ABSTRACT

Title of dissertation: ANALYTIC APPROACHES IN LAGRANGIAN GEOMETRY

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The focus of this thesis is two equations that arise in special Lagrangian geometry: the degenerate special Lagrangian equation (DSL) and the Lagrangian mean curvature flow (LMCF). A significant part of this focus centers on Dirichlet duality, subequations, and viscosity solutions, the analytic framework which we use to formulate and study both equations.

Given a Calabi–Yau manifold (X, ω, J, Ω) and a model manifold M, one can construct a kind of moduli space of Lagrangians in X called the space of positive Lagrangians. A Lagrangian $L \subset X$ belongs to this infinite-dimensional space if L is diffeomorphic to M and $\mathrm{Re}(\Omega|_L) > 0$. A Hamiltonian deformation class of the space of positive Lagrangians admits an L^2-type Riemannian metric which allows one to study this space from a geometric point of view. Geodesics in this space play a crucial role in a program initiated by Solomon [37, 36] to understand the existence and uniqueness of special Lagrangian submanifolds in Calabi–Yau manifolds. They also play a key role in a new approach to the Arnold conjecture put forth by Rubinstein–Solomon and in the development of a pluripotential theory for

Lagrangian graphs [29, 8]. The DSL arises as the geodesic equation in the space of positive graph Lagrangians when $X = \mathbb{C}^n$ and ω and Ω are associated to the Euclidean structure [29].

Building on the results of Rubinstein–Solomon [29], we show that the DSL induces a global equation on every Riemannian manifold, and that for certain associated geometries this equation governs, as it does in the Euclidean setting, geodesics in the space of positive Lagrangians. For example, geodesics in the space of positive Lagrangian sections of a smooth semi-flat Calabi–Yau torus fibration are governed by the Riemannian DSL on the product of the base manifold and an interval.

The geodesic endpoint problem in this setting thus corresponds to solving the Dirichlet problem for the DSL. However, the DSL is a degenerate-elliptic, fully nonlinear, second-order equation, and so the standard elliptic theory does not furnish solutions. Moreover, for Lagrangians with boundary the natural domains on which one would like to solve the Dirichlet problem are cylindrical and thus not smooth. These issues are resolved by Rubinstein–Solomon in the Euclidean setting by adapting the Dirichlet duality framework of Harvey–Lawson to domains with corners [29]. We further develop these analytic techniques, specifically modifications of the Dirichlet duality theory in the Riemannian setting to obtain continuous solutions to the Dirichlet problem for the Riemannian DSL and hence, in certain settings, continuous geodesics in the space of positive Lagrangians.

The uniqueness of solutions to the Dirichlet problem in the Euclidean formulation of Dirichlet duality theory relies on an important convex-analytic theorem of Slodkowski [34]. Motivated by the significance of this result and the technical, geometric nature of its proof, we provide a detailed exposition of the proof. We then study some of the quantities involved using the Legendre transform, offering a dual perspective on this theorem.

Given a Lagrangian submanifold in a Calabi–Yau, a fundamental and still open question is whether or not there is a special Lagrangian representative in its homology or Hamiltonian isotopy class. A natural approach to this problem is the Lagrangian mean curvature flow, which preserves not only the Lagrangian condition but also the homology and isotopy class. Assuming the flow exists for all time and converges, it will converge to a minimal (i.e., zero mean curvature) Lagrangian. In the Calabi–Yau setting these are precisely the special Lagrangian submanifolds. A major conjecture in this area is the Thomas–Yau conjecture [41], which posits certain stability conditions on the initial Lagrangian under which the LMCF will exist for all time and converge to the unique special Lagrangian in that isotopy class. Thomas–Yau stated a variant of their conjecture for a related, more tractable flow, called the almost Lagrangian mean curvature flow (ALMCF). In the setting of highly symmetric Lagrangian spheres in Milnor fibers, and under some additional technical assumptions, they made significant progress towards a proof of this variant of the conjecture [41]. We study the flow of 2-spheres from a slightly different perspective and provide a relatively short proof of the longtime existence of viscosity solutions under certain stability conditions, and their convergence to a special Lagrangian sphere.

