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MATHEMATICAL TASKS AND THE STUDENT

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Mathematics Education has at its core a conception of the mathematical performances that represent the aspirations of the mathematics classroom and curriculum. These performances are constituted through teacher and student participation in the activities stimulated by mathematical tasks selected by the teacher for the realization of an instructional purpose. In this nexus of activity, intention, interpretation and consequence, the mathematical task occupies a central place. This Research Forum provides an opportunity to explore and reflect upon the role that mathematical tasks play in the achievement of the goals of the international mathematics education community. Further, consistent with current curricular and theoretical priorities, the agency, attributes and activities of the student are foregrounded in the discussion of the instructional use of mathematical tasks. The contributors to this Research Forum represent a wide variety of theoretical perspectives and report research undertaken in different school systems and different cultures. These different perspectives offer a useful exploration of the theme: Mathematical Tasks and the Student.

RATIONALE

Attempts to model the complexity of the mathematics classroom have generated increased interest in theories capable of accommodating consideration of artifacts[1] as well as individuals. Theories such as Activity Theory (Engeström, 1987) and Distributed Cognition (Hutchins, 1995) foreground the mediational role of artifacts in facilitating learning, and locate tasks among those mediating artifacts.

> Mediating artifacts might be mathematics textbooks, digital technologies, as well as tasks and problems, [and] language. (Rezat & Strässer, 2012)

Rezat and Strässer (2012) identify the students' mathematics-related activity as an example of the Vygotskian conception of an instrumental act, where the student's interaction with mathematics is mediated by artifacts, such as mathematical tasks. Most importantly, recognizing the function of mathematical tasks as tools for the facilitation of student learning leads us to the further recognition that (à la Vygotsky) the use of a tool (i.e. a task) fundamentally affects the nature of the facilitated activity

[1] Either artifact or artefact are acceptable spellings to denote "arte factum" (Latin) as something made through the use of skill. We have employed Rezat and Strässer's (2012) spelling in this proposal, which also corresponds to North American usage.

(i.e. student learning). Rezat and Strässer (2012) have re-conceptualized the familiar didactical triangle (teacher-student-mathematics) as a socio-didactical tetrahedron, where the vertices are teacher, student, mathematics and mediating artifacts. This reconception of didactical relationships recognizes that the connections represented by the sides of the original didactical triangle require mediation. The vehicles of this mediation are artifacts, which include everything from textbooks and IT tools to tasks and language. Use of the socio-didactical tetrahedron provides us with an important tool by which to give recognition to the mediational role of tasks in the teaching and learning of mathematics.

One virtue of the socio-didactical tetrahedron is that it facilitates the separate consideration of the triangles forming each face of the tetrahedron and the vertices of each of those triangles. In this Research Forum, we focus attention on the task as mediating artifact and address the question of how the resultant socio-didactical tetrahedron (Fig. 1) might structure our consideration of research into the function of tasks in facilitating student learning and into the dynamic between student and task.

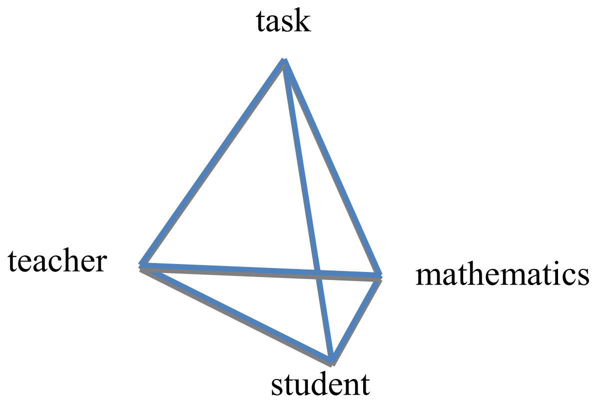

Figure 1: The socio-didactical tetrahedron (Rezat & Strässer, 2012)

To paraphrase Rezat and Strässer (2012, p. 645): Each of the triangular faces of the tetrahedron stands for a particular perspective on the role of tasks within mathematics education: the didactical role of the teacher is best described as an orchestrator of student mathematical activity as represented by the triangle *teacher-task-student* (Face A); the triangle *student-task-mathematics* represents the student's task-mediated activity of learning mathematics (Face B); the triangle *teacher-task-mathematics* depicts the teacher's task-mediated activity of representing mathematics in an instructional setting (Face C); the original didactical triangle constitutes the base of the model (i.e. *student-teacher-mathematics*) (Face D). The tetrahedral structure offers an important representation of the complexity of classroom teaching/learning that affords a level of detailed reflection on the didactical role of tasks. In utilizing this more complex conception of the instructional use of mathematical tasks, significant agency is accorded to *each* component (student, teacher, mathematics and task) in the determination of the actions and outcomes that find their nexus in the social situation for which the task provides the pretext.

Research into the design and use of mathematical tasks in instructional settings must accommodate student intentions, actions and interpretations to at least the same extent as those of the teacher. Research in this area is important, but fragmented. This Research Forum brings a variety of research studies together into a discussion intended to yield a more coherent picture and has been designed to assist in structuring the field of task-related research and to equip researchers to better situate the student within research on instructional task design.

Goals framing the Research Forum:

(i) To present research into the instructional use of mathematical tasks, with a specific focus on the associated student activity and the implications for task design, classroom practice and the mathematics curriculum internationally;
(ii) To focus attention specifically on the agency of the student during the completion of mathematical tasks in educational settings and examine the performative expression of this agency in different settings and in response to different task types;
(iii) To highlight, through the reporting of selected research studies, particular issues associated with the instructional use of mathematical tasks, including: teacher intentionality, student interpretation, implicit and actual task contexts, considerations of task sequence, and the distinction between the stated task and its realization as a social activity involving teacher and students;
(iv) To bring together researchers from a variety of countries, who share an interest in both the instructional use of mathematical tasks and the intended and resultant student activity;
(v) To draw to the attention of PME members some of the issues associated with the instructional use of mathematical tasks, particularly those arising from the assumptions implicit in different instructional theories, which may conceive the instructional purposes of mathematical tasks and optimal student activity very differently.

The Research Forum has been structured around the following issues:

(i) Differences in the instructional deployment and function of mathematical tasks and the nature of student task participation in different instructional settings;
(ii) Utilizing mathematical tasks to promote higher order thinking skills;
(iii) Differences in the theoretical frameworks by which the instructional use of mathematical tasks might be better understood (particularly from the perspective of the student) and thereby optimized;
(iv) The accommodation of student agency within the instructional use of mathematical tasks.

