ON k-CAPS IN $\mathbf{PG}(n,q)$, WITH q EVEN AND $n \geq 3$

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ABSTRACT. Let $m_2(n,q)$ be the maximum size of k for which there exists a k-cap in $\mathbf{PG}(n,q)$, and let $m_2'(n,q)$ be the second largest value of k for which there exists a complete k-cap in $\mathbf{PG}(n,q)$. In this paper Chao's upper bound q^2-q+5 for $m_2'(3,q)$, q even and $q \geq 8$, will be improved. As a corollary new bounds for $m_2(n,q)$, q even, $q \geq 8$ and $n \geq 4$, are obtained. Cao and Ou published a better bound but there seems to be a gap in their proof.

Keywords: projective space, finite field, k-cap

1. INTRODUCTION

A *k-arc* of $\mathbf{PG}(2,q)$ is a set of k points, no three of which are collinear; a *k-cap* of $\mathbf{PG}(n,q)$, $n \geq 3$, is a set of k points, no three of which are collinear. A k-arc or k-cap is *complete* if it is not contained in a $(k+1)$-arc or $(k+1)$-cap. The largest value of k for which a k-arc of $\mathbf{PG}(2,q)$, or a k-cap of $\mathbf{PG}(n,q)$ with $n \geq 3$, exists is denoted by $m_2(n,q)$. The size of the second largest complete k-arc of $\mathbf{PG}(2,q)$ or k-cap of $\mathbf{PG}(n,q)$, $n \geq 3$, is denoted by $m_2'(n,q)$.

Theorem 1.1. (i) $m_2(2,q) = q+2$, *q even* [7]*;*
(ii) $m_2(3,q) = q^2+1$, *q even, $q > 2$* [6, 1, 10]*;*
(iii) $m_2(n,2) = 2^n$ [1]*;*
(iv) $m_2(4,4) = 41$ [5]*;*
(v) $m_2'(n,2) = 2^{n-1} + 2^{n-3}$ [4]*;*
(vi) $m_2'(3,4) = 14$ [8].

Theorem 1.2 ([11, 13, 7])**.** *Let K be a k-arc of $\mathbf{PG}(2,q)$, q even and $q > 2$, with $q-\sqrt{q}+1 < k \leq q+1$. Then K can be uniquely extended to a $(q+2)$-arc of $\mathbf{PG}(2,q)$.*

For any k-arc K in $\mathbf{PG}(2,q)$ or k-cap K in $\mathbf{PG}(n,q)$, $n \geq 3$, a *tangent* of K is a line which has exactly one point in common with K. Let t be the number of tangents of K through a point P of K and let $\sigma_1(Q)$ be the number of tangents of K through a point $Q \notin K$. Then for a k-arc K $t+k = q+2$ and for a k-cap K $t+k = q^{n-1}+q^{n-2}+\cdots+q+2$.

Theorem 1.3 ([8])**.** *If K is a complete k-arc in $\mathbf{PG}(2,q)$, q even, or a complete k-cap in $\mathbf{PG}(n,q)$, $n \geq 3$ and q even, then $\sigma_1(Q) \leq t$ for each point Q not on K.*

Theorem 1.4 ([3])**.**

(1) $$m_2'(3,q) \leq q^2-q+5, q \text{ even}, q \geq 8.$$

To prove Theorem 1.4 J.-M. Chao relies on the following crucial lemma.

Lemma 1.5 ([3]). *Let K be a complete k-cap in $\mathbf{PG}(3,q)$ with q even. If Π is a plane such that $|\Pi \cap K| = x$, then*

$$t(t-1) \geq q(q+2-x)x. \tag{2}$$

In the underlying paper the following improvement of Chao's result will be obtained.

Theorem 1.6 (Main Theorem).

$$m_2'(3,q) < q^2 - (\sqrt{5}-1)q + 5, q \textit{ even}, q \geq 8. \tag{3}$$

As a corollary new bounds for $m_2(n,q)$, q even, $q \geq 8$ and $n \geq 4$, are obtained.

Combining the main theorem of [12] with Theorem 1.6, there is an immediate improvement of the upper bound for $m_2'(3,q)$, $q \geq 2048$. We thank T. Szőnyi for bringing reference [12] to our attention.

Theorem 1.7.

$$m_2'(3,q) < q^2 - 2q + 3\sqrt{q} + 2, q \textit{ even}, q \geq 2048. \tag{4}$$

2. A first improvement of Chao's bound

Theorem 2.1.

$$m_2'(3,q) \leq q^2 - q + 3, q \textit{ even}, q \geq 8. \tag{5}$$

Proof. Let K be a complete k-cap in $\mathbf{PG}(3,q)$, q even, $q \geq 8$ and $k < q^2+1$. Let Π be a plane of $\mathbf{PG}(3,q)$ for which $4 \leq |\Pi \cap K| \leq q-2$. Let $f(X) = q(q+2-X)X$. Then

$$t(t-1) \geq f(4) = f(q-2) = 4q(q-2), \tag{6}$$

by Lemma 1.5. So

$$t \geq \frac{1+\sqrt{1+16q(q-2)}}{2} \geq 2q - \frac{7}{4} \text{ for } q \geq 8. \tag{7}$$

Hence $k \leq q^2 + q + 2 - 2q + \frac{7}{4} = q^2 - q + \frac{15}{4}$, and consequently $k \leq q^2 - q + 3$.

So we may assume that either $|\Pi \cap K| \leq 3$, or $|\Pi \cap K| \geq q-1$, for any plane Π of $\mathbf{PG}(3,q)$. Let $l_1, l_2, ..., l_t$ be the t tangents of K through the point $P \in K$. We consider three cases depending on the number of planes containing l_i and intersecting K in at most 3 points.

(A) **There exists exactly one plane Π_{l_i} containing l_i such that $|\Pi_{l_i} \cap K| \leq 3, i = 1,2,...,t$. We will show that in this case $k \leq q^2 - q + 3$.**

Assume there is exactly one plane Π through P with $|\Pi \cap K| \leq 3$. Then for $i = 1,2,...,t$, $\Pi_{l_i} = \Pi$. Hence all tangents of K containing P are in Π. So $t \leq q+1$, a contradiction. Hence there are at least two planes Π_1, Π_2 through P such that $|\Pi_i \cap K| \leq 3, i = 1, 2$. Then $|\Pi_1 \cap \Pi_2 \cap K| = 2$. Consequently $t \geq 2(q-1)$, and so $k \leq q^2 + q + 2 - 2q + 2 = q^2 - q + 4$.

Assume, by way of contradiction, that $k = q^2 - q + 4$. So $t = 2(q-1)$. Then $|\Pi_1 \cap K| = |\Pi_2 \cap K| = 3$. All tangent lines at P are contained in Π_1 and Π_2. Let l be a tangent of K at P in Π_1, and consider the $q+1$ planes containing l. The plane Π_1 is the only of these planes which intersects K in 3 points, exactly $q-1$ planes through l contain 2 tangent lines at P and so intersect K in a q-arc and the remaining plane through l contains exactly one tangent line at P and so intersects K in a $(q+1)$-arc.

Let $\widetilde{\Pi}$ be the unique plane containing l which intersects K in a $(q+1)$-arc, let $\widetilde{\Pi} \cap K = O$, and let N be the kernel of O, that is, N is the unique point of $\widetilde{\Pi}$ which extends O to a $(q+2)$-arc of $\widetilde{\Pi}$. Clearly $N \in l$.

If K' is a k'-arc of a plane $\mathbf{PG}(2, q)$ and $P' \in \mathbf{PG}(2, q) \setminus K'$, then the parity of the number of tangents of K' through P' is the parity of k', see Chapter 1 of [7]. Hence, by considering O and the $q-1$ q-arcs whose planes contain l, we see that the number of tangents of K through N is at least $q + 1 + q - 1 = 2q$. As K is complete we have $2q \leq t$, so $k \leq q^2 + q + 2 - 2q = q^2 - q + 2$, a contradiction.

Consequently $k \leq q^2 - q + 3$.

(B) **Some tangent $l_i, 1 \leq i \leq t$, is contained in at least two planes having at most three points in common with K.**

First we will prove that $k \leq q^2 - q + 5$. For $k = q^2 - q + 5$ and $k = q^2 - q + 4$ a contradiction will be obtained; the case $k = q^2 - q + 4$ will be subdivided in two cases. Hence it follows that also in Case (B) we have $k \leq q^2 - q + 3$.

Counting the points of K on the $q+1$ planes containing l_i gives

(8) $$k - 1 \leq 2.2 + (q-1)q = q^2 - q + 4.$$

So $k \leq q^2 - q + 5$.

(B.1) **First, assume** $k = q^2 - q + 5$. Then two planes Π_1, Π_2 containing l_i intersect K in 3 points, while the remaining planes $\Pi_3, \Pi_4, ..., \Pi_{q+1}$ containing l_i intersect K in $q+1$ points. Let l be a tangent of K at P in Π_1, distinct from l_i. Any plane ζ containing l, with $\zeta \neq \Pi_1$, intersects each $(q+1)$-arc $\Pi_i \cap K, i = 3, 4, ..., q+1$, in exactly two points. Hence $|\zeta \cap K| \geq q$. Considering the lines $\zeta \cap \Pi_2$, we see that exactly two of the planes ζ, say ζ_1 and ζ_2, intersect K in $(q+1)$-arcs O_1 and O_2, while the $q-2$ other planes ζ, say $\zeta_3, \zeta_4, ..., \zeta_q$, intersect K in a q-arc.

