Regularity of ultrafilters, Boolean ultrapowers, and Keisler's order

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Abstract

This thesis investigates combinatorial properties of ultrafilters and their model-theoretic significance. Motivated by recent results on Keisler's order, we develop new tools for the study of Boolean ultrapowers, deepening our understanding of the interplay between set theory and model theory.

The main contributions can be summarized as follows. In Chapter 2, we undertake a systematic study of regular ultrafilters on Boolean algebras. In particular, we analyse two different notions of regularity which have appeared in the literature and compare their model-theoretic properties. We then apply our analysis to the study of cofinal types of ultrafilters; as an application, we answer a question of Brown and Dobrinen by giving two examples of complete Boolean algebras on which all ultrafilters have maximum cofinal type. In conclusion, we discuss the existence of non-regular ultrafilters and prove that, consistently, every decomposable ultrafilter on a complete Boolean algebra is regular.

Chapter 3 centres around the study of Keisler's order. We prove that good ultrafilters on Boolean algebras are precisely the ones which capture the maximum class in Keisler's order, solving a problem posed by Benda in 1974. We also show that, given a regular ultrafilter on a complete Boolean algebra satisfying a distributivity condition, the saturation of the Boolean ultrapower of a model of a complete theory does not depend on the choice of the particular model, but only on the theory itself. Motivated by this fact, we apply and expand the framework of 'separation of variables', recently developed by Malliaris and Shelah, to obtain a new characterization of Keisler's order via Boolean ultrapowers.

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Introduction

Over the last decade, Malliaris and Shelah proved a striking sequence of results in the intersection between set theory and model theory. They developed surprising connections between classification theory and general topology, and through this settled affirmatively the old question of whether the two cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$ are equal [44]. The starting point for their work is the study of *Keisler's order*, introduced originally in 1967 as a model-theoretic framework to classify theories using combinatorial objects called *regular ultrafilters*.

Even though Keisler's order is defined via regular ultrafilters over sets, Malliaris and Shelah developed the technique of 'separation of variables', which can be regarded as a paradigm shift towards building ultrafilters on Boolean algebras. These ultrafilters give rise to a model-theoretic construction called the *Boolean ultrapower*, a natural generalization of the classic ultrapower construction to the context of complete Boolean algebras. The saturation properties of Boolean ultrapowers have been examined in previous work [52].

On the other hand, regular ultrafilters and Boolean ultrapowers have found unexpected applications in the theory of forcing and cardinal invariants. Indeed, in recent work of Raghavan and Shelah [55], Boolean ultrapowers of forcing iterations are used to force inequalities between cardinal invariants both at and above the continuum. This technique culminates in the spectacular result of Goldstern, Kellner and Shelah [18] that, consistently, all entries in Cichoń's diagram can be pairwise different.

Despite these promising applications, there is not yet a systematic approach to the study of regular ultrafilters on Boolean algebras. In fact, two different notions of regularity for ultrafilters on Boolean algebras have appeared in the literature, both under the name of 'regular' [35, 23, 61]. This confusion motivates the results of Section 2.1, where those two notions are compared and shown not to be equivalent. We therefore propose the terminology 'regular' and 'quasiregular' to distinguish them.

In Section 2.4, we analyse κ-regular and κ-quasiregular ultrafilters in terms of model-theoretic properties of the corresponding Boolean ultrapowers. In particular, we focus on their cardinality, cofinality, and κ^+-universality: in each of the three cases, one notion of regularity behaves as expected, while the weaker notion is not well behaved. Furthermore, Lemma 2.2.2 highlights the relation between regularity and multiplicative functions, and we exploit this connection to establish in Proposition 2.2.4 that every $\aleph_1$-incomplete κ-OK ultrafilter is κ-regular.

Section 2.3 shows how our analysis of quasiregular ultrafilters can be applied to the study of *cofinal types* of ultrafilters, a topic which has been the focus of much attention in recent years [10]. More specifically, Brown and Dobrinen [6] have been investigating Boolean algebras where all ultrafilters have maximum cofinal type. Intuitively, this amounts to finding Boolean algebras where every ultrafilter is 'as complicated as possible'. After showing that every infinite free algebra has such a property [6, Fact 2.4], they naturally raised the following question: if $\mathbb{B}$ is an infinite Boolean algebra such that all ultrafilters on $\mathbb{B}$ have maximum cofinal type, is $\mathbb{B}$ necessarily a free algebra? [6, Question 4.2]. Using combinatorial properties of quasiregular ultrafilters, we give a negative answer: Theorem 2.3.10 and Theorem 2.3.13 provide two examples of non-free Boolean algebras on which all ultrafilters have maximum cofinal type.

We conclude Chapter 2 with a discussion of ultrafilters which are *not* regular. After introducing the notions of uniformity and κ-decomposability, we review some consistency results on the existence of uniform non-regular ultrafilters. Then, in Corollary 2.5.9 we show that, consistently, for every cardinal κ and every complete Boolean algebra $\mathbb{B}$, all κ-decomposable ultrafilters on $\mathbb{B}$ are κ-regular. This result extends a theorem of Donder [11] on regularity of ultrafilters in the core model.

Motivated by our previous work [52] on saturation of Boolean ultrapowers, in Chapter 3 we address the following key research question: what kind of classification can arise when we compare theories according to the saturation of Boolean ultrapowers of their models? Or, in other words, can we use Boolean ultrapowers to obtain new information on the structure of Keisler's order? For the power-set case, the importance of regular ultrafilters in the classification of theories lies a crucial result of Keisler [33, Corollary 2.1a], which implies that the saturation of the regular ultrapower of a structure $\mathfrak{M}$ depends solely on the complete theory of $\mathfrak{M}$. The purpose of Section 3.2 is to show that, under a distributivity assumption, Keisler's result can be extended to regular ultrafilters on complete Boolean algebras.

Section 3.3 is concerned with Boolean ultrapowers which are saturated. In 1964, Keisler [31] showed that $\aleph_1$-incomplete λ-good ultrafilters are precisely those ultrafilters which produce λ-saturated ultrapowers. On this account, good ultrafilters can be thought as the 'strongest' ultrafilters in terms of saturation of ultrapowers. The problem of finding an analogous characterization for the 'strongest' ultrafilters on Boolean algebras has proved to be elusive. Mansfield [46] defined a notion of 'goodness' for such ultrafilters, with the aim of generalizing Keisler's characterization. Three years later, Benda [3] tackled this problem, but observed that, with Mansfield's definition, Keisler's argument apparently does not fully generalize to the context of Boolean algebras. In Theorem 3.3.9 we finally settle this question by unifying Mansfield's and Benda's approaches, and providing a full generalization of Keisler's characterization to Boolean ultrapowers.

Malliaris and Shelah's method of separation of variables involves representing a complete Boolean algebra as a homomorphic image of a power-set algebra. This approach is briefly summarized in Section 3.4, while Theorem 3.5.5 reformulates the related notion of *morality* in terms of saturation of Boolean ultrapowers. Finally, in our main Theorem 3.5.6, we provide a new characterization of Keisler's order via Boolean ultrapowers. This result shows how regular ultrafilters on complete Boolean algebras are able to detect exactly the same model-theoretic properties as ultrafilters on power-set algebras, thus suggesting a fruitful way of classifying theories using Boolean ultrapowers.

Chapter 1

Basic material

1.1 Types and saturation

We assume the reader is already familiar with the basic concepts of model theory, such as languages, structures, and homomorphisms. All undefined notions can be found in any standard reference on the subject, such as Chang and Keisler [7].

A *theory* in a language L is a set of L-sentences. The *complete theory* of an L-structure $\mathfrak{M}$, denoted by $\mathrm{Th}(\mathfrak{M})$, is the set of L-sentences φ such that $\mathfrak{M} \models \varphi$. Two L-structures $\mathfrak{M}$ and $\mathfrak{N}$ are *elementarily equivalent*, in symbols $\mathfrak{M} \equiv \mathfrak{N}$, if $\mathrm{Th}(\mathfrak{M}) = \mathrm{Th}(\mathfrak{N})$.

Let $\mathfrak{M}$ be an L-structure and $A \subseteq M$. We define $L(A) = L \cup \{ c_a \mid a \in A \}$ to be the language obtained from L by adding a new constant symbol c_a for each $a \in A$. We may expand $\mathfrak{M}$ to $L(A)$ in a natural way: the interpretation of the symbol c_a is simply a. This expansion is denoted by $\mathfrak{M}_A$.

Definition 1.1.1. Let $\mathfrak{M}$ be an L-structure and $B \subseteq M$. Suppose that $\Sigma(x_1, \ldots, x_n)$ is a set of $L(B)$-formulae. We say that a n-tuple $\langle a_1, \ldots, a_n \rangle \in {}^nM$ *realizes* $\Sigma(x_1, \ldots, x_n)$ in $\mathfrak{M}$ if $\mathfrak{M}_B \models \varphi(a_1, \ldots, a_n)$ for all $\varphi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$.

Definition 1.1.2. Let T be a theory in a language L. An *n-type* of T is any set $p(x_1, \ldots, x_n)$ of L-formulae which is realized in some model of T.

An n-type $p(x_1, \ldots, x_n)$ is *complete* if for every formula $\varphi(x_1, \ldots, x_n)$ either $\varphi \in p(x_1, \ldots, x_n)$ or $\neg\varphi \in p(x_1, \ldots, x_n)$.

In most cases, T will be the complete theory of some structure, possibly with parameters. Instead of '$p(x_1, \ldots, x_n)$ is an n-type of $\mathrm{Th}(\mathfrak{M}_B)$' we shall say that '$p(x_1, \ldots, x_n)$ is an n-type over B in $\mathfrak{M}$'.

The next proposition is an immediate consequence of the compactness theorem for first-order logic.

Proposition 1.1.3. *Let $\mathfrak{M}$ be an L-structure and $B \subseteq M$. Suppose that $p(x_1, \ldots, x_n)$ is a set of $L(B)$-formulae. Then $p(x_1, \ldots, x_n)$ is an n-type over B in $\mathfrak{M}$ if and only if every finite subset of $p(x_1, \ldots, x_n)$ is realized in $\mathfrak{M}$.*

Definition 1.1.4 (Morley and Vaught [50])**.** Let λ be a cardinal. An L-structure $\mathfrak{M}$ is *λ-universal* if for every L-structure $\mathfrak{N}$, if $|N| < \lambda$ and $\mathfrak{N} \equiv \mathfrak{M}$ then there exists an elementary embedding $j\colon \mathfrak{N} \to \mathfrak{M}$.

An L-structure $\mathfrak{M}$ is *λ-saturated* if, for every $B \subseteq M$ with $|B| < \lambda$, all 1-types over B are realized in $\mathfrak{M}$.

Definition 1.1.5 (Shelah [57])**.** A theory T is *stable* if there exists an infinite cardinal κ such that for every model $\mathfrak{M} \models T$ and every $B \subseteq M$ with $|B| \leq \kappa$, there are at most κ complete 1-types over B in $\mathfrak{M}$.

1.2 Boolean algebras

In this section we introduce one of the central objects of our study, following Monk [48]. A *Boolean algebra* is a complemented distributive lattice, usually denoted as a structure $\langle \mathbb{B}, \vee, \wedge, \neg, \mathbb{0}, \mathbb{1} \rangle$. Homomorphisms and subalgebras are defined according to the language, in the usual model-theoretic fashion.

For any set I, the structure

$$\langle \mathcal{P}(I), \cup, \cap, \neg, \emptyset, I \rangle,$$

where $\neg b = I \setminus b$, is a Boolean algebra, called *power-set algebra.* A classic theorem of Stone [65, Theorem 67] implies that every Boolean algebra is isomorphic to a subalgebra of a power-set algebra.

If $\mathbb{B}$ is any Boolean algebra, we introduce a partial order on $\mathbb{B}$:

$$a \leq b \overset{\text{def}}{\iff} a \vee b = b.$$

With respect to this partial order, for a subset $X \subseteq \mathbb{B}$ we denote

$$\bigvee X = \sup(X) \quad \text{and} \quad \bigwedge X = \inf(X),$$

whenever they actually exist.

Finally, for each $b \in \mathbb{B}$, we let

$$\mathbb{B} \restriction b = \{ a \in \mathbb{B} \mid a \leq b \},$$

which is also a Boolean algebra equipped with the natural operations. Note that $\mathbb{B} \restriction b$ is not a subalgebra of $\mathbb{B}$, unless $b = \mathbb{1}$.

We now list a few basic notions which will be relevant to our discourse.

Definition 1.2.1. Let $\mathbb{B}$ be a Boolean algebra. A *filter* on $\mathbb{B}$ is a subset $F \subset \mathbb{B}$ such that:

- $\mathbb{1} \in F$ and $\mathbb{0} \notin F$;
- if $a \in F$ and $b \in F$, then $a \wedge b \in F$;
- if $a \in F$, $b \in \mathbb{B}$ and $a \leq b$, then $b \in F$.

An *ultrafilter* is a filter U that satisfies this additional property:

- for all $b \in \mathbb{B}$, either $b \in U$ or $\neg b \in U$.

Let λ be a cardinal. A filter F is *λ-complete* if for every $X \subseteq F$ with $|X| < \lambda$, if $\bigwedge X$ exists then $\bigwedge X \in F$. A filter is *λ-incomplete* if it is not λ-complete.

Definition 1.2.2. Let $\mathbb{B}$ be a Boolean algebra. An *antichain* of $\mathbb{B}$ is a subset $A \subseteq \mathbb{B} \setminus \{\mathbb{0}\}$ such that for all $a, b \in A$, if $\mathbb{0} < a \wedge b$ then $a = b$.

If $x \in \mathbb{B}$ and A is an antichain in $\mathbb{B}$, we say that x is *based* on A if for every $a \in A$ either $a \leq x$ or $a \wedge x = \mathbb{0}$.

Definition 1.2.3. Let λ be a cardinal; a Boolean algebra $\mathbb{B}$ is *λ-c.c.* if every antichain in $\mathbb{B}$ has cardinality less than λ. The *saturation* of $\mathbb{B}$, denoted by $\operatorname{sat}(\mathbb{B})$, is the least cardinal λ such that $\mathbb{B}$ is λ-c.c.

Theorem 1.2.4 (Erdős and Tarski [13, Theorem 1])**.** *If $\mathbb{B}$ is an infinite Boolean algebra, then $\operatorname{sat}(\mathbb{B})$ is an uncountable regular cardinal.*

The following strong form of the $\aleph_1$-c.c. will play a role in Section 2.3.

Definition 1.2.5 (Horn and Tarski [22])**.** A Boolean algebra $\mathbb{B}$ is *σ-bounded c.c.* if there are subsets $S_n \subseteq \mathbb{B}$, for $n < \omega$, such that $\mathbb{B} = \bigcup_{n<\omega} S_n$ and every antichain in S_n has at most n elements.

Definition 1.2.6. Let $\mathbb{B}$ be a Boolean algebra. A subset $D \subseteq \mathbb{B} \setminus \{\mathbb{0}\}$ is *dense* if for all $b \in \mathbb{B} \setminus \{\mathbb{0}\}$ there exists $d \in D$ such that $d \leq b$.

Definition 1.2.7 (Smith and Tarski [64])**.** Let κ be a cardinal; a Boolean algebra $\mathbb{B}$ is *$\langle\kappa, 2\rangle$-distributive* if for every function $b\colon \kappa \times 2 \to \mathbb{B}$ we have

$$\bigwedge_{\alpha<\kappa} \bigvee_{n<2} b(\alpha, n) = \bigvee_{f \in {}^\kappa 2} \bigwedge_{\alpha<\kappa} b(\alpha, f(\alpha)),$$

provided that all the relevant suprema and infima exist in $\mathbb{B}$.

Complete Boolean algebras

We turn our attention to a special class of Boolean algebras in which all suprema and infima are well defined.

Definition 1.2.8. Let λ be a cardinal; a Boolean algebra $\mathbb{B}$ is *λ-complete* if for all $X \subseteq \mathbb{B}$, if $|X| < \lambda$ then $\bigvee X$ and $\bigwedge X$ exist. A Boolean algebra is *complete* if it is λ-complete for every cardinal λ.

The next remark is straightforward, but will be useful in the proof of Proposition 2.1.4.

Remark 1.2.9. Suppose $\mathbb{B}$ is a complete Boolean algebra. Then, for every cardinal $0 < \kappa < \operatorname{sat}(\mathbb{B})$ there exists a maximal antichain $A \subset \mathbb{B}$ with $|A| = \kappa$. To prove this, we note that if $\kappa < \operatorname{sat}(\mathbb{B})$ then, by definition, $\mathbb{B}$ has an antichain of cardinality $\geq \kappa$. Using Zorn's lemma, we may extend this antichain to a maximal antichain W. Since $0 < \kappa \leq |W|$, it is possible to partition W into κ many non-empty disjoint parts: $W = \bigcup_{i<\kappa} W_i$. Then clearly $A = \{ \bigvee W_i \mid i < \kappa \}$ is a maximal antichain in $\mathbb{B}$ such that $|A| = \kappa$.

In what follows, we use the term *embedding* to refer to an injective homomorphism. A classic result of Sikorski shows that, from the point of view of category theory, complete Boolean algebras are *injective* objects:

Theorem 1.2.10 (Sikorski [62]). *Let $\mathbb{B}_0$, $\mathbb{B}$, $\mathbb{B}'$ be Boolean algebras; assume that $\mathbb{B}'$ is complete. Then, for every homomorphism $h\colon \mathbb{B}_0 \to \mathbb{B}'$ and every embedding $g\colon \mathbb{B}_0 \to \mathbb{B}$, there exists a homomorphism $h'\colon \mathbb{B} \to \mathbb{B}'$ such that $h = h' \circ g$.*

The next result provides a useful reformulation of $\langle \kappa, 2\rangle$-distributivity for κ^+-complete Boolean algebras.

Proposition 1.2.11 (Pierce [53]). *Let κ be a cardinal. Suppose $\mathbb{B}$ is a κ^+-complete Boolean algebra; then the following are equivalent:*

1. *$\mathbb{B}$ is $\langle \kappa, 2\rangle$-distributive;*

2. *for every family $\{\, x_\alpha \mid \alpha < \kappa \,\} \subseteq \mathbb{B}$ there exists a maximal antichain A in $\mathbb{B}$ such that for every $\alpha < \kappa$, x_α is based on A.*

Proof. ($1 \implies 2$) Let us assume that $\mathbb{B}$ is $\langle \kappa, 2\rangle$-distributive and let $\{\, x_\alpha \mid \alpha < \kappa \,\} \subseteq \mathbb{B}$. We define a function $b\colon \kappa \times 2 \to \mathbb{B}$ as follows: for every $\alpha < \kappa$,

$$b(\alpha, 0) = x_\alpha \quad \text{and} \quad b(\alpha, 1) = \neg x_\alpha.$$

Now let

$$A = \left\{ \bigwedge_{\alpha<\kappa} b(\alpha, f(\alpha)) \;\middle|\; f\colon \kappa \to 2 \right\} \setminus \{\mathbb{0}\};$$

first of all, it is clear that A is an antichain in $\mathbb{B}$. Furthermore, we can apply $\langle \kappa, 2\rangle$-distributivity to obtain that

$$\bigvee A = \bigvee_{f\in{}^\kappa 2} \bigwedge_{\alpha<\kappa} b(\alpha, f(\alpha)) = \bigwedge_{\alpha<\kappa} \bigvee_{n<2} b(\alpha, n) = \bigwedge_{\alpha<\kappa} \mathbb{1} = \mathbb{1}.$$

This shows that A is a maximal antichain. By definition of A, it follows that for every $\alpha < \kappa$ and every $a \in A$, either $a \leq b(\alpha, 0) = x_\alpha$ or $a \leq b(\alpha, 1) = \neg x_\alpha$, as desired.

($2 \implies 1$) Let $b\colon \kappa \times 2 \to \mathbb{B}$ be a function. By hypothesis, there exists a maximal antichain A in $\mathbb{B}$ such that for every $\langle \alpha, n\rangle \in \kappa \times 2$, $b(\alpha, n)$ is based on A. Let

$$u = \bigwedge_{\alpha<\kappa} \bigvee_{n<2} b(\alpha, n);$$

we aim to prove that u is the least upper bound of the set

$$X = \left\{ \bigwedge_{\alpha<\kappa} b(\alpha, f(\alpha)) \,\middle|\, f\colon \kappa \to 2 \right\}.$$

It is clear that u is an upper bound of X, so let v be another upper bound of X: we need to show that $u \leq v$. Towards a contradiction, suppose not. Then there exists $a \in A$ such that $\mathbb{0} < a \wedge u \wedge \neg v$. Since in particular $\mathbb{0} < a \wedge u$, for each $\alpha < \kappa$ we can choose some $f(\alpha) < 2$ such that $\mathbb{0} < a \wedge b(\alpha, f(\alpha))$, thus defining a function $f\colon \kappa \to 2$. For every $\alpha < \kappa$, we have that $b(\alpha, f(\alpha))$ is based on A, hence $a \leq b(\alpha, f(\alpha))$. Finally, using the fact that v is an upper bound of X,

$$a \leq \bigwedge_{\alpha<\kappa} b(\alpha, f(\alpha)) \leq v,$$

a contradiction. This shows that u is the least upper bound of X and completes the proof. □

We now turn to the related concept of complete homomorphism.

Definition 1.2.12. Let $f\colon \mathbb{B} \to \mathbb{C}$ be a homomorphism of Boolean algebras; we say that f is *complete* if for all $X \subseteq \mathbb{B}$, if $\bigvee^{\mathbb{B}} X$ exists then also $\bigvee^{\mathbb{C}} f[X]$ exists and

$$f\Big(\bigvee\nolimits^{\mathbb{B}} X\Big) = \bigvee\nolimits^{\mathbb{C}} f[X].$$

Proposition 1.2.13 (Sikorski [63])**.** *Let $\mathbb{B}$ be a subalgebra of $\mathbb{C}$. If $\mathbb{B} \setminus \{\mathbb{0}\}$ is dense in $\mathbb{C}$, then the inclusion map $i\colon \mathbb{B} \to \mathbb{C}$ is a complete embedding.*

Finally, we present a classic result which shows that every Boolean algebra can be 'completed' in a canonical way.

