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Complex multiplication of exactly solvable Calabi–Yau varieties

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Abstract

We propose a conceptual framework that leads to an abstract characterization for the exact solvability of Calabi–Yau varieties in terms of abelian varieties with complex multiplication. The abelian manifolds are derived from the cohomology of the Calabi–Yau manifold, and the conformal field theoretic quantities of the underlying string emerge from the number theoretic structure induced on the varieties by the complex multiplication symmetry. The geometric structure that provides a conceptual interpretation of the relation between geometry and conformal field theory is discrete, and turns out to be given by the torsion points on the abelian varieties.

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1. Introduction

Arithmetic properties of exactly solvable Calabi–Yau varieties encode string theoretic information of their underlying conformal field theory. Results in this direction address the issue of an intrinsic geometric description of the spectrum of the conformal field theory, and a geometric derivation of the characters of the partition function. The computations that have been performed so far depend on the explicit computation of the Hasse–Weil L-function of Fermat varieties, or more generally Brieskorn–Pham type spaces. The special feature of these manifolds, first observed by Weil [1,2] about fifty years ago, is that the cohomological L-function can be expressed in terms of number theoretic L-functions, defined by special kinds of so-called Größencharaktere, or algebraic Hecke characters. Weil's analysis of Fermat type L-functions in terms of Jacobi-sum Größencharaktere was generalized by Yui to the class of Brieskorn–Pham L-functions [3]. It was shown in [4] that the algebraic number field that emerges from the Hasse–Weil L-function of an exactly solvable Calabi–Yau variety leads to the fusion field of the underlying conformal field theory and thereby to the quantum dimensions. It was further proven in [5] that the modular form defined by the Mellin transform of the Hasse–Weil L-function of the Fermat torus arises from the characters of the underlying conformal field theory. This establishes a new connection between algebraic varieties and Kac–Moody algebras via their modular properties.

The basic ingredient of the investigations described in Refs. [4,5] is the Hasse–Weil L-function, an object which collects information of the variety at all prime numbers, therefore providing a 'global' quantity that is associated to Calabi–Yau varieties. The number theoretic interpretation which leads to the physical results proceeded in a somewhat experimental way, by observing the appearance of Jacobi-sum characters in [4], and that of affine theta functions in [5]. This leaves open the question whether these results depend on the special nature of the varieties under consideration, or whether it is possible to identify an underlying conceptual framework that explains the emergence of conformal field theoretic quantities from the discrete structure of the Calabi–Yau variety. It is this problem which we address in the present paper.

The physical question raised translates into a simply stated mathematical problem: provide a theorem that states the conditions under which the geometric Hasse–Weil L-function decomposes into a product of number theoretic L-functions. If such a statement were known one could ask whether the class of varieties that satisfies the stated conditions can be used to derive conformal field theoretic results, e.g., in the spirit of the results of [4,5]. It turns out that this question is very difficult. In dimension one it basically is the Shimura–Taniyama conjecture, which has recently been proven in full generality by Breuil et al. [6] by extending foundational results of Wiles and Taylor in the semistable case [7].

In higher dimensions much less is known. The problem is often summarized as the Langlands program, a set of conjectures, which might be paraphrased as the hope that certain conjectured geometric objects, called motives, lead to Hasse–Weil L-series that arise from automorphic representations [8]. At present very little is known in this direction as far as general structure theorems are concerned. There exists, however, a subclass of varieties for which interesting results have been known for some time, and which turns out to be useful in the present context. In dimension one this is the class of elliptic curves with complex multiplication (CM), i.e., curves which admit a symmetry algebra that is exceptionally

large. It was first shown by Deuring in the fifties [9], following a suggestion of Weil in [2], that for tori with CM the cohomological L-function becomes a number theoretic object. More precisely, he showed that associated to the complex multiplication field of the elliptic curve are algebraic Hecke characters which describe the Hasse–Weil L-function, much like Weil's Jacobi-sum Größencharaktere do in the case of the Fermat varieties. This provides an explicit description of the L-function for toroidal compactifications.

Complex multiplication is a group property, and it is not obvious what the most convenient physical generalization of this notion is for higher-dimensional Calabi–Yau varieties. One interesting attempt in this direction was recently made by Gukov and Vafa [10], who conjectured that exactly solvable Calabi–Yau varieties can be characterized in terms of a property of the intermediate Jacobian described in [11–13] (see also [14]). In the present paper we follow a different approach, which is motivated in part by the results of [5] and [15]. In [15] our focus was on properties of black hole attractor Calabi–Yau varieties with finite fundamental group. In an interesting paper Moore [16] had shown that attractor varieties with elliptic factors are distinguished by the fact that they admit complex multiplication. The aim of [15] was to introduce a framework in which the notion of complex multiplication can be generalized to non-toroidal Calabi–Yau varieties of arbitrary dimension via abelian varieties that can be derived from the cohomology. Abelian varieties are natural higher-dimensional generalizations of elliptic curves, and certain types admit complex multiplication. The link between Calabi–Yau manifolds and abelian varieties therefore allows us to generalize the elliptic analysis to the higher-dimensional abelian case.

In the most general context, the relation between exactly solvable Calabi–Yau varieties and complex multiplication very likely will go beyond abelian varieties, and involve the theory of motives with (potential) complex multiplication. The program of constructing a satisfactory framework of motives is incomplete at this point, despite much effort. In this paper we therefore focus on the simpler case of exactly solvable Calabi–Yau varieties that lead to motives derived from abelian varieties which admit complex multiplication. Within this context we provide a conceptual understanding of the results of [4], and thereby establish a framework that generalizes the analysis described there. Briefly, we identify two key ingredients of the exact solvability of Calabi–Yau varieties. The first is that just as in the case of an elliptic curve the Hasse–Weil L-function of an abelian variety with complex multiplication is a number theoretic object, described in terms of algebraic Hecke characters. The second is that the origin of these characters can be traced to the torsion points on the abelian variety, i.e., the points of finite order. This shows that it is the arithmetic structure of CM abelian varieties associated to Calabi–Yau manifolds which encodes the property of exact solvability.

The paper is organized as follows. In Sections 2 and 3 we very briefly recall the arithmetic and number theoretic concepts that will be used in the following parts. In Section 4 we discuss two examples of Fermat type varieties which illustrate the transition from geometry to number theory in an explicit way. These examples show how the arithmetic geometry of Calabi–Yau varieties provides non-trivial information about the underlying conformal field theory. Before showing how to characterize Calabi–Yau varieties in terms of abelian manifolds we briefly review in Section 5 the structure of higher-dimensional abelian varieties with complex multiplication and show how their L-functions can be expressed in terms of algebraic Hecke characters. We use an idèlic formulation because this

allows us to clearly identify the geometric structure that provides the conceptual basis of this result—the behavior of the discrete set of torsion points on the variety. In Section 6 we describe how one can associate abelian varieties to Calabi–Yau manifolds by tracing the cohomology of Calabi–Yau varieties to the Jacobians of curves [15]. In the last section we illustrate our framework by explicitly constructing the complex multiplication structure of an exactly solvable Calabi–Yau variety. In this example we apply our general framework to explain in a conceptual way some of the results described in Section 4.

2. Arithmetic L-functions

The first step in our construction is based on the observation that the Hasse–Weil L-function of an exactly solvable Calabi–Yau variety contains information about the underlying conformal field theory. In this section we briefly recall the notion of a geometric L-function and its key properties.

2.1. The Hasse–Weil L-function

The starting point of the arithmetic analysis is the set of Weil conjectures [1], the proof of which was completed by Deligne [17]. For algebraic varieties the Weil–Deligne result states a number of structural properties for the congruent zeta function at a prime number p defined as

$$Z(X/\mathbb{F}_p, t) \equiv \exp\left(\sum_{r\in\mathbb{N}} \#(X/\mathbb{F}_{p^r})\frac{t^r}{r}\right). \tag{1}$$

The motivation to arrange the numbers $N_{r,p} = \#(X/\mathbb{F}_{p^r})$ in this particular way, rather than a more naive generating function, like $\sum_r N_{r,p}t^r$, originates from the fact that they often show a simple behavior, as a result of which the zeta function can be shown to be a rational function. This was first shown by Artin in the 1920s for hyperelliptic function fields [18], and by Schmidt for curves of arbitrary genus [19,20]. Further experience by Hasse, Weil, and others led to the conjecture that this phenomenon is more general, culminating in the Weil conjectures, and Deligne's proof in the 1970s.

The part of the conjectures that is most important for the present context is that the rational factors of $Z(X/\mathbb{F}_p, t)$

$$Z(X/\mathbb{F}_p, t) = \frac{\prod_{j=1}^{d} \mathcal{P}_{2j-1}^{(p)}(t)}{\prod_{j=0}^{d} \mathcal{P}_{2j}^{(p)}(t)}, \tag{2}$$

can be written as

$$\mathcal{P}_0^{(p)}(t) = 1 - t, \qquad \mathcal{P}_{2d}^{(p)}(t) = 1 - p^d t \tag{3}$$

and for $1 \leqslant i \leqslant 2d-1$

$$\mathcal{P}_j^{(p)}(t) = \prod_{i=1}^{b_j}\left(1 - \beta_i^{(j)}(p)t\right), \tag{4}$$

with algebraic integers $\beta_i^{(j)}(p)$. The degree of the polynomials $\mathcal{P}_j^{(p)}(t)$ is given by the Betti numbers of the variety, $b_j = \dim \mathrm{H}^j_{\mathrm{DeRham}}(X)$. The rationality of the zeta function was first shown by Dwork [21] by adélic methods, and the form (2) was derived by Grothendieck [22]. More details of the Weil conjectures can be found in [4].

