings λ-complete). On the other hand, we show that every k-uniform hypergraph with the maximum degree $o(n^{1/(k+2)})$ has a complete colouring.

With this knowledge, it makes sense to ask how hard it is to decide if a given k-uniform hypergraph admits a complete colouring or a λ-complete colouring. In Section 4 we show that this problem is NP-complete for all $k \geq 3$ and all $0 < \lambda \leq 1$.

Next, we turn our attention to the so-called interpolation. Harary, Hedetniemi, and Prins [17] have shown that if a graph G has a complete colouring with r colours, and another with s colours for some $s > r$, then it has a complete colouring with t colours for every $r \leq t \leq s$. Thus since $\chi(G)$, $\psi(G)$ respectively are the least and greatest number of colours for which a complete colouring of G exists, G has a complete colouring with t colours if and only if $\chi(G) \leq t \leq \psi(G)$. In Section 5 we show that such an interpolation property does not hold in the case of uniform hypergraphs.

Finally, we consider the notion of λ-complete c-colourings of graph (i.e., 2-uniform hypergraphs). For $0 < \lambda \leq 1$ and a graph G, let $\psi_\lambda(G)$ denote the maximum number of colours, for which G has a λ-complete colouring. Clearly $\psi_1(G) = \psi(G)$ and $\psi_\lambda(G) \geq \psi_{\lambda'}(G)$ for all $\lambda' > \lambda$. We prove that the λ-complete colourings have the interpolation property: for every t, such that $\chi(G) \leq t \leq \psi_\lambda(G)$, the graph G has a λ-complete colouring with t colours. Moreover, we show that the problem of determining whether a tree with $\binom{c}{2}$ edges has a λ-complete c-colouring (i.e., whether $c \leq \psi_\lambda(G)$) is NP-complete for any $0 < \lambda \leq 1$, even if the radius of the input tree is 3.

The paper concludes with several open questions and possible directions for further research.

2. Preliminaries

A hypergraph H is a pair (V, E), where V denotes its *vertex set*, and elements of E, called *edges*, are non-empty subsets of V. A hypergraph H is *k-uniform* if every edge has exactly k-elements. Clearly graphs are exactly 2-uniform hypergraphs.

By the *degree* of a vertex v, denoted by $\deg v$, we mean the number of edges containing v. By $\Delta(H)$ we denote the maximum vertex degree in the hypergraph H. A hypergraph $H = (V, E)$ is *connected* if the graph $G(H) = (V, E')$, where $E' = \{uv \mid \{u, v\} \subseteq e \text{ for some } e \in E\}$, is connected.

A *strong colouring* of a k-uniform hypergraph $H = (V, E)$ is an assignment f of integers (called *colours*) to the vertices of H such that for every $e \in E$ and every distinct $v, w \in e$, $f(v) \neq f(w)$. By $\chi(H)$ we denote the minimum possible number of colours needed to strongly colour the hypergraph H. Observe that if H has at least one edge, then clearly $\chi(H) \geq k$.

A *complete colouring* of a k-uniform hypergraph H is a strong colouring of H in which every k-set of colours appears on some edge. If H has a complete colouring, then we denote by $\psi(H)$ its *achromatic number*, which is the maximum number of colours in a complete colouring of H. To make this parameter well-defined for all H, we set $\psi(H) := 0$ whenever H has no complete colouring.

We also consider the following relaxation of the notion of complete colouring. A c-colouring of a k-uniform hypergraph H is called λ-*complete* for $0 < \lambda \leq 1$, if it is a strong colouring and at least $\lambda\binom{c}{k}$ possible k-subsets of colours appear on edges of H. Clearly a 1-complete colouring is exactly a complete colouring.

3. Hypergraphs with and without a complete colouring

A *Steiner triple system* is a 3-uniform hypergraph, in which every pair of vertices appears in exactly one edge. Observe that if H is a Steiner triple system, then in any complete colouring of H, every vertex must receive a different colour (to make the colouring strong). However, since H is not a complete hypergraph, there are k-sets of colours which do not appear on any edge of H. Observe that $\deg v = \frac{n-1}{2}$ for any vertex v. Since a Steiner triple system with n vertices exists if and only if $n \equiv 1, 3 \pmod 6$, this gives us an infinite family of 3-uniform hypergraphs with maximum degree $\Theta(n)$, which do not have any complete colouring.

Below we extend this idea to construct an infinite family of k-uniform hypergraphs with significantly smaller maximum degrees, which do not have any λ-complete colouring (for any λ and any number of colours).

Theorem 1. *For any fixed $k \geq 3$ and any $\lambda \in (0, 1]$, there exist arbitrarily large connected k-uniform hypergraphs H with n vertices and $\Delta(H) = \Theta(n^{1/k})$ such that H does not have any λ-complete colouring.*

Proof. Let $k \geq 3$ and let c_0 be the minimum integer, such that $\lambda\binom{c_0}{k} \geq \binom{c_0}{2}/\binom{k}{2} + 1$. A celebrated theorem by Wilson [23] asserts that there are arbitrarily large integers $c \geq c_0$ such that the complete graph K_c can be decomposed into edge-disjoint copies of K_k. Replacing each copy of K_k by a k-edge, we obtain a k-uniform hypergraph H_0 with c vertices, and H_0 is regular of degree $\frac{c-1}{k-1}$. Also any pair of distinct vertices occur together in an edge, hence in any valid strong colouring, each vertex of H_0 receives a distinct colour. Note that $|E(H_0)| = \binom{c}{2}/\binom{k}{2}$. Now form a connected hypergraph H by adding a chain of $p = \lambda\binom{c}{k} - |E(H_0)| - 1$ extra edges to H_0. One can readily verify that $p \geq 0$ by the choice of c. Each new edge contains one vertex of the previous edge and $k-1$ extra vertices (the first new edge contains one vertex of H_0). Note that H has $|E(H_0)| + p = \lambda\binom{c}{k} - 1$ edges, and $n = c + (k-1)p = c + (k-1)(\lambda\binom{c}{k} - |E(H_0)| - 1)$ vertices.