ANALYTIC APPROACHES IN LAGRANGIAN GEOMETRY

by

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Dedication

To Olya, my family, and my friends.

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Chapter 1: Introduction

1.1 Background and motivation

This thesis is concerned with two partial differential equations in Lagrangian geometry: the degenerate special Lagrangian equation (DSL) and the Lagrangian mean curvature flow (LMCF), as well as the analytic methods used to study them. The motivation for studying these equations is a better understanding of Calabi–Yau manifolds through the existence and uniqueness of a particular class of submanifolds, called special Lagrangians.

1.1.1 Calabi–Yau manifolds and special Lagrangian submanifolds

Let (X, ω, J, Ω) be a *Calabi–Yau manifold*. That is, let (X, ω, J) be Kähler, with $g := \omega(\cdot, J\cdot)$ the Riemannian metric, and let Ω be a nowhere-vanishing holomorphic $(n, 0)$-form, satisfying the following compatibility condition:

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega}. \tag{1.1}$$

Condition (1.1) implies that the metric g is Ricci-flat. If this compatibility condition does not hold, (X, ω, J, Ω) is referred to as *almost Calabi–Yau*.

Although this will be the definition we use throughout, there are several other commonly used definitions of Calabi–Yau manifolds. For instance, in terms of Riemannian holonomy this is a Riemannian manifold (X, g) such that $\mathrm{hol}(g) \subset SU(n)$, where $SU(n)$ denotes the special unitary group of dimension n. In more complex-algebraic terms, Calabi–Yau implies vanishing first Chern class, $c_1(X) = 0$.

The existence of Calabi–Yau manifolds was conjectured by Calabi [5, 6] in 1954:

Conjecture 1.1.1 (Calabi Conjecture)**.** *Let (X, J) be a compact, complex manifold, and g a Kähler metric on X with Kähler form ω. Suppose that ρ' is a real, closed $(1,1)$-form on X with $[\rho'] = c_1(X)$. Then there exists a unique Kähler metric g' on X with Kähler form ω', such that $[\omega'] = [\omega] \in H^2(X, \mathbb{R})$, and the Ricci form of g' is ρ'.*

The Calabi Conjecture was proved by Yau [42], with previous partial results by Aubin [2, 3, 4]. The conjecture can be reformulated as a nonlinear, elliptic, second-order PDE of Monge–Ampère type, where the existence of the unique metric g' corresponds to a real-valued function on X being a unique smooth solution to this PDE. Yau's proof showed that this PDE does in fact admit unique, smooth solutions. Before the proof of the Calabi Conjecture it was still unclear whether or not there even existed compact Ricci-flat Riemannian manifolds that are not flat [13].

Shortly after their existence was confirmed, Calabi–Yau manifolds began to play a fundamental role in formulations of supersymmetric string theory [21], a physical theory seeking to quantize gravity. In this framework, particles are modelled as 1-dimensional objects ('strings') propogating in a 10-dimensional background space-time M, such that, locally, $M = \mathbb{R}^4 \times X$, where $\mathbb{R}^4$ is Minkowski space and X a compact, Calabi–Yau 3-fold ($\dim_{\mathbb{C}} X = 3$).

An important class of submanifolds that arise as a type of boundary condition for strings in X are *calibrated submanifolds*, introduced by Harvey–Lawson [17]. Given a Riemannian manifold (X, g) of dimension n, a calibration on X is a closed differential p-form ϕ (for some $0 \le p \le n$) such that for any $x \in X$ and any oriented p-dimensional subspace $\xi \subset T_x X$,

$$\phi|_\xi = \lambda \mathrm{vol}_\xi, \text{ with } \lambda \le 1,$$

where vol_ξ is induced from g on ξ, in other words take a g-orthonormal basis for ξ and use g to convert this to 1-forms and then wedge these 1-forms to obtain vol_ξ. We say that a p-dimensional submanifold $Y \subset X$ is calibrated with respect to ϕ if

$$\phi|_{T_y Y} = \mathrm{vol}_{T_y Y}, \text{ for all } y \in Y.$$

One of the significant features of calibrated submanifolds is that, by Stokes' theorem, they are volume-minimizing in their homology class [17].

