Cuspidal ℓ-modular representations of p-adic classical groups

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Abstract

For a classical group over a non-archimedean local field of odd residual characteristic p, we construct all cuspidal representations over an arbitrary algebraically closed field of characteristic different from p, as representations induced from a cuspidal type. We also give a fundamental step towards the classification of cuspidal representations, identifying when certain cuspidal types induce to equivalent representations; this result is new even in the case of complex representations. Finally, we prove that the representations induced from more general types are quasi-projective, a crucial tool for extending the results here to arbitrary irreducible representations.

1 Introduction

In recent years, congruences between automorphic representations have assumed a central importance in number theory. This has led to the desire to understand representations of reductive p-adic groups on vector spaces over fields of positive characteristic ℓ. There are vast differences between the cases $\ell = p$ and $\ell \neq p$, with the latter sharing many similarities with the theory of complex representations, including the existence of a Haar measure. However, there are also many important and interesting differences between the $\ell \neq p$ theory and the theory for complex representations, including the presence of compact open subgroups of measure zero, the non-semisimplicity of smooth representations of compact open subgroups, and that cuspidal representations can and do appear as subquotients of parabolically induced representations (in fact, all of these phenomena are related). In this article we focus on the $\ell \neq p$ case, and work with an arbitrary algebraically closed field of characteristic ℓ or zero.

The theory of (smooth) representations of a general reductive p-adic group over such fields was developed by Vignéras in [25]. However, many subsequent articles and fundamental results (for example, the unicity of supercuspidal support) focus just on the general linear group. One of the main reasons that this group has been more accessible for a modular theory, is that the Bushnell–Kutzko classification of irreducible complex representations via types extends in a natural way to ℓ-modular representations, which is the subject of the final chapter of [ibid.]. This classification, in favourable circumstances, allows one to reduce a problem to an analogous question in associated finite groups where hopefully it is either tractable to the pursuer, or already known. Recently, this approach has been adopted for other groups: Sécherre and

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Mínguez in [16] for inner forms of GL_n; and the first author in [13] for unramified $\mathrm{U}(2,1)$. In this article, we pursue this approach for p-adic classical groups G over locally compact non-archimedean local fields with odd residual characteristic.

Of particular importance in this approach is the construction of all irreducible cuspidal complex representations of general linear groups as compactly induced representations. We accomplish this for ℓ-modular representations in our main results:

Theorem A (Theorems 11.1, 11.2)**.** There is an explicit list of *cuspidal types*, consisting of certain pairs (J,λ), with J a compact open subgroup of G and λ an irreducible R-representation of J such that

(i) the compactly induced representation $\mathrm{ind}_J^G\,\lambda$ is irreducible and cuspidal;

(ii) every irreducible cuspidal representation arises as in (i), for some cuspidal type (J,λ).

See below for a more precise definition of cuspidal type. For complex representations this is the main result of [24]. But here we do more, giving an initial refinement of this exhaustive list of cuspidal types. Part of the data used to define a cuspidal type is a family of *skew semisimple characters*. In the case where two cuspidal types are defined relative to the same family (see below for a more precisely-worded condition), we obtain the following *intertwining implies conjugacy* result:

Theorem B (Theorem 11.3)**.** Let (J_1,λ_1), (J_2,λ_2) be cuspidal types defined relative to the same family of skew semisimple characters. Then $\mathrm{ind}_{J_1}^G\,\lambda_1 \simeq \mathrm{ind}_{J_2}^G\,\lambda_2$ if and only if there exists $g\in G$ such that $J_1^g = J_2$ and $\lambda_1^g \simeq \lambda_2$.

Note that λ_1^g here denotes the representation of $J_1^g = g^{-1}J_1 g$ given by $\lambda_1^g(j) = \lambda_1(gjg^{-1})$, for $j\in J_1$. In forthcoming joint work with Skodlerack, this theorem will be combined with work of the second author and Skodlerack to prove an intertwining implies conjugacy result without the condition on the skew semisimple characters. We now give more details and explain our approach.

Let G be a p-adic classical group with p odd, that is (the points of) a unitary, symplectic or special orthogonal group defined over a locally compact non-archimedean local field F of residual characteristic p. Let $\beta\in\mathrm{Lie}\,G$ be a semisimple element, and put $G_E = C_G(\beta)$ the G-centraliser of β. Let Λ be an $\mathfrak{o}_F$-lattice sequence corresponding to a point in the Bruhat–Tits building of G_E. From β and Λ we get a set of *self-dual semisimple characters* θ_Λ of a group H^1_Λ; and given another lattice sequence Υ as above, there is a canonical *transfer map* giving a corresponding self-dual semisimple character θ_Υ of H^1_Υ. Also write J_Λ for the normaliser of θ_Λ in the (non-connected) parahoric subgroup of G corresponding to Λ, and J^1_Λ for its pro-p radical. There is a unique irreducible representation η_Λ of J^1_Λ which contains θ_Λ on restriction. Our first major diversion from the earlier results of the second author is:

Theorem C (Theorems 3.10 & 4.1)**.** With notation as above.

(i) The intertwining of θ_Λ with θ_Υ is $J_\Upsilon G_E J_\Lambda$.

(ii) The intertwining spaces of η_Λ with η_Υ are at most one dimensional; more precisely:

$$\dim_R \mathrm{Hom}_{J^1_\Lambda\cap (J^1_\Upsilon)^g}(\eta_\Lambda,\eta_\Upsilon^g) = \begin{cases} 1 & \text{if } g\in J_\Upsilon G_E J_\Lambda; \\ 0 & \text{otherwise.}\end{cases}$$

This theorem is an asymmetric generalisation of [23, Propositions 3.27 & 3.31] (*cf.* also [17]) which deals with the case $\Lambda = \Upsilon$. It appears possible, and indeed it is already hinted at in [4, 1.5.12], that one could prove such an intertwining result by developing the theory *ab initio*, with lattice sequences such as these rather than just a single lattice sequence. However, our approach is more brief and elegant, utilising a construction for semisimple characters to relate the case of not necessarily conjugate lattice sequences to the case of conjugate lattice sequences in a larger group. This construction is inspired by a similar one for simple strata, in work of the second author with Broussous and Sécherre [3].

The next step is to extend η_Λ to a suitable representation of J_Λ, called a *β-extension*, which is accomplished in Section 5. While we have to change the proofs of [24] here, the changes are straightforward. That the formation of covers, of [24] and [17], is still valid in positive characteristic is proved in Sections 8 and 9. Let κ_Λ be a β-extension of η_Λ. The quotient J_Λ/J^1_Λ is a product of finite reductive groups and we write J° or the inverse image of the connected component. Let τ be an irreducible representation of J_Λ/J^1_Λ with cuspidal restriction to $J^\circ_\Lambda/J^1_\Lambda$, and put $\lambda = \kappa_\Lambda \otimes \tau$ and $J = J_\Lambda$. We call the pair (J, λ) a *type*; and if the centraliser G_E has compact centre, and the corresponding (connected) parahoric subgroup $J^\circ_\Lambda \cap G_E$ is maximal, we call the pair (J, λ) a *cuspidal type*.

Finally, we are able to extend the main result of the second author in [24] to ℓ-modular representations (see Theorem A). Our approach to proving Theorem A is different to [24] at the top level of the construction, relying on a *reduction to level zero* argument (see Section 7). Thanks to our work in this paper on asymmetric intertwining of semisimple characters and Heisenberg representations, this new approach allows us to compare cuspidal representations in this exhaustive list whose semisimple characters are in the same family (i.e. are related by the transfer map), and make an initial refinement of the exhaustive list (see Theorem B).

We now mention further results we prove with future work in mind. In the ℓ-modular setting, compactly induced representations from types may not be projective. This provides an obstruction to following Bushnell–Kutzko's approach via covers to the admissible dual, as the category of representations containing a type (J, λ), will not in general be equivalent to the the category of right modules over the algebra $\mathrm{End}_G(\mathrm{ind}_J^G \lambda)$. Following Mínguez–Sécherre we construct covers on pro-p groups (Theorem 9.3); these will have the advantage of providing such an equivalence of categories to the category of modules over an algebra as above. It may be that this algebra will prove unwieldy for classification purposes, but it can be related to a similar algebra in depth zero. For general linear groups, promising initial results in this direction have recently been obtained by Chinello in his thesis [6], while Dat has begun a detailed study of the depth zero subcategory in [8]. Writing λ° for an irreducible component of the restriction of λ to J°, we thus show:

Theorem D (Theorem 10.2)**.** The representation $\mathrm{ind}_{J^\circ}^G \lambda^\circ$ is quasi-projective.

Thanks to work of Vignéras and Arabia [26], this implies that the irreducible quotients of $\mathrm{ind}_{J^\circ}^G \lambda^\circ$ are in bijection with the simple right modules of $\mathrm{End}_G(\mathrm{ind}_{J^\circ}^G \lambda^\circ)$, (see Section 2 for details). As any irreducible representation of G is a quotient of such an induced representation, this result is the starting point of an approach to classifying all irreducible ℓ-modular representations of G.

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2 Notation and background

Let F_0 be a non-archimedean local field of odd residual characteristic p and let F be either F_0 or a quadratic extension of F_0. Let $\overline{}$ denote the generator of $\mathrm{Gal}(F/F_0)$. If E is a non-archimedean local field we denote by $\mathfrak{o}_E$ the ring of integers of E, by $\mathfrak{p}_E$ the unique maximal ideal of $\mathfrak{o}_E$, by k_E the residue field and by q_E the cardinality of k_E. We write $\mathfrak{o}_0 = \mathfrak{o}_{F_0}$, and similarly abbreviate $\mathfrak{p}_0, k_0, q_0$. We fix a uniformizer ϖ_F of F such that $\overline{\varpi_F} = -\varpi_F$ if F/F_0 is ramified and $\overline{\varpi_F} = \varpi_F$ otherwise. We fix a character ψ_0 of the additive group F_0 with conductor $\mathfrak{p}_0$ and let $\psi_F = \psi_0 \circ \mathrm{Tr}_{F/F_0}$.

Let V be an N-dimensional F-vector space equipped with a non-degenerate ε-hermitian form $h : V \times V \to F$ with $\varepsilon = \pm 1$. Let $A = \mathrm{End}_F(V)$ and $\widetilde{G} = \mathrm{Aut}_F(V)$. The group $G^+ = \{g \in G : h(gv, gw) = h(v, w) \text{ for all } v, w \in V\}$ is the F_0-points of a unitary, symplectic or orthogonal algebraic group $\mathbb{G}^+$ defined over F_0. We let G denote the F_0-points of the connected component of $\mathbb{G}^+$ and call G a *classical group*. Hence the special orthogonal group is a classical group whereas the full orthogonal group is not.

Let $\overline{}$ denote the adjoint (anti)-involution induced on A by h and let $A^- = \{a \in A : a + \overline{a} = 0\} \simeq \mathrm{Lie}(G)$. Let σ denote both the involution on $\widetilde{G}$ defined by $\sigma : g \mapsto \overline{g}^{-1}$, for $g \in \widetilde{G}$, and its derivative $a \mapsto -\overline{a}$, for $a \in A$. Let Σ be the cyclic group of order two generated by σ. Then $G^+ = \widetilde{G}^{\Sigma}$ and $A^- = A^{\Sigma}$. We have $A = A^- \oplus A^+$ where $A^+ = \{a \in A : a - \overline{a} = 0\}$. We let $\psi_A = \psi_F \circ \mathrm{Tr}_{A/F}$. If S is a subset of A, we let $S^* = \{x \in A : \psi_A(xS) = 1\}$.

We let R denote an algebraically closed field of characteristic ℓ different from p, allowing the case $\ell = 0$. For any locally compact topological group H, we denote by $\mathfrak{R}_R(H)$ the category of smooth R-representations of H.

2.1 Representations and Hecke algebras

For general results on representations of reductive p-adic groups over an algebraically closed field of characteristic different from p, we refer to Vignéras's book [25].

Let G be a reductive p-adic group. Let K, K_1, K_2 be compact open subgroups of G, $(\tau, \mathcal{W})$ be a smooth R-representation of K, and $(\tau_i, \mathcal{W}_i)$ be smooth R-representations of K_i, for $i = 1, 2$. For $g \in G$, the *g-intertwining space* of τ_1 with τ_2 is defined to be the set

$$I_g(\tau_1, \tau_2) = \mathrm{Hom}_{K_1 \cap K_2^g}(\tau_1, \tau_2^g),$$

and the *intertwining* of τ_1 with τ_2 in G is

$$I_G(\tau_1, \tau_2) = \{g \in G : I_g(\tau_1, \tau_2) \neq 0\},$$

where $K_2^g = g^{-1} K_2 g$ and $\tau_2^g(x) = \tau_2(gxg^{-1})$ for $x \in K_2^g$. For an R-representation $(\pi, \mathcal{V})$ of a locally profinite group we denote by $(\pi^\vee, \mathcal{V}^\vee)$ its contragredient representation.

Remark 2.1. The motivation for this definition is provided by the following decomposition

$$\mathrm{Hom}_G(\mathrm{ind}_{K_1}^G(\tau_1), \mathrm{ind}_{K_2}^G(\tau_2)) \simeq \bigoplus_{K_2 \backslash I_G(\tau_1,\tau_2)/K_1} I_g(\tau_1, \tau_2),$$

by reciprocity and Mackey theory. Note that, if $K = K_1 = K_2$, $\tau = \tau_1 = \tau_2$ and $g \in G$, for complex representations or if K is pro-p, the spaces $I_g(\tau) = \mathrm{Hom}_{K \cap K^g}(\tau, \tau^g)$ and $\mathrm{Hom}_{K \cap {}^gK}(\tau, {}^g\tau)$ are the same, as representations of $K \cap {}^gK = K \cap K^g$ are semisimple, so in previous works one sees intertwining defined in either way.

Suppose that K_1 and K_2 are normal open subgroups of K. Let $\mathcal{H}(G, \tau_1, \tau_2)$ be the R-vector space of compactly supported functions $f : G \to \mathrm{Hom}_R(\mathcal{W}_1, \mathcal{W}_2)$ which transform on the left by τ_2 and on the right by τ_1. Let $\mathcal{H}(G, \tau) = \mathcal{H}(G, \tau, \tau)$ denote the R-algebra consisting of compactly supported functions $f : G \to \mathrm{End}_R(\mathcal{W})$ which transform on the left and the right by τ together with the convolution product

$$f_1 \star f_2(h) = \sum_{g \in G/K} f_1(g) f_2(g^{-1}h),$$

for $f_1, f_2 \in \mathcal{H}(G, \tau)$. This algebra has a unit element if the index of every open subgroup in K is invertible in R (*i.e.* the pro-order of K is invertible in R). The K-invariant bilinear pairing $\langle \ , \ \rangle$ on $\mathcal{W} \times \mathcal{W}^\vee$ induces an anti-isomorphism $\mathcal{H}(G, \tau) \to \mathcal{H}(G, \tau^\vee)$ by $f \mapsto f^\vee$ with $f^\vee$ defined by $\langle w, f^\vee(g^{-1})\check{w}\rangle = \langle f(g)w, \check{w}\rangle$ for all $w \in \mathcal{W}$, $\check{w} \in \mathcal{W}^\vee$. Under convolution $\mathcal{H}(G, \tau_1, \tau_2)$ has an $(\mathcal{H}(G, \tau_1), \mathcal{H}(G, \tau_2))$-bimodule structure. If $g \in G$, we let $\mathcal{H}(G, \tau_1, \tau_2)_g$ denote the subspace of all functions with support $K_1 g K_2$.

Under composition, $\mathrm{End}_G(\mathrm{ind}_K^G \tau)$ has an R-algebra structure and $\mathrm{Hom}_G(\mathrm{ind}_{K_1}^G \tau_1, \mathrm{ind}_{K_2}^G \tau_2)$ is an $(\mathrm{End}_G(\mathrm{ind}_{K_1}^G \tau_1), \mathrm{End}_G(\mathrm{ind}_{K_2}^G \tau_2))$-bimodule. The proof of the following Lemma follows from the proofs contained in [25, §8.5, 8.6, & 8.10].

Lemma 2.2. (i) We have an isomorphism of algebras

$$\mathcal{H}(G, \tau) \simeq \mathrm{End}_G(\mathrm{ind}_K^G \tau).$$

(ii) For $i = 1, 2$, we identify $\mathcal{H}(G, \tau_i)$ with $\mathrm{End}_G(\mathrm{ind}_{K_i}^G \tau_i)$ by (i). We have an isomorphism of $(\mathcal{H}(G, \tau_1), \mathcal{H}(G, \tau_2))$-bimodules

$$\mathcal{H}(G, \tau_1, \tau_2) \simeq \mathrm{Hom}_G(\mathrm{ind}_{K_1}^G \tau_1, \mathrm{ind}_{K_2}^G \tau_2).$$

(iii) For $i = 1, 2$, let H_i be compact open subgroups of G containing K_i. We have an isomorphism of $(\mathcal{H}(G, \tau_1), \mathcal{H}(G, \tau_2))$-bimodules

$$\mathcal{H}(G, \mathrm{ind}_{K_1}^{H_1} \tau_1, \mathrm{ind}_{K_2}^{H_2} \tau_2) \simeq \mathcal{H}(G, \tau_1, \tau_2),$$

which restricts to give isomorphisms of vector spaces, for $g \in G$,

$$\mathcal{H}(G, \mathrm{ind}_{K_1}^{H_1} \tau_1, \mathrm{ind}_{K_2}^{H_2} \tau_2)_g \simeq \coprod_{\substack{h \in H_1 \backslash G / H_2 \\ K_1 h K_2 = K_1 g K_2}} \mathcal{H}(G, \tau_1, \tau_2)_h.$$

2.2 Lattice sequences and parahoric subgroups

An $\mathfrak{o}_F$*-lattice sequence* in V is a function

$$\Lambda : \mathbb{Z} \to \{\mathfrak{o}_F\text{-lattices in } V\}$$

which is *decreasing*, that is $\Lambda(n+1) \subseteq \Lambda(n)$, for all $n \in \mathbb{Z}$, and *periodic*, that is, there exists a positive integer $e(\Lambda)$ such that $\Lambda(n + e(\Lambda)) = \varpi_F \Lambda(n)$, for all $n \in \mathbb{Z}$.

The ε-hermitian form h defines a duality on the set of $\mathfrak{o}_F$-lattices; given an $\mathfrak{o}_F$-lattice L we let $L^\sharp = \{v \in V : h(v, L) \subseteq \mathfrak{p}_F\}$. An $\mathfrak{o}_F$-lattice sequence Λ is called *self-dual* if $\Lambda(k)^\sharp = \Lambda(1-k)$, for all $k \in \mathbb{Z}$.

An $\mathfrak{o}_F$-lattice sequence Λ induces a decreasing filtration on A by $\mathfrak{o}_F$-lattices $\mathfrak{A}_n(\Lambda)$ in A where

$$\mathfrak{A}_n(\Lambda) = \{x \in A : x\Lambda(m) = \Lambda(m+n), m \in \mathbb{Z}\}, \text{ for } n \in \mathbb{Z}.$$

This filtration induces a *valuation* on A defined by

$$\nu_\Lambda(x) = \begin{cases} \sup\{n \in \mathbb{Z} : x \in \mathfrak{A}_n(\Lambda)\} & \text{if } x \in A\backslash\{0\}; \\ \infty & \text{if } x = 0. \end{cases}$$

If Λ is self-dual, it induces a decreasing filtration on A^- by $\mathfrak{o}_F$-lattices $\mathfrak{A}_n^-(\Lambda)$ in A^- where

$$\mathfrak{A}_n^-(\Lambda) = \mathfrak{A}_n(\Lambda) \cap A^-, \text{ for } n \in \mathbb{Z}.$$

We let

$$\widetilde{P}^n(\Lambda) = \begin{cases} \mathfrak{A}_n(\Lambda)^\times & \text{if } n = 0; \\ 1 + \mathfrak{A}_n(\Lambda) & \text{if } n > 0. \end{cases}$$

Then $\widetilde{P}(\Lambda) = \widetilde{P}^0(\Lambda)$ is a compact open subgroup of $\widetilde{G}$ and $\widetilde{P}^n(\Lambda)$, $n > 0$, is a decreasing filtration of $\widetilde{P}(\Lambda)$ by normal open subgroups. If Λ is self-dual then $P(\Lambda) = \widetilde{P}(\Lambda) \cap G$ (resp. $P^+(\Lambda) = \widetilde{P}(\Lambda) \cap G^+$) is a compact open subgroup of G (resp. G^+) which has a decreasing filtration of normal compact open subgroups $P^n(\Lambda) = \widetilde{P}^n(\Lambda) \cap G$, $n > 0$. We have a short exact sequence

$$1 \to P^1(\Lambda) \to P(\Lambda) \xrightarrow{\pi} M(\Lambda) \to 1$$

where $M(\Lambda)$ is the k_0-points of a reductive group $\mathbb{M}$ defined over k_0. Let $M^\circ(\Lambda)$ denote the k_0-points of the connected component of $\mathbb{M}$ and let $P^\circ(\Lambda)$ be the inverse image of $M^\circ(\Lambda)$ under π. We call the subgroups $\widetilde{P}(\Lambda)$ of $\widetilde{G}$ and $P^\circ(\Lambda)$ of G *parahoric subgroups.*

In fact, by [2] and [14], the filtrations of parahoric subgroups defined here, by considering different (self-dual) lattice sequences in the vector space V, coincide with the Moy–Prasad filtrations.

Let Λ be an $\mathfrak{o}_F$-lattice sequence in V. For integers $a, b \in \mathbb{Z}$, we let $a\Lambda+b$ be denote the $\mathfrak{o}_F$-lattice sequence in V defined by

$$a\Lambda + b(r) = \Lambda\left(\lfloor (r-b)/a \rfloor\right),$$

for all $r \in \mathbb{Z}$. The *affine class* of Λ, is the set of lattices of the form $a\Lambda + b$ with $a, b \in \mathbb{Z}, a \geqslant 1$.

2.3 Semisimple strata and characters

A *stratum* in A is a quadruple $[\Lambda, n, r, \beta]$ where Λ is an $\mathfrak{o}_F$-lattice sequence in V, $n, r \in \mathbb{Z}$ with $n \geqslant r \geqslant 0$, and $\beta \in \mathfrak{A}_{-r}(\Lambda)$. A stratum $[\Lambda, n, r, \beta]$ is called *self-dual* if Λ is self-dual and $\beta \in A^-$. Two strata $[\Lambda, n, r, \beta_1]$ and $[\Lambda, n, r, \beta_2]$ are called equivalent if $\beta_1 - \beta_2 \in \mathfrak{A}_{-r}(\Lambda)$. If $n \geqslant r \geqslant \frac{n}{2} > 0$, an equivalence class of strata corresponds to a character of $\widetilde{P}_{r+1}(\Lambda)$, by

$$[\Lambda, n, r, \beta] \mapsto \psi_\beta$$

where $\psi_\beta(x) = \psi_A(\beta(x-1))$ for $x \in \widetilde{P}_{r+1}(\Lambda)$, while an equivalence class of self-dual strata corresponds to a character of $P_{r+1}(\Lambda)$, by

$$[\Lambda, n, r, \beta] \mapsto \psi_\beta^- = \psi_\beta \,|_{P_{r+1}(\Lambda)}\,.$$

If $F[\beta]$ is a field then we let $B = C_A(\beta)$ be the A-centraliser of β, $\widetilde{G}_E = B^\times$, $\mathfrak{B}_k(\Lambda) = \mathfrak{A}_k(\Lambda) \cap B$ and $\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{A}_0(\Lambda) : \beta x - x\beta \in \mathfrak{A}_k(\Lambda)\}$. We say $[\Lambda, n, r, \beta]$ is a *zero* stratum if $n = r$

and $\beta = 0$ and we call $[\Lambda, n, r, \beta]$ *simple* if it is either zero or $F[\beta]$ is a field, Λ is an $\mathfrak{o}_E$-lattice sequence, $\nu_\Lambda(\beta) = -n < -r$ and $\mathfrak{n}_{-r}(\beta, \Lambda) \subset \mathfrak{B}_0(\Lambda) + \mathfrak{A}_1(\Lambda)$.