Now H cannot have a λ-complete colouring, as such a colouring would have to have at least c colours, but H has fewer than $\lambda\binom{c}{k}$ edges, so any colouring cannot be λ-complete. However $n = \Theta(c^k)$ while $\Delta(H) = \frac{c-1}{k-1} + 1$, so we have $\Delta(H) = \Theta(n^{1/k})$, as required. □

On the other hand, we can show that if the maximum degree of the hypergraph is small enough, a complete colouring always exists.

Theorem 2. *Let Γ be a class of k-uniform hypergraphs with no isolated vertices and such that for $H \in \Gamma$, $\Delta(H) = o(n^{1/(k+2)})$, where $n = |V(H)|$. Then there exists n_0 such that every $H \in \Gamma$ with $n \geq n_0$ vertices has a complete colouring.*

Proof. Let ϵ be a positive number, whose value will be specified later and let n_0 be such that $\Delta(H) < \epsilon \cdot n^{1/(k+2)}$ for every $H \in \Gamma$ with $n \geq n_0$ vertices. Let H be such a hypergraph and let m be the number of its edges.

We shall use the following greedy procedure to choose a maximal set M of edges of H, such that for any edges $e_1, e_2 \in M$ there is no edge e of H, which intersects both e_1 and e_2 (note that in particular this implies that e_1, e_2 are disjoint). The procedure is iterative. Set $H_0 := H$ and $M_0 := \emptyset$. At every step $i \geq 1$, we select an arbitrary edge e of H_{i-1} and set $M_i := M_{i-1} \cup \{e\}$. Then we obtain H_i from H_{i-1} by removing the edge e, all edges intersecting e, and all edges intersected by an edge intersecting e. The procedure is terminated after step ℓ if H_ℓ has no edges. Set $M := M_\ell$.

At every step we add exactly one edge to M and discard at most $1+k\cdot(\Delta(H)-1)(1+k\cdot(\Delta(H)-1)) \leq k^2 \cdot \epsilon^2 \cdot n^{2/(k+2)}$ edges. Thus $|M| \cdot (k^2 \cdot \epsilon^2 \cdot n^{2/(k+2)}) \geq m$ and therefore $|M| \geq \frac{m}{k^2 \cdot \epsilon^2 \cdot n^{2/(k+2)}}$. Since H does not contain isolated vertices, we have $m \geq n/k$ and thus $|M| \geq \frac{n^{k/(k+2)}}{k^3 \cdot \epsilon^2}$.

Let c be equal to $k \cdot \lfloor \epsilon \cdot n^{1/(k+2)} \rfloor$. We shall show that H has a complete colouring with c colours. The number of k-elements subsets of colours is $\binom{c}{k} = \binom{k\cdot\lfloor \epsilon \cdot n^{1/(k+2)}\rfloor}{k} \leq \frac{k^k}{k!} \cdot \epsilon^k \cdot n^{k/(k+2)}$.

Choosing $\epsilon < \sqrt[k+2]{\frac{k!}{k^{k+3}}}$ we obtain that $\binom{c}{k} \leq |M|$. Thus for every k-set of colours C we can choose a distinct edge $e \in M$ and colour the vertices in e so that each of them receives a different colour from C.

The only thing left is to colour the remaining vertices, keeping in mind that no two vertices from a single edge should receive the same colour. We will do this in a greedy way. Note that no edge containing an uncoloured vertex intersects two edges of M, so such an edge cannot already have received the same colour on two distinct vertices. The number of forbidden colours for a vertex v is at most $\deg v \cdot (k-1) \leq (k-1) \cdot \lfloor \epsilon \cdot n^{1/(k+2)} \rfloor < c$. Thus there is always at least one colour, which can be used for v. □

4. Computational hardness

In this section we investigate the complexity of deciding if a given hypergraph has a λ-complete colouring. Let us start with the following auxiliary lemma.

Lemma 3 (Folklore)**.** *The problem of deciding if a given k-uniform hypergraph H has a strong colouring with at most $k+1$ colours is NP-complete for any fixed $k \geq 2$.*

Proof. It is clear that the problem is in NP. For $k = 2$ we obtain the 3-Colouring problem of graphs, which is NP-hard.

To show NP-hardness for $k \geq 3$, we shall use a reduction from 3-Colouring. For a graph $G = (V, E)$ with n' vertices, we define a k-uniform hypergraph H_G with the vertex set $V_H = V \cup \{u_1, u_2, \ldots, u_{k-2}\}$. The edge set of H_G is $E_H = \{e \cup \{u_1, u_2, \ldots, u_{k-2}\} | e \in E\}$. Clearly H_G is k-uniform. We claim that H_G has a strong colouring with $k+1$ colours if and only if G is 3-colourable.

Suppose that H_G has a strong colouring φ with $k+1$ colours. Without loss of generality assume that $\varphi(u_i) = i$ for all $i = 1, 2, \ldots, k-2$ (each of these vertices has to receive a different colour). We claim that $\varphi|_V$ is a proper colouring of G with colours $k-1, k, k+1$. Clearly there is no vertex $v \in V$ with $\varphi(v) \leq k-2$. Moreover, there is no edge $vw \in E$ such that $\varphi(v) = \varphi(w)$, since vw is contained in an edge of H_G.

Now suppose that G has a proper colouring φ with colours $1, 2, 3$. It is easy to see that we can obtain a proper strong $k+1$-colouring of H_G by setting $\varphi(u_i) = i+3$ for all $i = 1, 2, \ldots, k-2$. □

Now we are ready to show the main result of this section.

Theorem 4. *For every fixed $k \geq 3$ and every constant $0 < \lambda \leq 1$ it is NP-complete to decide whether the input k-uniform hypergraph H has a λ-complete colouring (for any number of colours).*

Proof. The problem is clearly in NP. We reduce from strong k-colouring $(k-1)$-uniform hypergraph, which is NP-hard (see Lemma 3). Let $A = (V, E)$ be a $(k-1)$-uniform hypergraph. Notice that we may assume that A contains a $(k-1)$-uniform k-clique, i.e., a subset S of k vertices, such that all $(k-1)$-element subsets of S belong to E (otherwise we can consider an equivalent problem of strong k-colouring of the hypergraph A' consisting of A and a $(k-1)$-uniform k-clique, disjoint with A).