Given a Calabi–Yau (X, J, ω, Ω), the real part of any rotation of the holomorphic $(n, 0)$-form is a calibration, i.e.,

$$\mathrm{Re}\, e^{-\sqrt{-1}c}\, \Omega, \quad c \in (-\pi, \pi],$$

and the corresponding calibrated submanifolds are called *special Lagrangian submanifolds*, a notion also introduced by Harvey–Lawson [17]. Equivalently, a Lagrangian $L \subset X$ (i.e., $\dim_{\mathbb{R}} L = n$ and $\omega|_L = 0$) is called special Lagragian of phase

c if there exists a constant $c \in (-\pi, \pi]$ such that

$$\mathrm{Im}\; e^{-\sqrt{-1}c}\,\Omega|_L = 0.$$

Special Lagrangians are also thought to play a fundamental role in Mirror symmetry—a relationship between pairs of Calabi–Yau manifolds proposed by string theorists. Although not fully understood, mirror symmetry has become a fundamental tool for doing calculations and has introduced deep connections between previously unrelated areas of mathematics [21]. It is conjectured by Strominger–Yau–Zaslow [40] that Calabi–Yau manifolds are actually "built" from special Lagrangian submanifolds and that mirror symmetry might be understood completely in terms of this Calabi–Yau sub-structure.

Conjecture 1.1.2 (SYZ Conjecture [40])**.**

(i) Any Calabi–Yau X has a structure of a (possibly singular) special Lagrangian torus fibration $\pi : X \to M$, and its mirror W is obtained as the dual special Lagrangian torus fibration $\pi : W \to M$.

(ii) There exists a fiberwise Fourier–Mukai transform which maps Lagrangian submanifolds of X to coherent sheaves on W.

1.1.2 The equations

Special Lagrangians, as both an important type of calibrated submanifold and the centerpiece of the SYZ conjecture, are thus of significant importance in understanding Calabi–Yau geometry. However, even the most basic questions one can ask about their existence and uniqueness are still largely open. For example, in a given homology or Hamiltonian isotopy class does there exists a unique special Lagrangian?

The DSL and LMCF can both be viewed as analytic approaches to this question, in the sense that they are a PDE formulation of a geometric process or phenomenon that can be used to find special Lagrangian submanifolds in a given class. Roughly speaking, the DSL paves the way to a new variational approach to finding special Lagrangians; while the LMCF represents a more classical gradient flow approach.

More specifically, in certain settings the DSL is the geodesic equation in a Hamiltonian deformation class of the space of positive Lagrangians of a fixed Calabi–Yau [29]. There exists a functional on this infinite-dimensional space of Lagrangians that is convex along geodesics and whose ciritical points are special Lagrangian submanifolds [37, 36]. Thus, special Lagrangians correspond to the minimizers of this functional. The LMCF is the flow of a Lagrangian submanifold so that the normal component of its velocity is its mean curvature vector, or, equivalently, the gradient flow of the area functional. The critical points of the area functional are submanifolds with zero mean, i.e., minimal submanifolds. In a Calabi–Yau manifold, the mean curvature flow not only preserves the Lagrangian condition [35], but is also a Hamiltonian deformation. Minimal (connected) Lagrangians are special Lagrangian, and so it is natural to consider the mean curvature flow to produce special Lagrangians in a given class. The major open conjecture in this area is the Thomas–Yau conjecture, stating roughly: Given a compact, zero Maslov class, Lagrangian satisfying certain stability conditions, the mean curvature flow exists for all time and converges smoothly to a special Lagrangian submanifold in the same Hamiltonian isotopy class.

1.1.3 Dirichlet duality

Our work on the DSL is formulated in terms of Harvey–Lawson's Dirichlet duality theory [18, 19], which we briefly describe now. A more thorough review can be found in Section 3.1.

