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There are two styles of proof of Zorn's lemma that are commonly found in texts. One of these is the style of proof that is given in [**1**] and [**2**], and the other uses ordinals and transfinite recursion. The purpose of this note is to suggest a proof which has some of the flavor of the ordinal proof, but which does not require ordinals.

Notation. If $\leqslant$ is a partial order in a set X, then a **chain** $C \subseteq X$ is a subset C of X that is totally ordered by the order $\leqslant$. Note that the empty set is a chain. If C is a chain in X and $x \in C$, then we define

$$P(C, x) = \{y \in C | y < x\}.$$

A subset of a chain C that has the form $P(C, x)$ is called an **initial segment** in C. An element x in a partially ordered set X is said to be **maximal** if there does not exist an element $y \in X$ such that $x < y$.

ZORN'S LEMMA (Hausdorff Maximal Principle). *Suppose that* $\leqslant$ *is a partial order in a set X and that every chain in X has an upper bound. Then X has a maximal element.*

To obtain a contradiction, suppose that X has no maximal member. If C is a chain in X, then by choosing an upper bound u of C and then choosing an element $x > u$, we can obtain an element $x \in X$ such that $y < x$ for every $y \in C$. Such an element x will be called a **strict upper bound** of C. Using the axiom of choice, we choose a function f that assigns to every chain $C \subseteq X$, a strict upper bound $f(C)$.

We shall say that a subset A of X is **conforming** if the following two conditions hold:

(a) The order $\leqslant$ is a well order of the set A.
(b) For every element $x \in A$, we have $x = f(P(A, x))$.

We now make an observation about the comparability of conforming subsets of X:

If A and B are conforming subsets of X and $A \neq B$, *then one of these two sets is an initial segment of the other.*

Proof. We may assume that $A \setminus B \neq \varnothing$. Define x to be the least member of $A \setminus B$. Then $P(A, x) \subseteq B$. We claim that $P(A, x) = B$. To obtain a contradiction, assume that $B \setminus P(A, x) \neq \varnothing$, and define y to be the least member of $B \setminus P(A, x)$. Given any element $u \in P(B, y)$ and any element $v \in A$ such that $v < u$, it is clear that $v \in P(B, y)$. Therefore if z is the least member of $A \setminus P(B, y)$, we have $P(A, z) = P(B, y)$. Note that $z \leqslant x$. But since

$$z = f(P(A, z)) = f(P(B, y)) = y,$$

and since $y \in B$, we cannot have $z = x$. Therefore $z < x$, and we conclude that $y = z \in P(A, x)$, contradicting the choice of y.

Using the property of comparability of conforming sets that we have just proved, we now observe that if A is a conforming subset of X and $x \in A$, then whenever $y < x$, either $y \in A$ or y does not belong to any conforming set. It now follows easily that the union U of all the conforming subsets of X is conforming, and we deduce from this fact that if $x = f(U)$, then the set $U \cup \{x\}$ is conforming. Therefore, $x \in U$, contradicting the fact that x is a strict upper bound of U.

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The Converse of Liouville's Theorem

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After seeing Liouville's theorem on the approximation of real algebraic numbers by rationals, it is natural to ask whether the converse is true. That is, if there are only finitely many rational numbers p/q such that $|\alpha - p/q| < c/q^k$ for some positive integer k and constant c, can we conclude that α is algebraic? There is a simple counterexample that isn't mentioned in the standard texts [**1**, **2**, **3**, **4**].

If $\mathbf{A} = \{[a_0, a_1, a_2, \dots];\ a_i = 1 \text{ or } 2\}$ is the set of numbers whose continued fraction representation contains only 1s and 2s then $\mathbf{A}$ is uncountable and so must contain transcendental numbers. However if $\alpha = [a_0, a_1, a_2, \dots] \in \mathbf{A}$ then α has no close rational approximation. To see this let p_m/q_m be the mth convergent. For any simple continued fraction we have

$$\left|\alpha - \frac{p_m}{q_m}\right| > \frac{1}{2}\left|\frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m}\right| = \frac{1}{2q_m q_{m+1}}.$$

But $q_{m+1} = q_m + q_{m-1}$ or $q_{m+1} = 2q_m + q_{m-1}$ since $a_m = 1$ or 2 so, for the numbers under consideration, we have $q_{m+1} \leqslant 3q_m$ and so

$$\left|\alpha - \frac{p_m}{q_m}\right| > \frac{1}{6q_m^2}.$$

Moreover, this inequality holds for any rational number p/q, for if $|\alpha - p/q| \leqslant 1/6q^2$ then $|\alpha - p/q| < 1/2q^2$, and this implies that p/q is a convergent; see [**4**, p. 219].

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