Let N_1 be the kernel of O_1; then $N_1 \in l$. The number of tangents of K containing N_1 is at least $q + 1 + q - 2 = 2q - 1$. As K is complete we have $2q - 1 \leq t$, so $k \leq q^2 + q + 2 - 2q + 1 = q^2 - q + 3$, a contradiction.

(B.2) **Next, assume** $k = q^2 - q + 4$. Then, considering all planes containing l_i, there are two cases to consider.

(B.2.1) **Two planes Π_1, Π_2 containing l_i intersect K in three points, the plane Π_3 containing l_i intersects K in q points, and the remaining planes $\Pi_4, \Pi_5, ..., \Pi_{q+1}$ containing l_i intersect K in $q+1$ points.** Let l be a tangent of K at P in Π_1, distinct from l_i. Any plane ζ containing l, distinct from Π_1, intersects each $(q+1)$-arc $\Pi_i \cap K, i = 4, 5, ..., q+1$, in exactly two points; $q-1$ of these planes ζ intersect $\Pi_3 \cap K$ in exactly two points. So for at least $q-1$ of these planes ζ we have $|\zeta \cap K| \geq q$, and for all planes ζ we have $|\zeta \cap K| \geq q-1$.

Assume that for all q planes ζ we have $|\zeta \cap K| \geq q$. Let s be the number of planes ζ for which $|\zeta \cap K| = q$ and let u be the number of planes ζ for which $|\zeta \cap K| = q+1$. Then

$$s(q-1) + uq + 3 = q^2 - q + 4, s + u = q. \tag{9}$$

So $s(q-1) + (q-s)q + 3 = q^2 - q + 4$, hence $s = q-1$ and $u = 1$. Let ζ be the plane which intersects K in a $(q+1)$-arc O, and let $N \in l$ be the nucleus of O. The number of tangents of K containing N is at least $q+1+q-1 = 2q$, so $k \leq q^2 + q + 2 - 2q = q^2 - q + 2$, a contradiction.

So we may assume that for exactly $q-1$ planes ζ we have $|\zeta \cap K| \geq q$ and that for exactly one plane ζ we have $|\zeta \cap K| = q-1$. Assume that for s planes ζ we have $|\zeta \cap K| = q$, and that for u planes ζ we have $|\zeta \cap K| = q+1$. Then

$$s(q-1) + uq + q - 2 + 3 = q^2 - q + 4, s + u = q - 1. \tag{10}$$

So $s(q-1) + (q-1-s)q + q + 1 = q^2 - q + 4$, hence $s = q-3$ and $u = 2$. Let ζ_1, ζ_2 be the planes containing l which intersect K in $(q+1)$-arcs O_1, O_2, let N_1, N_2 be the nuclei of O_1, O_2, and let $\Pi_1 \cap K = \{P, P_1, P_2\}$. Assume first that $N_1 \notin P_1P_2$. Then the number of tangents of K containing N_1 is at least $q+1+q-3+2 = 2q$, so $k \leq q^2 - q + 2$ a contradiction. Similarly if $N_2 \notin P_1P_2$. Hence we may assume that $N_1 = N_2 = P_1P_2 \cap l$. Then the number of tangents of K through N_1 is at least $q+1+q+q-3 = 3q-2$, so $k \leq q^2 + q + 2 - 3q + 2 = q^2 - 2q + 4$, again a contradiction.

(B.2.2) **One plane Π_1 containing l_i intersects K in three points, and one plane Π_2 containing l_i intersects K in two points**. Consequently the other $q-1$ planes $\Pi_3, \Pi_4, ..., \Pi_{q+1}$ containing l_i intersect K in $q+1$ points. Let l be a tangent of K at P in Π_1, distinct from l_i. Any plane ζ containing l, distinct from Π_1, intersects each $(q+1)$-arc $\Pi_i \cap K$, with $i = 3, 4, ..., q+1$, in exactly two points. As $k = q^2 - q + 4$ it easily follows that for $q-1$ of these planes ζ we have $|\zeta \cap K| = q$, while for the remaining plane ζ we have $|\zeta \cap K| = q+1$.

Let $\widetilde{\zeta}$ be the plane containing l which intersects K in a $(q+1)$-arc O, and let N be the nucleus of O. The number of tangents of K containing N is at least $q+1+q-1 = 2q$, so $k \leq q^2 - q + 2$, again a contradiction.

(C) **Some tangent l_i, with $1 \leq i \leq t$, is contained in no plane having at most three points in common with K.**

First we will prove that $k \leq q^2 - q + 5$. A contradiction will be obtained for $k \in \{q^2 - q + 5, q^2 - q + 4\}$; for $k = q^2 - q + 4$ two cases have to be considered. Hence again $k \leq q^2 - q + 3$.

Then $|\Pi_j \cap K| \geq q-1$ for each plane Π_j containing l_i, with $j = 1, 2, ..., q+1$. The arc $\Pi_j \cap K$ of Π_j can be completed to a $(q+2)$-arc of Π_j; see Theorem 1.2. This $(q+2)$-arc meets l_i in points P, P_j. As there are $q+1$ points P_j and $|l_i \setminus \{P\}| = q$, two of the points P_j coincide, say $P_1 = P_2$. The number of tangents of K containing P_1 is at least $2(q-2)+1 = 2q-3$, so $k \leq q^2 - q + 5$.

Now we make some observations on $(q-1)$-arcs of $\mathbf{PG}(2,q)$, q even. Let $\widetilde{K}$ be any $(q-1)$-arc of $\mathbf{PG}(2,q)$, q even, and let $\widetilde{l}$ be a tangent of $\widetilde{K}$ at $\widetilde{P} \in \widetilde{K}$. Let $\widetilde{C}$ be the unique $(q+2)$-arc which contains $\widetilde{K}$; see Theorem 1.2. Put $\widetilde{C} \cap \widetilde{l} = \{\widetilde{P}, \widetilde{N}\}$. Then it is easy to see that exactly $q-2$ points of $\widetilde{l} \setminus \{\widetilde{P}, \widetilde{N}\}$ are on exactly three tangents of $\widetilde{K}$, and that exactly one point $\widetilde{R}$ of $\widetilde{l} \setminus \{\widetilde{P}, \widetilde{N}\}$ is on exactly one tangent of $\widetilde{K}$; also, $\widetilde{R} = \widetilde{l} \cap \widetilde{N}'\widetilde{N}''$, with $\{\widetilde{N}, \widetilde{N}', \widetilde{N}''\} \cup \widetilde{K} = \widetilde{C}$.

(C.1) **First, assume** $k = q^2 - q + 5$. Then $\Pi_1 \cap K$ and $\Pi_2 \cap K$ are $(q-1)$-arcs of Π_1 and Π_2. Let r be the number of $(q-1)$-arcs $\Pi_j \cap K$, let s be the number of q-arcs $\Pi_j \cap K$ and let u be the number of $(q+1)$-arcs $\Pi_j \cap K$. Then

(11) $$r(q-2) + s(q-1) + uq + 1 = q^2 - q + 5, r + s + u = q + 1, \text{ with } r \geq 2.$$

So $r(q-2) + s(q-1) + (q+1-r-s)q + 1 = q^2 - q + 5$, hence $2r + s = 2q - 4$, with $r \geq 2$. If $s \geq 1$, then we have an extra tangent of K containing P_1, so $k \leq q^2 - q + 4$, a contradiction. Hence $s = 0, r = q-2, u = 3$.

As the number of tangents of K containing P_1 is exactly $2q-3$, the nuclei of the three $(q+1)$-arcs $\Pi_j \cap K$ are distinct from P_1. Let N be one of these nuclei. Also, P_1 is on exactly one tangent of each of the $q-4$ $(q-1)$-arcs $\Pi_j \cap K$, distinct from the $(q-1)$-arcs $\Pi_1 \cap K, \Pi_2 \cap K$. So N is on at least three tangents of each of these $q-4$ $(q-1)$-arcs $\Pi_j \cap K$. Hence the number of tangents of K containing N is at least $2(q-4) + q + 1 = 3q - 7 > 2q - 3$, a contradiction.

(C.2) **Finally, assume that** $k = q^2 - q + 4$. We have to consider two cases depending of the sizes of $\Pi_1 \cap K$ and $\Pi_2 \cap K$.

(C.2.1) **First, assume that** $\Pi_1 \cap K$ **and** $\Pi_2 \cap K$ **are** $(q-1)$**-arcs**. The tangents of K containing P_1 are the tangents of $\Pi_1 \cap K$ and $\Pi_2 \cap K$ containing P_1, and one extra tangent l. Assume that l is a tangent of $\Pi_3 \cap K$. If $\Pi_3 \cap K$ is a $(q+1)$-arc O, then P_1 is the nucleus of O, so there arise q extra tangents, a contradiction; if $\Pi_3 \cap K$ is a $(q-1)$-arc K', then P_1 is contained in at least three tangents of K', again a contradiction. Hence $\Pi_3 \cap K$ is a q-arc. Also, $\Pi_j \cap K$, with $j = 4, 5, ..., q+1$, cannot be a q-arc. Let r be the number of $(q-1)$-arcs $\Pi_j \cap K$, and let u be the number of $(q+1)$-arcs $\Pi_j \cap K$. Then

(12) $$r(q-2) + uq + q - 1 + 1 = q^2 - q + 4, r + u + 1 = q + 1.$$

So $r(q-2) + (q-r)q + q = q^2 - q + 4$, hence $r = q-2$ and $u = 2$. Let O_1, O_2 be the $(q+1)$-arcs $\Pi_j \cap K$, and let N_1, N_2 be the nuclei of O_1, O_2. Then $N_i \neq P_1, i = 1, 2$. Also P_1 is contained in exactly one tangent of each of the $q-4$ $(q-1)$-arcs $\Pi_j \cap K$, with $j \neq 1, 2$. Hence the number of tangents of K containing N_1 is at least $2(q-4) + q + 1 = 3q - 7 > 2q - 2$, clearly a contradiction.