Theorem 1.2.14 (MacNeille [39]; Tarski [67])**.** *Let $\mathbb{B}$ be a Boolean algebra; then there exists a complete Boolean algebra $\mathbb{C}$ such that $\mathbb{B}$ is a subalgebra of $\mathbb{C}$ and $\mathbb{B} \setminus \{\mathbb{0}\}$ is dense in $\mathbb{C}$. Furthermore, $\mathbb{C}$ is uniquely determined up to isomorphism, and is called the* completion *of $\mathbb{B}$.*

The following observation will be sometimes useful: if $\mathbb{C}$ is the completion of $\mathbb{B}$, then by density every element of $\mathbb{C}$ is the supremum of an antichain in $\mathbb{B}$. It follows that

$$|\mathbb{C}| \leq |\mathbb{B}|^{<\mathrm{sat}(\mathbb{B})}. \tag{1.1}$$

Free Boolean algebras

We introduce another class of Boolean algebras, whose importance for us lies in Theorem 2.1.11.

Definition 1.2.15. Let $\mathbb{F}$ be a Boolean algebra and $X \subseteq \mathbb{F}$. We say that $\mathbb{F}$ is *free* over X if for every Boolean algebra $\mathbb{B}$ and every function $f\colon X \to \mathbb{B}$ there exists a unique homomorphism $g\colon \mathbb{F} \to \mathbb{B}$ such that $g \restriction X = f$.

From the universal property above, it follows easily that if $\mathbb{F}$ is free over X and $\mathbb{F}'$ is free over X', then every bijection $f\colon X \to X'$ can be extended to a unique isomorphism $g\colon \mathbb{F} \to \mathbb{F}'$.

In the next proposition, as usual, $[X]^{<\aleph_0}$ denotes the set of finite subsets of X.

Proposition 1.2.16 (Stone [66])**.** *Let $\mathbb{F}$ be a Boolean algebra and $X \subseteq \mathbb{F}$. Then $\mathbb{F}$ is free over X if and only if X generates $\mathbb{F}$ and for every $S, T \in [X]^{<\aleph_0}$, if $\bigwedge S \leq \bigvee T$ then $S \cap T \neq \emptyset$.*

The characterization of Proposition 1.2.16 has a corollary which will be particularly useful for our purposes.

Corollary 1.2.17. *Let $\mathbb{F}$ be free over X. Then, for every infinite $Y \subseteq X$, we have*

$$\bigwedge Y = \mathbb{0} \quad \textit{and} \quad \bigvee Y = \mathbb{1}.$$

Proof. Let $Y \subseteq X$ be infinite; by contradiction, suppose the infimum of Y is not $\mathbb{0}$. Then, there exists some $b \in \mathbb{F} \setminus \{\mathbb{0}\}$ such that $b \leq y$ for all $y \in Y$. Since X generates $\mathbb{F}$, we can find disjoint finite subsets $S, T \in [X]^{<\aleph_0}$ such that

$$\bigwedge S \wedge \neg \bigvee T \leq b.$$

Using the fact that S is finite and Y is infinite, choose $y \in Y \setminus S$. It follows that

$$\bigwedge S \wedge \neg \bigvee T \leq b \leq y$$

and therefore

$$\bigwedge S \leq y \vee \bigvee T,$$

contradicting independence. This shows that $\bigwedge Y = \mathbb{0}$.

The fact that $\bigvee Y = \mathbb{1}$ follows from duality, reversing the role of $\vee$ and $\wedge$. □

The existence of free Boolean algebras is established by the next result.

Theorem 1.2.18 (Stone [66, Theorem 12])**.** *For every cardinal κ, let $\mathbb{F}_\kappa$ be the algebra of clopen subsets of the Cantor space ${}^\kappa 2$. Then $\mathbb{F}_\kappa$ is a free Boolean algebra over κ independent generators. Furthermore, if κ is infinite then $|\mathbb{F}_\kappa| = \kappa$.*

Theorem 1.2.19 (Marczewski [47])**.** *For every cardinal κ, $\mathbb{F}_\kappa$ is $\aleph_1$-c.c.*

The next result will play a role in Section 2.3.

Theorem 1.2.20 (Gaifman [16]; Hales [19])**.** *Let κ be a cardinal. Then $\mathbb{F}_\kappa$ is complete if and only if κ is finite.*

Thus, we are motivated to consider the completion of $\mathbb{F}_\kappa$, which will be denoted by $\mathbb{C}_\kappa$. From (1.1) and Theorem 1.2.19, it is not difficult to deduce that

$$|\mathbb{C}_\kappa| = \kappa^{\aleph_0} \tag{1.2}$$

whenever κ is infinite.

1.3 Ultrapowers

The ultrapower construction is due to Łoś [38]. Given an L-structure $\mathfrak{M}$ and a set I, let M^I be the set of functions $f\colon I \to M$. For an ultrafilter U over I, let $\equiv_U$ be the equivalence relation on M^I defined by

$$f \equiv_U g \overset{\text{def}}{\iff} \{\, i \in I \mid f(i) = g(i) \,\} \in U, \tag{1.3}$$

and let $[f]_U$ be the corresponding equivalence class of a function f. Quotienting the set M^I by the relation $\equiv_U$ gives rise to an L-structure $\mathfrak{M}^I/U$, the *ultrapower* of $\mathfrak{M}$ by U, which satisfies the following crucial property:

Theorem 1.3.1 (Łoś [38])**.** *Let $\mathfrak{M}$ be an L-structure and U an ultrafilter over a set I. For every L-formula $\varphi(x_1, \dots, x_n)$ and functions $f_1, \dots, f_n \in M^I$ we have*

$$\mathfrak{M}^I/U \models \varphi\big([f_1]_U, \dots, [f_n]_U\big) \iff \{\, i \in I \mid \mathfrak{M} \models \varphi\big(f_1(i), \dots, f_n(i)\big) \,\} \in U.$$

In this section, we briefly describe two generalizations of the classic ultrapower construction.

Boolean ultrapowers

The Boolean ultrapower construction dates back to Foster [14]. A detailed presentation of Boolean ultrapowers can be found in the standard reference of Mansfield [46]. However, to keep this thesis self-contained, we now recall the main points.

Let A and W be maximal antichains in a complete Boolean algebra $\mathbb{B}$. We say that W is a *refinement* of A if for every $w \in W$ there exists $a \in A$ such that $w \leq a$. Note that this element $a \in A$ is unique.

Definition 1.3.2 (Hamkins and Seabold [20])**.** Let M be a set, A a maximal antichain, and $\tau \colon A \to M$ a function. If W is a refinement of A, the *reduction* of τ to W is the function

$$\begin{aligned} (\tau \downarrow W) \colon W &\longrightarrow M \\ w &\longmapsto \tau(a) \end{aligned},$$

where a is the unique element of A such that $w \leq a$.

We remark that finitely many maximal antichains $A_0, \dots, A_{n-1}$ always admit a common refinement, namely the maximal antichain

$$\bigwedge_{i<n} A_i = \{\, a_0 \wedge \cdots \wedge a_{n-1} \mid a_i \in A_i \text{ for } i < n \,\} \setminus \{\mathbb{0}\}. \tag{1.4}$$

Let $\mathfrak{M}$ be an L-structure and $\mathbb{B}$ a complete Boolean algebra; we define first the set of *names*

$$M^{[\mathbb{B}]} = \{\, \tau \colon A \to M \mid A \subset \mathbb{B} \text{ is a maximal antichain} \,\}.$$

Remark 1.3.3. In the definition of names, we could equivalently reverse the arrows and consider functions from M to $\mathbb{B}$, as in Mansfield [46]. However, we find the above presentation more convenient for our purposes.

We then introduce a Boolean-valued semantic as follows: let $\varphi(x_1, \dots, x_n)$ be an L-formula and $\tau_1, \dots, \tau_n \in M^{[\mathbb{B}]}$. If W is any common refinement of $\mathrm{dom}(\tau_1), \dots, \mathrm{dom}(\tau_n)$, then

$$[\![\varphi(\tau_1, \dots, \tau_n)]\!]^{\mathfrak{M}^{[\mathbb{B}]}} = \bigvee \{\, w \in W \mid \mathfrak{M} \models \varphi\big((\tau_1 \downarrow W)(w), \dots, (\tau_n \downarrow W)(w)\big) \,\}.$$

It is easy to check that the *Boolean value* $[\![\varphi(\tau_1, \dots, \tau_n)]\!]^{\mathfrak{M}^{[\mathbb{B}]}}$ does not depend on the particular choice of the refinement W. From now on, when there is no danger of confusion, the superscript $\mathfrak{M}^{[\mathbb{B}]}$ will be omitted.

The following two results, sometimes called 'mixing property' and 'fullness' in the literature, will be useful in the sequel.

Lemma 1.3.4 (Mansfield [46, Theorem 1.3]). *Let $\mathfrak{M}$ be an L-structure and $\mathbb{B}$ a complete Boolean algebra. If $A \subset \mathbb{B}$ is an antichain and $\{\, \tau_a \mid a \in A \,\} \subseteq M^{[\mathbb{B}]}$, then there is $\tau \in M^{[\mathbb{B}]}$ such that $a \leq [\![\tau = \tau_a]\!]$ for all $a \in A$.*

Theorem 1.3.5 (Mansfield [46, Theorem 1.4]). *Let $\mathfrak{M}$ be an L-structure and $\mathbb{B}$ a complete Boolean algebra. Then, for every L-formula $\varphi(x, y_1, \dots, y_n)$ and $\sigma_1, \dots, \sigma_n \in M^{[\mathbb{B}]}$ there exists $\tau \in M^{[\mathbb{B}]}$ such that*

$$[\![\exists x \varphi(x, \sigma_1, \dots, \sigma_n)]\!] = [\![\varphi(\tau, \sigma_1, \dots, \sigma_n)]\!].$$

Given an ultrafilter U on B, let $\equiv_U$ be the equivalence relation on $M^{[\mathbb{B}]}$ defined analogously to (1.3):

$$\tau \equiv_U \sigma \overset{\text{def}}{\iff} [\![\tau = \sigma]\!] \in U.$$

Quotienting the set of names by the above equivalence relation gives rise to the L-structure $\mathfrak{M}^{[\mathbb{B}]}/U$, the *Boolean ultrapower* of $\mathfrak{M}$ by U, which satisfies the following analogue of Theorem 1.3.1.

Theorem 1.3.6 (Mansfield [46, Theorem 1.5]). *Let $\mathfrak{M}$ be an L-structure, $\mathbb{B}$ a complete Boolean algebra, and U an ultrafilter on $\mathbb{B}$. For every L-formula $\varphi(x_1, \ldots, x_n)$ and names $\tau_1, \ldots, \tau_n \in M^{[\mathbb{B}]}$ we have*

$$\mathfrak{M}^{[\mathbb{B}]}/U \models \varphi\big([\tau_1]_U, \ldots, [\tau_n]_U\big) \iff [\![\varphi(\tau_1, \ldots, \tau_n)]\!] \in U.$$

In particular, if for every $m \in M$ we define the name

$$\begin{aligned} \check{m} \colon \{\mathbb{1}\} &\longrightarrow M \\ \mathbb{1} &\longmapsto m \end{aligned},$$

then we obtain the following corollary.

Corollary 1.3.7. *Let $\mathfrak{M}$ be an L-structure, $\mathbb{B}$ a complete Boolean algebra, and U an ultrafilter on $\mathbb{B}$. Then the* natural embedding*, defined as*

$$\begin{aligned} d \colon M &\longrightarrow M^{[\mathbb{B}]}/U \\ m &\longmapsto [\check{m}]_U \end{aligned},$$

is an elementary embedding of $\mathfrak{M}$ into $\mathfrak{M}^{[\mathbb{B}]}/U$.

Remark 1.3.8. Suppose U is an ultrafilter over a set I. Then, for every structure $\mathfrak{M}$

$$\mathfrak{M}^{[\mathcal{P}(I)]}/U \cong \mathfrak{M}^I/U;$$

hence, Boolean ultrapowers are indeed a generalization of ultrapowers.

On the other hand, Koppelberg and Koppelberg [35] have shown that not all Boolean ultrapowers are isomorphic to classic ultrapowers.

Limit ultrapowers

We conclude this section with a brief remark on the limit ultrapower construction, which is due to Keisler [28]. Although we shall not be directly concerned with limit ultrapowers in this thesis, they will make a brief appearance in Section 3.5.

Definition 1.3.9. Let $\mathfrak{M}$ be an L-structure and I a set; for a function $f\colon I \to M$, let

$$\operatorname{eq}(f) = \{ \langle i, j\rangle \in I \times I \mid f(i) = f(j) \}.$$

Now, let U be an ultrafilter over I and F a filter over $I \times I$; we define

$$M^I_U|F = \left\{ [f]_U \;\middle|\; f \in M^I \text{ and } \operatorname{eq}(f) \in F \right\}.$$

It easy to verify that $M^I_U|F$ is a non-empty subset of M^I/U, closed under the functions and constants of the language. Hence, it is legitimate to define the *limit ultrapower* $\mathfrak{M}^I_U|F$ as the substructure of $\mathfrak{M}^I/U$ whose domain is $M^I_U|F$.

Limit ultrapowers enjoy many of the properties of classic ultrapowers; we refer the reader to Keisler [28] for a general discussion and to Keisler [27] for a justification of the terminology 'limit ultrapower'. A thorough analysis of model-theoretic properties of limit ultrapowers has been carried out in a sequence of three papers by Wierzejewski [72], Węglorz [71], and Pacholski [51].

Definition 1.3.10. Let $\mathfrak{M}$ and $\mathfrak{N}$ be L-structures. An elementary embedding $j\colon \mathfrak{M} \to \mathfrak{N}$ is a *complete embedding* if for every expansion $\mathfrak{M}'$ of $\mathfrak{M}$ to a language $L' \supset L$, there exists an expansion $\mathfrak{N}'$ of $\mathfrak{N}$ to L' such that $j\colon \mathfrak{M}' \to \mathfrak{N}'$ is an elementary embedding of L'-structures.

Remark 1.3.11. There is an unfortunate clash of terminology: the notion of complete embedding just introduced is unrelated to Definition 1.2.12.

The importance of limit ultrapowers lies in the fact that every complete extension of a structure can be realized as a limit ultrapower.

Theorem 1.3.12 (Keisler [28, Theorem 3.7])**.** *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two L-structures. Then $\mathfrak{N}$ is isomorphic to a limit ultrapower of $\mathfrak{M}$ if and only if there exists a complete embedding $j\colon \mathfrak{M} \to \mathfrak{N}$.*

Since our main focus in this work is on Boolean ultrapowers, we now outline a proof of the fact that every Boolean ultrapower of a structure is isomorphic to a limit ultrapower. This is a well-known result and appears, for instance, in Shelah [60, Exercise VI.3.34].

Proposition 1.3.13. *Let $\mathfrak{M}$ be an L-structure. Then, every Boolean ultrapower of $\mathfrak{M}$ is isomorphic to some limit ultrapower of $\mathfrak{M}$.*

Proof. Let U be an ultrafilter on a complete Boolean algebra $\mathbb{B}$; we need to show that the Boolean ultrapower $\mathfrak{M}^{[\mathbb{B}]}/U$ is isomorphic to some limit ultrapower of $\mathfrak{M}$. To do so, it is sufficient to prove that the natural embedding $d\colon \mathfrak{M} \to \mathfrak{M}^{[\mathbb{B}]}/U$ is a complete embedding and then apply Theorem 1.3.12.

Let $\mathfrak{M}'$ be an expansion of $\mathfrak{M}$ to a language $L' \supset L$. Obviously, $d\colon \mathfrak{M}' \to \mathfrak{M}'^{[\mathbb{B}]}/U$ is an elementary embedding of L'-structures. But the restriction of $\mathfrak{M}'^{[\mathbb{B}]}/U$ to L is simply $\mathfrak{M}^{[\mathbb{B}]}/U$; in other words, $\mathfrak{M}'^{[\mathbb{B}]}/U$ is an expansion of $\mathfrak{M}^{[\mathbb{B}]}/U$ to L' such that d remains an elementary embedding. Thus, d is a complete embedding. □

1.4 Rudin-Keisler ordering

This section presents one of the key tools in our study, the *Rudin-Keisler ordering*, introduced in the sixties by Mary Ellen Rudin and H. Jerome Keisler independently.

Definition 1.4.1 (Rudin [56]; Keisler [32])**.** Let D be an ultrafilter over I, and U an ultrafilter over J. We say that $D \leq_{\mathrm{RK}} U$ if there exists a function $f\colon J \to I$ such that for all $X \subseteq I$

$$X \in D \iff f^{-1}[X] \in U.$$

Informally, if D and U are ultrafilters, then $D \leq_{\mathrm{RK}} U$ in the Rudin-Keisler ordering if and only if U gives larger ultrapowers than D with respect to elementary embeddings. This intuition is made precise in the following theorem.

Theorem 1.4.2 (Blass [5, Proposition 11.7])**.** *Let D be an ultrafilter over I, and U an ultrafilter over J; then the following conditions are equivalent:*

- *$D \leq_{\mathrm{RK}} U$;*
- *for every structure $\mathfrak{M}$, there exists an elementary embedding $j\colon \mathfrak{M}^I/D \to \mathfrak{M}^J/U$.*

Jipsen, Pinus and Rose [26] developed a generalization of the Rudin-Keisler ordering which is especially useful for our study of ultrafilters on complete Boolean algebras.

To simplify the notation in the next definition, we let $\mathrm{Part}(\mathbb{B})$ denote the set of all maximal antichains of a Boolean algebra $\mathbb{B}$.

Definition 1.4.3 (Jipsen, Pinus and Rose [26]). Let $\mathbb{B}$ and $\mathbb{C}$ be complete Boolean algebras. For ultrafilters D on $\mathbb{B}$ and U on $\mathbb{C}$, we define $D \leq_{\mathrm{RK}} U$ if and only if there exist a function $g\colon \mathrm{Part}(\mathbb{B}) \to \mathrm{Part}(\mathbb{C})$ and, for each $A \in \mathrm{Part}(\mathbb{B})$, a function $f_A\colon g(A) \to A$ such that:

1. for every $A \in \mathrm{Part}(\mathbb{B})$ and every $X \subseteq A$,

$$\bigvee X \in D \iff \bigvee f_A^{-1}[X] \in U;$$

2. if $A, A' \in \mathrm{Part}(\mathbb{B})$ and A is a refinement of A', then for every common refinement W of $g(A)$ and $g(A')$ we have

$$\bigvee \{\, w \in W \mid (f_A \downarrow W)(w) \leq (f_{A'} \downarrow W)(w) \,\} \in U.$$

The two conditions of Definition 1.4.3 are quite complex, but the following sufficient condition will be often easier to verify.

Proposition 1.4.4 (Jipsen, Pinus and Rose [26, Proposition 1.2]). *Let $\mathbb{B}$ and $\mathbb{C}$ be complete Boolean algebras; suppose D is an ultrafilter on $\mathbb{B}$, and U is an ultrafilter on $\mathbb{C}$. If there exists a complete homomorphism $h\colon \mathbb{B} \to \mathbb{C}$ such that $h[D] \subseteq U$, then $D \leq_{\mathrm{RK}} U$.*

The next result establishes a parallel to Theorem 1.4.2.

Theorem 1.4.5 (Jipsen, Pinus and Rose [26, Theorem 2.4]). *Let $\mathbb{B}$ and $\mathbb{C}$ be complete Boolean algebras. For ultrafilters D on $\mathbb{B}$ and U on $\mathbb{C}$, the following conditions are equivalent:*

- *$D \leq_{\mathrm{RK}} U$;*
- *for every structure $\mathfrak{M}$, there exists an elementary embedding $j\colon \mathfrak{M}^{[\mathbb{B}]}/D \to \mathfrak{M}^{[\mathbb{C}]}/U$.*

Chapter 2

Regularity of ultrafilters

2.1 Two notions of regularity

Regular ultrafilters were first constructed by Frayne, Morel and Scott [15] to determine the possible cardinalities of ultrapowers. Two years later, Keisler coined the term 'regular ultrafilters' and initiated their systematic study.

Definition 2.1.1 (Keisler [30])**.** Let κ be a cardinal. A filter F over a set I is *κ-regular* if there exists a family $\{ X_\alpha \mid \alpha < \kappa \} \subseteq F$ such that for every infinite $I \subseteq \kappa$ we have $\bigcap_{\alpha \in I} X_\alpha = \emptyset$.

In this section, we shall present and compare two different definitions of regularity for filters on complete Boolean algebras. As we remarked in the Introduction, both notions have appeared in the literature under the name 'regular'. To avoid creating further confusion, we have decided to use the names 'regular' and 'quasiregular' to distinguish them.

We are ready to state the first main definition.

Definition 2.1.2 (Shelah [61])**.** Let κ be a cardinal. A filter F on a complete Boolean algebra $\mathbb{B}$ is *κ-regular* if there exist a family $\{ x_\alpha \mid \alpha < \kappa \} \subseteq F$ and a maximal antichain $A \subset \mathbb{B}$ such that:

- for every $\alpha < \kappa$, x_α is based on A;
- for every $a \in A$, the set $\{ \alpha < \kappa \mid a \leq x_\alpha \}$ is finite.

It follows immediately from Definition 2.1.2 that if F is a κ-regular filter and $\lambda \leq \kappa$, then F is also λ-regular.

Lemma 2.1.3. *Let F be a filter on a complete Boolean algebra $\mathbb{B}$. If a family $\{ x_\alpha \mid \alpha < \kappa \} \subseteq F$ and a maximal antichain $A \subset \mathbb{B}$ witness the κ-regularity of F, then for every $b \in F$*

$$\kappa \leq |\{ a \in A \mid \mathbb{0} < a \wedge b \}|.$$

Proof. Let $b \in F$; we need to show that the set

$$A_b = \{ a \in A \mid \mathbb{0} < a \wedge b \}$$

has cardinality $\geq \kappa$. Since A is a maximal antichain, for every $\alpha < \kappa$ we can choose some $a_\alpha \in A$ such that $\mathbb{0} < a_\alpha \wedge b \wedge x_\alpha$. We claim that, by κ-regularity, the map $\alpha \mapsto a_\alpha$ is finite-to-one from κ to A_b. Indeed, if $a_\alpha = a$ for infinitely many α's, then we would have $\mathbb{0} < a \leq x_\alpha$ for infinitely many α's, a contradiction. This shows that $\kappa \leq |A_b|$, as desired. □

We now present an existence result for regular ultrafilters; our argument is a simple modification of the original construction by Frayne, Morel and Scott [15, Theorem 1.17]. Also, we remark that more general existence results for regular ultrafilters will appear in Raghavan and Shelah [55].

Proposition 2.1.4. *Let κ be an infinite cardinal. For a complete Boolean algebra $\mathbb{B}$, the following conditions are equivalent:*

1. $\operatorname{sat}(\mathbb{B}) > \kappa$;
2. *there exists a κ-regular ultrafilter on $\mathbb{B}$.*

Proof. (2 $\Longrightarrow$ 1) We already know from Lemma 2.1.3 that if there exists a κ-regular ultrafilter on $\mathbb{B}$, then $\mathbb{B}$ has necessarily an antichain of cardinality $\geq \kappa$.