We see from the rationality of the zeta function that the basic information of this quantity is parametrized by the cohomology of the variety. More precisely, one can show that the jth polynomial $\mathcal{P}_j^{(p)}(t)$ is associated to the action induced by the Frobenius morphism on the jth cohomology group $\mathrm{H}^j(X)$. In order to gain insight into the arithmetic information encoded in these Frobenius actions it is useful to decompose the zeta function of the variety into pieces determined by its cohomology. This leads to the concept of a local L-function that is associated to the polynomials $\mathcal{P}_j^{(p)}(t)$ via the following definition.

Let $\mathcal{P}_j^{(p)}(t)$ be a polynomial determined by the rational congruent zeta function over the field $\mathbb{F}_p$. The jth local L-function of the variety X over $\mathbb{F}_p$ is defined via

$$L^{(j)}(X/\mathbb{F}_p, s) = \frac{1}{\mathcal{P}_j^{(p)}(p^{-s})}. \tag{5}$$

Such L-functions are of interest for a number of reasons. One of these is that often they can be modified by simple factors so that after analytic continuation they (are conjectured to) satisfy a functional equation.

2.2. Arithmetic via Jacobi sums

The simplest exactly solvable Calabi–Yau varieties are of Brieskorn–Pham type, defined via zero sets

$$X = \left\{ \sum_{i=0}^{r} z_i^{n_i} = 0 \right\} \subset \mathbb{P}_{k_0,\ldots,k_r}. \tag{6}$$

For this class it is possible to gain more insight into the structure of the L-function polynomials $\mathcal{P}_{r-1}^{(p)}(t)$. In the case of Fermat hypersurfaces an old result by Weil shows that the cardinalities of the variety can be expressed in terms of Jacobi sums of finite fields.

Theorem [1]. *Define the number $d = (n, q-1)$ and the set*

$$\mathcal{A}_r^{q,n} = \left\{ (\alpha_0, \ldots, \alpha_r) \in \mathbb{Q}^{r+1} \mid 0 < \alpha_i < 1,\ d\alpha_i = 0 \pmod 1,\ \sum_i \alpha_i = 0 \pmod 1 \right\}. \tag{7}$$

Then the number of solutions of the projective variety

$$X_{r-1} = \left\{ (z_0 : z_1 : \cdots : z_r) \in \mathbb{P}_r \mid \sum_{i=0}^{r} b_i z_i^n = 0 \right\} \subset \mathbb{P}_r \tag{8}$$

over the finite field $\mathbb{F}_q$ *is given by*

$$N_q(X_{r-1}) = 1 + q + q^2 + \cdots + q^{r-1} + \sum_{\alpha \in \mathcal{A}_r^{q,n}} j_q(\alpha) \prod \bar{\chi}_{\alpha_i}(b_i), \tag{9}$$

where

$$j_q(\alpha) = \frac{1}{q-1} \sum_{\substack{u_i \in \mathbb{F}_q \\ u_0 + \cdots + u_r = 0}} \chi_{\alpha_0}(u_0) \cdots \chi_{\alpha_r}(u_r), \tag{10}$$

with

$$\chi_{\alpha_i}(u_i) = e^{2\pi i \alpha_i m_i}, \tag{11}$$

where m_i *is determined via* $u_i = g^{m_i}$ *for any generator* $g \in \mathbb{F}_q$.

With these Jacobi sums $j_q(\alpha_0, \ldots, \alpha_r)$ one defines the polynomial

$$\mathcal{P}_{r-1}^{(q)}(t) = \prod_{\alpha \in \mathcal{A}_r^n} \left(1 - (-1)^{r-1} j_{q^f}(\alpha_0, \ldots, \alpha_r) \prod_i \bar{\chi}_{\alpha_i}(b_i) t^f\right)^{1/f} \tag{12}$$

with $f = f(\alpha)$, $\mathcal{A}_r^n$ is obtained from $\mathcal{A}_r^{n,q}$ by setting $d = r$. The associated Hasse–Weil L-function of the variety is defined as

$$L_{\mathrm{HW}}(X, s) = \prod_p \frac{1}{\mathcal{P}_{r-1}^{(p)}(p^{-s})}. \tag{13}$$

A slight modification of this result is useful even in the case of weighted projective Brieskorn–Pham varieties because it can be used to compute the factor of the zeta function coming from the invariant part of the cohomology, when viewing these spaces as quotient varieties of projective spaces [3].

3. L-functions of algebraic number fields

The surprising aspect of the Hasse–Weil L-function is that it is determined by another, a priori completely different kind of L-function that is derived not from a variety but from a number field. It is this possibility to interpret the cohomological Hasse–Weil L-function as a field theoretic L-function which establishes the connection that allows us to derive number fields K from algebraic varieties X. These in turn encode conformal field theoretic information of the underlying exactly solvable model.

In the present context the type of L-function that is important is that of a Hecke L-function determined by a Hecke character, more precisely an algebraic Hecke character. Following Weil we will see that the relevant field for Fermat type varieties is the cyclotomic field extension $\mathbb{Q}(\mu_m)$ of the rational field $\mathbb{Q}$ by roots of unity, generated by $\xi = e^{2\pi i/m}$ for some rational integer m. It turns out that these fields fit in very nicely with the conformal field theory point of view. In order to see how this works we first describe the concept of Hecke characters and then explain how the L-function fits into this framework.

There are many different definitions of algebraic Hecke characters, depending on the precise number theoretic framework. Originally this concept was introduced by Hecke [23] as Größencharaktere of an arbitrary algebraic number field. In the following Deligne's adaptation of Weil's Größencharaktere of type A_0 is used [24].

Definition. Let $\mathcal{O}_K \subset K$ be the ring of integers of the number field K, $\mathfrak{f} \subset \mathcal{O}_K$ an integral ideal, and F a field of characteristic zero. Denote by $\mathcal{I}_\mathfrak{f}(K)$ the set of fractional ideals of K that are prime to $\mathfrak{f}$ and denote by $\mathcal{I}_\mathfrak{f}^p(K)$ the principal ideals (α) of K for which $\alpha \equiv 1 \pmod{\mathfrak{f}}$. An algebraic Hecke character modulo $\mathfrak{f}$ is a multiplicative function χ defined on the ideals $\mathcal{I}_\mathfrak{f}(K)$ for which the following condition holds. There exists an element in the integral group ring $\sum n_\sigma \sigma \in \mathbb{Z}[\mathrm{Hom}(K, \bar{F})]$, where $\bar{F}$ is the algebraic closure of F, such that if $(\alpha) \in \mathcal{I}_\mathfrak{f}^p(K)$ then

$$\chi((\alpha)) = \prod_\sigma \sigma(\alpha)^{n_\sigma}. \tag{14}$$

Furthermore there is an integer $w > 0$ such that $n_\sigma + n_{\bar{\sigma}} = w$ for all $\sigma \in \mathrm{Hom}(K, \bar{F})$. This integer w is called the weight of the character χ.

Given any such character χ defined on the ideals of the algebraic number field K we can follow Hecke and consider a generalization of the Dirichlet series via the L-function

$$L(\chi, s) = \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \frac{1}{1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}} = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}, \tag{15}$$

where the sum runs through all the ideals. Here $\mathrm{N}\mathfrak{p}$ denotes the norm of the ideal $\mathfrak{p}$, which is defined as the number of elements in $\mathcal{O}_K/\mathfrak{p}$. The norm is a multiplicative function, hence it can be extended to all ideals via the prime ideal decomposition of a general ideal. If we can deduce from the Hasse–Weil L-function the particular Hecke character(s) involved we will be able to derive directly from the variety in an intrinsic way distinguished number field(s) K.

Insight into the nature of number fields can be gained by recognizing that for certain extensions K of the rational number $\mathbb{Q}$ the higher Legendre symbols provide the characters that enter the discussion above. Inspection then suggests that we consider the power residue symbols of cyclotomic fields $K = \mathbb{Q}(\mu_m)$ with integer ring $\mathcal{O}_K = \mathbb{Z}[\mu_m]$. The transition from the cyclotomic field to the finite fields is provided by the character which is determined for any algebraic integer $x \in \mathbb{Z}[\mu_m]$ prime to m by the map

$$\chi_\bullet(x) : \mathfrak{I}_m(\mathcal{O}_K) \to \mathbb{C}^\times, \tag{16}$$

which is defined on ideals $\mathfrak{p}$ prime to m by sending the prime ideal to the mth root of unity for which

$$\mathfrak{p} \mapsto \chi_\mathfrak{p}(x) = x^{\frac{\mathrm{N}\mathfrak{p}-1}{m}} \pmod{\mathfrak{p}}. \tag{17}$$

Using these characters one can define Jacobi-sums of rank r for any fixed element $a = (a_1, \ldots, a_r)$ by setting

$$J_a^{(r)}(\mathfrak{p}) = (-1)^{r+1} \sum_{\substack{u_i \in \mathcal{O}_K/\mathfrak{p} \\ \sum_i u_i = -1 \pmod{\mathfrak{p}}}} \chi_{\mathfrak{p}}(u_1)^{a_1} \cdots \chi_{\mathfrak{p}}(u_r)^{a_r} \tag{18}$$

for prime $\mathfrak{p}$. For non-prime ideals $\mathfrak{a} \subset \mathcal{O}_K$ the sum is generalized via prime decomposition $\mathfrak{a} = \prod_i \mathfrak{p}_i$ and multiplicativity $J_a(\mathfrak{a}) = \prod_i J_a(\mathfrak{p}_i)$. Hence we can interpret these Jacobi sums as a map $J^{(r)}$ of rank r

$$J^{(r)} : \mathfrak{I}_m\big(\mathbb{Z}[\mu_m]\big) \times (\mathbb{Z}/m\mathbb{Z})^r \to \mathbb{C}^\times, \tag{19}$$

where $\mathfrak{I}_m$ denotes the ideals prime to m. For fixed $\mathfrak{p}$ such Jacobi sums define characters on the group $(\mathbb{Z}/m\mathbb{Z})^r$. It can be shown that for fixed $a \in (\mathbb{Z}/m\mathbb{Z})^r$ the Jacobi sum $J_a^{(r)}$ evaluated at principal ideals (x) for $x \equiv 1 \pmod{m^r}$ is of the form $x^{S(a)}$, where

$$S(a) = \sum_{\substack{(\ell,m)=1 \\ \ell \bmod m}} \left[\sum_{i=1}^{r} \left\langle \frac{\ell a_i}{m} \right\rangle \right] \sigma_\ell^{-1}, \tag{20}$$

where $\langle x \rangle$ denotes the fractional part of x and $[x]$ describes the integer part of x.