For the construction H we will need to have a set of $p \geq k$ new vertices $c_1, c_2, \dots, c_p$, and a set of edges on $c_1, c_2, \dots, c_p$ so that any two distinct vertices lie in some edge; let the minimum number of edges necessary to do this be $f(p)$. Note that by the well-known Theorem of Wilson [23], we have $f(p) = O(p^2)$.

We pick integers p and s (depending on p) such that

$$p \geq k$$

and

$$pk + f(p) + s \geq \lambda \binom{p+k}{k}$$

and

$$p|E| + f(p) + s < \lambda \binom{p+k+1}{k}.$$

First, since $f(p) = O(p^2)$ and $k \geq 3$, it is clear that for any sufficiently large p, we have $\lambda\binom{p+k}{k} > pk + f(p)$, so set $s := \lceil \lambda\binom{p+k}{k} \rceil - (pk + f(p))$.

Then we need to choose p such that also

$$p|E| + f(p) + s < \lambda \binom{p+k+1}{k},$$

that is, such that

$$p(|E| - k) + \left\lceil \lambda \binom{p+k}{k} \right\rceil < \lambda \binom{p+k+1}{k}.$$

Thus we require that

$$p(|E| - k) < \lambda \binom{p+k+1}{k} - \left\lceil \lambda \binom{p+k}{k} \right\rceil = \lambda \binom{p+k}{k-1} - O(1).$$

Since $k - 1 \geq 2$ we can clearly choose p as required, and p (and s) will be polynomial in $|E|$.

Now to construct H we start with V (the vertices of A) and add the p new vertices $c_1, c_2, \dots, c_p$. Next, we add edges of the following form: $E_A = \{e \cup \{c_i\} \colon e \in E, i \in [p]\}$. Then we add $f(p)$ edges on the vertices $c_1, c_2, \dots, c_p$ so that each pair of these vertices occurs together in an edge. Let E_c be the set of these edges. Finally, we add s disjoint edges on new vertices, call them E_{free}. Note that $|E(H)| = p|E| + f(p) + s < \lambda\binom{k+p+1}{k}$.

We claim that A has a strong k-colouring if and only if H has a λ-complete colouring.

First suppose A is strongly k-colourable and fix such a colouring on vertices of V, call these colours $1, 2, 3, \dots, k$ (note that since A contains a k-clique, we use exactly k colours on A). For each $i \in [p]$, we colour the vertex c_i with the colour $k + i$. Thus we are using $c := k + p$ colours. Every k-subset of colours containing one colour from $[c] \setminus [k]$ and $k-1$ colours from $[k]$ appears on some edge from E_A. Also every edge in E_c has a distinct set of colours. Thus, in total, $pk + f(p)$ k-subsets of colours appear on edges of H. Finally, we colour the edges in E_{free} using s further k-tuples of colours not used previously. Summing up, the number of k-tuples of colours that appear on edges of H is $pk + f(p) + s = \lceil \lambda \binom{c}{k} \rceil \geq \lambda \binom{c}{k}$.

Now suppose there is a λ-complete c-colouring of H, for some integer c. Observe that $c \geq k + p$. Indeed, each of vertices $c_1, c_2, \ldots, c_p$ must receive a distinct colour. Moreover, since A contains a k-clique, we must use at least k colours on vertices of A. Finally, because of edges in E_A, we cannot assign the same colour to some c_i and a vertex in V.

However we cannot have $c > k + p$, because then $\lambda \binom{c}{k} \geq \lambda \binom{k+p+1}{k} > |E(H)|$. Thus $c = k + p$, and so exactly k colours appear on the vertices of V, which implies that A has a strong k-colouring. □

5. Interpolation

For an undirected graph G, it was shown in [17] that if G has a complete colouring with r colours, and another with s colours, $s > r$, then it has a complete colouring with t colours whenever $r \leq t \leq s$. Thus since $\chi(G)$, $\psi(G)$ respectively are the least and greatest number of colours for which a complete colouring of G exists, G has a complete colouring with t colours if and only if $\chi(G) \leq t \leq \psi(G)$.

We show below that this interpolation result fails in the case of uniform hypergraphs.

Theorem 5. *Let $k \geq 9$ be a positive integer. There exists a k-uniform hypergraph H which has a complete k-colouring and a complete r-colouring for some $r > k$, but no complete t-colouring for some t with $k < t < r$.*

Proof. We will define a k-uniform k-partite hypergraph with vertex sets $V_1, \ldots, V_k$, where $V_i = \{v_{i1}, \ldots, v_{ir}\}$ for each $i = 1, \ldots, k$. For a vertex v_{ij}, we will refer to i as the part of the vertex and to j as its position (see Figure 1). Each edge of the hypergraph will

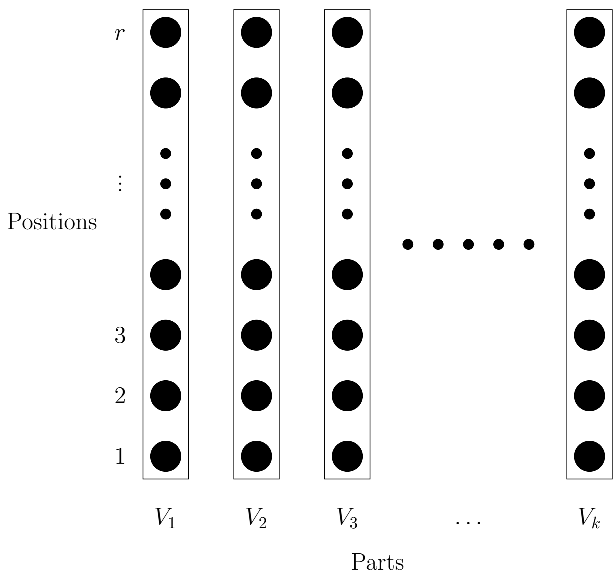

Figure 1: Parts and positions of vertices in the hypergraph H.

contain one vertex from each set V_i, thus each edge is of the form $\{v_{ip_i} \mid 1 \le i \le k\}$; we will denote this edge by the vector $(p_1, \dots, p_k)$.