Each issue can be usefully addressed in the form of a question.

Focus Question 1. What are the possible functions of a mathematical task in different instructional settings and how do these functions prescribe the nature of student task participation?

Focus Question 2. What contingencies affect the effectiveness of a mathematical task as a tool for promoting student higher order thinking skills?

Focus Question 3. How might we best theorize and research the learning processes and outcomes arising from the instructional use of any mathematical task or sequence of tasks from the perspective of the student?

Focus Question 4. What differences exist in the degree of agency accorded to students in the completion of different mathematical tasks and with what consequences?

The sequencing of the forum contributions constitutes a research narrative aligned with the issues listed above and structured by the socio-didactical tetrahedron already discussed. It is the construction of structure within substantial research diversity that provides a key motivation for this Research Forum.

ISSUE ONE: DIFFERENCES IN THE FUNCTION OF MATHEMATICAL TASKS AND THE NATURE OF STUDENT TASK PARTICIPATION IN DIFFERENT INSTRUCTIONAL SETTINGS

1: MAKING DISTINCTIONS IN TASK DESIGN AND STUDENT ACTIVITY

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The design principles below have developed during the time of our collaboration, over a period of fifteen years (e.g., Brown & Coles, 1997). The principles are drawn both from the enactivist theory of cognition and learning (Varela, Thompson, & Rosch, 1991) and the pedagogic ideas of Gattegno (1987). We developed these principles within a community centred around one school (School S) in the Bristol area of the UK. Laurinda made this school her main research site and visited, where possible, weekly. We focus on one particular community in the spirit of 'particularization' (Krainer, 2011, p. 52), to draw out general principles from an in-depth study of one case. Our data comes from transcripts of video recordings of lessons as well as the scheme of work of School S.

We believe task design that centres around activities that provoke differences in student response can allow the opportunity for students to make mathematical distinctions and for teachers to introduce new skills. Our task design principles are:

- starting with a closed activity (which may involve teaching a new skill).
- considering at least two contrasting examples (where possible, images) and collecting responses on a 'common board'.
- asking students to comment on what is the same or different about contrasting examples and/or to pose questions.

- having an open-ended challenge prepared in case no questions are forthcoming.
- introducing language and notation arising from student distinctions.
- opportunities for students to spot patterns, make conjectures and work on proving them (hence involving generalising and algebra).
- opportunities for the teacher to teach further new skills and for students to practice skills in different contexts.

Our data analysis indicates these design principles operate to inform: (1) teacher planning, (2) teaching actions in the classroom and (3) students' mathematical activity. Firstly, the principles inform teacher planning. For example, the offer of contrasting examples (principle 2) can be used to focus students on mathematical distinctions, from which questions and challenges can be generated that provoke further work with that distinction. Secondly, we have evidence from video recordings that, over time, our design principles inform teacher actions in the classroom. In particular, the principles seemed to support teachers in School S adapting tasks in the light of student responses. Thirdly, there is evidence from transcripts that the principles can inform (implicitly) student actions in the mathematics classroom; through making distinctions, students notice and extend patterns, they ask questions and generalize (principle 6).

There is a significant problem, identified in the literature, around the student experience of tasks compared to the intentions of the designer or teacher (Watson & Mason, 2007). Mason, Graham and Johnston-Wilder (2005, p. 131) raise the issue of how an expert's awarenesses get translated into instructions for the learner that do not lead to those same awarenesses.

Our results indicate that the making of distinctions within mathematics can become a habit and a normal way of engaging in tasks for students. Creating opportunities for students to make distinctions within mathematics can also become a habit for teachers and a normal way of both planning activity and informing decisions in the classroom. When this happens, there is a convergence of planned and actual activity. With a focus on distinctions, there is a potential route out of the problems highlighted by Mason et al. (2005) around the divergence of teacher intention and student activity. With a focus on distinctions, the expert (teacher) can plan, initially via the choice of examples, to support students in making the same distinctions as a mathematician, leading to the same awarenesses.

2: ORDER OF TASKS IN SEQUENCES OF EARLY ALGEBRA[2]

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Rationale

It is assumed that epistemic and cognitive aspects are fundamental to build sequences of tasks. We investigated different aspects that appear when we analyse the process as a teaching experiment and examined how teacher intentions evolved according to interactional and ecological suitability.

Our research focused on student-related aspects influencing the ordering of tasks and how student responses are accommodated, using the case of early algebra. It is well known that structured investigative activities provide opportunities for meaningful learning of mathematical concepts. We consider task design as a crucial element of the learning environment, and describe a teaching experiment in which class discussion introduces unexpected new perspectives to an initial a priori instructional scheme. Our perspective relates to Realistic Mathematics Education, where the designer conducts anticipatory thought experiments by envisioning both how proposed instructional activities might be realized in the classroom, and what students might learn as they engage in them.

Framework and Methodology

It is important for our design process, a task analysis, to identify difficulty factors providing frameworks for hypothetical designs inspired initially by developmental cognition according to levels of abstraction. We decided to choose an early algebra task as the basis for a situated study supporting the perspective in which algebraic reasoning could be strongly promoted as a tool intertwined with arithmetic building through their interconnection in order to promote success by developing both arithmetic and algebra together, one implicated in the development of the other (Smith, 2011). The study supporting this paper has been done with two classes of 8-9 years old students. The basis for building our sequence of tasks and test analysis was to promote algebraic thinking by overcoming relational apprehension and the use of patterns in connection with a search for order or structure. Therefore regularity, repetition and symmetry are frequently present because of their relevance to the development of abstraction, generalization and the establishment of relations. Next step concerns the experimental task design process based upon a refined sequence of tasks. The principles for our task design are the following: (1) ensure the possibility of using arithmetic number sense related to algebraic reasoning; (2) apply suitability criteria for analysing mathematical activities; (3) use mathematical examples, using relations and diversity of representations but not letters for the unknowns; (4) prioritise the voice of the students for analyzing and promoting mathematisation and retention. The tasks

[2] Work partially funded by Ministry of Economy & Competitivity of Spain. EDU2012-32644.

were meant to be diverse, some leading to an exploratory and investigative open activity to improve meaningful construction. In our study, we considered one class solving 6 sequential tasks and then six structural tasks and another class solving six structural tasks and then six sequential tasks (Palhares, Giménez, & Vieira, 2013). A typical sequential task would ask the student to “Observe carefully the sequence of numbers: 6, 10, 14, 18, 22, . . . What will be the 20th term of the sequence? . . . Explain how you found the 20th term of the sequence. Will the number 63 be part of this sequence of numbers? Justify your answer.” A typical structural task would ask students to “Observe carefully the four ‘number machines’ (shown below). Replace the question mark with a number that follows the rule of the other three machines.”