(C.2.2) **Consequently, we may assume that** $\Pi_1 \cap K$ **is a** $(q-1)$**-arc and that** $\Pi_2 \cap K$ **is a** q**-arc** . Let r be the number of $(q-1)$-arcs $\Pi_j \cap K$, let s be the number of q-arcs $\Pi_j \cap K$ and let u be the number of $(q+1)$-arcs $\Pi_j \cap K$. Then

(13) $$r(q-2) + s(q-1) + uq + 1 = q^2 - q + 4, r + s + u = q + 1, r \geq 1, s \geq 1.$$

So $2r+s=2q-3, r\geq 1, s\geq 1$. Clearly, $s=1$, as otherwise we have an extra tangent containing P_1, and then $k<q^2-q+4$. Hence $r=q-2, s=1, u=2$. The nuclei of the two $(q+1)$-arcs $\Pi_j\cap K$ are distinct from P_1. Let N be one of these nuclei. Also, P_1 is on exactly one tangent of each of the $q-3$ $(q-1)$-arcs $\Pi_j\cap K$ distinct from $\Pi_1\cap K$. So N is on at least three tangents of each of these $q-3$ $(q-1)$-arcs $\Pi_j\cap K$. Consequently the number of tangents of K containing N is at least $2(q-3)+q+1=3q-5>2q-2$, a final contradiction. ■

3. Main Theorem

Theorem 3.1.

$$m_2'(3,q)<q^2-(\sqrt{5}-1)q+5, q \textit{ even}, q\geq 8. \tag{14}$$

$$m_2'(3,4)=14. \tag{15}$$

Proof By [8] we have $m_2'(3,4)=14$, and by Theorem 2.1 we have $m_2'(3,8)\leq 59$, which proves Theorem 3.1 for $q=8$. So from now on we assume $q>8$.

Let K be a complete k-cap in $\mathbf{PG}(3,q)$, q even, $q>8$, and $k<q^2+1$. Let Π be a plane of $\mathbf{PG}(3,q)$ for which

$$5\leq|\Pi\cap K|\leq q-3. \tag{16}$$

Let $f(X)=q(q+2-X)X$. Then by Lemma 1.5 of Chao

$$t(t-1)\geq f(5)=f(q-3)=5q(q-3). \tag{17}$$

So

$$t\geq\frac{1+\sqrt{1+20q(q-3)}}{2}. \tag{18}$$

Put $\frac{1+\sqrt{1+20q(q-3)}}{2}\geq\sqrt{5}q-\alpha$, that is,

$$\sqrt{1+20q(q-3)}\geq 2\sqrt{5}q-2\alpha-1. \tag{19}$$

For $\alpha\leq\sqrt{5}q-(1/2)$ this is equivalent to

$$1+20q(q-3)\geq 20q^2+4\alpha^2+1-8\alpha\sqrt{5}q-4\sqrt{5}q+4\alpha, \tag{20}$$

or

$$0\geq 4\alpha^2+\alpha(-8\sqrt{5}q+4)+60q-4\sqrt{5}q, \tag{21}$$

or

$$0\geq\alpha^2+\alpha(-2\sqrt{5}q+1)+15q-\sqrt{5}q. \tag{22}$$

Put $\alpha=3$. Then there arises $0\geq 9+3(-2\sqrt{5}q+1)+15q-\sqrt{5}q$, that is, $0\geq 12+15q-7\sqrt{5}q$. This inequality is satisfied for $q>16$.

Hence for $q>16$ we have $t\geq\sqrt{5}q-3$, and so,

$$k\leq q^2+q+2-\sqrt{5}q+3, \tag{23}$$

that is,

$$k\leq q^2+(1-\sqrt{5})q+5. \tag{24}$$

For $q = 16$ it follows from (18) that $t > 32$ and so $k \leq 241$, which is equivalent to $k \leq q^2 + (1 - \sqrt{5})q + 5$ with $q = 16$.

From now on suppose that either $|\Pi \cap K| \leq 4$ or $|\Pi \cap K| \geq q - 2$ for any plane Π of $\mathbf{PG}(3,q)$. Let $l_1, l_2, ..., l_t$ be the t tangents of K containing the point $P \in K$. Assume, by way of contradiction, that $k > q^2 + (1 - \sqrt{5})q + 5$. We consider three cases depending on the number of planes containing l_i and intersecting K in at most 4 points. In each case a contradiction will be obtained .

(A) **Assume, by way of contradiction, that each l_i is contained in exactly one plane Π_{l_i} for which $|\Pi_{l_i} \cap K| \leq 4$, with $i = 1, 2, ..., t$.**

(A.1) **Assume that there is exactly one plane Π through P with $|\Pi \cap K| \leq 4$.** Then for $i = 1, 2, ..., t$ we have $\Pi_{l_i} = \Pi$. So $t \leq q + 1$, hence $k \geq q^2 + 1$, a contradiction.

(A.2) **There are at least two planes Π_1, Π_2 through P such that $|\Pi_i \cap K| \leq 4, i = 1, 2$.** Then $|\Pi_1 \cap \Pi_2 \cap K| = 2$. Consequently $t \geq 2(q - 2)$, and so $k \leq q^2 + q + 2 - 2q + 4 = q^2 - q + 6$.

The plane Π_1 intersects K in a m-arc, $m \leq 4$, and contains at least $q - 2$ tangents of K at P. Let $P_1 \in (K \cap \Pi_1) \setminus P$ and assume that PP_1 is contained in α planes Π with $|\Pi \cap K| \leq 4$. Then $t \geq \alpha(q - 2)$, so $k \leq q^2 + (1 - \alpha)q + 2 + 2\alpha$. Consequently

$$q^2 + (1 - \alpha)q + 2 + 2\alpha > q^2 + (1 - \sqrt{5})q + 5, \tag{25}$$

or

$$(\sqrt{5} - \alpha)q + 2\alpha - 3 > 0 \tag{26}$$

This gives a contradiction for $\alpha > 2$ with $q > 8$. So PP_1 is contained in at most two planes intersecting K in at most four points.

Assume, by way of contradiction, that for some plane Π of $\mathbf{PG}(3,q)$ we have $\Pi \cap K = \{P\}$. As there are at least two planes Π, Π' through P intersecting K in at most four points, we have $|\Pi \cap \Pi' \cap K| = 2$ and so $|\Pi \cap K| \geq 2$, a contradiction.

Let $\mathbf{PG}(2,q)$ be a plane of $\mathbf{PG}(3,q)$ not containing P and let σ be the projection of $\mathbf{PG}(3,q) \setminus \{P\}$ from P onto $\mathbf{PG}(2,q)$. Further, let $\mathcal{P}$ be the set of all images under σ of all points of $K \setminus \{P\}$ contained in planes Π, with $P \in \Pi$, for which $|\Pi \cap K| \leq 4$, and let $\mathcal{B}$ be the set of all images under σ of the sets $\Pi \setminus \{P\}$. Then there arises an incidence structure $(\mathcal{P}, \mathcal{B})$ of points and lines for which

(1) $|\mathcal{B}| \geq 2$,
(2) any two distinct lines in $\mathcal{B}$ have exactly one point in common,
(3) each point is contained in at most two lines,
(4) each line contains at most three points and at least one point.

It follows easily that $2 \leq |\mathcal{B}| \leq 4$. For each value of $|\beta|$ we will find a contradiction.

(α) $\underline{|\mathcal{B}| = 4}$
Then $t = 4(q - 2)$, so $k = q^2 + q + 2 - 4q + 8 = q^2 - 3q + 10$. Hence $q^2 - 3q + 10 > q^2 + (1 - \sqrt{5})q + 5$, or $5 > (4 - \sqrt{5})q$, a contradiction as $q > 8$.

(β) $|\mathcal{B}| = 3$

If $|\mathcal{P}| = 3$, then $t = 3(q-1)$, so $k = q^2-2q+5$. Hence $q^2-2q+5 > q^2+(1-\sqrt{5})q+5$, or $(3-\sqrt{5})q < 0$, a contradiction.

If $|\mathcal{P}| = 4$, then $t = 2(q-1)+q-2$, so $k = q^2-2q+6$. Hence $q^2+(1-\sqrt{5})q+5 < q^2-2q+6$, or $(3-\sqrt{5})q-1 < 0$, a contradiction.

If $|\mathcal{P}| = 5$, then $t = q-1+2(q-2)$, so $k = q^2-2q+7$. Hence $q^2+(1-\sqrt{5})q+5 < q^2-2q+7$, or $(3-\sqrt{5})q < 2$, a contradiction.

If $|\mathcal{P}| = 6$, then $t = 3(q-2)$, so $k = q^2-2q+8$. Hence $q^2+(1-\sqrt{5})q+5 < q^2-2q+8$, or $(3-\sqrt{5})q < 3$, a contradiction.

(γ) $|\mathcal{B}| = 2$

By Theorem 2.1 we may assume that $k \leq q^2-q+3$.

If $|\mathcal{P}| = 1$, then $t = 2q$, so $k = q^2-q+2$.