The hypergraph H contains all edges $(p_1, \dots, p_k)$ such that the p_i are all distinct, and the sequence $p_1, \dots, p_k$ contains at most four monotonic sections, i.e., it changes direction from increasing to decreasing or vice versa at most three times. More formally, $(p_1, \dots, p_k)$ is an edge of H if and only if

$$|\{i \mid 2 \le i \le k-1 \text{ and } (p_{i+1} - p_i)(p_i - p_{i-1}) < 0\}| \le 3.$$

We claim first that, provided r is large enough compared to k, any two vertices in different parts and at different positions are contained in some edge, i.e., if v_{ab} and v_{cd} are vertices with $a \ne c$ and $b \ne d$, then there is an edge containing both v_{ab} and v_{cd}.

To see this, we can assume without loss of generality that $a < c$.

Case 1: The parts a and c are not consecutive, so $c > a+1$. Pick z such that $a < z < c$.

Case 1(a): $|b - d| \le r/3$. Then clearly either $b, d \le 2r/3$ or $b, d \ge r/3$. If $b, d \le 2r/3$, then provided r is large enough compared to k, we can pick an edge $(p_1, \dots, p_k)$ such that $p_a = b$, $p_c = d$, and the sequence $p_1, \dots, p_k$ is decreasing on $p_1, \dots, p_a$, increasing on $p_a, \dots, p_z$, decreasing on $p_z, \dots, p_c$ and increasing on $p_c, \dots, p_k$.

The case when $b, d \ge r/3$ is similar, starting with an increasing sequence on $p_1, \dots, p_a$.

Case 1(b): $|b - d| > r/3$. The cases $b < d$ and $b > d$ are similar, so suppose $b < d$. Then we can pick an edge $(p_1, \dots, p_k)$ such that $p_a = b$, $p_c = d$, and the sequence $p_1, \dots, p_k$ is decreasing on $p_1, \dots, p_a$, increasing on $p_a, \dots, p_c$ and decreasing on $p_c, \dots, p_k$.

Case 2: The parts a and c are consecutive, so $c = a + 1$. Then we can pick an edge $(p_1, \dots, p_k)$ such that $p_a = b$, $p_c = d$, as follows: if $b \le r/2$, choose $p_1, \dots, p_a$ to be decreasing, otherwise increasing, similarly if $d \le r/2$, $p_c, \dots, p_k$ is increasing, otherwise decreasing.

Note that three changes of direction in the sequence $p_1, \dots, p_k$ are essential to ensure that any two vertices in different parts and at different positions are contained in some edge; for example, the vertices v_{21} and v_{42} do not occur together in any edge with at most two changes of direction.

Next we observe that H has a complete k-colouring and a complete r-colouring.

Complete k-colouring: We take a colour set $\{c_1, \dots, c_k\}$ and, for each i, colour every vertex of part V_i with colour c_i. This is clearly a valid colouring (because no edge contains two vertices from the same part), and every edge contains each of the colours, so the colouring is complete.

Complete r-colouring: We take a colour set $\{x_1, \dots, x_r\}$ and, for each j, colour every vertex in position j with colour x_j. Again this is clearly a valid colouring (because no edge contains two vertices in the same position), and it is also clearly complete because for any k-element subset of the colours, say $\{x_{p_1}, \dots, x_{p_k}\}$, where $x_{p_1} < \dots < x_{p_k}$, the edge $(p_1, \dots, p_k)$ contains each of these colours.

Now consider t such that $k < t < r$, and suppose that H has a complete t-colouring. Fix such a colouring. Let C be the set of colours which occur on only a single part of H, and X be the set of colours which occur on at least two parts.

First consider a colour $x \in X$. By assumption, x occurs in more than one part. If it also occurred in more than one position, there would be two vertices with the same colour both contained in some edge, and the colouring would not be proper. Thus all occurrences of x are in the same position. Also two colours in X cannot share the same position, because then no edge would contain both of these colours, and the colouring could not be complete.

Since $t < r$, there are less than r colours in X, and therefore less than r positions whose vertices can be coloured by colours from X. Thus at least one vertex in each part is coloured with a colour from C. Also at most one colour from C can occur on each part, since otherwise there would be two colours from C which do not occur together in any edge, and the colouring could not be complete. Hence there are exactly k colours in C; let the colour associated with part V_i be c_i.

Since $|C| = k$, we have $|X| = t - k$. So to summarise, (i) $t - k$ positions j have an associated colour $x_j \in X$ which occurs only in position j (in each case on at least two parts), and (ii) each part V_i also has an associated colour c_i which occurs only in part V_i (and occurs at least once). Each vertex v_{ij} is coloured either with the colour c_i or (if it exists) the colour x_j.

Note that there are $t - k$ positions which have an associated colour and therefore $r - t + k$ which do not. Thus provided that

$$t - k > 4(r - t + k + 1)$$

(or equivalently $t > \frac{4r+5k+4}{5}$), there exist five consecutive positions $J, \ldots, J+4$ all of which have an associated colour in X. Form a k-element set S of colours containing the five colours $x_J, \ldots, x_{J+4}$ and the $\lfloor k/2 \rfloor$ colours c_{2i} associated with the even numbered parts. Since $k \geq 9$, $k \geq 5 + \lfloor k/2 \rfloor$; the remaining colours in S can be chosen arbitrarily.

We claim that no edge has its vertices coloured with the set S. For suppose $(p_1, \ldots, p_k)$ is such an edge. The vertex in each even numbered part V_{2i} must be coloured with the associated colour c_{2i}, thus the other colours must occur in odd numbered parts. For each $j = J, \ldots, J+4$, there must be a vertex coloured x_j, and this vertex must be in position j (and in an odd numbered part). No other vertex can be in any of the positions $J, \ldots, J+4$, since no edge contains two vertices in the same position. Thus the vertex in each even numbered part has position outside the range $J, \ldots, J+4$. The five vertices coloured $x_J, \ldots, x_{J+4}$ can be in any five distinct odd numbered parts, but between any two consecutive ones there is a vertex in an even numbered part whose position is outside the range $J, \ldots, J+4$. Hence the sequence $p_1, \ldots, p_k$ has at least four changes of direction, a contradiction. Thus H has no complete t-colouring. $\square$

Observe that in fact $\chi(H) = k$ and $\psi(H) = r$. To see that we could not have a complete colouring with more than r colours, note that we can partition the colours into sets C and X as above, and each colour in X is associated with a unique position. If $|X| < r$, then as above there must be k colours in C, so since there are more than r colours, we must have $|X| > r - k$, and the argument above shows that the colouring cannot be complete. On the other hand, if $|X| = r$, then every position has an associated colour, and in addition there must be at least one colour in C, say colour c_i used on V_i. If c_i occurs fewer than k times on V_i, then no edge contains both c_i and the colours corresponding to the positions where c_i occurs. However if c_i occurs at least k times on V_i, then no edge can contain the set of colours corresponding to any k of these positions.