(1 $\Longrightarrow$ 2) Assume that $\operatorname{sat}(\mathbb{B}) > \kappa$; by Remark 1.2.9, we can find a maximal antichain $A = \{ a_i \mid i < \kappa \}$ in $\mathbb{B}$ such that $|A| = \kappa$. Let us fix an enumeration $[\kappa]^{<\aleph_0} = \{ S_i \mid i < \kappa \}$ and define for every $\alpha < \kappa$

$$x_\alpha = \bigvee \{ a_i \mid \alpha \in S_i \}.$$

Observe that for every $\alpha_1, \dots, \alpha_n < \kappa$ we have

$$x_{\alpha_1} \wedge \cdots \wedge x_{\alpha_n} = \bigvee \{\, a_i \mid \{\alpha_1, \dots, \alpha_n\} \subseteq S_i \,\} > \mathbb{0},$$

hence the family $\{\, x_\alpha \mid \alpha < \kappa \,\}$ has the finite intersection property, and so it generates an ultrafilter U on $\mathbb{B}$.

To show that U is κ-regular, we just observe that for each $\alpha < \kappa$ and every $i < \kappa$ we have the two implications

$$\mathbb{0} < a_i \wedge x_\alpha \implies \alpha \in S_i \implies a_i \leq x_\alpha.$$

From this, it follows immediately that the family $\{\, x_\alpha \mid \alpha < \kappa \,\} \subseteq U$ and the maximal antichain $A \subset \mathbb{B}$ satisfy the two conditions of Definition 2.1.2. □

We shall now present a second definition of regularity, which can be found in Koppelberg and Koppelberg [35] and Huberich [23]. This is arguably the most natural generalization of Definition 2.1.1 to the language of Boolean algebras; however, our choice of terminology 'quasiregular' is motivated by the results in Section 2.4, which demonstrate that this natural generalization is in fact not well behaved model theoretically.

Definition 2.1.5. Let κ be a cardinal and $\mathbb{B}$ a complete Boolean algebra. A filter F on $\mathbb{B}$ is *κ-quasiregular* if there exists a family $\{\, x_\alpha \mid \alpha < \kappa \,\} \subseteq F$ such that for every infinite $I \subseteq \kappa$ we have $\bigwedge_{\alpha \in I} x_\alpha = \mathbb{0}$.

Again, it follows from Definition 2.1.5 that if F is a κ-quasiregular filter and $\lambda \leq \kappa$, then F is also λ-quasiregular.

Remark 2.1.6. Let κ be an infinite cardinal. If F is a κ-quasiregular filter on $\mathbb{B}$, then for all $b \in F$, every dense subset of $\mathbb{B} \restriction b$ has cardinality at least κ. To see this, let $b \in F$; if $D \subset \mathbb{B} \restriction b$ is dense, then for every $\alpha < \kappa$ we can choose some $d_\alpha \in D$ such that $d_\alpha \leq b \wedge x_\alpha$. Hence, by κ-quasiregularity, the map $\alpha \mapsto d_\alpha$ is finite-to-one from κ to D.

The next proposition is straightforward, and justifies our choice of terminology.

Proposition 2.1.7. *Let κ be a cardinal; for any complete Boolean algebra $\mathbb{B}$, every κ-regular filter on $\mathbb{B}$ is also κ-quasiregular.*

Proof. Suppose F is a κ-regular filter on $\mathbb{B}$; this is witnessed by a family $\{ x_\alpha \mid \alpha < \kappa \} \subseteq F$ and a maximal antichain $A \subset \mathbb{B}$. We shall prove that for every infinite $I \subseteq \kappa$ we have $\bigwedge_{\alpha \in I} x_\alpha = \mathbb{0}$. To obtain a contradiction, suppose this is not the case. Then $\bigwedge_{\alpha \in I} x_\alpha > \mathbb{0}$, which implies the existence of some $a \in A$ with $a \wedge \bigwedge_{\alpha \in I} x_\alpha > \mathbb{0}$, since A is maximal. Therefore, for every $\alpha \in I$ we have $a \wedge x_\alpha > \mathbb{0}$, which implies $a \leq x_\alpha$ by the definition of κ-regularity. Thus, we have shown that there exists $a \in A$ such that $a \leq x_\alpha$ for infinitely many α's, a contradiction. □

Lemma 2.1.8. *Let κ be an infinite cardinal and $\mathbb{B}$ a complete Boolean algebra. If F is a κ-regular filter on $\mathbb{B}$, then the maximal antichain witnessing its regularity can be chosen to have cardinality κ.*

Proof. Suppose F is a κ-regular filter on $\mathbb{B}$; this is witnessed by a family $\{ x_\alpha \mid \alpha < \kappa \} \subseteq F$ and a maximal antichain $A \subset \mathbb{B}$. Consider the following antichain:

$$W = \left\{ \bigwedge_{\alpha \in I} x_\alpha \wedge \bigwedge_{\alpha \notin I} \neg x_\alpha \;\middle|\; I \subseteq \kappa \right\} \setminus \{\mathbb{0}\}.$$

From the definition of W, it follows that for every $\alpha < \kappa$ and every $w \in W$, either $w \leq x_\alpha$ or $w \wedge x_\alpha = \mathbb{0}$. Furthermore, for every $w \in W$ the set $\{ \alpha < \kappa \mid w \leq x_\alpha \}$ must be finite, otherwise it would contradict the κ-quasiregularity of F (Proposition 2.1.7).

To see that W is maximal, it suffices to observe for every $a \in A$ there exists a set $I \subseteq \kappa$ such that

$$a \leq \bigwedge_{\alpha \in I} x_\alpha \wedge \bigwedge_{\alpha \notin I} \neg x_\alpha.$$

Hence, $\mathbb{1} = \bigvee A \leq \bigvee W$ and so W is maximal.

From Lemma 2.1.3 we already know that $\kappa \leq |W|$. To see that $|W| = \kappa$, observe that whenever I is infinite we must have $\bigwedge_{\alpha \in I} x_\alpha =$

$\mathbb{0}$; therefore we have the equality

$$W = \left\{ \bigwedge_{\alpha \in I} x_\alpha \wedge \bigwedge_{\alpha \notin I} \neg x_\alpha \;\middle|\; I \in [\kappa]^{<\aleph_0} \right\} \setminus \{\mathbb{0}\},$$

which gives us $|W| \leq \kappa^{<\aleph_0} = \kappa$. □

Proposition 2.1.9. *Let $\mathbb{B}$ be a complete Boolean algebra. For an ultrafilter U on $\mathbb{B}$, the following conditions are equivalent:*

1. *U is $\aleph_0$-regular;*
2. *U is $\aleph_0$-quasiregular;*
3. *U is $\aleph_1$-incomplete.*

Proof. ($1 \implies 2$) Follows already from Proposition 2.1.7.

($2 \implies 3$) From the definition of $\aleph_0$-quasiregularity, we obtain the existence of some $X \subseteq U$ with $|X| = \aleph_0$ such that $\bigwedge X = \mathbb{0} \notin U$, as desired.

($3 \implies 1$) Suppose U is $\aleph_1$-incomplete; since U is an ultrafilter, this entails the existence of a countable subset $\{ x_n \mid n < \omega \} \subseteq U$ such that $\bigwedge_{n<\omega} x_n = \mathbb{0}$. Without loss of generality, we may assume that $x_{n+1} < x_n$ for all $n < \omega$, and $x_0 = \mathbb{1}$. Let us define for every $i < \omega$

$$a_i = x_i \wedge \neg x_{i+1};$$

it is clear that $A = \{ a_i \mid i < \omega \}$ is an antichain. Furthermore, A is maximal, because for all $i < \omega$

$$a_0 \vee \cdots \vee a_i = x_0 \wedge \neg x_{i+1} = \mathbb{1} \wedge \neg x_{i+1} = \neg x_{i+1},$$

and therefore

$$\bigvee A = \bigvee_{i<\omega} (a_0 \vee \cdots \vee a_i) = \bigvee_{i<\omega} \neg x_{i+1} = \neg \bigwedge_{i<\omega} x_{i+1} = \mathbb{1}.$$

To show that U is $\aleph_0$-regular, it is sufficient to observe that for all $i, n < \omega$ we have the two implications

$$\mathbb{0} < a_i \wedge x_n \implies n \leq i \implies a_i \leq x_n.$$

From this, we deduce that the family $\{ x_n \mid n < \omega \}$ and the maximal antichain A satisfy the two conditions of Definition 2.1.2. □

Thus, when $\kappa = \aleph_0$, both regularity properties coincide with $\aleph_1$-incompleteness. When κ is arbitrary, an additional distributivity assumption on $\mathbb{B}$ will also make the two properties coincide.

Proposition 2.1.10. *Let κ be an infinite cardinal. If $\mathbb{B}$ is a $\langle \kappa, 2 \rangle$-distributive complete Boolean algebra, then every κ-quasiregular filter on $\mathbb{B}$ is κ-regular.*

Proof. Suppose F is a κ-quasiregular filter on $\mathbb{B}$; by definition, there exists a family $\{ x_\alpha \mid \alpha < \kappa \} \subseteq F$ such that for every infinite $I \subseteq \kappa$ we have $\bigwedge_{\alpha \in I} x_\alpha = \mathbb{0}$.

By Proposition 1.2.11, there exists a maximal antichain A such that for every $\alpha < \kappa$, x_α is based on A. Clearly, for every $a \in A$ the set $\{ \alpha < \kappa \mid a \leq x_\alpha \}$ is finite, otherwise we would contradict the κ-quasiregularity of F. This shows that F is κ-regular. □

We conclude this section by investigating when *all* ultrafilters on a complete Boolean algebra are κ-quasiregular. The next result, in particular, will be applied in Section 2.3.

Theorem 2.1.11. *Let κ be a cardinal and $\mathbb{B}$ be a complete Boolean algebra. If there exists a complete homomorphism $f \colon \mathbb{F}_\kappa \to \mathbb{B}$, then every ultrafilter on $\mathbb{B}$ is κ-quasiregular.*

Proof. Let $\mathbb{F}_\kappa$ be free over $\{ x_\alpha \mid \alpha < \kappa \}$. Let U be any ultrafilter on $\mathbb{B}$; we need to show that U is κ-quasiregular.

For every $I \subseteq \kappa$, let us define

$$I_0 = \{ \alpha \in I \mid f(x_\alpha) \in U \} \quad \text{and} \quad I_1 = \{ \alpha \in I \mid f(x_\alpha) \notin U \}.$$

Now, we define a family $\{ y_\alpha \mid \alpha < \kappa \} \subseteq U$ as follows: for all $\alpha < \kappa$,

$$y_\alpha = \begin{cases} f(x_\alpha) & \text{if } \alpha \in \kappa_0, \\ f(\neg x_\alpha) & \text{if } \alpha \in \kappa_1. \end{cases}$$

We complete the proof by showing that for every infinite $I \subseteq \kappa$ we have $\bigwedge_{\alpha \in I} y_\alpha = \mathbb{0}$.

If $I \subseteq \kappa$ is infinite, then at least one of I_0 and I_1 must be infinite. Therefore, we use Corollary 1.2.17 and the fact that f is complete to deduce that

$$\bigwedge_{\alpha\in I} y_\alpha = \bigwedge_{\alpha\in I_0} y_\alpha \wedge \bigwedge_{\alpha\in I_1} y_\alpha = \bigwedge_{\alpha\in I_0} f(x_\alpha) \wedge \bigwedge_{\alpha\in I_1} f(\neg x_\alpha)$$
$$= f\left(\bigwedge_{\alpha\in I_0} x_\alpha \wedge \neg \bigvee_{\alpha\in I_1} x_\alpha\right) = f(\mathbb{0}) = \mathbb{0},$$

as desired. □

As a special case of Theorem 2.1.11, we obtain a known result due to Bernd and Sabine Koppelberg.

Corollary 2.1.12 (Koppelberg and Koppelberg [35])**.** *For every cardinal κ, every ultrafilter on $\mathbb{C}_\kappa$ is κ-quasiregular.*

Proof. By Proposition 1.2.13 and Theorem 2.1.11. □

2.2 OK ultrafilters

While the main focus of this chapter is on regular ultrafilters, we make a brief digression on *OK ultrafilters.* Our motivation here is to show that if an ultrafilter is $\aleph_1$-incomplete and κ-OK, then it is κ-regular in the sense of Definition 2.1.2. Moreover, we introduce some terminology on monotonic and multiplicative functions which will be helpful for the rest of the thesis.

Definition 2.2.1. Let κ be a cardinal, $\mathbb{B}$ a Boolean algebra, and $f\colon [\kappa]^{<\aleph_0} \to \mathbb{B}$.

- f is *monotonic* if for all $S, T \in [\kappa]^{<\aleph_0}$, $S \subseteq T$ implies $f(T) \leq f(S)$.
- f is *multiplicative* if for all $S, T \in [\kappa]^{<\aleph_0}$, $f(S \cup T) = f(S) \wedge f(T)$.

Note that every multiplicative function is monotonic. The next lemma highlights a property of multiplicative functions, which will be used to establish Proposition 2.2.4 and Theorem 3.5.5. As usual, $[\kappa]^n$ denotes the set of subsets of κ of cardinality n.

Lemma 2.2.2. *Let κ be a cardinal and $\mathbb{B}$ a complete Boolean algebra. For a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to \mathbb{B}$, the following two conditions are equivalent:*

1. $\bigwedge\Big\{ \bigvee g\big[[\kappa]^n\big] \;\Big|\; n < \omega \Big\} = \mathbb{0}.$
2. *There is a maximal antichain $A \subset \mathbb{B}$ such that:*
   - *for every $\alpha < \kappa$, $g(\{\alpha\})$ is based on A;*
   - *for every $a \in A$, the set $\{\, \alpha < \kappa \mid a \leq g(\{\alpha\}) \,\}$ is finite.*

Proof. ($1 \Longrightarrow 2$) The idea for the proof of this implication is already implicit in Mansfield [46, Theorem 4.1]. Let $g\colon [\kappa]^{<\aleph_0} \to \mathbb{B}$ be a multiplicative function satisfying (1). Without loss of generality, we may assume that $g(\emptyset) = \mathbb{1}$. Let D be the set of all $d \in \mathbb{B} \setminus \{\mathbb{0}\}$ such that: for every $\alpha < \kappa$, either $d \leq g(\{\alpha\})$ or $d \wedge g(\{\alpha\}) = \mathbb{0}$, and the set $\{\, \alpha < \kappa \mid d \leq g(\{\alpha\}) \,\}$ is finite. We shall show that D is dense, so that every maximal antichain $A \subseteq D$ will have the desired property.

Let $b \in \mathbb{B} \setminus \{\mathbb{0}\}$; we need to find some $d \in D$ such that $d \leq b$. For every $n < \omega$, let

$$c_n = \bigvee g\big[[\kappa]^n\big].$$

Note that $c_0 = g(\emptyset) = \mathbb{1}$ and, by hypothesis, $\bigwedge_{n<\omega} c_n = \mathbb{0}$. Furthermore, it is easy to verify that $c_{n+1} \leq c_n$ for all $n < \omega$. It follows that there exists some $i < \omega$ such that $\mathbb{0} < b \wedge c_i \wedge \neg c_{i+1}$. Therefore, by definition of c_i, there exists $S \in [\kappa]^i$ such that

$$d = b \wedge g(S) \wedge \neg c_{i+1} > \mathbb{0}.$$

Clearly $d \leq b$, so we shall conclude the proof by showing that $d \in D$. For every $\alpha < \kappa$, if $\alpha \in S$ then

$$d \leq g(S) \leq g(\{\alpha\});$$

otherwise, if $\alpha \notin S$, then by the multiplicativity of g

$$\begin{aligned} d \wedge g(\{\alpha\}) &= b \wedge g(S) \wedge g(\{\alpha\}) \wedge \neg c_{i+1} \\ &= b \wedge g(S \cup \{\alpha\}) \wedge \neg c_{i+1} \leq b \wedge c_{i+1} \wedge \neg c_{i+1} = \mathbb{0}. \end{aligned}$$

Therefore $d \in D$, as desired.

($2 \implies 1$) Suppose $g\colon [\kappa]^{<\aleph_0} \to \mathbb{B}$ satisfies (2); let A be the maximal antichain in $\mathbb{B}$ given by the hypothesis. We want to show that

$$\bigwedge \Big\{ \bigvee g\big[[\kappa]^n\big] \,\Big|\, n < \omega \Big\} = \mathbb{0}.$$

Suppose not; then there exists some $a \in A$ such that

$$a \wedge \bigwedge \Big\{ \bigvee g\big[[\kappa]^n\big] \,\Big|\, n < \omega \Big\} > \mathbb{0},$$

hence for every $n < \omega$ there exists some $S \in [\kappa]^n$ such that

$$\mathbb{0} < a \wedge g(S).$$

Using the fact that g is monotonic, we note that $g(S) \leq \bigwedge_{\alpha \in S} g(\{\alpha\})$ and consequently

$$\mathbb{0} < a \wedge \bigwedge_{\alpha \in S} g(\{\alpha\}).$$

Since each $g(\{\alpha\})$ is based on A, we conclude that for all $n < \omega$ there exists $S \in [\kappa]^n$ such that

$$a \leq \bigwedge_{\alpha \in S} g(\{\alpha\}),$$

but this contradicts our condition (2). □

OK ultrafilters were originally defined by Kunen [37] in the context of the topology of $\beta\omega$, the Stone-Čech compactification of the set of natural numbers. Five years later, Dow [12] rephrased Kunen's definition in terms of existence of multiplicative functions: this is the definition we employ here.

Definition 2.2.3. Let κ be an infinite cardinal. A filter F on a complete Boolean algebra $\mathbb{B}$ is *κ-OK* if for every monotonic function $f\colon [\kappa]^{<\aleph_0} \to F$ such that $|S| = |T|$ implies $f(S) = f(T)$, there exists a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to F$ with the property that $g(S) \leq f(S)$ for all $S \in [\kappa]^{<\aleph_0}$.

The model-theoretic relevance of OK ultrafilters lies in a property called *flexibility*, first isolated by Malliaris [41]. For more details about the connection between OK ultrafilters, flexibility, and Keisler's order we refer the reader to the work of Malliaris and Shelah [43].

Proposition 2.2.4. *Let κ be an infinite cardinal; suppose U is an $\aleph_1$-incomplete κ-OK ultrafilter on a complete Boolean algebra $\mathbb{B}$. Then U is κ-regular.*

Proof. Since U is an $\aleph_1$-incomplete ultrafilter, there exists a countable subset $\{ a_n \mid n < \omega \} \subseteq U$ such that $a_{n+1} \leq a_n$ for all $n < \omega$ and $\bigwedge_{n<\omega} a_n = \mathbb{0}$. Using this sequence, we can define a monotonic function as follows:

$$\begin{aligned} f\colon [\kappa]^{<\aleph_0} &\longrightarrow U \\ S &\longmapsto a_{|S|} \end{aligned}.$$

Since U is κ-OK, we can find a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to U$ such that $g(S) \leq f(S)$ for all $S \in [\kappa]^{<\aleph_0}$.

We wish to show that g satisfies condition (1) of Lemma 2.2.2. Let $n < \omega$; for all $S \in [\kappa]^n$ we have

$$g(S) \leq f(S) = a_n,$$

hence

$$\bigvee g\big[[\kappa]^n\big] \leq a_n.$$

It follows that

$$\bigwedge \Big\{ \bigvee g\big[[\kappa]^n\big] \;\Big|\; n < \omega \Big\} \leq \bigwedge_{n<\omega} a_n = \mathbb{0},$$

as desired.

By Lemma 2.2.2, there exists a maximal antichain $A \subset \mathbb{B}$ such that for every $\alpha < \kappa$, $g(\{\alpha\})$ is based on A, and for every $a \in A$, the set $\{ \alpha < \kappa \mid a \leq g(\{\alpha\}) \}$ is finite. This shows that U is κ-regular. □

Finally, we remark that Ulrich [70, Section 8] has independently proved Proposition 2.2.4.

2.3 Cofinal types of ultrafilters

In this section we apply our analysis of quasiregularity properties of ultrafilters, in particular Theorem 2.1.11, to answer a question posed by Brown and Dobrinen [6]. We begin by reviewing some terminology on directed sets.

Definition 2.3.1 (Moore and Smith [49])**.** A *directed set* $\langle D, \leq \rangle$ is a non-empty set D with a binary relation $\leq \subseteq D \times D$ such that:

- for every $d_0, d_1, d_2 \in D$, if $d_0 \leq d_1$ and $d_1 \leq d_2$, then $d_0 \leq d_2$;
- for every $d_0, d_1 \in D$, there exists $d \in D$ such that $d_0 \leq d$ and $d_1 \leq d$.

Remark 2.3.2. Over the years, the notion of directed set has evolved to include the condition that $\leq$ be a reflexive relation. However, since reflexivity plays no role in our discussion, we prefer to employ the original definition.

Definition 2.3.3 (Tukey [68])**.** Let $\left\langle D, \leq^D \right\rangle$ and $\left\langle E, \leq^E \right\rangle$ be directed sets. We say that $\left\langle D, \leq^D \right\rangle \leq_{\mathrm{T}} \left\langle E, \leq^E \right\rangle$ if and only if there exist functions $f\colon D \to E$ and $g\colon E \to D$ such that for all $d \in D$ and $e \in E$

$$f(d) \leq^E e \implies d \leq^D g(e).$$

Furthermore, we say that $\left\langle D, \leq^D \right\rangle$ and $\left\langle E, \leq^E \right\rangle$ have the same *cofinal type*, in symbols $\left\langle D, \leq^D \right\rangle \sim_{\mathrm{T}} \left\langle E, \leq^E \right\rangle$, if and only if both $\left\langle D, \leq^D \right\rangle \leq_{\mathrm{T}} \left\langle E, \leq^E \right\rangle$ and $\left\langle E, \leq^E \right\rangle \leq_{\mathrm{T}} \left\langle D, \leq^D \right\rangle$ hold.

Tukey realized that directed sets of the form $\left\langle [\kappa]^{<\aleph_0}, \subseteq \right\rangle$ provide a bound on the possible cofinal types of directed sets of cardinality $\leq \kappa$.