Suppose $V = \bigoplus_{i\in I} V^i$ is a decomposition of V into F-subspaces. We let $\Lambda^i = \Lambda \cap V^i$ and we let $\beta_i = \mathbf{e}^i \beta \mathbf{e}^i$, where $\mathbf{e}^i : V \to V^i$ is the projection with kernel $\bigoplus_{j\neq i} V^j$. The decomposition $V = \bigoplus_{i\in I} V^i$ of V is called a *splitting* of $[\Lambda, n, r, \beta]$ if $\beta = \sum_{i\in I} \beta_i$ and $\Lambda(k) = \bigoplus_{i\in I} \Lambda^i(k)$, for all $k \in \mathbb{Z}$. A stratum $[\Lambda, n, r, \beta]$ in A is called *semisimple* if it is zero or $\nu_\Lambda(\beta) = -n$ and there exists a splitting $\bigoplus_{i\in I} V^i$ for $[\Lambda, n, r, \beta]$ such that:

(i) for $i \in I$, the stratum $[\Lambda^i, q_i, r, \beta_i]$ in $\mathrm{End}_F(V^i)$ is simple, where

$$q_i = \begin{cases} r & \text{if } \beta_i = 0, \\ -\nu_{\Lambda^i}(\beta_i) & \text{otherwise;} \end{cases}$$

(ii) for $i, j \in I$ with $i \neq j$, the stratum $[\Lambda^i \oplus \Lambda^j, \max\{q_i, q_j\}, r, \beta_i + \beta_j]$ is not equivalent to a simple stratum in $\mathrm{End}_F(V^i \oplus V^j)$.

We write $E = F[\beta]$ and $E_i = F[\beta_i]$, hence $E = \bigoplus_{i\in I} E_i$ is a sum of fields. As in the case when E is a field, we write $B = C_A(\beta)$ and $\widetilde{G}_E = B^\times$. By abuse of notation, we will call a sum $\bigoplus_{i\in I} \Lambda_i$ of $\mathfrak{o}_{E_i}$-lattice sequences in V_i an $\mathfrak{o}_E$-lattice sequence in V. We write $\mathfrak{B}_k(\beta, \Lambda) = \mathfrak{A}_k(\Lambda) \cap B$ which gives the filtration on B by considering Λ as an $\mathfrak{o}_E$-lattice sequence. We write $\mathfrak{B}(\beta, \Lambda) = \mathfrak{B}_0(\beta, \Lambda)$, $\mathfrak{Q}(\beta, \Lambda) = \mathfrak{B}_1(\beta, \Lambda)$ and $\mathfrak{A}(\Lambda) = \mathfrak{A}_0(\Lambda)$.

Let $A^{ij} = \mathrm{Hom}_F(V^j, V^i)$ and $\mathcal{L} = \bigoplus_{i\in I} A^{ii}$, and write $\widetilde{L} = \mathcal{L}^\times = \prod_{i\in I} \widetilde{G}_i$, where $\widetilde{G}_i = \mathrm{Aut}_F(V^i)$. Also put $B_i = C_{A^{ii}}(\beta_i)$ and $\widetilde{G}_{E_i} = B_i^\times \subseteq \widetilde{G}_i$. Then $B = \bigoplus_{i\in I} B_i \subseteq \mathcal{L}$ and $\widetilde{G}_E = \prod_{i\in I} \widetilde{G}_{E_i} \subseteq \widetilde{L}$. We write Λ_E when we want to make it clear that we are considering Λ as an $\mathfrak{o}_E$-lattice sequence.

If $[\Lambda, n, 0, \beta]$ is a non-zero semisimple stratum we let

$$k_0(\beta, \Lambda) = -\min\{r \in \mathbb{Z} : [\Lambda, n, r, \beta] \text{ is not semisimple}\}$$

denote the *critical exponent* of $[\Lambda, n, 0, \beta]$ and $k_F(\beta) := \frac{1}{e(\Lambda)} k_0(\beta, \Lambda)$; by [23, §3.1], this is independent of Λ.

If $[\Lambda, n, r, \beta]$ is self-dual with associated splitting $V = \bigoplus_{i\in I} V^i$ then, for each $i \in I$, there exists a unique $\sigma(i) = j \in I$ such that $\overline{\beta_i} = -\beta_j$. We set $I_0 = \{i \in I : \sigma(i) = i\}$ and choose a set of representatives I^+ for the orbits of σ in $I \setminus I_0$. Then we let $I_- = \sigma(I_+)$ so that we have a disjoint union $I = I_+ \cup I_0 \cup I_-$.

A semisimple stratum $[\Lambda, n, r, \beta]$ is called *skew* if it is self-dual and the associated splitting $\bigoplus_{i\in I} V^i$ is orthogonal with respect to the ϵ-hermitian form h, *i.e.* $I = I_0$ in the notation above. In this case, we let $G_{E_i} = \widetilde{G}_{E_i} \cap G$ and $G_E = \prod_{i\in I} G_{E_i}$.

Associated to a semisimple stratum $[\Lambda, n, r, \beta]$ there are two $\mathfrak{o}_F$-orders $\mathfrak{H}(\beta, \Lambda)$ and $\mathfrak{J}(\beta, \Lambda)$ which are defined inductively in [23, §3.2]. These give rise to compact open subgroups $\widetilde{H}(\beta, \Lambda) = \mathfrak{H}(\beta, \Lambda) \cap \widetilde{P}(\Lambda)$ and $\widetilde{J}(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda) \cap \widetilde{P}(\Lambda)$ of $\widetilde{G}$ with decreasing filtrations $\widetilde{H}^i(\beta, \Lambda) = \mathfrak{H}(\beta, \Lambda) \cap \widetilde{P}_i(\Lambda)$ and $\widetilde{J}^i(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda) \cap \widetilde{P}_i(\Lambda)$, for $i \geqslant 1$ by compact open normal subgroups.

If $[\Lambda, n, r, \beta]$ is self-dual then the associated orders and groups are stable under the action of Σ and we write $\mathfrak{J}^-(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda) \cap A^-$, $J(\beta, \Lambda) = \widetilde{J}(\beta, \Lambda) \cap G$, $J^+(\beta, \Lambda) = \widetilde{J}(\beta, \Lambda) \cap G^+$, $J^i(\beta, \Lambda) = \widetilde{J}^i(\beta, \Lambda) \cap G$, for $i \geqslant 1$, and similarly define $\mathfrak{H}^-(\beta, \Lambda), H(\beta, \Lambda), H^i(\beta, \Lambda)$. We have $J(\beta,\Lambda) = P(\Lambda_E)J^1(\beta,\Lambda)$ and

$$J(\beta,\Lambda)/J^1(\beta,\Lambda) \simeq P(\Lambda_E)/P^1(\Lambda_E) \simeq M(\Lambda_E).$$

The group $M(\Lambda_E)$ is the group of points of a finite reductive group over k_F, and we denote by $J^\circ(\beta,\Lambda)$ the inverse image of the connected component $M^\circ(\Lambda_E)$ under the projection map.

By [23, Proposition 3.4], the stratum $[\Lambda, n, r{+}1, \beta]$ is equivalent to a semisimple stratum $[\Lambda, n, r{+}1, \gamma]$ with $\gamma \in \mathcal{L}$. In [23, Definition 3.13], for $0 \leqslant m < r+1$, a set of characters $\mathcal{C}(\Lambda, m, \beta)$ of $\widetilde{H}^{m+1}(\beta,\Lambda)$ is attached to $[\Lambda, n, r, \beta]$, depending on our initial choice of ψ_F. Precisely, $\mathcal{C}(\Lambda, m, \beta)$ consists of the characters $\widetilde{\theta}$ of $\widetilde{H}^{m+1}(\beta,\Lambda)$ which satisfy

(i) $\widetilde{\theta}\,|_{\widetilde{H}^{m+1}(\beta,\Lambda)\cap\widetilde{G}_i}$ is a simple character, in the sense of [4, Definition 3.2.3];

(ii) if $m' = \max\{m, \lceil r/2\rceil\}$ then there exists $\widetilde{\theta}_0 \in \mathcal{C}(\Lambda, m', \gamma)$ such that $\widetilde{\theta}\,|_{\widetilde{H}^{m'+1}(\beta,\Lambda)} = \widetilde{\theta}_0\psi_{\beta-\gamma}$.

If $[\Lambda, n, r, \beta]$ is self-dual then $\mathcal{C}(\Lambda, m, \beta)$ is preserved by the involution σ and, as in [23, § 3.6], one associates to $[\Lambda, n, r, \beta]$ the set $\mathcal{C}_-(\Lambda, m, \beta)$ of characters of $H^{m+1}(\beta,\Lambda)$ obtained by restriction from $\mathcal{C}(\Lambda, m, \beta)^\Sigma$.

The following results were proved in the case $R = \mathbb{C}$ but, since the groups involved are all pro-p, their proofs apply provided the characteristic of R is not p, as is the case here.

Theorem 2.3 ([23, Theorem 3.22])**.** Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum in A.

(i) If $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ then $I_{\widetilde{G}}(\widetilde{\theta}) = \widetilde{J}^1(\beta,\Lambda)\widetilde{G}_E\widetilde{J}^1(\beta,\Lambda)$.

(ii) Let $[\Lambda', n', 0, \beta]$ be another semisimple stratum in A. There is a bijection

$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}(\Lambda, 0, \beta) \to \mathcal{C}(\Lambda', 0, \beta),$$

called the transfer map, which takes $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ to the unique character $\widetilde{\theta}' \in \mathcal{C}(\Lambda', 0, \beta)$ such that $\widetilde{G}_E \subseteq I_{\widetilde{G}}(\widetilde{\theta}, \widetilde{\theta}')$.

Let $[\Lambda, n, r, \beta]$ be a semisimple stratum. The *affine class* of $[\Lambda, n, r, \beta]$ is the set of all (semisimple) strata of the form

$$[\Lambda', n', r', \beta],$$

where $\Lambda' = a\Lambda + b$ is in the affine class of Λ, $n' = an$ and r' is any integer such that $\lfloor r'/a \rfloor = r$. By induction on $k_F(\beta)$ (cf. [3, Lemma 2.2]), many objects associated to a semisimple stratum only depend on the affine class of the stratum. In particular, if $[\Lambda', n', r', \beta]$ is in the affine class of $[\Lambda, n, r, \beta]$, we have:

(i) $\widetilde{H}^{m'+1}(\beta', \Lambda') = \widetilde{H}^{m+1}(\beta, \Lambda)$;

(ii) $\mathcal{C}(\Lambda', m', \beta') = \mathcal{C}(\Lambda, m, \beta)$;

(iii) the transfer map $\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}(\Lambda, m, \beta) \to \mathcal{C}(\Lambda', m', \beta)$ is the identity.

If the associated strata are self-dual, then we have the following analogue of Theorem 2.3.

Theorem 2.4 ([17, Lemma 2.5])**.** Let $[\Lambda, n, 0, \beta]$ be a self-dual semisimple stratum in A.

(i) If $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ then $I_G(\theta) = J^1(\beta, \Lambda) G_E J^1(\beta, \Lambda)$.

(ii) Let $[\Lambda', n', 0, \beta]$ be another self-dual semisimple stratum in A. There is a bijection

$$\tau_{\Lambda,\Lambda',\beta} : \mathcal{C}_-(\Lambda, 0, \beta) \to \mathcal{C}_-(\Lambda', 0, \beta),$$

called the transfer map, which takes $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ to the unique character $\theta' \in \mathcal{C}_-(\Lambda', 0, \beta)$ such that $G_E \subseteq I_G(\theta, \theta')$.

Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum and $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$.

Theorem 2.5 ([23, Corollary 3.25])**.** *There exists a unique irreducible representation $\widetilde{\eta}$ of $\widetilde{J}^1(\beta, \Lambda)$ containing $\widetilde{\theta}$.*

If $[\Lambda, n, 0, \beta]$ is self-dual and $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, then we have the following analogue of Theorem 2.5.

Theorem 2.6 ([17, Lemma 2.5])**.** *There exists a unique representation η of $J^1(\beta, \Lambda)$ containing θ.*

We call the representations η and $\widetilde{\eta}$ of Theorems 2.5 and 2.6, *Heisenberg representations*. We define a bijection, which we also denote by $\tau_{\Lambda,\Lambda',\beta}$, between the set of Heisenberg representations of $\widetilde{J}^1(\beta, \Lambda)$ containing a semisimple character in $\mathcal{C}(\Lambda, 0, \beta)$ and the set of Heisenberg representations of $\widetilde{J}^1(\beta, \Lambda')$ containing a semisimple character in $\mathcal{C}(\Lambda', 0, \beta)$ which restricts to the transfer map, *i.e.* if $\widetilde{\eta}$ is the unique Heisenberg representation of $\widetilde{J}^1(\beta, \Lambda)$ containing $\widetilde{\theta} \in \mathcal{C}(\Lambda, 0, \beta)$ then $\tau_{\Lambda,\Lambda',\beta}(\widetilde{\eta})$ is the unique Heisenberg representation of $\widetilde{J}^1(\beta, \Lambda')$ containing $\tau_{\Lambda,\Lambda',\beta}(\widetilde{\theta})$. Similarly, we define a bijection $\tau_{\Lambda,\Lambda',\beta}$ between the set of Heisenberg representations of $J^1(\beta, \Lambda)$ containing a self-dual semisimple character in $\mathcal{C}_-(\Lambda, 0, \beta)$ and the set of Heisenberg representations of $J^1(\beta, \Lambda')$ containing a self-dual semisimple character in $\mathcal{C}_-(\Lambda', 0, \beta)$.

2.4 Double coset identities

We state mild generalisations of some results of [21], the proofs of which, [*op. cit.*, Lemmas 2.1, 2.2 and Theorem 2.3], still apply. The notation in this short subsection is independent of that in the rest of the paper. Let G be a group and Γ a group of automorphisms of G. If H is a Γ-stable subgroup of G we let H^Γ denote subgroup of fixed points of Γ.

Theorem 2.7. *Let U_1 and U_2 be Γ-stable subgroups of G.*

(i) *Suppose that, for all $g \in G$, the (non-abelian) cohomology pointed set $H^1(\Gamma, gU_1g^{-1} \cap U_2)$ is trivial. Then, for all $g \in G^\Gamma$, we have $(U_1 g U_2)^\Gamma = U_1^\Gamma g U_2^\Gamma$.*

(ii) *Suppose that Γ is a soluble group of order coprime to p, that U_1 and U_2 are Γ-stable pro-p subgroups of G, and that $g \in G$.*

  (a) *$(U_1 g U_2)^\Gamma \neq \emptyset$ if and only if $U_1 g U_2$ is stable under Γ.*

  (b) *Let H be a Γ-stable subgroup of G such that $U_1 h U_2 \cap H = (U_1 \cap H) h (U_2 \cap H)$, for all $h \in H$. Then $(U_1 H U_2)^\Gamma = U_1^\Gamma H^\Gamma U_2^\Gamma$.*

2.5 Modular representation theory techniques

As R-representations of compact open subgroups are not necessarily semisimple (unlike the case $R = \mathbb{C}$), we will need to use appropriate versions of some well known representation theory techniques. The first is the *simple criterion for irreducibility* of [27].

Lemma 2.8. Let λ be an irreducible representation of a compact open subgroup K of G. Suppose that $\mathrm{End}_G(\mathrm{ind}_K^G(\lambda)) \simeq R$ and, for any irreducible representation π of G, if λ is a subrepresentation of π then it is also a quotient of π. Then $\mathrm{ind}_K^G(\lambda)$ is irreducible.

A representation π of G is called *quasi-projective* if, for all representations π' of G and all surjective homomorphisms $\varphi : \pi \to \pi'$, the homomorphism $\mathrm{End}_G(\pi) \to \mathrm{Hom}_G(\pi, \pi')$, $\alpha \mapsto \alpha \circ \varphi$ for $\alpha \in \mathrm{End}_G(\pi)$, is surjective. The second modular representation theory criterion we make use of is the *simple criterion for quasi-projectivity* of [27] (*cf.* also [11, Proposition 3.15]).

Lemma 2.9. Let K be a compact open subgroup of G, λ an irreducible representation of K and $\pi = \mathrm{ind}_K^G(\lambda)$. If the λ-isotypic component of π is a direct summand of the restriction of π to K and no subquotient of its complement is isomorphic to λ then π is quasi-projective.

Let π, τ be R-representations of G. Then $\mathrm{Hom}_G(\pi, \tau)$ is a right $\mathrm{End}_G(\pi)$-module by pre-composition. In attempts to classify the irreducible representations of G, quasi-projective representations are particularly interesting due to the following theorem of Arabia.

Theorem 2.10 ([26, Appendix Théorème 10])**.** Suppose π is quasi-projective and finitely generated. Then the functor $\mathfrak{R}_R(G) \to \mathrm{End}_G(\pi)$-mod, $\tau \mapsto \mathrm{Hom}_G(\pi, \tau)$, induces a bijection between the irreducible quotients of π and the simple right $\mathrm{End}_G(\pi)$-modules.

Suppose that J is a compact open subgroup of G containing a compact open pro-p subgroup J^1 which is normal in J and that η is an irreducible representation of J^1 which extends to an irreducible representation κ of J. Then we have the following lemma, implicit in [27] (*cf.* [28, Proposition 4.2] and [16, Lemme 2.6] for a proof).

Lemma 2.11. The functor $\kappa \otimes -$ induces an equivalence of categories between $\mathfrak{R}_R(J/J^1)$ and the category $\mathfrak{R}_R(J, \eta)$ of η-isotypic representations of J.

The following lemma is a mild abstraction of [4, Proposition 5.3.2].

Lemma 2.12. Let X_1 and X_2 be subgroups of G, and X_1^1 (resp. X_2^1) be a subgroup of X_1 (resp. X_2). For $i = 1, 2$, let ζ_i be a representation of X_i trivial on X_i^1, and let μ_i be a representation of X_i. Suppose that

$$\mathrm{Hom}_{X_1 \cap X_2}(\mu_1, \mu_2) = \mathrm{Hom}_{X_1^1 \cap X_2^1}(\mu_1, \mu_2) \simeq R.$$

Then, for any non-zero $S \in \mathrm{Hom}_{X_1 \cap X_2}(\mu_1, \mu_2)$, the map

$$\begin{array}{ccc} \mathrm{Hom}_{X_1 \cap X_2}(\zeta_1, \zeta_2) & \longrightarrow & \mathrm{Hom}_{X_1 \cap X_2}(\mu_1 \otimes \zeta_1, \mu_2 \otimes \zeta_2) \\ T & \longmapsto & S \otimes T \end{array}$$

is an isomorphism of vector spaces.

Proof. It is easy to check that the map is well-defined, and it is clearly injective, so we need only check surjectivity. Let $f \in \mathrm{Hom}_{X_1 \cap X_2}(\mu_1 \otimes \zeta_1, \mu_2 \otimes \zeta_2)$ be non-zero. Write f as a finite sum $\sum_k S_k \otimes T_k$, with $S_k \in \mathrm{Hom}_R(\mu_1, \mu_2)$ non-zero and $T_k \in \mathrm{Hom}_R(\zeta_1, \zeta_2)$, such that $\{T_k\}$ is linearly independent over R. Let $x \in X_1^1 \cap X_2^1$; then $f \circ \mu_1 \otimes \zeta_1(x) = \mu_2 \otimes \zeta_2(x) \circ f$. Hence, as ζ_1, ζ_2 are trivial on $X_1^1 \cap X_2^1$, we have

$$\sum_k (S_k \mu_1(y) - \mu_2(y) S_k) \otimes T_k = 0,$$

for $y \in X_1^1 \cap X_2^1$. Thus $S_k \in \mathrm{Hom}_{X_1^1 \cap X_2^1}(\mu_1, \mu_2)$, by the linear independence of $\{T_k\}$. The intertwining spaces $\mathrm{Hom}_{X_1^1 \cap X_2^1}(\mu_1, \mu_2)$ and $\mathrm{Hom}_{X_1 \cap X_2}(\mu_1, \mu_2)$ are one-dimensional and equal by our hypotheses. Thus S_k is a scalar multiple of S and we can write $f = S \otimes T$ with $T \in \mathrm{Hom}_R(\zeta_1, \zeta_2)$. Furthermore,

$$S \otimes T(\mu_1 \otimes \zeta_1(y)v) = (\mu_2(y) S \otimes \zeta_2(y) T)(v)$$

and

$$S \otimes T(\mu_1 \otimes \zeta_1(y)v) = (S\mu_1(y) \otimes T\zeta_1(y))(v) = (\mu_2(y) S \otimes T\zeta_1(y))(v)$$

for all $y \in X_1 \cap X_2$ and v in the space of $\mu_1 \otimes \zeta_1$. Hence $T \in \mathrm{Hom}_{X_1 \cap X_2}(\zeta_1, \zeta_2)$ and, since $f = S \otimes T$, our map is surjective. □

3 Asymmetric generalisations via †-constructions

In this section we present a particularly useful construction: to an $\mathfrak{o}_F$-lattice sequence Λ in V, we associate a strict $\mathfrak{o}_F$-lattice sequence $\Lambda^\dagger$ of period $e(\Lambda)$ in a direct sum of $e(\Lambda)$ copies of V, whose associated hereditary order $\mathfrak{A}(\Lambda^\dagger)$ is principal and such that all the blocks $\mathfrak{A}^{ii}(\Lambda^\dagger) = \mathfrak{A}(\Lambda)$, for $0 \leqslant i \leqslant e(\Lambda)$. This construction becomes useful later when applied to two $\mathfrak{o}_F$-lattice sequences Λ and Υ in V, which, if necessary, after changing in their affine classes we assume $e(\Lambda) = e(\Upsilon)$; in this situation $\mathfrak{A}(\Lambda^\dagger)$ and $\mathfrak{A}(\Upsilon^\dagger)$ are principal orders in $V^\dagger$ of the same block size, hence are conjugate, yet when we restrict to a single block we find the not necessarily conjugate orders $\mathfrak{A}(\Lambda)$ and $\mathfrak{A}(\Upsilon)$. This construction originates in work of the second author with Broussous and Sécherre in [3]. The first part of this section is concerned with revisiting the construction of [ibid.] and generalising it to semisimple strata. Then we provide two new applications of †: a generalisation of the semisimple intersection property of [24] and an extension of the computation of the intertwining a semisimple character in [23] to the case of two semisimple characters related by transfer.

3.1 The †-construction

Let Λ be an $\mathfrak{o}_F$-lattice sequence in V of $\mathfrak{o}_F$-period $e(\Lambda)$. Let $V^\dagger = V \oplus \cdots \oplus V$ ($e(\Lambda)$ times). Following [3, Section 2], we define an $\mathfrak{o}_F$-lattice sequence $\Lambda^\dagger$ in $V^\dagger$ by

$$\Lambda^\dagger(r) = \bigoplus_{k=0}^{e(\Lambda)-1} \Lambda(r+k), \text{ for all } r \in \mathbb{Z}.$$

Then, for all $r \in \mathbb{Z}$,

$$\dim_{k_F}(\Lambda^\dagger(r)/\Lambda^\dagger(r+1)) = \sum_{k=0}^{e(\Lambda)-1} \dim_{k_F}(\Lambda(r+k)/\Lambda(r+k+1)) = \dim_F(V).$$

Therefore, $\Lambda^\dagger$ is a strict $\mathfrak{o}_F$-lattice sequence in $V^\dagger$ of period $e(\Lambda)$ whose associated order $\mathfrak{A}(\Lambda^\dagger)$ is principal.

Let $[\Lambda, n, r, \beta]$ be a semisimple stratum in A with associated splitting $V = \bigoplus_{i\in I} V^i$, and $e = e(\Lambda) = e(\Lambda_i)$. For each $i \in I$, let $V^{i,\dagger} = V^i \oplus \cdots \oplus V^i$ ($e(\Lambda)$ times), and let $\Lambda_i^\dagger$ be the $\mathfrak{o}_F$-lattice sequence in $V^{i,\dagger}$, defined as above. Let $V^\dagger = \bigoplus_{i\in I} V^{i,\dagger}$ and let $\Lambda^\dagger$ be the $\mathfrak{o}_F$-lattice sequence in V' defined by $\Lambda^\dagger = \bigoplus_{i\in I} \Lambda^{i,\dagger}$. Note that this is the same lattice sequence as that defined above (working directly with Λ within V). Let $A^\dagger = \mathrm{End}_F(V^\dagger)$ and $\widetilde{G}^\dagger = \mathrm{Aut}_F(V^\dagger)$.