If $|\mathcal{P}| = 2$, then $t = 2q-1$, so $k = q^2-q+3$.

If $|\mathcal{P}| = 3$, then $t = 2q-2$, so $k = q^2-q+4$, a contradiction.

If $|\mathcal{P}| = 4$, then $t = 2q-3$, so $k = q^2-q+5$, a contradiction.

If $|\mathcal{P}| = 5$, then $t = 2q-4$, so $k = q^2-q+6$, a contradiction.

Hence the cases $k = q^2-q+2$ and $k = q^2-q+3$ have still to be considered.

(γ.1) $k = q^2-q+2$

On K there are two points P, P_1 such that PP_1 is contained in two planes Π_1, Π_2 intersecting K in just $\{P, P_1\}$, and in $q-1$ planes $\Pi_3, \Pi_4, ..., \Pi_{q+1}$ intersecting K in a $(q+2)$-arc.

Let $P' \in (\Pi_3 \cap K) \setminus \{P, P_1\}$ and let l be a tangent of K at P'. Assume, by way of contradiction, that each plane containing l intersects K in a m-arc with $m > 4$, so $m \geq q-2$. These m-arcs K'_i, with $i = 1, 2, ..., q+1$, are extendable to $(q+2)$-arcs C_i. Let $C_i \cap l = \{N_i, P'\}, i = 1, 2, ..., q+1$. At least two of the points $N_1, N_2, ..., N_{q+1}$ coincide, say $N_1 = N_2$. A plane Π' containing l, but not containing P nor P_1, intersects each of the $(q+2)$-arcs $\Pi_i \cap K$, with $i = 3, 4, ..., q+1$, in either 0 or 2 points; so $|\Pi' \cap K|$ is even. A plane Π' containing l and either P or P_1 intersects K in q points. Hence each plane containing l intersects K in a m-arc, with m even. Counting tangents of K containing N_1, we obtain at least $2(q-3)+1+q-1 = 3q-6$ tangents. So $k \leq q^2+q+2-3q+6 = q^2-2q+8$, a contradiction for $q > 8$. We conclude that there is a plane Π' containing l with $|\Pi' \cap K| \leq 4$.

Assume, by way of contradiction, that l is contained in at least two planes Π', Π'' with $|\Pi' \cap K| \leq 4, |\Pi'' \cap K| \leq 4$. Then, by a previous argument, these intersections have an even number of points and so $|\Pi' \cap K| \in \{2, 4\}$ and $|\Pi'' \cap K| \in \{2, 4\}$. Now we count the points of K in planes containing l, and obtain $k \leq (q-1)(q-1)+7 = q^2-2q+8$, a contradiction for $q > 8$.

Hence l is contained in exactly one plane Π' for which $|\Pi' \cap K| \leq 4$. It follows that the roles of P and P' may be interchanged.

Let l' be a second tangent of K containing P', with $l' \not\subset \Pi'$. Let $\tilde{K} = K \cap \Pi_3, \Pi' \cap \tilde{K} = \{P', P'_1\}$. If $P'_1 \in \{P, P_1\}$, then $|\Pi' \cap K| = q$, a contradiction. Hence $P'_1 \notin \{P, P_1\}$. With P' there corresponds an incidence structure $(\mathcal{P}', \mathcal{B}')$ of points and lines. As $k = q^2 - q + 2$, we necessarily have $|\mathcal{P}'| = 1$ and $|\mathcal{B}'| = 2$. Hence $\Pi' \cap K = \{P', P_1'\}$. If $\tilde{\Pi}'$ is the unique plane containing l' and intersecting K in at most 4 points, then $\tilde{\Pi}' \cap K = \{P', P_1'\}$. Also, the roles of P and P_1', P' and P_1', P and P_1 can be interchanged.

Interchanging Π_3 and Π_i, $i \in \{3, 4, ..., q+1\}$, and interchanging P' with any point of $(\Pi_i \cap K) \setminus \{P, P_1\}$, we see that K is partitioned into $(q^2 - q + 2)/2$ pairs, where each pair is contained in two planes intersecting K in that pair and in $q-1$ planes intersecting K in a $(q+2)$-arc. Any other plane contains either 0 or q points of K. Each point Q of K is contained in $2q$ tangents; the two planes on Q intersecting K in two points each contain q of these tangents.

Now we count the planes intersecting K in a $(q+2)$-arc, and obtain

$$\frac{q^2 - q + 2}{2}.(q-1)/\frac{q+2}{2}. \tag{27}$$

Hence $q + 2|(q^2 - q + 2)(q - 1)$, so $q + 2|24$, that is $q \in \{2, 4\}$, a contradiction.

(γ.2) $\underline{k = q^2 - q + 3}$
Then on K there are points P, P_1 such that PP_1 is contained in two planes Π_1, Π_2 with $\Pi_1 \cap K = \{P, P_1\}, \Pi_2 \cap K = \{P, P_1, P_2\}$, and in $q-1$ planes $\Pi_3, \Pi_4, ..., \Pi_{q+1}$ intersecting K in a $(q+2)$-arc.

Let $P' \in (\Pi_3 \cap K) \setminus \{P, P_1\}$ and let l be a tangent of K at P'. Assume, by way of contradiction, that each plane containing l intersects K in a m-arc with $m > 4$, so $m \geq q-2$. These m-arcs K_i', with $i = 1, 2, ..., q+1$, are extendable to $(q+2)$-arcs C_i. Let $C_i \cap l = \{N_i, P'\}, i = 1, 2, ..., q+1$. At least two of the points $N_1, N_2, ..., N_{q+1}$ coincide, say $N_1 = N_2$. A plane Π' containing l, but not containing P nor P_1, intersects each of the $(q+2)$-arcs $\Pi_i \cap K$, with $i = 3, 4, ..., q+1$, in either 0 or 2 points. So if $P_2 \notin \Pi'$, then $|\Pi' \cap K|$ is even. A plane Π' containing l and either P or P_1, but not P_2, intersects K in q points. Hence q planes containing l intersect K in a m-arc, with m even. Counting tangents of K containing N_1, we obtain at least $2(q-3)+1+q-2 = 3q-7$ tangents. So $k \leq q^2+q+2-3q+7 = q^2-2q+9$, a contradiction for $q > 8$. We conclude that there is a plane Π' containing l with $|\Pi' \cap K| \leq 4$.

Assume, by way of contradiction, that l is contained in at least two planes Π', Π'' with $|\Pi' \cap K| \leq 4, |\Pi'' \cap K| \leq 4$. Now we count the points of K in planes containing l, and obtain $k \leq q^2 - 2q + 9$, a contradiction for $q > 8$.

Hence l is contained in exactly one plane Π' for which $|\Pi' \cap K| \leq 4$. As all tangents of K at P_1 are contained in $\Pi_1 \cup \Pi_2$, it follows that each tangent of K at P_1 is contained in exactly one plane intersecting K in at most 4 points. Hence all points of $K \setminus \{P_2\}$ play the same role.

Let l' be a second tangent of K containing P', with $l' \not\subset \Pi'$. Let $K \cap \Pi_3 = \tilde{K}, \Pi' \cap \tilde{K} = \{P', P_1'\}$. If $P_1' \in \{P, P_1\}$, then $|\Pi' \cap K| \geq q$, a contradiction. Hence $P_1' \notin \{P, P_1\}$. With P' there corresponds an incidence structure $(\mathcal{P}', \mathcal{B}')$ of points and lines (see first part of (A)).

As $k = q^2 - q + 3$, we necessarily have $|\mathcal{P}'| = 2$ and $|\mathcal{B}'| = 2$. Hence $|\Pi' \cap K| \in \{2, 3\}$ and $\Pi' \cap K \supset \{P', P_1'\}$. Let $\widetilde{\Pi}'$ be the unique plane containing l' and intersecting K in at most 4 points, and let $\widetilde{\Pi}' \cap \widetilde{K} = \{P', \widetilde{P}_1'\}$. If $P_1' \neq \widetilde{P}_1'$, then by the structure of $(\mathcal{P}', \mathcal{B}')$ we have $\{P_1', \widetilde{P}_1'\} \subset \Pi'$, clearly a contradiction. Hence $P_1' = \widetilde{P}_1'$, and so $\{P', P_1'\} \subset \widetilde{\Pi}' \cap K$.

Without loss of generality we may assume that $\Pi' \cap K = \{P', P_1', P_2'\}$ and $\widetilde{\Pi}' \cap K = \{P', P_1'\}$. As $|\Pi' \cap K|$ is odd, the set $\Pi' \cap K$ has to contain the point P_2. Consequently $P_2 = P_2'$.

Interchanging Π_3 and Π_i, $i \in \{3, 4, \cdots, q+1\}$, and interchanging P' with any point of $(\Pi_i \cap K) \setminus \{P, P_1\}$, we see that $K \setminus \{P_2\}$ is partitioned into $(q^2 - q + 2)/2$ pairs, where each pair is contained in one plane intersecting K in that pair, in one plane intersecting K in that pair together with P_2, and in $q-1$ planes intersecting K in a $(q+2)$-arc. Any other plane contains 0, 1, q or $q+1$ points of K.

Now we count the planes intersecting K in a $(q+2)$-arc and obtain

(28) $$\frac{q^2 - q + 2}{2}.(q-1)/\frac{q+2}{2}.$$

Hence $q + 2 | (q^2 - q + 2)(q - 1)$, so $q + 2 | 24$, that is $q \in \{2, 4\}$, a final contradiction.

We conclude that there is some tangent l_i containing P, with $i \in \{1, 2, ..., t\}$, which is contained in exactly $\theta > 1$ planes having at most 4 points in common with K.