Let Ω be a bounded domain in $\mathbb{R}^n$ and consider an equation of the form

$$F(D^2u) = 0 \text{ on } \Omega.$$

To any such equation F, Dirichlet duality associates a *subequation* $\mathcal{F}$. This is a closed proper subset of $\mathrm{Sym}^2(\mathbb{R}^n)$ that is invariant under translation by positive matrices. Roughly speaking,

$$F(D^2u) = 0 \longrightarrow \mathcal{F} := \{A \in \mathrm{Sym}^2(\mathbb{R}^n) \ : \ F(A) \geq 0\}.$$

In most cases, the set $\mathcal{F}$ will usually be a more regular proper subset of this set. Regardless, solutions $u \in C^2(\Omega)$ of F must satisfy $D^2_x u \in \partial\mathcal{F}$.

This gives rise to a natural notion of a subsolution to F. A C^2 function u is *$\mathcal{F}$–subharmonic* on Ω if

$$D^2_x u \in \mathcal{F}, \quad \forall x \in \Omega. \tag{1.2}$$

This definition extends to upper semi-continuous functions in a viscosity-like way via C^2 test functions, and these $\mathcal{F}$-subharmonic functions comprise the subsolutions.

The class of $\mathcal{F}$-subharmonic functions remarkably share most of the important properties that the classical subharmonic and convex functions satisfy. For example, closure under decreasing limits and taking maxima, decreasing limits, uniform limits, and upper envelopes. See Theorem 3.1.2.

For each subequation $\mathcal{F}$, there is an associated *dual* subequation $\widetilde{\mathcal{F}}$, defined as $\widetilde{\mathcal{F}} = -(\sim \text{Int}\mathcal{F})$. The importance of the dual subequation lies in the fact that

$$-\widetilde{\mathcal{F}} \cap \mathcal{F} = \partial \mathcal{F} \subset \{F = 0\}. \tag{1.3}$$

This immediately allows one to define a notion of weak solution. Let u be a C^2 function such that u is $\mathcal{F}$-subharmonic and $-u$ is $\widetilde{\mathcal{F}}$-subharmonic on Ω. Then, by (1.2) and (1.3), u satisfies

$$F(D^2u(x)) = 0 \quad \forall x \in \Omega.$$

A function u is said to be $\mathcal{F}$-*harmonic* if u is $\mathcal{F}$-subharmonic and $-u$ is $\widetilde{\mathcal{F}}$-subharmonic. These functions will comprise our weak solutions. Note that since both u and $-u$ are upper semi-continuous, u is automatically continuous.

Given a domain Ω and an equation F, the existence of continuous solutions to Dirichlet problem for F requires that $\partial\Omega$ is $\mathcal{F}$-*convex*, where $\mathcal{F}$ is a subequation associated to F. This subequation-specific convexity generalizes conventional convexity. The Perron method is used to construct solutions, and this convexity implies the existence of barrier functions. This explicit relationship between the equation and the necessary geometry of the boundary is one of the most attractive aspects of Dirichlet duality.

1.2 Summary of results

1.2.1 The degenerate special Lagrangian equation

Most of the results mentioned in this section were published in [10].

We begin with the observation that the DSL carries over to a global equation on every Riemannian manifold. For precise statements see Section 2.3 and Section 3.1.

Proposition 1.2.1. *Given any Riemannian manifold M, the degenerate special Lagrangian equation carries over (in the sense of Harvey–Lawson) to a global equation on $\mathbb{R}\times M$, locally modelled on the DSL. We refer to this equation as the Riemannian DSL on $\mathbb{R}\times M$.*

We would like to understand the geometric significance of the Riemannian DSL. In particular, does it also govern geodesics in the space of positive Lagrangians for certain geometries associated to M? In most cases, it will not. However, there are interesting settings where it does; for example, when the ambient Calabi–Yau manifold is the cotangent bundle T^*M of certain paralellizable manifolds or when it is a semi-flat Lagrangian torus fibration over M. See Section 2.4 for details and definitions.

