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HAMILTON DECOMPOSITIONS OF 6-REGULAR CAYLEY GRAPHS ON EVEN ABELIAN GROUPS WITH INVOLUTION-FREE CONNECTIONS SETS

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Abstract. Alspach conjectured that every connected Cayley graph on a finite Abelian group A is Hamilton-decomposable. Liu has shown that for $|A|$ even, if $S = \{s_1, \ldots, s_k\} \subset A$ is an inverse-free strongly minimal generating set of A, then the Cayley graph $\mathrm{Cay}(A; S^\star)$, is decomposable into k Hamilton cycles, where $S^\star$ denotes the inverse-closure of S. Extending these techniques and restricting to the 6-regular case, this article relaxes the constraint of strong minimality on S to require only that S be strongly a-minimal, for some $a \in S$ and the index of $\langle a \rangle$ be at least four. Strong a-minimality means that $2s \notin \langle a \rangle$ for all $s \in S \setminus \{a, -a\}$. Some infinite families of open cases for the 6-regular Cayley graphs on even order Abelian groups are resolved. In particular, if $|s_1| \geq |s_2| > 2|s_3|$, then $\mathrm{Cay}(A; \{s_1, s_2, s_3\}^\star)$ is Hamilton-decomposable.

1. Introduction

In 1969, Lovász [1] posed the following problem: "Let us construct a finite, connected, undirected graph which is [vertex-transitive] and has no simple path containing all vertices." From this problem, the following conjecture was posed: every finite, connected vertex-transitive graph has a Hamilton path. The conjecture has generated a vast and rich body of literature, yet there exist no known examples of connected vertex-transitive graphs which do not possess a Hamilton path. Furthermore, only four non-trivial connected vertex-transitive graphs are known which do not possess a Hamilton *cycle* (Bondy [6]). Even more interesting is the fact that none of these four are Cayley graphs. Ergo, it has been widely conjectured that connected Cayley graphs on three or more vertices have Hamilton cycles. This conjecture too is unresolved, though it has been verified for finite Abelian groups (Marušič [17]). If A is a finite Abelian group, and S is a subset of $A - \{0\}$, that is inverse-closed, i.e., $s \in S \Leftrightarrow -s \in S$, then the *Cayley graph* of A with *connection set* S, is the graph X, denoted $X = \mathrm{Cay}(A; S)$, with $V(X) = A$ and $E(X) = \{\{x, y\} : y - x \in S\}$. In fact, connected Cayley graphs of Abelian groups are rich with Hamilton cycles (see Chen-Quimpo [8]) and Alspach [2] conjectured in 1984 that this class of graphs is Hamilton-decomposable. Alspach's conjecture remains open in general (see the survey by Curran-Gallian [9]) despite being investigated from many vantage points: the valency, the group order, the group type, restrictions on the connection set, etc. The following theorem is a summary of complete results when only the valency is considered:

Theorem 1.1 (Alspach et al. [3], Bermond et al. [5], Dean [11], Fan et al. [12], Liu [14, 15], Westlund et al. [19])**.** *Every connected k-regular Cayley graph on a finite Abelian group A is Hamilton-decomposable if $k \leq 5$ or $k = 6$ and $|A|$ is odd.*

2. Preliminaries

A path (resp. cycle) that spans the vertices of a graph is called a *Hamilton path* (resp. *Hamilton cycle* or *HC*). A *Hamilton decomposition* of a graph is a partition of its edge set into Hamilton cycles if it has

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even valency, or Hamilton cycles and a single perfect matching if it has odd valency. A graph is *Hamilton-decomposable* (or *HD*) if it admits a Hamilton decomposition.

The Cayley graph $X = \mathrm{Cay}(A; S)$ is connected if and only if S is a generating set for A. If $y - x = s \in S$, then $\{x, y\}$ is *generated by* s, or $\{x, y\}$ is an *s-edge.* The subgraph Y of X is *generated by* s if $E(Y)$ consists of all s-edges of $E(X)$. If s is an involution of A, i.e., $s \neq 0$ and $s = -s$, then the subgraph generated by s is a 1-factor of X. If s is a non-involution, then the subgraph generated by s is a 2-factor of X.

Let $S \subset A$. S is *inverse-free* if $s \in S$ implies either $s = -s$ or $-s \notin S$. S is *involution-free* if no element of S is an involution. The *inverse-closure* of S, denoted $S^\star$, is the smallest superset of S that is inverse-closed. S is *minimal* if for all $s \in S$, the subgroup generated by the elements of $S \setminus \{s, -s\}$ does not contain s; and *strongly minimal* if for all $s \in S$, the subgroup generated by $S \setminus \{s, -s\}$ does not contain $2s$. Clearly, any strongly-minimal subset of a group and any minimal subset of an odd order group is involution-free.

Liu has proved some general results on Alspach's conjecture when restrictions on the connection set are considered.

Theorem 2.1 (Liu [15, 16])**.** *Let A be an Abelian group and S an inverse-free subset of $A - \{0\}$ that generates A. The Cayley graph $\mathrm{Cay}(A; S^\star)$, is HD if either one of the following hold:*

1. *The order of A is odd, and S is minimal.*
2. *The order of A is even, at least four, and S is strongly minimal.*

Theorem 2.2 ([16])**.** *Let A be an Abelian group of even order at least four, and $S \subset A$ be an inverse-free minimal generating set. If $\langle s \rangle$ has odd index for all $s \in S$, then $\mathrm{Cay}(A; S^\star)$ is HD.*

Alspach, Caliskan, and Kreher [4] have recently made significant advances in outlining new approaches to obtaining Hamilton decompositions via liftings, with a focus on Cayley graphs of odd order. This manuscript only examines the specific case of 6-regular even order Cayley graphs on finite Abelian groups. The following results apply to 6-regular Cayley graphs.

Theorem 2.3 ([12])**.** *If A is an Abelian group of odd order generated by $S = \{s_1, s_2, s_3\} \subset A - \{0\}$, which is inverse-free and involution-free, such that $|s_1| \geq |s_2| > |s_3|$, then $\mathrm{Cay}(A; S^\star)$ is HD.*

Theorem 2.4 (Dean [10, 11])**.** *Let A be a cyclic group generated by $S = \{s_1, s_2, s_3\} \subset A - \{0\}$, where S is inverse-free and involution-free. $\mathrm{Cay}(A; \{s_1, s_2, s_3\}^\star)$ is HD if either one of the following hold:*

1. *A has odd order.*
2. *A has even order, and $A = \langle s_3 \rangle$.*

Theorem 2.5 (Westlund [18])**.** *Let A be an Abelian group of even order generated by $S = \{s_1, s_2\} \subset A - \{0\}$, where S is involution-free and inverse-free. If there exists an $s_3 \in A$, such that $\langle s_3 \rangle$ has index two in A, then $\mathrm{Cay}(A; (S \cup \{s_3\})^\star)$ is HD.*

3. New Results

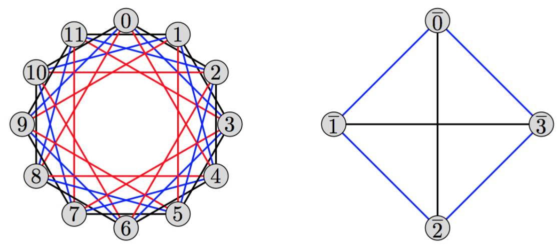
This article relaxes the strong minimality requirement of Theorem 2.1 (2) in the 6-regular case. The following definition will be used throughout:

Definition 3.1. A subset $S \subseteq A$ is *strongly a-minimal* for some fixed $a \in S$, if $2s \notin \langle a \rangle$ for all $s \in S \setminus \{a, -a\}$.

Clearly, if S is strongly minimal, then S is strongly a-minimal for all $a \in S$. The converse is not true. For example, $S = \{(1, 0), (0, 1), (3, 1)\} \subset \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is strongly a-minimal for all $a \in S$, yet S is not even minimal.

Given a subgroup $B \leq A$, and $a \in A$, use $\overline{a}$ to denote the coset $a + B$ in A, and $|A : B|$ the index of B in A. Cayley graphs of quotient groups have been used to find Hamilton cycles. If $X = \mathrm{Cay}(A; S^\star)$, and $B \leq A$, then $\overline{X} = \mathrm{Cay}(A/B; \overline{S}^\star)$, where $\overline{S}^\star = \{\overline{s} \colon s \in S^\star\}$, is called the *quotient* (multi)graph of X by B (see Figure 1). The next proposition follows immediately from the definitions so far.

Proposition 3.2. *If A is an Abelian group, and $S = \{s_1, \ldots, s_k\} \subset A - \{0\}$ is a generating set for A that is both inverse-free and involution-free, then $\mathrm{Cay}(A; S^\star)$ is $2k$-regular and connected. Furthermore, if S is strongly a-minimal, for some $a \in S$, then $\mathrm{Cay}(A/\langle a \rangle; \overline{S}^\star)$ is $2(k-1)$-regular.*



$X = \mathrm{Cay}(\mathbb{Z}_{12}; \{2,3,4\}^\star)$ $\qquad$ $\overline{X} = \mathrm{Cay}(\mathbb{Z}_{12}/\langle 4\rangle; \{\overline{2}, \overline{3}\}^\star)$

FIGURE 1. A 6-regular Cayley graph and one of its cubic quotient graphs. The quotient is not 4-regular because the connection set of X is not strongly 4-minimal. See Proposition 3.2.

The focus of this manuscript is on Cayley graphs $\mathrm{Cay}(A; S^\star)$, where $S = \{s_1, s_2, s_3\} \subset A - \{0\}$ and S is both involution-free and inverse-free. By Proposition 3.2 $\mathrm{Cay}(A; S^\star)$ is 6-regular, and each element of S generates a 2-factor. Note, that if S is not involution-free, then $|S^\star| \leq 5$, and so $X = \mathrm{Cay}(A; S^\star)$ is k-regular for some $k \in \{1,2,3,4,5\}$. By Theorem 1.1 X is HD. The following theorem, proved in [12], forms the basis for a portion of the proof of the Main Result.

Theorem 3.3 ([12])**.** *Let A be an Abelian group and $S = \{s_1, s_2, s_3\} \subset A - \{0\}$ be an inverse-free and involution-free generating set for A. If S is strongly s_3-minimal, where $|s_3|$ is odd, and $|A : \langle s_3\rangle| \geq 9$, then $\mathrm{Cay}(A; S^\star)$ is HD.*

This article resolves certain cases not covered by Theorems 3.3 or 2.5. The following Main Result is an extension of Theorem 3.3.

Theorem 3.4 (Main Result)**.** *Let A be an Abelian group and $S = \{s_1, s_2, s_3\} \subset A - \{0\}$ be an inverse-free and involution-free generating set for A. $\mathrm{Cay}(A; S^\star)$ is HD if S is strongly s_3-minimal and either one of the following holds:*

1. *$\langle s_3\rangle$ has index at least four in A.*
2. *$\langle s_3\rangle$ has index three in A and $\langle s_1\rangle$ and $\langle s_2\rangle$ have even index.*

Proof. By Lemma 5.13, X is a $D_3(m,n)$-graph (see Definition 5.2), where $m = |s_3| \geq 3$ and $n = |A : \langle s_3\rangle| \geq 3$. If $n \geq 4$, then X is HD by Theorem 5.11, and Lemmas 6.2–6.6), 7.2–7.7, which resolve the corresponding open cases of Remark 5.12. If $n = 3$, and $\langle s_1\rangle$ and $\langle s_2\rangle$ both have even index, then X is HD by Lemma 6.5. □

The hypotheses of Theorem 3.4 implicitly forbid $\langle s_3\rangle$ from having index two. The glaring omission to Theorem 3.4 is when $\langle s_3\rangle$ has index three, and at least one of $\langle s_i\rangle$ for $i \in \{1,2\}$ has odd index. Additionally, it remains to resolve the case when connection sets possess at least one involution. Even the case when there are involutions in the connection set of $\overline{X}$ is still open.

The following corollary shows that X is HD even in the case when the connection set is far from minimal. The hypotheses of Corollary 3.5 directly imply the conditions of Theorem 3.4.

Corollary 3.5. *If $X = \mathrm{Cay}(A; \{s_1, s_2, s_3\}^\star)$ is a 6-regular, Cayley graph on an Abelian group A of even order generated by both $\{s_2, s_3\}$ and $\{s_1, s_3\}$, and $\langle s_3\rangle$ has index at least four in A, then X is HD.*

Corollary 3.6 provides an even order analogue to Theorem 2.3.

Corollary 3.6. *If $X = \mathrm{Cay}(A; \{s_1, s_2, s_3\}^\star)$ is a connected, 6-regular, Cayley graph on an Abelian group A of even order, then X is HD if*

1. *$|s_1| \geq |s_2| > 2|s_3|$; or*
2. *$|s_1| \geq |s_2| > |s_3|$, and either*
   a. *$\langle s_3\rangle$ has odd index at least five, or*
   b. *$\langle s_3\rangle$ had index at least four, and $|s_1|$ and $|s_2|$ are odd.*

Proof. If (1) holds, then assume $|A| > |s_1|$, for the case of equality is resolved by Theorem 2.4. Clearly, the connection set is strongly s_3-minimal and $|A| \geq 2|s_1| > 4|s_3|$, so that $|A : \langle s_3 \rangle| > 4$. Then X is HD by Theorem 3.4. The proof of (2) follows similarly. □

4. Color-switching configurations

This section lists some techniques to create Hamilton cycles, relying considerably on the notation and techniques developed in [14] and [12]. Let $A_n = a_1 a_2 \cdots a_n a_1$ and $B_m = b_1 b_2 \cdots b_m b_1$ denote cycles of length n and m. Definitions 4.1–4.2 and Remarks 4.3–4.4 appeared in [14].

Definition 4.1. The *r-pseudo-cartesian product* of A_n and B_m, denoted $A_n \times_r B_m$, where $0 \leq r < m$, is the simple graph with vertex set $\{(a_i, b_j) : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and edge set consisting of *horizontal* and *vertical* edges.

$$\text{Horizontal edges} : \{\{(a_i, b_j), (a_{i+1}, b_j)\}, \{(a_n, b_j), (a_1, b_{j+r})\} : 1 \leq i < n, 1 \leq j \leq m\}$$
$$\text{Vertical edges} : \{\{(a_i, b_j), (a_i, b_{j+1})\}, \{(a_i, b_1), (a_i, b_m)\} : 1 \leq i \leq n,\ 1 \leq j < m\}$$

For a fixed i, where $1 \leq i \leq n$, the *a_i-column* is the subgraph induced on the vertices $\{(a_i, b_j) : 1 \leq j \leq m\}$. For a fixed j, where $1 \leq j \leq m$, the *b_j-row* is the subgraph induced on the vertices $\{(a_i, b_j) : 1 \leq i \leq n\}$.

E.g., the left graph in Figure 7 is a 6-psuedo-cartesian product of A_6 and B_8.

Definition 4.2. Color the vertical edges of $A_n \times_r B_m$ red and the horizontal edges green. For fixed integers i and j, where $1 \leq i < n$ and $1 \leq j \leq m$, define an $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-*color switch* as an operation that interchanges the color of the edges

$$\{\{(a_i, b_j), (a_{i+1}, b_j)\}, \{(a_i, b_{j+1}), (a_{i+1}, b_{j+1})\}\}$$

with the color of the edges

$$\{\{(a_i, b_j), (a_i, b_{j+1})\}, \{(a_{i+1}, b_j), (a_{i+1}, b_{j+1})\}\}.$$

Similarly, for a fixed integer j, where $1 \leq j \leq m$, an $\{a_n, a_1, b_j, b_{j+1}\}$-*color switch* interchanges the color of the edges

$$\{\{(a_n, b_j), (a_1, b_{j+r})\}, \{(a_n, b_{j+1}), (a_1, b_{j+1+r})\}\}$$

with the color of the edges

$$\{\{(a_n, b_j), (a_n, b_{j+1})\}, \{(a_1, b_{j+r}), (a_1, b_{j+1+r})\}\}.$$

For brevity, we shall denote this as $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS. A *color-switching configuration*, abbreviated CSC, is a set of color-switches that are pairwise edge-disjoint.

The effect of a CSC on the edge-colors is invariant with respects to the order in which the individual color-switches are applied.

Remark 4.3. The horizontal edges of $A_n \times_r B_m$ in the b_i and b_j-rows are on the same cycle if and only if $i \equiv j \pmod{\gcd(r, m)}$. Thus, if $\gcd(r, m) = t$, then the horizontal edges form a 2-factor, H, which consists of t cycles of length mn/t, and any consecutive t rows of $A_n \times_r B_m$ are on t different cycles of H.

The following fundamental facts are presented without proof.

Remark 4.4. Suppose C_1 and C_2 are vertex-disjoint cycles in a graph X, and $\{x_i, y_i\} \in E(C_i)$ for $i \in \{1, 2\}$. If $C = x_1 y_1 x_2 y_2$ is a cycle of length four in X, and the edges $\{y_1, x_2\}$ and $\{y_2, x_1\}$ are not in $E(C_1) \cup E(C_2)$, then the subgraph of X whose edge set is the symmetric difference

$$(E(C_1) \cup E(C_2)) \oplus E(C)$$

is a single cycle. In particular, suppose the vertical edges of $A_n \times_r B_m$ are colored red and the horizontal edges are colored green. If the a_i and a_{i+1}-columns are on vertex disjoint red cycles (respectively, the b_j and b_{j+1}-rows are on vertex disjoint green cycles), then applying an $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS, will join the two red cycles into a single red cycle (respectively, join the two green cycles into a single green cycle). (See the leftmost diagram of Figure 2.) If $\{(a_i, b_j), (a_i, b_{j+1})\}$ and $\{(a_{i+1}, b_j), (a_{i+1}, b_{j+1})\}$ lie on a common cycle C, of length d and color c, and are separated by at least two edges, then applying an $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS

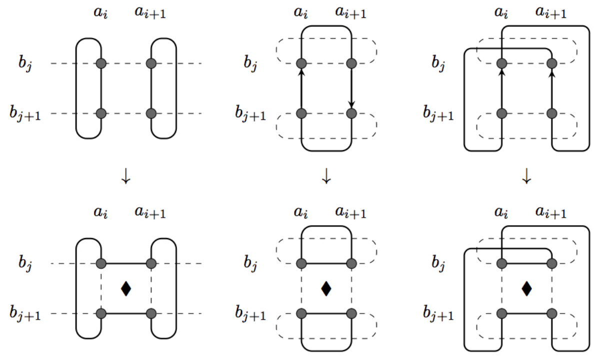


FIGURE 2. Color-switches (depicted using ♦) illustrating Lemma 4.4.

will produce a cycle also having length d and color c, if and only if, upon making C a directed cycle, the subsequence of vertices $(a_i, b_{j+1}), (a_i, b_j), (a_{i+1}, b_{j+1}), (a_{i+1}, b_j)$ appears on C in that order. (See the right two diagrams in Figure 2.)

Lemma 4.5. *The graph $A_n \times_r B_m$ is isomorphic to* $\mathrm{Cay}(A; \{a, b\}^\star)$, *via* $(a_i, b_j) \mapsto (i-1)a + (j-1)b$, *for all i and j satisfying $1 \le i \le n$ and $1 \le j \le m$, where $na = rb$.*

Theorem 4.6 ([12])**.** *For all integers n and m, where $n \ge 3$, $m \ge 3$, the graph $A_n \times_r B_m$ is HD.*

The following CSCs are an important tool in creating longer cycles (see Figures 3 and 4).

Definition 4.7. Let d be a fixed integer where $d = 2d' \ge 2$.

- *Left-alternating horizontal switch*, $\{a_i, a_{i+d}, b_j\}$-LAHS, defined by

$$\{\{a_{i+2x}, a_{i+1+2x}, b_j, b_{j+1}\}\text{-CS}, \{a_{i+1+2x}, a_{i+2+2x}, b_{j-1}, b_j\}\text{-CS} : 0 \le x \le (d-2)/2\}.$$

- *Right-alternating horizontal switch*, $\{a_i, a_{i+d}, b_j\}$-RAHS, defined by

$$\{\{a_{i+2x}, a_{i+1+2x}, b_{j-1}, b_j\}\text{-CS}, \{a_{i+1+2x}, a_{i+2+2x}, b_j, b_{j+1}\}\text{-CS} : 0 \le x \le (d-2)/2\}.$$

- *Left-alternating vertical switch*, $\{a_i, b_j, b_{j+d}\}$-LAVS, defined by

$$\{\{a_{i-1}, a_i, b_{j+2x}, b_{j+1+2x}\}\text{-CS}, \{a_i, a_{i+1}, b_{j+1+2x}, b_{j+2+2x}\}\text{-CS} : 0 \le x \le (d-2)/2\}.$$

- *Right-alternating vertical switch*, $\{a_i, b_j, b_{j+d}\}$-RAVS, defined by

$$\{\{a_i, a_{i+1}, b_{j+2x}, b_{j+1+2x}\}\text{-CS}, \{a_{i-1}, a_i, b_{j+1+2x}, b_{j+2+2x}\}\text{-CS} : 0 \le x \le (d-2)/2\}.$$

The rest of this section lists a summary of CSCs developed by Liu and Fan et al. and some simple extensions. The following theorem combines and slightly generalizes Lemmas 3.12 and 3.14 in [16].