(B) **Assume, by way of contradiction, that some tangent l of K is contained in no plane intersecting K in at most 4 points**.

Hence each plane Π_i containing l satisfies $|\Pi_i \cap K| \geq q-2$, with $i = 1, 2, ..., q+1$. By Theorem 1.2 the arc $\Pi_i \cap K$ can be extended to a $(q+2)$-arc C_i; let $C_i \cap l = \{N_i, P\}$ with $l \cap K = \{P\}$. For at least two planes Π_i, say Π_1 and Π_2, we have $N_1 = N_2$.

(B.1) **First we prove that N_1 is on a tangent of K not in $\Pi_1 \cup \Pi_2$; clearly N_1 is on at least $2q-5$ tangents of K contained in** $\Pi_1 \cup \Pi_2$. Assume the contrary. Then for any plane $\Pi_i \notin \{\Pi_1, \Pi_2\}$, the arc $\Pi_i \cap K$ must have an odd number of points. So $\Pi_i \cap K$ either is a $(q-1)$-arc or a $(q+1)$-arc, $i \in \{3, 4, ..., q+1\}$. Also, $N_i \neq N_1$ for $i = 3, 4, ..., q-1$. If $\Pi_i \cap K$ is a $(q-1)$-arc and $C_i \setminus (\Pi_i \cap K) = \{N_i, N_i', N_i''\}$, $i \in \{3, 4, ..., q+1\}$, then $N_1 \in N_i'N_i''$, as otherwise N_1N_i' and N_1N_i'' are tangents of $\Pi_i \cap K$.

Let r be the number of planes Π_i, with $i \neq 1, 2$, for which $\Pi_i \cap K$ is a $(q-1)$-arc, and let s be the number of planes Π_i, with $i \neq 1, 2$, for which $\Pi_i \cap K$ is a $(q+1)$-arc. The number of points of K is at least

(29) $$r(q-2) + sq + 2(q-3) + 1, \text{with } r + s = q - 1.$$

As K is complete, by Theorem 2.1

(30) $$r(q-2) + (q-1-r)q + 2(q-3) + 1 \leq q^2 - q + 3,$$

so

(31) $$r \geq q - 4.$$

We may assume that $\Pi_3 \cap K$ is a $(q-1)$-arc. The number of tangents of K containing N_3 is at least

(32) $$q - 1 + 2(r-1) \geq q - 1 + 2q - 10 = 3q - 11.$$

Hence

(33) $$k \leq q^2 + q + 2 - 3q + 11 = q^2 - 2q + 13.$$

So

(34) $$q^2 - 2q + 13 > q^2 + (1-\sqrt{5})q + 5,$$

a contradiction for $q > 8$.

Consequently N_1 is on a tangent l' of K not in $\Pi_1 \cup \Pi_2$.

(B.2) **Now we consider all planes Π_i' containing the tangent l', with $i = 1, 2, ..., q+1$. We will show that:**
(a) For each plane Π_i' such that $|\Pi_i' \cap K| \geq q-2$ the point N_1 does not extend the arc $\Pi_i' \cap K$.
(b) For each i we have $|\Pi_i' \cap K| \geq q-2$.

(a) Let $|\Pi_1 \cap K| = \alpha, q-2 \leq \alpha \leq q+1, |\Pi_2 \cap K| = \beta, q-2 \leq \beta \leq q+1$. Then N_1 is contained in at least $\alpha + \beta$ tangents of K. Now we consider all planes Π_i' containing the tangent l', with $i = 1, 2, ..., q+1$. Assume, by way of contradiction, that $m = |\Pi_i' \cap K| \geq q-2$ and that the $(q+2)$-arc C_i' extending $\Pi_i' \cap K$ intersects l' in $\{N_1, P'\}$, with $l' \cap K = \{P'\}, i \in \{1, 2, ..., q+1\}$. Then the number of tangents of K containing N_1 is at least

(35) $$\alpha + \beta + m - 3 \geq 2q - 4 + m - 3 \geq 3q - 9.$$

Hence

(36) $$k \leq q^2 + q + 2 - 3q + 9 = q^2 - 2q + 11.$$

So $q^2 - 2q + 11 > q^2 + (1-\sqrt{5})q + 5$, a contradiction. Consequently for $|\Pi_i' \cap K| \geq q-2$ we have $N_1 \notin C_i', i \in \{1, 2, ..., q+1\}$.

(b) Next, assume by way of contradiction that for at least one plane Π_i' containing l', say Π_1', we have $|\Pi_1' \cap K| \leq 4$. Let Π_2' be the plane ll'. Now we count the points of K in the planes Π_i', with $i = 1, 2, ..., q+1$. Let
θ_1 be the number of planes $\Pi_i', i \in \{3, 4, ..., q+1\}$, containing a tangent of $\Pi_1 \cap K$ through N_1 and a tangent of $\Pi_2 \cap K$ through N_1,
θ_2 be the number of planes $\Pi_i', i \in \{3, 4, ..., q+1\}$, containing a tangent of $\Pi_1 \cap K$ through N_1, but no tangent of $\Pi_2 \cap K$ through N_1,
θ_3 be the number of planes $\Pi_i', i \in \{3, 4, ..., q+1\}$, containing a tangent of $\Pi_2 \cap K$ through N_1, but no tangent of $\Pi_1 \cap K$ through N_1,
θ_4 be the number of planes $\Pi_i', i \in \{3, 4, ..., q+1\}$, containing no one of the tangents of $\Pi_1 \cap K$ or $\Pi_2 \cap K$ through N_1.
Then, as $N_1 \notin C_i'$ for $|\Pi_i' \cap K| \geq q-2$, we have
(37)
$$k \leq 4 + q - 1 + \theta_1(q-2) + \theta_2(q-1) + \theta_3(q-1) + \theta_4 q, \text{with } 2 + \theta_1 + \theta_2 + \theta_3 + \theta_4 = q+1.$$

Hence

(38) $$k \leq q(\theta_1 + \theta_2 + \theta_3 + \theta_4) - (2\theta_1 + \theta_2 + \theta_3) + q + 3,$$

so

(39) $$k \leq q(q-1) - (2\theta_1 + \theta_2 + \theta_3) + q + 3.$$

Now we have
$\theta_1 + \theta_2 \geq |\Pi_1 \cap K| - 2 \geq q - 4,$

$\theta_1 + \theta_3 \geq |\Pi_2 \cap K| - 2 \geq q - 4$.
Hence

(40) $$k \leq q(q-1) - 2q + 8 + q + 3 = q^2 - 2q + 11.$$

So $q^2 - 2q + 11 > q^2 + (1 - \sqrt{5})q + 5$, a contradiction.

Hence no plane Π'_i containing l' intersects K in a m-arc, with $m \leq 4, 1 \leq i \leq q+1$. Consequently, for each plane Π'_i containing l' we have $|\Pi'_i \cap K| \geq q - 2$. Also, we know that the $(q+2)$-arc C'_i extending $\Pi'_i \cap K$ does not contain N_1, with $i = 1, 2, ..., q+1$.

(B.3) **A final contradiction will be obtained by considering the possible intersections** $\Pi'_i \cap K, i = 1, 2, ..., q+1$. It is easy to see that at least $q - 6$ planes Π'_i containing l' intersect K in a m-arc having at least 3 tangents containing N_1; these planes are the planes containing l' passing through distinct tangents of $\Pi_1 \cap K$ and $\Pi_2 \cap K$ containing N_1. For any such plane Π'_i the arc $\Pi'_i \cap K$ is either a $(q-1)$-arc or a $(q-2)$-arc. Let
θ'_1 be the number of planes Π'_i, with $\Pi'_i \neq ll'$, containing a tangent of $\Pi_1 \cap K$ through N_1, a tangent of $\Pi_2 \cap K$ through N_1, where $\Pi'_i \cap K$ is a $(q-1)$-arc,
θ'_2 be the number of planes Π'_i, with $\Pi'_i \neq ll'$, containing a tangent of $\Pi_1 \cap K$ through N_1, a tangent of $\Pi_2 \cap K$ through N_1, where $\Pi'_i \cap K$ is a $(q-2)$-arc.
Let $C'_i \cap l' = \{P', N'_i\}$, with $l' \cap K = \{P'\}$ and C'_i the $(q+2)$-arc extending $\Pi'_i \cap K, i = 1, 2, ..., q+1$. Then $N'_i \neq N_1, i = 1, 2, ..., q+1$. We may assume that $N'_1 = N'_2$. Assume, by way of contradiction, that $N'_1 = N'_2 = N'_i$, with $i \in \{3, 4, ..., q+1\}$. Then N'_1 is on at least $3(q-3)+1$ tangents of K. So

(41) $$k \leq q^2 + q + 2 - 3q + 8 = q^2 - 2q + 10.$$

Hence

(42) $$q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 10,$$

that is,

(43) $$(3 - \sqrt{5})q < 5,$$

clearly a contradiction. Hence we may assume that $N'_1 = N'_2, N'_3 = N'_4, N'_1 \neq N'_3, N'_i \notin \{N'_1, N'_3\}$ for $i = 5, 6, ..., q+1$. At least $\theta'_1 - 4$ of the arcs $\Pi'_5 \cap K, \Pi'_6 \cap K, ..., \Pi'_{q+1} \cap K$ are $(q-1)$-arcs, say $\Pi'_5 \cap K, \Pi'_6 \cap K, ..., \Pi'_{\theta'_1} \cap K$ are $(q-1)$-arcs. The number of tangents of $\Pi'_i \cap K$ containing N'_j , with $j \in \{1, 3\}$, is either 1 or 3, with $i = 5, 6, ..., \theta'_1$; if N'_j is contained in one tangent of $\Pi'_i \cap K$, then N'_u is contained in 3 tangents of $\Pi'_i \cap K$, with $\{j, u\} = \{1, 3\}$ and $i \in \{5, 6, ..., \theta'_1\}$. So we may assume that at least $(\theta'_1 - 4)/2$ of the $(q-1)$-arcs $\Pi'_i \cap K, i = 5, 6, ..., \theta'_1$, have 3 tangents containing N'_1. Counting the tangents of K through N'_1, we obtain at least

(44) $$1 + (\theta'_1 - 4) + (\theta'_2 - 2) + 2(q - 3)$$

tangents. As $\theta'_1 + \theta'_2 \geq q - 6$, this number of tangents is at least $1 + q - 6 - 6 + 2q - 6 = 3q - 17$. Hence

(45) $$k \leq q^2 + q + 2 - 3q + 17 = q^2 - 2q + 19.$$

So

(46) $$q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q + 19,$$

or

$$(3-\sqrt{5})q < 14, \tag{47}$$

a contradiction for $q > 16$.