We recall that $\beta = \sum_{i\in I} \beta_i$, where $\beta_i = e_i\beta e_i$ and $e_i : V \to V^i$ is the projection map with kernel $\bigoplus_{j\neq i} V^j$. Let $\beta_i^\dagger$ denote the image of β_i under the diagonal embedding of $\mathrm{End}_F(V^i)$ into $\mathrm{End}_F(V^{i,\dagger})$, and $\beta^\dagger = \sum_{i\in I} \beta_i^\dagger$. Then $\Lambda^{i,\dagger}$ is an $\mathfrak{o}_{E_i}$-lattice sequence, whose associated hereditary $\mathfrak{o}_F$-order $\mathfrak{A}(\Lambda^{i,\dagger})$ is principal. Moreover, the stratum $[\Lambda^\dagger, n, r, \beta^\dagger]$ in $A^\dagger$ is semisimple, with associated splitting $V^\dagger = \bigoplus_{i\in I} V^{i,\dagger}$.

We recall also that $\widetilde{L}$ is the stabilizer in $\widetilde{G}$ of the decomposition $V = \bigoplus_{i\in I} V^i$. Let $\widetilde{Q} = \widetilde{L}\widetilde{U}_Q^+$ be a parabolic subgroup of $\widetilde{G}$ with Levi component $\widetilde{L}$, and opposite parabolic $\widetilde{Q}^- = \widetilde{L}\widetilde{U}_Q^-$ with respect to $\widetilde{L}$. Then, for any $m \geq 0$, the group $\widetilde{H}^{m+1}(\beta, \Lambda)$ has an Iwahori decomposition with respect to $(\widetilde{L}, \widetilde{Q})$ with

$$\widetilde{H}^{m+1}(\beta, \Lambda) \cap \widetilde{L} = \prod_{i\in I} \widetilde{H}^{m+1}(\beta_i, \Lambda^i). \tag{3.1}$$

Moreover, by [23, Lemma 3.15], any semisimple character $\widetilde{\theta} \in \mathcal{C}(\beta, m, \Lambda)$ is trivial on the unipotent parts $\widetilde{H}^{m+1}(\beta, \Lambda) \cap \widetilde{U}_Q^\pm$ and

$$\widetilde{\theta}|_{(\widetilde{H}^{m+1}(\beta,\Lambda)\cap\widetilde{L})} = \bigotimes_{i\in I} \widetilde{\theta}_i,$$

with $\widetilde{\theta}_i \in \mathcal{C}(\beta_i, m, \Lambda^i)$ a simple character. Analogously, we have the Levi subgroup $\widetilde{L}^\dagger$ which is the stabilizer of the decomposition $V^\dagger = \bigoplus_{i\in I} V^{i,\dagger}$ and $\widetilde{H}^{m+1}(\beta^\dagger, \Lambda^\dagger)$ has an Iwahori decomposition with respect to any parabolic subgroup $\widetilde{Q}^\dagger$ with Levi component $\widetilde{L}^\dagger$, with

$$\widetilde{H}^{m+1}(\beta^\dagger, \Lambda^\dagger) \cap \widetilde{L}^\dagger = \prod_{i\in I} \widetilde{H}^{m+1}(\beta_i^\dagger, \Lambda^{i,\dagger}).$$

Let $\mathcal{M}^\dagger$ denote the Levi subalgebra of $A^\dagger$ which is the stabilizer of the splitting $V^\dagger = V \oplus \cdots \oplus V$, and let $\widetilde{M}^\dagger$ be its group of units. Let Γ be the subgroup of $\widetilde{M}^\dagger$ consisting of elements with blocks $\pm\,\mathrm{Id}$. Let $\widetilde{P}^\dagger$ be any parabolic subgroup of $\widetilde{G}^\dagger$ with Levi factor $\widetilde{M}^\dagger$ and unipotent radical $\widetilde{U}^\dagger$, and let $\widetilde{P}^{-,\dagger}$ denote the opposite parabolic of $\widetilde{P}^\dagger$ with respect to $\widetilde{M}^\dagger$, with Levi decomposition $\widetilde{P}^{-,\dagger} = \widetilde{M}^\dagger \ltimes \widetilde{U}^{-,\dagger}$. Similarly, for each $i \in I$, we have a Levi subgroup $\widetilde{M}^{i,\dagger}$ of $\widetilde{G}_i^\dagger = \mathrm{Aut}_F(V^{i,\dagger})$.

For all $m \geqslant 0$, using [24, Proposition 5.2], we have an Iwahori decomposition

$$\widetilde{H}^{m+1}(\beta^\dagger, \Lambda^\dagger) = (\widetilde{H}^{m+1}(\beta^\dagger, \Lambda^\dagger) \cap \widetilde{U}^{-,\dagger})(\widetilde{H}^{m+1}(\beta^\dagger, \Lambda^\dagger) \cap \widetilde{M}^\dagger)(\widetilde{H}^{m+1}(\beta^\dagger, \Lambda^\dagger) \cap \widetilde{U}^\dagger), \tag{3.2}$$

$$\widetilde{H}^{m+1}(\beta^\dagger, \Lambda^\dagger) \cap \widetilde{M}^\dagger = \widetilde{H}^{m+1}(\beta, \Lambda) \times \cdots \times \widetilde{H}^{m+1}(\beta, \Lambda).$$

There are similar decompositions for $\widetilde{H}^{m+1}(\beta_i^\dagger, \Lambda^{i,\dagger})$.

Let $\widetilde{\theta} \in \mathcal{C}(\beta, m, \Lambda)$ be a semisimple character, corresponding to simple characters $\widetilde{\theta}_i \in \mathcal{C}(\beta_i, m, \Lambda^i)$ as in (3.1). Put $\widetilde{\theta}_i^\dagger = \tau_{\Lambda^i,\Lambda^{i,\dagger},\beta_i,\beta_i^\dagger}(\widetilde{\theta}_i)$, the transfer of $\widetilde{\theta}_i$ to $\mathcal{C}(\beta_i^\dagger, m, \Lambda_i^\dagger)$. By [3, Lemma 2.7], the restriction of $\widetilde{\theta}_i^\dagger$ to $\widetilde{H}^{m+1}(\beta_i^\dagger, \Lambda^{i,\dagger}) \cap \widetilde{M}^{i,\dagger}$ has the form $\widetilde{\theta}_i \otimes \cdots \otimes \widetilde{\theta}_i$; moreover, for $\widetilde{P}^{i,\dagger} = \widetilde{M}^{i,\dagger}\widetilde{U}^{i,\dagger}$ any parabolic subgroup of $\widetilde{G}^{i,\dagger}$ with Levi component $\widetilde{M}^{i,\dagger}$, the restriction of $\widetilde{\theta}_i^\dagger$ to the unipotent part $\widetilde{H}^{m+1}(\beta_i^\dagger, \Lambda^{i,\dagger}) \cap \widetilde{U}^{i,\dagger}$ is trivial.

Lemma 3.3. There is a unique semisimple character $\widetilde{\theta}^\dagger \in \mathcal{C}(\beta^\dagger, m, \Lambda^\dagger)$ such that

$$\widetilde{\theta}^\dagger|_{(\widetilde{H}^{m+1}(\beta^\dagger,\Lambda^\dagger)\cap\widetilde{L}^\dagger)} = \bigoplus_{i\in I} \widetilde{\theta}_i^\dagger;$$

Moreover, $\widetilde{\theta}^\dagger$ is trivial on the unipotent parts in (3.2), and

$$\widetilde{\theta}^\dagger\,|_{(\widetilde{H}^{m+1}(\beta^\dagger,\Lambda^\dagger)\cap\widetilde{M}^\dagger)} = \widetilde{\theta} \otimes \cdots \otimes \widetilde{\theta}.$$

Proof. The first part follows easily from the inductive definition of semisimple characters (see in particular [23, Lemma 3.15]). Moreover, for any parabolic subgroup $\widetilde{Q}^\dagger = \widetilde{L}^\dagger\widetilde{U}_Q^\dagger$ with Levi component $\widetilde{L}^\dagger$, the restriction of $\widetilde{\theta}^\dagger$ to $\widetilde{H}^{m+1}(\beta^\dagger, \Lambda^\dagger) \cap \widetilde{U}_Q^\dagger$ is trivial; the second statement follows from this, the corresponding statement in the simple case ([3, Lemma 2.7]) and the unicity in [23, Lemma 3.15] again. □

For $g \in \widetilde{G}$, let $g^\dagger$ denote its diagonal embedding in $\widetilde{G}^\dagger$.

Lemma 3.4. For $i = 1, 2$, let $\widetilde{\theta}_i$ be semisimple characters in $\mathcal{C}(\Lambda, m, \beta_i)$. If g intertwines $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$, then $g^\dagger$ intertwines $\widetilde{\theta}_1^\dagger$ and $\widetilde{\theta}_2^\dagger$.

Proof. For simple characters, it is shown in the proof of [3, Proposition 2.6] that this follows from [3, Lemma 2.7]. The proof in the semisimple case follows *mutatis mutandis* using Lemma 3.3 in place of [3, Lemma 2.7]. □

3.2 Applications of †

Let $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ be semisimple strata in A with splitting $V = \bigoplus_{i\in I} V^i$. Let e_Λ (resp. e_Υ) denote the $\mathfrak{o}_F$-period of Λ (resp. Υ), and hence of Λ^i (resp. Υ^i) for all $i \in I$. By changing $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ in their affine classes, we assume the $e = e_\Lambda = e_\Upsilon$. As remarked earlier, this does not change the objects (orders, groups, characters) associated to the semisimple strata.

For $i \in I$, we apply the construction of Section 3.1 to Λ^i and to Υ^i. Suppose that the $\mathfrak{o}_{E_i}$-period e_{E_i} of Λ, and hence of Υ, is related to the $\mathfrak{o}_F$-period e, by

$$e_{E_i} = m_i e,$$

so that m_i is the ramification index of E_i/F. Then, for all $r \in \mathbb{Z}$,

$$\dim_{k_{E_i}}(\Lambda^{i,\dagger}(r)/\Lambda^{i,\dagger}(r+1)) = \sum_{k=0}^{m_i e_{E_i}-1} \dim_{k_{E_i}}(\Lambda^i(r+k)/\Lambda^i(r+k+1)) = m_i \dim_{E_i}(V^i).$$

Hence, the lattice sequences $\Lambda^{i,\dagger}$ and $\Upsilon^{i,\dagger}$ are strict $\mathfrak{o}_{E_i}$-lattice sequences in $V^{i,\dagger}$ of $\mathfrak{o}_F$-period e (and $\mathfrak{o}_{E_i}$-period e_{E_i}). Furthermore, the associated hereditary $\mathfrak{o}_{E_i}$-orders $\mathfrak{B}(\beta_i, \Lambda^{i,\dagger})$and $\mathfrak{B}(\beta_i, \Upsilon^{i,\dagger})$ are principal $\mathfrak{o}_{E_i}$-orders with the same block size, hence there exist $x_i \in C_{\widetilde{G}_i^\dagger}(\beta_i^\dagger)$, such that

$$\Lambda^{i\,\dagger}(r) = x_i \cdot \Upsilon^{i\,\dagger}(r),$$

for all $r \in \mathbb{Z}$. Let $x = \sum_{i\in I} x_i$; then $x \in \widetilde{G}_E^\dagger$ and we have

$$\Lambda^\dagger = x \cdot \Upsilon^\dagger.$$

It follows that the data coming from the semisimple strata $[\Lambda^\dagger, n_\Lambda, 0, \beta^\dagger]$ and $[\Upsilon^\dagger, n_\Upsilon, 0, \beta^\dagger]$ are conjugate in $\widetilde{G}_E^\dagger$ and we get:

Lemma 3.5. In the situation above, there exists $x \in \widetilde{G}_E^\dagger$ such that

(i) $\mathfrak{J}(\beta^\dagger, \Lambda^\dagger) = \mathfrak{J}(\beta^\dagger, \Upsilon^\dagger)^x$ and $\mathfrak{H}(\beta^\dagger, \Lambda^\dagger) = \mathfrak{H}(\beta^\dagger, \Upsilon^\dagger)^x$;

(ii) $\widetilde{J}(\beta^\dagger, \Lambda^\dagger) = \widetilde{J}(\beta^\dagger, \Upsilon^\dagger)^x$ and $\widetilde{H}(\beta^\dagger, \Lambda^\dagger) = \widetilde{H}(\beta^\dagger, \Upsilon^\dagger)^x$;

(iii) conjugation by x defines a bijection $\mathcal{C}(\beta^\dagger, 0, \Lambda^\dagger) \to \mathcal{C}(\beta^\dagger, 0, \Upsilon^\dagger)$.

Throughout this section, "applying the †-construction" will mean applying it in the way just described.

3.3 Semisimple intersection property

In this section we generalise the semisimple intersection property of [24, Lemma 2.6].

Lemma 3.6. Let $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ be semisimple strata in A and $y \in \widetilde{G}_E$. Then

$$\widetilde{P}^1(\Upsilon) y \widetilde{P}^1(\Lambda) \cap \widetilde{G}_E = \widetilde{P}^1(\Upsilon_E) y \widetilde{P}^1(\Lambda_E).$$

Proof. Applying the †-construction, by Lemma 3.5 we have $x \in \widetilde{G}_E^\dagger$ such that

$$\widetilde{P}^1(\Upsilon^\dagger) y^\dagger \widetilde{P}^1(\Lambda^\dagger) = \widetilde{P}^1(\Upsilon^\dagger) y^\dagger x \widetilde{P}^1(\Upsilon^\dagger) x^{-1}.$$

By the semisimple intersection property in $\widetilde{G}_E^\dagger$ (*cf.* the proof of [24, Lemma 2.6]), because $x \in \widetilde{G}_E^\dagger$ we have

$$\widetilde{P}^1(\Upsilon^\dagger) y^\dagger x \widetilde{P}^1(\Upsilon^\dagger) \cap \widetilde{G}_E^\dagger = \widetilde{P}^1(\Upsilon_E^\dagger) y^\dagger x \widetilde{P}^1(\Upsilon_E^\dagger).$$

Hence

$$\widetilde{P}^1(\Upsilon^\dagger) y^\dagger \widetilde{P}^1(\Lambda^\dagger) \cap \widetilde{G}_E^\dagger = \widetilde{P}^1(\Upsilon_E^\dagger) y^\dagger \widetilde{P}^1(\Lambda_E^\dagger).$$

Recall, $\widetilde{M}^\dagger$ is the Levi subgroup of $\widetilde{G}^\dagger$ defined by the decomposition of $V^\dagger$ into a sum of copies of V, and Γ is the 2-subgroup of $\widetilde{M}^\dagger$ consisting of elements with blocks $\pm\,\mathrm{Id}$. Notice that, $\widetilde{M}^\dagger$ is equal to the fixed point subgroup of $\widetilde{G}^\dagger$ under the conjugation action of Γ. Hence, because Γ is a 2-group and $\widetilde{P}^1(\Lambda_E^\dagger)$ and $\widetilde{P}^1(\Upsilon_E^\dagger)$ are pro-p groups, with p odd, $H^1(\Gamma, y^\dagger \widetilde{P}^1(\Upsilon_E^\dagger)(y^\dagger)^{-1} \cap \widetilde{P}^1(\Lambda_E^\dagger)) = 1$ and we can apply Theorem 2.7(i) to find

$$\widetilde{P}^1(\Upsilon_E^\dagger) y^\dagger \widetilde{P}^1(\Lambda_E^\dagger) \cap \widetilde{M}^\dagger = (\widetilde{P}^1(\Upsilon_E^\dagger) \cap \widetilde{M}^\dagger) y^\dagger (\widetilde{P}^1(\Lambda_E^\dagger) \cap \widetilde{M}^\dagger).$$

We have $(\widetilde{P}^1(\Upsilon_E^\dagger) \cap \widetilde{M}^\dagger) = \prod_{i=1}^d \widetilde{P}^1(\Upsilon_E)$ and $(\widetilde{P}^1(\Lambda_E^\dagger) \cap \widetilde{M}^\dagger) = \prod_{i=1}^d \widetilde{P}^1(\Lambda_E)$. Thus, restricting to a single block in $\widetilde{M}^\dagger$ we recover the result. ◻

Corollary 3.7. Let $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ be self-dual semisimple strata in A. Then

$$P^1(\Upsilon) y P^1(\Lambda) \cap G_E^+ = P^1(\Upsilon_E) y P^1(\Lambda_E), \qquad \text{for } y \in G_E^+;$$
$$P^1(\Upsilon) y P^1(\Lambda) \cap G_E = P^1(\Upsilon_E) y P^1(\Lambda_E), \qquad \text{for } y \in G_E.$$

Proof. Applying Theorem 2.7(i), under the fixed points of the involution σ, we have

$$\widetilde{P}^1(\Upsilon_E)y\widetilde{P}^1(\Lambda_E)\cap G_E^+ = (\widetilde{P}^1(\Upsilon_E)\cap G_E^+)y(\widetilde{P}^1(\Lambda_E)\cap G_E^+).$$

Therefore, by Lemma 3.6, $P^1(\Upsilon)yP^1(\Lambda)\cap G_E^+ = P^1(\Upsilon_E)yP^1(\Lambda_E)$. The second equality follows by intersecting with G, since $P^1(\Lambda_E)\subseteq G_E$. □

A simple application of the semisimple intersection property gives us the following bijection of double cosets, where we note that $J_\Upsilon G_E J_\Lambda = J^1_\Upsilon G_E J^1_\Lambda$.

Lemma 3.8. Let $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ be self-dual semisimple strata in A. Let $J_\Upsilon = J(\beta,\Lambda)$ and $J_\Upsilon = J(\beta,\Upsilon)$. The following map is a bijection

$$\begin{array}{ccc} P(\Upsilon_E)\backslash G_E/P(\Lambda_E) & \longrightarrow & J_\Upsilon\backslash J_\Upsilon G_E J_\Lambda/J_\Lambda \\ X & \longmapsto & J_\Upsilon X J_\Lambda. \end{array}$$

Proof. Let $g\in G_E$. Considering Λ and Υ as $\mathfrak{o}_F$-lattice sequences, we have containments $J^1_\Upsilon\subseteq P^1(\Upsilon)$ and $J^1_\Lambda\subseteq P^1(\Lambda)$. Hence

$$J^1_\Upsilon(P(\Upsilon_E)gP(\Lambda_E))J^1_\Lambda\cap G_E\subseteq P^1(\Upsilon)(P(\Upsilon_E)gP(\Lambda_E))P^1(\Lambda)\cap G_E.$$

We choose a set of representatives for the finite double coset space $P^1(\Upsilon)\backslash(P(\Upsilon_E)gP(\Lambda_E))/P^1(\Lambda)$ and for each representative we apply the simple intersection property, Corollary 3.7, to find

$$P^1(\Upsilon)(P(\Upsilon_E)gP(\Lambda_E))P^1(\Lambda)\cap G_E = P(\Upsilon_E)gP(\Lambda_E).$$

Therefore $P(\Upsilon_E)gP(\Lambda_E) = J^1_\Upsilon(P(\Upsilon_E)gP(\Lambda_E))J^1_\Lambda\cap G_E$ and the map is a bijection. □

3.4 Intertwining of transfers

Let $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ be semisimple strata. Let $\widetilde{\theta}_\Lambda\in\mathcal{C}(\Upsilon,0,\beta)$ and $\widetilde{\theta}_\Upsilon = \tau_{\Lambda,\Upsilon,\beta}(\widetilde{\theta}_\Upsilon)$. We apply the †-construction and abbreviate $\widetilde{J}^1_\Lambda = \widetilde{J}^1(\beta,\Lambda)$ and $(\widetilde{J}^\dagger_\Lambda)^1 = \widetilde{J}^1(\beta^\dagger,\Lambda^\dagger)$, with similar notation for Υ, and also write $\tau = \tau_{\Lambda,\Upsilon,\beta}$ and $\tau^\dagger = \tau_{\Lambda^\dagger,\Upsilon^\dagger,\beta^\dagger}$.

Theorem 3.9. We have

$$I_{\widetilde{G}}(\widetilde{\theta}_\Lambda,\widetilde{\theta}_\Upsilon) = \widetilde{J}^1_\Upsilon\widetilde{G}_E\widetilde{J}^1_\Lambda.$$

Proof. Let $g\in I_{\widetilde{G}}(\widetilde{\theta}_\Lambda,\widetilde{\theta}_\Upsilon)$ and, as before, let $g^\dagger$ denote the diagonal embedding of g in $\widetilde{G}^\dagger$. By Lemma 3.4, we have

$$g^\dagger\in I_{\widetilde{G}}(\widetilde{\theta}^\dagger_\Lambda,\widetilde{\theta}^\dagger_\Upsilon).$$

Thus, as $G_E\in I_{\widetilde{G}}(\widetilde{\theta}_\Lambda,\widetilde{\theta}_\Upsilon)$ by Theorem 2.3 (ii), we have

$$\widetilde{G}^\dagger_E\in I_{\widetilde{G}}(\widetilde{\theta}^\dagger_\Lambda,\widetilde{\theta}^\dagger_\Upsilon),$$

hence $\widetilde{\theta}^\dagger_\Upsilon = \tau^\dagger(\widetilde{\theta}^\dagger_\Lambda)$, again by Theorem 2.3 (ii). Moreover, taking $x\in\widetilde{G}^\dagger_E$ such that $\Lambda^\dagger = x\cdot\Upsilon^\dagger$, as in Lemma 3.5, we have

$$\widetilde{G}^\dagger_E\subseteq I_{\widetilde{G}}(\widetilde{\theta}^\dagger_\Lambda,(\widetilde{\theta}^\dagger_\Lambda)^x),$$

as $\widetilde{G}^\dagger_E$ intertwines $\widetilde{\theta}^\dagger_\Lambda$ by Theorem 2.3 (i). Since $(\widetilde{\theta}^\dagger_\Lambda)^x\in\mathcal{C}(\Upsilon,0,\beta)$, we deduce that $\widetilde{\theta}^\dagger_\Upsilon = (\widetilde{\theta}^\dagger_\Lambda)^x$ by the unicity of the transfer in Theorem 2.3 (ii).