Theorem 2.3.4 (Tukey [68, Theorem II-5.1])**.** *Let κ be a cardinal and $\langle D, \leq \rangle$ a directed set. If $|D| \leq \kappa$, then $\langle D, \leq \rangle \leq_{\mathrm{T}} \left\langle [\kappa]^{<\aleph_0}, \subseteq \right\rangle$.*

If U is an ultrafilter on a Boolean algebra $\mathbb{B}$ then, by the second condition of Definition 1.2.1, $\langle U, \geq \rangle$ is a directed set (where, of course, $a \geq b$ simply means that $b \leq a$ in $\mathbb{B}$). What are the possible cofinal types of ultrafilters on $\mathbb{B}$? Clearly, if U is any such ultrafilter, then Theorem 2.3.4 gives

$$\langle U, \geq \rangle \leq_{\mathrm{T}} \left\langle [|\mathbb{B}|]^{<\aleph_0}, \subseteq \right\rangle.$$

Hence, we shall say that U has *maximum cofinal type* whenever

$$\langle U, \geq \rangle \sim_{\mathrm{T}} \left\langle [|\mathbb{B}|]^{<\aleph_0}, \subseteq \right\rangle.$$

The study of cofinal types of ultrafilters was originated by Isbell [24], who proved that power-set algebras always have an ultrafilter of maximum cofinal type.

Theorem 2.3.5 (Isbell [24, Theorem 5.4])**.** *For every infinite cardinal κ, there exists an ultrafilter U over κ such that $\langle U, \geq \rangle \sim_{\mathrm{T}} \left\langle [2^\kappa]^{<\aleph_0}, \subseteq \right\rangle$.*

The above theorem stimulated a fruitful line of research which is surveyed in Dobrinen [10]. More recently, Brown and Dobrinen [6] posed the problem of characterizing the class of Boolean algebras in which *all* ultrafilters have maximum cofinal type.

Proposition 2.3.6 (Brown and Dobrinen [6, Fact 2.4])**.** *If κ is an infinite cardinal, then every ultrafilter on $\mathbb{F}_\kappa$ has maximum cofinal type.*

Hence, the following question arises naturally:

Question 2.3.7 (Brown and Dobrinen [6, Question 4.2])**.** If $\mathbb{B}$ is an infinite Boolean algebra such that all ultrafilters on $\mathbb{B}$ have maximum cofinal type, is $\mathbb{B}$ necessarily a free algebra?

In the rest of this section, we shall give a negative answer to Question 2.3.7 by means of two examples, both using quasiregular ultrafilters.

The following simple observation highlights a crucial connection between quasiregularity of ultrafilters and cofinal types.

Proposition 2.3.8. *Let κ be an infinite cardinal. If U is a κ-quasiregular ultrafilter on a complete Boolean algebra $\mathbb{B}$, then $\left\langle [\kappa]^{<\aleph_0}, \subseteq \right\rangle \leq_{\mathrm{T}} \langle U, \geq \rangle$.*

Proof. By hypothesis, there exists a subset $X \subseteq U$ with $|X| = \kappa$ such that whenever $Y \subseteq X$ is infinite, we have $\bigwedge Y = \mathbb{0}$. Since κ is an infinite cardinal, there exists an injective function $f \colon [\kappa]^{<\aleph_0} \to X$.

For every $b \in U$, the set

$$\left\{ S \in [\kappa]^{<\aleph_0} \;\middle|\; f(S) \geq b \right\}$$

must be finite: otherwise, we would have an infinite $Y \subseteq X$ such that $\mathbb{0} < b \leq \bigwedge Y$, a contradiction. This allows us to define a function $g \colon U \to [\kappa]^{<\aleph_0}$ as follows: for every $b \in U$

$$g(b) = \bigcup \left\{ S \in [\kappa]^{<\aleph_0} \;\middle|\; f(S) \geq b \right\}.$$

We conclude that for all $S \in [\kappa]^{<\aleph_0}$ and $b \in U$

$$f(S) \geq b \implies S \subseteq g(b),$$

completing the proof. □

Remark 2.3.9. We note that the converse implication in Proposition 2.3.8 does not hold. For example, by Theorem 2.3.5 we can find an ultrafilter U over ω such that $\langle U, \geq \rangle \sim_{\mathrm{T}} \left\langle \left[2^{\aleph_0}\right]^{<\aleph_0}, \subseteq \right\rangle$, however U cannot be $2^{\aleph_0}$-quasiregular by Lemma 2.1.3 and Proposition 2.1.10.

We are ready to present our first example.

Theorem 2.3.10. *Let κ be a cardinal such that $\kappa^{\aleph_0} = \kappa$. Then every ultrafilter on $\mathbb{C}_\kappa$ has maximum cofinal type, but $\mathbb{C}_\kappa$ is not a free algebra.*

Proof. Firstly, from (1.2) we deduce that $|\mathbb{C}_\kappa| = \kappa^{\aleph_0} = \kappa$. Let U be an ultrafilter on $\mathbb{C}_\kappa$; by Corollary 2.1.12, U is κ-quasiregular, hence we may apply Proposition 2.3.8 to conclude that

$$\langle U, \geq \rangle \sim_{\mathrm{T}} \left\langle [\kappa]^{<\aleph_0}, \subseteq \right\rangle,$$

thus showing that U has maximum cofinal type. Finally, Theorem 1.2.20 implies that no infinite free Boolean algebra can be complete, hence $\mathbb{C}_\kappa$ is not a free algebra. □

Our second example relies on the following construction due to Galvin and Hajnal, the details of which can be found in Comfort and Negrepontis [8, Theorem 6.32].

Theorem 2.3.11 (Galvin and Hajnal [17]). *There exists a complete Boolean algebra* $\mathbb{B}_{\mathrm{GH}}$ *of cardinality* $2^{\aleph_0}$ *which is* $\aleph_1$*-c.c. but not* σ*-bounded c.c.*

Theorem 2.3.12 (Dobrinen [9, Theorem 2.5]). *There exists a complete embedding* $f\colon \mathbb{C}_{\mathrm{cf}(2^{\aleph_0})} \to \mathbb{B}_{\mathrm{GH}}$.

Our second example then follows from Theorem 2.3.12 under the assumption that $2^{\aleph_0}$ is a regular cardinal.

Theorem 2.3.13. *Assume that* $\mathrm{cf}(2^{\aleph_0}) = 2^{\aleph_0}$*; then every ultrafilter on* $\mathbb{B}_{\mathrm{GH}}$ *has maximum cofinal type, but* $\mathbb{B}_{\mathrm{GH}}$ *is not a free algebra.*

Proof. Since $\mathrm{cf}(2^{\aleph_0}) = 2^{\aleph_0}$, Theorem 2.3.12 gives us a complete embedding $f\colon \mathbb{C}_{2^{\aleph_0}} \to \mathbb{B}_{\mathrm{GH}}$. Furthermore, by Proposition 1.2.13, the inclusion map $i\colon \mathbb{F}_{2^{\aleph_0}} \to \mathbb{C}_{2^{\aleph_0}}$ is also a complete embedding. Composing gives us a complete embedding $(f \circ i)\colon \mathbb{F}_{2^{\aleph_0}} \to \mathbb{B}_{\mathrm{GH}}$.

Now let U be an ultrafilter on $\mathbb{B}_{\mathrm{GH}}$: Theorem 2.1.11 implies that U is $2^{\aleph_0}$-quasiregular, hence

$$\langle U, \geq \rangle \sim_{\mathrm{T}} \left\langle \left[2^{\aleph_0}\right]^{<\aleph_0}, \subseteq \right\rangle$$

follows from Proposition 2.3.8. This shows that U has maximum cofinal type. Finally, $\mathbb{B}_{\mathrm{GH}}$ is not a free algebra, again because of Theorem 1.2.20. □

Question 2.3.14. Is the conclusion of Theorem 2.3.13 still true without the assumption that $\mathrm{cf}(2^{\aleph_0}) = 2^{\aleph_0}$?

2.4 Model-theoretic properties

This section is dedicated to the analysis of model-theoretic properties of regular and quasiregular ultrafilters on complete Boolean algebras. The natural tool for this analysis is the Boolean ultrapower construction; in particular, we shall focus on the cardinality, cofinality, and universality of Boolean ultrapowers. In each case, one notion of regularity behaves as expected, while the other notion is not well behaved.

Cardinality

The problem of determining the possible cardinalities of the ultrapowers of a given structure starts with an simple observation: if U is an ultrafilter over I, then for every structure $\mathfrak{M}$

$$|M| \leq \left|M^I/U\right| \leq |M|^{|I|}. \tag{2.1}$$

Of course, if U is principal then $|M| = \left|M^I/U\right|$, hence the lower bound in (2.1) can be attained. Therefore, it is natural to ask whether or not the upper bound in (2.1) can be attained for some ultrafilter U over I. This question led Frayne, Morel and Scott to consider regular ultrafilters in [15].

Theorem 2.4.1 (Frayne, Morel and Scott [15, Theorem 1.26])**.** *Let κ be an infinite cardinal; suppose U is a κ-regular ultrafilter over a set I. For every infinite structure $\mathfrak{M}$, we have*

$$|M|^\kappa \leq \left|M^I/U\right|.$$

In particular, if $|I| = \kappa$ then the upper bound $|M|^{|I|}$ is always attained.

Motivated by this result, we can ask whether the same is true for Boolean ultrapowers. As we shall see, the parallel of Theorem 2.4.1 is true for regular ultrafilters on complete Boolean algebras, but can fail for quasiregular ultrafilters. First, we need to establish a bound analogous to (2.1).

Lemma 2.4.2. *Let U be an ultrafilter on a complete Boolean algebra $\mathbb{B}$. For every structure $\mathfrak{M}$, we have*

$$|M| \leq \left|M^{[\mathbb{B}]}/U\right| \leq |M|^{<\mathrm{sat}(\mathbb{B})} + |\mathbb{B}|^{<\mathrm{sat}(\mathbb{B})}. \tag{2.2}$$

Proof. The inequality $|M| \leq \left|M^{[\mathbb{B}]}/U\right|$ follows immediately from Corollary 1.3.7. On the other hand,

$$\begin{aligned}\left|M^{[\mathbb{B}]}/U\right| \leq \left|M^{[\mathbb{B}]}\right| &= |\{\, \tau\colon A \to M \mid A \subset \mathbb{B} \text{ is a maximal antichain} \,\}| \\ &\leq \left|\bigcup\left\{\, {}^X M \;\middle|\; X \in [\mathbb{B}]^{<\mathrm{sat}(\mathbb{B})} \,\right\}\right| = |M|^{<\mathrm{sat}(\mathbb{B})} + |\mathbb{B}|^{<\mathrm{sat}(\mathbb{B})},\end{aligned}$$

as desired. □

We now show that regular ultrafilters produce Boolean ultrapowers of large cardinality; the proof of this result is just a minor modification of the proof of Theorem 2.4.1.

Proposition 2.4.3. *Let κ be an infinite cardinal; suppose U is a κ-regular ultrafilter on a complete Boolean algebra $\mathbb{B}$. For every infinite structure $\mathfrak{M}$, we have*

$$|M|^\kappa \leq \left|M^{[\mathbb{B}]}/U\right|. \tag{2.3}$$

In particular, if $\mathbb{B}$ is a κ^+-c.c. Boolean algebra of cardinality $\leq 2^\kappa$, then the upper bound in (2.2) *is always attained.*

Proof. Since $|{}^{<\omega}M| = |M|$, it is sufficient to find an injective function $i\colon {}^\kappa M \to ({}^{<\omega}M)^{[\mathbb{B}]}/U$. Let the family $\{\, x_\alpha \mid \alpha < \kappa \,\} \subseteq U$ and the maximal antichain $A \subset \mathbb{B}$ witness the κ-regularity of U. Hence, for every $a \in A$ the set

$$S(a) = \{\, \alpha < \kappa \mid a \leq x_\alpha \,\}$$

is finite.

Given a function $f\colon \kappa \to M$, we define $\tau_f\colon A \to {}^{<\omega}M$ as follows. Fix $a \in A$; list all the elements of $S(a)$ increasingly as $\alpha_1 < \cdots < \alpha_n$ and define

$$\tau_f(a) = \langle f(\alpha_1), \ldots, f(\alpha_n)\rangle.$$

We now prove that the function

$$\begin{aligned} i\colon {}^\kappa M &\longrightarrow \left({}^{<\omega}M\right)^{[\mathbb{B}]}/U \\ f &\longmapsto [\tau_f]_U \end{aligned}$$

is injective. Let $f, g\colon \kappa \to M$; if $f \neq g$ then there exists some $\alpha < \kappa$ such that $f(\alpha) \neq g(\alpha)$. For all $a \in A$, if $a \leq x_\alpha$ then $\alpha \in S(a)$ and therefore, by construction, $\tau_f(a) \neq \tau_g(a)$. It follows that

$$[\![\tau_f \neq \tau_g]\!] = \bigvee \{\, a \in A \mid \tau_f(a) \neq \tau_g(a) \,\} \geq \bigvee \{\, a \in A \mid a \leq x_\alpha \,\} = x_\alpha \in U,$$

hence $[\![\tau_f \neq \tau_g]\!] \in U$, as required. This shows that $i\colon {}^{\kappa}M \to ({}^{<\omega}M)^{[\mathbb{B}]}/U$ is injective, establishing (2.3).

Now, if we assume further that $\mathbb{B}$ is a κ^+-c.c. Boolean algebra of cardinality $\leq 2^\kappa$, then for every infinite structure $\mathfrak{M}$

$$|M|^\kappa \leq \left|M^{[\mathbb{B}]}/U\right| \leq |M|^{<\operatorname{sat}(\mathbb{B})} + |\mathbb{B}|^{<\operatorname{sat}(\mathbb{B})} \leq |M|^\kappa + (2^\kappa)^\kappa = |M|^\kappa,$$

hence we have equality throughout. □

Conversely, if we assume (2.3), then we can deduce some information on the ultrafilter U.

Proposition 2.4.4. *Let κ be an infinite cardinal; suppose U is an ultrafilter on a complete Boolean algebra $\mathbb{B}$. If for every infinite structure $\mathfrak{M}$ we have*

$$|M|^\kappa \leq \left|M^{[\mathbb{B}]}/U\right|,$$

then for every $b \in U$, $\operatorname{sat}(\mathbb{B} \restriction b) > \kappa$.

Proof. To reach a contradiction, suppose that $\operatorname{sat}(\mathbb{B} \restriction b) \leq \kappa$ for some $b \in U$. By Theorem 1.2.4, $\operatorname{sat}(\mathbb{B} \restriction b)$ is a regular cardinal; hence it is possible to find a strong limit cardinal $\lambda \geq |\mathbb{B}|$ such that $\operatorname{cf}(\lambda) = \operatorname{sat}(\mathbb{B} \restriction b)$. Now, let $\mathfrak{M}$ be a structure with $|M| = \lambda$.

For every name $\tau \in \mathfrak{M}^{[\mathbb{B}]}$, we define a name $\tau_b \in \mathfrak{M}^{[\mathbb{B}\restriction b]}$ as follows:

$$\operatorname{dom}(\tau_b) = \{\, a \wedge b \mid a \in \operatorname{dom}(\tau) \,\} \setminus \{\mathbb{0}\}$$

and

$$\tau_b(a \wedge b) = \tau(a).$$

Note that, by definition, for all $\tau, \sigma \in \mathfrak{M}^{[\mathbb{B}]}$

$$[\![\tau_b = \sigma_b]\!]^{\mathfrak{M}^{[\mathbb{B}\restriction b]}} = b \wedge [\![\tau = \sigma]\!]^{\mathfrak{M}^{[\mathbb{B}]}}. \tag{2.4}$$

Since $b \in U$, clearly $U_b = U \cap (\mathbb{B} \restriction b)$ is an ultrafilter on $\mathbb{B} \restriction b$. Furthermore, it follows from (2.4) that the function

$$\begin{aligned} M^{[\mathbb{B}]}/U &\longrightarrow M^{[\mathbb{B}\restriction b]}/U_b \\ [\tau]_U &\longmapsto [\tau_b]_{U_b} \end{aligned}$$

is well defined and injective. Putting everything together and applying Lemma 2.4.2, we conclude that

$$\lambda < \lambda^\kappa \leq \left| M^{[\mathbb{B}]}/U \right| \leq \left| M^{[\mathbb{B}\restriction b]}/U_b \right| \leq \lambda^{<\mathrm{cf}(\lambda)} + |B|^{<\mathrm{cf}(\lambda)} = \lambda^{<\mathrm{cf}(\lambda)} = \lambda,$$

a contradiction. □

Using Proposition 2.4.4, we can find a counterexample for quasiregular ultrafilters.

Proposition 2.4.5. *Let κ be an uncountable cardinal. Then there exist a complete Boolean algebra $\mathbb{B}$ and a κ-quasiregular ultrafilter U on $\mathbb{B}$ such that, for some infinite structure $\mathfrak{M}$,*

$$\left| M^{[\mathbb{B}]}/U \right| < |M|^\kappa.$$

Proof. Let U be any ultrafilter on the complete Boolean algebra $\mathbb{C}_\kappa$; we know that U is κ-quasiregular by Corollary 2.1.12. Applying Theorem 1.2.19, we have

$$\mathrm{sat}(\mathbb{C}_\kappa) = \aleph_1 \leq \kappa,$$

so the conclusion follows immediately from Proposition 2.4.4. □

Cofinality

An important feature of regular ultrafilters is that they produce ultrapowers of large cofinality. We shall now investigate whether the same is true in the context of complete Boolean algebras and Boolean ultrapowers. Again, our results show that regular ultrafilters behave as expected, while quasiregular ultrafilters are not well behaved.

Proposition 2.4.6. *Let κ be an infinite cardinal; suppose U is a κ-regular ultrafilter over a set I. For every infinite cardinal λ, the ultrapower $\langle \lambda, < \rangle^I/U$ has cofinality $> \kappa$.*

The above result can be found in Benda and Ketonen [4, Theorem 1.3], where it is referred to as a 'standard fact'. It appears also in Koppelberg [34, Lemma 2].

By adapting the usual proof of Proposition 2.4.6, and using the mixing property of Lemma 1.3.4, we can establish the corresponding result for Boolean ultrapowers.

Proposition 2.4.7. *Let κ be an infinite cardinal; suppose U is a κ-regular ultrafilter on a complete Boolean algebra $\mathbb{B}$. For every infinite cardinal λ, the Boolean ultrapower $\langle\lambda,<\rangle^{[\mathbb{B}]}/U$ has cofinality $>\kappa$.*

Proof. Let the family $\{\, x_\alpha \mid \alpha<\kappa \,\} \subseteq U$ and the maximal antichain $A \subset \mathbb{B}$ witness the κ-regularity of U. In particular, this means that for every $a \in A$ the set

$$S(a) = \{\, \alpha < \kappa \mid a \leq x_\alpha \,\}$$

is finite.

Given any $\{\, \tau_\alpha \mid \alpha < \kappa \,\} \subset \lambda^{[\mathbb{B}]}$, we show that the sequence $\{\, [\tau_\alpha]_U \mid \alpha < \kappa \,\}$ is not cofinal in $\langle\lambda,<\rangle^{[\mathbb{B}]}/U$ by finding some $\sigma \in \lambda^{[\mathbb{B}]}$ such that $[\![\tau_\alpha \leq \sigma]\!] \in U$ for all $\alpha < \kappa$.

For every $a \in A$ we wish to define a name $\sigma_a \in \lambda^{[\mathbb{B}]}$ such that

$$\bigwedge_{\alpha\in S(a)} [\![\tau_\alpha \leq \sigma_a]\!] = \mathbb{1}. \tag{2.5}$$

To do so, let W_a be a common refinement of the finitely many maximal antichains $\{\, \mathrm{dom}(\tau_\alpha) \mid \alpha \in S(a) \,\}$. We define $\sigma_a \colon W_a \to \lambda$ as follows: for all $w \in W_a$

$$\sigma_a(w) = \max\{\, (\tau_\alpha \downarrow W_a)(w) \mid \alpha \in S(a) \,\}.$$

Clearly σ_a bounds each τ_α, for $\alpha \in S(a)$, with Boolean value $\mathbb{1}$, and so (2.5) is proved.

Now, use Lemma 1.3.4 to obtain a name $\sigma \in \lambda^{[\mathbb{B}]}$ such that $a \leq [\![\sigma = \sigma_a]\!]$ for each $a \in A$. Since $x_\alpha \in U$ for every $\alpha < \kappa$, to complete the proof it is sufficient to show that

$$x_\alpha \leq [\![\tau_\alpha \leq \sigma]\!]. \tag{2.6}$$

For all $a \in A$, if $a \leq x_\alpha$ then $\alpha \in S(a)$, hence $[\![\tau_\alpha \leq \sigma_a]\!] = \mathbb{1}$ and

$$a \leq [\![\sigma = \sigma_a]\!] = [\![\sigma = \sigma_a]\!] \wedge \mathbb{1} = [\![\sigma = \sigma_a]\!] \wedge [\![\tau_\alpha \leq \sigma_a]\!] \leq [\![\tau_\alpha \leq \sigma]\!].$$

Thus we have shown that, for all $a \in A$, if $a \leq x_\alpha$ then $a \leq [\![\tau_\alpha \leq \sigma]\!]$. Now (2.6) follows: for every $\alpha < \kappa$

$$[\![\tau_\alpha \leq \sigma]\!] \geq \bigvee \{\, a \in A \mid a \leq [\![\tau_\alpha \leq \sigma]\!] \,\} \geq \bigvee \{\, a \in A \mid a \leq x_\alpha \,\} = x_\alpha \in U,$$

thus showing that $[\![\tau_\alpha \leq \sigma]\!] \in U$. □

Since the cofinality of an ordered set is not greater than its cardinality, from the estimate of Lemma 2.4.2 we already obtain a counterexample for quasiregular ultrafilters. To see this, let κ be a cardinal such that $\kappa^{\aleph_0} = \kappa$. If U is any ultrafilter on $\mathbb{C}_\kappa$, then U is κ-quasiregular, however by Lemma 2.4.2

$$\operatorname{cf}\Big(\langle \kappa, <\rangle^{[\mathbb{C}_\kappa]}/U\Big) \leq \Big|\kappa^{[\mathbb{C}_\kappa]}/U\Big| \leq \kappa^{<\aleph_1} + \left(\kappa^{\aleph_0}\right)^{<\aleph_1} = \kappa^{\aleph_0} = \kappa.$$

Actually, we can prove a slightly more general result.

Proposition 2.4.8. *Let κ be a regular uncountable cardinal and $\mathbb{B}$ a κ-c.c. complete Boolean algebra. For every ultrafilter U on $\mathbb{B}$, the Boolean ultrapower $\langle \kappa, <\rangle^{[\mathbb{B}]}/U$ has cofinality κ.*

Proof. We observe first that for every $\tau \in \kappa^{[\mathbb{B}]}$ there exists some $\alpha < \kappa$ such that $[\![\tau \leq \check{\alpha}]\!] = \mathbb{1}$. Indeed, given a name τ, the κ-c.c. implies that $|\operatorname{dom}(\tau)| < \kappa$. Since κ is a regular cardinal, there exists an $\alpha < \kappa$ such that $\tau(a) \leq \alpha$ for all $a \in \operatorname{dom}(\tau)$, as required.

