12-2001

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Recommended Citation
Ellermeyer, S. F. & Robinson, D. G. (2001). Integrals of periodic functions. Mathematics Magazine, 74(5), 393-396.

Proof Without Words: The Weierstrass Substitution

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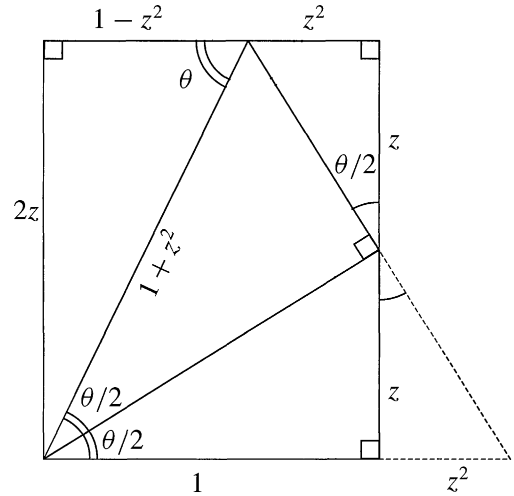

$$z = \tan\left(\frac{\theta}{2}\right) \quad \Rightarrow \quad \sin(\theta) = \frac{2z}{1+z^2}, \quad \cos(\theta) = \frac{1-z^2}{1+z^2}$$

Integrals of Periodic Functions

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Computing integrals of powers of the sine function is a standard exercise in calculus. Using integration by parts or some basic trigonometric identities, the student discovers that

$$\int \sin^2 t\, dt = -\frac{1}{2}\cos t \sin t + \frac{1}{2}t + C \tag{1}$$

and

$$\int \sin^3 t\, dt = -\frac{1}{3}\sin^2 t \cos t - \frac{2}{3}\cos t + C. \tag{2}$$

Further investigation reveals that all even powers of sine have integrals containing periodic terms and a (nontrivial) linear term, whereas all odd powers of sine have integrals with only periodic terms.

In this note, we show that the first integral is representative of the integral of any periodic function. Although this fact (Proposition 1) is not pointed out in any of the calculus textbooks we have studied, we feel that it is elementary enough and of sufficient usefulness in applications to be given attention in an elementary calculus course. As we show by examples, Proposition 1 can come in handy not only in knowing what to expect when we integrate a periodic function, but also in enabling us to introduce some qualitative aspects of differential equations (the existence of periodic solutions) at the elementary calculus level.

Throughout, when we say that f *is periodic of period* $T > 0$, we mean that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(t+T) = f(t)$ for all t, but not necessarily that T is the smallest positive period possessed by f. If f has period T, then we define the *average value of* f in the usual way, as

$$\overline{f} = \frac{1}{T} \int_t^{t+T} f(s)\, ds.$$

Since f has period T, the value of $\overline{f}$ does not depend on the choice of $t \in \mathbb{R}$.

PROPOSITION 1. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic of period $T > 0$, then*

$$\int f(t)\, dt = g(t) + \overline{f}t + C, \tag{3}$$

where g is a periodic function of period T.

To prove the proposition, we let

$$g(t) = \int_0^t f(s)\, ds - \overline{f}t.$$

Then

$$\frac{d}{dt}\left(g(t) + \overline{f}t\right) = f(t),$$

which implies (3). Also, for any $t \in \mathbb{R}$, we have

$$\begin{aligned} g(t+T) &= \int_0^{t+T} f(s)\, ds - \overline{f}T - \overline{f}t \\ &= \int_0^{t+T} f(s)\, ds - \int_t^{t+T} f(s)\, ds - \overline{f}t \\ &= g(t), \end{aligned}$$

which shows that g is periodic of period T. ■

Since Proposition 1 is qualitative (rather than quantitative) in nature, it should be expected that applications of the Proposition will yield mainly qualitative information.

Example 1 (Integrals of Powers of Sine): By Proposition 1, if n is a positive integer, then

$$\int \sin^n t\, dt = g(t) + mt + C,$$

where g is periodic of period 2π and m is the average value of $f(t) = \sin^n t$. Since even powers of sine have positive average value and odd powers of sine have zero average value, the constant m that appears in the linear term will be positive when n is even and zero when n is odd. This agrees with the results shown in (1) and (2).