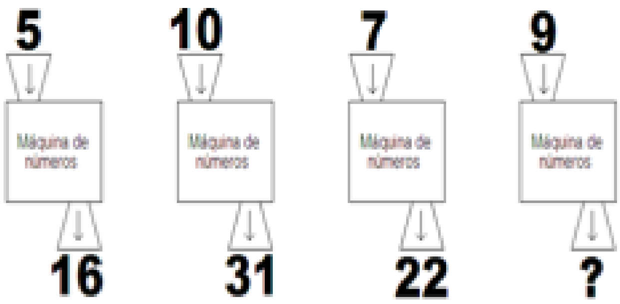

Figure 1

The research design focused directly on the consequences of task sequence.

Results and Final Comments

Statistical results show that there are significant differences starting with sequential or with structural tasks. Sequential tasks are better for starters and apparently provide a solid foundation for the work with structural tasks. The study is a first step for reconsidering the tasks for the next redesign stage in which a new cycle of testing could lead to small or big changes in task sequence. It is clear that students who started with the sequential tasks seemed to be capable of establishing broad generalizations, when the other group could not. These findings argue for redesigning in terms of stability and improving connectivity in self-regulation processes as synthesis activities. Also, the group that started with sequential tasks appeared to retain their performance more robustly as stable across time. The experiment did not consider any modelling situations from the real world. We assume that this would improve and enrich not only structural, but sequential examples in providing students with new learning experiences.

3: TASKS TO PROMOTE HOLISTIC FLEXIBLE REASONING ABOUT SIMPLE ADDITIVE STRUCTURES

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Our team is conducting a 3-year research project funded by the Quebec Ministry of Education on additive problem solving in early grades of elementary school. The goals of the project are: 1) to develop a pedagogical approach that would promote holistic and flexible reasoning about simple additive structures; 2) to design and test a set of tasks and didactical scenarios that implements the new approach; 3) to propose a related teacher professional development program. Our research team consists of two researchers (Savard and Freiman), a designer (Polotskaia), and a school board consultant responsible for the teachers' professional development (Gervais). We want to support teachers to guide their students on solving additive structures problems.

There are two paradigms in which additive problem solving can be seen. The *Operational Paradigm* puts the focus on addition and subtraction as arithmetic operations. From this position, additive word problems can be seen as exercises where the knowledge about arithmetic operations can be applied or further developed. Contemporary research (Thevenot, 2010) shows that some problems are particularly difficult because they require a flexible and holistic analysis of their mathematical structure while easy problems do not require such analysis.

The *Relational Paradigm*, appears in the work of Davydov (1982) and more recent studies (Iannece, Mellone, & Tortora, 2009). According to Davydov (1982), the concept of additive relationship is, "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229). Davydov (1982) advanced the premise that an adequate understanding of the additive relationship is the basis for the learning of addition and subtraction and should be taught prior to calculation. The analysis of the additive relationships present in the situation yields the following task design principles:

1. The task should be based on a situation involving a simple additive relationship between three quantities.
2. The task should involve students in the mathematical analysis of the described relationship as a whole. It should help students to discover different properties of the relationship, and to see how different arithmetic operations can be used in the described situation for different purposes.
3. The task should use a socio-cultural context in which students can identify themselves as active agents.
4. The task should not contain any explicit and immediate questions that could be answered by finding one particular number. This criterion is to prevent students from immediately calculating the answer. However, the task should include an

intriguing element, which would support students' natural interest and commitment.

5. The goal of the task, which is learning to analyze the situation, should be explicitly communicated to students.
6. The text of the task should be very short and should contain simple words and expressions that the students are familiar with.
7. The mathematical discussion of the situation should integrate appropriate graphical representations as a method of analysis.

We provide here one example of the task that we named 360° *situation* to highlight the main goal – holistic analysis of the mathematical structure of the situation. This is an example of a text proposed to students.

> Peter, Gabriel and Daniel are playing marbles. Peter says, "I have 5 marbles." Gabriel says, "I have 8 marbles." Daniel says, "Peter has 4 marbles less than Gabriel".

We introduce this text as a strange situation or as a situation where one of the persons made a mistake. Students are invited to explain why the text is unrealistic and how it can be corrected considering different quantities involved. The objective of the first is to make explicit the fact that all three quantities are related to each other and that the choice of two values implies one (and only one) third value. At the next step, we invite students to construct a graphical representation, which can support discovering of the appropriate arithmetic operations. Each quantity should be evaluated to figure out a correct numeric value in the condition where the other two quantities are fixed. At this step, the formal use of arithmetic operations can be discussed. Finally, the numbers in the text can be replaced with different ones to further generalise the initially discussed quantitative relations. This will complete the 360° tour around the situation.

The teachers we worked with had a tendency to return to the traditional teaching behaviours as soon as they start to work with traditional problems. For example, once the numerical answer was found for the problem, the discussion of the problem often ended abruptly. Thus, the focus of the activity was often shifted towards the use of the correct representation or the calculation of the numerical answer. A one year follow-up provided for each teacher-participant was needed for a sustainable change in teaching habits.

ISSUE TWO: UTILIZING MATHEMATICAL TASKS TO PROMOTE STUDENTS' HIGHER ORDER THINKING SKILLS

4: HYBRID TASKS: PROMOTING STUDENT STATISTICAL THINKING AND CRITICAL THINKING THROUGH THE SAME MATHEMATICAL ACTIVITIES

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In a well-known definition of Statistical Literacy by Gal (2004), a "critical stance" is included among the key attitudes for successful statistical thinking (ST) – hence, Gal includes such attitudes in his definition of statistical literacy. However, being critical in statistical contexts is not only an attitude. It is possible to describe specific abilities that have to be used in order to critically evaluate statistical data. Two key concepts or overarching ideas in statistical thinking relevant for a critical evaluation of data are *manipulation of data by reduction and dealing with statistical variation*.

Critical thinking (CT) skills rely on self-regulation of the thinking processes, construction of meaning, and detection of patterns in supposedly disorganized structures (Ennis, 1989). Critical thinking tends to be complex and requires the use of multiple, sometimes mutually contradictory criteria, and frequently concludes with uncertainty. This description of CT already suggests links with ST, such as dealing with uncertainty, contradictions and a critical evaluation of given claims. Dealing critically with information – a crucial aspect for both domains – demands critical/evaluative thinking based on rational thinking processes and decisions (Aizikovitsh-Udi, 2012).