By Theorem 2.3 (i), we have

$$I_{\widetilde{G}}(\widetilde{\theta}^{\dagger}_{\Lambda}, \widetilde{\theta}^{\dagger}_{\Lambda}) = (\widetilde{J}^{\dagger}_{\Lambda})^1 \widetilde{G}^{\dagger}_E (\widetilde{J}^{\dagger}_{\Lambda})^1.$$

If $y \in \widetilde{G}^{\dagger}$ then $y \in I_{\widetilde{G}}(\widetilde{\theta}^{\dagger}_{\Lambda}, \widetilde{\theta}^{\dagger}_{\Lambda})$ if and only if $xy \in I_{\widetilde{G}}(\widetilde{\theta}^{\dagger}_{\Lambda}, (\widetilde{\theta}^{\dagger}_{\Lambda})^x)$. Therefore

$$I_{\widetilde{G}}(\widetilde{\theta}^{\dagger}_{\Lambda}, \widetilde{\theta}^{\dagger}_{\Upsilon}) = x^{-1} I_{\widetilde{G}}(\widetilde{\theta}^{\dagger}_{\Lambda}, (\widetilde{\theta}^{\dagger}_{\Lambda})^x) = x^{-1} (\widetilde{J}^{\dagger}_{\Lambda})^1 \widetilde{G}^{\dagger}_E (\widetilde{J}^{\dagger}_{\Lambda})^1 = (\widetilde{J}^{\dagger}_{\Upsilon})^1 \widetilde{G}^{\dagger}_E (\widetilde{J}^{\dagger}_{\Lambda})^1.$$

Now, as in the proof of Lemma 3.6, let Γ be the group 2-subgroup of $\widetilde{M}^{\dagger}$ generated by blocks consisting of Id and $-\operatorname{Id}$. Because Γ is a 2-group and $(\widetilde{J}^{\dagger}_{\Lambda})^1$ and $(\widetilde{J}^{\dagger}_{\Upsilon})^1$ are pro-p groups, with p odd, the non-abelian cohomology pointed set $H^1(\Gamma, g(\widetilde{J}^{\dagger}_{\Upsilon})^1 g^{-1} \cap (\widetilde{J}^{\dagger}_{\Lambda})^1)$ is trivial, for all $g \in \widetilde{G}$. Hence, by Theorem 2.7,

$$\begin{aligned} ((\widetilde{J}^{\dagger}_{\Upsilon})^1 \widetilde{G}^{\dagger}_E (\widetilde{J}^{\dagger}_{\Lambda})^1) \cap \widetilde{M}^{\dagger} &= ((\widetilde{J}^{\dagger}_{\Upsilon})^1 \cap \widetilde{M}^{\dagger})(\widetilde{G}^{\dagger}_E \cap \widetilde{M}^{\dagger})((\widetilde{J}^{\dagger}_{\Lambda})^1 \cap \widetilde{M}^{\dagger}) \\ &= (\widetilde{J}^{\dagger}_{\Upsilon} \cap \widetilde{M}^{\dagger})(\widetilde{G}^{\dagger}_E \cap \widetilde{M}^{\dagger})(\widetilde{J}^{\dagger}_{\Lambda} \cap \widetilde{M}^{\dagger}). \end{aligned}$$

Finally, for $g^{\dagger} \in I_{\widetilde{G}}(\widetilde{\theta}^{\dagger}_{\Lambda}, \widetilde{\theta}^{\dagger}_{\Upsilon})$, we have an Iwahori decomposition

$$\begin{aligned} \widetilde{H}^1(\beta^{\dagger}, \Lambda^{\dagger}) \cap \widetilde{H}^1(\beta^{\dagger}, \Upsilon^{\dagger})^{g^{\dagger}} &= (\widetilde{H}^1(\beta^{\dagger}, \Lambda^{\dagger}) \cap \widetilde{H}^1(\beta^{\dagger}, \Upsilon^{\dagger})^{g^{\dagger}} \cap \widetilde{U}^{-,\dagger}) \\ &\quad (\widetilde{H}^1(\beta^{\dagger}, \Lambda^{\dagger}) \cap \widetilde{H}^1(\beta^{\dagger}, \Upsilon^{\dagger})^{g^{\dagger}} \cap \widetilde{M}^{\dagger})(\widetilde{H}^1(\beta^{\dagger}, \Lambda^{\dagger}) \cap \widetilde{H}^1(\beta^{\dagger}, \Upsilon^{\dagger})^{g^{\dagger}} \cap \widetilde{U}^{-,\dagger}), \end{aligned}$$

and, by Lemma 3.3, $\widetilde{\theta}^{\dagger}_{\Lambda}, \widetilde{\theta}^{\dagger}_{\Upsilon}$ are trivial on the unipotent parts of this decomposition. Hence, we have

$$I_{\widetilde{M}^{\dagger}}(\widetilde{\theta}^{\dagger}_{\Lambda} |_{\widetilde{M}^{\dagger}}, \widetilde{\theta}^{\dagger}_{\Upsilon} |_{\widetilde{M}^{\dagger}}) = I_{\widetilde{G}}(\widetilde{\theta}^{\dagger}_{\Lambda}, \widetilde{\theta}^{\dagger}_{\Upsilon}) \cap \widetilde{M}^{\dagger}.$$

Therefore

$$I_{\widetilde{M}^{\dagger}}(\widetilde{\theta}^{\dagger}_{\Lambda} |_{\widetilde{M}^{\dagger}}, \widetilde{\theta}^{\dagger}_{\Upsilon} |_{\widetilde{M}^{\dagger}}) = (\widetilde{J}^{\dagger}_{\Upsilon} \cap \widetilde{M}^{\dagger})(\widetilde{G}^{\dagger}_E \cap \widetilde{M}^{\dagger})(\widetilde{J}^{\dagger}_{\Lambda} \cap \widetilde{M}^{\dagger}).$$

Restricting this equality to a single block in $\widetilde{M}^{\dagger}$ we recover $I_{\widetilde{G}}(\widetilde{\theta}_{\Lambda}, \widetilde{\theta}_{\Upsilon}) = \widetilde{J}_{\Upsilon} \widetilde{G}_E \widetilde{J}_{\Lambda}$. □

Suppose further that $[\Lambda, n_{\Lambda}, 0, \beta]$ and $[\Upsilon, n_{\Upsilon}, 0, \beta]$ are self-dual. Let $\theta_{\Lambda} \in \mathcal{C}_-(\Lambda, 0, \beta)$ and $\theta_{\Upsilon} = \tau_{\Lambda,\Upsilon,\beta}(\theta_{\Lambda})$. Let $J_{\Lambda} = J(\beta, \Lambda)$ and $J_{\Upsilon} = J(\beta, \Upsilon)$.

Theorem 3.10. We have $I_G(\theta_{\Lambda}, \theta_{\Upsilon}) = J_{\Upsilon} G_E J_{\Lambda}$.

Proof. Let $\widetilde{\theta}_{\Lambda} \in \mathcal{C}(\Lambda, 0, \beta)$ and $\widetilde{\theta}_{\Upsilon} \in \mathcal{C}(\Upsilon, 0, \beta)$ be self-dual semisimple characters which restrict to θ_{Λ} and θ_{Υ} respectively. Since $\widetilde{\theta}_{\Upsilon}$ is the unique Σ-fixed semisimple character restricting to θ_{Υ}, we have $\widetilde{\theta}_{\Upsilon} = \tau(\widetilde{\theta}_{\Lambda})$. Furthermore, letting $\mathbf{g}$ denote the Glauberman correspondence (*cf.* [22, §2] and the references therein), $\theta_{\Lambda} = \mathbf{g}(\widetilde{\theta}_{\Lambda})$ and $\theta_{\Upsilon} = \mathbf{g}(\widetilde{\theta}_{\Upsilon})$. By [22, Corollary 2.5], $I_g(\widetilde{\theta}_{\Lambda}, \widetilde{\theta}_{\Upsilon}) \neq 0$ if and only if $I_g(\mathbf{g}(\widetilde{\theta}_{\Lambda}), \mathbf{g}(\widetilde{\theta}_{\Upsilon})) \neq 0$. Therefore,

$$I_G(\theta_{\Lambda}, \theta_{\Upsilon}) = I_G(\widetilde{\theta}_{\Lambda}, \widetilde{\theta}_{\Upsilon}) \cap G.$$

Furthermore, $I_G(\widetilde{\theta}_{\Lambda}, \widetilde{\theta}_{\Upsilon}) = \widetilde{J}_{\Upsilon} \widetilde{G}_E \widetilde{J}_{\Lambda}$ by Theorem 3.9, and $(\widetilde{J}_{\Upsilon} \widetilde{G}_E \widetilde{J}_{\Lambda}) \cap G = (\widetilde{J}^1_{\Upsilon} \widetilde{G}_E \widetilde{J}^1_{\Lambda}) \cap G = J^1_{\Upsilon} G_E J^1_{\Lambda} = J_{\Upsilon} G_E J_{\Lambda}$ by Theorem 2.7 and the semisimple intersection property Corollary 3.7. □

3.5 Some exact sequences

Let $[\Lambda, n_{\Lambda}, 0, \beta]$ be a semisimple stratum in A. We denote by a_{β} the adjoint map given by $a_{\beta}(x) = \beta x - x\beta$ for $x \in A$, and by s a *tame corestriction* on A relative to $F[\beta]/F$ (cf. [4, 1.3] and [23, Proposition 3.31]).

Lemma 3.11. (i) Let $[\Lambda, n_\Lambda, 0, \beta]$ be a semisimple stratum in A. The sequence

$$0 \longrightarrow \mathfrak{Q}(\beta,\Lambda) \longrightarrow \mathfrak{J}^1(\beta,\Lambda) \xrightarrow{a_\beta} \mathfrak{H}^1(\beta,\Lambda)^* \xrightarrow{s} \mathfrak{B}(\beta,\Lambda) \longrightarrow 0$$

is exact.

(ii) Let $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ be semisimple strata in A and $y \in \widetilde{G}_E$. The sequence

$$0 \longrightarrow \mathfrak{Q}(\beta,\Lambda) + (\mathfrak{Q}(\beta,\Upsilon))^y \longrightarrow \mathfrak{J}^1(\beta,\Lambda) + (\mathfrak{J}^1(\beta,\Upsilon))^y \xrightarrow{\alpha_\beta}$$
$$\mathfrak{H}^1(\beta,\Lambda)^* + (\mathfrak{H}^1(\beta,\Upsilon)^*)^y \xrightarrow{s} \mathfrak{B}(\beta,\Lambda) + (\mathfrak{B}(\beta,\Upsilon))^y \longrightarrow 0$$

is exact.

Proof. When $\Lambda = \Upsilon$, both parts follow from [23, Lemma 3.17] (*cf.* [*op. cit.*, Proposition 3.31]). Passing to $\dagger$ we have the second exact sequence for the semisimple strata $[\Lambda^\dagger, n_\Lambda, 0, \beta^\dagger]$ and $[\Upsilon^\dagger, n_\Lambda^\dagger, 0, \beta]$, by choosing $x \in \widetilde{G}_E^\dagger$ as in Lemma 3.5, and replacing y by xy in the exact sequence for $\Lambda^\dagger$. Intersecting with a single block we have (ii), while (i) is the special case $y = 1$. □

When we have a self-dual semisimple stratum $[\Lambda, n_\Lambda, 0, \beta]$, we may (and do) choose a tame corestriction s which commutes with the anti-involution σ on A (cf. [20, 2.1.1]). Then we get the self-dual analogue of Lemma 3.11.

Lemma 3.12. (i) Let $[\Lambda, n_\Lambda, 0, \beta]$ be a self-dual semisimple stratum in A. The sequence

$$0 \longrightarrow \mathfrak{Q}^-_\Lambda(\beta,\Lambda) \longrightarrow \mathfrak{J}^1_-(\beta,\Lambda) \xrightarrow{a_\beta} \mathfrak{H}^1_-(\beta,\Lambda)^* \xrightarrow{s} \mathfrak{B}^-(\beta,\Lambda) \longrightarrow 0$$

is exact.

(ii) Let $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ be self-dual semisimple strata in A and $y \in G_E^+$. The sequence

$$0 \longrightarrow \mathfrak{Q}^-(\beta,\Lambda) + (\mathfrak{Q}^-(\beta,\Upsilon))^y \longrightarrow \mathfrak{J}^1_-(\beta,\Lambda) + (\mathfrak{J}^1_-(\beta,\Upsilon))^y \xrightarrow{\alpha_\beta}$$
$$\mathfrak{H}^1_-(\beta,\Lambda)^* + (\mathfrak{H}^1_-(\beta,\Upsilon)^*)^y \xrightarrow{s} \mathfrak{B}^-(\beta,\Lambda) + (\mathfrak{B}^-(\beta,\Upsilon))^y \longrightarrow 0$$

is exact.

4 Intertwining of Heisenberg representations

While up to now, we have been generalising results for both $\widetilde{G}$ and G in this section we concern ourself only with representations of G. The same methods apply for representations of $\widetilde{G}$.

Let $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$ be self-dual semisimple strata in A. In this section we will abbreviate lattices in A^- without the superscript $^-$, to simplify the notation. Thus we write $\mathfrak{Q}_\Lambda = \mathfrak{Q}^-(\beta,\Lambda)$, $\mathfrak{H}_\Lambda = \mathfrak{H}^1_-(\beta,\Lambda)$, $\mathfrak{J}_\Lambda = \mathfrak{J}^1_-(\beta,\Lambda)$, and $\mathfrak{B}_\Lambda = \mathfrak{B}^-(\beta,\Lambda)$, using analogous notation for Υ.

(Note, in particular, that we are omitting the superscript [1] here.) We also write $H^1_\Lambda = H^1(\beta,\Lambda)$ and $J^1_\Lambda = J^1(\beta,\Lambda)$, with $H^1_\Upsilon, J^1_\Upsilon$ defined similarly.

Let $\theta_\Lambda \in \mathcal{C}_-(\Lambda,0,\beta)$ and $\theta_\Upsilon = \tau_{\Lambda,\Upsilon,\beta}(\theta_\Lambda)$. Let η_Λ be the unique Heisenberg representation containing θ_Λ and $\eta_\Upsilon = \tau_{\Lambda,\Upsilon,\beta}(\eta_\Lambda)$ the unique Heisenberg representation containing θ_Υ.

Theorem 4.1. The intertwining of η_Λ and η_Υ in G is given by

$$\dim_R(I_g(\eta_\Lambda,\eta_\Upsilon)) = \begin{cases} 1 & \text{if } g \in J_\Upsilon G_E J_\Lambda; \\ 0 & \text{otherwise.} \end{cases}$$

This theorem is an asymmetric generalisation of [4, Proposition 5.1.8] in the classical groups setting (see also [23, Proposition 3.31]) and we imitate those proofs.

Lemma 4.2. For any $y \in G_E^+$, we have

$$(J^1_\Lambda : J^1_\Lambda \cap yJ^1_\Upsilon y^{-1})(J^1_\Upsilon : y^{-1}J^1_\Lambda y \cap J^1_\Upsilon) = (H^1_\Lambda : H^1_\Lambda \cap yH^1_\Upsilon y^{-1})(H^1_\Upsilon : y^{-1}H^1_\Lambda y \cap H^1_\Upsilon).$$

Proof. We begin by recalling the following from [4]: let $0 \to V_1 \to V_2 \to V_3 \to V_4 \to 0$ be an exact sequence of finite-dimensional F-vector spaces and, for $1 \le i \le 4$, let μ_i be an F-Haar measure on V_i. By [4, Lemma 5.1.3], there is a constant $c \in F^\times$ such that, if the sequence restricts to an exact sequence $0 \to L_1 \to L_2 \to L_3 \to L_4 \to 0$ of $\mathfrak{o}_F$-lattices L_i in V_i, then

$$\frac{\mu_1(L_1)\mu_3(L_3)}{\mu_2(L_2)\mu_4(L_4)} = c.$$

Moreover, $\mu_1(L_1)\mu_1(L_1^*)$ is also independent of the $\mathfrak{o}_F$-lattice L_1, by [4, Lemma 5.1.5].

We have such an exact sequence

$$0 \longrightarrow B \xrightarrow{\ s\ } A \xrightarrow{\ a_\beta\ } A \xrightarrow{\ s\ } B \longrightarrow 0,$$

and, choosing F-Haar measures μ_A on A and μ_B on B, we denote by $c \in F^\times$ the invariant given by [4, Lemma 5.1.3], as above. Now we apply this to the rows of the following giant commutative diagram of $\mathfrak{o}_F$-lattices, which we get from Lemma 3.12(ii).

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{Q}_\Lambda \cap \mathfrak{Q}_\Upsilon^y & \longrightarrow & \mathfrak{J}_\Lambda \cap \mathfrak{J}_\Upsilon^y & \longrightarrow & \mathfrak{H}_\Lambda^* \cap (\mathfrak{H}_\Upsilon^*)^y & \longrightarrow & \mathfrak{B}_\Lambda \cap \mathfrak{B}_\Upsilon^y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{Q}_\Lambda \oplus \mathfrak{Q}_\Upsilon^y & \longrightarrow & \mathfrak{J}_\Lambda \oplus \mathfrak{J}_\Upsilon^y & \longrightarrow & \mathfrak{H}_\Lambda^* \oplus (\mathfrak{H}_\Upsilon^*)^y & \longrightarrow & \mathfrak{B}_\Lambda \oplus \mathfrak{B}_\Upsilon^y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{Q}_\Lambda + \mathfrak{Q}_\Upsilon^y & \longrightarrow & \mathfrak{J}_\Lambda + \mathfrak{J}_\Upsilon^y & \longrightarrow & \mathfrak{H}_\Lambda^* + (\mathfrak{H}_\Upsilon^*)^y & \longrightarrow & \mathfrak{B}_\Lambda + \mathfrak{B}_\Upsilon^y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Using the first row, we get

$$\frac{\mu_A(\mathfrak{H}_\Lambda^* \cap (\mathfrak{H}_\Upsilon^*)^y)}{\mu_A(\mathfrak{J}_\Lambda \cap \mathfrak{J}_\Upsilon^y)} = c\,\frac{\mu_B(\mathfrak{B}_\Lambda \cap \mathfrak{B}_\Upsilon^y)}{\mu_B(\mathfrak{Q}_\Lambda \cap \mathfrak{Q}_\Upsilon^y)}.$$

we have $\mu_B(\mathfrak{Q}_\Lambda \cap \mathfrak{Q}_\Upsilon^y) = \mu_B(\mathfrak{Q}_\Lambda)\mu_B(\mathfrak{Q}_\Upsilon)/\mu_B(\mathfrak{Q}_\Lambda + \mathfrak{Q}_\Upsilon^y)$, from the first column, and similarly for $\mu_A(\mathfrak{H}_\Lambda^* \cap (\mathfrak{H}_\Upsilon^*)^y)$, whence

$$\frac{\mu_A(\mathfrak{H}_\Lambda^*)\mu_A(\mathfrak{H}_\Upsilon^*)}{\mu_A(\mathfrak{J}_\Lambda \cap \mathfrak{J}_\Upsilon^y)\mu_A(\mathfrak{H}_\Lambda^* + (\mathfrak{H}_\Upsilon^*)^y)} = c\,\frac{\mu_B(\mathfrak{B}_\Lambda \cap \mathfrak{B}_\Upsilon^y)\mu_B(\mathfrak{Q}_\Lambda + \mathfrak{Q}_\Upsilon^y)}{\mu_B(\mathfrak{Q}_\Lambda)\mu_B(\mathfrak{Q}_\Upsilon)},$$

Since $(\mathfrak{H}_\Lambda \cap \mathfrak{H}_\Upsilon^y)^* = \mathfrak{H}_\Lambda^* + (\mathfrak{H}_\Upsilon^*)^y$, from [4, Lemma 5.1.5] we have

$$\mu_A(\mathfrak{H}_\Lambda^* + (\mathfrak{H}_\Upsilon^*)^y)\mu_A(\mathfrak{H}_\Lambda \cap \mathfrak{H}_\Upsilon^y) = (\mu_A(\mathfrak{H}_\Lambda)\mu_A(\mathfrak{H}_\Lambda^*)\mu_A(\mathfrak{H}_\Upsilon)\mu_A(\mathfrak{H}_\Upsilon^*))^{\frac{1}{2}},$$

with a similar result using $(\mathfrak{B}_\Lambda \cap \mathfrak{B}_\Upsilon^y)^* = \mathfrak{Q}_\Lambda + \mathfrak{Q}_\Upsilon^y$. Substituting, we get

$$\frac{\mu_A(\mathfrak{H}_\Lambda \cap \mathfrak{H}_\Upsilon^y)}{\mu_A(\mathfrak{J}_\Lambda \cap \mathfrak{J}_\Upsilon^y)}\left(\frac{\mu_A(\mathfrak{H}_\Lambda^*)\mu_A(\mathfrak{H}_\Upsilon^*)}{\mu_A(\mathfrak{H}_\Lambda)\mu_A(\mathfrak{H}_\Upsilon)}\right)^{\frac{1}{2}} = c\left(\frac{\mu_B(\mathfrak{B}_\Lambda)\mu_B(\mathfrak{B}_\Upsilon)}{\mu_B(\mathfrak{Q}_\Lambda)\mu_B(\mathfrak{Q}_\Upsilon)}\right)^{\frac{1}{2}}.$$

Finally, from Lemma 3.11(i), we have

$$\frac{\mu_A(\mathfrak{H}_\Lambda^*)}{\mu_A(\mathfrak{J}_\Lambda)} = c\,\frac{\mu_B(\mathfrak{B}_\Lambda)}{\mu_B(\mathfrak{Q}_\Lambda)},$$

and similarly for Υ, which gives

$$\frac{\mu_A(\mathfrak{H}_\Lambda \cap \mathfrak{H}_\Upsilon^y)}{\mu_A(\mathfrak{J}_\Lambda \cap \mathfrak{J}_\Upsilon^y)} = \left(\frac{\mu_A(\mathfrak{H}_\Lambda)\mu_A(\mathfrak{H}_\Upsilon)}{\mu_A(\mathfrak{J}_\Lambda)\mu_A(\mathfrak{J}_\Upsilon)}\right)^{\frac{1}{2}}.$$

Conjugating by y, we get the same formula for $\mu_A({}^y\mathfrak{H}_\Lambda \cap \mathfrak{H}_\Upsilon)/\mu_A({}^y\mathfrak{J}_\Lambda \cap \mathfrak{J}_\Upsilon)$. Multiplying these and rearranging, we get

$$\left(\frac{\mu_A(\mathfrak{H}_\Lambda)}{\mu_A(\mathfrak{H}_\Lambda \cap \mathfrak{H}_\Upsilon^y)}\right)\left(\frac{\mu_A(\mathfrak{H}_\Upsilon)}{\mu_A({}^y\mathfrak{H}_\Lambda \cap \mathfrak{H}_\Upsilon)}\right) = \left(\frac{\mu_A(\mathfrak{J}_\Lambda)}{\mu_A(\mathfrak{J}_\Lambda \cap \mathfrak{J}_\Upsilon^y)}\right)\left(\frac{\mu_A(\mathfrak{J}_\Upsilon)}{\mu_A({}^y\mathfrak{J}_\Lambda \cap \mathfrak{J}_\Upsilon)}\right).$$

The result follows from this additive statement since H_Λ^1 is the image under the Cayley transform of $\mathfrak{H}_\Lambda$, and similarly for the other groups involved. □

Lemma 4.3. For any $y \in G_E^+$, we have

$$\left|H_\Upsilon^1 \backslash J_\Upsilon^1 y J_\Lambda^1 / H_\Lambda^1\right| = (J_\Lambda^1 : H_\Lambda^1)^{\frac{1}{2}}(J_\Upsilon^1 : H_\Upsilon^1)^{\frac{1}{2}}.$$

Proof. Fix a F-Haar measure μ on G. Decomposing $J_\Upsilon^1 y J_\Lambda^1$ by right J_Υ^1-cosets, and by left J_Λ^1-cosets, and then multiplying, we have

$$\mu(J_\Upsilon^1 y J_\Lambda^1)^2 = \mu(J_\Lambda^1)\mu(J_\Upsilon^1)(J_\Lambda^1 : J_\Lambda^1 \cap y^{-1}J_\Upsilon^1 y)(J_\Upsilon^1 : yJ_\Lambda^1 y^{-1} \cap J_\Upsilon^1).$$

By normality of H_Λ^1 in J_Λ^1 and H_Υ^1 in J_Υ^1, for any $y' \in J_\Upsilon^1 y J_\Lambda^1$ we similarly have

$$\mu(H_\Upsilon^1 y' H_\Lambda^1)^2 = \mu(H_\Lambda^1)\mu(H_\Upsilon^1)(H_\Lambda^1 : H_\Lambda^1 \cap y^{-1}H_\Upsilon^1 y)(H_\Upsilon^1 : yH_\Lambda^1 y^{-1} \cap H_\Upsilon^1).$$

Therefore, we have $\left|H_\Lambda^1 \backslash J_\Lambda^1 g J_\Upsilon^1 / H_\Upsilon^1\right| = (J_\Lambda^1 : H_\Lambda^1)^{\frac{1}{2}}(J_\Upsilon^1 : H_\Upsilon^1)^{\frac{1}{2}}$, by Lemma 4.2. □

Proof of Theorem 4.1. By [23, Corollary 3.29] the induced representation $\mathrm{ind}_{H^1_\Lambda}^{J^1_\Lambda}(\theta_\Lambda)$ is a multiple of η_Λ, that multiple being $(J^1_\Lambda : H^1_\Lambda)^{\frac{1}{2}}$, and analogously for θ_Υ. Thus

$$\dim_R(I_g(\mathrm{ind}_{H^1_\Lambda}^{J^1_\Lambda}\theta_\Lambda, \mathrm{ind}_{H^1_\Upsilon}^{J^1_\Upsilon}\theta_\Upsilon)) = (J^1_\Lambda : H^1_\Lambda)^{\frac{1}{2}}(J^1_\Upsilon : H^1_\Upsilon)^{\frac{1}{2}}\dim_R(I_g(\eta_\Lambda, \eta_\Upsilon)).$$

By Lemma 2.2(iii),

$$\mathcal{H}(G, \mathrm{ind}_{H^1_\Lambda}^{J^1_\Lambda}\theta_\Lambda, \mathrm{ind}_{H^1_\Upsilon}^{J^1_\Upsilon}\theta_\Upsilon)_g \simeq \coprod_{\substack{h\in H^1_\Lambda\backslash G/H^1_\Upsilon\\ J^1_\Lambda h J^1_\Upsilon = J^1_\Lambda g J^1_\Upsilon}} \mathcal{H}(G, \theta_\Lambda, \theta_\Upsilon)_h.$$

Therefore, by Theorem 3.10 and Lemma 4.3, we have

$$\dim_R(I_g(\mathrm{ind}_{H^1_\Lambda}^{J^1_\Lambda}\theta_\Lambda, \mathrm{ind}_{H^1_\Upsilon}^{J^1_\Upsilon}\theta_\Upsilon)) = \begin{cases}(J^1_\Lambda : H^1_\Lambda)^{\frac{1}{2}}(J^1_\Upsilon : H^1_\Upsilon)^{\frac{1}{2}} & \text{if } g \in J_\Upsilon G_E J_\Lambda;\\ 0 & \text{otherwise,}\end{cases}$$

whence the result. □

Remark 4.4. In the setting of Theorem 3.10, we also have $I_{G^+}(\theta_\Lambda, \theta_\Upsilon) = J^1_\Upsilon G^+_E J^1_\Lambda$ by intersecting the intertwining of $I_{\widetilde{G}}(\widetilde{\theta}_\Lambda, \widetilde{\theta}_\Upsilon)$ with G^+ rather than G. Moreover, in the setting of Theorem 4.1 the same proof shows that the intertwining of η_Λ and η_Υ in G^+ is given by

$$\dim_R(I_g(\eta_\Lambda, \eta_\Upsilon)) = \begin{cases}1 & \text{if } g \in J^1_\Upsilon G^+_E J^1_\Lambda;\\ 0 & \text{otherwise.}\end{cases}$$

We will also make use of the following lemma of [24].