We therefore see that the L-function of a Brieskorn–Pham variety is determined by Hecke L-functions of cyclotomic fields.

4. Examples

In this section we illustrate the importance of the Hasse–Weil L-function for the connection between Calabi–Yau varieties and conformal field theories with two examples. In the first, the cubic elliptic curve, the field of quantum dimensions is trivial. Nonetheless, the L-function contains non-trivial information because it allows us to provide a geometric understanding of the key building blocks of the conformal field theoretic characters.

4.1. The elliptic Fermat curve

In [5] the elliptic curve defined by the plane cubic torus

$$C_3 = \big\{ (z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^3 + z_1^3 + z_2^3 = 0 \big\} \tag{21}$$

was analyzed in some detail.

The zeta function (1) simplifies for curves into the form

$$Z(X, s) = \prod_{\mathbb{Z} \ni p \text{ good prime}} \frac{\mathcal{P}^{(p)}(p^{-s})}{(1 - p^{-s})(1 - p^{1-s})} = \frac{\zeta(s)\zeta(s-1)}{L_{\mathrm{HW}}(X, s)}, \tag{22}$$

written in terms of the Hasse–Weil L-function defined as

$$L_{\mathrm{HW}}(X, s) = \prod_{\mathbb{Z} \ni p \text{ good prime}} \frac{1}{\mathcal{P}^{(p)}(p^{-s})}, \tag{23}$$

and the Riemann zeta function $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ of the rational field $\mathbb{Q}$.

The Hasse–Weil L-function can be determined via (1) by direct counting of the number of solutions of $C_3/\mathbb{F}_{p^r}$ over finite extensions $[\mathbb{F}_{p^r}\colon \mathbb{F}_p]$ of the finite fields $\mathbb{F}_p$ of prime order p. This results in

$$L_{\mathrm{HW}}(C_3, s) = 1 - \frac{2}{4^s} - \frac{1}{7^s} + \frac{5}{13^s} + \frac{4}{16^s} - \frac{7}{19^s} + \cdots, \tag{24}$$

leading to the Hasse–Weil q-expansion

$$f_{\mathrm{HW}}(C_3, q) = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots. \tag{25}$$

It turns out that this is a modular form of weight 2 and modular level 27, which can be written as a product of the theta function $\Theta(\tau)$ associated to the string function $c(\tau)$ of the affine $SU(2)$ Kac–Moody algebra at conformal level $k = 1$. More precisely, the following result was obtained.

Theorem [5]. *The Mellin transform of the Hasse–Weil L-function $L_{\mathrm{HW}}(C_3, s)$ of the cubic elliptic curve $C_3 \subset \mathbb{P}_2$ is a modular form $f_{\mathrm{HW}}(C_3, q) \in S_2(\Gamma_0(27))$ which factors into the product*

$$f_{\mathrm{HW}}(C_3, q) = \Theta\big(q^3\big)\Theta\big(q^9\big). \tag{26}$$

Here $\Theta(\tau) = \eta^3(\tau)c(\tau)$ is the Hecke modular form associated to the quadratic extension $\mathbb{Q}(\sqrt{3})$ of the rational field $\mathbb{Q}$, determined by the unique string function $c(\tau)$ of the affine Kac–Moody $SU(2)$-algebra at conformal level $k = 1$.

This establishes that it is possible to derive the modularity of the underlying string theoretic conformal field theory from the geometric target space and that the Hasse–Weil L-function admits a conformal field theoretic interpretation.

The number theoretic interpretation of the Hasse–Weil L-function is best seen from the expression for the polynomials $\mathcal{P}^{(p)}(t)$, which completely determine the congruent zeta function and the Hasse–Weil L-function of these plane curves, in terms of the finite field Jacobi sums. For curves this reduces to

$$\mathcal{P}^{(p)}(t) = \prod_{\alpha \in \mathcal{A}_2^p} \big(1 - j_{p^f}(\alpha) t^f\big)^{1/f} \tag{27}$$

with

$$j_q(\alpha) = \frac{1}{q-1} \sum_{\substack{u_i \in \mathbb{F}_q \\ u_0 + u_1 + u_2 = 0}} \chi_{\alpha_0}(u_0)\chi_{\alpha_1}(u_1)\chi_{\alpha_2}(u_2). \tag{28}$$

Computing values of $j_q(\alpha)$ at primes is in part easier than direct counting because the cardinalities of the sets $\mathcal{A}_2^{p,3}$ are easy to control. For the first few primes the results are collected in Table 1, in which the zeroes follow immediately from the structure of the sets $\mathcal{A}_2^{p,3}$

$$\mathcal{A}_2^{p,3} = \left\{ \begin{array}{cc} \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\} & (3, p-1) = 1 \\ \emptyset & (3, p-1) > 1 \end{array} \right\}. \tag{29}$$

Table 1
Finite field Jacobi sums of the elliptic cubic curve C_3 at the lower rational primes

q	2	3	5	7	11	13
$j_q(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	0	0	0	$2+3\xi_3^2$	0	$-1+3\xi_3^2$
$j_q(\frac{2}{3},\frac{2}{3},\frac{2}{3})$	0	0	0	$2+3\xi_3$	0	$-1+3\xi_3$

The number field theoretic interpretation of the Hasse–Weil L-function emerges as follows. For any rational prime p we can find a prime ideal $\mathfrak{p} \subset \mathbb{Z}[\mu_3]$ over p such that the finite field character χ_{α_i} defined over the finite field $\mathbb{F}_p$ can be expressed in terms of the 3rd power residue symbol $\chi_{\mathfrak{p}}$, where $\mathbb{F}_p$ is viewed as the residue field of the ring of integers $\mathbb{Z}[\mu_3]$ with respect to the ideal $\mathfrak{p}$. This allows to translate the finite field Jacobi-sums $j_p(\alpha_0,\alpha_1,\alpha_2)$ with $\alpha_i = a_i/3$ into Jacobi-sum type Hecke characters $J_{(a,a,a)}(\mathfrak{p})$ of the cyclotomic field $\mathbb{Q}(\mu_3)$, where $a \in \{1,2\}$. The field $\mathbb{Q}(\mu_3)$ is the fusion field of the underlying conformal field theory, and in turn determines the field of quantum dimensions, which in the present example is just $\mathbb{Q}$ [4]. This analysis furthermore shows that one can write the geometric Hasse–Weil L-function as a number theoretic object associated to the fusion field. Applied to the field $\mathbb{Q}(\mu_3)$ this procedure leads to the number theoretic representation of the Hasse–Weil L-function of the plane cubic curve as

$$L_{\mathrm{HW}}(E,s) = L_H(J_{(1,1,1)},s)L_H(J_{(2,2,2)},s). \tag{30}$$

4.2. The quintic threefold

Consider the Calabi–Yau variety defined by the Fermat quintic hypersurface in ordinary projective fourspace $\mathbb{P}_4$ defined by

$$X = \left\{ (x_0:\cdots:x_4) \in \mathbb{P}_4 \;\middle|\; \sum_{i=0}^{4} x_i^5 = 0 \right\}. \tag{31}$$

It follows from Lefshetz's hyperplane theorem that the cohomology below the middle dimension is inherited from the ambient space. Thus we have $h^{1,0} = 0 = h^{0,1}$ and $h^{1,1} = 1$ while $h^{2,1} = 101$ follows from counting monomials of degree five. For the smooth Fermat quintic the zeta function simplifies to the expression

$$Z(X/\mathbb{F}_p,t) = \frac{\mathcal{P}_3^{(p)}(t)}{(1-t)(1-pt)(1-p^2t)(1-p^3t)}, \tag{32}$$

where the numerator is given by the polynomial $\mathcal{P}_3^{(p)}(t) = \prod_{i=1}^{204}(1-\beta_i^{(3)}(p)t)$ which takes the form

$$\mathcal{P}_3^{(p)}(t) = \prod_{\alpha\in\mathcal{A}_4^p} \left(1 - j_{p^f}(\alpha)t^f\right)^{1/f}. \tag{33}$$