Theorem 4.8. *Partition the edges of $A_n \times_r B_m$ into t horizontal green cycles and n vertical red cycles, where $m = kt$. Fix integers i, ℓ, z, and d, that satisfy the inequalities: $2 \le i < z \le n$; $0 \le \ell < m$; and $1 \le d \le k$.*

1. *If $t = 2t'+1 \ge 3$, then an $\{a_i, b_{1+\ell}, b_{t+\ell}\}$-LAVS (resp. RAVS) will produce a green HC. Furthermore, the switch, $\{a_z, a_{z+1}, b_{dt+\ell}, b_{dt+1+\ell}\}$-CS will preserve the green HC.*
2. *If $t = 2t' \ge 4$, then an $\{a_i, b_{1+\ell}, b_{t-1+\ell}\}$-LAVS (resp. -RAVS) and either an*

$$\{a_z, a_{z+1}, b_{dt-1+\ell}, b_{dt+\ell}\}\text{-CS } \textit{or an } \{a_z, a_{z+1}, b_{m+\ell}, b_{1+\ell}\}\text{-CS}$$

will produce a green HC.

In either case, the red cycles in the a_{i-1}-, a_i-, and a_{i+1}-columns are joined into a single red cycle.

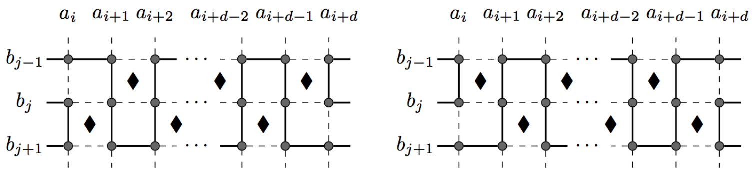


FIGURE 3. An $\{a_i, a_{i+d}, b_j\}$-LAHS (on left) and an $\{a_i, a_{i+d}, b_j\}$-RAHS (on right) from Definition 4.7.

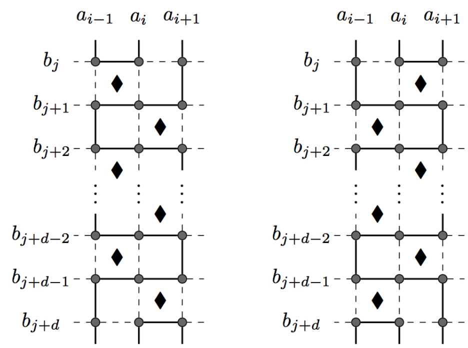


FIGURE 4. An $\{a_i, b_j, b_{j+d}\}$-LAVS (on left) and an $\{a_i, b_j, b_{j+d}\}$-RAVS (on right) from Definition 4.7.

Theorem 4.9 ([16])**.** *Partition the edges of $A_n \times_r B_m$ into t horizontal green cycles and n vertical red cycles, where $m = kt$ and $t = 2t'$. If the colors of the edge sets E_1 and E_2 are switched, where*

$$E_1 = \{(a_1, b_{2j-1})(a_1, b_{2j}) : 1 \leq j \leq m/2\} \cup \{(a_n, b_{2j})(a_n, b_{2j+1}) : 1 \leq j \leq m/2\}$$

and $E_2 = \{(a_n, b_i)(a_1, b_{i+r}) : 1 \leq i \leq m\}$, then the result is a green HC and a red cycle consisting of the vertices in the a_1- and a_n-columns. If this red cycle is oriented, all vertical edges have the same direction.

The following simple observation is obtained immediately by Remark 4.4.

Proposition 4.10. *If the* CS*-configuration of Theorem 4.9 is applied to $A_n \times_r B_m$, then for any positive integer j and even integer d, where $d \geq 2$, applying an $\{a_i, a_{i+d}, b_{2j+1}\}$-*RAHS *will preserve the green HC.*

(See Fig. 2 and 3 of [16] to help visualize the following CS-configuration.)

Theorem 4.11 ([16])**.** *Suppose that $n \geq 6$ and $t = \gcd(r, m) = 2k$ in $A_n \times_r B_m$. Apply an $\{a_i, a_{i+1}, b_i, b_{i+1}\}$-*CS *where $i = 1$ if $k = 1$; $i \in \{1, 2, 3\}$ if $k = 2$; or $i \in \{1, \ldots, 6\}$ if $k \geq 3$. Additionally, if $t \equiv 0 \pmod 4$ and $t \geq 8$, then apply an $\{a_2, b_{t-2}, b_t\}$-*LAVS *and the following* CSC*:*

$$\{\{a_1, a_2, b_j, b_{j+1}\}\text{-CS}, \{a_2, a_3, b_{j+1}, b_{j+2}\}\text{-CS}, \{a_3, a_4, b_{j+2}, b_{j+3}\}\text{-CS}, \{a_4, a_5, b_{j+3}, b_{j+4}\}\text{-CS}\},$$

where j takes on all integer values congruent to 2 modulo 4 in the interval,

$$6 \leq j \leq t - 4 \text{ if } t \equiv 2 \pmod 4 \text{ and } t \geq 10, \text{ or}$$
$$6 \leq j \leq t - 6 \text{ if } t \equiv 0 \pmod 4 \text{ and } t \geq 12.$$

The result produces a green HC and a red 2-factor that includes a cycle containing all red edges in any of the involved columns.

Definition 4.12. If $X = \{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS is a fixed color-switch in $A_n \times_r B_m$, then X is *incident* to the a_i- and a_{i+1}-columns and *incident* to the b_j- and b_{j+1}-rows. Individual edge colors are invariant upon applying X any even number of times. *Removing* a color-switch X means applying X any even number of times.

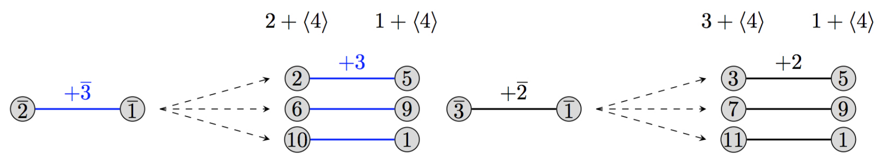


Figure 5. Edge lifts, $L_X\{\overline{2},\overline{1}\}$ and $L_X\{\overline{3},\overline{1}\}$ from Figure 1, illustrating Definition 5.1.

The following slight modification of Theorem 4.11 follows from Remarks 4.3 and 4.4.

Proposition 4.13. *If $n \geq 6$ and $t = \gcd(r,m) = 2k \geq 6$, then upon applying the* CSC *of Theorem 4.11 to $A_n \times_r B_m$, we may remove the $\{a_4, a_5, b_4, b_5\}$-*CS *and $\{a_5, a_6, b_5, b_6\}$-*CS*; and apply an $\{a_4, a_5, b_5, b_6\}$-*CS*; and an $\{a_5, a_6, b_4, b_5\}$-*CS *and the result will preserve the green HC and the red cycle on the a_i-columns, where $1 \leq i \leq 6$.*

Definition 4.14 ([12])**.** Let $K_1, K_2, \ldots, K_k$ be a set of edge-disjoint color switches in $A_n \times_r B_m$, where $k \geq 3$. $\{K_{k-1}, K_k\}$ is a *good pair* if K_{k-2} is to the right of K_i for all $i \leq k-3$, K_{k-1} is to the right of K_{k-2}, K_k is to the right of K_{k-1}, and there exists a positive integer y such that K_{k-2} and K_k are both incident with the b_y- and b_{y+1}-rows, and K_{k-1} is incident with either the b_y- and b_{y-1}-rows or the b_{y+1}- and b_{y+2}-rows.

Theorem 4.15 ([12])**.** *Suppose $K_1, K_2, \ldots, K_k$ is a set of edge-disjoint color-switches in $A_n \times_r B_m$, where $k \geq 3$, and $\{K_{k-1}, K_k\}$ is a good pair. If, after applying $K_1, \ldots, K_{k-2}$, there exists a green HC, then applying K_{k-1} and K_k will preserve the cycle.*

The following result is obtained by applying Definition 4.14 and Theorem 4.15.

Corollary 4.16. *If a green HC is created by applying a* CS*-configuration to $A_n \times_r B_m$ such that, for some $1 \leq i < n$ and $1 \leq j \leq m$, $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-*CS *is a rightmost color-switch, then an $\{a_{i+1}, a_{i+1+d}, b_j\}$-*RAHS *or an $\{a_{i+1}, a_{i+1+d}, b_{j+1}\}$-*LAHS*, for some $d \geq 2$ even, will preserve the HC. Furthermore, removing any two consecutive color-switches in the* RAHS *or* LAHS *will preserve the HC.*

5. Liu's Lifting Technique

Definition 5.1. Suppose $X = \mathrm{Cay}(A; S^\star)$ and $\overline{X} = \mathrm{Cay}(\overline{A}; \overline{S}^\star)$. If $\{\overline{x}, \overline{y}\} \in E(\overline{X})$ and $\overline{x} - \overline{y} = \overline{s}$, for some $\overline{s} \in \overline{S}^\star$, then the set

$$L_X\{\overline{x}, \overline{y}\} = \{\{u, v\} : \overline{u} = \overline{x},\, \overline{v} = \overline{y},\, u - v = s\} \subseteq E(X)$$

is called the *lift* $\{\overline{x}, \overline{y}\}$. Furthermore, given a subgraph $\overline{F}$ of $\overline{X}$, let F be the subgraph of X that is induced on the lifts of all edges of $\overline{F}$. Then F is called the *lift of the subgraph* $\overline{F}$, or we say F *is the subgraph of* X *that* $\overline{F}$ *lifts to.*

Liu established an important relationship between certain 6-regular Cayley graphs and $D_3(m,n)$-graphs, defined below (also see Example 5.5).

Definition 5.2 ([14])**.** For integers m and n, where $m \geq 3$ and $n \geq 3$, let G be a graph of order nm and size $3mn$, with vertex and edge sets:

$$V(G) = \{(a_i, b_j) : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$

$$E(G) = F \cup H_1 \cup H_2, \text{ where } F,\ H_1, \text{ and } H_2 \text{ are edge-disjoint 2-factors:}$$

$$F = \{\{(a_i, b_j), (a_i, b_{j+1})\}, \{(a_i, b_1), (a_i, b_m)\} : 1 \leq i \leq n, 1 \leq j < m\};$$

$$H_1 = \{\{(a_{\sigma_1(i)}, b_j), (a_{\sigma_1(i+1)}, b_j)\}, \{(a_{\sigma_1(n)}, b_j), (a_{\sigma_1(1)}, b_{j+r_1})\} : 1 \leq i < n, 1 \leq j \leq m\};$$

$$H_2 = \{\{(a^2_{\sigma_2(i)}, b_j), (a^2_{\sigma_2(i+1)}, b_j)\}, \{(a^2_{\sigma_2(n)}, b_j), (a^2_{\sigma_2(1)}, b_{j+r_2})\} : 1 \leq i < n, 1 \leq j \leq m\};$$

where r_k is an integer and $0 \leq r_k < m$; σ_1 and σ_2 are permutations of $[n]$; and (a^2_t, b_j) denotes the vertex (a_t, b_{j+h_t}) for some integer h_t, $0 \leq h_t < m$. A graph, X, isomorphic to G is a $D_3(m,n)$-*graph.*

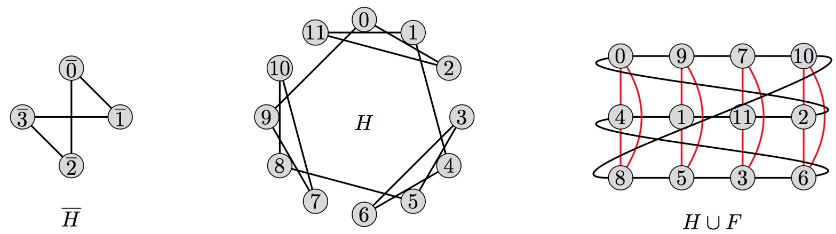


FIGURE 6. A Hamilton cycle $\overline{H}$, of $\overline{X}$ lifts to a 2-factor H of X from Figure 1 (in this case H is itself a HC), such that $H \cup F \cong A_4 \times_2 B_3$, where F is the 2-factor generated by $\langle 4 \rangle$. See Theorem 5.4.

Remark 5.3 ([14])**.** $D_3(m,n)$-graphs can be viewed as two pseudo-cartesian products that share a common 2-factor. If $B_m = b_1 b_2 \cdots b_m b_1$, $A_n^{(1)} = a_{\sigma_1(1)} a_{\sigma_1(2)} \cdots a_{\sigma_1(n)} a_{\sigma_1(1)}$, and $A_n^{(2)} = a^2_{\sigma_2(1)} a^2_{\sigma_2(2)} \cdots a^2_{\sigma_2(n)} a^2_{\sigma_2(1)}$, then $H_1 \cup F \cong A_n^{(1)} \times_{r_1} B_m$ and $H_2 \cup F \cong A_n^{(2)} \times_{r_2} B_m$.

The following result forms the basis for the graph structure used in the rest of this article (see Figure 6).

Theorem 5.4 ([14])**.** *If* $\overline{X} = \mathrm{Cay}(A/\langle s_3 \rangle; \{\overline{s_1}, \overline{s_2}\}^\star)$ *is HD into cycles*

$$\overline{H_1} = \overline{a_{\sigma_1(1)}}, \overline{a_{\sigma_1(2)}}, \ldots, \overline{a_{\sigma_1(n)}}, \overline{a_{\sigma_1(1)}} \quad \text{and} \quad \overline{H_2} = \overline{a_{\sigma_2(1)}}, \overline{a_{\sigma_2(2)}}, \ldots, \overline{a_{\sigma_2(n)}}, \overline{a_{\sigma_2(1)}},$$

then $X = \mathrm{Cay}(A; \{s_1, s_2, s_3\}^\star)$ *is a* $D_3(m,n)$*-graph, with* $m = |s_3|$, $n = |A : \langle s_3 \rangle|$, *where* H_i *is the 2-factor that* $\overline{H_i}$ *lifts to, and* F *the 2-factor generated by* s_3.

Throughout this article, t_1 and t_2 will denote the *number* of cycles in the 2-factors, H_1 and H_2, respectively, in a $D_3(m,n)$-graph. If $\sigma_1 = (1)$, the identity permutation, then A_n will be written in place of $A_n^{(1)}$. For the remainder of this article, $E(F)$ will be colored red, $E(H_1)$ colored blue, and $E(H_2)$ colored green. If $\sigma_2(k) = i$ in $A_n^{(2)} \times_{r_2} B_m$, then $\{a_i^2, b_j, b_{j+d}\}$-RAVS (resp., LAVS) will mean an alternating vertical color-switch between the $a^2_{\sigma_2(k-1)}$-, $a^2_{\sigma_2(k)}$-, and $a^2_{\sigma_2(k+1)}$-columns.

Example 5.5. Consider $X = \mathrm{Cay}(\mathbb{Z}_6 \oplus \mathbb{Z}_8; \{(3,1), (1,2), (5,7)\}^\star)$ (see Figure 7). If $B = \langle (3,1) \rangle$, then the quotient graph, $\overline{X}$, of X by B is HD into

$$\overline{H_1} = \overline{(0,0)}, \overline{(1,2)}, \overline{(5,2)}, \overline{(4,0)}, \overline{(3,6)}, \overline{(5,7)}, \overline{(0,0)}$$

$$\overline{H_2} = \overline{(0,0)}, \overline{(5,2)}, \overline{(3,6)}, \overline{(1,2)}, \overline{(5,7)}, \overline{(4,0)}, \overline{(0,0)}.$$

Here $\sigma_1 = (1)$ and $\sigma_2 = (23564)$. By Theorem 5.4, X is a $D_3(8,6)$-graph. Clearly, $H_1 \cup F \cong A_6 \times_6 B_8$ and $H_2 \cup F \cong A_6^{(2)} \times B_8$, where $B_8 = b_1 b_2 \cdots b_8 b_1$, $A_6^{(1)} = A_6 = a_1 a_2 a_3 a_4 a_5 a_6 a_1$, and $A_6^{(2)} = a_1^2 a_3^2 a_5^2 a_2^2 a_6^2 a_4^2 a_1^2$. Also, $(a_i^2, b_t) = (a_i, b_{t+h_i})$, where $h_i = 0$ for $i \in \{1,4,5\}$; $h_2 = 6$; $h_3 = 3$; and $h_6 = 5$. F consists of $n = 6$ red cycles; H_1 consists of $t_1 = \gcd(8,6) = 2$ blue cycles; H_2 consists of $t_2 = 8$ green cycles.

The location of the columns relative to one another in $H_1 \cup F$ and $H_2 \cup F$ is of great importance when applying color-switches. Thus the following definition:

Definition 5.6. Given a $D_3(m,n)$-graph, X, define the graph union, $\overline{G}_X = A_n^{(1)} \cup A_n^{(2)}$, where $A_n^{(1)} = a_{\sigma_1(1)} a_{\sigma_1(2)} \cdots a_{\sigma_1(n)} a_{\sigma_1(1)}$, $A_n^{(2)} = a^2_{\sigma_2(1)} a^2_{\sigma_2(2)} \cdots a^2_{\sigma_2(n)} a^2_{\sigma_2(1)}$, and the vertices $a_j = a_j^2$ for all j.

Clearly, $\overline{G}_X$ is isomorphic to $\overline{X}$. In Example 5.5, $\overline{G}_X = A_6 \cup A_6^{(2)} \cong \mathrm{Cay}(\mathbb{Z}_6; \{1,2\}^\star)$. The following properties are useful for finding Hamilton decompositions of $D_3(m,n)$-graphs.

Definition 5.7. Let G contain Hamilton cycles C and C'. We say G has a

1. $P^5_{3,3}$*-path* $u_1u_2u_3u_4u_5$, if $P_1 = u_1u_2u_3$ is on C and $P_2 = u_3u_4u_5$ is on C';
2. $P^6_{4,3}$*-path* $u_1u_2u_3u_4u_5u_6$, if $P_1 = u_1u_2u_3u_4$ is on C and $P_2 = u_4u_5u_6$ is on C'; and a
3. $P^8_{2,6}$*-path* $u_1u_2u_3u_4u_5u_6u_7u_8$, if $P_1 = u_1u_2$ is on C and $P_2 = u_3u_4u_5u_6u_7u_8$ is on C'.

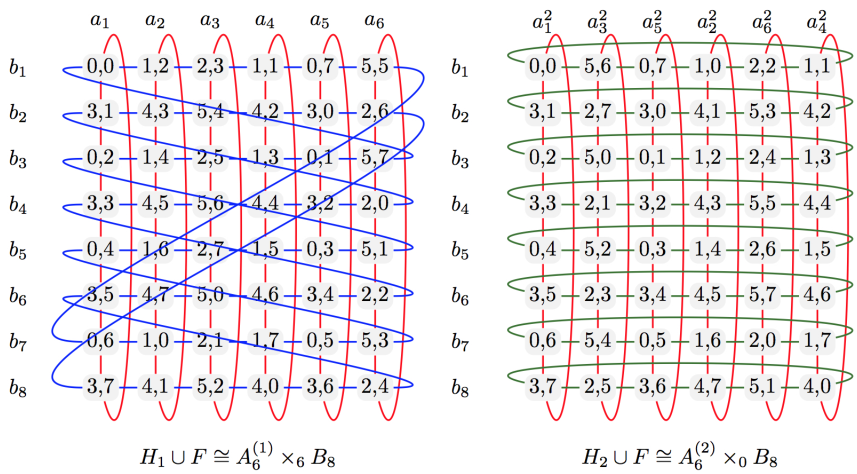


FIGURE 7. $\mathrm{Cay}(\mathbb{Z}_6 \oplus \mathbb{Z}_8; \{(3,1),(1,2),(5,7)\}^{\star})$, from Example 5.5 is a $D_3(8,6)$-graph.

Theorem 5.8 ([12], [14, 16])**.** *If X is a $D_3(m,n)$-graph, then $\overline{G}_X$ has a $P^5_{3,3}$-path if $n \geq 7$; a $P^6_{4,3}$-path if $n \geq 9$; and an $P^8_{2,6}$-path if $n \geq 13$.*

Theorem 5.9 ([16])**.** *If X is a $D_3(m,n)$-graph, where $n \geq 8$ is even, $m \geq 6$, $t_1 = 2k_1 \geq 2$, and $\overline{G}_X$ has a $P^8_{2,6}$-path, then X is HD.*

Theorem 5.10 ([12], [16])**.** *If X is a $D_3(m,n)$-graph, and $\overline{G}_X$ has a $P^6_{4,3}$-path, then X is HD if either one of the following holds:*

1. *$t_i = 2k_i + 1$, for $i = 1, 2$.*
2. *n is even, and $t_1 = 2k_1$ and $t_2 = 2k_2 + 1$.*

The following theorem summarizes which $D_3(m,n)$-graphs have been proven to be HD.