Hence we have the following slightly stronger theorem:

Theorem 6. *Let $k \geq 9$ be a positive integer. There exists a k-uniform hypergraph H which has a complete $\chi(H)$-colouring and a complete $\psi(H)$-colouring, but no complete t-colouring for some t with $\chi(H) < t < \psi(H)$.*

A minimum colouring of a graph is necessarily complete, but this is not the case for hypergraphs, even when some complete colouring exists (with more colours).

Theorem 7. *There exists a 3-uniform hypergraph H which has a complete r-colouring for some $r > \chi(H)$, but no complete $\chi(H)$-colouring.*

Proof. As above we construct a hypergraph with k parts and r positions, in this case with $k = 4$ and $r > 4$. The edge set contains every edge with 3 vertices in distinct parts and distinct positions; every edge has a vertex in part 4.

As before, it is easy to see that for any two vertices in different parts and different positions, there is an edge containing the two vertices. Hence for example, the 4 vertices v_{ii}, $i = 1, 2, 3, 4$, must all have distinct colours. Clearly $\chi(H) = 4$, since we can colour each part with a distinct colour. Also, colouring each position with a distinct colour clearly gives a complete r-colouring.

Now consider any colouring, with t colours say, and as before let C be the set of colours which occur on just one part, and X be the set of colours which occur on at least two parts. If $|X| < r$, then some position has no X colour, hence $C \geq 4$ and there is at least one colour associated with each part. So either $t > 4$, or we have the 4-colouring above, which is not complete since no edge has the colour set $\{c_1, c_2, c_3\}$. Thus there is no complete 4-colouring. $\square$

6. λ-complete colourings of graphs

In this section we investigate how the properties of complete colourings of graphs generalise to λ-complete colourings. In particular, we consider interpolation and computational complexity.

6.1. Interpolation

Recall that Harary, Hedetniemi, Prins [17] proved that for any graph G and any t, such that $\chi(G) \leq t \leq \psi(G)$, the graph G has a complete t-colouring. Now we show the following.

Theorem 8. *Let $0 < \lambda \leq 1$ be a fixed real, and let G be a graph. Then for any t, such that $\chi(G) \leq t \leq \psi_\lambda(G)$, the graph G has a λ-complete t-colouring.*

Proof. If $t \leq \psi(G)$, then G has a complete t-colouring, and thus a λ-complete colouring for any λ. By the assumption, the claim holds also for $t = \psi_\lambda(G)$. Let us assume that $\psi(G) < t < \psi_\lambda(G)$ and suppose G has a λ-complete $(t+1)$-colouring f. We will show that it has a λ-complete t-colouring.

For each colour $c \in \{1, 2, \ldots, t+1\}$, let $d(c)$ be the number of distinct colour pairs containing c, which appear on some edge of G under the colouring f, and choose $c' \in \{1, 2, \ldots, t+1\}$, such that $d(c')$ is minimum possible. Since $t + 1 > \psi(G)$, the colouring f is not complete, so there is a colour c'', such that the pair $\{c', c''\}$ does not appear on any edge of G. Let f' be the colouring of G defined as follows: $f'(v) = f(v)$ if $f(v) \neq c'$ and $f'(v) = c''$ if $f(v) = c'$. Clearly f' is a proper t-colouring of G. We claim that f' is λ-complete.

Let K (K', respectively) be the number of distinct colour pairs that appear on edges of G under f (f', respectively). Note that $K' \geq K - d(c')$. If $d(c') \leq \lambda t$, then

$$K' \geq K - d(c') \geq \lambda \binom{t+1}{2} - d(c') \geq \lambda \binom{t+1}{2} - \lambda t = \lambda \binom{t}{2},$$

and we are done. On the other hand, $K \geq (t+1) \cdot d(c')/2$, so if $d(c') > \lambda t$,

$$K' \geq K - d(c') \geq \frac{(t+1) \cdot d(c')}{2} - d(c') = \frac{(t-1) \cdot d(c')}{2} > \lambda \frac{t(t-1)}{2} = \lambda \binom{t}{2},$$

which completes the proof. $\square$

6.2. Computational complexity for trees

Let us start with recalling a result on harmonious colourings of star forests by Edwards and McDiarmid [13]. A *harmonious colouring* of a graph G is a proper C-colouring, in which every pair of distinct colours appears at most once on a edge of G.

Lemma 9 (Edwards, McDiarmid [13]). *Let F be a forest consisting of t stars $F_1, \ldots, F_t$ with sizes (i.e. number of edges) $m_1 \geq m_2 \geq \cdots \geq m_t$ respectively, and let $C \geq t$. Then F can be coloured harmoniously with C colours so that the centres of the stars all receive distinct colours if and only if*

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (C-i) \text{ for each } k = 1, \ldots, t.$$

Now we prove the main result of this section, which is a generalisation of the hardness of finding complete colouring of trees, shown by Cairnie and Edwards [4].

Theorem 10. *Let λ be a rational constant with $0 < \lambda \leq 1$. It is NP-complete to decide whether a tree T with $\binom{K}{2}$ edges has a λ-complete K-colouring, even if the radius of T is at most 3.*

Proof. The problem is obviously in NP. To prove completeness, we reduce from the NP-complete problem Numerical 3-Dimensional Matching [16]:

> **Instance** Disjoint sets W, X, and Y, each containing m elements, a size $s(a) \in \mathbb{Z}^+$ for each element $a \in W \cup X \cup Y$, and a bound $B \in \mathbb{Z}^+$.
>
> **Question** Can $W \cup X \cup Y$ be partitioned into m disjoint sets $A_1, \ldots, A_m$, such that each A_i contains exactly one element from each of W, X, and Y and such that, for $1 \leq i \leq m$, $\sum_{a \in A_i} s(a) = B$?