Lemma 4.5 ([24, Lemma 3.6])**.** We have $\dim(\eta_\Lambda)/\dim(\eta_\Upsilon) = (J^1_\Lambda : J^1_\Upsilon)/(P^1(\Lambda_E) : P^1(\Upsilon_E))$.

Conjugating if necessary, we assume that $\mathfrak{B}(\Lambda)$ and $\mathfrak{B}(\Upsilon)$ contain a common minimal self-dual hereditary order $\mathfrak{B}(\Gamma)$ corresponding to an $\mathfrak{o}_E$-lattice sequence Γ in V; thus $P^\circ(\Gamma_E)$ is an Iwahori subgroup of G_E. Let $\theta_\Gamma = \tau_{\Lambda,\Gamma,\beta}(\theta_\Lambda) = \tau_{\Upsilon,\Gamma,\beta}(\theta_\Upsilon) \in \mathcal{C}_-(\Gamma, 0, \beta)$. Let η_Γ be the unique Heisenberg representation containing θ_Γ and let $J_\Gamma = J(\beta, \Gamma)$. Since $P^1(\Gamma_E)$ normalises J^1_Λ and J^1_Υ we can form the groups $J^1_{\Gamma,\Lambda} = P^1(\Gamma_E)J^1_\Lambda$ and $J^1_{\Gamma,\Upsilon} = P^1(\Gamma_E)J^1_\Upsilon$.

Lemma 4.6 ([24, Proposition 3.7])**.** There exist unique irreducible representations $\eta_{\Gamma,\Lambda}$ of $J^1_{\Gamma,\Lambda}$ and $\eta_{\Gamma,\Upsilon}$ of $J^1_{\Gamma,\Upsilon}$ such that

(i) $\eta_{\Gamma,\Lambda}|_{J^1_\Lambda} = \eta_\Lambda$ and $\eta_{\Gamma,\Upsilon}|_{J^1_\Upsilon} = \eta_\Upsilon$;

(ii) $\eta_{\Gamma,\Lambda}$, $\eta_{\Gamma,\Upsilon}$ and η_Γ induce equivalent irreducible representations of $P^1(\Gamma)$.

We can now extend the intertwining result of [24, Proposition 3.7]. The proof is essentially the same as that of [4, Proposition 5.1.19].

Lemma 4.7. The intertwining of $\eta_{\Gamma,\Lambda}$ and $\eta_{\Gamma,\Upsilon}$ in G is given by

$$\dim_R(I_g(\eta_{\Gamma,\Lambda}, \eta_{\Gamma,\Upsilon})) = \begin{cases}1 & \text{if } g \in J^1_{\Gamma,\Upsilon} G_E J^1_{\Gamma,\Lambda};\\ 0 & \text{otherwise.}\end{cases}$$

We remark that $J^1_{\Gamma,\Upsilon}G_E J^1_{\Gamma,\Lambda} = J^1_\Upsilon G_E J^1_\Lambda$, and that we have a similar result for the intertwining in G^+.

Proof. We have $I_G(\eta_{\Gamma,\Lambda},\eta_{\Gamma,\Upsilon}) \subseteq I_G(\eta_\Lambda,\eta_\Upsilon) = J^1_\Upsilon G_E J^1_\Lambda$ and the non-zero intertwining spaces are one-dimensional by Lemma 4.1. If $x \in G_E$ then $x \in I_G(\eta_\Gamma)$, by Theorem 4.1, so $x \in I_G(\mathrm{Ind}_{J^1_\Gamma}^{P^1(\Gamma)}(\eta_\Gamma))$. Thus $x \in I_G(\mathrm{Ind}_{J^1_{\Gamma,\Lambda}}^{P^1(\Gamma)}(\eta_{\Gamma,\Lambda}), \mathrm{Ind}_{J^1_{\Gamma,\Upsilon}}^{P^1(\Gamma)}(\eta_{\Gamma,\Upsilon}))$ by Lemma 4.6. Therefore there exist $u,v \in P^1(\Gamma)$ such that $uxv \in I_G(\eta_{\Gamma,\Lambda},\eta_{\Gamma,\Upsilon})$; since this intertwining set is contained in $J^1_\Upsilon G_E J^1_\Lambda$, there exist $j_\Lambda \in J^1_\Lambda$ and $j_\Upsilon \in J^1_\Upsilon$ such that $j_\Upsilon uxvj_\Lambda \in G_E$. By Corollary 3.7, $P^1(\Gamma)xP^1(\Lambda)\cap G_E = P^1(\Gamma_E)xP^1(\Lambda_E)$. Therefore, we can find $u' \in P^1(\Gamma_E)$ and $v' \in P^1(\Lambda_E)$ such that $u'xv' = j_1uxvj_2$, whence $x \in I_G(\eta_{\Gamma,\Lambda},\eta_{\Gamma,\Upsilon})$. ☐

5 β-extensions

We generalise the definition of β-extensions for classical groups, as defined by the second author when $R = \mathbb{C}$ in [24]. As the J groups are not pro-p, the proofs of the corresponding statements need to be adapted in characteristic ℓ. However, as the J^1 groups are pro-p, these modifications are relatively simple.

Let $[\Lambda, n_\Lambda, 0, \beta]$ be a self-dual semisimple stratum, $\theta_\Lambda \in \mathcal{C}_-(\Lambda, 0, \beta)$ and η_Λ the unique Heisenberg representation containing θ_Λ. We will write $\mathfrak{B}(\Lambda_E) = \mathfrak{B}(\beta,\Lambda)$ for the hereditary $\mathfrak{o}_E$-order in B determined by the lattice sequence Λ, and will abbreviate $J^+_\Lambda = J^+(\beta,\Lambda)$, etc.

Theorem 5.1. Let Γ be any self-dual $\mathfrak{o}_E$-lattice sequence such that $\mathfrak{B}(\Gamma_E)$ is a minimal self-dual $\mathfrak{o}_E$-order in B contained in $\mathfrak{B}(\Lambda_E)$. There exists a representation κ^+_Λ of J^+_Λ extending $\eta_{\Gamma,\Lambda}$. Moreover, any two such extensions differ by a character of $P^+(\Lambda_E)/P^1(\Lambda_E)$ which is trivial on the subgroup generated by all its unipotent subgroups.

Proof. The proof follows *mutatis mutandis* the proof of [24, Theorem 4.1]. ☐

If $\mathfrak{B}(\Lambda_E)$ is a maximal self-dual $\mathfrak{o}_E$-order in B, we call an extension κ^+_Λ of η_Λ, as constructed in Theorem 5.1, a *β-extension*. In the case where $\mathfrak{B}(\Lambda_E)$ is not maximal, while Theorem 5.1, gives a collection of extensions of η_Λ it gives too many such extensions. As in the complex case, we define β-extensions in the non-maximal case by compatibility with β-extensions in the maximal case. Let $[\Upsilon, n_\Upsilon, 0, \beta]$ be a self-dual semisimple stratum such that $\mathfrak{B}(\Upsilon_E)$ is maximal and $\mathfrak{B}(\Lambda_E) \subseteq \mathfrak{B}(\Upsilon_E)$; let $\theta_\Upsilon = \tau_{\Lambda,\Upsilon,\beta}(\theta_\Lambda)$ and $\eta_\Upsilon = \tau_{\Lambda,\Upsilon,\beta}(\eta_\Lambda)$. Let $J^1_{\Lambda,\Upsilon} = P^1(\Lambda_E)J^1_\Upsilon$ and $J^+_{\Lambda,\Upsilon} = P^+(\Lambda_E)J^1_\Upsilon$.

Theorem 5.2. There is a canonical bijection

$$b_{\Lambda,\Upsilon} : \{\text{extensions } \kappa^+_\Lambda \text{ of } \eta_\Lambda \text{ to } J^+_\Lambda\} \to \{\text{extensions } \kappa^+_{\Lambda,\Upsilon} \text{ of } \eta_\Upsilon \text{ to } J^+_{\Lambda,\Upsilon}\}.$$

Furthermore, if $\mathfrak{A}(\Lambda) \subseteq \mathfrak{A}(\Upsilon)$ then $b_{\Lambda,\Upsilon}(\kappa^+_\Lambda)$ is the unique extension of η_Υ such that κ^+_Λ and $b_{\Lambda,\Upsilon}(\kappa^+_\Lambda)$ induce to equivalent irreducible representations of $P^+(\Lambda_E)P^1(\Lambda)$.

Proof. Assume that $\mathfrak{A}(\Lambda) \subseteq \mathfrak{A}(\Upsilon)$ and, as in the proof of [24, Lemma 4.3, Case (i)], we follow the argument of [4, Proposition 5.2.5]. Let κ^+_Λ be an extension of η^+_Λ to J^+_Λ and put

$$\lambda = \mathrm{ind}_{J^+_\Lambda}^{P^+(\Lambda_E)P^1(\Lambda)}(\kappa^+_\Lambda).$$

By Mackey Theory,

$$\mathrm{Res}^{P^+(\Lambda_E)P^1(\Lambda)}_{P^1(\Lambda)}(\lambda) \simeq \mathrm{ind}_{J^1_\Lambda}^{P^1(\Lambda)}(\eta_\Lambda),$$

which is irreducible, since $I_G(\eta_\Lambda) \cap P^1(\Lambda) = J^1_\Lambda$; in particular, λ is irreducible. Moreover, by Lemma 4.6,

$$\operatorname{Res}^{P^+(\Lambda_E)P^1(\Lambda)}_{P^1(\Lambda)}(\lambda) \simeq \operatorname{ind}^{P^1(\Lambda)}_{J^1_{\Lambda,\Upsilon}}(\eta_{\Lambda,\Upsilon}),$$

so there is an irreducible quotient $\kappa^+_{\Lambda,\Upsilon}$ of $\operatorname{Res}^{P^+(\Lambda_E)P^1(\Lambda)}_{J^+_{\Lambda,\Upsilon}} \lambda$ which contains $\eta_{\Lambda,\Upsilon}$; indeed, there is a unique such quotient, since $\eta_{\Lambda,\Upsilon}$ appears with multiplicity 1 in $\operatorname{Res}^{P^1(\Lambda)}_{J^1(\Lambda,\Upsilon)} \operatorname{ind}^{P^1(\Lambda)}_{J^1_{\Lambda,\Upsilon}}(\eta_{\Lambda,\Upsilon})$, by Lemma 4.7. Now put

$$\lambda' = \operatorname{ind}^{P^+(\Lambda_E)P^1(\Lambda)}_{J^+_{\Lambda,\Upsilon}} \kappa^+_{\Lambda,\Upsilon}.$$

Then, as above,

$$\operatorname{Res}^{P^+(\Lambda_E)P^1(\Lambda)}_{P^1(\Lambda)}(\lambda') \simeq \operatorname{ind}^{P^1(\Lambda)}_{J^1_{\Lambda,\Upsilon}}(\eta_{\Lambda,\Upsilon}),$$

so that λ' is also irreducible, and hence equivalent to λ. Comparing dimensions, using Lemma 4.5, we see that $\kappa^+_{\Lambda,\Upsilon}$ extends $\eta_{\Lambda,\Upsilon}$ as required.

The argument is reversible, giving the required bijection, and the remainder of the proof follows from this special case $\mathfrak{A}(\Lambda) \subseteq \mathfrak{A}(\Upsilon)$, exactly as in the proof of [24, Lemma 4.3]. □

An extension κ^+_Λ of η_Λ to J^+_Λ is called a *β-extension* if there exist a self-dual semisimple stratum $[\Upsilon, n_\Upsilon, 0, \beta]$ such that $\mathfrak{B}(\Upsilon_E)$ is a maximal self-dual $\mathfrak{o}_E$-order containing $\mathfrak{B}(\Lambda_E)$ and a β-extension κ^+_Υ of $\eta_\Upsilon = \tau_{\Lambda,\Upsilon,\beta}(\eta_\Lambda)$ such that $b_{\Lambda,\Upsilon}(\kappa^+_\Lambda) = \operatorname{Res}^{J^+_\Upsilon}_{J^+_{\Lambda,\Upsilon}}(\kappa^+_\Upsilon)$. More precisely, we say that such a representation κ^+_Λ is a *β-extension relative to Υ*.

There is a standard (non-canonical) choice for the self-dual $\mathfrak{o}_E$-lattice sequence Υ. Let

$$\mathfrak{M}^i_\Lambda(2r+s) = \begin{cases} \mathfrak{p}^r_{E_i}\Lambda^i(0) & \text{if } i \in I_+; \\ \mathfrak{p}^r_{E_i}\Lambda^i(s) & \text{if } i \in I_0; \\ \mathfrak{p}^r_{E_i}\Lambda^i(1) & \text{if } i \in I_-. \end{cases}$$

Then $\mathfrak{M}_\Lambda = \bigoplus_{i\in I} \mathfrak{M}^i_\Lambda$ is a self-dual $\mathfrak{o}_E$-lattice sequence in V such that $\mathfrak{A}(\mathfrak{M}_\Lambda) \cap B$ is a maximal self-dual hereditary $\mathfrak{o}_E$-order in B. A representation κ^+_Λ of J^+_Λ is called a *standard β-extension* of η_Λ if it is a β-extension relative to $\mathfrak{M}_\Lambda$.

If κ^+_Λ is a standard β extension and $[\Upsilon, n_\Upsilon, 0, \beta]$ is another self-dual semisimple stratum with $\mathfrak{M}_\Upsilon = \mathfrak{M}_\Lambda$, we say that the standard β-extension κ^+_Υ of J^+_Υ is *compatible* κ^+_Λ if they correspond to the same β-extension of $J^+_{\mathfrak{M}_\Lambda}$. In the case that $\mathfrak{A}(\Lambda) \subseteq \mathfrak{A}(\Upsilon)$, this is equivalent to saying that κ^+_Λ and $\operatorname{Res}^{J^+_\Upsilon}_{J^+_{\Lambda,\Upsilon}} \kappa^+_\Upsilon$ induce to equivalent (irreducible) representations of $P^+(\Lambda_E)P^1(\Lambda)$.

We also call the restriction from J^+_Λ to J_Λ (resp. J°_Λ) of a (standard) β-extension a (standard) β-extension and denote the restriction of κ^+_Λ to J_Λ (resp. J°_Λ) by κ_Λ (resp. κ°_Λ), and speak of compatibility for these standard β-extensions.

Remark 5.3. Being smooth representations of a compact group, all $\overline{\mathbb{Q}}_\ell$-beta extensions are integral. When $\mathfrak{B}(\Lambda_E)$ is a maximal self-dual $\mathfrak{o}_E$-order in B, it is straightforward to check that reduction modulo-ℓ defines a surjective map from the set of $\overline{\mathbb{Q}}_\ell$-beta extensions to the set of $\overline{\mathbb{F}}_\ell$-beta extensions. Moreover, the bijections $b_{\Lambda,\Upsilon}$, for $\overline{\mathbb{Q}}_\ell$-representation and $\overline{\mathbb{F}}_\ell$-representations, defined by Theorem 5.2 commute with reduction modulo-ℓ; thus reduction modulo-ℓ defines a surjective map from the set of $\overline{\mathbb{Q}}_\ell$-beta extensions to the set of $\overline{\mathbb{F}}_\ell$-beta extensions in all cases. Moreover, the reduction modulo-ℓ of a standard $\overline{\mathbb{Q}}_\ell$-beta extension is a standard $\overline{\mathbb{F}}_\ell$-beta extension.

5.1 Induction functors for classical groups

Now suppose that $[\Lambda, n, 0, \beta]$ is a skew semisimple stratum in A. Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$, let η be the unique Heisenberg extension of θ to $J^1(\beta, \Lambda)$ and κ be a β-extension of η to $J(\beta, \Lambda)$. Recall that we have an exact sequence

$$1 \to J^1(\beta, \Lambda) \to J(\beta, \Lambda) \to M(\Lambda_E) \to 1,$$

with $M(\Lambda_E)$ a (possibly disconnected) finite reductive group.

We have a functor $\mathrm{I}_\kappa : \mathfrak{R}_R(M(\Lambda_E)) \to \mathfrak{R}_R(G)$, which we call κ-induction, given by

$$\mathrm{I}_\kappa(-) = \mathrm{ind}_{J(\beta,\Lambda)}^{G}(\kappa \otimes \mathrm{infl}_{M(\Lambda_E)}^{J(\beta,\Lambda)}(-))$$

where $\mathrm{infl}_{M(\Lambda_E)}^{J(\beta,\Lambda)} : \mathfrak{R}_R(M(\Lambda_E)) \to \mathfrak{R}_R(J(\beta, \Lambda))$ is the functor defined by trivial inflation to $J^1(\beta, \Lambda)$. The functor I_κ possesses a right adjoint $\mathrm{R}_\kappa : \mathfrak{R}_R(G) \to \mathfrak{R}_R(M(\Lambda_E))$, which we call κ-restriction, given by

$$\mathrm{R}_\kappa(-) = \mathrm{Hom}_{J^1(\beta,\Lambda)}(\kappa, -).$$

If π is a smooth representation of G, the action of $M(\Lambda_E)$ on $\mathrm{R}_\kappa(\pi)$ is given as follows: if $f \in \mathrm{R}_\kappa(\pi)$, $m \in M(\Lambda_E)$ and $j \in J(\beta, \Lambda)$ is any representative for m, then $m \cdot f = \pi(j) \circ f \circ \kappa(j^{-1})$. The functors of κ-induction and κ-restriction are exact functors as $J^1(\beta, \Lambda)$ is pro-p.

Now let $[\Upsilon, n_\Upsilon, 0, \beta]$ be another self-dual semisimple stratum with $\mathfrak{M}_\Upsilon = \mathfrak{M}_\Lambda$ and $\mathfrak{A}(\Lambda) \subseteq \mathfrak{A}(\Upsilon)$, and let θ_Υ be the transfer of θ. Let κ be a β-extension and let κ_Υ be a compatible β-extension of $J(\beta, \Upsilon)$. Set $P_{\Lambda,\Upsilon}^E = P(\Lambda_E)/P^1(\Upsilon_E)$, a parabolic subgroup of $M(\Upsilon_E)$ with Levi factor $M(\Lambda_E)$; we write $i_{P_{\Lambda,\Upsilon}^E}^{M(\Upsilon_E)}$ for the parabolic induction functor and $r_{P_{\Lambda,\Upsilon}^E}^{M(\Upsilon_E)}$ for its adjoint. By transitivity of induction, an exercise shows that we have isomorphisms of functors

$$\mathrm{I}_{\kappa_\Upsilon} \circ i_{P_{\Lambda,\Upsilon}^E}^{M(\Upsilon_E)} \simeq \mathrm{I}_\kappa \text{ and } r_{P_{\Lambda,\Upsilon}^E}^{M(\Upsilon_E)} \circ \mathrm{R}_{\kappa_\Upsilon} \simeq \mathrm{R}_\kappa,$$

where the latter follows from the former by unicity of the adjoint.

We also have the special case of these functors when the stratum is zero, which we can apply in G_E. Thus, since Λ_E is an $\mathfrak{o}_E$-lattice chain, we have a level zero *parahoric induction* functor $\mathrm{I}_{\Lambda_E} : \mathfrak{R}_R(M(\Lambda_E)) \to \mathfrak{R}_R(G_E)$ attached to $[\Lambda, n, 0, \beta]$ given by

$$\mathrm{I}_{\Lambda_E}(-) = \mathrm{ind}_{P(\Lambda_E)}^{G_E}(\mathrm{infl}_{M(\Lambda_E)}^{P(\Lambda_E)}(-))$$

where $\mathrm{infl}_{M(\Lambda_E)}^{P(\Lambda_E)} : \mathfrak{R}_R(M(\Lambda_E)) \to \mathfrak{R}_R(P(\Lambda_E))$ is the functor defined by trivial inflation to $P^1(\Lambda_E)$. The functor I_{Λ_E} possesses a right adjoint, which we call level zero *parahoric restriction*, $\mathrm{R}_{\Lambda_E} : \mathfrak{R}_R(G_E) \to \mathfrak{R}_R(M(\Lambda_E))$ given by the functor of $P^1(\Lambda_E)$-invariants

$$\mathrm{R}_{\Lambda_E}(-) = (-)^{P^1(\Lambda_E)},$$

with the group $P(\Lambda_E)/P^1(\Lambda_E) \simeq M(\Lambda_E)$ acting naturally. Level zero parahoric induction and restriction are exact functors.

6 Level zero interlude

In this section we recall some results of Morris [18] and Vignéras [27] on level zero representations of G (cf. also [29, §4]). Later, we will apply them to G_E, which will be a product of groups like G over extensions of F. The results of this section apply in the greater generality of [18], and we retain the notation of [ibid.] as it is much more convenient here, as such, the notation of this section is independent of that of the rest of the paper. We recall this notation briefly below and explain how to translate to our notation in the rest of the paper.

Let $\mathbb{G}$ be a connected reductive group over F, $\mathbb{T}$ be a maximal F-split torus in $\mathbb{G}$, and $\mathbb{N} = N_{\mathbb{G}}(\mathbb{T})$. We write $G = \mathbb{G}(F)$, $T = \mathbb{T}(F)$, and $N = \mathbb{N}(F)$ for the respective groups of F-points. Let B be an Iwahori subgroup of G. Following [ibid.], (G, B, N) is called a *generalised affine BN-pair*, and, associated to this data, we have a *generalised affine Weyl group* $W = N/B \cap N$. According to [ibid.], we have a decomposition $W = \Omega \ltimes W'$ with W' the affine Weyl group of some split affine root system. Let S be a set of fundamental reflections in W'.

If $J \subset S$ is a proper subset of S, we let W_J be the subgroup of W generated by the reflections in J. The standard parahoric subgroups of G correspond to proper subsets of S, via $J \subset S$ maps to $P_J = BN_JB$ for N_J any set of representatives of W_J in G. Given a parahoric subgroup P_J, we write U_J for its pro-p unipotent radical and $M_J = P_J/U_J$ the points of a connected reductive group over a finite field. We write U_B for the pro-p unipotent radical of $B = P_\emptyset$.

Let J, K be proper subsets of S. A set of double coset representatives $\overline{D}_{J,K}$ for $W_J\backslash W/W_K$ is called *distinguished* if each representative has minimal length in its double coset, (cf. [ibid., §3.10]). A set of double coset representatives $D_{J,K}$ for $P_J\backslash G/P_K$ is called *distinguished* if its projection to W is a set of distinguished double coset representatives for $W_J\backslash W/W_K$. Let $D_{J,K}$ be a set of distinguished set of double coset representatives for $P_J\backslash G/P_K$. Let $d \in D_{J,K}$ and w be its projection in W. By [ibid., Lemma 3.19, Corollary 3.20, Lemma 3.21], we have

(i) $P_{J\cap wK} = U_J(P_J \cap {}^nP_K)$ with unipotent radical $U_{J\cap wK} = U_J(P_J \cap {}^nU_K)$.