Theorem 5.11 ([5, 10, 11, 12, 13, 14, 15, 16, 19])**.** *A $D_3(m,n)$-graph is HD if any one of the following are true:*

1. *$m, n \geq 3$ and $t_1 = 1$ or $t_2 = 1$.*
2. *$m \geq 3$ is odd, and $n \in \{3,5,7\}$ or $n \geq 9$.*
3. *$m \geq 4$ is even, and $n \geq 9$, and t_1 and t_2 are odd.*
4. *$m \geq 4$ is even, and $n \geq 10$ is even, t_1 is even, t_2 is odd.*
5. *$m \geq 6$ is even, and $n \geq 14$ is even, t_1 and t_2 are even.*

This article uses a variety of CSCs to obtain Hamilton decompositions of $D_3(m,n)$-graphs not covered by Theorem 5.11. The following remark lists the remaining open cases.

Remark 5.12 (Open Cases)**.** The following cases are not covered in Theorem 5.11:

1. $m \geq 3$; $3 \leq n \leq 8$; and mn is even.
2. $m \geq 4$ is even; $9 \leq n \leq 12$ or $n \geq 13$ is odd; and t_1 and t_2 are both even.
3. $m \geq 4$ is even; $n \geq 9$ is odd; and t_1 is even, t_2 is odd.

The (high order) Cases (2) and (3) are resolved in Section 7 and the (low order) Case (1), with a few exceptions, is resolved in Sections 6.

The following lemma is immediately obtained from Theorem 5.4 and Proposition 3.2.

Lemma 5.13. *Let A be an Abelian group and let $S = \{s_1, s_2, s_3\} \subset A - \{0\}$ be a generating set for A that is inverse-free and involution-free. If S is strongly s_3-minimal, then $X = \mathrm{Cay}(A; \{s_1, s_2, s_3\}^{\star})$ is a $D_3(m,n)$-graph, where $m = |s_3| \geq 3$ and $n = |A : \langle s_3 \rangle| \geq 3$.*

Remark 5.14 (Assumptions)**.** For the remainder of this article, A will always denote an Abelian group of even order, $S = \{s_1, s_2, s_3\} \subset A - \{0\}$ will be a generating set for A that is inverse-free, involution-free, and strongly s_3-minimal. We shall use X and $\overline{X}$ to represent $\mathrm{Cay}(A; S^\star)$ and $\mathrm{Cay}(A/\langle s_3\rangle; \overline{S}^\star)$, respectively.

Proposition 5.15. *If $\{\overline{H_1}, \overline{H_2}\}$ is a Hamilton decomposition of $\overline{X}$, then the following are equivalent:*

1. *There exists an $i \in \{1, \ldots, n\}$ such that the edge $\{\overline{a_{\sigma_1(i)}}, \overline{a_{\sigma_1(i+1)}}\}$ is a multi-edge.*
2. *There exists an $i \in \{1, \ldots, n\}$ such the $a_{\sigma_1(i)}$ and $a_{\sigma_1(i+1)}$-columns are adjacent in $A_n^{(1)} \times_{r_1} B_m$ and the $a^2_{\sigma_1(i)}$- and $a^2_{\sigma_1(i+1)}$-columns are adjacent in $A_n^{(2)} \times_{r_2} B_m$.*
3. $\overline{s_1} \in \{\overline{s_2}, -\overline{s_2}\}$.

6. Hamilton Decompositions for Low-Order Quotient Graphs

This section obtains Hamilton decompositions for Cayley graphs with low-order quotient graphs described in Case (1) of Remark 5.12. There are a few cases that are not resolved using theoretical means. These cases (up to graph isomorphism) are resolved by computational methods and are stated as Lemma 6.6. The overall technique used in the proofs of Lemmas 6.2–6.4 is similar to that used [19] and again draws heavily on the CSCs of [16] and [12]. The following corollary of Theorem 2.5 is useful.

Corollary 6.1 ([18])**.** *If $\overline{X} = \mathrm{Cay}(A/\langle a\rangle; \{\overline{b}, \overline{c}\}^\star)$ is a quotient graph of $X = \mathrm{Cay}(A; \{a, b, c\}^\star)$; and $\overline{b}$ generates a HC in $\overline{X}$; and $\langle c\rangle$ has index two in A, then X is HD.*

Lemma 6.2. *If $m = 2d \geq 4$ and $n \in \{5, 7\}$, then X is HD.*

Proof. As $\overline{X}$ is a circulant graph of prime order, and $t_i = |A : \langle s_i\rangle|$ for $i \in \{1, 2\}$. We may use the technique in [19]. The remainder of the proof is omitted. □

Lemma 6.3. *If $m \geq 4$, and $n = 6$, then X is HD.*

Proof. By Remark 5.14, $|\overline{s_i}| \in \{3, 6\}$ for $i \in \{1, 2\}$. By Theorem 5.11, it suffices to show the result when $t_2 \geq t_1 \geq 2$. Furthermore, if $m = 4$, then $|A| = 24$, which is resolved by Lemma 6.6. Proceed with the case $m \geq 6$. The quotient graph $\overline{X} \cong Y \in \{Y_1, Y_2\}$ where $Y_1 = \mathrm{Cay}(\mathbb{Z}_6; \{1, 1\}^\star)$ and $Y_2 = \mathrm{Cay}(\mathbb{Z}_6; \{1, 2\}^\star)$. If $\overline{X} \cong Y_1$, then $\sigma_1 = \sigma_2 = (1)$, and by Corollary 6.1, it is assumed $t_1 \geq 3$. If $\overline{X} \cong Y_2$, then assume $\sigma_1 = (1)$, $\sigma_2 = (2365)$, and consider the parity of t_1 and t_2.

Case 1: $t_2 = 2k_2 \geq 2k_1 = t_1$.

a. $\overline{X} \cong Y_1$. Apply the CSC of Theorem 4.9 to $A_6 \times_{r_1} B_m$ to obtain a blue HC. Suppose a -RAVS or -LAVS was applied between the a_1^2- and a_3^2-columns, followed by color-switches $\mathcal{X}_1$ and $\mathcal{X}_2$, where $\mathcal{X}_1$ is incident to the a_3^2- and a_4^2-columns, and $\mathcal{X}_2$ is incident to the a_4^2- and a_5^2-columns. Clearly, a red HC would be formed, and upon orientation, all red edges in the a_i^2-columns are ↑-edges for all $i \in \{1, 3, 5\}$, and all red edges in the a_i^2-columns are ↓-edges, for all $i \in \{2, 4\}$. By Theorem 4.9, the red edges in the a_6^2-column are also ↑-edges. Hence, applying a color-switch $\mathcal{X}_3$ incident with the a_5^2- and a_6^2-columns, will preserve the red HC. The definition of $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$ follows. Choose $x \in \mathbb{Z}$ so that $\{(a_1^2, b_{1+x}), (a_1^2, b_{2+x})\}$ is red, and let $e = \{(a_6^2, b_{t_2-1+x}), (a_6^2, b_{t_2+x})\}$. If e is red, then apply an $\{a_2^2, b_{1+x}, b_{t_2-1+x}\}$-LAVS and define $\mathcal{X}_1 = \{a_3^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS. By Theorem 4.15, define

$$\mathcal{X}_2 = \{a_4^2, a_5^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS and } \mathcal{X}_3 = \{a_5^2, a_6^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to produce a Hamilton decomposition of X. If e is non-red, then $\{(a_6^2, b_{t_2-2+x}), (a_6^2, b_{t_2-1+x})\}$ is red. Hence, apply an $\{a_2^2, b_{m+x}, b_{t_2-2+x}\}$-RAVS and define $\mathcal{X}_1 = \{a_3^2, a_4^2, b_{t_2-2+x}, b_{t_2-1+x}\}$-CS and

$$\mathcal{X}_2 = \{a_4^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS and } \mathcal{X}_3 = \{a_5^2, a_6^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS}$$

to produce a Hamilton decomposition of X.

b. $\overline{X} \cong Y_2$. If $t_2 = 2$, then apply an $\{a_1, a_2, b_1, b_2\}$-CS to obtain a blue HC. By Theorem 4.15, the application of a $\{a_2, a_6, b_2\}$-LAHS will preserve the blue cycle, and create a red HC whose column-direction pattern would be: ↑↓↑↓↑↓. As $m \geq 4$, there exists some integer x such that both $e = \{(a_1^2, b_{1+x}), (a_1^2, b_{2+x})\}$ and $f = \{(a_3^2, b_{1+x}), (a_3^2, b_{2+x})\}$ are red edges. Furthermore, e and f have the same direction, and the application of an $\{a_1^2, a_3^2, b_{1+x}, b_{2+x}\}$-CS yields the result. If $t_2 \geq 4$, then relocate the r_1-jump in $A_6 \times_{r_1} B_m$ to between the a_5- and a_6-columns. Apply the

CSC of Theorem 4.9 to $A_6 \times_{r_1} B_m$ to obtain a blue HC and connect the red edges in the a_5- and a_6-columns into one cycle. Fix an $x \in \mathbb{Z}$ so that $\{(a_6^2, b_{1+x}), (a_6^2, b_{2+x})\}$ is red, and consider $e = \{(a_5^2, b_{t_2-1+x}), (a_5^2, b_{t_2+x})\}$. Subdivide into cases on the color of e.

i. If e is red, then apply an $\{a_3^2, b_{1+x}, b_{t_2-1+x}\}$-RAVS and an $\{a_6^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS to produce a green HC. Applying a color-switch incident with the a_4^2- and a_2^2-columns will produce a red HC and break the green cycle. Upon orienting the red cycle, by Theorem 4.9, the red edges in the a_6^2- and a_5^2-columns are $\uparrow$-edges. It follows that the red edges in the a_4^2-column are $\downarrow$-edges, and thus the red edges in the a_2^2-column are forced to be $\uparrow$-edges. By Remark 4.4, an additional color-switch incident with the a_2^2- and a_5^2-columns will preserve the red HC. Hence, apply an

$$\{a_4^2, a_2^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS and an } \{a_2^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS.}$$

By Theorem 4.15 the result is obtained.

ii. If e is non-red, then relocate the r_2-jump to be between the a_6^2- and a_3^2-columns. In this manner, view $H_2 \cup F$ as $A_6^{(2)} \times_{r_2} B_m$ where $A_6^{(2)} = a_6^2 a_4^2 a_2^2 a_5^2 a_1^2 a_3^2 a_6^2$. Apply an $\{a_6^2, a_4^2, b_{1+x}, b_{2+x}\}$-CS and an $\{a_4^2, a_2^2, b_{2+x}, b_{3+x}\}$-CS to connect all red edges in the a_i^2-columns, where $i \in \{6, 4, 2, 5\}$, into one cycle. Apply an $\{a_1^2, b_{3+x}, b_{t_2-1+x}\}$-RAVS to create a red HC. As before, making the red cycle a directed cycle, it is clear the the vertical red edges in the a_i^2-columns take the form: $\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow$. Hence, apply an additional switch, $\mathcal{X}$, incident to the a_3^2 and a_6^2-columns, to preserve the red cycle by Remark 4.4. Let e_1, e_2, e_3, and e_4 be the edges between consecutive vertices on the 4-cycle

$$(a_3^2, b_{t_2-1+x}), (a_6^2, b_{t_2-1+r_2+x}), (a_6^2, b_{t_2+r_2+x}), (a_3^2, b_{t_2+x}), (a_3^2, b_{t_2-1+x}).$$

Note e_4 is red, and e_1 and e_3 are green. By the choice of x, and parity of r_2, e_2 is never blue. If e_2 is red, then let $\mathcal{X} = \{a_3^2, a_6^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS to obtain a green HC. If e_2 is non-red, then $(a_6^2, b_{t_2-1+r_2+x}) = (a_6^2, b_{1+x})$, and hence, it is the only green edge in the a_6^2-column. In this case, remove the $\{a_1^2, b_{3+x}, b_{t_2-1+x}\}$-RAVS, and apply an $\{a_1^2, b_{4+x}, b_{t_2+x}\}$-LAVS. Define $\mathcal{X} = \{a_3^2, a_6^2, b_{3+x}, b_{4+x}\}$-CS to obtain a Hamilton decomposition of X.

Case 2: $t_2 = 2k_2 > 2k_1 + 1 = t_1$.

a. $\overline{X} \cong Y_1$. Apply the CSC of Theorem 4.15 to $A_6^{(2)} \times_{r_2} B_m$ and fix $x \in \mathbb{Z}$ so that the edge $\{(a_1, b_{1+x}), (a_2, b_{2+x})\}$ is red. Apply an $\{a_2, b_{1+x}, b_{t_1+x}\}$-LAVS and an $\{a_3, a_5, b_{t_1+x}\}$-LAHS to obtain the result.

b. $\overline{X} \cong Y_2$. Apply an $\{a_2, b_1, b_{t_1}\}$-LAVS to obtain a blue HC. There exists a path of $t_1 - 1 < m$ blue edges in the a_2-column. Again, fix $x \in \mathbb{Z}$ such that $\{(a_2^2, b_{t_2-1+x}), (a_2^2, b_{t_2+x})\}$ is red and apply an $\{a_2^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS. Finally, apply the following CSC to obtain the result:

$$\begin{cases} \{a_6^2, b_{1+x}, b_{t_2-1+x}\}\text{-LAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is red} \\ \{a_6^2, b_{1+x}, b_{t_2-1+x}\}\text{-RAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is not red.} \end{cases}$$

The cases where $t_2 = 2k_2 + 1 > 2k_1 = t_1$ and $t_2 = 2k_2 + 1 \geq 2k_1 + 1 = t_1$ follow very similarly using the technique of Case (2). □

Lemma 6.4. *If $m \geq 5$, and $n \in \{4, 8\}$, then X is HD.*

Proof. If $n = 4$, then $|\overline{s_1}| = |\overline{s_2}| = 4$, thus $\overline{X} \cong \mathrm{Cay}(\mathbb{Z}_4; \{1, 1\}^\star)$. Without loss of generality, $\sigma_1 = \sigma_2 = (1)$ and H_i is just the 2-factor generated by s_i for $i \in \{1, 2\}$. By Theorem 2.4 or Corollary 6.1 it suffices to show the result when $t_2 \geq t_1 \geq 3$. Subdivide into cases on the parity of t_1 and t_2 and follow the technique of Lemma 6.3.

If $n = 8$, then $|\overline{s_i}| \in \{4, 8\}$ for $i \in \{1, 2\}$. By Theorem 5.11, it suffices to show the result, without loss of generality, when $t_2 \geq t_1 \geq 2$. If $|\overline{s_i}| = 8$, for some $i \in \{1, 2\}$, then $A/B \cong \mathbb{Z}_8$, and, $\overline{X} \cong Y_1 = \mathrm{Cay}(\mathbb{Z}_8; \{1, 1\}^\star)$, so that $\sigma_1 = \sigma_2 = (1)$; or $\overline{X} \cong Y_2 = \mathrm{Cay}(\mathbb{Z}_8; \{1, 2\}^\star)$, so that $\sigma_1 = (1)$ and $\sigma_2 = (2356487)$. If $|\overline{s_1}| = |\overline{s_2}| = 4$, then $A/B \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, and $\overline{X} \cong Y_3 = \mathrm{Cay}(\mathbb{Z}_4 \oplus \mathbb{Z}_2; \{(1, 0), (1, 1)\}^\star)$, so that $\sigma_1 = (1)$ and $\sigma_2 = (24)(37)(68)$. As before, subdivide into cases depending $\overline{X}$ and the parity of t_1 and t_2. As before, the proof follows exactly using the technique of Lemma 6.3. □

Lemma 6.5. *If $m = 2m' \geq 4$; $n = 3$; $t_1 = 2k_1 \geq 2$; $t_2 = 2k_2 \geq 2$, then X is HD.*

Proof. X is a $D_3(m,3)$-graph and it is assumed that $\sigma_1 = \sigma_2 = (1)$. The case $m \in \{4,6,8\}$ is resolved by Lemma 6.6, so assume $m \geq 10$. By Corollary 6.1, assume $t_2 \geq t_1 \geq 4$. Apply the CS-configuration of Theorem 4.9 to $H_1 \cup F$ to obtain a blue HC and join the a_1- and a_3-columns into one red cycle. Relocate the r_2-jump in $H_2 \cup F$ so that it is between the a_2^2 and a_3^2-columns. In this manner, view $H_2 \cup F$ as $A_3^{(2)} \times_{r_2} B_m$ where $A_3^{(2)} = a_3^2\, a_1^2\, a_2^2\, a_3^2$. Let h be any integer such that $(a_3^2, b_{1+h})(a_3^2, b_{2+h})$ is red. Now, apply the CS-configuration of Theorem 4.9 to $H_2 \cup F$ to create a green HC, and obtain a Hamilton decomposition of X. □

The techniques outlined so far were insufficient for several small cases. To resolve this, the set of all connected 6-regular Cayley graphs on Abelian groups of order 12, 18, 24, and 32 with involution-free connection sets was pruned for isomorphism using Magma [7], and listed in Table 1. By Theorem 2.4, only Cayley graphs of the form $X = \mathrm{Cay}(A;\{s_1,s_2,s_3\}^\star)$ where $|A| > |s_i| \geq 3$ for all $i \in \{1,2,3\}$ are listed. Hamilton decompositions of each of the seventy-six graphs in Table 1 were obtained using a program written by Donald L. Kreher of Michigan Technological University and have been omitted in the interest of length. The author maintains this list of Hamilton decompositions at:

`http://math.kennesaw.edu/~ewestlun/computational_appendix.pdf`

This establishes the following result:

Lemma 6.6. *If X is a $D_3(m,n)$-graph and $(m,n) \in \{(3,4),(3,6),(3,8),(4,3),(4,6),(4,8)\}$, then X is HD.*

7. Hamilton Decompositions for High-Order Quotient Graphs

This section addresses the case when X has a quotient of order at least nine. The following lemma is obtained immediately from the Pigeonhole Principle.

Lemma 7.1. *Suppose X is a $D_3(m,n)$-graph, and $\overline{G}_X$ has no $P^8_{2,6}$-path.*

1. *If $n = 12$ then there exists an edge in $A_n^{(1)}$ that divides $A_n^{(2)}$ into two paths, each with six edges, so without loss of generality, $\sigma_1(1) = \sigma_2(6) = 1$ and $\sigma_1(12) = \sigma_2(12) = 12$.*
2. *If $n = 10$, then there exists an edge in $A_n^{(1)}$ that divides $A_n^{(2)}$ into either*
   a. *one path on six vertices and one path on four vertices, so without loss of generality, $\sigma_1(1) = \sigma_2(6) = 1$, and $\sigma_1(10) = \sigma_2(10) = 10$; or*
   b. *two paths on five vertices, so without loss of generality, $\sigma_1(1) = \sigma_2(5)$ and $\sigma_1(10) = \sigma_2(10)$.*

Lemmas 7.2–7.7 address Case (2) of Remark 5.12.

Lemma 7.2. *If $m \geq 6$; $n = 12$; $t_1 = 2k_1 \geq 2$; $t_2 = 2k_2 < m$, then X is HD.*

Proof. The proof closely follows the technique of ([16], Lemma 3.18). By Lemma 5.13, X is a $D_3(m,12)$-graph. If $\overline{G}_X$ has a $P^8_{2,6}$-path, the proof concludes by Theorem 5.9. Otherwise, by Lemma 7.1,

$$A_{12}^{(2)} = a^2_{\sigma_2(1)} \cdots a^2_{\sigma_2(5)}\, a_1^2\, a^2_{\sigma_2(7)} \cdots a^2_{\sigma_2(11)}\, a^2_{12}.$$

Apply the CSC of Lemma 4.9 to $A_{12} \times_{r_1} B_m$ to obtain a blue HC and join the red edges in the a_1- and a_{12}-columns into a single cycle. Assume that $t_2 \geq 6$ for the case when $t_2 \in \{2,4\}$ follows similarly, giving even more freedom to define a CSC. Apply the CSC of Lemma 4.11 to $A_{12}^{(2)} \times_{r_2} B_m$ by *starting* with an $\{a^2_{\sigma_2(1)}, a^2_{\sigma_2(2)}, b_{1+\ell}, b_{2+\ell}\}$-CS, where ℓ is an integer such that the edge $e = \{(a_1^2, b_{5+\ell}), (a_1^2, b_{6+\ell})\}$ is red. By Remark 4.4 and Lemma 4.11 there now exists a green HC, and a red cycle hereafter called the ★-cycle, consisting of red edges in the a_1^2-, a_{12}^2-, and $a^2_{\sigma_2(i)}$-columns, for all $i \in \{1,2,3,4,5\}$. Every column except the a_1^2- and a_{12}^2-columns has a red path of at least three edges. For any integers i and j, where $i \in \{2,3,\ldots,11\}$ and $j \in \{1,2,\ldots,m\}$, at least one of the two edges $e_j = \{(a_i,b_j),(a_i,b_{j+1})\}$ and $e_{j+2} = \{(a_i,b_{j+2}),(a_i,b_{j+3})\}$ is red. Additional color-switches in $A_{12} \times_{r_1} B_m$ are now defined to create a red HC. Let

$$\{x_1,x_2,x_3,x_4,x_5\} = \{\sigma_2(i) : 7 \leq i \leq 11\}, \text{ where } 1 < x_1 < x_2 < x_3 < x_4 < x_5 < 12.$$

Define $D_i = \{a_{x_i}, a_{x_i+1}, b_{w_i}, b_{w_i+1}\}$-CS and $\mathcal{D} = \{D_i : 1 \leq i \leq 5\}$. By Remark 4.4, if the CSC $\mathcal{D}$ was applied to $A_{12} \times_{r_1} B_m$, a red HC would be created. Upon orientation, all vertical red edges in any column have the same direction, the a_{x_i}- and a_{x_i+1}-columns have opposite directions, and by Lemma 4.9, the a_1- and a_{12}-columns have the *same* direction. Thus, there exists an $x \in \{1,\ldots,11\}$, such that the a_x- and

a_{x+1}-columns have the *same direction*. By Remark 4.4, any color-switch between these two columns will preserve the red cycle. Thus, let

$$\{x, x_1, x_2, x_3, x_4, x_5\} = \{y_i : 1 \le i \le 6\}, \text{ where } 1 \le y_1 < y_2 < y_3 < y_4 < y_5 < y_6 < 12,$$

and define $Y_i = \{a_{y_i}, a_{y_i+1}, b_{z_i}, b_{z_i+1}\}$-CS and $\mathcal{Y} = \{Y_i : 1 \le i \le 6\}$. Good pairs $\{Y_1, Y_2\}$, $\{Y_3, Y_4\}$, and $\{Y_5, Y_6\}$ are now defined so that applying the CSC $\mathcal{Y}$ will preserve the blue HC, and yield the result.

