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These matrices have the same easily computable characteristic polynomials as N-matrices and their entries may be selected so as to yield results analogous to the N-matrix cases. For 3×3 matrices one obtains the same six cases (1)–(6) discussed above, but in each of the cases there are slight differences in the *form of the eigenvectors*.

A TRULY ELEMENTARY APPROACH TO THE BOUNDED CONVERGENCE THEOREM

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The bounded convergence theorem follows trivially from the Lebesgue dominated convergence theorem, but at the level of an introductory course in analysis, when the Riemann integral is being studied, how hard is the bounded convergence theorem? For an answer, we might look at Bartle and Sherbert [**2**], page 203: *The proof of this result is quite delicate and will be omitted*. Or we might look at Apostol [**1**], page 228: *The proof of Arzela's theorem is considerably more difficult than ... and will not be given here*. Walter Rudin in [**4**] ignores the theorem altogether in his chapter on Riemann integration, presenting it only as a corollary to the Lebesgue dominated convergence theorem several chapters later, and in [**5**], in an interesting problem in Chapter Two, Rudin refers his readers to [**3**]. In [**3**], Eberlein does present a proof which from some points of view is elementary. Certainly, his proof does not require any notions of measurability, but it is hardly elementary from the point of view of a student who is first learning the Riemann integral. So the answer to the above question seems to be: very hard! But this is not so. In this paper, we present the proof of the bounded convergence theorem in a truly elementary setting, and in such a way that it could be included for the first time in an introductory course.

We begin by defining an elementary set. A bounded subset E of R is said to be *elementary* if E is a finite union of bounded intervals, or equivalently, if χ_E is a step function. One can define the Lebesgue measure $m(E)$ of an elementary set E to be $\int_a^b \chi_E$, where $[a, b]$ is an interval including E, and one can show simply that on the family of elementary sets (which is closed under union, intersection and differences), Lebesgue measure is finitely additive and finitely subadditive. Given a Riemann integrable function f on an interval $[a, b]$, and an elementary subset E of $[a, b]$, we define $\int_E f = \int_a^b f\chi_E$. If E and F are mutually disjoint elementary sets, then one may show easily that $\int_{E \cup F} f = \int_E f + \int_F f$, and if $|f(x)| \leqslant K$ for every point x in E, then $|\int_E f| \leqslant Km(E)$. One may also prove simply that if E is an elementary set and $\varepsilon > 0$, then one can find a closed elementary subset H of E such that $m(H) > m(E) - \varepsilon$.

LEMMA. *Suppose (A_n) is a contracting sequence of bounded subsets of R, with an empty intersection. For each n, define*

$$\alpha_n = \sup\{\, m(E) \,|\, E \text{ is an elementary subset of } A_n \}.$$

Then $\alpha_n \to 0$ as $n \to \infty$.

Proof. The sequence (α_n) is clearly decreasing. Now, to obtain a contradiction, assume that this sequence does not converge to 0, and choose $\delta > 0$, such that $\alpha_n > \delta$ for all n. For each n, choose a closed elementary subset E_n of A_n such that

$$m(E_n) > \alpha_n - \delta/2^n,$$

and define

$$H_n = \bigcap_{i=1}^{n} E_i.$$

Since (H_n) is a contracting sequence of closed bounded sets, we can obtain the desired contradiction by showing that each set H_n is non-empty; for then the intersection of all the sets H_n would be non-empty even though the larger sets A_n have an empty intersection. For this purpose, we make the following two observations: Firstly, for any n, if E is an elementary subset of $A_n \setminus E_n$, then since

$$m(E) + m(E_n) = m(E \cup E_n) \leqslant \alpha_n \quad \text{and} \quad m(E_n) > \alpha_n - \delta/2^n,$$

it follows that $m(E) < \delta/2^n$. Secondly, for any n, if E is an elementary subset of $A_n \setminus H_n$, then since

$$E = (E \setminus E_1) \cup (E \setminus E_2) \cup (E \setminus E_3) \cup \cdots \cup (E \setminus E_n)$$

and since $E \setminus E_i$ is an elementary subset of $A_i \setminus E_i$ for every $i = 1, 2, \dots n$, it follows that $m(E) < \delta$.

But for every n, because $\alpha_n > \delta$, the set A_n must have an elementary subset E such that $m(E) > \delta$, and so it follows that each set H_n is non-empty.