This expression involves the following ingredients. Define $d = (5, p-1)$ and rational numbers α_i via $d\alpha_i \equiv 0 \pmod 1$. The set $\mathcal{A}_4^{p,5}$ then takes the form

$$\mathcal{A}_4^{p,5} = \left\{ \alpha = (\alpha_0, \ldots, \alpha_4) \mid 0 < \alpha_i < 1,\ d\alpha_i \equiv 0 \pmod 1,\ \sum_i \alpha_i = 0 \pmod 1 \right\}. \tag{34}$$

Defining the characters $\chi_{\alpha_i} \in \hat{\mathbb{F}}_p$ in the dual of $\mathbb{F}_p$ as $\chi_{\alpha_i}(u_i) = \exp(2\pi i \alpha_i s_i)$ with $u_i = g^{s_i}$ for a generating element $g \in \mathbb{F}_p$, the factor $j_p(\alpha)$ finally is determined as

$$j_p(\alpha) = \frac{1}{p-1} \sum_{\sum_i u_i = 0} \prod_{i=0}^{4} \chi_{\alpha_i}(u_i). \tag{35}$$

We thus see that the congruent zeta function leads to the Hasse–Weil L-function associated to a Calabi–Yau threefold

$$L_{\mathrm{HW}}(X, s) = \prod_{p \in P(X)} \prod_{\alpha \in \mathcal{A}_4^p} \left(1 - \frac{j_{p^f}(\alpha)}{p^{fs}} \right)^{-1/f}, \tag{36}$$

ignoring the bad primes. In the case of the quintic threefold we can proceed along the lines described above to find that the cyclotomic characters associated to rational primes are defined via prime ideals in the cyclotomic field $\mathbb{Q}(\mu_5)$. This again is the fusion field of the underlying conformal field theory, and leads to the field of quantum dimensions, given by $\mathbb{Q}(\sqrt{5})$ [4], a result which we will explain in Section 7.

We will see further below that it is the complex multiplication structure, underlying not only the elliptic curve C_3, but also the quintic threefold, which is responsible for the results obtained for these two examples. This view will lead to a more conceptual understanding of the relation between geometry and conformal field theory that allows to generalize the analysis to broader classes of varieties.

5. Abelian varieties

The main problem we are addressing in this paper is the question how the conformal field theoretic results that are encoded in the number theoretic form of the Hasse–Weil L-function of the previous sections can be formulated in a more conceptual framework that allows us to generalize the characterization of exact solvability to arbitrary Calabi–Yau manifolds. It is known in mathematics that L-functions of abelian varieties with complex multiplication are number theoretic in nature. We can therefore use the theory of such manifolds if we are able to recover abelian manifolds from Calabi–Yau spaces. In this section we briefly review the background of abelian varieties that is necessary for our discussion further below. In the next section we will show how such abelian varieties emerge from Calabi–Yau manifolds.

5.1. General definition

An abelian variety over some number field K is a smooth, geometrically connected, projective variety, which is also an algebraic group, with the group law $A \times A \to A$ defined over K. A concrete way to construct such manifolds is via complex tori $\mathbb{C}^n/\Lambda$ with respect to some lattice $\Lambda \subset \mathbb{C}^n$, or, put differently, via an exact sequence

$$0 \to \Lambda \to \mathbb{C}^n \xrightarrow{f} A \to 0, \tag{37}$$

where f is a holomorphic map. The lattice Λ is not necessarily integral and admits a Riemann form, which is defined as an $\mathbb{R}$-bilinear form $\langle\,,\rangle$ on $\mathbb{C}^n$ such that the following hold:

(1) $\langle x, y\rangle$ takes integral values for all $x, y \in \Lambda$;
(2) $\langle x, y\rangle = -\langle y, x\rangle$;
(3) $\langle x, iy\rangle$ is a symmetric and positive definite form in x, y.

The result then is that a complex torus $\mathbb{C}^n/\Lambda$ has the structure of an abelian variety if and only if there exists a non-degenerate Riemann form on $\mathbb{C}^n/\Lambda$.

5.2. Abelian varieties of CM type

A special class of abelian varieties are those of complex multiplication (CM) type. These are varieties which admit automorphism groups that are larger than those of general abelian manifolds. The reason why CM type varieties are special is because certain number theoretic questions can be addressed in a systematic fashion for this class. The first to discover this was Weil [2] in the context of Fermat type hypersurfaces. The fact that this relation can be traced to the property of CM for abelian varieties was first shown by Deuring in the context of elliptic curves, following a suggestion by Weil. This was later generalized conditionally to higher dimensions by Taniyama and Shimura [26,27], Serre and Tate [28], and Shimura [25,29].

Consider a number field F over the rational numbers $\mathbb{Q}$ and denote by $[F:\mathbb{Q}]$ the degree of the field F over $\mathbb{Q}$, i.e., the dimension of F over the subfield $\mathbb{Q}$. An abelian variety A of dimension n is called a CM-variety if there exists an algebraic number field F of degree $[F:\mathbb{Q}] = 2n$ over the rational numbers $\mathbb{Q}$ which can be embedded into the endomorphism algebra $\mathrm{End}(A) \otimes \mathbb{Q}$ of the variety. More precisely, a CM-variety is a triplet (A, θ, F), where

$$\theta : F \to \mathrm{End}(A) \otimes \mathbb{Q} \tag{38}$$

describes the embedding of F. It follows from this that the field F necessarily is a CM field, i.e., a totally imaginary quadratic extension of a totally real field. The important ingredient here is that the restriction to $\theta(F) \subset \mathrm{End}(A) \otimes \mathbb{Q}$ is equivalent to the direct sum of n isomorphisms $\varphi_1, \ldots, \varphi_n \in \mathrm{Iso}(F, \mathbb{C})$ such that $\mathrm{Iso}(F, \mathbb{C}) = \{\varphi_1, \ldots, \varphi_n, \rho\varphi_1, \ldots, \rho\varphi_n\}$, where ρ denotes complex conjugation. These considerations suggest calling the pair

$(F, \{\varphi_i\})$ a CM-type. In principle we can think of the CM type as an abstract representation defined by some matrix

$$\Phi(a) = \begin{pmatrix} a^{\varphi_1} & & \\ & \ddots & \\ & & a^{\varphi_n} \end{pmatrix}, \quad \text{for } a \in F, \tag{39}$$

but in the present context $(F, \Phi = \{\varphi_i\})$ describes the CM-type of a CM-variety (A, θ, F).

It is possible to prescribe the CM structure and construct an abelian variety with that given structure by constructing a diagram of the following form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & F_{\mathbb{R}} & \longrightarrow & F_{\mathbb{R}}/\mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow u & & \downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}^n & \xrightarrow{f} & A & \longrightarrow & 0 \end{array} \tag{40}$$

where u is the map

$$\begin{aligned} u &: F_{\mathbb{R}} \to \mathbb{C}^n \\ a &\mapsto \begin{pmatrix} a^{\varphi_1} \\ \vdots \\ a^{\varphi_n} \end{pmatrix}, \end{aligned} \tag{41}$$

defined as an $\mathbb{R}$-linear extension on F, and $\mathfrak{a}$ is the preimage of u of the lattice Λ. The abelian variety is thereby obtained as the quotient $F_{\mathbb{R}}/\mathfrak{a}$ of $F_{\mathbb{R}} = F \otimes_\rho \mathbb{R}$, with ρ denoting complex conjugation, by an ideal in F, with a complex structure determined by u, and an embedding $\theta : F \to \mathrm{End}(A) \otimes \mathbb{Q}$ given by (39).

Concrete examples of these concepts, which have been discussed in [4] in the context of the Calabi–Yau/conformal field theory relation, are varieties which have complex multiplication by a cyclotomic field $F = \mathbb{Q}(\mu_n)$, where μ_n denotes the cyclic group generated by a non-trivial nth root of unity ξ_n. The degree of $\mathbb{Q}(\mu_n)$ is given by $[\mathbb{Q}(\mu_n) : \mathbb{Q}] = \phi(n)$, where $\phi(n) = \#\{m \in \mathbb{N} \mid m < n,\ \gcd(m, n) = 1\}$ is the Euler function. Hence the abelian varieties encountered have complex dimension $\dim A = \phi(n)/2$.

The simplest examples of abelian CM varieties are elliptic curves with complex multiplication. These occur in the context of higher dimensional Calabi–Yau varieties via the Shioda–Katsura decomposition of the cohomology of Fermat type manifolds. Further below we briefly describe the reduction of the cohomology of more general Brieskorn–Pham varieties to that generated by curves and then analyze the structure of the resulting weighted curve Jacobians.

5.3. *L-function of abelian varieties with complex multiplication*

In this section we describe the generalization of the conceptual framework underlying the number theoretic interpretation of the zeta function of abelian varieties with complex multiplication. Our goal is to detail the underlying structure that explains this phenomenon. As in the case of elliptic curves the main objects that provide the transition from the discrete geometry of the variety to number theory are the torsion points on the variety, i.e., the points

in the kernel of a multiplication map $n : A \to A$, analogous to the corresponding map on the elliptic curves.

The general concept of a geometric L-function is derived from the reduction of a variety over discrete fields $\mathbb{F}_q$ of order q. One way to think about this structure is by considering the fields $\mathbb{F}_q$ as residue fields $\mathcal{O}_K/\mathfrak{p}$, generated by prime ideals $\mathfrak{p}$ in the ring of integers $\mathcal{O}_K$ of some algebraic number field K. Denoting the residue field produced by $\mathfrak{p}$ as $K(\mathfrak{p})$ and the reduced variety by $X(\mathfrak{p})$ one can define the local zeta function as

$$Z(X, \mathfrak{p}, s) := Z\big(X(\mathfrak{p})/K(\mathfrak{p}),\ t = \mathrm{N}\mathfrak{p}^{-s}\big). \tag{42}$$

By combining these local zeta functions for all prime ideals one obtains the global zeta function

$$Z(X/K, s) = \prod_{\mathfrak{p} \subset \mathcal{O}_K} Z(X, \mathfrak{p}, s) \tag{43}$$

of the variety X defined over the number field K.