Case 1: $x \notin \{y_{2j-1}, y_{2j}\}$. The $a_{y_{2j-1}}$-column has at most one non-red edge, the $a_{y_{2j}}$-column has all red edges, and $1 < y_{2j-1} < 11$. Let y be an integer such that $y \equiv 0 \pmod 2$ and there exists a red 3-path in the $a_{y_{2j-1}}$-column between the b_y- and b_{y+3}-rows. Let

$$e_y = \{(a_{y_{2j-1}+1}, b_y), (a_{y_{2j-1}+1}, b_{y+1})\} \quad \text{and} \quad f_y = \{(a_{y_{2j}+1}, b_y), (a_{y_{2j}+1}, b_{y+1})\}.$$

At least one of e_y and e_{y+2} is red. Define the good pair $\{Y_{2j-1}, Y_{2j}\}$ as follows:

$$Y_{2j-1} = \begin{cases} \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_y, b_{y+1}\}\text{-CS} & \text{if } e_y \text{ is red} \\ \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y+2}, b_{y+3}\}\text{-CS} & \text{if } e_y \text{ is not red} \end{cases} \tag{7.2.1}$$

$$Y_{2j} = \begin{cases} \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y+1}, b_{y+2}\}\text{-CS} & \text{if } f_{y+1} \text{ is red} \\ \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y-1}, b_y\}\text{-CS} & \text{if } f_{y+1} \text{ is not red and } e_y \text{ is red} \\ \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y+3}, b_{y+4}\}\text{-CS} & \text{if } f_{y+1} \text{ is not red and } e_y \text{ is not red} \end{cases} \tag{7.2.2}$$

Case 2: $x \in \{y_{2j-1}, y_{2j}\}$. Clearly, $x \in \{1, \sigma_2(1), \ldots, \sigma_2(5)\}$, the a_x-column has at most one blue edge, and the a_{x+1}-column has no blue edges. Furthermore, by (7.2.2), any blue edge in the a_x-column lies between the b_z- and b_{z+1}-rows where z is some integer and $z \equiv 1 \pmod 2$. Consider two subcases:

1. $x = y_{2j-1}$. If $x = 1$, select an integer y, where $y \equiv 0 \pmod 2$, and both $\{(a_1, b_y), (a_1, b_{y+1})\}$ and $\{(a_1, b_{y+2}), (a_1, b_{y+3})\}$ are red. Define Y_{2j-1} as in (7.2.1). If $x \ne 1$, let P be a *longest* path of red edges in the a_{x+1}-column. Because $x+1 \notin \{1, 12\}$, and $m \ge 2t_2$, it follows that P has length at least four. Therefore, there exists an integer y, where $y \equiv 0 \pmod 2$, and both $\{(a_{x+1}, b_y), (a_{x+1}, b_{y+1})\}$ and $\{(a_{x+1}, b_{y+2}), (a_{x+1}, b_{y+3})\}$ lie on P. If $e_y = \{(a_x, b_y), (a_x, b_{y+1})\}$ is red, then define $Y_{2j-1} = \{a_x, a_{x+1}, b_y, b_{y+1}\}$-CS. If e_y is green, then define $Y_{2j-1} = \{a_x, a_{x+1}, b_{y+2}, b_{y+3}\}$-CS. Having already defined Y_{2j-1}, it is clear that the $a_{y_{2j}}$-column contains at most one non-red edge. Define Y_{2j} similarly to (7.2.2) to obtain a good pair $\{Y_{2j-1}, Y_{2j}\}$.
2. $x = y_{2j}$. There exists an integer y, where $y \equiv 1 \pmod 2$, and both $\{(a_{y_{2j}}, b_y), (a_{y_{2j}}, b_{y+1})\}$ and $\{(a_{y_{2j}+1}, b_y), (a_{y_{2j}+1}, b_{y+1})\}$ are red. Define $Y_{2j} = \{a_{y_{2j}}, a_{y_{2j}+1}, b_y, b_{y+1}\}$-CS. Again, the $a_{y_{2j-1}}$-column contains all red edges, except possibly one blue edge $\{(a_{y_{2j-1}}, b_z), (a_{y_{2j-1}}, b_{z+1})\}$, for some $z \equiv 1 \pmod 2$. Thus, either

$$Y_{2j-1} = \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y-1}, b_y\}\text{-CS} \quad \text{or} \quad Y_{2j-1} = \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y+1}, b_{y+2}\}\text{-CS}$$

   will make $\{Y_{2j-1}, Y_{2j}\}$ a good pair.

Note that the blue HC is preserved upon applying the good pair $\{Y_{2j-1}, Y_{2j}\}$. To see this, after applying Y_{2j-1}, the blue HC is broken into one cycle

$$(a_{y_{2j-1}+1}, b_{\hat{y}})(a_{y_{2j-1}+1}, b_{\hat{y}+1})(a_{y_{2j-1}+2}, b_{\hat{y}+1}) \cdots (a_{12}, b_{\hat{y}+1})(a_{12}, b_{\hat{y}})(a_{11}, b_{\hat{y}}) \cdots (a_{y_{2j-1}+1}, b_{\hat{y}}),$$

for some $\hat{y} \equiv 0 \pmod 2$, and one cycle on the remaining vertices. By Lemma 4.4, Y_{2j} restores the blue HC, and the result follows. □

The proof of the following lemma is very similar to that of Lemma 7.2, with only slight modifications, and is therefore omitted.

Lemma 7.3. *If $m = t_2 = 2k_2 \ge 6$; $n = 12$; $t_1 = 2k_1 \ge 2$, then X is HD.*

Lemma 7.4. *If $m \ge 6$; $n = 10$; and t_1 and t_2 are both even, then X is HD.*

Proof. X is a $D_3(m, 10)$-graph and $A/\langle s_3 \rangle$ is a cyclic group of order 10, so that $\overline{X} \cong \mathrm{Cay}(\mathbb{Z}_{10}; \{x, y\}^\star)$. By Remark 5.14, $\langle x, y \rangle = \mathbb{Z}_{10}$, and $|x|, |y| \ge 3$. Without loss of generality, up to a relabeling of the vertices,

$$\{x, y\} \in \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}.$$

If $\{x,y\}=\{1,1\}$, then $\overline{X}$ is a multigraph and $\overline{H_1}=\overline{H_2}$. Trivially, the path $P=0,1,2,3,4,5,6,7$ forms a $P^8_{2,6}$-path, and Theorem 5.9 produces the result. If $\{x,y\}=\{1,3\}$, then

$$\overline{H_1}:=0,1,2,3,4,5,6,7,8,9,0 \quad \text{and} \quad \overline{H_2}:=0,3,6,9,2,5,8,1,4,7,0$$

is a Hamilton decomposition, and $P=0,3,4,5,6,7,8,9$ forms a $P^8_{2,6}$-path. Similarly, if $\{x,y\}=\{1,2\}$, then

$$\overline{H_1}:=0,2,1,3,4,5,6,7,8,9,0 \quad \text{and} \quad \overline{H_2}:=7,5,3,2,4,6,8,0,1,9,7$$

is a Hamilton decomposition, and $P=0,9,7,5,3,2,4,6$ forms a $P^8_{2,6}$-path. If $\{x,y\}=\{2,3\}$, then

$$\overline{H_1}:=0,2,9,1,3,5,8,6,4,7,0 \quad \text{and} \quad \overline{H_2}:=5,2,4,1,8,0,3,6,9,7,5$$

is a Hamilton decomposition of $\overline{X}$, and by Lemma 7.1 assume $\sigma_1(1)=\sigma_2(6)=1$ and $\sigma_1(10)=\sigma_2(10)=10$. Use the technique of Lemma 7.2 to find a Hamilton decomposition of X. The result is obtained similarly for $\{x,y\}=\{1,4\}$ and $\{x,y\}=\{3,4\}$. □

Lemma 7.5. *If $m\geq 6$ is even, $n\geq 9$ is odd, and t_1 and t_2 are both even, then X is HD.*

Proof. The proof follows the technique ([14], Lemma 3.14). Only the case when $\overline{s_1}\notin\{\overline{s_2},-\overline{s_2}\}$ is considered, for the case when $\overline{s_1}\in\{\overline{s_2},-\overline{s_2}\}$ differs in only minor technicalities and is much easier. Without loss of generality, assume $t_2\geq t_1$. By Theorem 5.8, $\overline{X}$ has a $P^6_{4,3}$-path, call it P, where

$$P=\overline{a_{\sigma_1(1)}}\,\overline{a_{\sigma_1(2)}}\,\overline{a_{\sigma_1(3)}}\,\overline{a_{\sigma_1(4)}}\,\overline{a_{\sigma_2(2)}}\,\overline{a_{\sigma_2(3)}}.$$

Again without loss of generality, $\sigma_1(4)=\sigma_2(1)=p$, and $\{\sigma_2(2),\sigma_2(3)\}=\{q,n\}$, such that $4\leq p<q<n-1$; where the rightmost inequality follows from Proposition 5.15. Thus,

$$A_n^{(1)}=a_1\cdots a_p\cdots a_q\cdots a_{n-1}a_n, \text{ and } A_n^{(2)}=a_p^2a^2_{\sigma_2(2)}a^2_{\sigma_2(3)}\cdots a^2_{\sigma_2(n)},$$

Apply the following CSC to $A_n^{(1)}\times_{r_1}B_m$:

$$(7.5.1)\qquad \begin{cases}\{a_1,a_2,b_2,b_3\}\text{-CS}, \{a_2,a_3,b_1,b_2\}\text{-CS}, \{a_3,a_4,b_2,b_3\}\text{-CS} & \text{if } t_1=2\\ \{a_2,b_1,b_{t_1-1}\}\text{-RAVS}, \{a_3,a_4,b_{t_1-1},b_{t_1}\}\text{-CS} & \text{if } t_1\geq 4\end{cases}$$

Upon applying (7.5.1), by Remark 4.4, and either Lemma 4.15 or Theorem 4.8 (2), a blue HC is created, and the red edges in the a_i-columns are joined into a single cycle for all $i\in\{1,2,3,4\}$. Apply the CSC

$$(7.5.2)\qquad \begin{cases}\{a_4,a_{n-1},b_1\}\text{-RAHS or an } \{a_4,a_{n-1},b_2\}\text{-LAHS} & \text{if } t_1=2\\ \{a_4,a_{n-1},b_{t_1-1}\}\text{-RAHS or an } \{a_4,a_{n-1},b_{t_1}\}\text{-LAHS} & \text{if } t_1\geq 4\end{cases}$$

which preserves the blue HC by Lemma 4.15 and the parity of n. Remove the two color-switches that are incident to the a_q-column. The blue HC is preserved by Corollary 4.16, and there now exists a red 2-factor consisting of the following four cycles:

- **♢-cycle:** all red edges in the a_i-columns for all $i\in\{1,\ldots,q-1\}$;
- **♣-cycle:** all red edges in the $a_{\sigma_2(2)}$-column;
- **♡-cycle:** all red edges in the $a_{\sigma_2(3)}$-column;
- **♠-cycle:** all red edges in the a_i-columns for all $i\in\{q+1,\ldots,n-1\}$.

The a_p-column either contains one blue edge if $p=q-1$, (which by Proposition 5.15, is only possible if $q=\sigma_2(3)$) or two consecutive blue edges, if $p\neq q-1$. In the latter case, these blue edges are either $\{e_1,e_2\}$ if a LAHS was applied in (7.5.2) or $\{e_1,e_3\}$ if a RAHS was applied in (7.5.2), where

$$e_1=\{(a_p,b_y),(a_p,b_{y+1})\},\quad e_2=\{(a_p,b_{y+1}),(a_p,b_{y+2})\},\quad e_3=\{(a_p,b_{y-1}),(a_p,b_y)\},$$

and $y=1$ if $t_1=2$ or $y=t_1-1$ if $t_1\geq 4$. A CSC for $A_n^{(2)}\times_{r_2}B_m$ is now defined. Without loss of generality, draw $A_n^{(2)}\times_{r_2}B_m$ so that (a_p^2,b_{m-1}) corresponds to the vertex (a_p,b_y) in $A_n^{(1)}\times_{r_1}B_m$. Therefore, the edges e_1, e_2, and e_3 are:

$$e_1=\{(a_p^2,b_{m-1}),(a_p^2,b_m)\},\quad e_2=\{(a_p^2,b_m),(a_p^2,b_1)\},\quad e_3=\{(a_p^2,b_{m-2}),(a_p^2,b_{m-1})\}.$$

Apply the following CSC (if $t_2 = m$ and e_3 is blue we *must* pick a LAVS):

$$\begin{cases} \{a^2_{\sigma_2(2)}, b_1, b_3\}\text{-LAVS or -RAVS} & \text{if } t_2 = 2 \\ \{a^2_{\sigma_2(2)}, b_1, b_{t_2-1}\}\text{-LAVS or -RAVS} & \text{if } t_2 \geq 4. \end{cases} \tag{7.5.3}$$

By Lemma 4.4, the switches of (7.5.3) create a green 2-factor of two cycles; and a red 2-factor consisting of the ♠-cycle, and a cycle on the vertices contained in the ♢-, ♣-, and ♡-cycles; hereafter referred to as the ★-cycle. Let z be the smallest integer in $\{3, \ldots, n-1\}$ such that the red edges in the $a^2_{\sigma_2(z)}$-column are in the ★-cycle, and the red edges in the $a^2_{\sigma_2(z+1)}$-column are in the ♠-cycle. Any column whose red edges are on the ♠-cycle have either one blue path of length two; one blue edge (if the column is a_{q+1}- or a_{n-1}-); or zero blue edges (only if $q = n-2$). Likewise, for all $i \in \{4, \ldots, n-1\}$, the $a^2_{\sigma_2(i)}$-columns whose red edges are on the ★-cycle have either one or two consecutive blue edges, unless it is the case that $\sigma_2(i) \in \{1, 2, 3\}$ and $t_1 \geq 4$. A final color-switch D, between the $a^2_{\sigma_2(z)}$- and $a^2_{\sigma_2(z+1)}$-columns is now defined. Consider three cases:

Case 1: the $\boldsymbol{a^2_{\sigma_2(z)}}$-column has either one or two consecutive blue edges. Consider $m > t_2 \geq 4$ and $z \neq 3$, for the case when $t_2 = 2$ or $z = 3$ is very similar. Define

$$g_i = \{(a^2_{\sigma_2(z)}, b_{t_2-1+i}), (a^2_{\sigma_2(z)}, b_{t_2+i})\} \quad \text{and} \quad h_i = \{(a^2_{\sigma_2(z+1)}, b_{t_2-1+i}), (a^2_{\sigma_2(z+1)}, b_{t_2+i})\}.$$

If both g_0 and h_0 are red, then define $D = \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_{t_2-1}, b_{t_2}\}$-CS. If both g_0 and h_0 are blue, then define $D = \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_m, b_1\}$-CS to obtain a Hamilton decomposition, by Lemma 4.4. If g_0 is blue and h_0 is red (similarly for the case g_0 is red and h_0 is blue), then both g_2 and $\{(a^2_{\sigma_2(z)}, b_m), (a^2_{\sigma_2(z)}, b_1)\}$ are red. Furthermore, at least one of h_2 or $w = \{(a^2_{\sigma_2(z+1)}, b_m), (a^2_{\sigma_2(z+1)}, b_1)\}$ is red. If w is red, define $D = \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_m, b_1\}$-CS and if w is not red, define $D = \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_{t_2+1}, b_{t_2+2}\}$-CS and replace the -RAVS or -LAVS of (7.5.3) with an $\{a^2_{\sigma_2(2)}, b_3, b_{t_2+1}\}$-LAVS or -RAVS. A HD is now obtained by Lemma 4.4.

When $m = t_2$ it may be assumed, without loss of generality, that e_1 and e_2 are blue edges in the a^2_p-column. Define g_i and h_i as before.

$$D = \begin{cases} \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_{m-1}, b_m\}\text{-CS} & \text{if } g_0 \text{ and } h_0 \text{ are both red} \\ \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_m, b_1\}\text{-CS} & \text{if } g_1 \text{ and } h_1 \text{ are both red} \end{cases} \tag{7.5.4}$$

If g_{-1} and h_{-1} are both red, define $D = \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_{m-2}, b_{m-1}\}$-CS, remove the -LAVS or -RAVS from (7.5.3) and apply an $\{a^2_{\sigma_2(2)}, b_m, b_{m-2}\}$-RAVS. Similarly, if g_2 and h_2 are both red, define $D = \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_1, b_2\}$-CS, remove the -LAVS or -RAVS from (7.5.3) and apply an $\{a^2_{\sigma_2(2)}, b_2, b_m\}$-LAVS. Now, if there does *not* exist an integer $i \in \{-1, 0, 1, 2\}$, such that both g_i and h_i are red, then all edges in both the $a^2_{\sigma_2(z)}$- and $a^2_{\sigma_2(z+1)}$-columns that are between the b_2- and b_{m-2}-rows are red. In this case, define $D = \{a^2_{\sigma_2(z)}, a^2_{\sigma_2(z+1)}, b_3, b_4\}$-CS, and remove the -LAVS or -RAVS from (7.5.3), and apply an $\{a^2_{\sigma_2(2)}, b_1, b_3\}$-LAVS and an $\{a^2_{\sigma_2(2)}, b_4, b_m\}$-LAVS to obtain a Hamilton decomposition of X.

In Cases (2) and (3) below, the following statements are all true:

- The a^2_1-column has a path of length $t_1 - 2$ that alternates between red and blue edges.
- The a^2_2-column has a blue edge path of length $t_1 - 2$.
- The a^2_3-column has a path of length t_1 that alternates between red and blue edges.

If $t_2 = 2$, then $t_1 = 2$ and use the technique of Case (1) above. Thus, it suffices to show the result when $t_2 \geq 4$. Again, without loss of generality, assume that e_1 and e_2 are blue edges in the a^2_p-column.

Case 2: the $\boldsymbol{a^2_{\sigma_2(z)}}$-column is the $\boldsymbol{a^2_1}$ or $\boldsymbol{a^2_3}$-column. Again, define g_i and h_i as in Case (1). The $a^2_{\sigma_2(z)}$-column contains an *alternating path* of red and blue edges and because the $a^2_{\sigma_2(z+1)}$-column contains at most two consecutive blue edges, there exists at least one integer $i \in \{-1, 0, 1-t_2, 2-t_2\}$ such that both g_i and h_i are red. Use the technique of Case (1) to define D.

Case 3: the $\boldsymbol{a^2_{\sigma_2(z)}}$-column is the $\boldsymbol{a^2_2}$-column. First consider the case $m > t_2$ and let j be the smallest integer in $\{1, \ldots, t_2+1\}$ such that both $\{(a^2_2, b_j), (a^2_2, b_{j+1})\}$ and $\{(a^2_{\sigma_2(z+1)}, b_j), (a^2_{\sigma_2(z+1)}, b_{j+1})\}$ are red edges. The a^2_2-column has a blue-edge path of length $d = t_1 - 2$. Note that $d \leq t_2 - 2$ and there are at most two blue edges in the $a^2_{\sigma_2(z+1)}$-column, so we are guaranteed at least one such value of j exists. Define $D = \{a^2_2, a^2_{\sigma_2(z+1)}, b_j, b_{j+1}\}$-CS and remove the -LAVS or -RAVS of (7.5.3). Apply an $\{a^2_{\sigma_2(2)}, b_{j+1}, b_{j+t_2-1}\}$-LAVS. Because $j + t_2 - 1 \le t_2 + 1 + t_2 - 1 = 2t_2 \le m$ and e_3 is red, this -LAVS always defines a valid CSC. By Lemma 4.4, the result is a Hamilton decomposition.