The Main Result. *Suppose (f_n) is a sequence of Riemann integrable functions on $[a, b]$, suppose f is a Riemann integrable function on $[a, b]$, that $f_n \to f$ pointwise on $[a, b]$ and that for some constant $K > 0$, we have $|f_n| \leqslant K$ for every n. Then we have*

$$\int_a^b f_n \to \int_a^b f.$$

Proof. There is no loss of generality in assuming that $f_n \geqslant 0$ for each n and that $f = 0$. Let $\varepsilon > 0$, and for each n, define

$$A_n = \left\{ x \in [a, b] \,|\, f_i(x) \geqslant \frac{\varepsilon}{2(b-a)} \quad \text{for at least one natural } i \geqslant n \right\}.$$

We now apply the lemma to (A_n) to choose a natural N such that whenever $n \geqslant N$, and E is an elementary subset of A_n, we have $m(E) < \varepsilon/2K$, and the proof will be complete when we have shown that whenever $n \geqslant N$, we have $\int_a^b f_n \leqslant \varepsilon$. Let $n \geqslant N$. Since the integral of a Riemann integrable function is the same as its lower integral, in order to show that $\int_a^b f_n \leqslant \varepsilon$, it is sufficient to show that whenever s is a step function and $0 \leqslant s \leqslant f_n$, we have $\int_a^b s \leqslant \varepsilon$. Let s be such a step function and define

$$E = \left\{ x \in [a, b] \,|\, s(x) \geqslant \frac{\varepsilon}{2(b-a)} \right\} \quad \text{and} \quad F = [a, b] \setminus E.$$

Then E and F are elementary sets, and since $E \subseteq A_n$, we have $m(E) < \varepsilon/2K$. Therefore

$$\int_a^b s = \int_E s + \int_F s \leqslant \int_E K + \int_F \frac{\varepsilon}{2(b-a)} \leqslant \int_E K + \int_a^b \frac{\varepsilon}{2(b-a)}$$

$$= Km(E) + \frac{\varepsilon}{2(b-a)}(b-a) < \varepsilon.$$

And that is all there is to it. Notice that while the above proof employs some of the notation and conveys some of the atmosphere of more advanced treatments of integration, it keeps well away from anything hard: Lebesgue measure is needed only for elementary sets; and all the measure is in this case is the sum of the lengths of the finitely many component intervals that make up an elementary set. The proof is accessible to students who have never seen countability and never seen infinite series. They don't even need the Heine Borel theorem if they know that a bounded sequence of real numbers must have a partial limit (cluster point) and that, consequently, a contracting sequence of non empty closed bounded sets must have a non empty intersection.

Incidentally, it is easy to adapt the above proof to show that even if it is not assumed that the limit function f is Riemann integrable, because $(f_n(x))$ is a Cauchy sequence for each x, the sequence of integrals $\int_a^b f_n$ must be a Cauchy sequence and must therefore converge. This may be used to give a revealing explanation of the inadequacy of the Riemann integral.

References

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EXPLAINING SIMPLE COMBINATORIAL ANSWERS

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This note illustrates the principle: *If the answer to a problem turns out to be simple, there is probably a good explanation for it!* A simple answer should motivate us to try to derive that answer in a way which makes it obvious, or at least clarifies the underlying reason for its simplicity. Simplicity and clarity are of course subjective measures, but ones which are still useful. The practice of mathematics is an art as well as a science.

Consider combinatorics. Here it is recognized that simple answers are often satisfyingly explicable in terms of correspondences. This theme was taken up in [3], for example, from the viewpoint that counting the elements of a relatively unfamiliar set X can be satisfyingly achieved if we establish a correspondence between the elements of X and those of some relatively familiar set A. The correspondence constitutes the desired explanation. In this note we take up the theme from the viewpoint that explanations in terms of correspondences can also be achieved between two sets X and A of equally familiar structure. We illustrate this with several examples, most of which "explain" a well-known identity, and are therefore suitable for classroom use.

We shall use lower case symbols to denote natural numbers, including zero, and $I(n)$ will denote the set comprising the first n natural numbers (that is, the natural numbers less than n). The family of k-subsets of $I(n)$ will be denoted by $I(n, k)$. We regard the binomial coefficients as the cardinalities of such sets, by definition:

$$\binom{n}{k} := |I(n,k)|.$$

EXAMPLE 1 (*Symmetry of Pascal's Triangle*). Let $A := I(n, k)$ and $X := I(n, n-k)$. Pairing each k-subset of $I(n)$ with its complement gives a one-to-one correspondence $X \leftrightarrow A$. Hence $|X| = |A|$, so

$$\binom{n}{n-k} = \binom{n}{k}.$$

EXAMPLE 2 (*Pascal's Identity*, sometimes called *Vandermonde's Identity*). Let $A := I(n+1, k+1)$ and $X := X_0 \cup X_1$, where $X_0 := I(n, k)$ and $X_1 := I(n, k+1)$. Any $(k+1)$-subset of $I(n+1)$ either contains the element n or it does not. In the former case, pair it with the k-subset of $I(n)$ obtained by deleting the n, while in the latter case simply pair it with itself, now regarded as a $(k+1)$-subset of $I(n)$. This gives a one-to-one correspondence $X \leftrightarrow A$, since X_0 and X_1 are disjoint. Hence $|X| = |X_0| + |X_1| = |A|$, so

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

EXAMPLE 3 (*Arithmetic Series Identity*). The sum of natural numbers up to n, inclusive, is $\frac{1}{2}n(n+1)$, which is a barely-disguised binomial coefficient. How can we explain the binomial