Note that Numerical 3-Dimensional Matching is NP-complete in the strong sense [16] thus we can restrict our attention to instances for which B and each $s(a)$ is at most $p(m)$ for some fixed polynomial p. It is helpful to assume that m is odd, this is easy since we can just add new elements with sizes $B-2$, 1, 1 to W, X, and Y if necessary. Also note that we can assume that each $s(a) \leq B$ and that each $s(a)$, and B, is a multiple of m, for otherwise we can replace each $s(a)$ by $ms(a)$ and B by mB. Finally we assume that the sum of the sizes of all the elements in $W \cup X \cup Y$ is mB. Let $W = \{w_1, \ldots, w_m\}$, and similarly for X and Y.

Now given an instance I of Numerical 3-Dimensional Matching, satisfying these constraints, we choose K and construct a tree T with radius 3 and $\binom{K}{2}$ edges as follows:

We choose the integer C to be the least integer greater than or equal to $6B + m + 7$, such that

$$\lambda \binom{C+3}{2} \geq mC - \frac{1}{2}m(m+1) + 3(B+1)(C-m) + 3C + 3.$$

Note that in the case $\lambda = 1$, this gives (with equality) $C = 6B + m + 7$. Note also that the difference

$$D = \lambda\binom{C+3}{2} - (mC - \frac{1}{2}m(m+1) + 3(B+1)(C-m) + 3C + 3)$$

can be at most $\binom{C+3}{2} - \binom{C+2}{2} = C + 2$.

We will take the number of colours K to be $C + 3$.

Now we construct the tree T, shown in Figure 2. First we construct 3 sets S_W, S_X, S_Y, each containing C stars. The sets S_X and S_Y each contain one star S_a of size $s(a)$ for each a in X and Y respectively, and $C - m$ stars of size $B + 1$. The set S_W contains a star S_a of size $s(a) + C - \frac{1}{2}(m+1) - B$ for each $a \in W$, and $C - m$ stars of size $B + 1$. (Recall that m is assumed to be odd.) Later we will add one more leaf to some of the stars of size $B + 1$. In Figure 2 the quantity $s(a) + C - \frac{1}{2}(m+1) - B$ is shown as $s(a)^+$. Denote by V_W, V_X, V_Y the set of centres of the stars in S_W, S_X, S_Y respectively; these vertices are shown as open circles in Figure 2.

Now add 3 new vertices r_W, r_X, and r_Y, and join r_W to each element of V_W, r_X to each element of V_X and r_Y to each element of V_Y. In addition we join r_W to r_X and r_Y, and add one further new vertex q_X, joined to r_X.

Note that the number of edges of the tree constructed so far is

$$\sum_{a \in W \cup X \cup Y} s(a) + mC - \frac{1}{2}m(m+1) - mB + 3(C-m)(B+1) + 3C + 3$$
$$= mC - \frac{1}{2}m(m+1) + 3(C-m)(B+1) + 3C + 3.$$

We now add one extra leaf to $\lceil D \rceil$ of the stars of size $B + 1$, so that we have $\lceil D \rceil$ stars of size $B + 2$. Note that the number of stars of size $B + 1$ was $3(C - m)$, and we have $3(C - m) = 3C - 3m \geq C + 12B + 2m + 14 - 3m \geq C + 11m + 14 > C + 2 \geq \lceil D \rceil$ since $B \geq m$ by assumption, hence there will be enough stars available. We add the extra leaf first to stars in S_W, then S_X, then S_Y as long as necessary. Call the current tree T', with edge set E'. Note that $|E'| = \lceil \lambda\binom{K}{2} \rceil$.

Now add a set Q_W of $\binom{K}{2} - |E'|$ new vertices, each joined to r_W (these vertices and edges are represented by the dotted triangle in Figure 2). This completes the construction of the tree T. Note that T is easily constructed from I in polynomial time.

We now claim that the tree T has a λ-complete K-colouring if and only if I has a solution. First suppose that T has such a colouring.

Let E_W be the set of all edges incident with r_W; these are the dashed and dotted edges in Figure 2. Let $E^- = E \setminus E_W = E' \setminus E_W$ (these edges are shown solid in Figure 2), and let V^- be the set of endpoints of the edges in E^-. Then $V^- = V \setminus (\{r_W\} \cup Q_W)$. Note that E' consists of E^- together with the C edges joining r_W to the elements of V_W and the edges (r_W, r_X) and (r_W, r_Y) (thus E' consists of the solid and dashed edges in Figure 2). We now establish a number of properties of the colouring. We may assume that vertex r_W has colour $C + 1$.

1. The number of colour pairs which appear on T is at most $|E'|$. To see this note that the number of colour pairs which can appear on E^- is at most $|E^-| = |E'| - (C+2)$. However the number of pairs which can appear on E_W is clearly at most $C+2$ (since each pair contains the colour $C + 1$), so the result follows.