Secondly, consider the case $m = t_2$. Here, the a^2_2-column contains at most $m - 2$ blue edges, and so it is possible that there exists no pair of edges $g = \{(a^2_2, b_x), (a^2_2, b_{x+1})\}$ and $h = \{(a^2_{\sigma_2(z+1)}, b_x), (a^2_{\sigma_2(z+1)}, b_{x+1})\}$ that are both red. However, this is only possible if no pair g and h are both blue, as there are at most two blue edges in the $a^2_{\sigma_2(z+1)}$-column. In this case, we may remove the -LAHS from (7.5.2) and apply the -RAHS of (7.5.2) so that e_1 and e_3 are now blue. Clearly, the switch reflect does not change the position of the blue edges in the a^2_2-column, but it does shift the blue edges in the $a^2_{\sigma_2(z+1)}$-column up or down one row, thereby creating a pair g and h that are *both red*. The conclusion now follows. □

The proofs of the following two lemmas, which address Case (3) of Remark 5.12, are very similar to the proof of Lemma 7.5, requiring only a few technical modifications, and are both omitted.

Lemma 7.6. *If $m = 4$ and $n \ge 9$, then X is HD.*

Lemma 7.7. *If $m \ge 6$ is even, $n \ge 9$ is odd, and exactly one of t_1 and t_2 is even, then X is HD.*

Example 7.8. The proof of Lemma 7.5 is illustrated with $X = \mathrm{Cay}(\mathbb{Z}_8 \oplus \mathbb{Z}_{11}; \{(4,1),(4,2),(1,0)\}^\star)$. Note that $|(4,1)| = |(4,2)| = 22$, and $|(1,0)| = 8$. It is straightforward to check that none of Theorems 1.1–2.5 can be applied to find a Hamilton decomposition of X. Let $B = \langle(1,0)\rangle$, and note that $S = \{(4,1),(4,2),(1,0)\}$ is inverse-free, involution-free, and strongly $(1,0)$-minimal because $\{2(4,1), 2(4,2)\} = \{(0,2),(0,4)\} \cap B = \emptyset$. The quotient graph $\overline{X} = \mathrm{Cay}((\mathbb{Z}_8 \oplus \mathbb{Z}_{11})/B; \{\overline{(4,1)}, \overline{(4,2)}\}^\star)$ is a circulant graph of prime order $|\overline{(4,1)}| = |\overline{(4,2)}| = 11$, and $\overline{(4,1)} \notin \{\overline{(4,2)}, -\overline{(4,2)}\}$. A Hamilton decomposition of $\overline{X}$ is $\{\overline{H_1}, \overline{H_2}\}$ where

$$\overline{H_1} = \overline{(4,3)}, \overline{(0,4)}, \overline{(4,5)}, \overline{(0,6)}, \overline{(4,7)}, \overline{(0,8)}, \overline{(4,9)}, \overline{(0,10)}, \overline{(1,0)}, \overline{(4,1)}, \overline{(0,2)} = \overline{a_1}, \overline{a_2}, \ldots, \overline{a_{11}};$$

$$\overline{H_2} = \overline{(0,6)}, \overline{(0,8)}, \overline{(0,10)}, \overline{(4,1)}, \overline{(4,3)}, \overline{(4,5)}, \overline{(4,7)}, \overline{(4,9)}, \overline{(1,0)}, \overline{(0,2)}, \overline{(0,4)}.$$

In this case, the lift of $\overline{H_1}$ and $\overline{H_2}$ is just the 2-factor generated by $(4,1)$ and $(4,2)$, respectively. A $P^6_{4,3}$-path is

$$P = \overline{(4,3)}, \overline{(0,4)}, \overline{(4,5)}, \overline{(0,6)}, \overline{(0,8)}, \overline{(0,10)} = \overline{a_4}, \overline{a_5}, \overline{a_6}, \overline{a_7}, \overline{a_9}, \overline{a_{11}}.$$

Hence, X is a $D_3(8,11)$-graph with $H_1 \cup F \cong A^{(1)}_{11} \times_4 B_8$ and $H_2 \cup F \cong A^{(2)}_{11} \times_4 B_8$. For visual convenience, relocate the r_1-jump ($r_1 = 4$) between the columns whose vertices correspond to elements of the cosets $\overline{(0,10)}$ and $\overline{(1,0)}$. Thus, $A^{(1)}_{11} = a_1, a_2, \ldots, a_{11}$, and $A^{(2)}_{11} = a^2_7, a^2_9, a^2_{11}, a^2_2, a^2_4, a^2_6, a^2_8, a^2_{10}, a^2_1, a^2_3, a^2_5$, $r_1 = r_2 = 4$; $t_1 = t_2 = 4$; $p = 7$, $q = \sigma_2(2) = 9$, and $n = \sigma_2(3) = 11$. Let $\mathcal{S}_1$ denote the CSC of (7.5.1) and (7.5.2) including the removal of the color-switches incident to a_9-column. I.e.,

$$\mathcal{S}_1 = \{\{a_2, b_1, b_3\}\text{-RAVS}, \{a_3, a_4, b_3, b_4\}\text{-CS}, \{a_4, a_8, b_3\}\text{-RAHS}\}$$

(Figure 8 shows these switches and the ♢-, ♣-, ♡-, and ♠-cycles, as described in Lemma 7.5.) X is drawn so that $e_1 = \{(a_7, b_3), (a_7, b_4)\}$ and $e_3 = \{(a_7, b_2), (a_7, b_3)\}$ correspond to $\{(a^2_7, b_7), (a^2_7, b_8)\}$ and $\{(a^2_7, b_6), (a^2_7, b_7)\}$, respectively (thus, $\ell = 0$). Next, apply the CSC $\mathcal{S}_2 = \{\{a^2_9, b_1, b_3\}\text{-LAVS}\}$ to X as described in (7.5.3) (see Figure 9). The effect is a green 2-factor consisting of two cycles, depicted as $\circ$ and $\triangleright$ on the rows in Figure 9 (rightmost graph). Finally determine that $z = 7$ and seek a final color-switch between the $a^2_{\sigma_2(7)}$- and $a^2_{\sigma_2(8)}$-columns (corresponding to the a^2_8- and a^2_{10}-columns). Following Case 1 of Lemma 7.5, this final switch is $\mathcal{S}_3 = \{\{a^2_8, a^2_{10}, b_8, b_1\}\text{-CS}\}$; as shown in Figure 10. The result is a Hamilton decomposition of X into blue, red, and green Hamilton cycles.

8. Acknowledgements

The author thanks Donald L. Kreher for writing the program used to find the Hamilton decompositions of the seventy-six small cases in Table 1 and for personal communication that improved the results of this article. Additionally, the author thanks the referees for their time and many helpful suggestions.

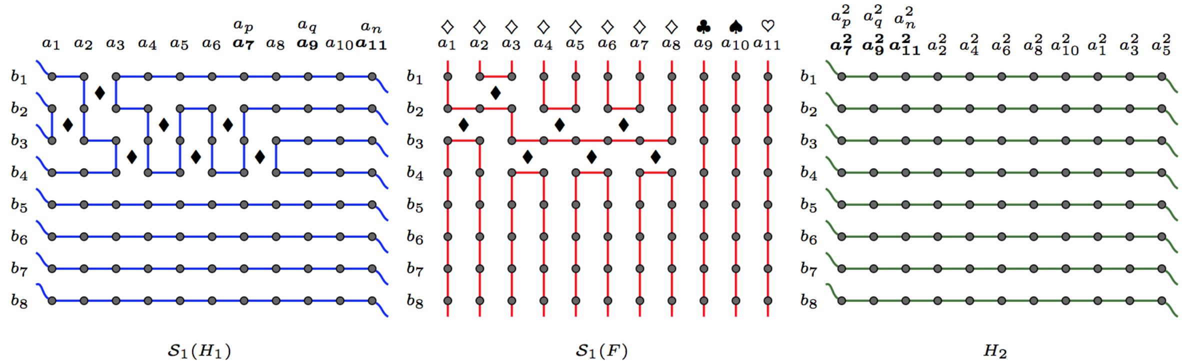


FIGURE 8. A blue HC; red 2-factor with 4 cycles; green 2-factor with 4 cycles.

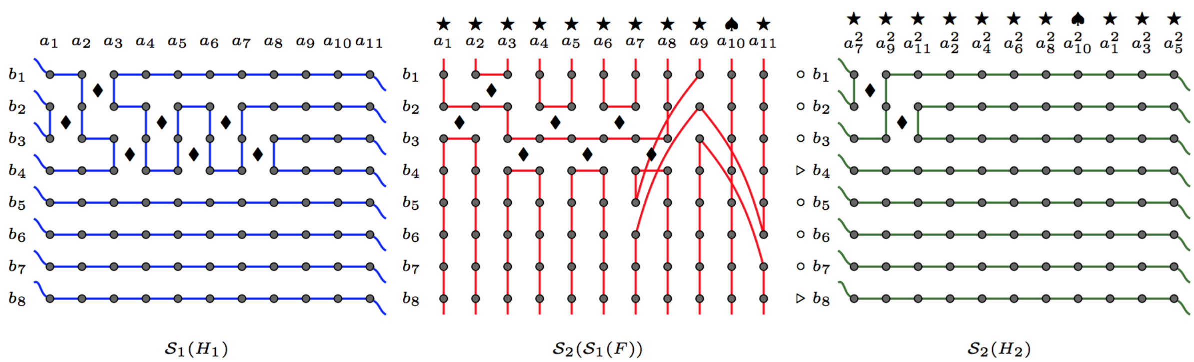


FIGURE 9. A blue HC; red 2-factor with 2 cycles (the ★-cycle and the ♠-cycle); green 2-factor with 2 cycles (the ○-cycle and the ▷-cycle).

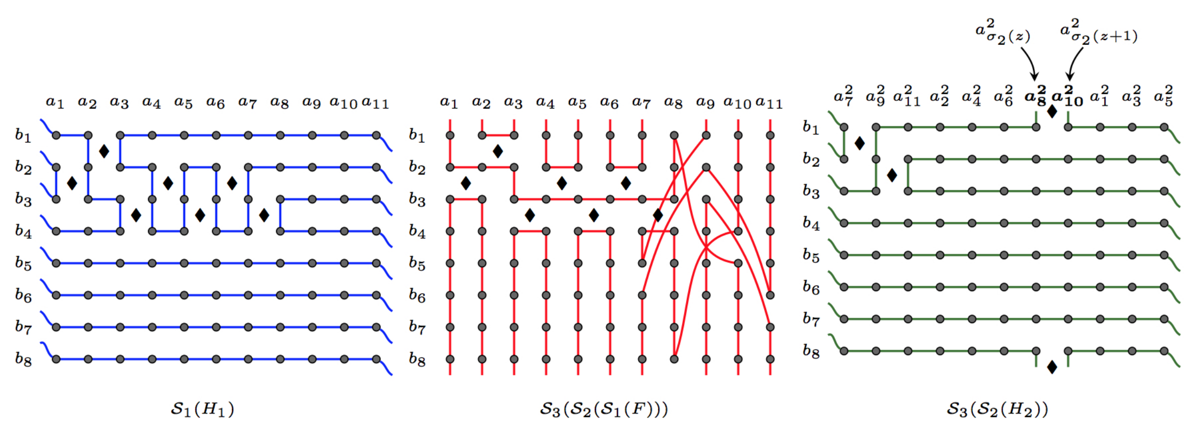


FIGURE 10. A blue HC; a red HC; a green HC.

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Order	Group A	Connection Sets S
12	$\mathbb{Z}_{12}$	$\{2,3,4\}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$	$\{(0,1),(0,2),(1,1)\}, \{(0,1),(1,1),(1,2)\}$
18	$\mathbb{Z}_{18}$	$\{2,3,4\}, \{3,4,6\}$
	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$	$\{(1,3),(0,1),(1,2)\}, \{(1,3),(2,1),(1,1)\}, \{(0,1),(2,2),(1,0)\}, \{(2,1),(1,1),(1,2)\}, \{(2,1),(1,0),(1,2)\}$
24	$\mathbb{Z}_{24}$	$\{3,9,10\}, \{2,6,9\}, \{3,4,10\}, \{2,9,10\}, \{6,8,9\}, \{3,4,9\}, \{2,3,8\}, \{3,8,9\}, \{4,8,9\}, \{4,6,9\}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$	$\{(1,4),(1,5),(0,3)\}, \{(1,3),(1,4),(0,3)\}, \{(0,1),(0,2),(1,1)\}, \{(1,3),(0,4),(1,2)\}, \{(1,3),(1,5),(0,3)\}, \{(0,4),(0,5),(1,2)\}, \{(0,2),(1,5),(1,2)\}, \{(1,3),(0,1),(1,5)\}, \{(0,1),(0,3),(1,2)\}, \{(1,3),(0,1),(0,4)\}, \{(0,2),(0,3),(1,2)\}, \{(1,4),(1,1),(1,2)\}, \{(1,4),(1,5),(0,5)\}, \{(1,3),(0,1),(0,5)\}, \{(0,2),(0,3),(1,1)\}, \{(0,1),(0,4),(1,1)\}, \{(0,1),(1,4),(0,5)\}, \{(1,3),(0,3),(0,4)\}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$	$\{(0,0,1),(1,1,1),(1,0,1)\}$
32	$\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$	$\{(0,2),(1,5),(0,3)\}, \{(1,5),(1,6),(1,1)\}, \{(1,3),(0,5),(1,2)\}, \{(1,3),(0,7),(1,1)\}, \{(1,3),(0,1),(1,5)\}, \{(1,4),(0,3),(1,7)\}, \{(1,4),(0,2),(0,5)\}, \{(0,1),(1,6),(0,7)\}, \{(1,5),(0,4),(1,2)\}, \{(0,1),(1,4),(1,2)\}, \{(1,5),(1,6),(1,2)\}, \{(1,3),(0,7),(1,2)\}, \{(1,3),(1,4),(0,4)\}, \{(1,3),(0,5),(1,1)\}, \{(0,2),(1,6),(0,7)\}, \{(0,1),(1,5),(0,6)\}, \{(1,3),(1,4),(0,5)\}, \{(0,1),(1,5),(1,6)\}, \{(1,4),(1,5),(1,7)\}, \{(1,3),(1,4),(1,5)\}, \{(1,6),(0,6),(1,1)\}, \{(1,3),(0,1),(0,4)\}, \{(0,1),(1,4),(0,2)\}$
	$\mathbb{Z}_4 \oplus \mathbb{Z}_8$	$\{(3,1),(3,2),(2,3)\}, \{(3,1),(2,1),(2,2)\}, \{(3,1),(1,1),(1,2)\}, \{(1,4),(0,2),(2,3)\}, \{(0,1),(2,2),(1,0)\}, \{(3,1),(1,0),(2,3)\}, \{(0,1),(1,4),(1,2)\}, \{(1,3),(3,3),(2,1)\}, \{(1,4),(3,3),(1,0)\}, \{(3,2),(2,1),(2,3)\}, \{(0,1),(1,4),(2,1)\}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$	$\{(1,0,3),(0,0,3),(0,1,2)\}, \{(0,0,1),(1,0,3),(1,1,1)\}, \{(1,0,2),(0,1,3),(1,1,2)\}$

Table 1. The seventy-six exceptional cases from Lemma 6.6.

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COMPUTATIONAL APPENDIX: HAMILTON DECOMPOSITIONS OF 6-REGULAR CAYLEY GRAPHS ON EVEN ABELIAN GROUPS WITH INVOLUTION-FREE CONNECTION SETS

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This appendix accompanies the manuscript Westlund [1] (Lemma 6.6). Hamilton decompositions (denoted $\{H_0, H_1, H_2\}$ in this document) of each of the seventy-six Cayley graphs in Table 1 were obtained using a program written by Donald L. Kreher of Michigan Technological University.

Order	Group	Connection Sets
12	$\mathbb{Z}_{12}$	$\{2,3,4\}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$	$\{(0,1),(0,2),(1,1)\},\{(0,1),(1,1),(1,2)\}$
18	$\mathbb{Z}_{18}$	$\{2,3,4\},\{3,4,6\}$
	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$	$\{(1,3),(0,1),(1,2)\},\{(1,3),(2,1),(1,1)\},\{(0,1),(2,2),(1,0)\},\{(2,1),(1,1),(1,2)\},$ $\{(2,1),(1,0),(1,2)\}$
24	$\mathbb{Z}_{24}$	$\{3,9,10\},\{2,6,9\},\{3,4,10\},\{2,9,10\},\{6,8,9\},\{3,4,9\},\{2,3,8\},\{3,8,9\},$ $\{4,8,9\},\{4,6,9\}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$	$\{(1,4),(1,5),(0,3)\},\{(1,3),(1,4),(0,3)\},\{(0,1),(0,2),(1,1)\},\{(1,3),(0,4),(1,2)\},$ $\{(1,3),(1,5),(0,3)\},\{(0,4),(0,5),(1,2)\},\{(0,2),(1,5),(1,2)\},\{(1,3),(0,1),(1,5)\},$ $\{(0,1),(0,3),(1,2)\},\{(1,3),(0,1),(0,4)\},\{(0,2),(0,3),(1,2)\},\{(1,4),(1,1),(1,2)\},$ $\{(1,4),(1,5),(0,5)\},\{(1,3),(0,1),(0,5)\},\{(0,2),(0,3),(1,1)\},\{(0,1),(0,4),(1,1)\},$ $\{(0,1),(1,4),(0,5)\},\{(1,3),(0,3),(0,4)\}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$	$\{(0,0,1),(1,1,1),(1,0,1)\}$
32	$\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$	$\{(0,2),(1,5),(0,3)\},\{(1,5),(1,6),(1,1)\},\{(1,3),(0,5),(1,2)\},\{(1,3),(0,7),(1,1)\},$ $\{(1,3),(0,1),(1,5)\},\{(1,4),(0,3),(1,7)\},\{(1,4),(0,2),(0,5)\},\{(0,1),(1,6),(0,7)\},$ $\{(1,5),(0,4),(1,2)\},\{(0,1),(1,4),(1,2)\},\{(1,5),(1,6),(1,2)\},\{(1,3),(0,7),(1,2)\},$ $\{(1,3),(1,4),(0,4)\},\{(1,3),(0,5),(1,1)\},\{(0,2),(1,6),(0,7)\},\{(0,1),(1,5),(0,6)\},$ $\{(1,3),(1,4),(0,5)\},\{(0,1),(1,5),(1,6)\},\{(1,4),(1,5),(1,7)\},\{(1,3),(1,4),(1,5)\},$ $\{(1,6),(0,6),(1,1)\},\{(1,3),(0,1),(0,4)\},\{(0,1),(1,4),(0,2)\}$
	$\mathbb{Z}_4 \oplus \mathbb{Z}_8$	$\{(3,1),(3,2),(2,3)\},\{(3,1),(2,1),(2,2)\},\{(3,1),(1,1),(1,2)\},\{(1,4),(0,2),(2,3)\},$ $\{(0,1),(2,2),(1,0)\},\{(3,1),(1,0),(2,3)\},\{(0,1),(1,4),(1,2)\},\{(1,3),(3,3),(2,1)\},$ $\{(1,4),(3,3),(1,0)\},\{(3,2),(2,1),(2,3)\},\{(0,1),(1,4),(2,1)\}$
	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$	$\{(1,0,3),(0,0,3),(0,1,2)\},\{(0,0,1),(1,0,3),(1,1,1)\},\{(1,0,2),(0,1,3),(1,1,2)\}$

Table 1. The seventy-six exceptional cases from Westlund [1] (Lemma 6.6).

$\mathrm{Cay}(\mathbb{Z}_{12};\{2,3,4\}^{\star})$

H_0:9,6,2,0,4,1,5,3,11,8,10,7

H_1:4,7,3,0,8,5,2,11,9,1,10,6

H_2:1,3,6,8,4,2,10,0,9,5,7,11

Date: June 2, 2014.