Consequently, the natural embedding

$$\begin{aligned} d\colon \kappa &\longrightarrow \kappa^{[\mathbb{B}]}/U \\ \alpha &\longmapsto [\check{\alpha}]_U \end{aligned}$$

is strictly increasing and cofinal in $\langle \kappa, <\rangle^{[\mathbb{B}]}/U$. Hence, the cofinality of $\langle \kappa, <\rangle^{[\mathbb{B}]}/U$ is κ. □

We conclude by mentioning a related result for Boolean ultrapowers of $\langle \omega, <\rangle$.

Proposition 2.4.9 (Koppelberg and Koppelberg [35, Lemma 3])**.** *Let κ be a regular cardinal with $\kappa^{\aleph_0} = \kappa$. Then there exists an ultrafilter U on $\mathbb{C}_\kappa$ such that*

$$\mathrm{cf}\Big(\langle \omega, < \rangle^{[\mathbb{C}_\kappa]}/U\Big) = \Big|\omega^{[\mathbb{C}_\kappa]}/U\Big| = \kappa.$$

Starting from Proposition 2.4.9, the topic of the possible cardinality and cofinality of a Boolean ultrapower of $\langle \omega, < \rangle$ was further explored by Koppelberg [34] and Jin and Shelah [25].

Universality

The third model-theoretic property we consider in this section is *universality.* The following characterization of regularity is implicit in Frayne, Morel and Scott [15] and appears explicitly in Keisler [33, Theorem 1.5a].

Theorem 2.4.10. *Let κ be an infinite cardinal; for an ultrafilter U over a set I, the following conditions are equivalent:*

1. *U is κ-regular;*
2. *for every L-structure $\mathfrak{M}$, with $|L| \leq \kappa$, the ultrapower $\mathfrak{M}^I/U$ is κ^+-universal.*

Again, we can adapt the proof of Theorem 2.4.10 to establish a similar characterization for ultrafilters on complete Boolean algebras in terms of universality and the Rudin-Keisler ordering.

Theorem 2.4.11. *Let κ be an infinite cardinal. For an ultrafilter U on a complete Boolean algebra $\mathbb{B}$, the following conditions are equivalent:*

1. *U is κ-regular;*
2. *there exists a κ-regular ultrafilter D over κ such that $D \leq_{\mathrm{RK}} U$;*
3. *for every L-structure $\mathfrak{M}$, with $|L| \leq \kappa$, the Boolean ultrapower $\mathfrak{M}^{[\mathbb{B}]}/U$ is κ^+-universal.*

Proof. ($1 \Longrightarrow 2$) Let the family $\{ x_\alpha \mid \alpha < \kappa \} \subseteq U$ and the maximal antichain $A \subset \mathbb{B}$ witness the κ-regularity of U. By Lemma 2.1.8, we can assume A has cardinality κ, so let $A = \{ a_i \mid i < \kappa \}$.

Let us define

$$\begin{aligned} h\colon \mathcal{P}(\kappa) &\longrightarrow \mathbb{B} \\ X &\longmapsto \bigvee\{\, a_i \mid i \in X \,\} \end{aligned};$$

by a routine verification, h is a homomorphism. We show that h is complete: for any family $\{\, X_t \mid t \in T \,\} \subseteq \mathcal{P}(\kappa)$ we have

$$\begin{aligned} h\left(\bigcup_{t\in T} X_t\right) = \bigvee\left\{\, a_i \;\middle|\; i \in \bigcup_{t\in T} X_t \,\right\} &= \bigvee \bigcup_{t\in T} \{\, a_i \mid i \in X_t \,\} \\ &= \bigvee_{t\in T} \bigvee \{\, a_i \mid i \in X_t \,\} = \bigvee_{t\in T} h(X_t), \end{aligned}$$

as desired.

Let $D = h^{-1}[U]$; clearly, D is an ultrafilter over κ. By Proposition 1.4.4 we have $D \leq_{\mathrm{RK}} U$. To conclude the proof, it is sufficient to show that D is κ-regular. For every $\alpha < \kappa$, the set

$$X_\alpha = \{\, i < \kappa \mid a_i \leq x_\alpha \,\}$$

is such that $h(X_\alpha) = x_\alpha \in U$ and, whenever $I \subseteq \kappa$ is infinite,

$$\bigcap_{\alpha\in I} X_\alpha = \left\{\, i < \kappa \;\middle|\; a_i \leq \bigwedge_{\alpha\in I} x_\alpha \,\right\} = \{\, i < \kappa \mid a_i = \mathbb{0} \,\} = \emptyset.$$

Hence, the family $\{\, X_\alpha \mid \alpha < \kappa \,\} \subseteq D$ shows that D is κ-regular.

($2 \Longrightarrow 3$) Let D be a κ-regular ultrafilter over κ such that $D \leq_{\mathrm{RK}} U$. Let $\mathfrak{M}$ be an L-structure with $|L| \leq \kappa$. By Theorem 2.4.10, the ultrapower $\mathfrak{M}^\kappa/D$ is κ^+-universal. By Theorem 1.4.5, there exists an elementary embedding $j\colon \mathfrak{M}^\kappa/D \to \mathfrak{M}^{[\mathbb{B}]}/U$, hence $\mathfrak{M}^{[\mathbb{B}]}/U$ is also κ^+-universal.

($3 \Longrightarrow 1$) Let $\mathfrak{M} = \left\langle [\kappa]^{<\aleph_0}, \subseteq, \langle\, \{\alpha\} \mid \alpha < \kappa \,\rangle \right\rangle$ be the structure in the language L with a binary relation symbol for the inclusion and constant symbols for the singletons $\{\alpha\} \in [\kappa]^{<\aleph_0}$, for $\alpha < \kappa$.

We now define a set of L-formulae

$$\Sigma(x) = \{\, \{\alpha\} \subseteq x \mid \alpha < \kappa \,\},$$

and we show that $\Sigma(x)$ is realized in $\mathfrak{M}^{[\mathbb{B}]}/U$.

Since every finite subset of $\Sigma(x)$ is realized in $\mathfrak{M}$, by compactness there exists a model $\mathfrak{N}$ of the theory of $\mathfrak{M}$ in which $\Sigma(x)$ is realized. Since $|L| = \kappa$, by Löwenheim-Skolem we may assume that $|N| = \kappa$. We have $\mathfrak{N} \equiv \mathfrak{M} \equiv \mathfrak{M}^{[\mathbb{B}]}/U$, and $\mathfrak{M}^{[\mathbb{B}]}/U$ is κ^+-universal by hypothesis, therefore there exists an elementary embedding $j\colon \mathfrak{N} \to \mathfrak{M}^{[\mathbb{B}]}/U$. So, if $n \in N$ realizes $\Sigma(x)$ in $\mathfrak{N}$, then by elementarity $j(n)$ realizes $\Sigma(x)$ in $\mathfrak{M}^{[\mathbb{B}]}/U$. This completes the proof that $\Sigma(x)$ is realized in $\mathfrak{M}^{[\mathbb{B}]}/U$.

Now, let $\tau\colon A \to [\kappa]^{<\aleph_0}$ be such that $[\tau]_U$ realizes $\Sigma(x)$ in $\mathfrak{M}^{[\mathbb{B}]}/U$. For each $\alpha < \kappa$ define

$$x_\alpha = \bigvee \{\, a \in A \mid \mathfrak{M} \models \{\alpha\} \subseteq \tau(a) \,\},$$

and note that $x_\alpha \in U$ by Theorem 1.3.6. To show that U is κ-regular, we just observe that for each $\alpha < \kappa$ and every $a \in A$ we have the two implications

$$\mathbb{0} < a \wedge x_\alpha \implies \alpha \in \tau(a) \implies a \leq x_\alpha.$$

From this, we conclude that the family $\{\, x_\alpha \mid \alpha < \kappa \,\}$ and the maximal antichain A satisfy the two conditions of Definition 2.1.2. □

Koppelberg and Koppelberg [35] showed the existence of a κ-quasiregular ultrafilter U on $\mathbb{C}_\kappa$ such that, for some L-structure $\mathfrak{M}$, with $|L| = \aleph_1$, the Boolean ultrapower $\mathfrak{M}^{[\mathbb{C}_\kappa]}/U$ is not $\aleph_2$-universal.

With Theorem 2.4.11 available to us, we can give an immediate proof of this fact. Let κ be any uncountable cardinal and let U be an ultrafilter on $\mathbb{C}_\kappa$. We already know (Corollary 2.1.12) that U is κ-quasiregular, however U cannot be $\aleph_1$-regular, due to the $\aleph_1$-c.c. Therefore, by Theorem 2.4.11 there exists some L-structure $\mathfrak{M}$, with $|L| \leq \aleph_1$, such that the Boolean ultrapower $\mathfrak{M}^{[\mathbb{C}_\kappa]}/U$ is not $\aleph_2$-universal.

2.5 Non-regularity and decomposability

In this section we focus on ultrafilters which are *not* regular. We discuss the related notion of decomposability and obtain in Corollary 2.5.9 that, consistently, all κ-decomposable ultrafilters are κ-regular.

Definition 2.5.1. An ultrafilter U over a set I is *uniform* if for all $X \in U$, $|X| = |I|$.

The notion of decomposability was originally defined by Keisler for ultrafilters over sets, and later extended to subsets of Boolean algebras in general by Balcar and Simon. By Theorem 2.5.4 below, decomposability can be thought as a generalization of uniformity.

Definition 2.5.2 (Balcar and Simon [2])**.** Let κ be a cardinal. An ultrafilter U on a Boolean algebra $\mathbb{B}$ is *κ-decomposable* if there exists an antichain $A \subset \mathbb{B}$ such that for every $b \in U$

$$|\{ a \in A \mid \mathbb{0} < a \wedge b \}| = \kappa.$$

Proposition 2.5.3. *Let κ be an infinite cardinal; for any complete Boolean algebra $\mathbb{B}$, every κ-regular ultrafilter on $\mathbb{B}$ is κ-decomposable.*

Proof. By Lemma 2.1.3 and Lemma 2.1.8. □

The next result provides a characterization of decomposability parallel to Theorem 2.4.11.

Theorem 2.5.4. *Let κ be an infinite cardinal. For an ultrafilter U on a complete Boolean algebra $\mathbb{B}$, the following conditions are equivalent:*

1. *U is κ-decomposable;*
2. *there exists a uniform ultrafilter D over κ such that $D \leq_{\mathrm{RK}} U$;*

Proof. ($1 \implies 2$) This implication is essentially due to Prikry [54]. Suppose U is κ-decomposable; let $\{ a_i \mid i < \kappa \} \subset \mathbb{B}$ be an antichain of cardinality κ such that for every $b \in U$

$$|\{ i < \kappa \mid \mathbb{0} < a_i \wedge b \}| = \kappa.$$

Without loss of generality we can assume that $\{ a_i \mid i < \kappa \}$ is maximal: indeed, if $\bigvee_{i<\kappa} a_i < \mathbb{1}$ then we can simply add $\neg \bigvee_{i<\kappa} a_i$ to obtain a maximal antichain of cardinality κ with the same property.

Exactly as in the proof of Theorem 2.4.11, we have a complete homomorphism

$$\begin{aligned} h\colon \mathcal{P}(\kappa) &\longrightarrow \mathbb{B} \\ X &\longmapsto \bigvee\{\, a_i \mid i \in X \,\} \end{aligned}$$

such that $D = h^{-1}[U]$ is an ultrafilter over κ with $D \leq_{\mathrm{RK}} U$. To conclude the proof, it is sufficient to show that D is uniform. If $X \in D$, then

$$\bigvee\{\, a_i \mid i \in X \,\} = h(X) \in U$$

hence, by κ-decomposability,

$$\kappa = \left|\left\{\, j < \kappa \;\middle|\; \mathbb{0} < a_j \wedge \bigvee\{\, a_i \mid i \in X \,\} \,\right\}\right| = |\{\, j < \kappa \mid j \in X \,\}| = |X|,$$

thus showing that D is uniform.

($2 \Longrightarrow 1$) Conversely, suppose D is a uniform ultrafilter over κ such that $D \leq_{\mathrm{RK}} U$. By Definition 1.4.3, there exist a maximal antichain $W \subset \mathbb{B}$ and a function $f\colon W \to \kappa$ such that, for every $X \subseteq \kappa$,

$$X \in D \iff \bigvee f^{-1}[X] \in U. \tag{2.7}$$

To show that U is κ-decomposable, it is sufficient to prove that for every $b \in U$

$$\left|\left\{\, \alpha < \kappa \;\middle|\; \mathbb{0} < b \wedge \bigvee f^{-1}[\{\alpha\}] \,\right\}\right| = \kappa.$$

To do so, let $b \in U$ be fixed and define

$$X = \left\{\, \alpha < \kappa \;\middle|\; \mathbb{0} < b \wedge \bigvee f^{-1}[\{\alpha\}] \,\right\}.$$

Firstly, it is clear that for each $w \in W$ we have $w \leq \bigvee f^{-1}[\{f(w)\}]$. Consequently,

$$\begin{aligned} \bigvee f^{-1}[X] = \bigvee\left\{\, w \in W \;\middle|\; \mathbb{0} < b \wedge \bigvee f^{-1}[\{f(w)\}] \,\right\} \\ \geq \bigvee\{\, w \in W \mid \mathbb{0} < b \wedge w \,\} \geq b \in U. \end{aligned}$$

It follows that $\bigvee f^{-1}[X] \in U$ and, by (2.7), that $X \in D$. Since D is a uniform ultrafilter over κ, we have $|X| = \kappa$, as desired. □

If λ is a measurable cardinal, there are obviously uniform ultrafilters over λ which are not even $\aleph_0$-regular. More precisely, we have the following result.

Proposition 2.5.5. *Let U be a λ-complete uniform ultrafilter over a measurable cardinal λ. Then U is λ-decomposable, but U is not κ-decomposable for any $1 < \kappa < \lambda$.*

Proof. Clearly, U is λ-decomposable by Theorem 2.5.4. Arguing by contradiction, suppose U is κ-decomposable for some $1 < \kappa < \lambda$. Hence, there exists an antichain $\{ A_i \mid i < \kappa \} \subset \mathcal{P}(\lambda)$ of cardinality κ such that for every $X \in U$

$$|\{ i < \kappa \mid A_i \cap X \neq \emptyset \}| = \kappa.$$

For each $i < \kappa$ we have

$$|\{ j < \kappa \mid A_j \cap A_i \neq \emptyset \}| = |\{A_i\}| = 1 < \kappa$$

and therefore $A_i \notin U$. Since $\kappa < \lambda$ and U is a λ-complete ultrafilter, we have $\lambda \setminus \bigcup_{i<\kappa} A_i \in U$. By κ-decomposability again, we conclude that

$$\kappa = \left| \left\{ j < \kappa \;\middle|\; A_j \cap \lambda \setminus \bigcup_{i<\kappa} A_i \neq \emptyset \right\} \right| = |\emptyset| = 0,$$

a contradiction. □

It is more challenging to determine whether an accessible cardinal can carry a uniform non-regular ultrafilter. A classic result of Magidor gives a consistent positive answer.

Theorem 2.5.6 (Magidor [40, Corollary 7])**.** *Assuming the existence of a huge cardinal, it is consistent that there is a uniform ultrafilter over $\aleph_2$ which is not $\aleph_2$-regular.*

On the other hand, Donder showed, using the core model, that the opposite scenario is also consistent.

Theorem 2.5.7 (Donder [11])**.** *It is consistent that for every cardinal κ, all uniform ultrafilters over κ are κ-regular.*

Combining Theorem 2.4.11 and Theorem 2.5.4, we can easily generalize Donder's result to the context of κ-decomposable ultrafilters on complete Boolean algebras.

Theorem 2.5.8. *Let κ be a cardinal. If all uniform ultrafilters over κ are κ-regular, then for every complete Boolean algebra $\mathbb{B}$, all κ-decomposable ultrafilters on $\mathbb{B}$ are κ-regular.*

Proof. Suppose all uniform ultrafilters over a cardinal κ are κ-regular; let U be a κ-decomposable ultrafilter on a complete Boolean algebra $\mathbb{B}$. By Theorem 2.5.4, there exists a uniform ultrafilter D over κ such that $D \leq_{\mathrm{RK}} U$. By hypothesis, D is κ-regular, hence U is also κ-regular by Theorem 2.4.11. □

Corollary 2.5.9. *It is consistent that for every cardinal κ and every complete Boolean algebra $\mathbb{B}$, all κ-decomposable ultrafilters on $\mathbb{B}$ are κ-regular.*

Proof. By Theorem 2.5.7 and Theorem 2.5.8. □

Chapter 3

Keisler's order

3.1 Overview

The intuitive idea behind Keisler's order is simple: a theory T_0 is 'less complicated' than a theory T_1 if the ultrapowers of models of T_0 are more easily saturated than the ultrapowers of models of T_1. As Malliaris and Shelah [43] put it, Keisler's order classifies 'theories through the lens of ultrafilters'.

The class of regular ultrafilters is particularly suitable for this classification work, due to a fundamental result of Keisler.

Theorem 3.1.1 (Keisler [33, Corollary 2.1a]). *Let κ be an infinite cardinal; suppose U is a κ-regular ultrafilter over a set I. If two L-structures $\mathfrak{M}$ and $\mathfrak{N}$ are elementarily equivalent and $|L| \leq \kappa$, then*

$$\mathfrak{M}^I/U \text{ is } \kappa^+\text{-saturated} \iff \mathfrak{N}^I/U \text{ is } \kappa^+\text{-saturated.}$$

In other words, Theorem 3.1.1 implies that the saturation of the regular ultrapower of a model of a complete theory does not depend on the choice of the particular model, but only on the theory itself. This naturally suggests a relation on the class of complete theories, which is now known as Keisler's order.

Definition 3.1.2 (Keisler [33]). Let T_0 and T_1 be complete countable theories and κ a cardinal. We define $T_0 \trianglelefteq_\kappa T_1$ if for every κ-regular ultrafilter U over κ and models $\mathfrak{M}_0 \models T_0$, $\mathfrak{M}_1 \models T_1$, if ${\mathfrak{M}_1}^\kappa/U$ is κ^+-saturated then ${\mathfrak{M}_0}^\kappa/U$ is κ^+-saturated.

Keisler's order is then defined as follows:

$$T_0 \trianglelefteq T_1 \stackrel{\text{def}}{\iff} \forall\kappa(T_0 \trianglelefteq_\kappa T_1).$$

Remark 3.1.3. We observe that $\trianglelefteq$ is a reflexive and transitive relation, which divides the complete countable theories into equivalence classes.

In 1972, Shelah used a model-theoretic property introduced by Keisler [33], called *finite cover property*, to completely determine Keisler's order on stable theories.

Theorem 3.1.4 (Shelah [59, Theorem 4.1])**.** *Keisler's order on stable theories has exactly two equivalence classes: the (minimum) class of theories without the finite cover property, and the class of stable theories with the finite cover property.*

A few years later, Shelah [60, Problem VI.0.1] stated that 'it would be very desirable to prove' that Keisler's order has five classes, and that this task 'will complete the model-theoretic share of investigating Keisler's order for countable theories'. In the following years little progress was made, until Malliaris and Shelah finally shed some light on the structure of Keisler's order on unstable theories. Their investigation revealed a surprising complexity which is highlighted by the following striking result.

Theorem 3.1.5 (Malliaris and Shelah [45, Theorem 6.6])**.** *There is an infinite strictly descending sequence of theories in Keisler's order.*

The rest of this chapter will be dedicated to the study of some aspects of Keisler's order, focusing in particular on saturation of Boolean ultrapowers.

3.2 Elementary equivalence

The purpose of this section is to show that, under a distributivity assumption, Theorem 3.1.1 can be extended to quasiregular ultrafilters on complete Boolean algebras.

A preliminary remark on notation: when we introduce a formula as $\varphi(\boldsymbol{x})$, we mean that $\boldsymbol{x}$ is a finite tuple of variables including the ones appearing free in φ. If we then write $\varphi(\boldsymbol{a})$, we shall implicitly assume that $\boldsymbol{a}$ is a finite tuple of parameters of the same length as the tuple $\boldsymbol{x}$. By abuse of notation, tuples of functions will be sometimes treated as single functions, with the convention that if $\boldsymbol{\tau} = \langle \tau_1, \ldots, \tau_n \rangle$, then $\boldsymbol{\tau}(b) = \langle \tau_1(b), \ldots, \tau_n(b) \rangle$.

The following lemma is very easy, but will simplify some arguments in the proof of Theorem 3.2.2.

Lemma 3.2.1. *Let $\mathbb{B}$ be a Boolean algebra, $A \subset \mathbb{B}$ a maximal antichain, and $x \in \mathbb{B}$. Suppose that x is based on A. Then, for any $b \in \mathbb{B}$ the following conditions are equivalent:*

- *$x \leq b$;*
- *for every $a \in A$, if $a \leq x$ then $a \leq b$.*

Proof. If $x \nleq b$, then there exists $a \in A$ such that $\mathbb{0} < a \wedge x \wedge \neg b$. Since x is based on A, we obtain $a \leq x$; on the other hand, clearly $a \nleq b$. The other implication is obvious. □

Before we present our Theorem 3.2.2, we first remark that, in his independent work, Ulrich [69, Theorem 5.9] has proved the same result for regular ultrafilters, with no distributivity assumption on $\mathbb{B}$.

Theorem 3.2.2. *Let κ be an infinite cardinal. Suppose $\mathbb{B}$ is a $\langle \kappa, 2 \rangle$-distributive complete Boolean algebra and U is a κ-quasiregular ultrafilter on $\mathbb{B}$. If two L-structures $\mathfrak{M}$ and $\mathfrak{N}$ are elementarily equivalent and $|L| \leq \kappa$, then*

$$\mathfrak{M}^{[\mathbb{B}]}/U \text{ is } \kappa^+\text{-saturated} \iff \mathfrak{N}^{[\mathbb{B}]}/U \text{ is } \kappa^+\text{-saturated.}$$

Proof. We assume that $\mathfrak{N}^{[\mathbb{B}]}/U$ is κ^+-saturated and we prove that $\mathfrak{M}^{[\mathbb{B}]}/U$ is κ^+-saturated. Let

$$p(x) = \{ \varphi_\alpha(x, [\boldsymbol{\tau}_\alpha]_U) \mid \alpha < \kappa \}$$

be a 1-type over some set of parameters from $M^{[\mathbb{B}]}/U$; our purpose is to show that $p(x)$ is realized in $\mathfrak{M}^{[\mathbb{B}]}/U$.