Example 2 (The Harmonic Oscillator): The motion of a unit mass attached to a spring is described by the differential equation

$$y'' + ay' + by = 0, \tag{4}$$

where $y(t)$ is the position of the spring at time t, $-by$ is the force exerted on the mass by the spring, and $-ay'$ the damping force exerted on the mass by the medium through which it is moving. (No other force, including gravity, is assumed to be present.) In elementary differential equations courses, the model (4) is often used to motivate the study of linear differential equations and linear systems. The usual approach first handles the case $a = 0$ (the undamped case), showing that the mass exhibits sustained oscillations about its rest position (assuming also that $b > 0$). Once this is done, it is shown that when $a > 0$, the mass eventually comes to rest (possibly via decaying oscillations) due to the damping force. In the setting of a differential equations course, the fact that the real parts of the roots of the characteristic equation of (4) are negative shows that the mass approaches equilibrium when $a > 0$. In what follows, we use Proposition 1 to explain why equation (4) cannot have nontrivial periodic solutions if $a \neq 0$. Our argument does not delve into the specific forms of solutions of (4) and hence does not yield the quantitative information that would follow from a more detailed analysis, but our approach does reveal the qualitative effect of a damping force.

If y is a periodic solution of (4) with period $T > 0$, then y' is also periodic of period T. Multiplying both sides of (4) by y', we obtain

$$y'y'' + a\left(y'\right)^2 + byy' = 0.$$

Integrating this from time 0 to any positive time t gives

$$\left(y'(t)\right)^2 - \left(y'(0)\right)^2 + 2a\int_0^t \left(y'(s)\right)^2 ds + b\,(y(t))^2 - b\,(y(0))^2 = 0.$$

Since $(y')^2$ is periodic of period T, Proposition 1 implies

$$\int_0^t \left(y'(s)\right)^2 ds = g(t) - g(0) + mt,$$

where g is periodic of period T and m is the average value of $(y')^2$. This gives us

$$\left(y'(t)\right)^2 - \left(y'(0)\right)^2 + 2ag(t) - 2ag(0) + b\,(y(t))^2 - b\,(y(0))^2 = -2amt. \tag{5}$$

Since the left-hand side of (5) is periodic of period T, then so is $-2amt$. Of course, this is only possible if either $a = 0$ or $m = 0$. If a isn't zero, then m is, and in fact y' is identically zero (because m is the average value of $(y')^2$, which is nonnegative). This in turn implies that y is constant (i.e., y is trivially periodic of period T). But then (4) reduces to $by = 0$. Hence, if we assume in addition that $b \neq 0$ (which would be reasonable in a spring problem), then it follows that $y(t) = 0$ is the only periodic solution of (4).

Finally, if $a = 0$, $b \neq 0$, and y is a nontrivial periodic solution of (4), then y cannot be of constant sign for all t (because integration of equation (4) from 0 to T would yield a contradiction). Hence, we may assume without loss of generality that $y(0) > 0$

is the maximum value of y. Equation (5) then gives us

$$(y(t))^2 = (y(0))^2 - \frac{1}{b}\left(y'(t)\right)^2.$$

If $b < 0$, then we would have $(y(t))^2 > 0$ for all t, which would mean that y cannot change sign. We conclude that the differential equation (4) has nontrivial periodic solutions only if $a = 0$ and $b > 0$.

The same approach can be extended to the study of similar nonlinear differential equations such as the unforced *Duffing's equation*,

$$y'' + ay' + by + cy^3 = 0,$$

which is usually considered in more advanced differential equations courses. (See, for example, Hale [**2**, p. 168].)

Example 3 (Linear Systems with Constant Coefficients): We conclude by using Proposition 1 to show that if $abcd < 0$, then the linear system of differential equations

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{6}$$

has no nontrivial periodic solutions. Conditions that determine the qualitative nature of solutions of (6) are usually given in differential equations courses in terms of the trace $(a + d)$ and the determinant $(ad - bc)$ of the coefficient matrix of (6) [**1**, p. 312]. These are obtained by analyzing the characteristic equation of the system. As a contrast, we argue as we did for the harmonic oscillator.

If we assume that $abcd < 0$ and that (x, y) is a periodic solution of (6) of period T, then we can write

$$\begin{aligned} cxx' &= acx^2 + bcxy \\ -byy' &= -bcxy - bdy^2 \end{aligned}$$

to obtain

$$cxx' - byy' = acx^2 - bdy^2.$$

Integration of both sides of the above equation from 0 to t gives

$$c\,(x(t))^2 - c\,(x(0))^2 - b\,(y(t))^2 + b\,(y(0))^2 = g(t) - g(0) + mt, \tag{7}$$

where g is periodic of period T and m is the average value of $2acx^2 - 2bdy^2$. Since all terms in equation (7) must be periodic of period T, we conclude that $m = 0$. Also, since $(ac)(-bd) > 0$, then ac and $-bd$ must have the same sign, which means that x and y must both be identically 0 on $[0, T]$, and hence on all of $\mathbb{R}$.

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2. J. K. Hale, *Ordinary Differential Equations*, Robert E. Krieger, Malabar, FL, 1980.