$\mathrm{Cay}(\mathbb{Z}_2\oplus\mathbb{Z}_6;\{(0,1),(0,2),(1,1)\}^\star)$
H_0:(1,1),(1,0),(0,1),(0,0),(0,4),(0,2),(0,3),(0,5),(1,4),(1,2),(1,3),(1,5)
H_1:(1,2),(0,3),(0,1),(0,2),(0,0),(1,5),(1,4),(1,0),(0,5),(0,4),(1,3),(1,1)
H_2:(0,1),(1,2),(1,0),(1,5),(0,4),(0,3),(1,4),(1,3),(0,2),(1,1),(0,0),(0,5)

$\mathrm{Cay}(\mathbb{Z}_2\oplus\mathbb{Z}_6;\{(0,1),(1,1),(1,2)\}^\star)$
H_0:(0,2),(1,4),(1,5),(0,0),(1,2),(0,3),(0,4),(1,0),(1,1),(0,5),(1,3),(0,1)
H_1:(1,0),(0,2),(0,3),(1,4),(1,3),(1,2),(1,1),(0,0),(0,5),(0,4),(1,5),(0,1)
H_2:(1,0),(0,5),(1,4),(0,0),(0,1),(1,2),(0,4),(1,3),(0,2),(1,1),(0,3),(1,5)

$\mathrm{Cay}(\mathbb{Z}_{18};\{2,3,4\}^\star)$
H_0:0,14,12,9,7,3,6,4,1,17,15,11,13,10,8,5,2,16
H_1:5,3,17,14,10,12,16,1,15,13,9,6,2,0,4,8,11,7
H_2:4,2,17,13,16,14,11,9,5,1,3,0,15,12,8,6,10,7

$\mathrm{Cay}(\mathbb{Z}_{18};\{3,4,6\}^\star)$
H_0:11,5,9,3,0,4,10,6,2,8,14,17,13,16,12,15,1,7
H_1:4,8,11,17,2,5,1,13,7,3,6,12,9,15,0,14,10,16
H_2:9,6,0,12,8,5,17,3,15,11,14,2,16,1,4,7,10,13

$\mathrm{Cay}(\mathbb{Z}_3\oplus\mathbb{Z}_6;\{(1,3),(0,1),(1,2)\}^\star)$
H_0:(1,5),(1,0),(2,2),(2,1),(2,0),(1,3),(0,0),(0,1),(2,5),(2,4),(2,3),(1,1),(1,2),(0,5),(0,4),(0,3),(0,2),(1,4)
H_1:(2,1),(0,3),(1,0),(0,4),(1,1),(2,4),(0,0),(2,3),(0,5),(2,2),(1,5),(0,2),(2,0),(2,5),(1,2),(1,3),(0,1),(1,4)
H_2:(2,0),(0,3),(1,5),(2,1),(0,4),(2,2),(2,3),(1,0),(1,1),(0,5),(0,0),(1,2),(2,4),(0,1),(0,2),(2,5),(1,3),(1,4)

$\mathrm{Cay}(\mathbb{Z}_3\oplus\mathbb{Z}_6;\{(1,3),(2,1),(1,1)\}^\star)$
H_0:(0,0),(2,1),(0,4),(1,1),(2,0),(0,1),(2,2),(0,5),(1,2),(0,3),(1,0),(2,3),(1,4),(2,5),(0,2),(1,5),(2,4),(1,3)
H_1:(0,3),(2,2),(1,5),(0,0),(2,3),(0,2),(1,1),(2,4),(0,1),(1,2),(2,1),(1,4),(0,5),(1,0),(2,5),(0,4),(1,3),(2,0)
H_2:(0,4),(1,5),(2,0),(0,5),(2,4),(0,3),(1,4),(0,1),(1,0),(2,1),(0,2),(1,3),(2,2),(1,1),(0,0),(2,5),(1,2),(2,3)

$\mathrm{Cay}(\mathbb{Z}_3\oplus\mathbb{Z}_6;\{(0,1),(2,2),(1,0)\}^\star)$
H_0:(0,0),(1,0),(2,0),(0,4),(0,5),(2,1),(0,1),(1,5),(1,4),(1,3),(1,2),(0,2),(0,3),(1,1),(2,5),(2,4),(2,3),(2,2)
H_1:(1,3),(2,3),(0,1),(0,0),(2,0),(1,2),(1,1),(1,0),(1,5),(0,5),(2,5),(0,3),(0,4),(1,4),(2,4),(0,2),(2,2),(2,1)
H_2:(0,4),(1,2),(2,2),(1,4),(0,0),(0,5),(1,3),(0,3),(2,3),(1,5),(2,5),(2,0),(2,1),(1,1),(0,1),(0,2),(1,0),(2,4)

$\mathrm{Cay}(\mathbb{Z}_3\oplus\mathbb{Z}_6;\{(2,1),(1,1),(1,2)\}^\star)$
H_0:(0,1),(2,0),(1,4),(0,5),(1,0),(2,2),(1,1),(0,2),(2,1),(0,0),(1,2),(2,3),(0,4),(1,5),(0,3),(2,4),(1,3),(2,5)
H_1:(0,1),(1,3),(0,2),(1,4),(0,3),(2,1),(1,2),(2,4),(0,5),(2,3),(1,1),(2,0),(1,5),(0,0),(2,5),(1,0),(0,4),(2,2)
H_2:(0,4),(2,5),(1,4),(2,3),(0,2),(2,0),(0,5),(1,1),(0,0),(2,4),(1,5),(2,1),(1,0),(0,1),(1,2),(0,3),(2,2),(1,3)

$\mathrm{Cay}(\mathbb{Z}_3\oplus\mathbb{Z}_6;\{(2,1),(1,0),(1,2)\}^\star)$
H_0:(0,5),(1,4),(0,2),(2,0),(1,0),(0,0),(2,4),(0,3),(2,1),(1,2),(2,2),(0,4),(1,3),(2,3),(1,1),(0,1),(2,5),(1,5)
H_1:(0,0),(1,2),(2,4),(0,4),(1,0),(2,2),(0,1),(2,1),(1,1),(0,2),(2,3),(1,4),(2,0),(0,5),(2,5),(1,3),(0,3),(1,5)
H_2:(2,5),(1,0),(0,1),(1,3),(2,2),(0,2),(1,2),(0,3),(2,3),(0,5),(1,1),(2,0),(0,0),(2,1),(1,5),(2,4),(1,4),(0,4)

$\mathrm{Cay}(\mathbb{Z}_{24};\{3,9,10\}^\star)$
H_0:6,9,18,4,7,16,2,5,8,22,19,10,1,11,14,17,20,23,13,3,0,15,12,21
H_1:15,1,16,19,5,14,4,13,10,0,21,7,22,12,9,23,2,11,20,6,3,17,8,18
H_2:2,12,3,18,21,11,8,23,14,0,9,19,4,1,22,13,16,6,15,5,20,10,7,17

$\mathrm{Cay}(\mathbb{Z}_{24};\{2,6,9\}^\star)$
H_0:15,0,6,4,10,1,3,9,7,22,20,14,16,18,12,21,19,17,2,8,23,5,11,13
H_1:15,6,12,10,16,7,1,19,4,13,22,0,18,9,11,2,20,5,3,21,23,14,8,17
H_2:12,3,18,20,11,17,23,1,16,22,4,2,0,9,15,21,6,8,10,19,13,7,5,14

$\mathrm{Cay}(\mathbb{Z}_{24};\{3,4,10\}^\star)$
H_0:19,5,9,13,17,20,6,10,14,0,4,18,15,11,1,21,7,3,23,2,16,12,8,22
H_1:8,5,1,15,12,2,22,18,14,17,3,6,9,19,16,13,23,20,10,0,21,11,7,4
H_2:10,13,3,0,20,16,6,2,5,15,19,23,9,12,22,1,4,14,11,8,18,21,17,7

$\mathrm{Cay}(\mathbb{Z}_{24};\{2,9,10\}^\star)$
H_0:7,21,6,20,10,8,17,3,13,11,1,15,0,9,23,14,5,19,4,18,16,2,12,22
H_1:13,15,6,8,22,0,10,1,16,7,17,19,9,18,3,5,20,11,2,4,14,12,21,23

H_2:0,2,17,15,5,7,9,11,21,19,10,12,3,1,23,8,18,20,22,13,4,6,16,14

Cay($\mathbb{Z}_{24}$;{6,8,9}*)

H_0:0,9,1,10,4,20,5,23,14,8,16,7,13,22,6,15,21,12,3,19,11,17,2,18

H_1:3,18,10,2,8,0,6,12,4,19,13,21,5,11,20,14,22,16,1,7,15,23,17,9

H_2:3,11,2,20,12,18,9,15,0,16,10,19,1,17,8,23,7,22,4,13,5,14,6,21

Cay($\mathbb{Z}_{24}$;{3,4,9}*)

H_0:5,20,0,3,12,15,6,2,23,19,10,7,4,8,11,14,17,21,1,16,13,22,18,9

H_1:3,6,21,18,14,23,8,5,1,10,13,17,20,16,12,9,0,4,19,15,11,2,22,7

H_2:22,1,4,13,9,6,10,14,5,2,17,8,12,21,0,15,18,3,23,20,11,7,16,19

Cay($\mathbb{Z}_{24}$;{2,3,8}*)

H_0:19,22,6,3,1,4,2,18,16,14,17,9,11,13,15,23,20,12,10,7,5,8,0,21

H_1:16,0,2,5,3,11,14,6,8,10,13,21,23,7,4,12,9,1,22,20,18,15,17,19

H_2:18,10,2,23,1,17,20,4,6,9,7,15,12,14,22,0,3,19,11,8,16,13,5,21

Cay($\mathbb{Z}_{24}$;{3,8,9}*)

H_0:2,18,15,0,3,11,20,12,21,6,9,17,14,5,13,10,1,22,7,4,19,16,8,23

H_1:5,2,17,8,0,9,1,4,12,3,6,22,14,11,19,10,18,21,13,16,7,15,23,20

H_2:0,16,1,17,20,4,13,22,19,3,18,9,12,15,6,14,23,7,10,2,11,8,5,21

Cay($\mathbb{Z}_{24}$;{4,8,9}*)

H_0:6,21,13,4,8,12,3,7,22,2,10,18,9,17,1,16,0,15,23,19,11,20,5,14

H_1:10,19,15,7,11,2,17,13,5,21,1,9,0,4,12,20,16,8,23,3,18,14,22,6

H_2:7,16,12,21,17,8,0,20,4,19,3,11,15,6,2,18,22,13,9,5,1,10,14,23

Cay($\mathbb{Z}_{24}$;{4,6,9}*)

H_0:0,18,3,23,8,12,16,1,19,10,4,22,2,6,15,21,17,11,7,13,9,5,14,20

H_1:16,7,22,18,12,3,9,15,11,5,23,14,10,1,21,6,0,4,19,13,17,8,2,20

H_2:23,17,2,11,20,5,1,7,3,21,12,6,10,16,22,13,4,8,14,18,9,0,15,19

Cay($\mathbb{Z}_2\oplus\mathbb{Z}_{12}$;{(1,4),(1,5),(0,3)}*)

H_0:(0,2),(1,6),(0,11),(1,4),(1,1),(0,8),(1,3),(1,0),(1,9),(0,1),(0,4),(1,8),(0,3),(1,7),(0,0),(0,9),(1,5),(0,10),(1,2),(0,7),(1,11),(0,6),(1,10),(0,5)

H_1:(0,3),(1,11),(0,4),(0,7),(1,3),(1,6),(0,10),(0,1),(1,5),(1,8),(0,0),(1,4),(0,8),(1,0),(0,5),(1,9),(0,2),(0,11),(1,7),(1,10),(1,1),(0,9),(1,2),(0,6)

H_2:(0,1),(1,8),(1,11),(1,2),(1,5),(0,0),(0,3),(1,10),(0,2),(1,7),(1,4),(0,9),(0,6),(1,1),(0,5),(0,8),(0,11),(1,3),(0,10),(0,7),(1,0),(0,4),(1,9),(1,6)

Cay($\mathbb{Z}_2\oplus\mathbb{Z}_{12}$;{(1,3),(1,4),(0,3)}*)

H_0:(0,8),(0,5),(1,2),(0,6),(0,3),(1,7),(0,11),(1,3),(1,0),(1,9),(0,0),(1,4),(0,1),(0,4),(1,1),(0,10),(0,7),(1,10),(0,2),(1,6),(0,9),(1,5),(1,8),(1,11)

H_1:(1,8),(0,11),(0,2),(0,5),(1,1),(0,9),(0,0),(0,3),(1,11),(1,2),(1,5),(0,1),(0,10),(1,7),(1,10),(0,6),(1,9),(1,6),(1,3),(0,7),(1,4),(0,8),(1,0),(0,4)

H_2:(0,1),(1,9),(0,5),(1,8),(0,0),(1,3),(0,6),(0,9),(1,0),(0,3),(1,6),(0,10),(1,2),(0,11),(0,8),(1,5),(0,2),(1,11),(0,7),(0,4),(1,7),(1,4),(1,1),(1,10)

Cay($\mathbb{Z}_2\oplus\mathbb{Z}_{12}$;{(0,1),(0,2),(1,1)}*)

H_0:(0,6),(0,7),(1,8),(1,10),(1,0),(1,1),(0,2),(0,4),(0,3),(0,1),(1,2),(1,3),(1,4),(0,5),(1,6),(1,5),(1,7),(1,9),(0,10),(1,11),(0,0),(0,11),(0,9),(0,8)

H_1:(1,3),(0,2),(0,0),(0,1),(1,0),(0,11),(1,10),(0,9),(0,10),(0,8),(0,7),(1,6),(1,7),(1,8),(1,9),(1,11),(1,1),(1,2),(1,4),(0,3),(0,5),(0,4),(0,6),(1,5)

H_2:(1,9),(0,8),(1,7),(0,6),(0,5),(0,7),(0,9),(1,8),(1,6),(1,4),(1,5),(0,4),(1,3),(1,1),(0,0),(0,10),(0,11),(0,1),(0,2),(0,3),(1,2),(1,0),(1,11),(1,10)

Cay($\mathbb{Z}_2\oplus\mathbb{Z}_{12}$;{(1,3),(0,4),(1,2)}*)

H_0:(0,1),(1,4),(0,6),(1,8),(1,0),(0,2),(1,11),(0,9),(1,6),(0,4),(0,0),(1,3),(1,7),(0,10),(1,1),(1,9),(0,7),(0,11),(0,3),(1,5),(0,8),(1,10),(1,2),(0,5)

H_1:(0,1),(1,3),(0,5),(1,8),(1,4),(0,7),(0,3),(1,1),(0,4),(1,7),(0,9),(1,0),(0,10),(0,6),(0,2),(1,5),(1,9),(0,11),(1,2),(1,6),(1,10),(0,0),(0,8),(1,11)

H_2:(0,5),(1,7),(1,11),(1,3),(0,6),(1,9),(0,0),(1,2),(0,4),(0,8),(1,6),(0,3),(1,0),(1,4),(0,2),(0,10),(1,8),(0,11),(1,1),(1,5), (0,7),(1,10),(0,1),(0,9)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(1,3),(1,5),(0,3)}*)

H_0:(0,4),(0,1),(1,4),(1,7),(0,2),(1,5),(0,8),(1,11),(1,2),(0,5),(1,8),(0,11),(1,6),(1,3),(0,6),(0,9),(1,0),(0,7),(0,10),(1,1),
(1,10),(0,3),(0,0),(1,9)

H_1:(0,7),(1,2),(0,9),(1,6),(0,3),(0,6),(1,1),(0,4),(1,7),(1,10),(0,1),(0,10),(1,3),(0,0),(1,5),(1,8),(1,11),(0,2),(1,9),(1,0),
(0,5),(0,8),(0,11),(1,4)

H_2:(0,8),(1,3),(1,0),(0,3),(1,8),(0,1),(1,6),(1,9),(0,6),(1,11),(0,4),(0,7),(1,10),(0,5),(0,2),(0,11),(1,2),(1,5),(0,10),(1,7),
(0,0),(0,9),(1,4),(1,1)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(0,4),(0,5),(1,2)}*)

H_0:(0,3),(0,8),(0,0),(0,4),(0,11),(0,6),(1,8),(0,10),(1,0),(0,2),(0,9),(1,7),(1,2),(1,9),(1,4),(1,11),(0,1),(0,5),(1,3),(1,10),
(1,6),(1,1),(1,5),(0,7)

H_1:(0,1),(1,3),(1,7),(1,0),(1,4),(1,8),(1,1),(0,3),(1,5),(1,9),(0,11),(0,7),(0,2),(0,6),(0,10),(0,5),(0,0),(1,2),(1,10),(0,8),
(0,4),(1,6),(1,11),(0,9)

H_2:(1,8),(1,0),(1,5),(1,10),(0,0),(0,7),(1,9),(1,1),(0,11),(0,3),(0,10),(0,2),(1,4),(0,6),(0,1),(0,8),(1,6),(1,2),(0,4),(0,9),
(0,5),(1,7),(1,11),(1,3)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(0,2),(1,5),(1,2)}*)

H_0:(1,2),(0,7),(1,0),(0,5),(0,3),(1,8),(1,6),(0,4),(0,2),(0,0),(1,10),(0,8),(0,6),(1,1),(1,3),(0,10),(1,5),(1,7),(1,9),(0,11),
(0,1),(1,11),(0,9),(1,4)

H_1:(1,11),(0,6),(1,4),(1,6),(0,8),(1,3),(0,5),(1,7),(0,2),(1,0),(1,10),(0,3),(0,1),(1,8),(0,10),(0,0),(1,5),(0,7),(1,9),(0,4),
(1,2),(0,9),(0,11),(1,1)

H_2:(1,1),(0,3),(1,5),(1,3),(0,1),(1,6),(0,11),(1,4),(0,2),(1,9),(1,11),(0,4),(0,6),(1,8),(1,10),(0,5),(0,7),(0,9),(1,7),(0,0),
(1,2),(1,0),(0,10),(0,8)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(1,3),(0,1),(1,5)}*)

H_0:(0,3),(1,6),(0,1),(0,2),(1,11),(1,10),(1,9),(0,6),(0,5),(0,4),(1,7),(0,0),(1,3),(1,4),(0,7),(1,0),(1,1),(0,8),(1,5),(0,10),
(0,9),(1,2),(0,11),(1,8)

H_1:(1,0),(0,3),(0,2),(1,9),(0,4),(1,1),(1,2),(0,7),(0,6),(1,3),(0,10),(0,11),(0,0),(0,1),(1,10),(0,5),(1,8),(1,7),(1,6),(1,5),
(1,4),(0,9),(0,8),(1,11)

H_2:(0,5),(1,2),(1,3),(0,8),(0,7),(1,10),(0,3),(0,4),(1,11),(0,6),(1,1),(0,10),(1,7),(0,2),(1,5),(0,0),(1,9),(1,8),(0,1),(1,4),
(0,11),(1,6),(0,9),(1,0)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(0,1),(0,3),(1,2)}*)

H_0:(1,9),(0,7),(1,5),(1,2),(1,11),(1,0),(0,10),(1,8),(1,7),(1,4),(0,2),(0,11),(1,1),(0,3),(0,0),(1,10),(0,8),(0,9),(0,6),(0,5),
(1,3),(0,1),(0,4),(1,6)

H_1:(1,5),(1,8),(0,6),(0,7),(0,4),(0,3),(0,2),(0,5),(1,7),(1,6),(0,8),(0,11),(0,0),(0,1),(0,10),(0,9),(1,11),(1,10),(1,9),(1,0),
(1,1),(1,2),(1,3),(1,4)

H_2:(1,6),(1,3),(1,0),(0,2),(0,1),(1,11),(1,8),(1,9),(0,11),(0,10),(0,7),(0,8),(0,5),(0,4),(1,2),(0,0),(0,9),(1,7),(1,10),(1,1),
(1,4),(0,6),(0,3),(1,5)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(1,3),(0,1),(0,4)}*)

H_0:(1,1),(1,2),(0,5),(0,1),(1,4),(1,5),(0,8),(0,7),(0,6),(1,9),(0,0),(1,3),(1,11),(1,10),(1,6),(0,9),(1,0),(0,3),(0,2),(0,10),
(0,11),(1,8),(1,7),(0,4)

H_1:(0,10),(1,1),(1,5),(0,2),(0,1),(0,9),(0,5),(0,4),(0,3),(1,6),(1,2),(1,10),(1,9),(1,8),(1,0),(1,4),(0,7),(0,11),(0,0),(0,8),
(1,11),(1,7),(1,3),(0,6)

H_2:(1,5),(1,6),(1,7),(0,10),(0,9),(0,8),(0,4),(0,0),(0,1),(1,10),(0,7),(0,3),(0,11),(1,2),(1,3),(1,4),(1,8),(0,5),(0,6),(0,2),
(1,11),(1,0),(1,1),(1,9)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(0,2),(0,3),(1,2)}*)

H_0:(0,0),(0,3),(0,1),(1,3),(1,5),(1,2),(0,4),(0,2),(1,4),(1,1),(0,11),(0,9),(0,6),(0,8),(1,6),(1,8),(1,11),(1,9),(1,7),(0,5),
(0,7),(0,10),(1,0),(1,10)

H_1:(1,5),(0,3),(0,6),(0,4),(0,7),(0,9),(0,0),(0,2),(1,0),(1,9),(1,6),(1,4),(1,2),(1,11),(1,1),(1,3),(0,5),(0,8),(0,11),(0,1),
(0,10),(1,8),(1,10),(1,7)

H_2:(1,8),(1,5),(0,7),(1,9),(0,11),(0,2),(0,5),(0,3),(1,1),(1,10),(0,8),(0,10),(0,0),(1,2),(1,0),(1,3),(1,6),(0,4),(0,1),(1,11),
(0,9),(1,7),(1,4),(0,6)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(1,4),(1,1),(1,2)}*)

H_0:(1,6),(0,4),(1,2),(0,6),(1,7),(0,5),(1,3),(0,2),(1,4),(0,3),(1,1),(0,9),(1,5),(0,7),(1,8),(0,10),(1,9),(0,1),(1,11),(0,0),
(1,10),(0,11),(1,0),(0,8)

H_1:(0,3),(1,11),(0,7),(1,3),(0,4),(1,8),(0,6),(1,5),(0,1),(1,0),(0,10),(1,6),(0,5),(1,9),(0,11),(1,1),(0,2),(1,10),(0,9),(1,7),(0,8),(1,4),(0,0),(1,2)

H_2:(0,0),(1,1),(0,5),(1,4),(0,6),(1,10),(0,8),(1,9),(0,7),(1,6),(0,2),(1,0),(0,4),(1,5),(0,3),(1,7),(0,11),(1,3),(0,1),(1,2),(0,10),(1,11),(0,9),(1,8)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(1,4),(1,5),(0,5)}*)