In order to explore thinking processes related to tasks in the domains of both Statistical Thinking and Critical Thinking, individual semi-structured interviews were conducted with mathematics teachers. By using mathematics teachers as subjects, basic content competence can be assumed and it becomes possible to examine their content-related higher order thinking skills, both in terms of statistical thinking and critical thinking. The interviews focused on thinking-aloud when solving tasks and each lasted about 40–50 minutes. Figure 1 shows a sample task.

Looking at both CT and ST, the interviews appeared to highlight how elements of CT can contribute to ST, for example when evaluating data, its presentation and analysis, planning data collection, etc. Conversely, aspects of ST like dealing with statistical variation and uncertainty were shown to contribute to CT, especially when it comes to decisions in non-determinist situations, where full data is unavailable. This study has shown that both ST and CT skills can be evoked by the same task. We suggest that this models authentic and useful thinking practice more effectively than a more closed task that stimulated only statistical thinking and the application of taught procedures.

Connections clearly exist between Statistical Thinking and Critical Thinking at the level of individual reasoning practices. We suggest that an instructional program of hybrid tasks could provide the opportunity to employ Statistical Thinking, while simultaneously introducing students to the practices and structure of Critical Thinking.

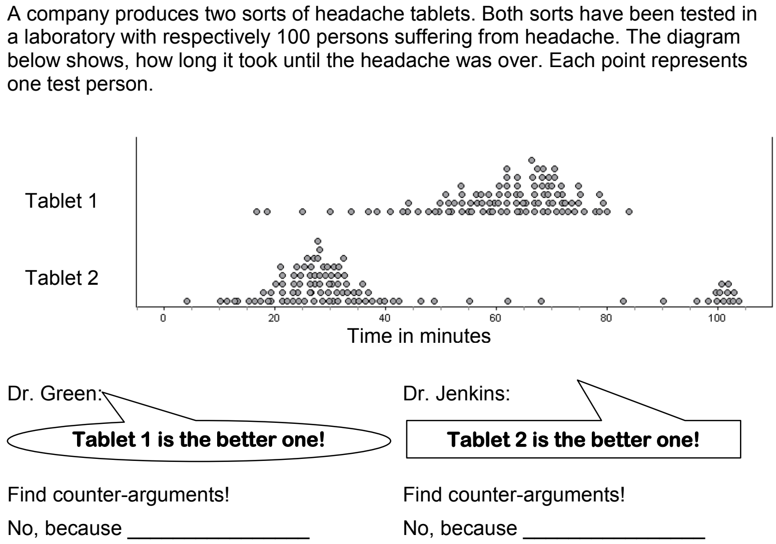

Figure 1: Task “tablets” (Kuntze, Lindmeier, & Reiss, 2008)

5: DESIGNING COVARIATION TASKS TO SUPPORT STUDENTS’ REASONING ABOUT QUANTITIES INVOLVED IN RATE OF CHANGE

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Researchers using mathematical tasks involving dynamic representations of covarying quantities have supported secondary students’ forming and interpreting relationships between changing quantities (e.g., Johnson, 2012; Saldanha & Thompson, 1998). Taking into account students’ emergent conceptions of rates of change, the design of this covariation task sequence provided opportunities for students to use non-numerical quantitative reasoning in situations involving constant and varying rates of change. By covariation tasks, I mean tasks that involve forming and interpreting relationships between changing quantities.

Adapting the bottle problem to design covariation tasks

I designed covaration tasks by adapting Thompson, Byerly, and Hatfield’s (2013) version of the well-known bottle problem (see Figure 1).

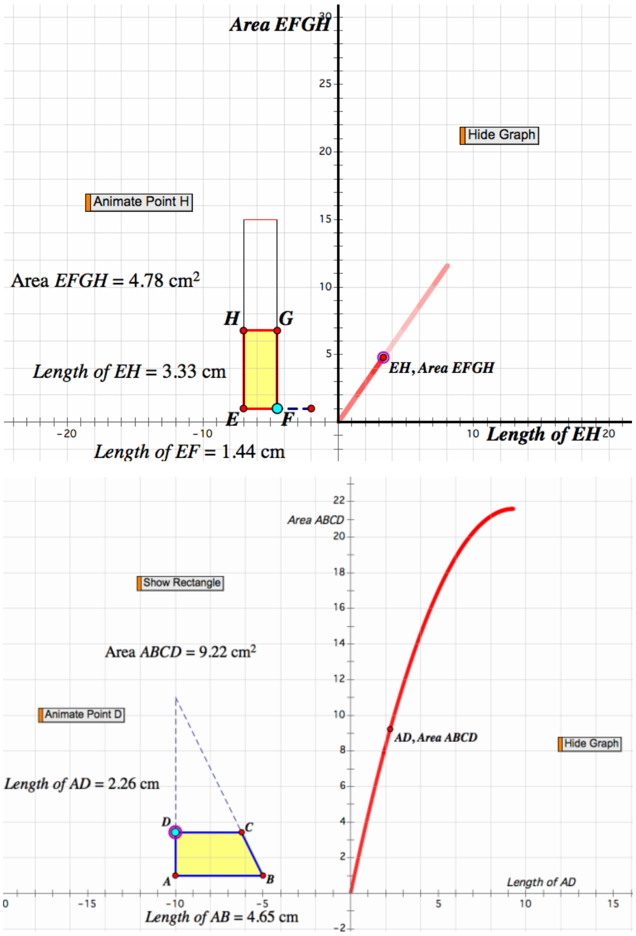

Figure 1: Filling Rectangle and Filling Triangle Sketches

The adaptation for middle school students resulted in a sequence of tasks. To accompany each task, I developed dynamic sketches linking a rectangle or right triangle “filling” with area to a graph representing shaded (“filled”) area as a function of height (Figure 1). Students could vary the height of the rectangle or triangle by animating or dragging points H (Figure 1, top) or D (Figure 1, bottom), respectively, then predict and create a corresponding graph representing shaded area as a function of height. Additionally, students could drag point F (Figure1, top) to vary the width of the rectangle. Anticipating that students might interpret linked graphs iconically (Leinhardt, Zaslavsky, & Stein, 1990), in particular that graphs would represent pictures of filling rectangles or triangles, I chose to represent the height of the shaded region on the horizontal rather than the vertical axis. By affording students’ manipulation of dynamically linked representations, the dynamic sketches provided opportunities for students to form and interpret relationships between quantities.