When the variety has complex multiplication with respect to some number field F the zeta function admits a number theoretic interpretation which generalizes the results of Deuring for elliptic curves with complex multiplication. Associated to the field F are Größencharaktere χ_i, $i = 1, \ldots, n$ which lead to Hecke L-functions $L(\chi_i, s)$. The zeta function of the abelian variety with complex multiplication then is described by these Hecke L-functions.

Theorem [29]. *Let the abelian CM-variety (A, θ, F) be defined over an algebraic number field K of finite degree. Then the zeta function of A over K coincides exactly with the product*

$$\prod_{i=1}^{n} L(\chi_i, s) L(\bar{\chi}_i, s), \tag{44}$$

where the χ_i are Größencharaktere, and $\bar{\chi}_i$ is the complex conjugate of χ_i.

This result was first shown in a conditional formulation by Taniyama and Shimura, and Serre and Tate. This shows that our framework applies to any Calabi–Yau variety to which we can associate abelian manifolds.

5.4. Character construction from abelian varieties

The character construction from higher-dimensional abelian manifolds differs somewhat from that of elliptic curves because of the emergence of the reflex type, denoted here by $(\hat{F}, \hat{\Phi} = \{\hat{\varphi}_i\})$ of the complex multiplication type $(F, \Phi = \{\varphi_i\})$. This reflex field is defined by adjoining to F all traces determined by the CM type of F, i.e., $\hat{F} = F(\{\sum_i x^{\varphi_i} \mid x \in F\})$. To define the reflex type $\hat{\Phi}$ consider a Galois extension $L/\mathbb{Q}$ over the rationals that contains the CM field F. Denote by S the subset of all those elements of the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ of L that induce some φ_i on F and define fur-

ther

$$S^{-1} = \{\sigma^{-1} \mid \sigma \in S\},$$
$$H = \{\gamma \in \mathrm{Gal}(L/\mathbb{Q}) \mid \gamma S^{-1} = S^{-1}\}. \tag{45}$$

Then the reflex type of (F, Φ) is completed by defining the maps

$$\hat{\varphi}_i : \hat{F} \to \mathbb{C} \tag{46}$$

as those that are obtained from S^{-1}. In the one-dimensional case the discussion simplifies because one has $\hat{F} = F$.

Consider an abelian variety A/K of dimension n defined over a number field K with complex multiplication, i.e., with an embedding $\theta : F \to \mathrm{End}(A) \otimes \mathbb{Q}$. Denote this structure by $(A/K, \theta, F)$ with type $(F, \Phi = \{\varphi_i\}_{i=1,\ldots,n})$. The appearance of $\hat{F}$ leads to a modification of the norm map that appears in the elliptic construction of the character. To simplify the discussion we assume that[1] $\hat{F} \subset K$. The construction of the algebraic Hecke character is now a two-step procedure. The first ingredient is a map α constructed as follows. For any finite extension L/F an idèlic norm map

$$\mathrm{N}_F^L : \mathbb{A}_L^\times \to \mathbb{A}_F^\times, \tag{47}$$

can be defined by specifying what the vth component is of the image idèle, where v runs through the finite primes as well as the infinite primes, which are associated to the embeddings of the number field. For $x \in \mathbb{A}_K^\times$ one sets

$$\left(\mathrm{N}_F^L x\right)_v = \prod_{w|v} \mathrm{N}_{F_v}^{L_w} x_w, \tag{48}$$

where L_w and F_v are completions of the fields L and F at the primes w and v, respectively. Let further $F^\times$ denote the invertible elements of F.

Next, we compose the norm map with the determinant map

$$\delta : \mathbb{A}_{\hat{F}}^\times \to \mathbb{A}_F^\times, \tag{49}$$

defined as the continuous extension of the determinant of the reflex type

$$\delta(x) = \det \hat{\Phi}(x), \quad \forall x \in \hat{F}^\times. \tag{50}$$

The composition $g := \delta \circ \mathrm{N}_{\hat{F}}^K$ of the norm map and the determinant map provides a map from the K-idèles to the F-idèles

$$g : \mathbb{A}_K^\times \to \mathbb{A}_F^\times, \tag{51}$$

generalizing the norm map in the elliptic case.

The construction of the character is based on the action of the idèlic Artin symbol $[x, K]$ for $x \in \mathbb{A}_K^\times$ on the torsion points. The main result of the theory of complex multiplication in the case of abelian varieties that are relevant to us can now be summarized as follows.

[1] This condition generalizes the assumption in the elliptic case that the CM field is contained in the field of definition.

Theorem. *Let (F,Φ) be a CM-type, $(\hat{F},\hat{\Phi})$ its reflex, and $\mathfrak{a}$ a lattice in F. Let further (A,θ) be of type (F,Φ), $u: F_{\mathbb{R}} \to \mathbb{C}^n$ the map in the diagram (40), and f the map that defines A via*

$$0 \to \Lambda \to \mathbb{C}^n \xrightarrow{f} A \to 0. \tag{52}$$

Further let $\sigma \in \mathrm{Aut}(\mathbb{C}/\hat{F})$, $x \in \mathbb{A}_{\hat{F}}^{\times}$ be an idèle of the reflex field such that

$$\sigma|_{\hat{F}_{\mathrm{ab}}} = [x, \hat{F}], \tag{53}$$

and $g = \delta \circ \mathrm{N}_{\hat{F}}^{K}: \mathbb{A}_K^{\times} \to \mathbb{A}_F^{\times}$ the map defined by the idèlic extension of the determinant map. Then there is an exact sequence

$$0 \to u\big(g(x)^{-1}\mathfrak{a}\big) \to \mathbb{C}^n \xrightarrow{f'} A^{\sigma} \to 0 \tag{54}$$

such that

$$f\big(u(v)\big)^{\sigma} = f'\big(u\big(g(x)^{-1}v\big)\big) \quad \forall v \in F/\mathfrak{a}, \tag{55}$$

i.e., there exists a commutative diagram

$$\begin{array}{ccc} F/\mathfrak{a} & \xrightarrow{g(x)^{-1}} & F/g(x)^{-1}\mathfrak{a} \\ \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega'} \\ A_{\mathrm{tor}} & \xrightarrow{\sigma} & A_{\mathrm{tor}}^{\sigma} \end{array} \tag{56}$$

where $\omega = f \circ u$ and $\omega' = f' \circ u$.

The construction of the algebraic Hecke character associated to the torsion points of A is now achieved by constructing an idèlic map α of K in the following way.

Theorem. *For the map $\omega: F_{\mathbb{R}} \to A$ defined by $\omega = f \circ u$ with $F_{\mathbb{R}} = F \otimes_{\rho} \mathbb{R}$ and ρ denotes complex conjugation, there exists a map*

$$\alpha: \mathbb{A}_K^{\times} \to F^{\times} \tag{57}$$

such that

$$\omega(v)^{[x,K]} = \omega\big(\alpha(x)g(x)^{-1}v\big) \quad \forall x \in \mathbb{A}_K^{\times},\ v \in F/\mathfrak{a}, \tag{58}$$

which is determined uniquely by the following properties

$$\alpha(x)g(x)^{-1}\mathfrak{a} = \mathfrak{a}, \qquad \alpha(x)\alpha(x)^{\rho} = \mathrm{N}(x), \tag{59}$$

where (x) is the ideal associated to x. Furthermore the kernel of α is open in the idèles.

We can now define characters ψ_i on the idèles by picking appropriate components from the map that describes the Artin symbol on the elements of $F/\mathfrak{a}$. More precisely, define

$$\psi_i(x) = \big(\alpha(x)g(x)^{-1}\big)_{\infty i}, \quad i = 1, \ldots, n, \tag{60}$$

via the infinite primes of the complex multiplication field F.

6. Abelian varieties from Brieskorn–Pham type hypersurfaces

We now have described all the key concepts we need in our formulation of a characterization of exact solvability of Calabi–Yau manifolds. What remains is to establish the existence of abelian varieties with complex multiplication in the context of Calabi–Yau spaces. In the present section we will consider explicitly the class of Brieskorn–Pham hypersurfaces in weighted projective space. It will become clear from the discussion that the construction can be extended to more general types of polynomials. The basic idea is to first reduce the intermediate cohomology of the Calabi–Yau via the Shioda–Katsura construction to the cohomology spanned by curves embedded in the manifold, and then to use the results of Faddeev, Gross, Rohrlich, and others, to decompose the Jacobian varieties derived from these curves to find factors that admit complex multiplication.

6.1. The Shioda–Katsura decomposition

The decomposition of the intermediate cohomology of projective hypersurfaces was first described by Shioda and Katsura [30] and Deligne [31]. Their analysis can be generalized to weighted hypersurfaces, in particular the class of Brieskorn–Pham varieties, perhaps the simplest class of exactly solvable Calabi–Yau manifolds. This generalization works because the cohomology $\mathrm{H}^3(X)$ for these varieties decomposes into the monomial part and the part coming from the resolution. The monomial part of the intermediate cohomology can easily be obtained from the cohomology of a projective hypersurface of the same degree by realizing the weighted projective space as a quotient variety with respect to a product of discrete groups determined by the weights of the coordinates.