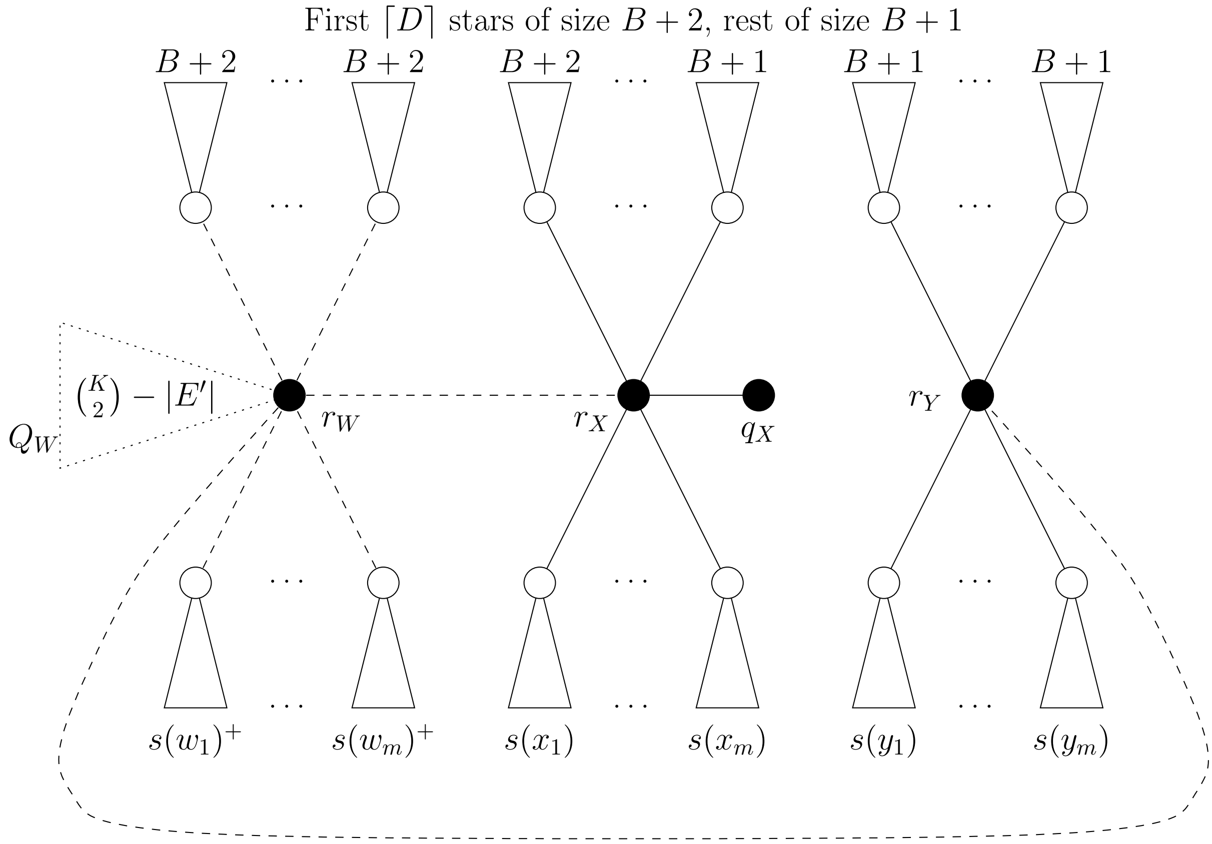

Figure 2: The tree T. Triangles represent stars, with the number of leaves given by the label.

2. Since the number of colour pairs is at least $\lceil \lambda \binom{K}{2} \rceil = |E'|$, it follows that exactly $|E'|$ colour pairs occur on T. Thus both E^- and E_W have the maximum possible number of colour pairs, and these sets of pairs are disjoint. Thus (i) every edge of E^- has a different colour pair, (ii) every pair involving colour $C+1$ occurs on E_W, (iii) the colour $C+1$ does not occur on V^-. It also follows that for any colour c, the number of edges in E^- with an endpoint coloured c cannot be more than $C+1$.

3. The vertices r_X and r_Y must have distinct colours, since if r_X and r_Y have the same colour c, then at least $2C+1 > C+1$ edges of E^- have an endpoint coloured c.

4. Hence r_W, r_X, r_Y have distinct colours, and so we can assume that r_X and r_Y are coloured $C+2$ and $C+3$ respectively.

5. Colour $C+3$ cannot be used on any vertex $v \in V_X$, because then the number of edges in E^- with an endpoint coloured $C+3$ would be at least $C + d(v) > C+1$.

6. From above, $C+1$ does not appear on any vertex in V^-, in particular it cannot be used on V_X. Also, to avoid repeated colour pairs, the vertices in V_X must all have distinct colours, so we conclude that they have the colours $1, \ldots, C$ in some order. Also, the vertex q_X must have colour $C+3$.

7. The edge incident to q_X uses the colour pair $(C+2, C+3)$, hence the vertices in V_Y cannot use colours $C+1$ or $C+2$, and must all have distinct colours. Thus these vertices also use the colours $1, \ldots, C$ in some order.

8. No vertex of V_W can use colour $C+2$ or $C+3$ because then the number of edges of E^- with an endpoint coloured with one of these colours would be too large. Hence the vertices of V_W all have colours from the set $\{1, \ldots, C\}$.

9. No two of the vertices $S_{w_1}, \ldots, S_{w_m}$ can have the same colour c, because if S_{w_i}, S_{w_j} have colour c, then the number of edges of E^- with an endpoint coloured c is at least $s(w_i) + s(w_j) + 2C - (m+1) - 2B > C + 1$. Hence we can assume that the vertices $S_{w_1}, \ldots, S_{w_m}$ have colours $1, \ldots, m$ respectively. Note that we do not claim that all the vertices of V_W have distinct colours.

10. The leaves attached to the vertices in $V_W \cup V_X \cup V_Y$ must all be coloured with colours from $\{1, \ldots, C\}$, since $C+1$ does not occur on V^- and all pairs involving $C+2$ and $C+3$ are used elsewhere in E^-.

We now focus attention on the stars $S_W \cup S_X \cup S_Y$ with centres in $V_W \cup V_X \cup V_Y$. As noted above, we can assume that the centre of the star S_{w_i} is coloured i for $i = 1, \ldots, m$. Now the total size of the stars with centres coloured $1, \ldots, m$ can be at most the number of colour pairs which could occur on their edges, which is

$$\binom{m}{2} + m(C - m) = m(C - \frac{1}{2}(m+1)),$$

while the total size of the stars S_{w_i}, $i = 1, \ldots, m$ is

$$m(C - \frac{1}{2}(m+1) - B) + \sum_{i=1}^{m} s(w_i).$$

Hence the total size of the stars from S_X and S_Y with centres coloured $1, \ldots, m$ is at most

$$mB - \sum_{i=1}^{m} s(w_i).$$

Since the total size of the objects in $X \cup Y$ is exactly this number, and all of the stars not of the form S_a have size $B+1$ or $B+2$ which is greater than any element of $W \cup X \cup Y$, then the only way in which this is possible is if the stars S_{x_i}, $i = 1, \ldots, m$ have centres coloured $1, \ldots, m$ in some order and similarly for the stars S_{y_i}. But then let

$$A_i = \{a | \text{ centre of } S_a \text{ has colour } i\}.$$

Then for each $i = 1, \ldots, m$, A_i contains exactly one element from each of W, X, and Y, and since the leaves of the stars with centres coloured i all have distinct colours, then

$$C - \frac{1}{2}(m+1) - B + \sum_{a \in A_i} s(a) \leq C - 1$$

hence

$$\sum_{a \in A_i} s(a) \leq B - 1 + \frac{1}{2}(m+1).$$

But since m divides B and each $s(a)$, then the sum on the left must in fact be at most B, and since the sum of all the elements in $W \cup X \cup Y$ is mB, the sums must be equal to B. Hence the instance I of Numerical 3-Dimensional Matching has a solution.