Task design principles

In designing the task sequence, I provided students with opportunities to demonstrate that they conceived of rate of change as some attribute of a situation that could be measured. In the case of the filling rectangle and triangle situations, such a conception of rate of change could entail a student being able to envision the filling area as increasing in relationship to another changing quantity.

To investigate how students might conceive of rate of change in the context of a filling rectangle or triangle situation, I began by asking students what changed and what stayed the same. This prompt provided students the opportunity to identify different attributes of the situation that could be measured. Once students demonstrated evidence of attending to a rate of change as something that could be measured in the context of the situation, I provided students with representations of constituent quantities (e.g., a graph representing area as a function of height) that could be used to quantify the measurable attribute students had just described.

Task implementation results

Students reasoning about area as a result of a numerical calculation interpreted variable increase as if it were constant. These students made sense of unfamiliar graphs by connecting shapes of objects to shapes of graphs such that rectangles elicit one type of graph and triangles elicit another type. Students' work suggests that iconic interpretations of graphs extend to dynamic graphs such that dynamic graphs are pictures in motion. Students reasoning about area as a measurable attribute of a rectangle or triangle attended to variable increase in area when interpreting and/or predicting features of a graph relating area and side length. These students attended to variation in amounts of change in area, identified sections with different kinds of increases in area, and described variation in how area could increase as side length continually changed. Students attending to variable increase in area also interpreted dynamic sketches and graphs as relationships between quantities.

Concluding remarks

Using non-numerical quantitative reasoning, students can make predictions and create representations indicating how quantities might change together. Although representations included in the tasks explicitly indicate quantities of area and height, students may interpret the graphs shown in Fig. 1 as representing a relationship between area and elapsing time rather than area and height. The possibility for such interpretation highlights the complexity of designing tasks to provide students with opportunities to engage in rate-related reasoning. Future iterations of implementation and analysis could provide further explanation as to how students' non-numerical reasoning develops when constructing relationships between quantities.

ISSUE THREE: THEORETICAL FRAMEWORKS BY WHICH STUDENT PARTICIPATION IN MATHEMATICAL TASKS MIGHT BE BETTER UNDERSTOOD AND OPTIMIZED

6: APPLYING THE PHENOMENOGRAPHIC APPROACH TO STUDENTS' CONCEPTIONS OF TASKS

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Tasks serve a communicative purpose between teacher and student, by conveying the teacher's intent for learning and the student's conception of that intent. Often, responses or work produced by students from a task reveal a disconnect between the teacher's learning expectation and the true depth of knowledge attained by the student. By applying the descriptions of an outcomes space from a phenomenographic inquiry to student work samples, I will discuss how this approach informs a framework for connecting a student's conception of learning to the quality of the individual's task engagement.

Phenomenography is a research methodology with its own theoretical framework that accounts for the qualitatively different ways people experience learning. From this theoretical stance, the impact a task has on learning may be analysed using the outcome space of student conceptions about the learning. By analysing a student's conception of, and approach to learning, the relationship between focal awareness and task performance is further documented. The analysis is guided by the question: "What do students focus on when assigned a task, and in what way does the work produce communicate to the teacher the student's personal epistemology of the content to be learned?"

Learning is defined as perceiving, conceptualizing, or understanding something in a new way by discerning it from and relating it to a context. Furthermore, learning involves two aspects: i) what is to be learned, and ii) how one goes about learning (Marton & Booth, 1997). The learner's perspective of what is to be learned is derived from the student's definition of the direct object of learning. How the learner assigns meaning to the learning object is determined by the learning strategies the student utilizes to meet personal learning goals.

To maintain consistency with the phenomenographic definition of learning, a task is characterized by its relationship to the structural and referential aspects of the learning experience. A task is a situation requiring the learner to experience the object of learning in such a way that the learner must discern components of the situation and how they are related (structural aspect), then assign a meaning to the situation (referential aspect). The task analysed in the study assessed student understanding of descriptive statistics and data analysis.

Since the student's conception is the unit of analysis, an explanation of what a student is attentive to when engaged in completing a task is warranted. The basic components of awareness are appresentation, discernment, and simultaneity (Marton & Booth, 1997). *Appresentation* refers to being conscious of a perceptual or sensual experience in the presence of concrete or abstract entities; *discernment* involves recognizing a foreground-background structure of a situation; *simultaneity* means knowing how the discerned parts are related to the whole structure. The structure of a student's focal awareness directly informs the way the student understands content, which leads the student to perceive that something has been learned.

Collectively, the various levels of student performance in the class fell into the first three conceptions of the learning of statistics outcome space. The majority of the students met the level of knowledge attainment deemed acceptable to teacher. This finding supports the proposition that the meaning and purpose a student assigns to a task seem to be aligned with the student's meaning of learning, approaches to learning, and capabilities sought as a result of learning.

7: THE MILIEU AND THE MATHEMATICAL KNOWLEDGE AIMED AT IN A TASK

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Context and theoretical background

The research question addressed in the paper is: How does the *milieu* devolved to the students for algebraic generalisation of shape patterns influence their mathematical activity? A gap between the teacher's intention with a task and the students' mathematical activity is explained in terms of a lacking coordination between the knowledge aimed at (an equivalence statement) and the milieu (Brousseau, 1997) devolved to the students.

Participants in the reported research are two groups of three student teachers enrolled at a teacher education programme for primary and lower secondary school in Norway, and a teacher educator who teaches mathematics to these students. The data are a mathematical task and transcripts from video-recorded small-group sessions where the students engage with the task. The theory of didactical situations in mathematics (Brousseau, 1997) has been used to analyse the empirical material.

A shape pattern in elementary algebra is usually instantiated by some consecutive geometric configurations in an alignment imagined as continuing until infinity. According to Måsøval (2011), there are two types of shape patterns: *arbitrary patterns* (Figure 1), *and conjectural patterns* (Figure 2).

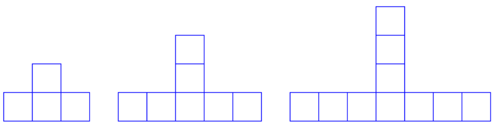

Figure 1: An arbitrary pattern

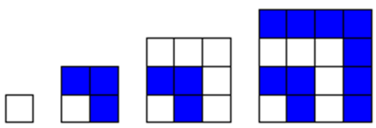

Figure 2: A conjectural pattern

These patterns correspond respectively to two different mathematical objects aimed at in the process of generalising (Måsøval, 2011): *formula* (for the general member of the sequence mapped from the shape pattern; e.g., $a_n = 3n+1$ in Figure 1), and *theorem* (a general numerical statement; e.g., $1+3+5+\cdots \qquad 1 = n^2$ in Figure 2).