If at least one of the arcs $\Pi'_1 \cap K, \Pi'_2 \cap K$ is a m-arc with $m > q-2$, then (44) becomes

$$1 + (\theta'_1 - 4) + (\theta'_2 - 1) + (q-3) + (q-2), \tag{48}$$

which is at least $3q-15$. Hence $k \leq q^2-2q+17$. For $q=16$ this gives $k \leq 241$. But for $q = 16$ the inequality $k > q^2 + (1-\sqrt{5})q + 5$ yields $k \geq 242$, a contradiction.

Finally we assume that $\Pi'_1 \cap K$ and $\Pi'_2 \cap K$ are $(q-2)$-arcs. Then at least $\theta'_1 - 2$ of the arcs $\Pi'_i \cap K$, with $i = 5, 6, ..., q+1$, are $(q-1)$-arcs, say $\Pi'_5 \cap K, \Pi'_6 \cap K, ..., \Pi'_{\theta'_1+2} \cap K$. So at least $(\theta'_1 - 2)/2$ of the $(q-1)$-arcs $\Pi'_i \cap K$, with $i = 5, 6, ..., \theta'_1 + 2$, have 3 tangents containing either N'_1 or N'_3. First, assume that this is the case for N'_3. If at least one of the arcs $\Pi'_3 \cap K, \Pi'_4 \cap K$ is a m-arc with $m > q-2$, then the number of tangents of K containing N'_3 is at least

$$1 + (\theta'_1 - 2) + (\theta'_2 - 1) + (q-3) + (q-2), \tag{49}$$

which is at least $3q - 13$. Hence $k \leq q^2 - 2q + 15$, and so $q^2 + (1-\sqrt{5})q + 5 < q^2 - 2q + 15$, that is, $(3-\sqrt{5})q < 10$, a contradiction. Hence the arcs $\Pi'_3 \cap K$ and $\Pi'_4 \cap K$ are $(q-2)$-arcs. Then the number of tangents of K containing N'_3 is at least

$$1 + \theta'_1 + (\theta'_2 - 2) + 2(q-3), \tag{50}$$

which is at least $3q - 13$. This yields again a contradiction. Consequently at least $(\theta'_1 - 2)/2$ of the $(q-1)$-arcs $\Pi'_i \cap K$, with $i = 5, 6, ..., \theta'_1 + 2$, have 3 tangents containing N'_1. But then in (44) $\theta'_1 - 4$ may be replaced by $\theta'_1 - 2$, yielding at least $3q - 15$ tangents of K containing N'_1. Hence $k \leq q^2 - 2q + 17$, which is a final contradiction.

We conclude that each tangent l of K is contained in at least one plane intersecting K in at most four points.

(C) **Assume, by way of contradiction, that there is a tangent l of K which is contained in at least two planes Π_1, Π_2 intersecting K in a m-arc, with $m \leq 4$.**

Assume that $l \cap K = \{P\}$ and that $\Pi_1 \cup \Pi_2$ contains $2q + \delta$ tangents of K through P. We have $-5 \leq \delta \leq 1$.

(C.1) **Here we will show that $2q + \delta$ is the total number of tangents of K containing P; as a corollary it will follow that $k \in \{q^2-q+1, q^2-q+2, q^2-q+3\}$.** Assume, by way of contradiction, that there is a tangent l' of K containing P with $l' \not\subset \Pi_1 \cup \Pi_2$. If $|ll' \cap K| \leq 4$, then the number of tangents of K containing P is at least $2q + \delta + q - 3 = 3q + \delta - 3 \geq 3q - 8$, so $k \leq q^2 + q + 2 - 3q + 8 = q^2 - 2q + 10$. Hence

$$q^2 + (1-\sqrt{5})q + 5 < q^2 - 2q + 10, \tag{51}$$

or $(3-\sqrt{5})q < 5$, a contradiction. Now we consider all planes containing l'. By (B) at least one of these planes intersects K in a m-arc, with $m \leq 4$. If at least two planes containing l' intersect K in at most 4 points, then P is contained in at least

$2q-5+2(q-5)+1 = 4q-14$ tangents of K. Hence $k \leq q^2+q+2-4q+14 = q^2-3q+16$, so

$$q^2+(1-\sqrt{5})q+5 < q^2-3q+16, \tag{52}$$

that is, $(4-\sqrt{5})q < 11$, clearly a contradiction. Consequently exactly one plane Π' containing l' intersects K in at most 4 points. Now we count the points of K in the planes containing l'. Let

θ_1 be the number of planes , distinct from ll' and Π', containing l', containing a tangent of K in Π_1 and containing a tangent of K in Π_2,

θ_2 be the number of planes containing l', distinct from Π', containing a tangent of K in Π_1 and containing no tangent of K in Π_2,

θ_3 be the number of planes containing l', distinct from Π', containing a tangent of K in Π_2 and containing no tangent of K in Π_1,

θ_4 be the number of planes , distinct from Π', containing l' and containing no tangent of K in Π_1 or Π_2.

Then

$$k \leq 1+(q-1)+\theta_1(q-2)+\theta_2(q-1)+\theta_3(q-1)+\theta_4 q+3, \tag{53}$$

with

$$\theta_1+\theta_2+\theta_3+\theta_4 = q-1 \text{and } \theta_2+\theta_3+2\theta_4 \leq 6. \tag{54}$$

So

$$k \leq q+3+(q-1-\theta_2-\theta_3-\theta_4)(q-2)+\theta_2(q-1)+\theta_3(q-1)+\theta_4 q, \tag{55}$$

that is,

$$k \leq q^2-2q+5+(\theta_2+\theta_3+2\theta_4), \tag{56}$$

hence

$$k \leq q^2-2q+5+6 = q^2-2q+11. \tag{57}$$

Consequently

$$q^2+(1-\sqrt{5})q+5 < q^2-2q+11, \tag{58}$$

or $(3-\sqrt{5})q < 6$, a contradiction.

It follows that $2q+\delta$ is the total number of tangents of K containing P and so $k = q^2+q+2-2q-\delta = q^2-q+2-\delta$. As $k \leq q^2-q+3$ by Theorem 2.1, we have $-1 \leq \delta \leq 1$.

(C.2) **Some further properties of** K. Let l'' be any tangent of K not containing P and let $K \cap l'' = \{P'\}$. By (B) l'' is contained in a plane Π'' with $|\Pi'' \cap K| \leq 4$. There is a tangent n of K at P' not contained in Π''. The tangent n is contained in a plane ρ with $|\rho \cap K| \leq 4$. Let $2q+\delta'$ be the number of tangents of K at P' in $\rho \cup \Pi''$. Then $\delta' \leq \delta$ and if $\rho \cap \Pi''$ is a tangent, then by the foregoing section we have $\delta' = \delta$. Assume, by way of contradiction, that $\rho \cap \Pi''$ is not a tangent of K and that $\delta' < \delta$. Then there is a tangent n' of K at P' not contained in $\rho \cup \Pi''$. The tangent n' is contained in a plane ρ' with $|\rho' \cap K| \leq 4$. If $\rho \cap \rho'$ is a tangent of K, then the $2q+\delta$ tangents of K at P' are contained in $\rho \cup \rho'$, a contradiction. So $\rho \cap \rho'$ is not a tangent; similarly $\rho' \cap \Pi''$ is not a tangent. Hence the number of tangents of K at P' is at least $3(q-2)$, so $2q+\delta \geq 3q-6$, hence $\delta \geq q-6$, a contradiction. We conclude that $\delta' = \delta$ and that all tangents of K at P' are contained in $\rho \cup \Pi''$.

Hence, given any point $Q \in K$ there are two planes α_1 and α_2 containing all tangents of K at Q; also $|\alpha_1 \cap K| \leq 4$ and $|\alpha_2 \cap K| \leq 4$. These planes are uniquely defined by Q, and so is $\alpha_1 \cap \alpha_2$. By Section (A) the line $\alpha_1 \cap \alpha_2$ is a tangent of K at Q. Let $\widetilde{\Pi}$ be any plane containing Q, with $\widetilde{\Pi} \notin \{\alpha_1, \alpha_2\}$. Then $\widetilde{\Pi} \cap K$ contains at most two tangents at Q, so $|\widetilde{\Pi} \cap K| \geq q$. It follows that K contains no $(q-2)$-arcs and no $(q-1)$-arcs.