H_0:(1,9),(1,2),(0,7),(0,2),(1,10),(0,3),(1,8),(0,0),(0,5),(1,0),(1,7),(0,11),(0,4),(1,11),(0,6),(1,1),(0,8),(1,3),(0,10),(1,6),(0,1),(1,5),(0,9),(1,4)

H_1:(1,4),(0,8),(1,0),(0,7),(0,0),(1,5),(0,10),(1,2),(0,6),(1,10),(1,3),(0,11),(1,6),(1,1),(0,5),(1,9),(0,1),(1,8),(0,4),(0,9),(0,2),(1,7),(0,3),(1,11)

H_2:(1,9),(0,2),(1,6),(1,11),(0,7),(1,3),(1,8),(1,1),(0,9),(1,2),(1,7),(0,0),(1,4),(0,11),(0,6),(0,1),(0,8),(0,3),(0,10),(0,5),(1,10),(1,5),(1,0),(0,4)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(1,3),(0,1),(0,5)}*)

H_0:(1,4),(0,1),(0,2),(0,9),(1,6),(1,5),(1,0),(1,1),(1,2),(1,3),(0,0),(0,5),(0,6),(0,11),(0,4),(0,3),(0,10),(1,7),(1,8),(1,9),(1,10),(0,7),(0,8),(1,11)

H_1:(1,0),(1,7),(1,2),(0,11),(0,0),(0,1),(0,6),(1,9),(1,4),(0,7),(0,2),(0,3),(0,8),(1,5),(1,10),(1,3),(1,8),(0,5),(0,4),(0,9),(0,10),(1,1),(1,6),(1,11)

H_2:(1,10),(0,1),(0,8),(0,9),(1,0),(0,3),(1,6),(1,7),(0,4),(1,1),(1,8),(0,11),(0,10),(0,5),(1,2),(1,9),(0,0),(0,7),(0,6),(1,3),(1,4),(1,5),(0,2),(1,11)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(0,2),(0,3),(1,1)}*)

H_0:(0,7),(0,4),(1,3),(0,2),(0,5),(0,3),(1,2),(1,5),(0,6),(1,7),(1,10),(1,0),(0,11),(0,1),(0,10),(0,8),(1,9),(1,11),(0,0),(1,1),(1,4),(1,6),(1,8),(0,9)

H_1:(1,2),(1,4),(0,3),(0,6),(0,8),(1,7),(1,9),(1,6),(0,5),(0,7),(1,8),(1,5),(1,3),(1,0),(0,1),(0,4),(0,2),(1,1),(1,10),(0,11),(0,9),(0,0),(0,10),(1,11)

H_2:(0,9),(0,6),(0,4),(1,5),(1,7),(1,4),(0,5),(0,8),(0,11),(0,2),(0,0),(0,3),(0,1),(1,2),(1,0),(1,9),(0,10),(0,7),(1,6),(1,3),(1,1),(1,11),(1,8),(1,10)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(0,1),(0,4),(1,1)}*)

H_0:(1,2),(0,3),(1,4),(1,3),(0,4),(0,5),(0,1),(1,0),(1,1),(0,0),(0,8),(1,7),(1,11),(1,10),(0,9),(1,8),(1,9),(1,5),(0,6),(0,2),(0,10),(0,11),(0,7),(1,6)

H_1:(1,1),(0,2),(0,3),(0,7),(0,6),(1,7),(1,6),(0,5),(0,9),(0,1),(0,0),(0,4),(0,8),(1,9),(0,10),(1,11),(1,3),(1,2),(1,10),(0,11),(1,0),(1,8),(1,4),(1,5)

H_2:(1,11),(0,0),(0,11),(0,3),(0,4),(1,5),(1,6),(1,10),(1,9),(1,1),(1,2),(0,1),(0,2),(1,3),(1,7),(1,8),(0,7),(0,8),(0,9),(0,10),(0,6),(0,5),(1,4),(1,0)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(0,1),(1,4),(0,5)}*)

H_0:(0,6),(0,1),(0,8),(0,3),(1,11),(0,7),(0,2),(1,6),(1,5),(1,10),(1,3),(0,11),(0,0),(1,4),(1,9),(0,5),(1,1),(1,8),(1,7),(1,0),(0,4),(0,9),(0,10),(1,2)

H_1:(0,4),(0,3),(0,2),(0,9),(1,5),(0,1),(0,0),(0,5),(0,6),(0,7),(1,3),(1,2),(1,7),(0,11),(0,10),(1,6),(1,1),(1,0),(0,8),(1,4),(1,11),(1,10),(1,9),(1,8)

H_2:(0,2),(0,1),(1,9),(1,2),(1,1),(0,9),(0,8),(0,7),(0,0),(1,8),(1,3),(1,4),(1,5),(1,0),(1,11),(1,6),(1,7),(0,3),(0,10),(0,5),(0,4),(0,11),(0,6),(1,10)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$;{(1,3),(0,3),(0,4)}*)

H_0:(1,6),(1,2),(0,5),(0,1),(0,10),(1,1),(1,4),(1,8),(1,0),(0,3),(0,7),(0,11),(0,2),(1,11),(1,3),(0,6),(0,9),(0,0),(1,9),(1,5),(0,8),(0,4),(1,7),(1,10)

H_1:(0,5),(0,8),(0,11),(1,2),(1,5),(1,8),(1,11),(1,7),(1,3),(1,0),(1,4),(0,1),(0,4),(0,0),(0,3),(0,6),(0,2),(0,10),(0,7),(1,10),(1,1),(1,9),(1,6),(0,9)

H_2:(1,3),(1,6),(0,3),(0,11),(1,8),(0,5),(0,2),(1,5),(1,1),(0,4),(0,7),(1,4),(1,7),(0,10),(0,6),(1,9),(1,0),(0,9),(0,1),(1,10),(1,2),(1,11),(0,8),(0,0)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$;{(0,0,1),(1,1,1),(1,0,1)}*)

H_0:(0,0,4),(0,0,5),(0,0,0),(0,0,1),(1,1,2),(0,1,3),(0,1,4),(1,0,3),(0,0,2),(1,0,1),(0,1,0),(0,1,1),(1,0,2),(0,0,3),(1,0,4),(1,0,5),(1,0,0),(0,1,5),(1,1,0),(1,1,1),(0,1,2),(1,1,3),(1,1,4),(1,1,5)

H_1:(1,0,3),(0,0,4),(1,1,3),(0,1,4),(1,1,5),(0,0,0),(1,0,5),(0,1,0),(0,1,5),(1,0,4),(0,0,5),(1,0,0),(0,0,1),(1,1,0),(0,1,1),(1,1,2),(1,1,1),(0,0,2),(0,0,3),(1,1,4),(0,1,3),(1,0,2),(1,0,1),(0,1,2)

H_2:(0,1,1),(0,1,2),(0,1,3),(1,0,4),(1,0,3),(1,0,2),(0,0,1),(0,0,2),(1,1,3),(1,1,2),(0,0,3),(0,0,4),(1,0,5),(0,1,4),(0,1,5), (1,1,4),(0,0,5),(1,1,0),(1,1,5),(0,1,0),(1,1,1),(0,0,0),(1,0,1),(1,0,0)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;$\{(0,2),(1,5),(0,3)\}^\star$)

H_0:(1,7),(1,4),(1,6),(1,3),(1,5),(0,10),(0,8),(0,6),(0,3),(0,1),(0,4),(0,7),(0,9),(0,11),(0,13),(0,0),(0,14),(0,12),(0,15),(0,2),(0,5),(1,10),(1,8),(1,11),(1,13),(1,0),(1,2),(1,15),(1,1),(1,14),(1,12),(1,9)

H_1:(1,6),(0,1),(1,12),(1,10),(1,13),(0,8),(0,5),(1,0),(0,11),(0,14),(1,3),(1,1),(1,4),(0,15),(0,13),(1,2),(0,7),(0,10),(1,15),(0,4),(0,6),(0,9),(0,12),(1,7),(0,2),(0,0),(1,5),(1,8),(0,3),(1,14),(1,11),(1,9)

H_2:(0,3),(0,0),(1,11),(0,6),(1,1),(0,12),(0,10),(0,13),(1,8),(1,6),(0,11),(0,8),(1,3),(1,0),(1,14),(0,9),(1,4),(1,2),(1,5),(1,7),(1,10),(0,15),(0,1),(0,14),(1,9),(0,4),(0,2),(1,13),(1,15),(1,12),(0,7),(0,5)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;$\{(1,5),(1,6),(1,1)\}^\star$)

H_0:(0,2),(1,8),(0,9),(1,14),(0,3),(1,9),(0,8),(1,2),(0,13),(1,7),(0,1),(1,11),(0,6),(1,1),(0,0),(1,10),(0,4),(1,5),(0,10),(1,4),(0,15),(1,0),(0,11),(1,12),(0,7),(1,13),(0,12),(1,6),(0,5),(1,15),(0,14),(1,3)

H_1:(0,6),(1,5),(0,11),(1,10),(0,5),(1,0),(0,1),(1,6),(0,7),(1,1),(0,12),(1,2),(0,3),(1,8),(0,13),(1,3),(0,4),(1,14),(0,15),(1,9),(0,10),(1,11),(0,0),(1,15),(0,9),(1,4),(0,14),(1,13),(0,8),(1,7),(0,2),(1,12)

H_2:(1,15),(0,4),(1,9),(0,14),(1,8),(0,7),(1,2),(0,1),(1,12),(0,13),(1,14),(0,8),(1,3),(0,9),(1,10),(0,15),(1,5),(0,0),(1,6),(0,11),(1,1),(0,2),(1,13),(0,3),(1,4),(0,5),(1,11),(0,12),(1,7),(0,6),(1,0),(0,10)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;$\{(1,3),(0,5),(1,2)\}^\star$)

H_0:(1,14),(0,1),(1,3),(0,5),(1,2),(0,0),(0,11),(1,8),(0,6),(1,9),(0,7),(0,2),(1,4),(1,15),(0,13),(0,8),(1,11),(0,9),(1,7),(0,4),(1,6),(0,3),(1,1),(0,15),(1,13),(0,10),(1,12),(0,14),(1,0),(1,5),(1,10),(0,12)

H_1:(0,1),(1,4),(0,6),(1,3),(0,0),(0,5),(1,7),(0,10),(1,8),(1,13),(0,11),(1,14),(1,9),(0,12),(0,7),(1,5),(0,8),(0,3),(0,14),(1,1),(1,12),(0,15),(1,2),(0,4),(0,9),(1,6),(1,11),(1,0),(0,2),(0,13),(1,10),(1,15)

H_2:(0,8),(1,6),(1,1),(0,4),(0,15),(0,10),(0,5),(1,8),(1,3),(1,14),(0,0),(1,13),(1,2),(1,7),(1,12),(0,9),(0,14),(1,11),(0,13),(1,0),(0,3),(1,5),(0,2),(1,15),(0,12),(0,1),(0,6),(0,11),(1,9),(1,4),(0,7),(1,10)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;$\{(1,3),(0,7),(1,1)\}^\star$)

H_0:(1,6),(0,5),(1,2),(0,15),(1,12),(1,3),(0,4),(1,7),(1,0),(0,1),(1,4),(0,7),(0,14),(1,1),(0,2),(1,5),(0,8),(1,9),(0,6),(0,13),(1,14),(0,11),(1,10),(0,9),(1,8),(1,15),(0,0),(1,13),(0,10),(1,11),(0,12),(0,3)

H_1:(1,13),(0,12),(1,9),(1,0),(0,3),(1,2),(0,1),(0,10),(1,7),(0,8),(0,15),(1,14),(1,5),(0,6),(1,3),(0,2),(0,9),(0,0),(1,1),(0,4),(0,11),(1,12),(0,13),(1,10),(0,7),(1,8),(0,5),(1,4),(1,11),(0,14),(1,15),(1,6)

H_2:(0,6),(1,7),(1,14),(0,1),(0,8),(1,11),(1,2),(1,9),(0,10),(0,3),(1,4),(1,13),(0,14),(0,5),(0,12),(1,15),(0,2),(0,11),(1,8),(1,1),(1,10),(1,3),(0,0),(0,7),(1,6),(0,9),(1,12),(1,5),(0,4),(0,13),(1,0),(0,15)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;$\{(1,3),(0,1),(1,5)\}^\star$)

H_0:(0,0),(0,15),(0,14),(1,1),(0,12),(1,15),(1,0),(0,11),(0,10),(1,7),(0,4),(0,3),(1,6),(0,9),(1,14),(1,13),(0,8),(0,7),(0,6),(0,5),(1,8),(1,9),(1,10),(0,13),(1,2),(1,3),(1,4),(1,5),(0,2),(0,1),(1,12),(1,11)

H_1:(0,5),(1,2),(0,7),(1,4),(0,1),(1,6),(1,7),(0,2),(0,3),(1,14),(0,11),(1,8),(0,13),(0,12),(1,9),(0,6),(1,3),(0,14),(1,11),(1,10),(0,15),(1,12),(0,9),(0,8),(1,5),(0,0),(1,13),(0,10),(1,15),(0,4),(1,1),(1,0)

H_2:(0,10),(1,5),(1,6),(0,11),(0,12),(1,7),(1,8),(0,3),(1,0),(0,13),(0,14),(1,9),(0,4),(0,5),(1,10),(0,7),(1,12),(1,13),(0,2),(1,15),(1,14),(0,1),(0,0),(1,3),(0,8),(1,11),(0,6),(1,1),(1,2),(0,15),(1,4),(0,9)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;$\{(1,4),(0,3),(1,7)\}^\star$)

H_0:(1,10),(0,1),(1,5),(1,8),(0,4),(1,11),(0,7),(1,3),(0,12),(0,9),(1,0),(1,13),(0,6),(1,2),(0,11),(0,14),(1,7),(0,0),(1,4),(0,13),(1,9),(0,5),(0,2),(0,15),(1,6),(0,10),(1,14),(1,1),(0,8),(1,12),(1,15),(0,3)

H_1:(0,2),(1,6),(1,9),(1,12),(0,0),(0,3),(0,6),(1,10),(1,7),(1,4),(1,1),(0,13),(0,10),(1,3),(0,15),(1,8),(0,12),(1,0),(0,7),(0,4),(0,1),(1,13),(0,9),(1,5),(0,14),(1,2),(1,15),(0,11),(0,8),(0,5),(1,14),(1,11)

H_2:(1,3),(1,0),(0,4),(1,13),(1,10),(0,14),(0,1),(1,8),(1,11),(0,15),(0,12),(1,5),(1,2),(0,9),(0,6),(1,15),(0,8),(1,4),(0,11),(1,7),(0,3),(1,12),(0,5),(1,1),(0,10),(0,7),(1,14),(0,2),(1,9),(0,0),(0,13),(1,6)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;$\{(1,4),(0,2),(0,5)\}^\star$)

H_0:(0,5),(0,0),(0,2),(0,4),(0,6),(0,1),(0,3),(1,15),(1,1),(0,13),(1,9),(1,4),(1,6),(0,10),(0,8),(1,12),(1,7),(0,11),(0,9),(1,5),(1,0),(1,14),(1,3),(0,15),(1,11),(1,13),(1,2),(0,14),(1,10),(1,8),(0,12),(0,7)

H_1:(1,0),(1,2),(1,7),(0,3),(0,14),(0,12),(0,1),(0,15),(0,10),(0,5),(1,1),(1,3),(1,5),(1,10),(1,15),(1,13),(0,9),(0,7),(0,2),(0,13),(0,11),(0,6),(0,8),(1,4),(0,0),(1,12),(1,14),(1,9),(1,11),(1,6),(1,8),(0,4)

H_2:(1,14),(0,2),(1,6),(1,1),(1,12),(1,10),(0,6),(1,2),(1,4),(1,15),(0,11),(0,0),(0,14),(0,9),(0,4),(0,15),(0,13),(0,8),(0,3),(0,5),(1,9),(1,7),(1,5),(0,1),(1,13),(1,8),(1,3),(0,7),(1,11),(1,0),(0,12),(0,10)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;$\{(0,1),(1,6),(0,7)\}^\star$)

H_0:(1,6),(0,0),(0,1),(0,2),(0,9),(1,3),(1,10),(1,1),(0,11),(0,4),(0,3),(1,9),(1,2),(0,8),(0,7),(0,6),(0,15),(1,5),(1,4),(0,10),(1,0),(1,7),(0,13),(0,14),(1,8),(1,15),(1,14),(1,13),(1,12),(1,11),(0,5),(0,12)

H_1:(1,4),(1,11),(0,1),(0,8),(1,14),(1,7),(1,6),(1,15),(1,0),(0,6),(0,5),(0,4),(0,13),(1,3),(1,2),(1,1),(1,8),(0,2),(1,12),
(1,5),(0,11),(0,12),(0,3),(0,10),(0,9),(0,0),(1,10),(1,9),(0,15),(0,14),(0,7),(1,13)

H_2:(0,13),(0,6),(1,12),(1,3),(1,4),(0,14),(0,5),(1,15),(0,9),(0,8),(0,15),(0,0),(0,7),(1,1),(1,0),(1,9),(1,8),(1,7),(0,1),
(0,10),(0,11),(0,2),(0,3),(1,13),(1,6),(1,5),(1,14),(0,4),(1,10),(1,11),(1,2),(0,12)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,5),(0,4),(1,2)}*)

H_0:(0,14),(1,3),(0,1),(0,5),(1,10),(0,12),(0,0),(1,14),(0,3),(0,15),(1,1),(0,6),(1,8),(0,13),(1,2),(0,4),(1,15),(1,11),(1,7),
(0,9),(1,4),(1,0),(0,2),(1,13),(0,8),(1,6),(0,11),(1,9),(1,5),(0,7),(1,12),(0,10)

H_1:(0,11),(0,15),(1,4),(0,6),(0,10),(1,8),(1,12),(0,14),(0,2),(1,7),(1,3),(0,8),(1,10),(1,14),(0,12),(1,1),(1,13),(1,9),(0,4),
(0,0),(1,5),(0,3),(0,7),(1,2),(1,6),(0,1),(1,15),(0,13),(1,11),(0,9),(0,5),(1,0)

H_2:(1,9),(0,7),(0,11),(1,13),(0,15),(1,10),(1,6),(0,4),(0,8),(0,12),(1,7),(0,5),(1,3),(1,15),(0,10),(1,5),(1,1),(0,3),(1,8),
(1,4),(0,2),(0,6),(1,11),(0,0),(1,2),(1,14),(0,9),(0,13),(0,1),(1,12),(1,0),(0,14)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(0,1),(1,4),(1,2)}*)

H_0:(0,5),(1,1),(1,2),(0,4),(1,0),(1,15),(0,1),(0,2),(1,14),(0,0),(1,12),(1,13),(0,9),(1,5),(0,3),(1,7),(0,11),(1,9),(1,8),
(0,10),(1,6),(0,8),(1,4),(1,3),(0,15),(0,14),(0,13),(0,12),(1,10),(1,11),(0,7),(0,6)

H_1:(1,1),(0,15),(0,0),(0,1),(1,5),(1,6),(0,2),(1,4),(0,6),(1,10),(1,9),(0,5),(0,4),(0,3),(1,15),(0,13),(1,11),(0,9),(1,7),
(1,8),(0,12),(0,11),(1,13),(1,14),(0,10),(1,12),(0,8),(0,7),(1,3),(1,2),(0,14),(1,0)

H_2:(1,0),(0,2),(0,3),(1,1),(0,13),(1,9),(0,7),(1,5),(1,4),(0,0),(1,2),(0,6),(1,8),(0,4),(1,6),(1,7),(0,5),(1,3),(0,1),
(1,13),(0,15),(1,11),(1,12),(0,14),(1,10),(0,8),(0,9),(0,10),(0,11),(1,15),(1,14),(0,12)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,5),(1,6),(1,2)}*)

H_0:(1,3),(0,1),(1,15),(0,10),(1,8),(0,2),(1,12),(0,14),(1,0),(0,11),(1,13),(0,3),(1,1),(0,6),(1,4),(0,9),(1,14),(0,4),(1,2),
(0,7),(1,9),(0,15),(1,5),(0,0),(1,6),(0,12),(1,7),(0,13),(1,11),(0,5),(1,10),(0,8)

H_1:(0,3),(1,5),(0,10),(1,4),(0,14),(1,3),(0,5),(1,7),(0,1),(1,11),(0,9),(1,15),(0,4),(1,9),(0,11),(1,6),(0,8),(1,14),(0,0),
(1,10),(0,15),(1,13),(0,2),(1,0),(0,6),(1,12),(0,7),(1,1),(0,12),(1,2),(0,13),(1,8)

H_2:(0,6),(1,11),(0,0),(1,2),(0,8),(1,13),(0,7),(1,5),(0,11),(1,1),(0,15),(1,4),(0,2),(1,7),(0,9),(1,3),(0,13),(1,15),(0,5),
(1,0),(0,10),(1,12),(0,1),(1,6),(0,4),(1,10),(0,12),(1,14),(0,3),(1,9),(0,14),(1,8)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,3),(0,7),(1,2)}*)