A priori analysis: the milieu

The pattern in Task 3 (with which the students engaged) is intended to be a conjectural pattern, aiming at the formulation of a theorem. It is made of a first milieu (Shape pattern 1, in Figure 3) that evolves (Shape pattern 2 with white squares, in Figure 4).

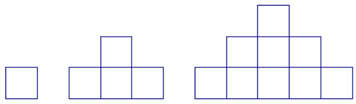

Figure 3: Shape pattern 1

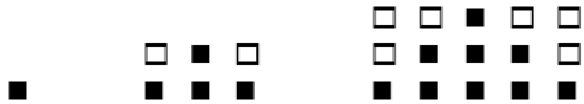

Figure 4: Shape pattern 2

For the teacher, the role of Shape pattern 1 is to provide students with the elements to formulate the theorem "the sum of the first *n* odd numbers is equal to the square of *n*", first in words and then algebraically: $1+3+5+\cdots \qquad 1 = n^2$. It is important to notice that the solution of the problem (proof of the theorem) can be reached without the algebraic formulation by direct manipulation the elements of the pattern. A generic example of this manipulation (made by me) is shown in Figure 5.

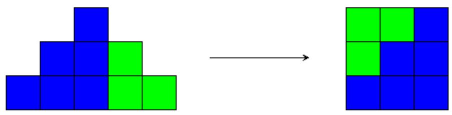

Figure 5: The third element manipulated into a 3x3 square

An alternative shape pattern that would illustrate that the *n*-th square number is equivalent to the sum of the first *n* odd numbers is the pattern shown in Figure 2 above (where the relationship is visualised directly). The pattern would then play the role of a "real milieu" in the sense of Brousseau (1997).

Because of that, the algebraic formulation $1+3+5+\cdots \qquad 1) = n^2$ does not appear as a necessary tool to construct the proof of the theorem; it is just a way to formulate a mathematical statement with symbols. In this respect, the pattern is a real milieu when it is considered as a geometrical representation of an arithmetical sequence, in that the elements of the pattern can be represented arithmetically $(1=1^2,\ 1+3=2^2,\ 1+3+5=3^2,\ \text{etc.})$ and serve as a "model" that can guide a process of algebraic thinking that aims at the equivalence statement $1+3+5+\cdots+ \quad -1 = n^2$. Here, the elements of the pattern serve as referents for first arithmetic and then algebraic symbols, the algebraic formulation being here only a tool to state the equivalence.

Results from the analysis of the transcript data show that: 1) The students produce adequate solutions to subtasks, but this does not constitute a milieu for the formulation of the mathematical statement aimed at. This is consistent with the *a priori* analysis presented above. 2) There is a weakness in the milieu caused by missing clarification of the concept of mathematical statement.

Task 3 is focused on calculations (how many), but the intended knowledge is *theoretical*. Hence the focus should be on *why* the sum of the first n odd numbers is equal to the square of n. This question has potential to create the need to use algebra.

ISSUE FOUR: ACCOMMODATING STUDENT RESPONSES AND STUDENT AGENCY WITHIN THE INSTRUCTIONAL USE OF MATHEMATICAL TASKS

8: WRITING THE STUDENT INTO THE TASK: AGENCY AND VOICE

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The classroom performance of a task is ultimately a unique synthesis of task, teacher, students and situation. Task selection by teachers initiates an instructional process that includes task enactment (collaboratively by teacher and student) and the interpretation of the consequences of this enactment (again, by teacher and student). In undertaking this study, we examined the function of mathematical tasks in classrooms in five countries. A three-camera method of video data generation (see Clarke, 2006), was supplemented by post-lesson video-stimulated reconstructive interviews with teacher and students, and by teacher questionnaires and copies of student work. Our analysis characterized the tasks employed in each classroom with respect to intention, action and interpretation and related the instructional purpose that guided the teacher's task selection and use to student interpretation and action, and, ultimately, to the learning that post-lesson interviews encouraged us to associate with each task.

The eighth-grade mathematics classrooms that provided the sites for our analysis were drawn from the data set generated by the Learner's Perspective Study (LPS) (Clarke, 2006). Our initial goal in the analysis of mathematical tasks undertaken in these classrooms was the selection of tasks that could legitimately be described as distinctive because of the character of the mathematical activity or because of the teachers' didactical moves in utilising the tasks to facilitate student learning.

The tasks were selected for their disparity across the key attributes: mathematics invoked (both content category and level of sophistication); figurative context (real-world or decontextualised); resources utilised in task completion (diagrams and other representations); and the nature of the role of the task participants. Two examples are noteworthy:

Japan School 3 – Lesson 1 (the Long Task)

In this task, the seemingly simple pair of simultaneous equations $5x + 2y = 9$ and $-5x + 3y = 1$ engaged the class for a fifty-minute lesson (and indeed was the discussion point for the first fifteen minutes of the following lesson). A feature of the performance of this task was the extent to which student suggestions, responses and the articulation of their thinking were regarded as instruments for developing understanding.

Shanghai School 3 – Lesson 7 (the Train Task)

In relation to mathematical tasks, Clarke and Helme (1998) distinguished the social context in which the task is undertaken from any 'figurative context' that might be an element of the way the task is posed. In this sense, the task:

> Siu Ming's family intends to travel to Beijing by train during the national holiday, so they have booked three adult tickets and one student ticket, totalling \$560. After hearing this, Siu Ming's classmate Siu Wong would like to go to Beijing with them. As a result they buy three adult tickets and two student tickets for a total of \$640. Can you calculate the cost of each adult and student ticket?

has a figurative context that integrates elements such as the family's need to travel by train and the familiar difference in cost between an adult and a student ticket. The social context, however, could take a wide variety of forms, including: an exploratory instructional activity undertaken in small collaborative groups; the focus of a whole class discussion, orchestrated by the teacher to draw out existing student understandings; or, an assessment task to be undertaken individually. In each case, the manner in which the task will be performed is likely to be quite different, even though we can conceive of the same student as participant in each setting.

Students were given a significant "voice" in the completion of each task, but the nature of their participation reflected differences in the extent and character of the distribution of responsibility for knowledge constructed in the course of task completion. This distribution of responsibility (or enhanced agency) is a consequence of each teacher's strategic decision, moment by moment, of how best to orchestrate student work on the task. We see task performance as the iterative culmination in the joint construction, not only of the task solution, but of the mathematical principles of which the task is model and purveyor.