For projective varieties

$$X_d^n = \left\{(z_0, \ldots, z_{n+1}) \in \mathbb{P}_{n+1} \mid z_0^d + \cdots + z_{n+1}^d = 0\right\} \subset \mathbb{P}_{n+1} \tag{61}$$

the intermediate cohomology can be determined by lower-dimensional varieties in combination with Tate twists by reconstructing the higher dimensional variety X_d^n of degree d and dimension n in terms of lower dimensional varieties X_d^r and X_d^s of the same degree with $n = r + s$. Briefly, this works as follows. The decomposition of X_d^n is given as

$$X_d^{r+s} \cong B_{Z_1,Z_2}\big(\big(\pi_Y^{-1}\big(X_d^r \times X_d^s\big)\big)/\mu_d\big), \tag{62}$$

which involves the following ingredients.

(1) $\pi_Y^{-1}(X_d^r \times X_d^s)$ denotes the blow-up of $X_d^r \times X_d^s$ along the subvariety

$$Y = X_d^{r-1} \times X_d^{s-1} \subset X_d^r \times X_d^s. \tag{63}$$

The variety Y is determined by the fact that the initial map which establishes the relation between the three varieties X_d^{r+s}, X_d^r, X_d^s is defined on the ambient spaces as

$$\big((x_0, \ldots, x_{r+1}), (y_0, \ldots, y_{s+1})\big) \mapsto (x_0 y_{s+1}, \ldots, x_r y_{s+1}, x_{r+1} y_0, \ldots, x_{r+1} y_s). \tag{64}$$

This map is not defined on the subvariety Y;

(2) $\pi_Y^{-1}(X_d^r \times X_d^s)/\mu_d$ denotes the quotient of the blow-up $\pi_Y^{-1}(X_d^r \times X_d^s)$ with respect to the action of

$$\mu_d \ni \xi : \big((x_0, \ldots, x_r, x_{r+1}), (y_0, \ldots, y_s, y_{s+1})\big) \\ \mapsto \big((x_0, \ldots, x_r, \xi x_{r+1}), (y_0, \ldots, y_s, \xi y_{s+1})\big);$$

(3) $B_{Z_1,Z_2}((\pi_Y^{-1}(X_d^r \times X_d^s))/\mu_d)$ denotes the blow-down in $\pi_Y^{-1}(X_d^r \times X_d^s)/\mu_d$ of the two subvarieties

$$Z_1 = \mathbb{P}_r \times X_d^{s-1}, \qquad Z_2 = X_d^{r-1} \times \mathbb{P}_s.$$

This construction leads to an iterative decomposition of the cohomology which takes the following form. Denote the Tate twist by

$$\mathrm{H}^i(X)(j) := \mathrm{H}^i(X) \otimes W^{\otimes j} \tag{65}$$

with $W = \mathrm{H}^2(\mathbb{P}_1)$ and let X_d^{r+s} be a Fermat variety of degree d and dimension $r+s$. Then

$$\mathrm{H}^{r+s}\big(X_d^{r+s}\big) \oplus \sum_{j=1}^{r} \mathrm{H}^{r+s-2j}\big(X_d^{r-1}\big)(j) \oplus \sum_{k=1}^{s} \mathrm{H}^{r+s-2k}\big(X_d^{s-1}\big)(k) \\ \cong \mathrm{H}^{r+s}\big(X_d^r \times X_d^s\big)^{\mu_d} \oplus \mathrm{H}^{r+s-2}\big(X_d^{r-1} \times X_d^{s-1}\big)(1). \tag{66}$$

This allows us to trace the cohomology of higher-dimensional varieties to that of curves.

Weighted projective hypersurfaces can be viewed as resolved quotients of hypersurfaces embedded in ordinary projective space. The resulting cohomology has two components, the invariant part coming from the projection of the quotient, and the resolution part. As described in [32], the only singular sets on arbitrary weighted hypersurface Calabi–Yau threefolds are either points or curves. The resolution of singular points contributes to the even cohomology group $\mathrm{H}^2(X)$ of the variety, but does not contribute to the middle-dimensional cohomology group $H^3(X)$. Hence we need to be concerned only with the resolution of curves (see, e.g., [33]). This can be described for general CY hypersurface threefolds as follows. If a discrete symmetry group $\mathbb{Z}/n\mathbb{Z}$ of order n acting on the threefold leaves invariant a curve then the normal bundle has fibres $\mathbb{C}_2$ and the discrete group induces an action on these fibres which can be described by a matrix

$$\begin{pmatrix} \alpha^{mq} & 0 \\ 0 & \alpha^m \end{pmatrix}, \tag{67}$$

where α is an nth root of unity and (q, n) have no common divisor. The quotient $\mathbb{C}_2/(\mathbb{Z}/n\mathbb{Z})$ by this action has an isolated singularity which can be described as the singular set of the surface in $\mathbb{C}_3$ given by the equation

$$S = \big\{(z_1, z_2, z_3) \in \mathbb{C}_3 \mid z_3^n = z_1 z_2^{n-q}\big\}. \tag{68}$$

The resolution of such a singularity is completely determined by the type (n, q) of the action by computing the continued fraction of $\frac{n}{q}$

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots_{-\frac{1}{b_s}}}} \equiv [b_1, \ldots, b_s]. \tag{69}$$

The numbers b_i specify completely the plumbing process that replaces the singularity and in particular determine the additional generator to the cohomology $\mathrm{H}^*(X)$ because the number of $\mathbb{P}_1$s introduced in this process is precisely the number of steps needed in the evaluation of $\frac{n}{q} = [b_1, \ldots, b_s]$. This can be traced to the fact that the singularity is resolved by a bundle which is constructed out of $s+1$ patches with s transition functions that are specified by the numbers b_i. Each of these gluing steps introduces a sphere, which in turn supports a $(1,1)$-form. The intersection properties of these 2-spheres are described by Hirzebruch–Jung trees, which for a $\mathbb{Z}/n\mathbb{Z}$ action is just an $SU(n+1)$ Dynkin diagram, while the numbers b_i describe the intersection numbers. We see from this that the resolution of a curve of genus g introduces s additional generators to the second cohomology group $\mathrm{H}^2(X)$, and $g \times s$ generators to the intermediate cohomology $\mathrm{H}^3(X)$.

Hence we see that the cohomology of weighted hypersurfaces is determined completely by the cohomology of curves. Because the Jacobian variety is the basic geometric invariant of a smooth projective curve this says that for weighted hypersurfaces the main cohomological structure is carried by their embedded curves.

6.2. *Cohomology of weighted curves*

For smooth algebraic curves C of genus g the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(C)$ decomposes (over the complex number field $\mathbb{C}$) as

$$\mathrm{H}^1_{\mathrm{dR}}(C) \cong \mathrm{H}^0\big(C, \Omega^1\big) \oplus \mathrm{H}^1(C, \mathcal{O}). \tag{70}$$

The Jacobian $J(C)$ of a curve C of genus g can be identified with

$$J(C) = \mathbb{C}^g/\Lambda, \tag{71}$$

where Λ is the period lattice

$$\Lambda := \left\{ \left(\ldots, \int_a \omega_i, \ldots \right)_{i=1,\ldots,g} \middle| \, a \in \mathrm{H}_1(C, \mathbb{Z}),\ \omega_i \in \mathrm{H}^0\big(C, \Omega^1\big) \right\}, \tag{72}$$

where the ω_i form a basis. Given a fixed point $p_0 \in C$ on the curve there is a canonical map from the curve to the Jacobian, called the Abel–Jacobi map

$$\Psi : C \to J(C), \tag{73}$$

defined as

$$p \mapsto \left(\ldots, \int_{p_0}^{p} \omega_i, \ldots \right) \mathrm{mod}\ \Lambda. \tag{74}$$

We are interested in curves of Brieskorn–Pham type, i.e., curves of the form

$$C_d = \big\{ x^d + y^a + z^b = 0 \big\} \in \mathbb{P}_{(1,k,\ell)}[d], \tag{75}$$

such that $a = d/k$ and $b = d/\ell$ are positive rational integers. Without loss of generality we can assume that $(k, \ell) = 1$. The genus of these curves is given by

$$g(C_d) = \frac{1}{2}(2 - \chi) = \frac{(d-k)(d-\ell) + (k\ell - d)}{2k\ell}. \tag{76}$$

For non-degenerate curves in the configurations $\mathbb{P}_{(1,k,\ell)}[d]$ the set of forms

$$\begin{aligned}&\mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{P}_{(1,k,\ell)}[d]\big)\\&\quad=\left\{\omega_{r,s,t}=y^{s-1}z^{t-d/\ell}\,dy\;\middle|\;r+ks+\ell t=0 \bmod d,\;\begin{pmatrix}1\leqslant r\leqslant d-1,\\1\leqslant s\leqslant \frac{d}{k}-1,\\1\leqslant t\leqslant \frac{d}{\ell}-1\end{pmatrix}\right\}\end{aligned}\tag{77}$$

defines a basis for the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(C_d)$ whose Hodge split is given by

$$\begin{aligned}&\mathrm{H}^0\big(C_d,\Omega^1_{\mathbb{C}}\big)=\{\omega_{r,s,t}\mid r+ks+\ell t=d\},\\&\mathrm{H}^1(C_d,\mathcal{O}_{\mathbb{C}})=\{\omega_{r,s,t}\mid r+ks+\ell t=2d\}.\end{aligned}\tag{78}$$