H_0:(0,9),(0,2),(1,4),(0,1),(1,3),(0,0),(1,13),(0,11),(0,4),(0,13),(1,0),(1,7),(1,14),(0,12),(1,10),(0,8),(1,6),(1,15),(1,8),
(1,1),(0,3),(0,10),(1,12),(1,5),(0,7),(1,9),(0,6),(0,15),(1,2),(0,5),(0,14),(1,11)

H_1:(0,1),(0,10),(1,7),(0,5),(1,3),(0,6),(1,8),(0,11),(1,14),(0,0),(1,2),(0,4),(1,6),(0,9),(1,12),(0,14),(0,7),(1,4),(1,13),
(0,15),(1,1),(1,10),(0,13),(1,11),(0,8),(1,5),(0,3),(0,12),(1,9),(1,0),(0,2),(1,15)

H_2:(1,5),(0,2),(0,11),(1,9),(1,2),(1,11),(1,4),(0,6),(0,13),(1,15),(0,12),(0,5),(1,8),(0,10),(1,13),(1,6),(0,3),(1,0),(0,14),
(1,1),(0,4),(1,7),(0,9),(0,0),(0,7),(1,10),(1,3),(1,12),(0,15),(0,8),(0,1),(1,14)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,3),(1,4),(0,4)}*)

H_0:(0,0),(1,12),(1,0),(0,12),(0,8),(1,5),(0,2),(1,6),(0,3),(1,15),(1,3),(0,15),(1,2),(1,14),(0,11),(1,8),(1,4),(0,7),(1,10),
(0,6),(1,9),(0,5),(0,1),(0,13),(0,9),(1,13),(0,10),(1,7),(1,11),(0,14),(1,1),(0,4)

H_1:(1,6),(1,2),(0,6),(1,3),(1,7),(0,3),(1,0),(0,4),(0,8),(1,4),(0,1),(1,5),(1,1),(0,13),(1,9),(1,13),(0,0),(0,12),(1,8),
(0,5),(0,9),(1,12),(0,15),(1,11),(0,7),(0,11),(1,15),(0,2),(1,14),(0,10),(0,14),(1,10)

H_2:(1,5),(0,9),(1,6),(0,10),(0,6),(0,2),(0,14),(1,2),(0,5),(1,1),(1,13),(0,1),(1,14),(1,10),(0,13),(1,0),(1,4),(0,0),(1,3),
(0,7),(0,3),(0,15),(0,11),(1,7),(0,4),(1,8),(1,12),(0,8),(1,11),(1,15),(0,12),(1,9)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,3),(0,5),(1,1)}*)

H_0:(1,6),(0,3),(0,8),(1,11),(0,12),(0,1),(0,6),(1,3),(0,0),(0,11),(1,12),(0,9),(0,4),(1,5),(1,0),(0,15),(0,10),(1,7),(1,2),
(0,5),(1,4),(1,9),(1,14),(0,13),(0,2),(1,15),(1,10),(0,7),(1,8),(1,13),(0,14),(1,1)

H_1:(0,10),(0,5),(1,6),(0,7),(0,2),(1,5),(0,8),(1,7),(0,4),(1,3),(1,8),(0,9),(1,10),(0,11),(0,6),(1,9),(0,12),(1,13),(0,0),
(1,1),(1,12),(0,13),(1,0),(0,3),(1,2),(0,15),(1,14),(0,1),(1,4),(1,15),(0,14),(1,11)

H_2:(1,15),(0,0),(0,5),(1,8),(0,11),(1,14),(1,3),(0,2),(1,1),(0,4),(0,15),(1,12),(1,7),(0,6),(1,5),(1,10),(0,13),(0,8),(1,9),
(0,10),(1,13),(1,2),(0,1),(1,0),(1,11),(1,6),(0,9),(0,14),(0,3),(1,4),(0,7),(0,12)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(0,2),(1,6),(0,7)}*)

H_0:(0,6),(1,0),(1,2),(1,9),(1,11),(1,13),(0,3),(0,1),(0,10),(1,4),(1,6),(1,8),(0,2),(0,0),(1,10),(0,4),(1,14),(1,12),(1,3),
(0,9),(1,15),(0,5),(0,12),(0,14),(0,7),(1,1),(0,11),(0,13),(1,7),(1,5),(0,15),(0,8)

H_1:(1,9),(1,7),(1,0),(0,10),(0,8),(1,14),(1,5),(0,11),(0,2),(0,4),(0,6),(1,12),(1,10),(1,1),(1,3),(0,13),(0,15),(0,1),(1,11),
(0,5),(0,14),(1,8),(1,15),(1,6),(0,0),(0,9),(0,7),(1,13),(1,4),(1,2),(0,12),(0,3)

H_2:(1,12),(0,2),(0,9),(0,11),(0,4),(0,13),(0,6),(0,15),(1,9),(1,0),(1,14),(1,7),(0,1),(0,8),(1,2),(1,11),(1,4),(0,14),(0,0), (0,7),(0,5),(0,3),(0,10),(0,12),(1,6),(1,13),(1,15),(1,1),(1,8),(1,10),(1,3),(1,5)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(0,1),(1,5),(0,6)}*)

H_0:(1,2),(1,1),(1,7),(0,12),(0,11),(1,0),(1,6),(1,5),(0,10),(0,4),(0,5),(0,6),(0,0),(0,1),(0,2),(0,8),(0,9),(0,3),(0,13),
(1,8),(1,9),(1,15),(1,14),(1,13),(1,3),(0,14),(0,15),(1,4),(1,10),(1,11),(1,12),(0,7)

H_1:(0,3),(1,8),(1,2),(1,12),(0,1),(0,7),(0,8),(0,14),(0,13),(0,12),(1,1),(0,6),(1,11),(1,5),(0,0),(0,10),(1,15),(0,4),(1,9),
(1,3),(1,4),(1,14),(0,9),(0,15),(1,10),(1,0),(0,5),(0,11),(1,6),(1,7),(1,13),(0,2)

H_2:(1,11),(0,0),(0,15),(0,5),(1,10),(1,9),(0,14),(0,4),(0,3),(1,14),(1,8),(1,7),(0,2),(0,12),(0,6),(0,7),(0,13),(1,2),(1,3),
(0,8),(1,13),(1,12),(1,6),(0,1),(0,11),(0,10),(0,9),(1,4),(1,5),(1,15),(1,0),(1,1)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,3),(1,4),(0,5)}*)

H_0:(0,12),(1,0),(0,3),(0,14),(0,9),(1,5),(1,10),(0,6),(1,2),(0,5),(0,0),(1,13),(0,1),(1,4),(1,9),(0,13),(1,1),(1,6),(1,11),
(0,8),(1,12),(0,15),(1,3),(1,8),(0,4),(1,7),(0,10),(1,14),(0,11),(1,15),(0,2),(0,7)

H_1:(0,0),(1,3),(1,14),(0,1),(1,5),(0,2),(0,13),(1,0),(1,11),(0,14),(1,10),(0,7),(1,4),(0,8),(0,3),(1,15),(0,12),(1,8),(0,5),
(1,9),(0,6),(0,11),(1,7),(1,2),(1,13),(0,9),(1,6),(0,10),(0,15),(0,4),(1,1),(1,12)

H_2:(0,7),(1,3),(0,6),(0,1),(0,12),(1,9),(1,14),(0,2),(1,6),(0,3),(1,7),(1,12),(0,9),(0,4),(1,0),(1,5),(0,8),(0,13),(1,10),
(1,15),(1,4),(0,0),(0,11),(1,8),(1,13),(0,10),(0,5),(1,1),(0,14),(1,2),(0,15),(1,11)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(0,1),(1,5),(1,6)}*)

H_0:(0,7),(0,8),(1,3),(0,13),(0,12),(1,2),(1,1),(0,11),(1,0),(0,6),(0,5),(0,4),(1,10),(1,11),(0,1),(1,6),(1,5),(0,0),(0,15),
(1,4),(0,9),(0,10),(1,15),(1,14),(0,3),(1,9),(0,14),(1,8),(1,7),(0,2),(1,12),(1,13)

H_1:(0,10),(1,4),(0,14),(0,13),(1,2),(1,3),(0,9),(1,14),(0,8),(1,13),(0,3),(0,4),(1,9),(1,8),(0,2),(0,1),(0,0),(1,10),(0,15),
(1,5),(0,11),(1,6),(1,7),(0,12),(1,1),(0,6),(0,7),(1,12),(1,11),(0,5),(1,15),(1,0)

H_2:(0,8),(1,2),(0,7),(1,1),(1,0),(0,5),(1,10),(1,9),(0,15),(0,14),(1,3),(1,4),(1,5),(0,10),(0,11),(0,12),(1,6),(0,0),(1,11),
(0,6),(1,12),(0,1),(1,7),(0,13),(1,8),(0,3),(0,2),(1,13),(1,14),(0,4),(1,15),(0,9)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,4),(1,5),(1,7)}*)

H_0:(1,1),(0,8),(1,12),(0,3),(1,7),(0,11),(1,0),(0,4),(1,11),(0,0),(1,4),(0,13),(1,2),(0,14),(1,5),(0,1),(1,6),(0,2),(1,9),
(0,5),(1,10),(0,15),(1,8),(0,12),(1,3),(0,7),(1,14),(0,9),(1,13),(0,6),(1,15),(0,10)

H_1:(1,1),(0,12),(1,0),(0,7),(1,2),(0,6),(1,10),(0,14),(1,3),(0,10),(1,5),(0,9),(1,4),(0,15),(1,11),(0,2),(1,7),(0,0),(1,9),
(0,4),(1,8),(0,3),(1,14),(0,5),(1,12),(0,1),(1,13),(0,8),(1,15),(0,11),(1,6),(0,13)

H_2:(1,0),(0,9),(1,2),(0,11),(1,4),(0,8),(1,3),(0,15),(1,6),(0,10),(1,14),(0,2),(1,13),(0,4),(1,15),(0,3),(1,10),(0,1),(1,8),
(0,13),(1,9),(0,14),(1,7),(0,12),(1,5),(0,0),(1,12),(0,7),(1,11),(0,6),(1,1),(0,5)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,3),(1,4),(1,5)}*)

H_0:(1,6),(0,1),(1,4),(0,8),(1,11),(0,0),(1,3),(0,6),(1,9),(0,4),(1,1),(0,14),(1,2),(0,15),(1,12),(0,7),(1,10),(0,5),(1,0),
(0,13),(1,8),(0,11),(1,7),(0,12),(1,15),(0,10),(1,5),(0,2),(1,13),(0,9),(1,14),(0,3)

H_1:(0,9),(1,5),(0,1),(1,12),(0,0),(1,4),(0,15),(1,3),(0,8),(1,13),(0,10),(1,7),(0,4),(1,0),(0,12),(1,9),(0,14),(1,10),(0,6),
(1,11),(0,7),(1,2),(0,13),(1,1),(0,5),(1,8),(0,3),(1,15),(0,2),(1,14),(0,11),(1,6)

H_2:(1,0),(0,3),(1,7),(0,2),(1,6),(0,10),(1,14),(0,1),(1,13),(0,0),(1,5),(0,8),(1,12),(0,9),(1,4),(0,7),(1,3),(0,14),(1,11),
(0,15),(1,10),(0,13),(1,9),(0,5),(1,2),(0,6),(1,1),(0,12),(1,8),(0,4),(1,15),(0,11)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,6),(0,6),(1,1)}*)

H_0:(1,9),(0,3),(1,2),(0,12),(0,2),(1,3),(0,4),(0,14),(1,13),(0,7),(1,1),(1,7),(0,13),(1,12),(1,6),(1,0),(0,6),(0,0),(0,10),
(1,4),(1,10),(0,11),(0,1),(1,11),(0,5),(0,15),(1,5),(1,15),(0,9),(1,8),(1,14),(0,8)

H_1:(0,7),(1,6),(0,0),(1,1),(0,11),(0,5),(1,4),(0,3),(0,9),(0,15),(1,14),(0,4),(1,10),(1,0),(0,1),(1,7),(0,8),(0,2),(1,12),
(1,2),(1,8),(0,14),(1,15),(1,9),(0,10),(1,11),(1,5),(0,6),(0,12),(1,13),(1,3),(0,13)

H_2:(1,1),(0,2),(1,8),(0,7),(0,1),(1,2),(0,8),(0,14),(1,4),(1,14),(0,13),(0,3),(1,13),(1,7),(0,6),(1,12),(0,11),(1,5),(0,4),
(0,10),(1,0),(0,15),(1,9),(1,3),(0,9),(1,10),(0,0),(1,15),(0,5),(1,6),(0,12),(1,11)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(1,3),(0,1),(0,4)}*)

H_0:(1,5),(0,8),(0,7),(1,4),(0,1),(0,5),(1,8),(1,7),(1,3),(0,0),(0,4),(0,3),(0,15),(1,12),(1,0),(1,15),(0,2),(0,6),(1,9),
(1,10),(1,11),(0,14),(1,1),(1,2),(1,14),(1,13),(0,10),(0,11),(0,12),(0,13),(0,9),(1,6)

H_1:(0,2),(1,5),(1,4),(1,0),(1,1),(0,4),(0,8),(0,9),(0,5),(1,2),(1,6),(1,10),(0,13),(0,1),(1,14),(0,11),(0,15),(0,14),(0,10),
(1,7),(1,11),(1,12),(1,8),(1,9),(1,13),(0,0),(0,12),(1,15),(1,3),(0,6),(0,7),(0,3)

H_2:(1,7),(0,4),(0,5),(0,6),(0,10),(0,9),(1,12),(1,13),(1,1),(1,5),(1,9),(0,12),(0,8),(1,11),(1,15),(1,14),(1,10),(0,7),(0,11),
(1,8),(1,4),(1,3),(1,2),(0,15),(0,0),(0,1),(0,2),(0,14),(0,13),(1,0),(0,3),(1,6)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_{16}$;{(0,1),(1,4),(0,2)}*)

H_0:(1,1),(1,15),(1,0),(1,2),(1,3),(0,7),(0,6),(0,4),(0,2),(0,0),(1,12),(1,13),(1,14),(0,10),(0,9),(0,8),(1,4),(1,5),(1,7),
(1,6),(1,8),(0,12),(0,11),(0,13),(0,14),(1,10),(1,9),(1,11),(0,15),(0,1),(0,3),(0,5)

H_1:(0,1),(1,5),(1,3),(1,1),(1,2),(1,4),(0,0),(0,14),(0,15),(0,13),(1,9),(0,5),(0,4),(1,8),(1,7),(0,11),(0,9),(0,7),(0,8),(0,6),(1,10),(1,11),(1,12),(1,14),(1,0),(0,12),(0,10),(1,6),(0,2),(0,3),(1,15),(1,13)

H_2:(0,4),(1,0),(1,1),(0,13),(0,12),(0,14),(1,2),(0,6),(0,5),(0,7),(1,11),(1,13),(0,9),(1,5),(1,6),(1,4),(1,3),(0,15),(0,0),(0,1),(0,2),(1,14),(1,15),(0,11),(0,10),(0,8),(1,12),(1,10),(1,8),(1,9),(1,7),(0,3)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;$\{(3,1),(3,2),(2,3)\}^\star$)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;$\{(3,1),(2,1),(2,2)\}^\star$)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;$\{(3,1),(1,1),(1,2)\}^\star$)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;$\{(1,4),(0,2),(2,3)\}^\star$)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;$\{(0,1),(2,2),(1,0)\}^\star$)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;$\{(3,1),(1,0),(2,3)\}^\star$)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;$\{(0,1),(1,4),(1,2)\}^\star$)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5), (3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;{(1,3),(3,3),(2,1)}*)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),
(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),
(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),
(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;{(1,4),(3,3),(1,0)}*)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),
(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),
(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),
3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;{(3,2),(2,1),(2,3)}*)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),
(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),
(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),
(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_4 \oplus \mathbb{Z}_8$;{(0,1),(1,4),(2,1)}*)

H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),
(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)

H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),
(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),
(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$;{(1,0,3),(0,0,3),(0,1,2)}*)

H_0:(0,1,0),(0,0,6),(0,0,3),(0,0,0),(0,0,5),(0,0,2),(0,0,7),(0,1,1),(0,1,6),(0,1,3),(1,1,6),(1,0,4),(1,0,7),(0,0,4),(0,0,1),
(0,1,7),(1,1,2),(0,1,5),(1,1,0),(1,1,3),(1,0,1),(1,0,6),(1,1,4),(1,0,2),(1,0,5),(1,0,0),(1,0,3),(1,1,1),(0,1,4),
(1,1,7),(0,1,2),(1,1,5)

H_1:(0,1,6),(1,1,3),(0,1,0),(0,0,2),(0,1,4),(0,0,6),(0,0,1),(0,1,3),(0,0,5),(1,0,0),(0,0,3),(0,1,5),(0,0,7),(1,0,4),(1,0,1),
(0,0,4),(0,1,2),(0,1,7),(1,1,4),(0,1,1),(1,1,6),(1,1,1),(1,0,7),(1,0,2),(1,1,0),(1,0,6),(1,0,3),(1,1,5),(1,1,2),
(1,1,7),(1,0,5),(0,0,0)

H_2:(1,1,2),(1,0,0),(1,1,6),(1,1,3),(1,0,5),(0,0,2),(1,0,7),(1,1,5),(1,1,0),(0,1,3),(0,1,0),(0,1,5),(0,1,2),(0,0,0),(1,0,3),
(0,0,6),(1,0,1),(1,1,7),(1,1,4),(1,1,1),(0,1,6),(0,0,4),(0,0,7),(1,0,2),(0,0,5),(0,1,7),(0,1,4),(0,1,1),(0,0,3),
(1,0,6),(0,0,1),(1,0,4)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$;{(0,0,1),(1,0,3),(1,1,1)}*)

H_0:(1,1,7),(0,0,6),(0,0,7),(0,0,0),(1,0,3),(0,1,2),(0,1,3),(1,0,2),(1,0,1),(0,1,0),(0,1,7),(1,0,0),(0,1,1),(1,1,4),(0,0,5),
(1,1,6),(1,1,5),(0,0,4),(0,0,3),(0,0,2),(0,0,1),(1,1,0),(0,1,5),(1,0,4),(1,0,5),(1,0,6),(1,0,7),(0,1,6),(1,1,3),
(1,1,2),(1,1,1),(0,1,4)

H_1:(0,0,2),(1,0,7),(0,1,0),(1,1,3),(0,0,4),(0,0,5),(0,0,6),(1,0,3),(1,0,2),(0,1,1),(1,1,6),(1,1,7),(1,1,0),(0,0,7),(1,0,4),
(0,1,3),(0,1,4),(0,1,5),(1,0,6),(0,0,1),(1,1,2),(0,0,3),(1,0,0),(1,0,1),(0,1,2),(1,1,5),(1,1,4),(0,1,7),(0,1,6),
(1,0,5),(0,0,0),(1,1,1)

H_2:(1,0,3),(0,1,4),(1,0,5),(0,0,2),(1,1,3),(1,1,4),(0,0,3),(1,0,6),(0,1,7),(1,1,2),(0,1,5),(0,1,6),(1,1,1),(1,1,0),(0,1,3),
(1,1,6),(0,0,7),(1,0,2),(0,0,5),(1,0,0),(1,0,7),(0,0,4),(1,0,1),(0,0,6),(1,1,5),(0,1,0),(0,1,1),(0,1,2),(1,1,7),
(0,0,0),(0,0,1),(1,0,4)

Cay($\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$;{(1,0,2),(0,1,3),(1,1,2)}*)

H_0:(1,0,2),(0,0,0),(0,1,3),(1,0,1),(0,0,3),(0,1,6),(1,0,0),(0,0,6),(1,0,4),(0,0,2),(0,1,7),(0,0,4),(1,1,2),(1,0,5),(1,1,0),
(1,0,3),(0,1,5),(1,0,7),(1,1,4),(0,1,2),(0,0,5),(1,1,7),(0,1,1),(1,1,3),(0,0,1),(0,1,4),(1,1,6),(0,1,0),(1,0,6),
(1,1,1),(0,0,7),(1,1,5)

Computational Appendix

H_1:(0,1,0),(0,0,3),(1,1,1),(0,1,3),(1,1,5),(0,1,7),(1,0,5),(0,0,7),(0,1,2),(1,0,4),(0,1,6),(1,1,0),(0,0,6),(0,1,1),(1,0,3),(0,0,1),(1,0,7),(0,0,5),(1,1,3),(1,0,0),(0,0,2),(1,1,4),(1,0,1),(1,1,6),(0,0,4),(1,0,6),(0,1,4),(1,0,2),(1,1,7),(0,1,5),(0,0,0),(1,1,2)

H_2:(1,0,3),(0,0,5),(0,1,0),(1,0,2),(0,0,4),(0,1,1),(1,0,7),(1,1,2),(0,1,4),(0,0,7),(1,0,1),(0,1,7),(1,1,1),(1,0,4),(1,1,7),(0,0,1),(0,1,6),(1,1,4),(0,0,6),(0,1,3),(1,0,5),(0,0,3),(1,1,5),(1,0,0),(0,1,2),(1,1,0),(0,0,2),(0,1,5),(1,1,3),(1,0,6),(0,0,0),(1,1,6)

References

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