(ii) $P^J_{J\cap wK} = P_{J\cap wK}/U_J$ is a parabolic subgroup of $M_J = P_J \cap U_J$.

We can form the following lattice of groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U_J & \longrightarrow & P_J & \longrightarrow & M_J & \longrightarrow & 1 \\
 & & \Vert & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & U_J & \longrightarrow & P_{J\cap wK} & \longrightarrow & P^J_{J\cap wK} & \longrightarrow & 1 \\
 & & \downarrow & & \Vert & & \uparrow & & \\
1 & \longrightarrow & U_J(P_J \cap {}^nU_K) & \longrightarrow & P_{J\cap wK} & \longrightarrow & M_{J\cap wK} & \longrightarrow & 1
\end{array}$$

Furthermore, as $D_{J,K}^{-1}$ is a set of distinguished double coset representatives for $P_K\backslash G/P_J$, the group $P^K_{w^{-1}J\cap K}$ is a parabolic subgroup of M_K and we can form an analogous diagram for $P^K_{w^{-1}J\cap K}$. Note also that $M_{J\cap wK} = (M_{w^{-1}J\cap K})^w$.

This section collects results based upon the following theorem of Vignéras. Before we state it, we must recall the parahoric induction/restriction functors in this notation; let $I_J : \mathfrak{R}_R(M_J) \to \mathfrak{R}_R(G)$ denote the parahoric induction functor

$$I_J(-) = \operatorname{ind}_{P_J}^G(\operatorname{infl}_{M_J}^{P_J}(-)),$$

and $R_J : \mathfrak{R}_R(G) \to \mathfrak{R}_R(M_J)$ denote, its right adjoint, the parahoric restriction functor

$$R_J(-) = (-)^{U_J}.$$

The normaliser $N_G(P_J)$ of P_J in G normalises U_J, and $M_J^+ = N_G(P_J)/U_J$ contains M_J as a normal subgroup. We write $I_J^+ : \mathfrak{R}_R(M_J^+) \to \mathfrak{R}_R(G)$ for the functor

$$I_J^+(-) = \mathrm{ind}_{N_G(P_J)}^G(-),$$

and $R_J^+ : \mathfrak{R}_R(G) \to \mathfrak{R}_R(M_J^+)$ its right adjoint, again given by U_J-invariants.

Theorem 6.1 ([27, Basic decomposition 5.1])**.** We have an isomorphism of functors

$$\mathrm{R}_J \circ \mathrm{I}_K \simeq \bigoplus_{w \in \overline{D}_{J,K}} i_{P^J_{J\cap wK}}^{M_J} \circ \left(r_{P^K_{w^{-1}J\cap K}}^{M_K} \right)^w .$$

Corollary 6.2. Let τ be a cuspidal R-representation of M_K.

(i) The representation $\mathrm{R}_K \circ \mathrm{I}_K(\tau)$ is a direct sum of conjugates of τ.

$$\mathrm{R}_K \circ I_K(\tau) \simeq \bigoplus_{\substack{w\in W_{K,K}\\ wK=K}} \tau^w.$$

Moreover, if P_K is a maximal and τ^+ is an irreducible R-representation of M_K^+ with cuspidal restriction to M_K, then

$$\mathrm{R}_K^+ \circ \mathrm{I}_K^+(\tau^+) = \tau^+.$$

(ii) Suppose that P_K is maximal and P_J is not conjugate to P_K in G. Then

$$\mathrm{R}_J \circ \mathrm{I}_K(\tau) = 0.$$

Proof. All statements are straightforward applications of the theorem. Part (i) is [27, Corollaries 5.2 & 5.3], and part (ii) follows as if P_J is not conjugate to P_K, then $P^K_{w^{-1}J\cap K}$ is a proper parabolic subgroup of M_K, for any $w \in W_{J,K}$, and hence $r^{M_K}_{P^K_{w^{-1}J\cap K}}(\tau^w) = 0$ by cuspidality. □

Remark 6.3. In case (i) of Lemma 6.2, the direct sum can be infinite. Indeed this is the case when K is empty (and the building of G is not a point).

Finally, we will need the following variant of [18, Proposition 4.13], (cf. [24, Lemma 1.1]), which requires a different proof in our setting.

Lemma 6.4. Let J, K be proper subsets of S, and D be a set of distinguished double coset representatives for $P_K\backslash G/P_J$. Let τ be a representation of M_J with cuspidal restriction to M_J°, and let $n \in D$. If n lies in the support of $\mathcal{H}(G, \tau\,|_{U_B})$, i.e.

$$\mathrm{Hom}_{U_B \cap U_B^n}(\tau, \tau^n) \neq 0,$$

then $wK = J$, where $w \in W$ is the projection of n.

Proof. By [18, Lemma 3.21], we have $P_J \cap U_K^n \subseteq U_{J\cap wK} \subseteq U_B$. Hence, as $U_K^n \subseteq U_B^n$, we have

$$\mathrm{Hom}_{U_B\cap U_B^n}(\tau, \tau^n) \subseteq \mathrm{Hom}_{P_J\cap U_K^n}(\tau, \tau^n) = \mathrm{Hom}_{P_J\cap U_K^n}(\tau, \dim(\tau)1).$$

But, by [ibid.], $P_J \cap U_K^n$ is the unipotent radical of the parabolic subgroup $P_{J\cap wK}/U_J$ of M_J. Hence by cuspidality of τ, we must have $wK = J$. □

6.1 Level zero Hecke algebras

Let $P^\circ(\Upsilon)$ be a parahoric subgroup of G associated to the $\mathfrak{o}_F$-lattice sequence Υ with pro-p unipotent radical $P^1(\Upsilon)$ and connected finite reductive quotient $M^\circ(\Upsilon)$.

Remark 6.5. By conjugating if necessary, we the parahoric subgroup $P^\circ(\Upsilon)$ will be equal to a standard parahoric subgroup P_J considered above, and we will interchange notations freely.

Let $Q^\circ(\Lambda)$ be a parabolic subgroup of $M^\circ(\Upsilon)$ with Levi decomposition $Q^\circ(\Lambda) = M^\circ(\Lambda) \ltimes U^\circ(\Lambda)$, and denote by $P^\circ(\Lambda)$ the parahoric subgroup which is the preimage of $Q^\circ(\Lambda)$ under the projection map $P^\circ(\Upsilon) \to M^\circ(\Upsilon)$. Thus the quotient of $P^\circ(\Lambda)$ by its pro-p unipotent radical $P^1(\Lambda)$ is $M^\circ(\Lambda)$. Let τ be an irreducible cuspidal representation of $M^\circ(\Lambda)$ and $\widetilde{\tau}$ denote both its inflation to $Q^\circ(\Lambda)$ and to $P^\circ(\Lambda)$. The following Lemma follows immediately from the definitions.

Lemma 6.6. We have a support preserving isomorphism of Hecke algebras $\mathcal{H}(M^\circ(\Upsilon), \widetilde{\tau}) \simeq \mathcal{H}(P^\circ(\Lambda), \widetilde{\tau})$: if $f \in \mathcal{H}(M^\circ(\Upsilon), \widetilde{\tau})$ is supported on $Q^\circ(\Lambda)yQ^\circ(\Lambda)$ for $y \in M^\circ(\Upsilon)$ then the corresponding element $f' \in \mathcal{H}(P^\circ(\Lambda), \widetilde{\tau})$ is supported on $P^\circ(\Lambda)yP^\circ(\Lambda)$.

Let $W(M^\circ(\Lambda), \tau)$ denote the *inertia group* of τ, that is, the elements of the relative Weyl group of $M^\circ(\Lambda)$ in $M^\circ(\Upsilon)$ which normalize τ (see [10, Proposition 4.2.11]). We can give a presentation of the algebra $\mathcal{H}(M^\circ(\Upsilon), \widetilde{\tau})$ due to Howlett–Lehrer [12] when $R = \mathbb{C}$ and to Geck–Hiss–Malle [9] in general.

Theorem 6.7 ([10, Theorem 4.2.12])**.** There are a Coxeter system (W_1, S_1) and a finite group Ω acting on (W_1, S_1) such that $W(M^\circ(\Lambda), \tau) \simeq \Omega \ltimes W_1$; furthermore $\mathcal{H}(M^\circ(\Upsilon), \widetilde{\tau})$ has a basis $\{T_w : w \in W(M^\circ(\Lambda), \tau)\}$ which gives a presentation of the algebra with the following rules for multiplication:

(i) for all $w \in W$ and $w' \in \Omega$,

$$T_w \star T_{w'} = \mu(w, w')T_{ww'} \quad \text{and} \quad T_{w'} \star T_w = \mu(w', w)T_{w'w},$$

for some 2-cocycle $\mu : W(M^\circ(\Lambda), \tau) \times W(M^\circ(\Lambda), \tau) \to R^\times$;

(ii) for $s \in S_1$, there are $p_s \in R\backslash\{0, 1\}$, such that,

$$T_s \star T_w = \begin{cases} T_{sw} & \text{if } l_1(sw) > l_1(w), \\ p_s T_{sw} + (p_s - 1)T_w & \text{if } l_1(sw) < l_1(w), \end{cases}$$

for all $s \in S_1$ and $w \in W_1$, where l_1 is the length function on W_1.

7 Reduction to level zero

Let $[\Upsilon, n_\Upsilon, 0, \beta]$ and $[\Lambda, n_\Lambda, 0, \beta]$ be self-dual semisimple strata in A. By conjugating by an element of G_E, if necessary, we assume that Υ_E and Λ_E lie in the closure of a common chamber in the building of G_E, corresponding to an $\mathfrak{o}_E$-lattice sequence Γ_E in V. As before, let $\theta_\Upsilon \in \mathcal{C}_-(\Upsilon, 0, \beta)$ and $\theta_\Lambda = \tau_{\Upsilon,\Lambda,\beta}(\theta_\Upsilon)$. Let η_Υ be the unique Heisenberg representation containing θ_Υ and $\eta_\Lambda = \tau_{\Upsilon,\Lambda,\beta}(\eta_\Upsilon)$ the unique Heisenberg representation containing θ_Λ. Let κ_Υ be a standard β-extension of η_Υ and κ_Λ be a standard β-extension of η_Λ.

We will abbreviate $J_\Upsilon = J(\beta, \Upsilon)$, and also $P_\Upsilon = P(\Upsilon_E)$ and $M_\Upsilon = M(\Upsilon_E)$, with analogous notation for Λ and Γ. We also write $J^1_{\Gamma,\Upsilon} = P^1_\Gamma J^1_\Upsilon$, etc.

Lemma 7.1. The intertwining of η_Λ and κ_Υ in G is given by

$$\dim_R(\mathrm{Hom}_{J^1_\Lambda\cap J^g_\Upsilon}(\kappa_\Lambda,\kappa^g_\Upsilon)) = \begin{cases} 1 & \text{if } g\in J^1_\Upsilon G_E J^1_\Lambda;\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We have $J^1_\Lambda\cap J^g_\Upsilon\subseteq J^1_\Lambda\cap K^g$ for some Sylow p-subgroup K of J_Υ. All Sylow p-subgroups of J_Υ are conjugate to $J^1_{\Gamma,\Upsilon}$ so $K=(J^1_{\Gamma,\Upsilon})^j$ for some $j\in J_\Upsilon$. Thus $\mathrm{Res}^{J_\Upsilon}_K\kappa_\Upsilon\simeq\eta^j_{\Gamma,\Upsilon}$ and, as vector spaces, we have

$$\mathrm{Hom}_{J^1_\Lambda\cap J^g_\Upsilon}(\kappa_\Lambda,\kappa^g_\Upsilon)\simeq\mathrm{Hom}_{J^1_\Lambda\cap(J^1_{\Gamma,\Upsilon})^{jg}}(\eta_\Lambda,\eta^{jg}_{\Gamma,\Upsilon}).$$

As $\eta_{\Gamma,\Upsilon}$ extends η_Υ, the result now follows by applying Lemma 4.7 and Theorem 4.1. □

Let τ be a representation of M_Υ which we identify with a representation of J_Υ trivial on J^1_Υ and with a representation of P_Υ trivial on P^1_Υ. By Mackey's restriction-induction formula and exactness of κ_Λ-restriction we have the following lemma.

Lemma 7.2. We have isomorphisms of representations of M_Λ

$$\mathrm{R}_{\kappa_\Lambda}\circ\mathrm{I}_{\kappa_\Upsilon}(\tau)\simeq\bigoplus_{J_\Upsilon\backslash G/J_\Lambda}\mathrm{Hom}_{J^1_\Lambda}\left(\kappa_\Lambda,\mathrm{ind}^{J_\Lambda}_{J_\Lambda\cap J^g_\Upsilon}((\kappa_\Upsilon\otimes\tau)^g)\right);$$
$$\mathrm{R}^E_\Lambda\circ\mathrm{I}^E_\Upsilon(\tau)\simeq\bigoplus_{P_\Upsilon\backslash G_E/P_\Lambda}\mathrm{Hom}_{P^1_\Lambda}\left(1,\mathrm{ind}^{P_\Lambda}_{P_\Lambda\cap P^g_\Upsilon}(\tau^g)\right).$$

Lemma 7.3. Let $g\in G$. If $\mathrm{Hom}_{J^1_\Lambda}\left(\kappa_\Lambda,\mathrm{ind}^{J_\Lambda}_{J_\Lambda\cap J^g_\Upsilon}(\kappa_\Upsilon\otimes\tau)^g\right)\neq0$ then $g\in J^1_\Upsilon G_E J^1_\Lambda$.

Proof. Consider $\mathrm{Hom}_{J^1_\Lambda}\left(\kappa_\Lambda,\mathrm{ind}^{J_\Lambda}_{J_\Lambda\cap J^g_\Upsilon}(\kappa_\Upsilon\otimes\tau)^g\right)$ as an abstract R-vector space. We have

$$\mathrm{Hom}_{J^1_\Lambda}\left(\mathrm{Res}^{J_\Lambda}_{J^1_\Lambda}\kappa_\Lambda,\mathrm{Res}^{J_\Lambda}_{J^1_\Lambda}\circ\mathrm{ind}^{J_\Lambda}_{J_\Lambda\cap J^g_\Upsilon}(\kappa_\Upsilon\otimes\tau)^g\right)\simeq\bigoplus_{h\in(J_\Lambda\cap J^g_\Upsilon)\backslash J_\Lambda/J^1_\Lambda}\mathrm{Hom}_{J^1_\Lambda\cap J^{gh}_\Upsilon}\left(\eta_\Lambda,(\kappa_\Upsilon\otimes\tau)^{gh}\right)$$

by Mackey's restriction-induction formula and Frobenius reciprocity. We have an injection of vector spaces

$$\mathrm{Hom}_{J^1_\Lambda\cap J^{gh}_\Upsilon}\left(\eta_\Lambda,(\kappa_\Upsilon\otimes\tau)^{gh}\right)\hookrightarrow\mathrm{Hom}_{J^1_\Lambda\cap(J^1_\Upsilon)^{gh}}\left(\eta_\Lambda,(\kappa_\Upsilon\otimes\tau)^{gh}\right)$$

and on $(J^1_\Upsilon)^{gh}$ we have $\kappa^{gh}_\Upsilon=\eta^{gh}_\Upsilon$ and τ^{gh} is a multiple of the trivial representation. Thus $gh\in\mathrm{I}_G(\eta_\Lambda,\eta_\Upsilon)=J^1_\Upsilon G_E J^1_\Lambda$, by Theorem 4.1, and we deduce that $g\in J^1_\Upsilon G_E J_\Lambda=J^1_\Upsilon G_E J^1_\Lambda$. □

Lemma 7.4. (i) Let $g\in G_E$. If $\mathrm{Hom}_{P^1_\Lambda}(1,\mathrm{ind}^{P_\Lambda}_{P_\Lambda\cap P^g_\Upsilon}(\tau^g))=0$, then $\mathrm{Hom}_{J^1_\Lambda}(\kappa_\Lambda,\mathrm{ind}^{J_\Lambda}_{J_\Lambda\cap J^g_\Upsilon}(\kappa_\Upsilon\otimes\tau)^g)=0$.

(ii) As representations of $M(\Upsilon_E)$, we have isomorphisms

$$\mathrm{Hom}_{J^1_\Upsilon}(\kappa_\Upsilon,\kappa_\Upsilon\otimes\tau)\simeq\mathrm{Hom}_{P^1_\Upsilon}(1,\tau)\simeq\tau.$$

Proof. As an abstract vector space, by Mackey theory, we have

$$\mathrm{Hom}_{J^1_\Lambda}(\kappa_\Lambda,\mathrm{ind}^{J_\Lambda}_{J_\Lambda\cap J^g_\Upsilon}(\kappa_\Upsilon\otimes\tau)^g)\simeq\bigoplus_{h\in(J_\Lambda\cap J^g_\Upsilon)\backslash J_\Lambda/J^1_\Lambda}\mathrm{Hom}_{J^1_\Lambda\cap J^{gh}_\Upsilon}(\eta_\Lambda,(\kappa_\Upsilon\otimes\tau)^{gh}).$$

By Lemma 7.1 gh intertwines η_Λ with κ_Υ for every $h \in J_\Lambda$. Hence by Lemma 2.12 (applied with $X_1 = X_1^1 = J_\Lambda^1$, $X_2 = J_\Upsilon^{gh}$, $X_2^1 = (J_\Upsilon^1)^{gh}$, $\mu_1 = \eta_\Lambda$, $\mu_2 = \kappa^{gh}$, $\zeta_1 = 1$, and $\zeta_2 = \tau^{gh}$) for each summand, we have an isomorphism of vector spaces

$$\mathrm{Hom}_{J_\Lambda^1 \cap J_\Upsilon^{gh}}\left(\eta_\Lambda, (\kappa_\Upsilon \otimes \tau)^{gh}\right) \simeq \mathrm{Hom}_{J_\Lambda^1 \cap J_\Upsilon^{gh}}(1, \tau^{gh}).$$

Moreover, as $J_\Lambda^1 \cap J_\Upsilon^{gh}$ contains $P_\Lambda^1 \cap P_\Upsilon^{gh}$, we have

$$\mathrm{Hom}_{J_\Lambda^1 \cap J_\Upsilon^{gh}}(1, \tau^{gh}) \subseteq \mathrm{Hom}_{P_\Lambda^1 \cap P_\Upsilon^{gh}}(1, \tau^{gh})$$

But, the right hand side is isomorphic as a vector space to a direct summand of the representation

$$\mathrm{Hom}_{P_\Lambda^1}(1, \mathrm{ind}_{P_\Lambda \cap P_\Upsilon^g}^{P_\Lambda}(\tau)^g) \simeq \bigoplus_{h' \in (P_\Lambda \cap P_\Upsilon^g)\backslash P_\Lambda / P_\Lambda^1} \mathrm{Hom}_{P_\Lambda^1 \cap P_\Upsilon^{gh'}}(1, \tau^{gh'}),$$

where the above decomposition is again an isomorphism of abstract vector spaces obtained by Mackey theory. However, by our hypotheses $\mathrm{Hom}_{P_\Lambda^1}(1, \mathrm{ind}_{P_\Lambda \cap P_\Upsilon^g}^{P_\Lambda}(\tau^g))$ is trivial, whence all the summands $\mathrm{Hom}_{J_\Lambda^1 \cap J_\Upsilon^{gh}}(\eta_\Lambda, (\kappa_\Upsilon \otimes \tau)^{gh})$ are trivial and, thus, so is $\mathrm{Hom}_{J_\Lambda^1}(\kappa_\Lambda, \mathrm{ind}_{J_\Lambda \cap J_\Upsilon^g}^{J_\Lambda}(\kappa_\Upsilon \otimes \tau)^g)$ and we have shown case (i).

For the second part, we can take $S \in \mathrm{Hom}_{J_\Upsilon}(\kappa_\Upsilon, \kappa_\Upsilon)$ to be the identity element. By Lemma 2.12 (applied with $X_1 = X_1^1 = X_2 = X_2^1 = J_\Upsilon^1$, $\mu_1 = \mu_2 = \eta_\Upsilon$, $\zeta_1 = 1$, and $\zeta_2 = \tau \mid_{J^1} = \dim(\tau)1$) we have an isomorphism of vector spaces $\mathrm{Hom}_{J_\Upsilon^1}(1, \tau) \to \mathrm{Hom}_{J_\Upsilon^1}(\kappa_\Upsilon, \kappa_\Upsilon \otimes \tau)$ given by $T \mapsto S \otimes T$. The action of $M(\Upsilon_E)$ on $\mathrm{Hom}_{J_\Upsilon^1}(\kappa_\Upsilon, \kappa_\Upsilon \otimes \sigma)$ induced from the action of $M(\Upsilon_E)$ on $\mathrm{R}_{\kappa_\Upsilon} \circ \mathrm{I}_{\kappa_\Upsilon}(\tau)$ is given by $m \cdot \Phi = \kappa_\Upsilon \otimes \tau(j) \circ \Phi \circ \kappa_\Upsilon(j^{-1})$, for $m \in M(\Upsilon_E)$, $\Phi \in \mathrm{Hom}_{J_\Upsilon^1}(\kappa_\Upsilon, \kappa_\Upsilon \otimes \tau)$ and j any representative of m in J. Thus, we have

$$\begin{aligned} m \cdot S \otimes T &= \kappa \otimes \tau(j) \circ (S \otimes T) \circ \kappa(j^{-1}) \\ &= \kappa(j) \circ S \circ \kappa(j^{-1}) \otimes \tau(j) \circ T. \end{aligned}$$

However, as $S \in \mathrm{Hom}_J(\kappa_\Upsilon, \kappa_\Upsilon^g)$, whence $\kappa_\Upsilon(j) \circ S \circ \kappa_\Upsilon(j^{-1}) = S$. Therefore, we have $m \cdot S \otimes T = S \otimes m \cdot T$, the isomorphism of vector spaces is an isomorphism of representations of $M(\Upsilon_E)$. Moreover,

$$\mathrm{Hom}_{J_\Upsilon^1}(1, \tau) \simeq \mathrm{Hom}_{P_\Upsilon^1}(1, \tau) \simeq \tau.$$

□

Corollary 7.5. Let τ be a representation of M_Υ.

(i) If $\mathrm{R}_\Lambda^E \circ \mathrm{I}_\Upsilon^E(\tau)$ is trivial then so is $\mathrm{R}_{\kappa_\Lambda} \circ \mathrm{I}_{\kappa_\Upsilon}(\tau)$.

(ii) Suppose τ is irreducible with cuspidal restriction to M_Υ°. If G_E has compact centre and $P^\circ(\Upsilon_E)$ is a maximal parahoric subgroup of G_E then

$$\mathrm{R}_{\kappa_\Upsilon} \circ \mathrm{I}_{\kappa_\Upsilon}(\tau) \simeq \tau.$$

Proof. By Lemmas 7.2 and 7.3, we have isomorphisms of representations of M_Λ

$$\mathrm{R}_{\kappa_\Lambda} \circ \mathrm{I}_{\kappa_\Upsilon}(\tau) \simeq \bigoplus_{J_\Upsilon \backslash J_\Upsilon^1 G_E J_\Lambda^1 / J_\Lambda} \mathrm{Hom}_{J_\Lambda^1}\left(\kappa_\Lambda, \mathrm{ind}_{J_\Lambda \cap J_\Upsilon^g}^{J_\Lambda}((\kappa_\Upsilon \otimes \tau)^g)\right);$$

$$\mathrm{R}_\Lambda^E \circ \mathrm{I}_\Upsilon^E(\tau) \simeq \bigoplus_{P_\Upsilon \backslash G_E / P_\Lambda} \mathrm{Hom}_{P_\Lambda^1}\left(1, \mathrm{ind}_{P_\Lambda \cap P_\Upsilon^g}^{P_\Lambda}(\tau^g)\right).$$

We choose a set of distinguished double coset representatives for $P_\Upsilon\backslash G_E/P_\Lambda$, which by with the bijection of Lemma 3.8, fixes a set of double coset representatives of $J_\Upsilon\backslash J^1_\Upsilon G_E J^1_\Lambda/J_\Lambda$ in G_E. We can now compare the summands of both isomorphisms on the right. Part (i) follows from Lemma 7.4 part (i), and Lemma 6.2 Part (ii). For Part (ii) notice by Lemma 7.4 parts (i) and (ii), and Lemma 6.2 Part (i), that the only summands which contribute correspond to distinguished double cosets $P_\Upsilon n P_\Upsilon$ where n has projection w in the extended affine Weyl group satisfying $wK = K$ for K the proper subset of fundamental reflections of the affine Weyl group corresponding to P°_Υ. However, as P°_Υ is maximal $wK = K$ implies that $n \in N_{G_E}(P^\circ_\Upsilon) = P_\Upsilon$ by [18, Appendix]. Thus, Part (ii) follows from Lemma 7.4 Part (ii). □

8 Skew covers

This section is concerned with revisiting and making the necessary changes to the second authors construction of covers in [24] so that the *same* construction works in positive characteristic ℓ. The construction follows *mutatis mutandis* the constructions of the second author for complex representations and rather than go through all the proofs, which are lengthy, we introduce all the the notation of *op. cit.* and indicate where changes need to be made to the proofs.