In order to show this we view the weighted projective space as the quotient of projective space with respect to the actions $\mathbb{Z}_k:[0,1,0]$ and $\mathbb{Z}_\ell:[0,0,1]$, where we use the abbreviation $\mathbb{Z}_k=\mathbb{Z}/k\mathbb{Z}$ and for any group $\mathbb{Z}_r$ the notation $[a,b,c]$ indicates the action

$$[a,b,c]:(x,y,z)\mapsto\big(\gamma^a x,\gamma^b y,\gamma^c z\big),\tag{79}$$

where γ is a generator of the group. This allows us to view the weighted curve as the quotient of a projective Fermat type curve

$$\mathbb{P}_{(1,k,\ell)}[d]=\mathbb{P}_2[d]/\mathbb{Z}_k\times\mathbb{Z}_\ell:\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}.\tag{80}$$

These weighted curves are smooth and hence their cohomology is determined by considering those forms on the projective curve $\mathbb{P}_2[d]$ which are invariant with respect to the group actions. A basis for $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}_2[d])$ is given by the set of forms

$$\begin{aligned}\mathrm{H}^1_{\mathrm{dR}}\big(\mathbb{P}_2[d]\big)=\big\{\omega_{r,s,t}=y^{s-1}z^{t-d}\,dy\mid 0<r,s,t<d,\\ r+s+t=0\ (\mathrm{mod}\ d),\ r,s,t\in\mathbb{N}\big\}.\end{aligned}\tag{81}$$

Denote the generator of the $\mathbb{Z}_k$ action by α and consider the induced action on $\omega_{r,s,t}$

$$\mathbb{Z}_k:\omega_{r,s,t}\mapsto\alpha^s\omega_{r,s,t}.\tag{82}$$

It follows that the only forms that descend to the quotient with respect to $\mathbb{Z}_k$ are those for which $s=0\ (\mathrm{mod}\ k)$. Similarly we denote by β the generator of the action $\mathbb{Z}_\ell$ and consider the induced action on the forms $\omega_{r,s,t}$

$$\mathbb{Z}_\ell:\omega_{r,s,t}\mapsto\beta^{t-d}\omega_{r,s,t}.\tag{83}$$

We see that the only forms that descend to the quotient are those for which $t=0\ (\mathrm{mod}\ \ell)$.

6.3. *Abelian varieties from weighted Jacobians*

Jacobian varieties in general are not abelian varieties with complex multiplication. The question we can ask, however, is whether the Jacobians of the curves that determine the cohomology of the Calabi–Yau varieties can be decomposed such that the individual factors admit complex multiplication by an order of a number field. In this section we show

that this is indeed the case and therefore we can define the complex multiplication type of a Calabi–Yau variety in terms of the CM types induced by the Jacobians of its curves.

It was shown by Faddeev [34][2] that the Jacobian variety $J(C_d)$ of Fermat curves $C_d \subset \mathbb{P}_2$ splits into a product of abelian factors $A_{\mathcal{O}_i}$

$$J(C_d) \cong \prod_{\mathcal{O}_i \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{\mathcal{O}_i}, \tag{84}$$

where the set $\mathcal{I}$ provides a parametrization of the cohomology of C_d, and the sets $\mathcal{O}_i$ are orbits in $\mathcal{I}$ of the multiplicative subgroup $(\mathbb{Z}/d\mathbb{Z})^\times$ of the group $\mathbb{Z}/d\mathbb{Z}$. More precisely it was shown that there is an isogeny

$$i : J(C_d) \to \prod_{\mathcal{O}_i \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{\mathcal{O}_i}, \tag{85}$$

where an isogeny $i : A \to B$ between abelian varieties is defined to be a surjective homomorphism with finite kernel. In the parametrization used in the previous subsection $\mathcal{I}$ is the set of triplets (r, s, t) in (81) and the periods of the Fermat curve have been computed by Rohrlich [37] to be

$$\int_{\mathcal{A}^j \mathcal{B}^k \kappa} \omega_{r,s,t} = \frac{1}{d} B\left(\frac{s}{d}, \frac{t}{d}\right)\left(1 - \xi^s\right)\left(1 - \xi^t\right)\xi^{js+kt}, \tag{86}$$

where ξ is a primitive dth root of unity, and

$$B(u, v) = \int_0^1 t^{u-1}(1 - v)^{v-1} dt \tag{87}$$

is the classical beta function. $\mathcal{A}$, $\mathcal{B}$ are the two automorphism generators

$$\mathcal{A}(1, y, z) = (1, \xi y, z), \qquad \mathcal{B}(1, y, z) = (1, y, \xi z) \tag{88}$$

and κ is the generator of $\mathrm{H}_1(C_d)$ as a cyclic module over $\mathbb{Z}[\mathcal{A}, \mathcal{B}]$. The period lattice of the Fermat curve therefore is the span of

$$\left(\ldots, \xi^{jr+ks}\left(1 - \xi^r\right)\left(1 - \xi^s\right)\frac{1}{d} B\left(\frac{r}{d}, \frac{s}{d}\right), \ldots\right)_{\substack{1 \leqslant r,s,t \leqslant d-1 \\ r+s+t=d}}, \quad \forall 0 \leqslant j, k \leqslant d - 1. \tag{89}$$

The abelian factor $A_{[(r,s,t)]}$ associated to the orbit $\mathcal{O}_{r,s,t} = [(r, s, t)]$ can be obtained as the quotient

$$A_{[(r,s,t)]} = \mathbb{C}^{\varphi(d_0)/2}/\Lambda_{r,s,t}, \tag{90}$$

where $d_0 = d/\gcd(r, s, t)$ and the lattice $\Lambda_{r,s,t}$ is generated by elements of the form

$$\sigma_a(z)\left(1 - \xi^{as}\right)\left(1 - \xi^{at}\right)\frac{1}{d} B\left(\frac{\langle as\rangle}{d}, \frac{\langle at\rangle}{d}\right), \tag{91}$$

[2] More accessible references on the subject are [35–37].

where $z \in \mathbb{Z}[\mu_{d_0}]$, $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\mu_{d_0})/\mathbb{Q})$ runs through subgroups of the Galois group of the cyclotomic field $\mathbb{Q}(\mu_{d_0})$ and $\langle x \rangle$ is the smallest integer $0 \leqslant x < 1$ congruent to x mod d.

Alternatively, the abelian variety $A_d^{r,s,t}$ can be constructed in a more geometric way as follows. Consider the orbifold of the Fermat curve C_d with respect to the group defined as

$$G_d^{r,s,t} = \left\{(\xi_1, \xi_2, \xi_3) \in \mu_d^3 \;\middle|\; \xi_1^r \xi_2^s \xi_3^t = 1\right\}. \tag{92}$$

The quotient $C_d/G_d^{r,s,t}$ can be described algebraically via projections

$$\begin{aligned} &T_d^{r,s,t} : C_d \to C_d^{r,s,t}, \\ &(x, y) \mapsto \left(x^d, x^r y^s\right) =: (u, v), \end{aligned} \tag{93}$$

which map C_d into the curves

$$C_d^{r,s,t} = \left\{v^d = u^r (1-u)^s\right\}. \tag{94}$$

For prime degrees the abelian varieties $A_d^{r,s,t}$ can be defined simply as the Jacobians $J(C_d^{r,s,t})$ of the projections $C_d^{r,s,t}$. When d has nontrivial divisors $m|d$, this definition must be modified as follows. Consider the projected Fermat curves

$$\begin{aligned} &C_d \to C_m, \\ &(x, y) \mapsto (\bar{x}, \bar{y}) := \left(x^{\frac{d}{m}}, y^{\frac{d}{m}}\right), \end{aligned} \tag{95}$$

whose Jacobians can be embedded as $e : J(C_m) \to J(C_d)$. Composing the projection $T_d^{r,s,t}$ as

$$J(C_m) \xrightarrow{e} J(C_d) \xrightarrow{T_d^{r,s,t}} J\left(C_d^{r,s,t}\right) \tag{96}$$

for all proper divisors $m|d$ leads to a collection of subvarieties $\bigcup_{m|d} T_d^{r,s,t}(e(J(C_m)))$. The abelian variety of interest then is defined as

$$A_d^{r,s,t} = J\left(C_d^{r,s,t}\right) / \bigcup_{m|d} T_d^{r,s,t}\left(e\left(J(C_m)\right)\right). \tag{97}$$

The abelian varieties $A_d^{r,s,t}$ are not necessarily simple but it can happen that they in turn can be factored. This question can be analyzed via a criterion of Shimura–Taniyama, described in [27]. Applied to the $A_d^{r,s,t}$ discussed here the Shimura–Taniyama criterion involves computing for each set $H_d^{r,s,t}$ defined as

$$\mathrm{H}_d^{r,s,t} := \left\{a \in (\mathbb{Z}/d\mathbb{Z})^\times \mid \langle ar \rangle + \langle aks \rangle + \langle a\ell t \rangle = d\right\} \tag{98}$$

another set $W_d^{r,s,t}$ defined as

$$W_d^{r,s,t} = \left\{a \in (\mathbb{Z}/d\mathbb{Z})^\times \mid a H_d^{r,s,t} = H_d^{r,s,t}\right\}. \tag{99}$$

If the order $|W_d^{r,s,t}|$ of $W_d^{r,s,t}$ is unity then the abelian variety $A_d^{r,s,t}$ is simple, otherwise it splits into $|W_d^{r,s,t}|$ factors [38].