Concluding Remarks

Of particular interest in our analysis were differences in the function of mathematically similar tasks, dealing with similar mathematical content (those relating to systems of linear equations), when employed by different teachers, in different classrooms, for different instructional purposes, with different students. The "entry point" for our analysis was a tabulation of the details related to the social performance of the task. Using these tables, our analysis drew on the video-stimulated, post-lesson interview data to identify intention and interpretation and relate both to social performance of the task.

The significance of differences between social, cultural and curricular settings, together with differences between participating classroom communities, challenges any reductionist attempts to characterize instructional tasks independent of these considerations. The attention given by competent teachers to student voice and student agency, and the mathematical tasks that they employ to catalyse that voice and agency, support our belief that the maximization of student agency and voice in the performative enactment of a mathematical task should be recognized as a key principle of task design and delivery.

9: EMERGENT TASKS: SPONTANEOUS DESIGN SUPPORTING IN-DEPTH LEARNING

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According to Bruder (2000), a task can be regarded as a triplet of an *initial state*, a *final state* and a *transformation* that transforms the initial state into a final one. Even adaptive mathematical tasks such as self-differentiating tasks designed before the lesson can only support optimal learning if the teacher also is able to spontaneously transform the situation into a fruitful epistemic process (Prediger & Scherres, 2012). How can such transformations be achieved? This question is addressed by the concept of *emergent tasks*. Emergent tasks are ad-hoc tasks created by the teacher when the teacher conceives the mathematical potential of a learning opportunity and translates it into a task, so that

- the students' interest present in the situation is taken up and
- acute mathematical problems and questions are addressed adaptively.

Our investigation of emergent tasks aims at elucidating how the gap between the students' epistemic needs and the affordances of a task can be bridged.

In order to identify emergent tasks in empirical situations, four types of tasks are distinguished (see Vogt, 2012, p. 35):

Task type	Students express interest	The teacher formulates an adaptive task for a situation
prepared task	-	-
spontaneous task	-	yes
missed emergent task	yes	-
emergent task	yes	yes

Table 1: Types of tasks

A *prepared task* is constructed before the lesson, it may or may not be adaptive or meet the students' interests. A *spontaneous task* is acutely created by the teacher in order to support a specific learning situation, it is not a requirement that it meets the students' interest. However, if a student shows interest in a problem but the teacher does not take this opportunity up to transform the situation into a suitable learning opportunity the teacher has missed setting an emergent task, in such a case we observe a *missed emergent task*.

Emergent tasks often appear when the initial and/or the final state of a problem are not clear to the students. If a student expresses epistemic interest for clarification, the teacher may translate this task into a more adaptive one, thus creating an emergent task. The students' may also *explicitly* express a different epistemic need, in this case the teacher has the chance to set an adaptive, hence, emergent task. If the students' epistemic need is *implicit*, the teacher may act in a sensitive way for instance by prompts ("please tell us what you mean") to make the student's problem visible and then formulate an emergent task. In addition, we found *emergent tasks that unveiled an epistemic gap* that initially remained unnoticed by the students.

Our investigations of emergent tasks has yielded two results: (1) An emergent task has the tendency to initiate further emergent tasks leading to a sequence of fruitful learning opportunities that sometimes shape more than one lesson; (2) based on an initial emergent task we gained five design principles for building a task sequence on learning a procedure: *emergent task* (the teacher is reacting to a student's interest), *presenting and questioning* (the students' solutions of the task are presented and questioned), *using and checking* (an interesting student solution is used and checked by the other students), *expanded use and application* (the potential in use is evaluated by an expanded task), and *institutionalization* ((individual) textualization of the procedure).

On the part of the teacher our studies point to the following conditions that enable the teacher to perform appropriate translations of learning situations into emergent tasks: "The teacher must

- have mathematical knowledge that extends the content of the lesson,
- show interest in the students' learning processes,
- and be open for unusual ways on the part of the students. She or he must be willing to abstain from the planned course" (cf. Bikner-Ahsbahs & Janßen, 2013, p. 162).

MATHEMATICAL TASKS AND THE STUDENT – MOVING FORWARD

The didactical relationship between the student as learner of mathematics and the mathematical task as facilitating that learning

The research reports present complementary perspectives on the student-task relationship and demonstrate just how diverse are the considerations affecting the instructional deployment of tasks and their role in facilitating student participation in particular types of mathematical activity Furthermore, considerable diversity is evident in the descriptions of the positioning of students within that mathematical activity, particularly with respect to the agency afforded to students to determine the nature of their participation. The socio-didactical tetrahedron provides a reflective structure within which to discuss the various research reports.

Teacher-student-task (*Face A*): In the mathematics classroom, the teacher, the student and the tasks provide the key structural elements through which the classroom's social activity is constituted. There has long been a tacit assumption that the completion of mathematical tasks chosen or designed by the teacher will result in the student learning the intended mathematics. This view is persistent despite research that suggests this is not a direct relationship (Margolinas, 2004, 2005).

Student-task-mathematics (*Face B*): For some time, theories of learning have viewed cognitive activity as not simply occurring in a social context, but as being constituted in and by social interaction (e.g., Hutchins, 1995). From this perspective, the activity that arises as a consequence of a student's completion of a task is itself a constituent element of the learning process and the artifacts (both conceptual and physical) employed in the completion of the task serve simultaneous purposes as scaffolds for cognition, repositories of distributed cognition and as cognitive products.

Teacher-task-mathematics (*Face C*): Task development, selection and sequencing by teachers represents the initiation of an instructional process that includes task performance (collaboratively by teacher and student) and the interpretation of the consequences of this enactment (again, by teacher and student).

Teacher-mathematics-student (*Face D – base*): The original didactical triangle has the virtue of connecting the classroom participants with the knowledge domain that provides the pretext for their interaction. As noted, however, the connections represented by the sides of the original didactical triangle require mediation by artifacts; in this case, tasks. The theory of didactical situations (Brousseau, 1997) provides a conceptualisation of the didactic relationship between the teacher, the mathematics and the student. Here, the mathematical task is part of the *milieu*, which models the elements of the material and intellectual reality on which the students act.

One of the dangers for both research and instructional design lies in the disconnection of the elements of the socio-didactical tetrahedron for separate, typically pairwise, study. For example, analysis of student response to a particular task independent of the instructional/learning context in which the task is encountered could understate the complexity of the activity under investigation by backgrounding considerations central to task completion, such as teacher intention, student interpretation, and curricular and organisational context. During the process of task completion, the effectiveness of the task in promoting learning will also be contingent on student intention (with respect to the task) and teacher interpretation (with respect to the students' activity). These socio-mathematical considerations are central to any attempt to understand (and thereby optimize) the function of tasks in catalyzing student mathematical activity and consequent learning in institutionalized settings such as mathematics classrooms.