8.1 Iwahori decompositions

Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum with associated splitting $V = \bigoplus_{i\in I} V^i$. A decomposition $V = \bigoplus_{j=1}^m W^{(j)}$ of V is called *subordinate* to $[\Lambda, n, 0, \beta]$ if

(i) each $W^{(j)} \cap V^i$ is an E_i-subspace of V^i;

(ii) $W^{(j)} = \bigoplus_{i\in I}(W^{(j)} \cap V^i)$;

(iii) $\Lambda(r) = \bigoplus_{j=1}^m(\Lambda(r) \cap W^{(j)})$, for all $r \in \mathbb{Z}$;

It is called *properly subordinate* to $[\Lambda, n, 0, \beta]$ if it is subordinate and, also,

(iv) for each $r \in \mathbb{Z}$ and $i \in I$, there is at most one j such that
$$(\Lambda(r) \cap W^{(j)} \cap V^i)) \supsetneq (\Lambda(r+1) \cap W^{(j)}j \cap V^i).$$

If $[\Lambda, n, 0, \beta]$ is a semisimple stratum and $V = \bigoplus_{j=1}^m W^{(j)}$ is a decomposition which is subordinate to $[\Lambda, n, 0, \beta]$ then we put $\Lambda^{(j)}$ to be the $\mathfrak{o}_E$-lattice sequence in $W^{(j)}$ given by $\Lambda^{(j)}(r) = \Lambda(r) \cap W_j$ and put $\beta^{(j)} = e^{(j)}\beta e^{(j)}$ where $e^{(j)}$ is the orthogonal projection $V \to W_j$. Then there is an integer $n^{(j)}$ such that $[\Lambda^{(j)}, n^{(j)}, 0, \beta^{(j)}]$ is a semisimple stratum in $A^{(j)} = \mathrm{End}_F(W^{(j)})$ with splitting $W^{(j)} = \bigoplus_{i\in I}(W^{(j)} \cap V^i)$. We put $B^{(j)} = C_{A^{(j)}}(\beta^{(j)})$.

Let $\widetilde{M}$ denote the Levi subgroup of $\widetilde{G}$ equal to the stabiliser of the decomposition $V = \bigoplus_{j=1}^m W^{(j)}$ and let $\widetilde{P}$ be any parabolic subgroup of $\widetilde{G}$ with Levi factor $\widetilde{M}$ and Levi decomposition $\widetilde{P} = \widetilde{M} \ltimes \widetilde{U}$.

Lemma 8.1 ([24, Propositions 5.2 and 5.4])**.** If $V = \bigoplus_{j=1}^m W^{(j)}$ is subordinate to $[\Lambda, n, 0, \beta]$ then $\widetilde{J}^1(\beta, \Lambda)$ and $\widetilde{H}^1(\beta, \Lambda)$ have Iwahori decompositions with respect to $(\widetilde{M}, \widetilde{P})$. Moreover
$$\widetilde{H}^1(\beta, \Lambda) \cap \widetilde{M} = \prod_{j=1}^m \widetilde{H}^1(\beta^{(j)}, \Lambda^{(j)}),$$

there is a similar decomposition for $\widetilde{J}^1(\beta,\Lambda)\cap\widetilde{M}$, and we can form the groups

$$\widetilde{H}^1_{\widetilde{P}}=\widetilde{H}^1(\beta,\Lambda)(\widetilde{J}^1(\beta,\Lambda)\cap\widetilde{U}),\quad \widetilde{J}^1_{\widetilde{P}}=\widetilde{H}^1(\beta,\Lambda)(\widetilde{J}^1(\beta,\Lambda)\cap\widetilde{P})$$

which have Iwahori decompositions with respect to any parabolic subgroup with Levi factor $\widetilde{M}$. If the decomposition $V=\bigoplus_{j=1}^m W^{(j)}$ is properly subordinate to $[\Lambda,n,0,\beta]$ then $\widetilde{J}(\beta,\Lambda)$ also has an Iwahori decomposition with respect to $(\widetilde{M},\widetilde{P})$, we also have

$$\widetilde{J}(\beta,\Lambda)\cap\widetilde{M}=\prod_{j=1}^m\widetilde{J}(\beta^{(j)},\Lambda^{(j)}),$$

and we can form the group $\widetilde{J}_{\widetilde{P}}=\widetilde{H}^1(\beta,\Lambda)(\widetilde{J}(\beta,\Lambda)\cap\widetilde{P})$ which has an Iwahori decomposition with respect to any parabolic subgroup with Levi factor $\widetilde{M}$.

Let $[\Lambda,n,0,\beta]$ be a self-dual semisimple stratum. A decomposition $V=\bigoplus_{j=-m}^m W^{(j)}$ is called self-dual if, for $-m\leqslant j\leqslant m$, the orthogonal complement of $W^{(j)}$ is $\bigoplus_{k\neq\pm j}W^{(k)}$. Put $M=\widetilde{M}\cap G$ a Levi subgroup of G and $M^+=\widetilde{M}\cap G^+$ a Levi subgroup of G^+. Choosing a σ-stable parabolic subgroup $\widetilde{P}$ of G with Levi factor $\widetilde{M}$, we have $P=\widetilde{P}\cap G$ a parabolic subgroup of G with Levi factor M and $P^+=\widetilde{P}\cap G^+$ a parabolic subgroup of G^+ with Levi factor M^+.

Lemma 8.2 ([24, Corollaries 5.10 and 5.11] (*cf.* [7, Fait 8.10]))**.** If $V=\bigoplus_{j=-m}^m W^{(j)}$ is a self-dual subordinate decomposition to $[\Lambda,n,0,\beta]$, then the groups $H^1(\beta,\Lambda)$ and $J^1(\beta,\Lambda)$ have Iwahori decompositions with respect to (M,P),

$$H^1(\beta,\Lambda)\cap M\simeq H^1(\beta^{(0)},\Lambda^{(0)})\times\prod_{j=1}^m\widetilde{H}^1(\beta^{(j)},\Lambda^{(j)}),$$

there is a similar decomposition for $J^1(\beta,\Lambda)$, and we can form the groups

$$H^1_P=H^1(\beta,\Lambda)(J^1(\beta,\Lambda)\cap U),\quad J^1_P=H^1(\beta,\Lambda)(J^1(\beta,\Lambda)\cap P).$$

Moreover, if the decomposition is properly subordinate to $[\Lambda,n,0,\beta]$ then $J^+(\beta,\Lambda)$ has an Iwahori decomposition with respect to (M^+,P^+), $J(\beta,\Lambda)$ and $J^\circ(\beta,\Lambda)$ have Iwahori decompositions with respect to (M,P),

$$J(\beta,\Lambda)\cap M\simeq J(\beta^{(0)},\Lambda^{(0)})\times\prod_{j=1}^m\widetilde{J}(\beta^{(j)},\Lambda^{(j)}),$$

there are similar decompositions for $J^+(\beta,\Lambda)\cap M^+$ and $J^\circ(\beta,\Lambda)\cap M$, and we can form the groups

$$J^+_P=H^1(\beta,\Lambda)(J^+(\beta,\Lambda)\cap P),\quad J_P=H^1(\beta,\Lambda)(J(\beta,\Lambda)\cap P),\quad J^\circ_P=H^1(\beta,\Lambda)(J^\circ(\beta,\Lambda)\cap P).$$

Let $\widetilde{\theta}\in\mathcal{C}(\Lambda,n,0,\beta)$ and $\widetilde{\eta}$ be the unique Heisenberg representation of $\widetilde{J}^1(\beta,\Lambda)$ containing $\widetilde{\theta}$. By Lemma [24, Lemma 5.6], we can define a character of $\widetilde{H}^1_{\widetilde{P}}$ by

$$\widetilde{\theta}_{\widetilde{P}}(hj)=\widetilde{\theta}(h),$$

for $h\in\widetilde{H}^1(\beta,\Lambda)$ and $j\in\widetilde{J}^1(\beta,\Lambda)\cap\widetilde{U}$.

Lemma 8.3 ([24, Corollary 5.7 and Lemma 5.8])**.** There exists a unique irreducible representation of $\widetilde{J}^1_{\widetilde{P}}$ containing $\widetilde{\theta}_{\widetilde{P}}$. Moreover $\widetilde{\eta} = \mathrm{ind}_{\widetilde{J}^1_{\widetilde{P}}}^{\widetilde{J}^1(\beta,\Lambda)}(\widetilde{\eta}_{\widetilde{P}})$ and for each $y \in \widetilde{G}_E$, there is a unique $(\widetilde{J}^1_{\widetilde{P}}, \widetilde{J}^1_{\widetilde{P}})$-double coset in $\widetilde{J}^1(\beta,\Lambda)y\widetilde{J}^1(\beta,\Lambda)$ which intertwines $\widetilde{\eta}_{\widetilde{P}}$ and $I_{\widetilde{G}}(\widetilde{\theta}_{\widetilde{P}}) = I_{\widetilde{G}}(\widetilde{\eta}_{\widetilde{P}}) = \widetilde{J}^1_{\widetilde{P}}\widetilde{G}_E\widetilde{J}^1_{\widetilde{P}}$.

Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and η be the unique Heisenberg representation of $J^1(\beta,\Lambda)$ containing θ. We can define a character θ_P of H^1_P by

$$\theta_P(hj) = \theta(h),$$

for $h \in H^1(\beta,\Lambda)$ and $j \in J^1(\beta,\Lambda) \cap U$. Then $\theta_P = \mathbf{g}(\widetilde{\theta}_{\widetilde{P}})$ is the Glauberman transfer of $\widetilde{\theta}_{\widetilde{P}}$ (as $\widetilde{\theta}_{\widetilde{P}}$ is a character the Glauberman transfer here is just restriction to H^1_P).

We let $\eta_P = \mathbf{g}(\widetilde{\eta}_{\widetilde{P}})$. Using properties of the Glauberman correspondence the following Lemma is proved in [24].

Lemma 8.4. The representation η_P is the unique irreducible representation of J^1_P which contains θ_P, $\eta = \mathrm{ind}_{J^1_P}^{J^1(\beta,\Lambda)}(\eta_P)$. Moreover for each $y \in G_E$, there is a unique (J^1_P, J^1_P)-double coset in $J^1(\beta,\Lambda)yJ^1(\beta,\Lambda)$ which intertwines η_P and $\dim_R(I_g(\eta_P))$ is 1 if $g \in J^1_P G^+_E J^1_P$ and 0 otherwise.

Let κ^+ be a standard β-extension of η. We can form the natural representation κ^+_P of J^+_P on the space of $(J^1 \cap U)$-fixed vectors in κ^+ by normality. Then $\mathrm{Res}^{J^+_P}_{J^1_P}(\kappa^+_P) = \eta_P$, hence κ_P is irreducible. The Mackey restriction formula as in [24, Proposition 5.13] shows that $\mathrm{ind}^{J^+}_{J^+_P}(\kappa^+_P) \simeq \kappa^+$. We can also define representations of κ_P of J_P and κ°_P of J°_P, for which analogous statements hold and $\mathrm{Res}^{J^+_P}_{J_P}(\kappa^+_P) = \kappa_P$, $\mathrm{Res}^{J^+_P}_{J^\circ_P}(\kappa^+_P) = \kappa^\circ_P$.

In the next Lemma we identify $H^1(\beta,\Lambda) \cap M$ with $H^1(\beta^{(0)}, \Lambda^{(0)}) \times \prod_{j=1}^m \widetilde{H}^1(\beta^{(j)}, \Lambda^{(j)})$ using Lemma 8.2, and use the similar identifications for $J^1(\beta,\Lambda) \cap M$ and $J(\beta,\Lambda) \cap M$.

Lemma 8.5 ([24, Section 5])**.** If $V = \bigoplus_{j=-m}^m W^{(j)}$ is a self-dual subordinate decomposition, then

$$\theta_P \mid_{H^1(\beta,\Lambda)\cap M} = \theta^{(0)} \otimes \bigotimes_{j=1}^m \left(\widetilde{\theta}^{(j)}\right)^2,$$

with $\theta^{(0)} \in \mathcal{C}_-(\Lambda^{(0)}, 0, \beta^{(0)})$ and $\widetilde{\theta}^{(j)} \in \mathcal{C}(\Lambda^{(j)}, 0, \beta^{(j)})$. Similarly we have

$$\eta_P \mid_{J^1(\beta,\Lambda)\cap M} = \eta^{(0)} \otimes \bigotimes_{j=1}^m \widetilde{\eta}^{(j)},$$

where $\eta^{(0)}$ is the unique irreducible representation of $J^1(\beta^{(0)}, \Lambda^{(0)})$ containing $\theta^{(0)}$ and $\widetilde{\eta}^{(j)}$ is the unique irreducible representation of $\widetilde{J}^1(\beta^{(j)}, \Lambda^{(j)})$ containing $\left(\widetilde{\theta}^{(j)}\right)^2$. Moreover, if $V = \bigoplus_{j=-m}^m W^{(j)}$ is a self-dual properly subordinate decomposition,

$$\kappa_P \mid_{J(\beta,\Lambda)\cap M} = \kappa^+_{(0)} \otimes \bigotimes_{j=1}^m \widetilde{\kappa}_{(j)},$$

with $\kappa^+_{(0)}$ an extension of $\eta^{(0)}$ to $J^+(\beta^{(0)}, \Lambda^{(0)})$ and $\widetilde{\kappa}_{(j)}$ an extension of $\widetilde{\eta}_{(j)}$ to $\widetilde{J}(\beta^{(j)}, \Lambda^{(j)})$.

Lemma 8.6 ([24, Lemma 6.1])**.** Let K be a compact open subgroup of $J^+(\beta,\Lambda)$ containing $J^1(\beta,\Lambda)$ which has an Iwahori decomposition with respect to (M^+,P^+) with $K\cap M^+ = K^{(0)}\times\prod\widetilde{K}^{(j)}$. Let ρ be the inflation to K of an irreducible representation of $K/J^1(\beta,\Lambda)$, $\lambda = \mathrm{Res}_K^{J^+(\beta,\Lambda)}(\kappa^+)\otimes\rho$ and λ_P the representation of $K_P = H^1(\beta,\Lambda)(K\cap P)$ on the space of $J^1(\beta,\Lambda)\cap U$-fixed vectors in λ. Then

(i) λ_P is irreducible and $\lambda = \mathrm{ind}_{K_P}^K\lambda_P$.

(ii) $\lambda_P \simeq \kappa_P\otimes\rho$ considering ρ as a representation of $K_P/J_P^1 \simeq K/J^1(\beta,\Lambda)$.

(iii) $\lambda_P\mid_{K\cap M} = \lambda^{(0)}\otimes\bigotimes_{j=1}^m\widetilde{\lambda}^{(j)}$, where $\widetilde{\lambda}^{(0)} = \kappa_{(0)}\mid_{K^{(0)}}\otimes\rho^{(0)}$ is a representation of $K^{(0)}$ and $\widetilde{\lambda}^{(j)} = \widetilde{\kappa}^+_{(j)}\mid_{\widetilde{K}^{(j)}}\otimes\widetilde{\rho}^{(j)}$ is a representation of $\widetilde{K}^{(j)}$, for $1\leqslant j\leqslant m$.

(iv) There is a support preserving algebra homomorphism $\mathcal{H}(G^+,\lambda_P)\simeq\mathcal{H}(G^+,\lambda)$; if $\phi\in\mathcal{H}(G^+,\lambda)$ has support KyK for some $y\in G_E^+$ then the corresponding $\phi_P\in\mathcal{H}(G^+,\lambda_P)$ has support K_PyK_P.

Proof. The proof follows *mutatis mutandis* the proof of [24, Lemma 6.1], making use of the results quoted in this section and Lemma 2.11 for parts (i), (ii), and (iii). ☐

The self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is *exactly subordinate* to $[\Lambda,n,0,\beta]$, in the sense of [24, Definition 6.5], if $P^\circ(\Lambda_E)\cap M$ is a maximal parahoric subgroup of $G_E\cap M$ and, for each $j\neq 0$, there is an i such that $W^{(j)}$ is contained in V^i and $\mathfrak{A}(\Lambda^{(j)})\cap B^{(j)}$ is a maximal $\mathfrak{o}_{E^i}$-order in $B^{(j)}$, or equivalently, if it is minimal amongst all self-dual decompositions which are properly subordinate to $[\Lambda,n,0,\beta]$.

For the rest of this section, we suppose that the self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is *exactly subordinate* to $[\Lambda,n,0,\beta]$. For $j,k>0$, in [24, Section 6.2] a collection of Weyl group element $s_{j,k}, s_j$, and s_j^ϖ, all of which lie in G_E^+, of G is defined. The element $s_{j,k}$ exchanges the blocks $e^{(j)}Ae^{(j)}$ and $e^{(k)}Ae^{(k)}$, and the blocks $e^{(-j)}Ae^{(-j)}$ and $e^{(-k)}Ae^{(-k)}$. The elements s_j and s_j^ϖ exchange the blocks $e^{(j)}Ae^{(j)}$ and $e^{(-j)}Ae^{(-j)}$. Let Λ^M be a $\mathfrak{o}_E$-lattice sequence in V such that $\mathfrak{A}(\Lambda_E^M)$ is a maximal $\mathfrak{o}_E$-order containing $\mathfrak{A}(\Lambda_E)$. For $j,k>0$, $W^{(j)}$ and $W^{(k)}$ are called *companion with respect to* Λ^M if $s_{j,k}\in P^+(\Lambda_E^M)$, while $W^{(j)}$ and $W^{(-j)}$ are called *companion with respect to* Λ^M if s_j or s_j^ϖ lies in $P^+(\Lambda_E^M)$. Following these definitions in *op. cit.* an involution σ_j is defined on $\widetilde{G}_j = \{(\overline{g}^{-1},g)\in\widetilde{G}^{(-j)}\times\widetilde{G}^{(j)}\}$ by conjugation by s_j. Furthermore, by [24, Lemma 6.9], the group $\widetilde{J}(\beta^{(j)},\Lambda^{(j)})$ is stable under σ_j, and , if $1\leqslant j<k\leqslant m$ and $W^{(j)}\simeq W^{(k)}$ as E_i-spaces for some i, then conjugation by $s_{j,k}$ induces an isomorphism $\widetilde{J}(\beta^{(j)},\Lambda^{(j)})\simeq\widetilde{J}(\beta^{(k)},\Lambda^{(k)})$.

Lemma 8.7 ([24, Proposition 6.3, Corollary 6.10])**.** Suppose the self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ is *exactly subordinate* to $[\Lambda,n,0,\beta]$. Then $\kappa_{(0)}^+$ is a standard $\beta^{(0)}$-extension of $\eta^{(0)}$ to $J^+(\beta^{(0)},\Lambda^{(0)})$ and $\widetilde{\kappa}_{(j)}$ is a standard $2\beta^{(j)}$-extension of $\widetilde{\eta}^{(j)}$ to $\widetilde{J}(\beta^{(j)},\Lambda^{(j)})$. Furthermore, for $1\leqslant j\leqslant m$, conjugation by s_j induces and equivalence $\widetilde{\kappa}_{(j)}\circ\sigma_j\simeq\widetilde{\kappa}_{(j)}$, and, if $1\leqslant j<k\leqslant m$ and $W^{(j)}\simeq W^{(k)}$ as E_i-spaces for some i, then conjugation by $s_{j,k}$ induces an equivalence $\widetilde{\kappa}_{(j)}\simeq\widetilde{\kappa}_{(k)}$.

This lemma together with the comparison of β-extensions leads to the following observation, as in *op. cit.* Let $\Lambda^M,\Lambda^{M'}$ be self-dual $\mathfrak{o}_E$-lattice sequences such that the associated $\mathfrak{o}_E$-orders are maximal and contain $\mathfrak{A}(\Lambda_E)$. Let κ be a β-extension of η relative to Λ^M and κ' be a β-extension of η relative to $\Lambda^{M'}$. There are σ_i-invariant characters $\widetilde{\chi}^{(j)}$ of $k_{E_i}^\times$ and a character $\chi^{(0)}$

of $M^\circ(\Lambda_E)$ such that, setting $\chi = \chi^{(0)} \otimes \bigotimes_{j=1}^m \widetilde{\chi}^{(j)} \circ \det^{(j)}$, we have

$$\kappa' = \operatorname{ind}_{J_P^\circ}^{J^\circ(\beta,\Lambda)}(\kappa_P \otimes \chi).$$

8.2 κ_P-induction and restriction

We have functors $\mathrm{I}_{\kappa_P} : \mathfrak{R}_R(M^\circ(\Lambda_E)) \to \mathfrak{R}_R(G)$ and $\mathrm{I}_{\kappa_L} : \mathfrak{R}_R(M^\circ(\Lambda_E)) \to \mathfrak{R}_R(L)$ with right adjoint functors $\mathrm{R}_{\kappa_P} : \mathfrak{R}_R(G) \to \mathfrak{R}_R(M^\circ(\Lambda_E))$ and $\mathrm{R}_{\kappa_L} : \mathfrak{R}_R(L) \to \mathfrak{R}_R(M^\circ(\Lambda_E))$; defined analogously to I_κ and R_κ in Section 5.1. In fact, as $\operatorname{ind}_{J_P}^J(\kappa_P) \simeq \kappa$, we have natural isomorphisms of functors $\mathrm{I}_\kappa \simeq \mathrm{I}_{\kappa_P}$ and $\mathrm{R}_\kappa \simeq \mathrm{R}_{\kappa_P}$.

8.3 Bounding $I_G(\kappa_P)$

Suppose $P^\circ(\Lambda_E)$ is not maximal. Let N_E denote the normaliser in G_E of the product of maximal E_i-split tori T_{E_i} in G_{Ei}, chosen relative to a certain E_i-basis of V^i as in [24, Section 6]. Let $N_\Lambda = \{w \in N_E : w \text{ normalises } P^\circ(\Lambda_E) \cap M\}$ and $N_\Lambda(\rho) = \{n \in N_\Lambda : \rho^n \simeq \rho\}$.

Lemma 8.8 ([24, Corollary 6.16])**.** The intertwining of κ_P° is given by

$$I_G(\kappa_P^\circ) \supseteq J_P^\circ N_\Lambda(\rho) J_P^\circ,$$

and the intertwining of $\lambda_P^\circ = \lambda_P \mid_{J_P^\circ}$ is given by

$$I_G(\lambda_P^\circ) = J_P^\circ N_\Lambda(\rho) J_P^\circ.$$

The proof follows exactly as in *op. cit.* with one caveat: we replace the use of [24, Proposition 1.1] with Lemma 6.4.