We adapt this discussion to the weighted case. Denote the index set of triples (r, s, t) parametrizing the one-forms of the weighted curves $C_d \in \mathbb{P}_{1,k,\ell}[d]$ again by $\mathcal{I}$. The cyclic

group $(\mathbb{Z}/d\mathbb{Z})^\times$ again acts on $\mathcal{I}$ and produces a set of orbits

$$\mathcal{O}_{r,s,t} = \big[(r,s,t)\big] \in \mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times. \tag{100}$$

Each of these orbits leads to an abelian variety $A_{[(r,s,t)]}$ of dimension

$$\dim A_{[(r,s,t)]} = \frac{1}{2}\varphi(d_0), \tag{101}$$

where φ is the Euler function $\varphi(n) = \#\{m \mid (m,n) = 1\}$, and complex multiplication with respect to the field $F_{[(r,s,t)]} = \mathbb{Q}(\mu_{d_0})$, where $d_0 = d/\gcd(r, ks, \ell t)$. This leads to an isogeny

$$i : J(C_d) \to \prod_{[(r,s,t)]\in\mathcal{I}/(\mathbb{Z}/d\mathbb{Z})^\times} A_{[(r,s,t)]}. \tag{102}$$

The complex multiplication type of the abelian factors $A_{r,s,t}$ of the Jacobian $J(C)$ can be identified with the set $H_d^{r,s,t}$ via a homomorphism from $\mathrm{H}_d^{r,s,t}$ to the Galois group. More precisely, the CM type is determined by the subgroup $G_d^{r,s,t}$ of the Galois group of the cyclotomic field that is parametrized by $\mathrm{H}_d^{r,s,t}$

$$G_d^{r,s,t} = \big\{\sigma_a \in \mathrm{Gal}\big(\mathbb{Q}(\mu_{d_0})/\mathbb{Q}\big) \mid a \in \mathrm{H}_d^{r,s,t}\big\} \tag{103}$$

by considering

$$\big(F, \{\phi_a\}\big) = \big(\mathbb{Q}(\mu_{d_0}), \big\{\sigma_a \mid \sigma_a \in G_d^{r,s,t}\big\}\big). \tag{104}$$

7. The Fermat quintic threefold

7.1. CM type

Consider the projective threefold embedded in projective 4-space and defined by

$$X_5 = \big\{(z_0 : z_1 : \cdots : z_5) \in \mathbb{P}_4 \mid z_0^5 + \cdots + z_4^5 = 0\big\}. \tag{105}$$

We can split $d = 3 = 1 + 2 = r + s$ and apply the Shioda–Katsura construction to obtain the decompositions

$$\mathrm{H}^3(X_5) \oplus \mathrm{H}^1(C_5)(1) \cong \mathrm{H}^3(C_5 \times S_5)^{\mu_5} \oplus \mathrm{H}^1\big(X_5^0 \times C_d\big)(1) \tag{106}$$

and

$$\mathrm{H}^2(S_5) \cong \mathrm{H}^2(C_5 \times C_5)^{\mu_5} \oplus d(d-2)\mathrm{H}^2(\mathbb{P}_1) \tag{107}$$

in terms of the cohomology groups of the Fermat curve

$$C_5 = \big\{x^5 + y^5 + z^5 = 0\big\} \subset \mathbb{P}_2 \tag{108}$$

and the Fermat surface S_5.

From this we see that the basic building block of the cohomology decomposition is given by the plane projective curve C_5 which has genus $g(C_5) = 6$. The index set $\mathcal{I}$

$$\mathcal{I} = \big\{(1,1,3),(1,3,1),(3,1,1),(1,2,2),(2,1,2),(2,2,1);\\ (2,4,4),(4,2,4),(4,4,2),(3,3,4),(3,4,3),(4,3,3)\big\}$$

parametrizes a basis of the first cohomology group of C_5, which can be written as

$$\mathrm{H}^1_{\mathrm{dR}}(C_5) = \big\{\omega_{r,s,t} = x^{r-1}y^{s-5}\,dx \mid (r,s,t) \in \mathcal{I}\big\}. \tag{109}$$

The action of $(\mathbb{Z}/5\mathbb{Z})^\times$ leads to the orbits

$$\begin{aligned}
\mathcal{O}_{1,1,3} &= \big\{(1,1,3),(2,2,1),(3,3,4),(4,4,2)\big\},\\
\mathcal{O}_{1,3,1} &= \big\{(1,3,1),(2,1,2),(3,4,3),(4,2,4)\big\},\\
\mathcal{O}_{3,1,1} &= \big\{(3,1,1),(1,2,2),(4,3,3),(2,4,4)\big\}.
\end{aligned} \tag{110}$$

Hence the Jacobian decomposes into a product of three abelian varieties

$$J(C_5) = \prod_{\mathcal{O}_{r,s,t} \in \mathcal{I}/(\mathbb{Z}/5\mathbb{Z})^\times} A_{r,s,t} = A_{1,1,3} \times A_{1,3,1} \times A_{3,1,1}, \tag{111}$$

each of dimension $\varphi(5)/2 = 2$, which arise from the Jacobians of the genus two curves

$$\begin{aligned}
C_5^{1,1,3} &= \big\{v^5 - u(1-u) = 0\big\},\\
C_5^{1,3,1} &= \big\{v^5 - u(1-u)^3 = 0\big\},\\
C_5^{3,1,1} &= \big\{v^5 - u^3(1-u) = 0\big\},
\end{aligned} \tag{112}$$

obtained via the maps $T_5^{r,s,t}$.

In order to check the simplicity of the abelian factors we can use the criterion of Shimura–Taniyama, described above. Computing the sets $W_5^{r,s,t}$ for any of the triplets (r,s,t) shows that the order of these groups is unity, hence all three factors are in fact simple.

For the complex multiplication type we find from

$$\mathrm{H}_5^{1,1,3} = \big\{a \in (\mathbb{Z}/5\mathbb{Z})^\times = \{1,2,3,4\} \mid \langle a\rangle + \langle a\rangle + \langle 3a\rangle = 5\big\} = \{1,2\} \tag{113}$$

that $G_5^{1,1,3} = \{\sigma_1, \sigma_2\}$ and therefore the complex multiplication type of $A_{1,1,3}$ is given by

$$\big(\mathbb{Q}(\mu_5), \{\varphi\} = \{\sigma_1, \sigma_2\}\big). \tag{114}$$

The remaining factors are described in the same way.

More explicitly, we can use the maps $T_5^{r,s,t}$ to express the differentials of C_d invariant under the action of $G_5^{r,s,t}$ in terms of the (u,v) coordinates of $C_5^{r,s,t}$ and observe their transformation behavior under the map

$$(u,v) \mapsto (u, \xi_5 v). \tag{115}$$

7.2. Fusion field and quantum dimensions

The field of complex multiplication derived for the quintic is given by the cyclotomic field $\mathbb{Q}(\mu_5)$ and embedded in this field is the real subfield $\mathbb{Q}(\sqrt{5})$, generated by the elements $(\xi_5 + \xi_5^{-1})$. To compare this to the number field determined by the string we briefly recall some facts about the corresponding Gepner model [39,40].

The underlying exactly solvable model of the quintic threefold is determined by the affine Kac–Moody algebra $SU(2)$ at conformal level $k = 2$. The central charge $c(k) = 3k/(k+2)$ at level k then leads to $c = 9/5$, leading to a product of five models to make a theory of total charge $c = 9$. The physical spectrum of this model is constructed from world sheet operators of the individual $SU(2)$ factors with the anomalous dimensions

$$\Delta_j^{(k)} = \frac{j(j+2)}{4(k+2)}, \quad j = 0, \ldots, k, \tag{116}$$

leading in the case $k = 3$ to $\Delta_j^{(3)} \in \{0, \frac{3}{20}, \frac{2}{5}, \frac{3}{4}\}$.

These anomalous dimensions can be mapped into the quantum dimensions Q_{ij} via the Rogers dilogarithm. Denote by Li_2 Euler's classical dilogarithm

$$\mathrm{Li}_2(z) = \sum_{n \in \mathbb{N}} \frac{z^n}{n^2}, \tag{117}$$

and by $L(z)$ the Rogers dilogarithm

$$L(z) = \mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z). \tag{118}$$

Then there exist relations between anomalous dimensions and the quantum dimensions Q_{ij} [41–43]

$$\frac{1}{L(1)} \sum_{i=1}^{k} L\left(\frac{1}{Q_{ij}^2}\right) = \frac{3k}{k+2} - 24\Delta_j^{(k)} + 6j, \tag{119}$$

where the Q_{ij} are defined as

$$Q_{ij} = \frac{S_{ij}}{S_{0j}}, \tag{120}$$

in terms of the modular S-matrix

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(i+1)(j+1)\pi}{k+2}\right), \quad 0 \leqslant i, j \leqslant k \tag{121}$$

describe the modular behavior of the $SU(2)$ affine characters

$$\chi_i\left(-\frac{1}{\tau}, \frac{u}{\tau}\right) = e^{\pi i k u^2/2} \sum_j S_{ij} \chi_j(\tau, u). \tag{122}$$

Applying this map to the theory at conformal level three leads to the quantum dimensions $Q_i = Q_{i0}$

$$Q_i\big(SU(2)_3\big) \in \left\{1, \frac{1}{2}(1+\sqrt{5})\right\} \subset \mathbb{Q}(\sqrt{5}). \tag{123}$$

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