Since U is κ-quasiregular, there exists a family $\{ x_\alpha \mid \alpha < \kappa \} \subseteq U$ such that for every infinite $I \subseteq \kappa$ we have $\bigwedge_{\alpha \in I} x_\alpha = \mathbb{0}$. Using $\langle \kappa, 2 \rangle$-distributivity and Proposition 1.2.11, we can find a maximal antichain $A \subset \mathbb{B}$ such that:

- for every $\alpha < \kappa$, x_α is based on A;
- for every $S \in [\kappa]^{<\aleph_0}$, $[\![\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha)]\!]^{\mathfrak{M}^{[\mathbb{B}]}}$ is based on A.

For every $b \in \mathbb{B} \setminus \{\mathbb{0}\}$, define $S(b) = \{ \alpha < \kappa \mid b \leq x_\alpha \}$, and note that $S(b)$ is finite by our quasiregularity assumption.

Claim 1. *For every $a \in A$, there exists a sequence of tuples of names $\langle \boldsymbol{\sigma}^a_\alpha \mid \alpha < \kappa \rangle$ in $N^{[\mathbb{B}]}$ such that for every $S \subseteq S(a)$,*

$$\left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha)\right]\!\!\right]^{\mathfrak{M}^{[\mathbb{B}]}} = \left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\sigma}^a_\alpha)\right]\!\!\right]^{\mathfrak{N}^{[\mathbb{B}]}}. \tag{3.1}$$

Proof of Claim 1. Fix $a \in A$, and let W_a be a common refinement of

$$\{ \operatorname{dom}(\boldsymbol{\tau}_\alpha) \mid \alpha \in S(a) \}.$$

For every $\alpha < \kappa$, we proceed to define a name $\boldsymbol{\sigma}^a_\alpha \colon W_a \to N$ such that (3.1) is satisfied. To do so, let $w \in W_a$ and define

$$\Gamma_w = \left\{ \pm \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{x}_\alpha) \;\middle|\; S \subseteq S(a) \right\},$$

where each $\boldsymbol{x}_\alpha$ is a new tuple of variables of the same length as $\boldsymbol{\tau}_\alpha$, and

$$\pm\psi(\boldsymbol{x}_\alpha) = \begin{cases} \psi(\boldsymbol{x}_\alpha) & \text{if } \mathfrak{M} \models \psi\big((\boldsymbol{\tau}_\alpha \downarrow W_a)(w)\big), \\ \neg\psi(\boldsymbol{x}_\alpha) & \text{if } \mathfrak{M} \models \neg\psi\big((\boldsymbol{\tau}_\alpha \downarrow W_a)(w)\big). \end{cases}$$

Let $\boldsymbol{x}$ be the finite tuple made of all the $\boldsymbol{x}_\alpha$'s appearing here. Then clearly $\mathfrak{M} \models \exists \boldsymbol{x} \bigwedge \Gamma_w$, but $\mathfrak{M} \equiv \mathfrak{N}$, therefore $\mathfrak{N} \models \exists \boldsymbol{x} \bigwedge \Gamma_w$. This allows us to define $\boldsymbol{\sigma}^a_\alpha(w)$ for every $\alpha \in S(a)$. Otherwise, if $\alpha \notin S(a)$, we can define $\boldsymbol{\sigma}^a_\alpha(w)$ arbitrarily. Now it is immediate to check that the desired property holds: for every $S \subseteq S(a)$

$$\left[\!\!\left[\exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha)\right]\!\!\right]^{\mathfrak{M}^{[\mathbb{B}]}} = \bigvee\left\{ w \in W_a \;\middle|\; \mathfrak{M} \models \exists x \bigwedge_{\alpha\in S} \varphi_\alpha\big(x, (\boldsymbol{\tau}_\alpha \downarrow W_a)(w)\big) \right\}$$

$$= \bigvee\left\{ w \in W_a \;\middle|\; \mathfrak{N} \models \exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\sigma}^a_\alpha(w)) \right\} = \left[\!\!\left[\exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\sigma}^a_\alpha)\right]\!\!\right]^{\mathfrak{N}^{[\mathbb{B}]}},$$

as desired. □

For every $\alpha < \kappa$, use Lemma 1.3.4 to define a name $\boldsymbol{\sigma}_\alpha \in N^{[\mathbb{B}]}$ with the property that for all $a \in A$, $a \leq [\![\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}^a_\alpha]\!]^{\mathfrak{N}^{[\mathbb{B}]}}$. We aim to show that

$$q(x) = \{\, \varphi_\alpha(x, [\boldsymbol{\sigma}_\alpha]_U) \mid \alpha < \kappa \,\}$$

is a type over the set of parameters $\{\, [\boldsymbol{\sigma}_\alpha]_U \mid \alpha < \kappa \,\} \subseteq N^{[\mathbb{B}]}/U$.

This will be established once we prove that, for every finite subset $S \subset \kappa$,

$$\left[\!\!\left[\exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha)\right]\!\!\right]^{\mathfrak{M}^{[\mathbb{B}]}} \wedge \bigwedge_{\alpha\in S} x_\alpha \leq \left[\!\!\left[\exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\sigma}_\alpha)\right]\!\!\right]^{\mathfrak{N}^{[\mathbb{B}]}}. \tag{3.2}$$

Indeed, since the left-hand side of (3.2) belongs to U, the same will be true also for the right-hand side, implying that $q(x)$ is finitely satisfiable in $\mathfrak{N}^{[\mathbb{B}]}/U$ by Theorem 1.3.6.

Let $S \subset \kappa$ be finite. Since the left-hand side of (3.2) is based on A (by our choice of A), it will be sufficient to prove that for every $a \in A$,

$$a \leq \left[\!\!\left[\exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha)\right]\!\!\right]^{\mathfrak{M}^{[\mathbb{B}]}} \wedge \bigwedge_{\alpha\in S} x_\alpha \implies a \leq \left[\!\!\left[\exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\sigma}_\alpha)\right]\!\!\right]^{\mathfrak{N}^{[\mathbb{B}]}}$$

and then apply Lemma 3.2.1. But if $a \leq [\![\exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha)]\!]^{\mathfrak{M}^{[\mathbb{B}]}} \wedge \bigwedge_{\alpha\in S} x_\alpha$ then in particular $S \subseteq S(a)$ and therefore (3.1) holds. Hence, putting everything together, we obtain that

$$a \leq \left[\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right]\!\right]^{\mathfrak{M}^{[\mathbb{B}]}} \wedge \bigwedge_{\alpha \in S} x_\alpha \wedge \bigwedge_{\alpha \in S} [\![\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}^a_\alpha]\!]^{\mathfrak{N}^{[\mathbb{B}]}}$$
$$\leq \left[\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\sigma}^a_\alpha) \right]\!\right]^{\mathfrak{N}^{[\mathbb{B}]}} \wedge \bigwedge_{\alpha \in S} [\![\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}^a_\alpha]\!]^{\mathfrak{N}^{[\mathbb{B}]}} \leq \left[\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\sigma}_\alpha) \right]\!\right]^{\mathfrak{N}^{[\mathbb{B}]}}.$$

This completes the proof that $q(x)$ is a type in $\mathfrak{N}^{[\mathbb{B}]}/U$.

Since $\mathfrak{N}^{[\mathbb{B}]}/U$ is κ^+-saturated, there is some $\sigma \in N^{[\mathbb{B}]}$ such that $[\sigma]_U$ realizes $q(x)$ in $\mathfrak{N}^{[\mathbb{B}]}/U$. Using $\langle \kappa, 2 \rangle$-distributivity and Proposition 1.2.11 again, let $W \subset \mathbb{B}$ be a maximal antichain which refines $\mathbb{B}$ and such that for every $\alpha < \kappa$, $[\![\varphi_\alpha(\sigma, \sigma^\alpha)]\!]^{\mathfrak{N}^{[\mathbb{B}]}}$ is based on W. Hence, for every $w \in W$ there exists a unique $a_w \in A$ such that $w \leq a_w$; observe that $S(w) = S(a_w)$ thanks to our choice of A.

For every $w \in W$, define

$$T(w) = \left\{ \alpha \in S(w) \;\middle|\; w \leq [\![\varphi_\alpha(\sigma, \boldsymbol{\sigma}_\alpha)]\!]^{\mathfrak{N}^{[\mathbb{B}]}} \right\}$$

and then, by Theorem 1.3.5, choose a name $\tau_w \in M^{[\mathbb{B}]}$ such that

$$\left[\!\left[\exists x \bigwedge_{\alpha \in T(w)} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right]\!\right]^{\mathfrak{M}^{[\mathbb{B}]}} = \left[\!\left[\bigwedge_{\alpha \in T(w)} \varphi_\alpha(\tau_w, \boldsymbol{\tau}_\alpha) \right]\!\right]^{\mathfrak{M}^{[\mathbb{B}]}}.$$

Finally, let $\tau \in M^{[\mathbb{B}]}$ be such that for all $w \in W$, $w \leq [\![\tau = \tau_w]\!]^{\mathfrak{M}^{[\mathbb{B}]}}$. We shall complete the proof by showing that $[\tau]_U$ realizes $p(x)$ in $\mathfrak{M}^{[\mathbb{B}]}/U$.

To do so, first we observe that for each $w \in W$

$$w = w \wedge a_w \leq \left[\!\left[\bigwedge_{\alpha \in T(w)} \varphi_\alpha(\sigma, \boldsymbol{\sigma}_\alpha) \right]\!\right]^{\mathfrak{N}^{[\mathbb{B}]}} \wedge \bigwedge_{\alpha \in T(w)} [\![\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}^{a_w}_\alpha]\!]^{\mathfrak{N}^{[\mathbb{B}]}}$$
$$\leq \left[\!\left[\bigwedge_{\alpha \in T(w)} \varphi_\alpha(\sigma, \boldsymbol{\sigma}^{a_w}_\alpha) \right]\!\right]^{\mathfrak{N}^{[\mathbb{B}]}} \leq \left[\!\left[\exists x \bigwedge_{\alpha \in T(w)} \varphi_\alpha(x, \boldsymbol{\sigma}^{a_w}_\alpha) \right]\!\right]^{\mathfrak{N}^{[\mathbb{B}]}}$$
$$= \left[\!\left[\exists x \bigwedge_{\alpha \in T(w)} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right]\!\right]^{\mathfrak{M}^{[\mathbb{B}]}} = \left[\!\left[\bigwedge_{\alpha \in T(w)} \varphi_\alpha(\tau_w, \boldsymbol{\tau}_\alpha) \right]\!\right]^{\mathfrak{M}^{[\mathbb{B}]}},$$

where in the penultimate equality we applied (3.1) to $T(w) \subseteq S(a_w)$.

From the above inequality, it follows that for all $w \in W$

$$w \leq \left[\!\left[\bigwedge_{\alpha \in T(w)} \varphi_\alpha(\tau_w, \boldsymbol{\tau}_\alpha) \right]\!\right]^{\mathfrak{M}^{[\mathbb{B}]}} \wedge [\![\tau = \tau_w]\!]^{\mathfrak{M}^{[\mathbb{B}]}} \leq \left[\!\left[\bigwedge_{\alpha \in T(w)} \varphi_\alpha(\tau, \boldsymbol{\tau}_\alpha) \right]\!\right]^{\mathfrak{M}^{[\mathbb{B}]}}.$$

In other words, this is what we obtained:

$$(\forall \alpha < \kappa)(\forall w \in W)\Big(\alpha \in T(w) \implies w \leq [\![\varphi_\alpha(\tau, \boldsymbol{\tau}_\alpha)]\!]^{\mathfrak{M}^{[\mathbb{B}]}}\Big). \tag{3.3}$$

But $\alpha \in T(w)$ is just equivalent to $w \leq [\![\varphi_\alpha(\sigma, \boldsymbol{\sigma}_\alpha)]\!]^{\mathfrak{N}^{[\mathbb{B}]}} \wedge x_\alpha$, therefore we can rephrase (3.3) as follows:

$$(\forall \alpha < \kappa)(\forall w \in W)\Big(w \leq [\![\varphi_\alpha(\sigma, \boldsymbol{\sigma}_\alpha)]\!]^{\mathfrak{N}^{[\mathbb{B}]}} \wedge x_\alpha \implies w \leq [\![\varphi_\alpha(\tau, \boldsymbol{\tau}_\alpha)]\!]^{\mathfrak{M}^{[\mathbb{B}]}}\Big).$$

Now it is sufficient to observe that $[\![\varphi_\alpha(\sigma, \boldsymbol{\sigma}_\alpha)]\!]^{\mathfrak{N}^{[\mathbb{B}]}} \wedge x_\alpha$ is based on W, and then apply Lemma 3.2.1 to conclude that for every $\alpha < \kappa$

$$[\![\varphi_\alpha(\sigma, \boldsymbol{\sigma}_\alpha)]\!]^{\mathfrak{N}^{[\mathbb{B}]}} \wedge x_\alpha \leq [\![\varphi_\alpha(\tau, \boldsymbol{\tau}_\alpha)]\!]^{\mathfrak{M}^{[\mathbb{B}]}}. \tag{3.4}$$

Since the left-hand side of (3.4) belongs to U, the same will be true also for the right-hand side, thus showing that $[\tau]_U$ realizes $p(x)$ in $\mathfrak{M}^{[\mathbb{B}]}/U$ by Theorem 1.3.6. □

It follows from the results of Section 2.4 that, without the distributivity assumption, Theorem 3.2.2 may be false in general.

Proposition 3.2.3. *Let κ be a cardinal such that $\kappa^{\aleph_0} = \kappa$. Then there exist a complete Boolean algebra $\mathbb{B}$, a κ-quasiregular ultrafilter U on $\mathbb{B}$, and two elementarily equivalent $\emptyset$-structures $\mathfrak{M} \equiv \mathfrak{N}$ such that $\mathfrak{M}^{[\mathbb{B}]}/U$ is κ^+-saturated, but $\mathfrak{N}^{[\mathbb{B}]}/U$ is not κ^+-saturated.*

Proof. Given a cardinal κ with $\kappa^{\aleph_0} = \kappa$, let U be any ultrafilter on $\mathbb{C}_\kappa$. We know from Corollary 2.1.12 that U is κ-quasiregular.

First, we observe that if $\mathfrak{M}$ is an infinite $\emptyset$-structure, then $\mathfrak{M}$ is κ^+-saturated if and only if $\kappa < |M|$. Now, let $\mathfrak{N} = \kappa$; by Lemma 2.4.2 we have

$$\left|\kappa^{[\mathbb{C}_\kappa]}/U\right| \leq \kappa^{<\aleph_1} + \left(\kappa^{\aleph_0}\right)^{<\aleph_1} = \kappa^{\aleph_0} = \kappa,$$

hence $\mathfrak{N}^{[\mathbb{C}_\kappa]}/U$ is not κ^+-saturated.

On the other hand, let $\mathfrak{M}$ be an $\emptyset$-structure of cardinality $> \kappa$ such that $\mathfrak{M} \equiv \mathfrak{N}$. Then

$$\kappa < |M| \leq \left|M^{[\mathbb{C}_\kappa]}/U\right|,$$

which means that $\mathfrak{M}^{[\mathbb{C}_\kappa]}/U$ is κ^+-saturated. □

Thus, not only Theorem 3.2.2 can fail in this context, but also the failure is due trivially to the cardinality of the Boolean ultrapowers and not to their saturation properties.

3.3 Good ultrafilters

Keisler's investigation started by identifying a class of ultrafilters which saturate every ultrapower. In 1964, he named those ultrafilters 'good'.

Definition 3.3.1 (Keisler [29]). Let λ be a cardinal. A filter F over a set I is λ-*good* if for every $\kappa < \lambda$ and every monotonic function $f\colon [\kappa]^{<\aleph_0} \to F$, there exists a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to F$ with the property that $g(S) \subseteq f(S)$ for all $S \in [\kappa]^{<\aleph_0}$.

Remark 3.3.2. Let κ be a cardinal and F a filter over a set I. Comparing Definition 2.2.3 and Definition 3.3.1, it is immediate to see that if F is κ^+-good, then F is κ-OK.

The next result implies, in particular, that every filter is $\aleph_1$-good.

Theorem 3.3.3 (Keisler [29, Theorem 4.1]). *Let κ be a cardinal and F a filter over a set I. If F is κ-complete, then F is κ^+-good.*

In our context, however, we shall be concerned with λ-good ultrafilters which are $\aleph_1$-incomplete. This is due to the following characterization, which shows that such ultrafilters are precisely the ones which yield λ-saturated ultrapowers.

Theorem 3.3.4 (Keisler [31]). *Let λ be an uncountable cardinal. For an ultrafilter U over a set I, the following conditions are equivalent:*

- *U is $\aleph_1$-incomplete and λ-good;*

- *for every L-structure $\mathfrak{M}$ with $|L| < \lambda$, the ultrapower $\mathfrak{M}^I/U$ is λ-saturated.*

We remark that Keisler [29, Theorem 4.4] proved that, for every infinite cardinal κ, if $2^\kappa = \kappa^+$ then there exists an $\aleph_1$-incomplete κ^+-good ultrafilter over κ. Later, Kunen [36, Theorem 3.2] removed the assumption that $2^\kappa = \kappa^+$.

In the remainder of this section, we shall consider whether a similar characterization can be found for ultrafilters on complete Boolean algebras. In fact, the problem of finding a translation of Theorem 3.3.4 for Boolean ultrapowers was first considered by Mansfield [46], who defined good ultrafilters in a way formally analogous to Definition 3.3.1.

Definition 3.3.5 (Mansfield [46])**.** Let λ be a cardinal. An ultrafilter U on a complete Boolean algebra $\mathbb{B}$ is λ-*good* if for every $\kappa < \lambda$ and every monotonic function $f\colon [\kappa]^{<\aleph_0} \to U$, there exists a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to U$ with the property that $g(S) \leq f(S)$ for all $S \in [\kappa]^{<\aleph_0}$.

Using this definition, he was able to generalize one of the two implications of Theorem 3.3.4.

Theorem 3.3.6 (Mansfield [46, Theorem 4.1])**.** *Let λ be an uncountable cardinal, $\mathbb{B}$ a complete Boolean algebra, and U an ultrafilter on $\mathbb{B}$. If U is $\aleph_1$-incomplete and λ-good, then for every L-structure $\mathfrak{M}$ with $|L| < \lambda$, the Boolean ultrapower $\mathfrak{M}^{[\mathbb{B}]}/U$ is λ-saturated.*

Three years later, Benda [3] observed that, using Mansfield's definition, 'it is not straightforward to prove the other implication'. To get around this problem, Benda introduced another class of ultrafilters, which he also called 'λ-good'. To avoid a clash of terminology, we shall temporarily rename such ultrafilters 'λ-Benda'. Also, to simplify the definition, we shall use the notation introduced in (1.4).

Definition 3.3.7 (Benda [3])**.** Let λ be a cardinal. An ultrafilter U on a complete Boolean algebra $\mathbb{B}$ is λ-*Benda* if for every $\kappa < \lambda$, for every function $f\colon [\kappa]^{<\aleph_0} \to U$, and every family of maximal antichains $\{\, A_\alpha \mid \alpha < \kappa \,\}$, if for all $S \in [\kappa]^{<\aleph_0}$ $f(S)$ is based on $\bigwedge_{\alpha\in S} A_\alpha$, then there exist a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to U$ and a maximal antichain A satisfying:

1. for all $S \in [\kappa]^{<\aleph_0}$, $g(S)$ is based on $A \wedge \bigwedge_{\alpha \in S} A_\alpha$;
2. for all $a \in A$, the set $\{ \alpha < \kappa \mid \mathbb{0} < a \wedge g(\{\alpha\}) \}$ is finite;
3. for all $S \in [\kappa]^{<\aleph_0}$, $g(S) \leq f(S)$.

The above definition, although quite complex, was specifically designed to establish the following equivalence.

Theorem 3.3.8 (Benda [3])**.** *Let λ be an uncountable cardinal. For an ultrafilter U on a complete Boolean algebra $\mathbb{B}$, the following conditions are equivalent:*

- *U is λ-Benda;*
- *for every L-structure $\mathfrak{M}$ with $|L| < \lambda$, the Boolean ultrapower $\mathfrak{M}^{[\mathbb{B}]}/U$ is λ-saturated.*

Clearly, the combination of Theorem 3.3.6 and Theorem 3.3.8 shows that if an ultrafilter U is $\aleph_1$-incomplete and λ-good, then it is λ-Benda. However, the question whether Benda's notion is actually weaker remained open.

In 1982, Balcar and Franek [1] discussed the existence of independent families in complete Boolean algebras. They acknowledged the existence of two different definitions of goodness for ultrafilters on complete Boolean algebras, and noted that Mansfield's definition 'is apparently stronger than Benda's for it implies that even the Boolean valued model of set theory modulo a κ-good ultrafilter is κ-saturated'.

Motivated by Benda's question and Balcar and Franek's remark, we decided to further investigate this problem. In the main result of this section, we finally settle this question by showing that Mansfield's definition, although apparently stronger, is in fact equivalent to Benda's definition.

Theorem 3.3.9. *Let λ be an uncountable cardinal, $\mathbb{B}$ be a complete Boolean algebra, and U an ultrafilter on $\mathbb{B}$. Then U is λ-Benda if and only if U is $\aleph_1$-incomplete and λ-good.*

Proof. As we have already observed, one implication follows immediately combining Theorem 3.3.6 and Theorem 3.3.8.

For the other implication, suppose U is λ-Benda: we shall show that U is $\aleph_1$-incomplete and λ-good. Let $\kappa < \lambda$ be a cardinal which, without loss of generality, we may assume to be infinite.

Firstly, consider the constant function $c\colon [\kappa]^{<\aleph_0} \to U$ defined by $c(S) = \mathbb{1}$ for all $S \in [\kappa]^{<\aleph_0}$. By taking $A_\alpha = \{\mathbb{1}\}$ for each $\alpha < \kappa$, it is trivial that for all $S \in [\kappa]^{<\aleph_0}$ $c(S)$ is based on $\bigwedge_{\alpha\in S} A_\alpha = \{\mathbb{1}\}$. Therefore, we can use the hypothesis that U is λ-Benda to obtain a multiplicative function $x\colon [\kappa]^{<\aleph_0} \to U$ and a maximal antichain $A \subset \mathbb{B}$ such that:

1. for all $S \in [\kappa]^{<\aleph_0}$, $x(S)$ is based on A;
2. for all $a \in A$, the set $S(a) = \{\, \alpha < \kappa \mid a \le x(\{\alpha\}) \,\}$ is finite.