8.3.1 A Hecke algebra injection

Let $[\Lambda, n, 0, \beta]$ and $[\Lambda', n', 0, \beta]$ be skew semisimple strata with $\mathfrak{A}(\Lambda_E) \subseteq \mathfrak{A}(\Lambda'_E)$. Let $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and $\theta' = \tau_{\Lambda,\Lambda',\beta}(\theta)$ be semisimple characters, κ and κ' compatible β-extensions of θ and θ', and ρ denote the inflation of an irreducible cuspidal representation $\overline{\rho}$ of $M^\circ(\beta,\Lambda)$ to the groups $J^\circ(\beta,\Lambda)$, $J^\circ_{\Lambda,\Lambda'}$ and $P^\circ(\Lambda_E)$. We put $\lambda = \kappa \otimes \rho$ and $\lambda' = \kappa' \mid_{J^\circ_{\Lambda,\Lambda'}} \otimes \rho$. We have a canonical support preserving isomorphism $\mathcal{H}(G,\lambda) \simeq \mathcal{H}(G,\lambda')$ as in [24, Proposition 7.1], this follows essentially by transitivity of induction and our results on β-extensions. Exactly as in *op. cit.* Proposition 7.2, we have a support preserving isomorphism of algebras $\mathcal{H}(J(\beta,\Lambda'),\lambda') \simeq \mathcal{H}(P(\Lambda'_E),\rho)$. The composition of these isomorphisms with the natural injection $\mathcal{H}(J(\beta,\Lambda'),\lambda') \hookrightarrow \mathcal{H}(G,\lambda')$, gives us an injective map

$$\mathcal{H}(P(\Lambda'_E),\rho) \hookrightarrow \mathcal{H}(G,\lambda),$$

which preserves support; if $\phi \in \mathcal{H}(P(\Lambda'_E),\rho)$ has support $P^\circ(\Lambda_E) y P^\circ(\Lambda_E)$ for $y \in P(\Lambda'_E)$ then the corresponding $\phi_G \in \mathcal{H}(G,\lambda)$ has support $J_P^\circ y J_P^\circ$.

8.3.2 Skew covers

Let π be an irreducible cuspidal representation of G, and consider the set of all such pairs $([\Lambda, n, 0, \beta], \theta)$ such that $[\Lambda, n, 0, \beta]$ is a skew semisimple strata, $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and π contains θ. Choose a pair in this set whose parahoric subgroup $P^\circ(\Lambda_E)$ is minimal under containment relative to all other pairs in the set. Since there is a unique irreducible representation η of $J^1(\beta,\Lambda)$ containing θ, π must also contain η. Hence, by Lemma 2.11, π contains a representation $\lambda = \kappa^\circ \otimes \rho$ of $J^\circ(\beta,\Lambda)$ where κ° is a standard β-extension of η and ρ is an irreducible representation of $J^\circ(\beta,\Lambda)/J^1(\beta,\Lambda)$. As $P^\circ(\Lambda_E)$ is minimal, it follows that ρ is cuspidal (*cf.* [24, Lemma 7.4]).

Suppose that either $P^\circ(\Lambda_E)$ is not a maximal parahoric subgroup in G_E or G_E does not have compact centre.

Theorem 8.9 ([24, Proposition 7.13] (*cf.* [17, Appendix A]))**.** There exists an exactly subordinate self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$ to $[\Lambda, n, 0, \beta]$ such that the pair $(J_P^\circ, \lambda_P^\circ)$ is a G-cover of $(J_P^\circ \cap M, \lambda_P \mid_M)$, where $J^x \circ_P$ is as constructed in Lemma 8.2 and κ_P as in Lemma 8.5.

The construction follows *mutatis mutandis* that of *op. cit.*, noting that:

(i) We use the results for β-extensions in positive characteristic from Section 5, and use Lemma 2.11 (the characteristic zero version of which is obvious).

(ii) In the construction of *op. cit.* for a parahoric subgroup $P^\circ(\mathfrak{M})$ containing $P^\circ(\Lambda_E)$, the proof requires knowledge of the structure of $\mathcal{H}(P(\mathfrak{M}), \rho^\circ)$ (*cf.* Section 7.2.2 of *op. cit.*) given by the results of [18]. Here we must appeal to Geck–Hiss–Malle's generalisation of the description of the structure of $\mathcal{H}(P(\mathfrak{M}), \rho^\circ)$ to positive characteristic (see Lemma 6.7).

(iii) The proof of *op. cit.* requires the construction of covers in general linear groups, namely it uses [19, Proposition 6.7]. For general linear groups, the analogous proposition holds in positive characteristic (see [16, Remarque 2.25]).

(iv) In the definition of *lies over* (cf. [24, Definition 7.6]), the use of the word component should be replaced with quotient.

9 Self-dual and pro-p covers

This section generalises the construction of covers we have give for skew strata to semisimple strata, following [17]. Also, inspired by [15, Lemme 5.19], we define pro-p covers at the level of the J^1 groups. These results will not be used in the rest of the paper, and are included with future work in mind.

Let M be a Levi subgroup of G which is the stabiliser of the self-dual decomposition $V = \bigoplus_{j=-m}^m W^{(j)}$. Letting $\widetilde{G}^{(j)} = \mathrm{Aut}_F(W^{(j)})$ and $G^{(0)} = \mathrm{Aut}_F(W^{(0)}) \cap G$ we have $M = G^{(0)} \times \prod_{j=1}^m \widetilde{G}^{(j)}$. Let $\tau = \tau^{(0)} \otimes \bigotimes_{j=1}^m \widetilde{\tau}^{(j)}$ be a cuspidal irreducible representation of M. Let $\mathcal{M}$ denote the stabiliser of $V = \bigoplus_{j=-m}^m W^{(j)}$ in A.

Lemma 9.1 ([7, Proposition 8.10], [17, Proposition 5.1])**.** There are a self-dual semisimple stratum $[\Lambda, n, 0, \beta]$ with $\beta \in \mathcal{M}$ and a self-dual semisimple character θ of $H^1(\beta,\Lambda)$ such that $V = \bigoplus_{j=-m}^m W^{(j)}$ is properly subordinate to $[\Lambda, n, 0, \beta]$ and

$$\theta \mid_{H^1(\beta,\Lambda)\cap M} = \theta^{(0)} \otimes \bigotimes_{j=1}^m \left(\widetilde{\theta}^{(j)}\right)^2,$$

with $\theta^{(0)}$ contained in $\tau^{(0)}$ and, for each $j > 0$, $\left(\widetilde{\theta}^{(j)}\right)^2$ contained in $\widetilde{\tau}_j$ where we have identified $H^1(\beta,\Lambda) \cap M$ with $H^1(\beta^{(0)}, \Lambda^{(0)}) \times \prod_{j=1}^m \widetilde{H}^1(\beta^{(j)}, \Lambda^{(j)})$ as in Lemma 8.5.

Let ρ be an irreducible cuspidal representation of $M^\circ(\Lambda_E) = J_P^\circ/J_P^1 \simeq J_L^\circ/J_L^1$. We can form the representations $\lambda_P^\circ = \kappa_P \otimes \rho$ of J_P° and $\lambda_L^\circ = \kappa_L \otimes \rho$ of J_L° by inflation.

Theorem 9.2 ([17, Theorem 4.3] (*cf.* also [24, Proposition 7.13])). The pair $(J_P^\circ, \lambda_P^\circ)$ is a G-cover of $(J_L^\circ, \lambda_L^\circ)$ relative to P.

The proof generalises to positive characteristic with the same adaptions as commented on the proof of Theorem 8.9.

Theorem 9.3. The pair (J_P^1, η_P) is a G-cover of (J_L^1, η_L) relative to P.

Proof. By [1, Page 246, (0.5)], it is equivalent to show that; for all smooth R-representations π of G the map of vector spaces

$$\Phi : \mathrm{R}_{\kappa_P}(\pi) \to \mathrm{R}_{\kappa_L}(r_P^G(\pi)),$$

given by $\Phi(f) = r_P^G \circ f$ for $f \in \mathrm{R}_{\kappa_P}(\pi)$, is injective. This map is easily checked to be a homomorphism of representations of $M^\circ(\Lambda_E)$. Assume $\ker(\Phi)$, the kernel of Φ, is non-zero and let ϕ be an irreducible subrepresentation of $\ker(\Phi)$. Let $(\overline{\tau}, \overline{L})$ be in the cuspidal support of ϕ, here we mean that ϕ is a quotient of $i_P^G(\overline{\tau})$.

Thus $\overline{L}$ is a Levi subgroup of $M^\circ(\Lambda_E)$ (we allow the case $\overline{L} = M^\circ(\Lambda_E)$). Let $\overline{P}$ be the standard parabolic subgroup of $M^\circ(\Lambda_E)$ containing $\overline{L}$ with Levi decomposition $\overline{P} = \overline{L}\,\overline{U}$. Choose a self-dual $\mathfrak{o}_E$-lattice sequence Λ' such that $P^\circ(\Lambda'_E)$ is equal to the preimage of $\overline{P}$ under the projection $P^\circ(\Lambda_E) \to M_P^\circ$ and such that $P^\circ(\Lambda) \supseteq P^\circ(\Lambda')$ (considering Λ and Λ' as $\mathfrak{o}_F$-lattice sequences), this is possible by [24, Lemma 2.8]. Let $\kappa' = b_{\Lambda,\Lambda'}(\kappa)$. The decomposition of $V = \bigoplus_{j=-m}^{m} W_j$ is exactly subordinate to the $[\Lambda', n', 0, \beta]$. Hence we can form the groups

$$J'_P = H^1(\beta, \Lambda')(J^\circ(\beta, \Lambda') \cap P), \quad J'_L = J'_P \cap L$$

and the representations κ'_P of J'_P (the natural representation on the $(J^\circ(\beta, \Lambda') \cap U)$-fixed vectors of κ') and $\kappa'_L = \kappa'_P \mid_{J'_L}$.

We have the left exact sequence

$$0 \to \omega \to \mathrm{R}_{\kappa_P}(\pi) \to \mathrm{R}_{\kappa_L}(r_P^G(\pi)).$$

We apply the Jacquet functor $r_{\overline{P}}^{M(\Lambda_E)}$ (which is exact) and have

$$0 \to r_{\overline{P}}^{M(\Lambda_E)}(\omega) \to \mathrm{R}_{\kappa'_P}(\pi) \to \mathrm{R}_{\kappa'_L}(r_P^G(\pi)),$$

as $r_{\overline{P}}^{M(\Lambda_E)} \circ \mathrm{R}_{\kappa_P}(\pi) \simeq \mathrm{R}_{\kappa'_P}(\pi)$ and $r_{\overline{P}}^{M(\Lambda_E)} \circ \mathrm{R}_{\kappa_L}(r_P^G(\pi)) \simeq \mathrm{R}_{\kappa'_L}(r_P^G(\pi))$ by compatibility of κ and κ'. Then, taking the $\overline{\tau}$-isotypic components (which is a left exact functor) we have an exact sequence

$$0 \to \mathrm{Hom}_{\overline{L}}(\overline{\tau}, r_{\overline{P}}^{M(\Lambda_E)}(\omega)) \to \mathrm{Hom}_{\overline{L}}(\overline{\tau}, \mathrm{R}_{\kappa'_P}(\pi)) \to \mathrm{Hom}_{\overline{L}}(\overline{\tau}, \mathrm{R}_{\kappa'_L}(r_P^G(\pi))).$$

By right adjointness of $\mathrm{R}_{\kappa'_P}$ and $\mathrm{R}_{\kappa'_L}$ with $\mathrm{I}_{\kappa'_P}$ and $\mathrm{I}_{\kappa'_L}$ and right adjointness of restriction with compact induction this is isomorphic to the exact sequence

$$0 \to \mathrm{Hom}_{\overline{L}}(\overline{\tau}, r_{\overline{P}}^{M(\Lambda_E)}(\omega)) \to \mathrm{Hom}_{J_P^\circ}(\kappa'_P \otimes \overline{\tau}, \pi) \to \mathrm{Hom}_{J_L^\circ}(\kappa'_L \otimes \overline{\tau}, r_P^G(\pi))$$

As ω contains a subrepresentation with cuspidal support $\overline{\tau}$, $\mathrm{Hom}_{\overline{L}}(\overline{\tau}, r_{\overline{P}}^{M(\Lambda_E)}(\omega)) \neq 0$. However, by Theorem 9.2, $(J'_P, \kappa'_P \otimes \overline{\tau})$ is a G-cover of $(J'_L, \kappa'_L \otimes \overline{\tau})$ relative to P. Hence, by [1, Page 246, (0.5)], the map $\mathrm{Hom}_{J_P^\circ}(\kappa'_P \otimes \overline{\tau}, \pi) \to \mathrm{Hom}_{J_L^\circ}(\kappa'_L \otimes \overline{\tau}, r_P^G(\pi))$ is injective, a contradiction. □

10 Quasi-projectivity of types

This section shows that the types we consider are quasi-projective, so that Theorem 2.10 applies.

Lemma 10.1. Suppose that n is a distinguished double coset representative of $P^\circ_\Upsilon\backslash G_E/P^\circ_\Upsilon$ with projection w in the affine Weyl group of G_E such that, if P°_Υ corresponds to the subset K of the fundamental reflections in the affine Weyl group W' (cf. Section 6), then $wK = K$. Let τ be a representation of $M^\circ(\Upsilon_E)$. Then, we have an isomorphism of vector spaces

$$\mathrm{Hom}_{J^1_\Upsilon}(\kappa^\circ_\Upsilon, \mathrm{ind}_{J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n}^{J^\circ_\Upsilon}(\kappa^\circ_\Upsilon\otimes\tau)^n) \simeq \tau^n,$$

which is an isomorphism of representations if $n \in I_G(\kappa^\circ_\Upsilon)$.

Proof. Observe that we have $J^\circ_\Upsilon = J^1_\Upsilon(J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n) \supseteq J^1_\Upsilon(P^\circ_\Upsilon\cap(P^\circ_\Upsilon)^n)$ and moreover $J^\circ_\Upsilon/J^1_\Upsilon = J^1_\Upsilon(P^\circ_\Upsilon\cap(P^\circ_\Upsilon)^n)/J^1_\Upsilon$, as $wK = K$ (and using Section 6 (i)). Therefore

$$J^\circ_\Upsilon = J^1_\Upsilon(P^\circ_\Upsilon\cap(P^\circ_\Upsilon)^n).$$

Thus, by Mackey theory, we have

$$\mathrm{Res}^{J^\circ_\Upsilon}_{J^1_\Upsilon}(\mathrm{ind}_{J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n}^{J^\circ_\Upsilon}(\kappa^\circ_\Upsilon\otimes\tau)^n) \simeq \mathrm{ind}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}^{J^1_\Upsilon}(\kappa^\circ_\Upsilon\otimes\tau)^n.$$

Therefore, we have isomorphisms of vector spaces

$$\begin{aligned}\mathrm{Hom}_{J^1_\Upsilon}(\kappa^\circ_\Upsilon, \mathrm{ind}_{J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n}^{J^\circ_\Upsilon}(\kappa^\circ_\Upsilon\otimes\tau)^n) &\simeq \mathrm{Hom}_{J^1_\Upsilon}(\kappa^\circ_\Upsilon, \mathrm{ind}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}^{J^1_\Upsilon}(\kappa^\circ_\Upsilon\otimes\tau)^n)\\ &\simeq \mathrm{Hom}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}(\kappa^\circ_\Upsilon, (\kappa^\circ_\Upsilon)^n\otimes\tau^n)\end{aligned}$$

which, checking actions, is actually an isomorphism of representations of M°_Υ, where the action of

$$M^\circ_\Upsilon = J^1_\Upsilon(J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n)/J^1_\Upsilon \simeq (J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n)/(J^1_\Upsilon\cap(J^\circ_\Upsilon)^n)$$

on homomorphisms in $\mathrm{Hom}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}(\kappa^\circ_\Upsilon, (\kappa^\circ_\Upsilon)^n\otimes\tau^n)$ is given in the usual way by pre-composition with $(\kappa^\circ_\Upsilon)^{-1}$ and post-composition with $(\kappa^\circ_\Upsilon)^n\otimes\tau^n$. By Lemma 7.1 we can choose $S \in \mathrm{Hom}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}(\kappa^\circ_\Upsilon, (\kappa^\circ_\Upsilon)^n)$ nonzero, and $\mathrm{Hom}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}(\kappa^\circ_\Upsilon, (\kappa^\circ_\Upsilon)^n) = \mathrm{Hom}_{J^1_\Upsilon\cap(J^1_\Upsilon)^n}(\kappa^\circ_\Upsilon, (\kappa^\circ_\Upsilon)^n) \simeq R$ by Theorem 4.1. Hence, by Lemma 2.12 (applied with $X_1 = X_1^1 = J^1_\Upsilon$, $X_2 = J^n_\Upsilon$, $X_2^1 = (J^1_\Upsilon)^n$, $\mu_1 = \eta_\Upsilon$, $\mu_2 = \kappa^n_\Upsilon$, $\zeta_1 = 1$, and $\zeta_2 = \tau^n$) we have an isomorphism of vector spaces

$$\mathrm{Hom}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}(1, \tau^n) \to \mathrm{Hom}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}(\kappa^\circ_\Upsilon, (\kappa^\circ_\Upsilon)^n\otimes\tau^n),$$

given by the tensor product with S which is an isomorphism if $S \in \mathrm{Hom}_{J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n}(\kappa^\circ_\Upsilon, (\kappa^\circ_\Upsilon)^n)$, which will be the case if $\mathrm{Hom}_{J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n}(\kappa^\circ_\Upsilon, (\kappa^\circ_\Upsilon)^n) \neq 0$, i.e. if $n \in I_G(\kappa^\circ_\Upsilon)$. Moreover, as a representation of $M^\circ_\Upsilon = (J^\circ_\Upsilon\cap(J^\circ_\Upsilon)^n)/(J^1_\Upsilon\cap(J^\circ_\Upsilon)^n)$,

$$\mathrm{Hom}_{J^1_\Upsilon\cap(J^\circ_\Upsilon)^n}(1, \tau^n) \simeq \tau^n.$$

□

It seems likely that the elements n considered in Lemma 10.1 do intertwine κ°_Υ, we do not prove this here as it is not needed for our application.

Theorem 10.2. Suppose τ is cuspidal. The representation $\mathrm{I}_{\kappa^\circ_\Upsilon}(\tau)$ is quasi-projective.

Proof. Notice that, as J^1_Υ is pro-p, the η-isotypic component of $\mathrm{I}_{\kappa^\circ_\Upsilon}(\tau)$ is a summand of the restriction of $\mathrm{I}_{\kappa^\circ_\Upsilon}(\tau)$ to J_Υ, and no representation in its complement contains η, whence cannot be isomorphic to $\lambda = \kappa^\circ_\Upsilon \otimes \tau$. However, we have $\mathrm{I}_{\kappa^\circ_\Upsilon}(\tau)^\eta \simeq \kappa^\circ_\Upsilon \otimes \mathrm{R}_{\kappa^\circ_\Upsilon} \circ \mathrm{I}_{\kappa^\circ_\Upsilon}(\tau)$ (*cf.* [16, Lemme 2.6]). We can decompose $\mathrm{R}_{\kappa^\circ_\Upsilon} \circ \mathrm{I}_{\kappa^\circ_\Upsilon}(\tau)$ as a direct sum and choose distinguished double cosets for each summand as in the proof of Theorem 7.5. By Lemmas 7.4 and 10.1, the summands are either zero (when the distinguished coset representative projects to an element w with $wK \neq K$), or have the same dimension of τ. Hence the $\kappa^\circ_\Upsilon \otimes \tau$-isotypic component must be a direct summand of the η-isotypic component of $\mathrm{I}_{\kappa^\circ_\Upsilon}(\tau)$ and, by Lemma 2.9, the representation $\mathrm{I}_{\kappa^\circ_\Upsilon}(\tau)$ is quasi-projective. □

11 Exhaustion

We show how Corollary 7.5 can be used to show certain representations of G we have constructed are irreducible and cuspidal. Moreover, with Theorem 8.9, we show that this construction exhausts all irreducible cuspidal representations of G. In the complex case this construction is the same as [24, Corollary 6.19]. However, in addition to extending this construction to ℓ-modular representations, Corollary 7.5 allows us to make some comparisons between certain irreducible cuspidal representations in our exhaustive lists.

We call a skew semisimple stratum $[\Lambda, n, 0, \beta]$ *cuspidal* if G_E has compact centre and $P^\circ(\Lambda_E)$ is a maximal parahoric subgroup. A *type* for G is a pair $(J, \kappa\otimes\tau)$ where $J = J(\beta,\Lambda)$ for some self-dual semisimple stratum $[\Lambda, n, 0, \beta]$, κ is a β-extension of the unique Heisenberg representation η containing $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ and τ is an irreducible representation of J/J^1 with cuspidal restriction to J°/J^1. We call a type $(J, \kappa\otimes\tau)$ *cuspidal* if $[\Lambda, n, 0, \beta]$ is a cuspidal stratum. We call a cuspidal type $(J, \kappa \otimes \tau)$ *supercuspidal* if τ is supercuspidal on restriction to J°/J^1.

Theorem 11.1. Let $(J, \kappa \otimes \tau)$ be a cuspidal type for G relative to the skew semisimple stratum $[\Lambda, n, 0, \beta]$, then $\mathrm{I}_\kappa(\tau)$ is irreducible and cuspidal.

Proof. The conditions on $[\Lambda, n, 0, \beta]$ guarantee that $P(\Lambda_E)$ is its own normaliser. By Corollary 7.5, $\mathrm{End}_G(\mathrm{I}_\kappa(\tau)) \simeq R$. Let π be an irreducible R-representation of G such that $\kappa \otimes \tau$ is a subrepresentation of π (hence π is a quotient of $\mathrm{I}_\kappa(\tau)$). We must show that $\kappa \otimes \tau$ is also a quotient of π in order to apply Lemma 2.8. As J^1 is pro-p, we can decompose $\pi \simeq \pi^\eta \oplus \pi(\eta)$ where π^η denotes the η-isotypic component of π and no subquotient of $\pi(\eta)$ contains η. By Corollary 7.5, we have $\mathrm{I}_\kappa(\tau)^\eta \simeq \kappa \otimes \tau$, and hence by exactness $\pi^\eta \simeq \kappa \otimes \tau$ (or zero which it can't be as $\kappa \otimes \tau$ is a subrepresentation of π). Therefore, by Lemma 2.8, $\mathrm{I}_\kappa(\tau)$ is irreducible. Cuspidality follows from a classical argument (*cf.* [5, §1] and [25, §2, 2.7]). □

Theorem 11.2. Every irreducible cuspidal representation of G contains a cuspidal type.

Proof. Let π be an irreducible cuspidal representation of G. By [23, Theorem 5.1], the proof of which applies in positive characteristic $\ell \neq p$, there exist a skew semisimple stratum $[\Lambda, n, 0, \beta]$ and $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ such that π contains θ. Thus π contains the unique extension η of θ to J^1. Let κ be a standard β-extension of η. By Lemma 2.11, the functor $\kappa\otimes -$ identifies the category of R-representations of $M(\Lambda_E)$ with the category of η-isotypic representations of J. Thus π contains $\kappa\otimes\tau$ for some irreducible representation τ of J/J^1. The proof now follows, using [26, II 10.1], from Theorem 8.9 (*cf.* [17, Appendix A] and [24, Theorem 7.14]). □

A consequence of the of Corollary 7.5 is the following intertwining implies conjugacy theorem:

Theorem 11.3. Suppose $(J_\Lambda, \kappa_\Lambda \otimes \tau_\Lambda)$ and $(J_\Upsilon, \kappa_\Upsilon \otimes \tau_\Upsilon)$ are cuspidal types associated to the semisimple strata $[\Lambda, n_\Lambda, 0, \beta]$ and $[\Upsilon, n_\Upsilon, 0, \beta]$. If $I_{\kappa_\Lambda}(\tau_\Lambda) \simeq \mathrm{I}_{\kappa_\Upsilon}(\tau_\Upsilon)$, then there exists $g \in G_E$ such that $(J_\Upsilon^g, \kappa_\Upsilon^g \otimes \tau_\Upsilon^g) = (J_\Lambda, \kappa_\Lambda \otimes \tau_\Lambda)$.

Proof. By Corollary 7.5 (i) and Corollary 6.2 (ii), the lattice sequences Λ_E and Υ_E are in the same G_E-orbit. Hence, by conjugating by an element of G_E if necessary, we can assume $\Lambda = \Upsilon$. Hence the groups of the cuspidal types coincide, and by twisting τ_Λ by a character χ of $M(\Lambda_E)$ if necessary, we can assume $\kappa_\Lambda = \kappa_\Upsilon$. By Corollary 7.5 (ii) and adjointness, we have

$$\mathrm{Hom}_{M(\Upsilon_E)}(I_{\kappa_\Upsilon}(\chi \otimes \tau_\Lambda), \mathrm{I}_{\kappa_\Upsilon}(\tau_\Upsilon)) = \mathrm{Hom}_{M(\Upsilon_E)}(\chi \otimes \tau_\Lambda, \tau_\Upsilon),$$

which is non-zero by hypothesis. Hence $\chi \otimes \tau_\Lambda \simeq \tau_\Upsilon$ and thus the cuspidal types are conjugate by an element of G_E. □

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