Notice that $|\alpha_1 \cap K| + |\alpha_2 \cap K| + \delta = 3$ and remind that $-1 \leq \delta \leq 1$.

Let $\widetilde{\Pi}$ be a plane containing Q, with $\widetilde{\Pi} \notin \{\alpha_1, \alpha_2\}$. The arc $\widetilde{\Pi} \cap K$contains always at least one tangent of K at Q, except when $\delta = -1, k = q^2 - q + 3, |\alpha_1 \cap K| = |\alpha_2 \cap K| = 2$. So if $k \in \{q^2 - q + 1, q^2 - q + 2\}$ and if $k = q^2 - q + 3$ with $|\alpha_1 \cap K| = |\alpha_2 \cap K| + 2 = 3$ or $|\alpha_2 \cap K| = |\alpha_1 \cap K| + 2 = 3$, then $\widetilde{\Pi} \cap K$ is not a $(q+2)$-arc. If $|\alpha_1 \cap K| = |\alpha_2 \cap K| = 2, k = q^2 - q + 3$, then there is excactly one plane $\widetilde{\Pi}$ containing Q for which $\widetilde{\Pi} \cap K$ is a $(q+2)$-arc.

(C.3) $\underline{k = q^2 - q + 1}$

Then $\delta = 1$ and $|\Pi_1 \cap K| = |\Pi_2 \cap K| = 1$. Let $U_1, U_2 \in K$, with $U_1 \neq U_2$, and let ξ_1, ξ_2 be the planes containing U_1 intersecting K in at most 4 points. If $U_2 \in \xi_1 \cup \xi_2$, then $\delta \leq 0$, a contradiction. Hence $U_2 \notin \xi_1 \cup \xi_2$. Consequently any plane containing the line U_1U_2 has more than 4 points in common with K.

Now we count the points of K in planes containing the line U_1U_2. Let θ_1 be the number of planes containing U_1U_2 intersecting K in a q-arc, and let θ_2 be the number of planes containing U_1U_2 intersecting K in a $(q+1)$-arc. Then

(59) $$\theta_1(q-2) + \theta_2(q-1) + 2 = q^2 - q + 1, \text{with } \theta_1 + \theta_2 = q + 1.$$

So

(60) $$\theta_1(q-2) + (q + 1 - \theta_1)(q-1) + 2 = q^2 - q + 1,$$

that is $\theta_1 = q$ and $\theta_2 = 1$.

Now we count the number of $(q+1)$-arcs on K, and obtain

(61) $$\frac{(q^2 - q + 1)(q^2 - q)}{(q+1)q}.$$

So $q + 1 | (q^2 - q + 1)(q - 1)$, so $q + 1 | 6$, a contradiction.

(C.4) $\underline{k = q^2 - q + 2}$

Then $\delta = 0$ and $\{|\Pi_1 \cap K|, |\Pi_2 \cap K|\} = \{1, 2\}$. Let Q be any point of K and let l_Q be the tangent of K which is the intersection of the two planes α_1 and α_2 containing the $2q$ tangents of K at Q. Let $(\alpha_1 \cup \alpha_2) \cap K = \{Q, Q'\}$. Starting with Q' and $l_{Q'}$, we find the same pair $\{Q', Q\}$. It follows that K is partitioned into pairs of type $\{Q, Q'\}$. Let $\mathcal{L}$ be the set of these $(q^2 - q + 2)/2$ pairs.

Let $\{Q, Q'\} \in \mathcal{L}$, let α_1 and α_2 be the planes containing the $2q$ tangents of K at Q, and assume that $Q' \in \alpha_1$. Then $\alpha_1 = l_Q l_{Q'}$. Let Π be a plane containing QQ', distinct from α_1. As Π contains a tangent of K at Q, we have $|\Pi \cap K| \leq q + 1$. Counting the points of K in the planes containing QQ', we obtain $|\Pi \cap K| = q + 1$. By an easy counting we see that the planes containing l_Q, but distinct from α_1 and α_2, intersect K in $(q+1)$-arcs. This way there arise $q-1$ $(q+1)$-arcs $K_1, K_2, ..., K_{q-1}$, having kernels $N_1, N_2, ..., N_{q-1}$ on $l_Q \setminus \{Q\}$. Assume, by way of contradiction, that

$N_i = N_j, i \neq j$ and $i, j \in \{1, 2, ..., q-1\}$. Then N_i is on at least $2q+1$ tangents of K, hence $k \leq q^2 - q + 1$, a contradiction. Let $l_Q \setminus \{N_1, N_2, ..., N_{q-1}\} = N_Q$.

Assume, by way of contradiction, that $l_Q \cap l_{Q'} \neq N_Q$. Let $l_Q \cap l_{Q'} = N_i, i \in \{1, 2, ..., q-1\}$, and let $R \in K_i \setminus \{Q\}$. Then $l_{Q'}R \cap K$ is a $(q+1)$-arc with kernel N_i. Hence N_i is on at least q^2+2 tangents, a contradiction. Consequently $l_Q \cap l_{Q'} = N_Q$; similarly, $l_Q \cap l_{Q'} = N_{Q'}$.

Assume, by way of contradiction, that $l_Q \cap l_S \neq \emptyset$, with $Q \neq S$ and $\{Q, S\} \notin \mathcal{L}$. Let $\{Q, Q'\}$ and $\{S, S'\}$ be elements of $\mathcal{L}$. Now we count the number of tangents of K containing $l_Q \cap l_S = M$. The arc $l_Q l_S \cap K$ is a $(q+1)$-arc with kernel M, so $l_Q l_S$ contains $q+1$ tangents of K through M; the arc $l_Q S' \cap K$ is a $(q+1)$-arc, and as the line MS' of the plane $l_S S'$ is a tangent of K, the point M is the kernel of $l_Q S' \cap K$, so $l_Q S'$ contains $q+1$ tangents of K through M; similarly the plane $l_S Q'$ contains $q+1$ tangents of K through M. Hence M is contained in more than $2q$ tangents of K, clearly a contradiction. It follows that if $l_Q \cap l_S \neq \emptyset$, with $Q \neq S$, then $\{Q, S\} \in \mathcal{L}$.

Let $\{Q, S\} \notin \mathcal{L}$, with Q and S distinct points of K. Then $l_Q \cap l_S = \emptyset$. Now we count the points of K in the planes containing the line QS. Let θ_1 be the number of planes which contain QS and intersect K in a q-arc, and let θ_2 be the number of planes which contain QS and intersect K in a $(q+1)$-arc. Hence

$$(62) \qquad \theta_1(q-2) + \theta_2(q-1) + 2 = q^2 - q + 2, \text{with } \theta_1 + \theta_2 = q + 1.$$

So $\theta_1 = q - 1$ and $\theta_2 = 2$. The 2 planes containing QS and intersecting K in a $(q+1)$-arc are the planes $l_Q S$ and $l_S Q$.

Let $\{Q, S\} \in \mathcal{L}$ and let $l_Q \cap l_S = N$. Then N is kernel of no one of the $q-1$ $(q+1)$-arcs defined by planes containing the tangent l_Q and of no one of the $q-1$ $(q+1)$-arcs defined by planes containing the tangent l_S. So for any line $n \notin \{l_Q, l_S\}$ containing N we have $|n \cap K| \in \{0, 2\}$. Let $n \cap K = \{U, U'\}$.

First, assume that $\{U, U'\} \notin \mathcal{L}$. Then $|l_U U' \cap K| = |l_{U'} U \cap K| = q + 1$. As $|l_Q U \cap K| = |l_S U \cap K| = q+1$, the planes $l_Q U$ and $l_S U$ are the two planes containing UU' and intersecting K in a $(q+1)$-arc. Hence $\{l_Q U, l_S U\} = \{l_U U', l_{U'} U\}$. So we may assume that $l_Q U = l_U U'$ and $l_S U = l_{U'} U$. Consequently $l_Q \cap l_U \neq \emptyset$ and $l_S \cap l_{U'} \neq \emptyset$, that is, $\{Q, U\} \in \mathcal{L}$ and $\{S, U'\} \in \mathcal{L}$. Hence $|l_Q l_U \cap K| = |l_S l_{U'} \cap K| = 2$, clearly a contradiction as $Q, U, U' \in l_Q l_U$.

It follows that $\{U, U'\} \in \mathcal{L}$. So for any pair $\{T, T'\} \in \mathcal{L}$, with $\{T, T'\} \neq \{Q, S\}$, we have $N \in TT'$. Let n', n'' be distinct lines containing N with $n' \neq n \neq n''$ and $n', n'' \notin \{l_Q, l_S\}$. Assume also that $n' \cap K = \{V, V'\}$ and $n'' \cap K = \{W, W'\}$. Then $\{V, V'\} \in \mathcal{L}$ and $\{W, W'\} \in \mathcal{L}$. By the foregoing the lines VV', WW', QS contain N, clearly a contradiction.

(C.5) $\underline{k = q^2 - q + 3}$
Let P be any point of K and let l_P be the tangent of K which is the intersection of the two planes Π_1, Π_2 containing the $2q-1$ tangents of K at P. Two cases are considered.