To show that U is $\aleph_1$-incomplete, we recall that κ is infinite and observe that

$$\bigwedge_{n<\omega} x(\{n\}) = \mathbb{0}.$$

Indeed, if we had $\bigwedge_{n<\omega} x(\{n\}) > \mathbb{0}$ then we could find some $a \in A$ such that $a \wedge \bigwedge_{n<\omega} x(\{n\}) > \mathbb{0}$. From property (1) we would have $a \le \bigwedge_{n<\omega} x(\{n\})$, contradicting property (2). Therefore, U is $\aleph_1$-incomplete.

To show that U is λ-good, let $f\colon [\kappa]^{<\aleph_0} \to U$ be a monotonic function. We claim that there exists a maximal antichain W such that for each $S \in [\kappa]^{<\aleph_0}$, $f(S) \wedge x(S)$ is based on W. Our claim will be proved once we show that the set

$$D = \left\{ d \in \mathbb{B} \setminus \{\mathbb{0}\} \;\middle|\; \text{for all } S \in [\kappa]^{<\aleph_0}, \text{ either } d \le f(S) \wedge x(S) \text{ or } d \wedge f(S) \wedge x(S) = \mathbb{0} \right\}$$

is dense in $\mathbb{B}$: any maximal antichain $W \subseteq D$ will then have the desired property. Let $b \in \mathbb{B} \setminus \{\mathbb{0}\}$; we shall find some $d \in D$ such that $d \le b$. First, we can find some $a \in A$ such that $\mathbb{0} < a \wedge b$. Now let P be a common refinement of the finitely many maximal antichains $\{f(S), \neg f(S)\}$ for $S \subseteq S(a)$. Let $p \in P$ be such that $\mathbb{0} < p \wedge a \wedge b$; then it is clear that $d = p \wedge a \wedge b$ is such that $d \le b$. To see that $d \in D$, suppose $S \in [\kappa]^{<\aleph_0}$ has the property that $\mathbb{0} < d \wedge f(S) \wedge x(S)$. In particular, we have $\mathbb{0} < a \wedge x(S)$, but $x(S)$ is based on A and therefore $a \leq x(S)$. This implies, since x is monotonic, that $S \subseteq S(a)$. We deduce that $f(S)$ is based on P and therefore $p \leq f(S)$. Putting everything together, we conclude that

$$d \leq p \wedge a \leq f(S) \wedge x(S).$$

This shows that $d \in D$ and completes the proof of the claim.

Now, letting $W_\alpha = W$ for each $\alpha < \kappa$, we deduce from our claim that for all $S \in [\kappa]^{<\aleph_0}$, $f(S) \wedge x(S)$ is based on $\bigwedge_{\alpha \in S} W_\alpha = W$. Hence, we can use the hypothesis that U is λ-Benda to obtain a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to U$ such that for all $S \in [\kappa]^{<\aleph_0}$, $g(S) \leq f(S) \wedge x(S) \leq f(S)$, as desired. Therefore, U is λ-good and the proof is complete. □

In conclusion, Definition 3.3.5 can be regarded as an appropriate generalization of the notion of goodness to the context of complete Boolean algebras, for it implies that the parallel of Theorem 3.3.4 holds in full generality for Boolean ultrapowers.

3.4 Homomorphisms and the extension theorem

In this section we briefly discuss the possibility of representing a complete Boolean algebra as a homomorphic image of a power-set algebra. In particular, Theorem 3.4.4 introduces a powerful technique, due to Malliaris and Shelah [42], which will be useful in the proof of our Theorem 3.5.6.

Remark 3.4.1. Suppose κ is a cardinal and $\mathbb{B}$ is a Boolean algebra; if $j\colon \mathcal{P}(\kappa) \to \mathbb{B}$ is a surjective homomorphism, then $j^{-1}[\{\mathbb{1}\}]$ is a filter over κ. Furthermore, the correspondence $F \mapsto j^{-1}[F]$ is a bijection from the set of filters on $\mathbb{B}$ to the set of filters over κ which include $j^{-1}[\{\mathbb{1}\}]$.

As we shall see, independent families provide a useful method to construct surjective homomorphisms. The method relies on a classic result of Hausdorff.

Theorem 3.4.2 (Hausdorff [21])**.** *For every infinite cardinal κ there exists a family $X \subseteq \mathcal{P}(\kappa)$, with $|X| = 2^\kappa$, such that for every $S, T \in$*

$[X]^{<\aleph_0}$, *if* $\bigcap S \subseteq \bigcup T$ *then* $S \cap T \neq \emptyset$.

We illustrate the construction of a surjective homomorphism $\mathcal{P}(\kappa) \to \mathbb{B}$ through a simple example. This is a special case of Balcar and Franek [1, Corollary 3].

Proposition 3.4.3. *Let κ be a cardinal and $\mathbb{B}$ a complete Boolean algebra. If $|\mathbb{B}| \leq 2^\kappa$, then there exists a surjective homomorphism $j\colon \mathcal{P}(\kappa) \to \mathbb{B}$.*

Proof. Let $\mathbb{B}$ be a complete Boolean algebra with $|\mathbb{B}| \leq 2^\kappa$. By Theorem 3.4.2 there exists a family $X \subseteq \mathcal{P}(\kappa)$, with $|X| = 2^\kappa$, such that for every $S, T \in [X]^{<\aleph_0}$, if $\bigcap S \subseteq \bigcup T$ then $S \cap T \neq \emptyset$. Let $f\colon X \to \mathbb{B}$ be a surjective function.

Let $\mathbb{F}$ be the subalgebra of $\mathcal{P}(\kappa)$ generated by X: it follows from Proposition 1.2.16 that $\mathbb{F}$ is free over X. By Definition 1.2.15 there exists a homomorphism $g\colon \mathbb{F} \to \mathbb{B}$ such that $g \restriction X = f$. Hence, g is also surjective.

Finally, by Theorem 1.2.10 there exists a homomorphism $j\colon \mathcal{P}(\kappa) \to \mathbb{B}$ such that $j \restriction \mathbb{F} = g$. Therefore, j is the desired surjective homomorphism. □

This method can be fine-tuned and applied to the construction of regular ultrafilters. For instance, suppose we can construct a surjective homomorphism $j\colon \mathcal{P}(\kappa) \to \mathbb{B}$ with the additional property that $j^{-1}[\{\mathbb{1}\}]$ is a κ-regular filter. Then, by Remark 3.4.1, for every ultrafilter U on $\mathbb{B}$ the inverse image $j^{-1}[U]$ will automatically be a κ-regular ultrafilter. Furthermore, if $j^{-1}[\{\mathbb{1}\}]$ has the additional property of being κ^+-good, then the correspondence $U \mapsto j^{-1}[U]$ preserves the model-theoretic properties of U, in a sense which is made precise in Theorem 3.5.3 below.

The following result, usually referred to as the 'existence theorem', shows that, under some conditions on the Boolean algebra $\mathbb{B}$, a surjective homomorphism as described above does indeed exist.

Theorem 3.4.4 (Malliaris and Shelah [42, Theorem 8.1])**.** *Let κ be a cardinal; if $\mathbb{B}$ is a κ^+-c.c. complete Boolean algebra of cardinality $\leq 2^\kappa$, then there exists a surjective homomorphism $j\colon \mathcal{P}(\kappa) \to \mathbb{B}$ such that $j^{-1}[\{\mathbb{1}\}]$ is a κ-regular κ^+-good filter over κ.*

Remark 3.4.5. Malliaris and Shelah originally stated Theorem 3.4.4, and similarly Theorem 3.5.3 below, in terms of *excellent* filters. However, by their [42, Theorem 5.2], a filter is κ^+-excellent if and only if it is κ^+-good. Hence, our formulation is equivalent to theirs.

3.5 Keisler's order via Boolean ultrapowers

In this section, we provide a new characterization of Keisler's order in terms of saturation of Boolean ultrapowers. To do so, we apply and expand the framework of 'separation of variables' recently developed by Malliaris and Shelah [42]. The results of this section, together with those of Section 3.2, were first presented at the Logic Colloquium 2017 in Stockholm.

We begin by introducing the crucial concept of *morality*, which can be thought of as a 'local' version of goodness. Namely, we do not require to be able to refine all monotonic functions into the ultrafilter U, but just those relative to some theory T. The meaning of 'relative to T' is made precise in the definition of *possibility*.

Definition 3.5.1. Let κ be a cardinal, $\mathbb{B}$ a complete Boolean algebra, T a complete countable theory, and $\varphi = \langle\, \varphi_\alpha(x, \boldsymbol{y}_\alpha) \,|\, \alpha < \kappa \,\rangle$ a sequence of formulae in the language of T.

A $\langle \kappa, \mathbb{B}, T, \varphi \rangle$-*possibility* is a monotonic function $f \colon [\kappa]^{<\aleph_0} \to \mathbb{B} \setminus \{\mathbb{0}\}$ such that: for all $S_* \in [\kappa]^{<\aleph_0}$ and $a \in \mathbb{B} \setminus \{\mathbb{0}\}$ which satisfy:

- for every $S \subseteq S_*$ either $a \leq f(S)$ or $a \wedge f(S) = \mathbb{0}$,
- $S_* \subseteq \{\, \alpha < \kappa \mid a \leq f(\{\alpha\}) \,\}$,

there exist a model $\mathfrak{M} \models T$ and $\{\, \boldsymbol{b}_\alpha \mid \alpha \in S_* \,\}$ in M such that for all $S \subseteq S_*$

$$a \leq f(S) \iff \mathfrak{M} \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{b}_\alpha). \tag{3.5}$$

Following our convention on notation, in Definition 3.5.1 it is implicitly assumed that each $\boldsymbol{b}_\alpha$ is a finite tuple from M of the same length as $\boldsymbol{y}_\alpha$.

Definition 3.5.2 (Malliaris and Shelah [42]). Let κ be a cardinal, $\mathbb{B}$ a complete Boolean algebra, and T a complete countable theory. An ultrafilter U on $\mathbb{B}$ is $\langle \kappa, \mathbb{B}, T \rangle$-*moral* if for every sequence of formulae $\varphi = \langle\, \varphi_\alpha(x, \boldsymbol{y}_\alpha) \mid \alpha < \kappa \,\rangle$ and every $\langle \kappa, \mathbb{B}, T, \varphi \rangle$-possibility $f\colon [\kappa]^{<\aleph_0} \to U$, there exists a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to U$ with the property that $g(S) \leq f(S)$ for all $S \in [\kappa]^{<\aleph_0}$.

Moral ultrafilters have recently played a crucial role in the study of Keisler's order, due to Malliaris and Shelah's technique of 'separation of variables', which yields that a problem of saturation can be translated to a problem of morality via a surjective homomorphism of Boolean algebras.

Theorem 3.5.3 (Malliaris and Shelah [42, Theorem 6.13]). *Let κ be an infinite cardinal, $\mathbb{B}$ a complete Boolean algebra, and T a complete countable theory. Suppose $j\colon \mathcal{P}(\kappa) \to \mathbb{B}$ is a surjective homomorphism with the property that $j^{-1}[\{\mathbb{1}\}]$ is a κ-regular κ^+-good filter over κ. Then, for an ultrafilter U on $\mathbb{B}$ the following conditions are equivalent:*

- *U is $\langle \kappa, \mathbb{B}, T \rangle$-moral;*
- *for a model $\mathfrak{M} \models T$, the ultrapower $\mathfrak{M}^\kappa / j^{-1}[U]$ is κ^+-saturated.*

Note that, as we remarked in Section 3.4, $j^{-1}[U]$ is a κ-regular ultrafilter over κ.

In order to formulate the main results of this section, it is convenient to introduce a natural concept of saturation for ultrafilters on complete Boolean algebras.

Definition 3.5.4. Let λ be a cardinal and $\mathbb{B}$ a complete Boolean algebra. Suppose U is an ultrafilter on $\mathbb{B}$; we say that U λ-*saturates* a complete theory T if for every λ-saturated model $\mathfrak{M} \models T$, the Boolean ultrapower $\mathfrak{M}^{[\mathbb{B}]}/U$ is λ-saturated.

Shelah [61, Claim 3.4] has first established a connection between morality of ultrafilters and saturation of Boolean ultrapowers. However, his result is framed in the context of atomic saturation in the infinitary logic $\mathbb{L}_{\theta,\theta}$ and only the case where $\mathbb{B}$ is a power-set algebra is proved explicitly. In the next theorem we present a detailed explanation of the equivalence; our proof relies on Lemma 2.2.2.

Theorem 3.5.5. *Let κ be an infinite cardinal, $\mathbb{B}$ a complete Boolean algebra, and T a complete countable theory. If U is a κ-regular ultrafilter on $\mathbb{B}$, then the following conditions are equivalent:*

- *U is $\langle \kappa, \mathbb{B}, T\rangle$-moral;*
- *U κ^+-saturates T.*

Proof. Let the family $\{ x_\alpha \mid \alpha < \kappa \} \subseteq U$ and the maximal antichain $A \subset \mathbb{B}$ witness the κ-regularity of U. In particular, for every $b \in \mathbb{B}\setminus\{\mathbb{0}\}$ the set $S(b) = \{ \alpha < \kappa \mid b \leq x_\alpha \}$ is finite.

Suppose U is $\langle \kappa, \mathbb{B}, T\rangle$-moral. Let $\mathfrak{M}$ be a model of T and

$$p(x) = \left\{ \varphi_\alpha\big(x, [\boldsymbol{\tau}_\alpha]_U\big) \mid \alpha < \kappa \right\}$$

be a type in $\mathfrak{M}^{[\mathbb{B}]}/U$, where each $\boldsymbol{\tau}_\alpha$ is a finite tuple from $M^{[\mathbb{B}]}$. We shall show that $p(x)$ is realized in $\mathfrak{M}^{[\mathbb{B}]}/U$.

By Theorem 1.3.6, for every $S \in [\kappa]^{<\aleph_0}$ we have

$$\left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right]\!\!\right] \in U.$$

This allows us to define a monotonic function $f\colon [\kappa]^{<\aleph_0} \to U$ by letting for every $S \in [\kappa]^{<\aleph_0}$

$$f(S) = \left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right]\!\!\right] \wedge \bigwedge_{\alpha \in S} x_\alpha.$$

Define $\varphi = \langle\, \varphi_\alpha(x, \boldsymbol{y}_\alpha) \,|\, \alpha < \kappa \,\rangle$, where each $\boldsymbol{y}_\alpha$ is a new tuple of variables of the same length as $\boldsymbol{\tau}_\alpha$; we aim to show that f is a $\langle \kappa, \mathbb{B}, T, \varphi\rangle$-possibility. Let $S_* \in [\kappa]^{<\aleph_0}$ and $a \in \mathbb{B} \setminus \{\mathbb{0}\}$ be fixed, and assume that

- for every $S \subseteq S_*$ either $a \leq f(S)$ or $a \wedge f(S) = \mathbb{0}$;
- $S_* \subseteq \{ \alpha < \kappa \mid a \leq f(\{\alpha\}) \}$.

Choose a maximal antichain D with the following property: for each $\alpha \in S_*$, D is a refinement of the domain of each name in the tuple $\boldsymbol{\tau}_\alpha$.

Then, there exists $d \in D$ such that $\mathbb{0} < d \wedge a$. For every $\alpha \in S_*$, let

$$\boldsymbol{b}_\alpha = (\boldsymbol{\tau}_\alpha \downarrow D)(d);$$

we show that this choice satisfies (3.5). Let $S \subseteq S_*$; if $a \leq f(S)$, then $\mathbb{0} < d \wedge f(S) \leq d \wedge \llbracket \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \rrbracket$, therefore

$$\mathfrak{M} \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha\big(x, (\boldsymbol{\tau}_\alpha \downarrow D)(d)\big). \tag{3.6}$$

Conversely, if (3.6) holds then $d \leq \llbracket \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \rrbracket$ and therefore

$$\mathbb{0} < a \wedge \left\llbracket \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right\rrbracket.$$

From the second assumption above we have $a \leq \bigwedge_{\alpha \in S_*} x_\alpha \leq \bigwedge_{\alpha \in S} x_\alpha$, so we deduce $\mathbb{0} < a \wedge f(S)$ and finally, from the first assumption above, $a \leq f(S)$. This concludes the proof that f is a $\langle \kappa, \mathbb{B}, T, \varphi \rangle$-possibility.

Since we are assuming that U is $\langle \kappa, \mathbb{B}, T \rangle$-moral, there exists a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to U$ with the property that $g(S) \leq f(S)$ for all $S \in [\kappa]^{<\aleph_0}$. We wish to show that g satisfies condition (1) of Lemma 2.2.2. Suppose not; then there exists some $a \in A$ such that

$$a \wedge \bigwedge \Big\{ \bigvee g\big[[\kappa]^n\big] \ \Big|\ n < \omega \Big\} > \mathbb{0},$$

hence for every $n < \omega$ there exists some $S \in [\kappa]^n$ such that

$$\mathbb{0} < a \wedge g(S) \leq a \wedge f(S) \leq a \wedge \bigwedge_{\alpha \in S} x_\alpha,$$

but this contradicts our regularity assumption that $S(a)$ is finite.

Consequently, we may apply Lemma 2.2.2 to find a maximal antichain $W \subset \mathbb{B}$ such that:

- for every $\alpha < \kappa$, $g(\{\alpha\})$ is based on W;
- for every $w \in W$, the set $T(w) = \{\, \alpha < \kappa \mid w \leq g(\{\alpha\}) \,\}$ is finite.

For each $w \in W$, use Theorem 1.3.5 to choose a name $\tau_w \in M^{[\mathbb{B}]}$ such

that
$$\left\llbracket \exists x \bigwedge_{\alpha\in T(w)} \varphi_\alpha(x,\boldsymbol{\tau}_\alpha) \right\rrbracket = \left\llbracket \bigwedge_{\alpha\in T(w)} \varphi_\alpha(\tau_w,\boldsymbol{\tau}_\alpha) \right\rrbracket.$$

Finally, by Lemma 1.3.4, let $\tau \in M^{[\mathbb{B}]}$ be such that for all $w \in W$, $w \leq \llbracket \tau = \tau_w \rrbracket$. We shall show that $[\tau]_U$ realizes the type $p(x)$ in $\mathfrak{M}^{[\mathbb{B}]}/U$.

For every $w \in W$, by multiplicativity of g we have

$$\begin{aligned} w \leq \bigwedge_{\alpha\in T(w)} g(\{\alpha\}) = g(T(w)) \leq f(T(w)) \\ \leq \left\llbracket \exists x \bigwedge_{\alpha\in T(w)} \varphi_\alpha(x,\boldsymbol{\tau}_\alpha) \right\rrbracket = \left\llbracket \bigwedge_{\alpha\in T(w)} \varphi_\alpha(\tau_w,\boldsymbol{\tau}_\alpha) \right\rrbracket, \end{aligned}$$

whence

$$w \leq \left\llbracket \bigwedge_{\alpha\in T(w)} \varphi_\alpha(\tau_w,\boldsymbol{\tau}_\alpha) \right\rrbracket \wedge \llbracket \tau = \tau_w \rrbracket \leq \left\llbracket \bigwedge_{\alpha\in T(w)} \varphi_\alpha(\tau,\boldsymbol{\tau}_\alpha) \right\rrbracket.$$

If follows that for every $\alpha < \kappa$

$$\begin{aligned} \llbracket \varphi_\alpha(\tau,\boldsymbol{\tau}_\alpha) \rrbracket &\geq \bigvee \{\, w \in W \mid \alpha \in T(w) \,\} \\ &= \bigvee \{\, w \in W \mid w \leq g(\{\alpha\}) \,\} = g(\{\alpha\}) \in U, \end{aligned}$$

thus showing that $\llbracket \varphi_\alpha(\tau,\boldsymbol{\tau}_\alpha) \rrbracket \in U$. This completes the proof that $\mathfrak{M}^{[\mathbb{B}]}/U$ is κ^+-saturated.

For the other implication, suppose that U κ^+-saturates T. Let $\varphi = \langle\, \varphi_\alpha(x, \boldsymbol{y}_\alpha) \mid \alpha < \kappa \,\rangle$ be a sequence of formulae; for a $\langle \kappa, \mathbb{B}, T, \varphi \rangle$-possibility $f\colon [\kappa]^{<\aleph_0} \to U$ we shall find a multiplicative function $g\colon [\kappa]^{<\aleph_0} \to U$ with the property that $g(S) \leq f(S)$ for all $S \in [\kappa]^{<\aleph_0}$.

Claim 2. *There exists a refinement W of A with the property that for every $w \in W$ and every $S \subseteq S(w)$ either $w \leq f(S)$ or $w \wedge f(S) = \mathbb{0}$.*

Proof of Claim 2. Let D be the set of all $d \in \mathbb{B} \setminus \{\mathbb{0}\}$ which are below some element of A, and such that for every $S \subseteq S(d)$ either $d \leq f(S)$ or $d \wedge f(S) = \mathbb{0}$. We shall show that D is dense in $\mathbb{B}$, so that every maximal antichain $W \subseteq D$ will have the desired property. The same argument as the proof of Theorem 3.3.9 will work: indeed, suppose $b \in \mathbb{B} \setminus \{\mathbb{0}\}$. We can find some $a \in A$ such that $\mathbb{0} < a \wedge b$. Now let P be a common refinement of the finitely many maximal antichains $\{f(S), \neg f(S)\}$ for $S \subseteq S(a)$. Let $p \in P$ be such that $\mathbb{0} < p \wedge a \wedge b$; then it is clear that $d = p \wedge a \wedge b$ is such that $d \leq b$ and $d \in D$. □

Now, for each $a \in W$ let

$$S_*(a) = \{\, \alpha \in S(a) \mid a \leq f(\{\alpha\}) \,\},$$

and note that:

- for every $S \subseteq S_*(a)$ either $a \leq f(S)$ or $a \wedge f(S) = \mathbb{0}$;
- $S_*(a) \subseteq \{\, \alpha < \kappa \mid a \leq f(\{\alpha\}) \,\}$.

Since f is a $\langle \kappa, \mathbb{B}, T, \varphi \rangle$-possibility, for each $a \in W$ there exist a model $\mathfrak{M}_a \models T$ and parameters $\{\, \boldsymbol{b}_\alpha(a) \mid \alpha \in S_*(a) \,\}$ in M_a such that for all $S \subseteq S_*(a)$

$$a \leq f(S) \iff \mathfrak{M}_a \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{b}_\alpha(a)). \tag{3.7}$$

Now let $\mathfrak{M}$ be a κ^+-saturated model of T.