Conversely suppose that we have a solution $A_1, \dots, A_m$ to the instance I. Colour r_W, r_X, r_Y with colours $C+1, C+2, C+3$ respectively, q_X with colour $C+3$, and the elements of Q_W arbitrarily.

Then let the forest F_i consist of the stars S_a, $a \in A_i$, for each $i = 1, \dots, m$, and consist of one star of size $B+1$ or $B+2$ from each of S_W, S_X, and S_Y for $i = m+1, \dots, C$. Let the size of F_i be M_i, so that

$$M_i = C - \frac{1}{2}(m+1)$$

for $i = 1, \dots, m,$. For $i \geq m+1$, each M_i is the sum of three integers, each $B+1$ or $B+2$. By reordering if necessary, we can assume that $M_{m+1} \geq \cdots \geq M_C$, and so since $C - \frac{1}{2}(m+1) > 3B+6$, we have $M_1 \geq \cdots \geq M_C$ Also because of the way we added the extra leaves, we can assume that for each $i \geq m+1$, $M_i - M_{i+1} \leq 1$.

Note that the total size of all the M_i is

$$\begin{aligned}
\sum_{i=1}^{C} M_i &= mC - \frac{1}{2}m(m+1) + 3(C-m)(B+1) + \lceil D \rceil \\
&= \lambda \binom{C+3}{2} - (3C+3) \\
&\leq \binom{C+3}{2} - (3C+3) \\
&= \binom{C}{2} \\
&= \sum_{i=1}^{C} (C-i).
\end{aligned}$$

Then by Lemma 9 the stars in $S_W \cup S_X \cup S_Y$ can be harmoniously coloured with C colours such that all of the centres in F_i have colour i, provided that for each $k = 1, \dots, C$,

$$\sum_{i=1}^{k} M_i \leq \sum_{i=1}^{k} (C-i).$$

Now if $k \leq m$, then

$$\begin{aligned}
\sum_{i=1}^{k} M_i &= k(C - \frac{1}{2}(m+1)) \\
&= kC - \frac{1}{2}k(m+1) \\
&\leq kC - \frac{1}{2}k(k+1) \\
&= \sum_{i=1}^{k} (C-i),
\end{aligned}$$

with equality when $k = m$. Now consider the case where $m < k \leq C$. Write $f(k) =$

$\sum_{i=1}^{k} M_i - \sum_{i=1}^{k}(C-i)$. Then

$$\begin{aligned} f(k) &= \sum_{i=1}^{k} M_i - \sum_{i=1}^{k}(C-i) \\ &= \sum_{i=1}^{m} M_i + \sum_{i=m+1}^{k} M_i - \sum_{i=1}^{m}(C-i) - \sum_{i=m+1}^{k}(C-i) \\ &= \sum_{i=m+1}^{k} M_i - (C-i). \end{aligned}$$

Then clearly $f(m) = 0$ and from above $f(C) \leq 0$. Since $M_i - M_{i+1} \leq 1$, the terms $M_i - (C-i)$ in the sum are non-decreasing, and it follows that $f(k) \leq 0$ whenever $m < k \leq C$.

Using this harmonious colouring to colour the vertices of the stars gives a K-colouring of T in which every edge of E' has a distinct colour pair. Hence the colouring uses at least $E' = \lceil \lambda \binom{K}{2} \rceil$ colour pairs, so is λ-complete. □

7. Conclusion and open problems

Let us conclude the paper by making a few suggestions for further investigations. First, there is a gap between the exponent $\frac{1}{k+2}$ in the function of $\Delta(H)$, which guarantees the existence of a complete colouring, and the exponent $\frac{1}{k}$ for which a hypergraph exists with no λ-complete colouring (see Section 3). It would be of interest to close this gap.

Problem 1. For any $k \geq 3$, find a constant δ_k, such that:

(a) for every class Γ of k-uniform hypergraphs with maximum degree $o(n^{1/\delta_k})$, every sufficiently large $H \in \Gamma$ admits a complete colouring,
(b) there exists an infinite family of k-uniform hypergraphs with maximum degree $\Theta(n^{1/\delta_k})$ and no λ-complete colouring for any $0 < \lambda \leq 1$.

In Section 4 we have shown determining whether a k-uniform hypergraph has a λ-complete colouring is NP-complete, so there is little hope to find a polynomial algorithm for this problem. The complexity of the trivial brute-force algorithm is $n^n \cdot n^{O(1)} = 2^{n \log n} \cdot n^{O(1)}$. It is interesting to know if this naive approach can be significantly improved.

Problem 2. Design an algorithm deciding if a given n-vertex uniform hypergraph admits a λ-complete colouring in time $2^{O(n)}$, or show that existence of such an algorithm contradicts some widely accepted complexity assumption (like ETH, see [5]).

Finally, in Section 5, for any $k \geq 9$, we presented a construction of a k-uniform hypergraph H with a complete $\chi(H)$-colouring and a complete $\psi(H)$-colouring, but no complete t-colouring for *some* t satisfying $\chi(H) < t < \psi(H)$. There are two natural directions in which this result could be strengthened.

Problem 3. Does there exist a 3-uniform example of a hypergraph for which interpolation fails?

Problem 4. Does there exist a hypergraph H with a complete $\chi(H)$-colouring and a complete $\psi(H)$-colouring (where $\psi(H) \geq \chi(H) + 2$) but no complete t-colouring for *any* t satisfying $\chi(H) < t < \psi(H)$?

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