(C.5.1) $\underline{\Pi_1 \cap K = \{P, P', P''\}, \Pi_2 \cap K = \{P\}}$
Then K contains no plane $(q+2)$-arcs containing P. Let l be a tangent of K at P, with l in Π_1 and $l \neq l_P$. We count the points of K in planes containing l. Let θ_1 be

the number of planes containing l and intersecting K in a $(q+1)$-arc, and let θ_2 be the number of planes containing l and intersecting K in a q-arc. Then

(63) $$\theta_1 q + \theta_2(q-1) + 3 = q^2 - q + 3, \text{with } \theta_1 + \theta_2 = q.$$

Hence $\theta_1 = 0$ and $\theta_2 = q$. Let $\widetilde{\Pi}_1, \widetilde{\Pi}_2, ..., \widetilde{\Pi}_q$ be the planes containing l and intersecting K in a q-arc, let $\widetilde{\Pi}_i \cap K = K_i$, let C_i be the $(q+2)$-arc extending K_i and let $C_i \cap l = \{P, N_i\}$, with $i = 1, 2, ..., q$. Assume that for some $i \in \{1, 2, ..., q\}$ we have $N_i \notin P'P''$. The number of tangents of K containing N_i is at least

(64) $$q + (q-1) + 2 = 2q + 1,$$

a contradiction. Hence $N_1 = N_2 = \cdots = N_q = l \cap P'P''$. Then the number of tangents of K containing N_1 is at least

(65) $$q(q-1) + 1 = q^2 - q + 1,$$

again a contradiction.

(C.5.2) $\underline{\Pi_1 \cap K = \{P, P'\}, \Pi_2 \cap K = \{P, P''\}}$
By (C.5.1), for each point $Q \in K$ the two planes α_1, α_2 through Q intersecting K in at most four points, intersect K in exactly two points. If $\alpha_1 \cap K = \{Q, Q'\}$ and $\alpha_2 \cap K = \{Q, Q''\}$, then the plane $QQ'Q''$ is the only plane on Q intersecting K in a $(q+2)$-arc. Hence the $(q+2)$-arcs on K partition K. So

(66) $$q+2 | q^2 - q + 3, \text{so } q+2 | q-7, \text{so } q+2 | 9,$$

a contradiction.

Now the theorem is proved. ■

4. Corollaries

We are grateful to T. Szőnyi for bringing reference [12] to our attention which, in combination with Theorem 1.6, gives the following considerable improvement of the bound in Theorem 1.6; see also Remark 4.4.

Theorem 4.1.

(67) $$m_2'(3,q) < q^2 - 2q + 3\sqrt{q} + 2, q \textit{ even}, q \geq 2048.$$

Proof. In [12] it is proved that there does not exist a complete k-cap in $\mathbf{PG}(3,q)$, q even, $q \geq 64$, for which

(68) $$k \in [q^2 - (a-1)q + a\sqrt{q} + 2 - a + \frac{a(a-1)}{2}, q^2 - (a-2)q - a^2\sqrt{q}]$$

where a is an integer which satisfies

(69) $$2 \leq a \leq \frac{-2\sqrt{q} + 3 + \sqrt{16q\sqrt{q} + 12q - 44\sqrt{q} - 7}}{4\sqrt{q} + 2}.$$

Putting $a = 3$, the desired result immediately follows from Theorem 1.6. ■

Theorem 4.2. (i) $m_2(4,4) = 41$,
(ii) $m_2(4,8) \leq 479$,
(iii) $m_2(4,q) < q^3 - q^2 + 2\sqrt{5}q - 8$, *$q$ even*, $q > 8$.

Proof. For $q = 4$, see [5]. Assume, by way of contradiction, that K is a k-cap of $\mathbf{PG}(4, 8)$ with $k > 479$, or a k-cap of $\mathbf{PG}(4, q)$, q even and $q > 8$, with $k > q^3 - q^2 + 2\sqrt{5}q - 8$. At each of its points the cap K has $t = q^3 + q^2 + q + 2 - k$ tangents. Hence we assume that $t < 107$ for $q = 8$ and $t < 2q^2 + (1 - 2\sqrt{5})q + 10$ for $q > 8$. We obtain a contradiction in several stages.

I K **contains no plane** q**-arc**
Similar to the reasoning in Section I in the proof of Theorem 6.27 in [9].

II **There exists no solid** δ **such that** $q^2 + 1 > |\delta \cap K| > q^2 + (1 - \sqrt{5})q + 5$
Suppose δ exists. Let $\delta \cap K = K'$. Then K' can be completed to an ovoid O of δ, by Theorem 3.1. Let $N \in O \setminus K'$ and let $N' \in K'$. Consider the $q + 1$ planes of δ through NN'. Since each of these planes meets O in a $(q + 1)$-arc, each plane meets K' in at most a q-arc. By I, there is no q-arc on K; so each plane meets K' in at most a $(q - 1)$-arc.

Assume, by way of contradiction, that none of these intersections is a $(q - 1)$-arc. Therefore a count of the points on K' gives

(70) $$|K'| \leq (q + 1)(q - 3) + 1,$$

whence

(71) $$q^2 + (1 - \sqrt{5})q + 5 < q^2 - 2q - 2,$$

so

(72) $$(3 - \sqrt{5})q + 7 < 0,$$

a contradiction.

So we may assume that for one of the planes δ through NN', say Π, we have $|\Pi \cap K'| = q - 1$. Now we consider all solids of $\mathbf{PG}(4, q)$ containing the plane Π. Let θ be the number of solids Π' for which $|\Pi' \cap K| > q^2 + (1 - \sqrt{5})q + 5$, so $q + 1 - \theta$ is the number of solids Π'' for which $|\Pi'' \cap K| < q^2 + (1 - \sqrt{5})q + 5$. We have $\theta \geq 1$.

First, assume $\theta \geq 2$. So there are at least two solids Π'_1, Π'_2 containing Π such that $|\Pi'_i \cap K| > q^2 + (1 - \sqrt{5})q + 5$, with $i = 1, 2$. By Theorem 3.1 $\Pi'_i \cap K$ can be completed to an ovoid O_i of $\Pi'_i, i = 1, 2$. So $O_i \cap \Pi$ is a $(q + 1)$-arc $(\Pi \cap K') \cup \{N'_i, N''_i\}, i = 1, 2$. Since $\Pi \cap K'$ can be contained in no more than three $(q + 1)$-arcs, contained in a common $(q + 2)$-arc, we have $|\{N'_1, N''_1\} \cap \{N'_2, N''_2\}| \geq 1$. Assume $N'_1 = N'_2$. So the number of tangents of K containing N'_1 is at least

(73) $$2(q^2 + (1 - \sqrt{5})q + 5 - q + 1) + q - 1,$$

so

(74) $$2q^2 + (1 - 2\sqrt{5})q + 11,$$

a contradiction.

Finally, assume that $\theta = 1$. Counting the points of K in the $q + 1$ solids, we obtain

(75) $$k < q(q^2 + (1 - \sqrt{5})q + 5 - q + 1) + (q^2 - 1),$$

that is,

(76) $$k < q^3 + (1 - \sqrt{5})q^2 + 6q - 1.$$

Hence, for $q > 8$,

(77) $$q^3 - q^2 + 2\sqrt{5}q - 8 < q^3 + (1 - \sqrt{5})q^2 + 6q - 1,$$

so

$$0 < (2-\sqrt{5})q^2 + (6-2\sqrt{5})q + 7, \tag{78}$$

a contradiction. For $q = 8$, there arises $479 < 479$, a contradiction.

III **For a point N not in K, there do not exist planes Π_1 and Π_2 such that $\Pi_1 \cap \Pi_2 = N$ and such that $\Pi_i \cap K$ is a $(q+1)$-arc with nucleus N for $i = 1, 2$**
Similar to the reasoning in Section III in the proof of Theorem 6.27 in [9].

IV **The tangents through any point Q off K lie in a solid**
Similar to the reasoning in Section IV in the proof of Theorem 6.27 in [9].

V **The final contradiction is obtained by counting the tangents of K**
Similar to the reasoning in Section V in the proof of Theorem 6.27 in [9]. ∎

Theorem 4.3. *For q even, $q > 2$, $n \geq 5$,*

(i) $m_2(n,4) \leq \frac{118}{3}.4^{n-4} + \frac{5}{3}$
(ii) $m_2(n,8) \leq 478.8^{n-4} - 2(8^{n-5} + \cdots + 8 + 1) + 1$,
(iii) $m_2(n,q) < q^{n-1} - q^{n-2} + 2\sqrt{5}q^{n-3} - 9q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1$, *for* $q > 8$.

Proof This follows directly from Theorem 1.1, Theorem 4.2 and Theorem 6.14(ii) in [9]. ∎

Remark 4.4. The bound in Theorem 4.1 leads to considerable improvements of Theorem 4.2 and Theorem 4.3. We just mention these bounds, but the proofs are the theme of a subsequent paper.

For q even, $q \geq 2048$,

$$m_2(4,q) < q^3 - 2q^2 + 3q\sqrt{q} + 8q - 9\sqrt{q} - 6. \tag{79}$$

For q even, $q \geq 2048$, $n \geq 5$,

$$m_2(n,q) < q^{n-1} - 2q^{n-2} + 3q^{n-3}\sqrt{q} + 8q^{n-3} - 9q^{n-4}\sqrt{q} - 7q^{n-4} - 2(q^{n-5} + \cdots + q + 1) + 1. \tag{80}$$

5. Remark

The bound in the MAIN THEOREM is better than the bound of Chao, see [3]. In 2014 Cao and Ou, see [2], published the bound $k < q^2 - 2q + 8$ (q even and $q \geq 128$), which is better than ours. I did not follow some reasoning in their proof, so I sent two mails to one of the authors explaining why I think Section 1.3 of the proof is not correct. Unfortunately I never received an answer.

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