Claim 3. *For every $a \in W$ there exists a sequence $\langle\, \boldsymbol{\tau}_\alpha(a) \mid \alpha < \kappa \,\rangle$ in M such that for every $S \subseteq S_*(a)$*

$$a \leq f(S) \iff \mathfrak{M} \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha(a)).$$

Proof of Claim 3. We proceed analogously to the proof of Claim 1. Let us fix $a \in W$ and define

$$\begin{aligned}\Gamma_a = &\left\{ \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{y}_\alpha) \;\middle|\; S \subseteq S_*(a) \text{ and } a \leq f(S) \right\} \\ &\cup \left\{ \neg\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{y}_\alpha) \;\middle|\; S \subseteq S_*(a) \text{ and } a \wedge f(S) = \mathbb{0} \right\}.\end{aligned}$$

Let $\boldsymbol{y}$ be the finite tuple of variables made of all the $\boldsymbol{y}_\alpha$ appearing in Γ_a. Then (3.7) implies that $\mathfrak{M}_a \models \exists \boldsymbol{y} \bigwedge \Gamma_a$, but $\mathfrak{M}_a \equiv \mathfrak{M}$ by completeness of T, therefore $\mathfrak{M} \models \exists \boldsymbol{y} \bigwedge \Gamma_a$. This allows us to define $\boldsymbol{\tau}_\alpha(a)$ in M for every $\alpha \in S_*(a)$. Otherwise, if $\alpha \notin S_*(a)$, we can define $\boldsymbol{\tau}_\alpha(a)$ arbitrarily. $\square$

We have thus defined a sequence of tuples of names $\langle \boldsymbol{\tau}_\alpha \mid \alpha < \kappa \rangle$ in $M^{[\mathbb{B}]}$. We aim to prove that

$$p(x) = \left\{ \varphi_\alpha\big(x, [\boldsymbol{\tau}_\alpha]_U\big) \;\middle|\; \alpha < \kappa \right\}$$

is a type in $\mathfrak{M}^{[\mathbb{B}]}/U$. To do so, we shall show that for each $S \in [\kappa]^{<\aleph_0}$

$$\left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right]\!\!\right] \wedge \bigwedge_{\alpha \in S} \big(f(\{\alpha\}) \wedge x_\alpha\big) = f(S) \wedge \bigwedge_{\alpha \in S} x_\alpha \in U \quad (3.8)$$

and then conclude using Theorem 1.3.6. First of all, note that both sides of (3.8) are based on W, due to our choice of W in Claim 2. Hence,

$$\begin{aligned}
&\left[\!\!\left[\exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right]\!\!\right] \wedge \bigwedge_{\alpha \in S} \big(f(\{\alpha\}) \wedge x_\alpha\big) \\
&= \bigvee \left\{ a \in W \;\middle|\; \mathfrak{M} \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha(a)) \right\} \wedge \bigwedge_{\alpha \in S} \big(f(\{\alpha\}) \wedge x_\alpha\big) \\
&= \bigvee \left\{ a \wedge \bigwedge_{\alpha \in S} \big(f(\{\alpha\}) \wedge x_\alpha\big) \;\middle|\; a \in W,\ \mathfrak{M} \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha(a)) \right\} \\
&= \bigvee \left\{ a \in W \;\middle|\; a \leq \bigwedge_{\alpha \in S} \big(f(\{\alpha\}) \wedge x_\alpha\big),\ \mathfrak{M} \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha(a)) \right\} \\
&= \bigvee \left\{ a \in W \;\middle|\; S \subseteq S_*(a),\ \mathfrak{M} \models \exists x \bigwedge_{\alpha \in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha(a)) \right\} \\
&= \bigvee \left\{ a \in W \;\middle|\; a \leq f(S) \wedge \bigwedge_{\alpha \in S} x_\alpha \right\} = f(S) \wedge \bigwedge_{\alpha \in S} x_\alpha
\end{aligned}$$

thus showing, in particular, that $p(x)$ is finitely satisfiable, hence a type in $\mathfrak{M}^{[\mathbb{B}]}/U$.

Since we are assuming that U κ^+-saturates T, let $\tau \in M^{[\mathbb{B}]}$ be a name such that $[\tau]_U$ realizes $p(x)$ in $\mathfrak{M}^{[\mathbb{B}]}/U$. We define a function

$g\colon [\kappa]^{<\aleph_0} \to U$ as follows: for $S \in [\kappa]^{<\aleph_0}$,

$$g(S) = \bigwedge_{\alpha\in S} \big([\![\varphi_\alpha(\tau, \boldsymbol{\tau}_\alpha)]\!] \wedge f(\{\alpha\}) \wedge x_\alpha \big).$$

Then clearly g is multiplicative and, for every $S \in [\kappa]^{<\aleph_0}$, we may apply (3.8) to obtain

$$\begin{aligned} g(S) = \left[\!\!\left[\bigwedge_{\alpha\in S} \varphi_\alpha(\tau, \boldsymbol{\tau}_\alpha) \right]\!\!\right] \wedge \bigwedge_{\alpha\in S} \big(f(\{\alpha\}) \wedge x_\alpha \big) \\ \leq \left[\!\!\left[\exists x \bigwedge_{\alpha\in S} \varphi_\alpha(x, \boldsymbol{\tau}_\alpha) \right]\!\!\right] \wedge \bigwedge_{\alpha\in S} \big(f(\{\alpha\}) \wedge x_\alpha \big) \leq f(S). \end{aligned}$$

This completes the proof that U is $\langle \kappa, \mathbb{B}, T\rangle$-moral. □

We now move on to present our main result in this section, which follows from Theorem 3.5.5 and Malliaris and Shelah's technique of separation of variables.

Theorem 3.5.6. *Let κ be an infinite cardinal and T_0, T_1 complete countable theories. Then the following are equivalent:*

- *$T_0 \trianglelefteq_\kappa T_1$;*
- *for every κ^+-c.c. complete Boolean algebra $\mathbb{B}$ of cardinality $\leq 2^\kappa$, and every κ-regular ultrafilter U on $\mathbb{B}$, if U κ^+-saturates T_1 then U κ^+-saturates T_0.*

Proof. Suppose that $T_0 \trianglelefteq_\kappa T_1$. Let $\mathbb{B}$ be a κ^+-c.c. complete Boolean algebra with $|\mathbb{B}| \leq 2^\kappa$, and let U be a κ-regular ultrafilter on $\mathbb{B}$ which κ^+-saturates T_1. By Theorem 3.5.5, we know that U is $\langle \kappa, \mathbb{B}, T_1\rangle$-moral.

By Theorem 3.4.4, there exists a surjective homomorphism $j\colon \mathcal{P}(\kappa) \to \mathbb{B}$ such that $j^{-1}[\{\mathbb{1}\}]$ is a κ-regular κ^+-good filter over κ. Therefore, $j^{-1}[U]$ is a κ-regular ultrafilter over κ, which κ^+-saturates T_1 by Theorem 3.5.3. But $T_0 \trianglelefteq_\kappa T_1$, therefore $j^{-1}[U]$ also κ^+-saturates T_0. By Theorem 3.5.3 again, we deduce that U is $\langle \kappa, \mathbb{B}, T_0\rangle$-moral, and finally we conclude that U is κ^+-saturates T_0 by Theorem 3.5.5.

For the reverse implication, it is sufficient to observe that $\mathcal{P}(\kappa)$ is a κ^+-c.c. complete Boolean algebra of cardinality $\leq 2^\kappa$. □

Working independently, Ulrich [69] has obtained another formulation of Keisler's order using Boolean-valued models. Compared to our Theorem 3.5.6, his characterization holds for all ultrafilters on κ^+-c.c. complete Boolean algebras $\mathbb{B}$, without the assumption $|\mathbb{B}| \leq 2^\kappa$. The following question, however, remains open.

Question 3.5.7. Does the equivalence of Theorem 3.5.6 still hold without the κ^+-c.c. assumption on $\mathbb{B}$?

Remark 3.5.8. The characterization of Theorem 3.5.6 suggests a fruitful way of showing that a theory T_0 is *not* less than a theory T_1 in Keisler's order. Indeed, to do so it is now sufficient to construct, for some cardinal κ, a κ-regular ultrafilter on a suitable Boolean algebra, which κ^+-saturates T_1 but does not κ^+-saturate T_0. The advantage of this approach is clear: complete Boolean algebras in general have a much richer combinatorial structure than power-set algebras, allowing us to build ultrafilters with a greater degree of accuracy. We expect to exploit this advantage in the future.

We recall that, by Proposition 1.3.13, the Boolean ultrapower is a special case of a more general construction, namely the limit ultrapower. Therefore, it is natural to ask the following question: can the characterization of Theorem 3.5.6 be further extended to encompass limit ultrapowers? As we discuss below, the answer is negative.

In 1972, Shelah introduced the following generalization of Keisler's order using limit ultrapowers.

Definition 3.5.9 (Shelah [59]). Let κ be a cardinal and let T_0 and T_1 be complete countable theories. We say that $T_0 \trianglelefteq^*_\kappa T_1$ if and only if: for every set I, every $\aleph_1$-incomplete ultrafilter U over I, every filter F over $I \times I$, and $(\kappa^+ + |I|^+)$-saturated models $\mathfrak{M}$, $\mathfrak{N}$ of T_0, T_1 respectively,

$$\mathfrak{N}^I_U|F \text{ is } \kappa^+\text{-saturated} \implies \mathfrak{M}^I_U|F \text{ is } \kappa^+\text{-saturated}.$$

Furthermore, we define

$$T_0 \trianglelefteq^* T_1 \overset{\text{def}}{\iff} \forall\kappa(T_0 \trianglelefteq^*_\kappa T_1).$$

We already know from Theorem 3.1.4 that Keisler's order on stable theories has exactly two equivalence classes. However, the order $\trianglelefteq^*$ exhibits a different behaviour, as the next result shows.

Theorem 3.5.10 (Shelah [58]). *The order $\trianglelefteq^*$ on stable theories has exactly four equivalence classes, including incomparable classes.*

In conclusion, the use of Boolean ultrapowers is optimal in the sense that it yields the same classification of theories as Keisler's order, but allows for more flexibility in the construction of ultrafilters. On the other hand, limit ultrapowers do give a different classification, already at the level of stable theories.

References

[1] B. Balcar and F. Franek. 'Independent families in complete Boolean algebras'. In: *Transactions of the American Mathematical Society* 274.2 (Dec. 1982), pp. 607–618.

[2] B. Balcar and P. Simon. 'Strong decomposability of ultrafilters I'. In: *Logic Colloquium '80*. Ed. by D. V. Dalen, D. Lascar and T. J. Smiley. Studies in Logic and the Foundations of Mathematics 108. North-Holland, 1982, pp. 1–10.

[3] M. Benda. 'Note on Boolean ultrapowers'. In: *Proceedings of the American Mathematical Society* 46.2 (Nov. 1974), pp. 289–293.

[4] M. Benda and J. Ketonen. 'Regularity of ultrafilters'. In: *Israel Journal of Mathematics* 17.3 (Sept. 1974), pp. 231–240.

[5] A. R. Blass. 'Orderings of ultrafilters'. PhD thesis. Harvard University, May 1970.

[6] J. A. Brown and N. Dobrinen. 'Spectra of Tukey types of ultrafilters on Boolean algebras'. In: *Algebra Universalis* 75.4 (July 2016), pp. 419–438.

[7] C. C. Chang and H. J. Keisler. *Model Theory*. Third Edition. Studies in Logic and the Foundations of Mathematics 73. North-Holland, 1990.

[8] W. W. Comfort and S. Negrepontis. *Chain conditions in topology*. Cambridge tracts in mathematics 79. Cambridge University Press, 1982.

[9] N. Dobrinen. 'Complete embeddings of the Cohen algebra into three families of c.c.c., non-measurable Boolean algebras'. In: *Pacific Journal of Mathematics* 214.2 (Apr. 2004), pp. 223–244.

[10] N. Dobrinen. 'Survey on the Tukey theory of ultrafilters'. In: *Zbornik Radova* 17(25) (2015), pp. 53–80.

[11] H.-D. Donder. 'Regularity of ultrafilters and the core model'. In: *Israel Journal of Mathematics* 63.3 (Oct. 1988), pp. 289–322.

[12] A. Dow. 'Good and OK ultrafilters'. In: *Transactions of the American Mathematical Society* 290.1 (July 1985), pp. 145–160.

[13] P. Erdős and A. Tarski. 'On families of mutually exclusive sets'. In: *Annals of Mathematics* 44.2 (Apr. 1943), pp. 315–329.

[14] A. L. Foster. 'Generalized "Boolean" theory of universal algebras. Part I. Subdirect sums and normal representation theorem'. In: *Mathematische Zeitschrift* 58 (Dec. 1953), pp. 306–336.

[15] T. Frayne, A. C. Morel and D. S. Scott. 'Reduced direct products'. In: *Fundamenta Mathematicae* 51.3 (1962), pp. 195–228.

[16] H. Gaifman. 'Infinite Boolean polynomials I'. In: *Fundamenta Mathematicae* 54.3 (1964), pp. 229–250.

[17] F. Galvin and A. Hajnal. 'On the relative strength of chain conditions'. Unpublished. 1976.

[18] M. Goldstern, J. Kellner and S. Shelah. 'Cichoń's maximum'. In: *Annals of Mathematics* 190.1 (July 2019), pp. 113–143.

[19] A. W. Hales. 'On the non-existence of free complete Boolean algebras'. In: *Fundamenta Mathematicae* 54.1 (1964), pp. 45–66.

[20] J. D. Hamkins and D. E. Seabold. 'Well-founded Boolean ultrapowers as large cardinal embeddings'. In: *ArXiv e-prints* (June 2012), pp. 1–40. arXiv: `1206.6075 [math.LO]`.

[21] F. Hausdorff. 'Über zwei Sätze von G. Fichtenholz und L. Kantorovitch'. In: *Studia Mathematica* 6 (1936), pp. 18–19.

[22] A. Horn and A. Tarski. 'Measures in Boolean algebras'. In: *Transactions of the American Mathematical Society* 64.3 (Nov. 1948), pp. 467–497.

[23] M. Huberich. 'Non-regular ultrafilters'. In: *Israel Journal of Mathematics* 87 (Feb. 1994), pp. 275–288.

[24] J. R. Isbell. 'The category of cofinal types. II'. In: *Transactions of the American Mathematical Society* 116 (Apr. 1965), pp. 394–416.

[25] R. Jin and S. Shelah. 'Possible size of an ultrapower of ω'. In: *Archive for Mathematical Logic* 38.1 (Jan. 1999), pp. 61–77.

[26] P. Jipsen, A. Pinus and H. Rose. 'Rudin-Keisler Posets of Complete Boolean Algebras'. In: *Mathematical Logic Quarterly* 47.4 (Nov. 2001), pp. 447–454.

[27] H. J. Keisler. 'On the class of limit ultrapowers of a relational system'. In: *Notices of the American Mathematical Society* 7.7 (Dec. 1960), pp. 878–879.

[28] H. J. Keisler. 'Limit Ultrapowers'. In: *Transactions of the American Mathematical Society* 107.3 (June 1963), pp. 382–408.

[29] H. J. Keisler. 'Good Ideals in Fields of Sets'. In: *Annals of Mathematics* 79.2 (Mar. 1964), pp. 338–359.

[30] H. J. Keisler. 'On cardinalities of ultraproducts'. In: *Bulletin of the American Mathematical Society* 70.4 (July 1964), pp. 644–647.

[31] H. J. Keisler. 'Ultraproducts and Saturated Models'. In: *Indagationes Mathematicae (Proceedings).* A 67 (1964), pp. 178–186.

[32] H. J. Keisler. Mimeographed Lecture Notes. University of California, Los Angeles. 1967.

[33] H. J. Keisler. 'Ultraproducts which are not saturated'. In: *The Journal of Symbolic Logic* 32.1 (Mar. 1967), pp. 23–46.

[34] B. Koppelberg. 'Ultrapowers and Boolean ultrapowers of ω and ω_1'. In: *Archiv für mathematische Logik und Grundlagenforschung* 20.3-4 (Sept. 1980), pp. 147–153.

[35] B. Koppelberg and S. Koppelberg. 'A Boolean ultrapower which is not an ultrapower'. In: *The Journal of Symbolic Logic* 41.1 (Mar. 1976), pp. 245–249.

[36] K. Kunen. 'Ultrafilters and independent sets'. In: *Transactions of the American Mathematical Society* 172 (Oct. 1972), pp. 299–306.

[37] K. Kunen. 'Weak P-Points in N^*'. In: *Topology*. Vol. II. Colloquia Mathematica Societatis János Bolyai 23. Budapest (Hungary), 1980, pp. 741–749.

[38] J. Łoś. 'Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres'. In: *Mathematical Interpretation of Formal Systems*. Ed. by T. Skolem et al. Studies in Logic and the Foundations of Mathematics 16. Amsterdam: North-Holland, 1955, pp. 98–113.

[39] H. M. MacNeille. 'Partially ordered sets'. In: *Transactions of the American Mathematical Society* 42.3 (Nov. 1937), pp. 416–460.

[40] M. Magidor. 'On the existence of nonregular ultrafilters and the cardinality of ultrapowers'. In: *Transactions of the American Mathematical Society* 249.1 (Apr. 1979), pp. 97–111.

[41] M. E. Malliaris. 'Hypergraph sequences as a tool for saturation of ultrapowers'. In: *The Journal of Symbolic Logic* 77.1 (Mar. 2012), pp. 195–223.

[42] M. Malliaris and S. Shelah. 'A dividing line within simple unstable theories'. In: *Advances in Mathematics* 249 (Dec. 2013), pp. 250–288.

[43] M. Malliaris and S. Shelah. 'Constructing regular ultrafilters from a model-theoretic point of view'. In: *Transactions of the American Mathematical Society* 367.11 (Nov. 2015), pp. 8139–8173.

[44] M. Malliaris and S. Shelah. 'Cofinality spectrum theorems in model theory, set theory, and general topology'. In: *Journal of the American Mathematical Society* 29.1 (Jan. 2016), pp. 237–297.

[45] M. Malliaris and S. Shelah. 'Keisler's order has infinitely many classes'. In: *Israel Journal of Mathematics* 224 (Apr. 2018), pp. 189–230.

[46] R. Mansfield. 'The theory of Boolean ultrapowers'. In: *Annals of Mathematical Logic* 2.3 (Jan. 1971), pp. 297–323.

[47] E. Marczewski. 'Séparabilité et multiplication cartésienne des espaces topologiques'. In: *Fundamenta Mathematicae* 34 (1947), pp. 127–143.

[48] J. D. Monk, ed. *Handbook of Boolean Algebras*. In collab. with R. Bonnet. North-Holland, 1989.

[49] E. H. Moore and H. L. Smith. 'A general theory of limits'. In: *American Journal of Mathematics* 44.2 (Apr. 1922), pp. 102–121.

[50] M. Morley and R. Vaught. 'Homogeneous universal models'. In: *Mathematica Scandinavica* 11 (1962), pp. 37–57.

[51] L. Pacholski. 'Homogeneity, universality and saturatedness of limit reduced powers III'. In: *Fundamenta Mathematicae* 95.2 (1977), pp. 85–94.

[52] F. Parente. 'Boolean valued models, saturation, forcing axioms'. Master's thesis. Università di Pisa, July 2015.

[53] R. S. Pierce. 'Distributivity in Boolean algebras'. In: *Pacific Journal of Mathematics* 7.1 (Jan. 1957), pp. 983–992.

[54] K. Prikry. 'On descendingly complete ultrafilters'. In: *Cambridge Summer School in Mathematical Logic*. Ed. by A. R. D. Mathias and H. Rogers. Lecture Notes in Mathematics 337. Springer-Verlag, 1973, pp. 459–488.

[55] D. Raghavan and S. Shelah. 'Boolean ultrapowers and iterated forcing'. In preparation.

[56] M. E. Rudin. 'Types of Ultrafilters'. In: *Topology Seminar, Wisconsin, 1965*. Annals of Mathematics Studies 60. Princeton University Press, 1966, pp. 147–151.

[57] S. Shelah. 'Stable theories'. In: *Israel Journal of Mathematics* 7.3 (Sept. 1969), pp. 187–202.

[58] S. Shelah. 'Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory'. In: *Annals of Mathematical Logic* 3.3 (Oct. 1971), pp. 271–362.

[59] S. Shelah. 'Saturation of ultrapowers and Keisler's order'. In: *Annals of Mathematical Logic* 4.1 (Mar. 1972), pp. 75–114.

[60] S. Shelah. *Classification theory and the number of non-isomorphic models*. Revised edition. Studies in Logic and the Foundations of Mathematics 92. North-Holland, 1990.

[61] S. Shelah. 'Atomic saturation of reduced powers'. In: *ArXiv e-prints* (Jan. 2016), pp. 1–38. arXiv: 1601.04824 [math.LO].

[62] R. Sikorski. 'A theorem on extension of homomorphisms'. In: *Annales de la Société Polonaise de Mathématique* 21 (1948), pp. 332–335.

[63] R. Sikorski. 'Cartesian Products of Boolean Algebras'. In: *Fundamenta Mathematicae* 37 (1950), pp. 25–54.

[64] E. C. Smith Jr. and A. Tarski. 'Higher degrees of distributivity and completeness in Boolean algebras'. In: *Transactions of the American Mathematical Society* 84.1 (Jan. 1957), pp. 230–257.

[65] M. H. Stone. 'The theory of representations for Boolean algebras'. In: *Transactions of the American Mathematical Society* 40.1 (July 1936), pp. 37–111.

[66] M. H. Stone. 'Applications of the theory of Boolean rings to general topology'. In: *Transactions of the American Mathematical Society* 41.3 (May 1937), pp. 375–481.

[67] A. Tarski. 'Über additive und multiplikative Mengenkörper und Mengenfunktionen'. In: *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Classe III* 30 (1937), pp. 151–181.

[68] J. W. Tukey. *Convergence and uniformity in topology*. Annals of Mathematics Studies 2. Princeton University Press, 1940.

[69] D. Ulrich. 'Keisler's Order and Full Boolean-Valued Models'. In: *ArXiv e-prints* (Oct. 2018), pp. 1–32. arXiv: 1810.05335 [math.LO].

[70] D. Ulrich. 'Cardinal Characteristics of Models of Set Theory'. In: *ArXiv e-prints* (Nov. 2018), pp. 1–35. arXiv: 1811.05444 [math.LO].

[71] B. Węglorz. 'Homogeneity, universality and saturatedness of limit reduced powers (II)'. In: *Fundamenta Mathematicae* 94.1 (1977), pp. 59–64.

[72] J. Wierzejewski. 'Homogeneity, universality and saturatedness of limit reduced powers I'. In: *Fundamenta Mathematicae* 94.1 (1977), pp. 35–39.