Some of these considerations can be summarised in the form of questions:

- What problem does the student think she is solving?
- What student-related factors determine the optimal selection and sequencing of tasks for instructional purposes?
- What are the student-related considerations affecting the use of mathematical tasks to promote students' higher order thinking skills?
- What contribution does the student make to the performative shaping of the task and how is this contribution accommodated within available theoretical frameworks?
- What degree of agency can the student realistically be afforded in the framing and performance of a mathematical task, if the teacher's instructional agenda is to be achieved?

These questions have been addressed to varying degrees in the papers that comprise this Research Forum. It is useful to review some of the key points made by each contribution.

A recurrent theme in the framing of this Research Forum was the tension between the teacher's instructional intentions and consequent student activity. Coles and Brown suggest that an emphasis on making distinctions foregrounds the targeted mathematical awarenesses that are otherwise only indirectly prompted by instruction based on different principles. This reduces the possibility of divergence of teacher intention and student activity by actively stimulating those student capabilities directly. Giménez, Palhares and Vieira investigated the role of task order in promoting algebraic thinking, by making comparison between instruction that commenced with sequential or structural tasks. This sensitivity to task sequence rather than simply to the quality or effectiveness of the individual tasks per se, introduces an additional consideration to the question of how best to utilise tasks to promote student learning. Savard, Polotskaia, Freiman and Gervais examined the contemporary premise that some problems (or tasks) are particularly difficult because they require a flexible and holistic analysis of their mathematical structure while easy problems do not require such analysis. The emphasis on the capacity of tasks to facilitate student consideration of mathematical relationships rather than simply mathematical operations introduces additional considerations in the design of instructional tasks.

In combination, these three studies usefully demonstrate the diversity of considerations invoked by the different aspirations pertaining to specific organisational and curricular settings. The interplay of these considerations can be seen in the significance of the students' response to a task and the sensitivity of that response to task characteristics, including task order. This interplay is most evident in the implicit compromise between prescription and devolution, undertaken in order to provide opportunities for the expression of student agency, while still holding out some hope that student activity and learning might resemble the teacher's instructional intentions.

The papers by Aizikovitsh-Udi et al. and Johnson identify some of the challenges faced by task designers hoping to elicit something more sophisticated than the replication of a taught procedure. The dynamic between promoting the development of mathematics-specific skills and modes of thought and meeting the more encompassing aims of contemporary curricula is presented as potentially a productive symbiosis by Aizikovitsh-Udi and her co-authors. Johnson's investigation of mathematical tasks involving dynamic representations of covarying quantities necessarily also documents student hypothesis formulation and associated mathematical reasoning. The capacity of her tasks to frame, shape and facilitate sophisticated student reasoning mirrors the capacity of the hybrid tasks of Aizikovitsh-Udi et al. to simultaneously stimulate statistical and critical thinking. Given the aspirations of contemporary curricula towards promoting higher order thinking skills, these two papers provide cause for optimism.

Our use of the socio-didactical tetrahedron to frame this Research Forum has already placed a Vygotskian slant on our conception of the process of mathematics learning and the role of instructional tasks in facilitating that learning process. Without wishing to be theoretically exclusive, we would argue that recognizing the function of mathematical tasks as tools for the facilitation of student learning leads us to the further useful recognition that the use of a tool (i.e. a task) fundamentally affects the nature of the facilitated activity (i.e. student learning). This does not preclude the use of other theoretical perspectives in the analysis and optimisation of task use in instruction. Phenomenographic approaches, as illustrated by Gardner, precisely capture the reflexive connection between the teacher's use of tasks and the students' conceptions of those tasks. The prioritisation of student perception of the object of learning aligns Gardner's perspective with aspects of the paper by Coles and Brown. However, Gardner adds a layer of sophistication in her consideration of the student's perception of and response to a given task as the social enactment of the student's conception of learning. This perspective accords a level of significance to student intellectual agency that both complicates and enhances our consideration of the student-task axis and its significance within the socio-didactical tetrahedron. The paper by Strømskag draws together several considerations: the tension between intention and activity, and the role of the task in creating a mileu (Brousseau, 1997) conducive to the promotion and use of the targeted mathematical knowledge. The conditions governing the teacher's capacity to orchestrate the creation of a milieu suitable for the development of the targeted mathematical knowledge are a direct consequence of the choice of instructional task.

The research narrative concludes by directing attention to student agency. Examination by Mesiti and Clarke of task functionality through the lens of international comparison highlights differences in instructional purpose and curricular context, which shape the particular activity arising from the instructional use of a task in differently situated classrooms. In the paper by Bikner-Ahsbahs, tasks encompass initial and final states [of knowing] and their connecting transformation. Emergent tasks appear, fractal-like, where the learning situation requires the revision, refinement, or elaboration of the intended task, including the insertion into the lesson of an entirely unintended task, called upon in response to the demands of the particular didactical situation. In an interesting way, emergent tasks embody the teacher's pedagogical agency through their incarnation of the teacher's response to an instructional situation not anticipated in the lesson's original planning. The implication is that teacher agency is best expressed in reflexive relation to student agency, but also in the provision of opportunities for the expression of that student agency. This recognition returns us to the assertion by Mesiti and Clarke that "the classroom performance of a task is ultimately a unique synthesis of task, teacher, students and situation" and reinvokes the socio-didactical tetrahedron.

As a final recapitulation: There is a tension between the teacher's instructional intentions (and associated actions) and the students' consequent activity (and ultimate learning). This tension is probably inevitable and even productive. The existence of this tension should reassure us that student agency has not been precluded entirely from our classrooms.

Equally, the tension is not one of opposition, but rather the recognition of the need for continual mutual adjustment. Both teacher and students are complicit in the construction of classroom practice; if the teacher appears to exert the greater control through task selection, the students can, by their responses, significantly determine the nature of consequent classroom activity. Within this process of incremental and iterative adjustment, the task serves as the frame for activity, while the activity constitutes the performance of the task.

In the preceding discussion and the research narrative constituted through the various research reports, we have attempted to examine the instructional use of mathematical tasks, the roles played by students in the performance of those tasks, and the anticipation of those roles by teachers and task designers. The results of several of these analyses have been interpreted as indicating principles for instructional (task) design. Tasks and their social performance provide both a window into the practices of mathematics classrooms internationally and the means to realise our curricular ambitions.

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