Theorem 1.2.2. *Let M be integrably parallelizable. Then T^*M admits a Calabi–Yau structure, and the Riemannian DSL on $[0,1]\times M$ governs geodesics in the space of positive graph Lagrangians in T^*M.*

Theorem 1.2.3. *Let X be a smooth Calabi–Yau torus fibration over M. Then the Riemannian DSL on $[0,1] \times M$ governs geodesics in the space of positive Lagrangian sections in X.*

This motivates the third goal of this thesis which is to solve the Dirchlet problem for the Riemannian DSL, as this corresponds to the endpoint problem for geodesics. More specifically, we aim to solve the Dirichlet problem on domains of the form

$$\mathcal{D} = [0,1] \times D \subset \mathbb{R} \times M,$$

where $D \subset M$ is a bounded domain. This is accomplished by following the approach of Rubinstein–Solomon. In particular, we extend the Dirichlet duality theory of Harvey–Lawson to include certain domains with corners in Riemannian manifolds, such as $\mathcal{D}$ when $D \subset M$ has boundary. This extension is contained in Theorem 3.2.5.

Under appropriate boundary conditions, Theorem 3.2.5 provides continuous solutions to the Dirichlet problem for the Riemannian DSL on $\mathcal{D}$ and thus continuous geodesics in the space of positive Lagrangians. A special case of this result is the following theorem. By strictly convex we mean that all of the eigenvalues of the second fundamental form $\mathrm{II}_{\partial D}$ are strictly positive, and by admissible we mean that this local frame for TM is part of a family of frames whose transition maps are $O(n)$-valued (see Appendix 3.1).

Theorem 1.2.4. *Let (M, g) be a complete simply-connected Riemannian manifold with non-positive sectional curvature, and let $D \subset M$ be a bounded strictly convex* *domain. For $i = 0, 1$, let $\phi_i \in C^2(D)$ satisfy*

$$\operatorname{tr}\tan^{-1}(\text{Hess } \phi_i(e,e)) \in (c - \pi/2, c + \pi/2), \tag{1.4}$$

where $e = (e_1, \cdots, e_n)$ is any admissible local frame for the tangent bundle. Then there exists a unique solution $u \in C^0(\overline{\mathcal{D}})$ to the Dirichlet problem for the Riemannian DSL of phase θ (where $c = \theta \bmod 2\pi$) with $u|_{\{i\}\times D} = \phi_i$ and $u|_{[0,1]\times\partial D}$ affine in t.

In certain settings (see Section 2.4 and [29, Section 1]), condition (1.4) is equivalent to the condition that the graph of $d\phi_i$ in the cotangent bundle is a positive Lagrangian.

1.2.2 Dirichlet duality

In order to solve the Dirichlet problem on the natural domains that arise for the Riemannian DSL in Chapter 2, we need extend Dirichlet duality to domains with corners. To do this we develop the approach of [29] in the Riemannian setting, so that, assuming a weakened form of boundary convexity appropriate for domains with corners in Riemannian manifolds, the Dirichlet problem admits unique continuous solutions. This result is contained in Theorem 3.2.5. The results of this chapter were published in [10].

1.2.3 Theorem of Slodkowski

As mentioned earlier, the uniqueness of solutions to the Dirichlet problem in the Euclidean formulation of Dirichlet duality theory relies on an important convex-analytic theorem of Slodkowski. Motivated by the significance of this result and the technical, geometric nature of its proof, we provide a detailed exposition of the proof. We then study some of the quantities involved using the Legendre transform, offering a dual perspective on this theorem. The main result here is Theorem 4.1.9. The results of this chapter were published in [9]

1.2.4 Lagrangian mean curvature flow

Given a Lagrangian submanifold in a Calabi–Yau, a fundamental and still open question is whether or not there is a special Lagrangian representative in its homology or Hamiltonian isotopy class. A natural approach to this problem is the Lagrangian mean curvature flow, which preserves not only the Lagrangian condition but also the homology and isotopy class. Assuming the flow exists for all time and converges, it will converge to a minimal (i.e., zero mean curvature) Lagrangian. In the Calabi–Yau setting these are precisely the special Lagrangian submanifolds. A major conjecture in this area is the Thomas–Yau conjecture [41], which posits certain stability conditions on the initial Lagrangian under which the LMCF will exist for all time and converge to the unique special Lagrangian in that isotopy class. Thomas–Yau stated a variant of their conjecture for a related, more tractable flow, called the almost Lagrangian mean curvature flow (ALMCF). In the setting of highly symmetric Lagrangian spheres in Milnor fibers, and under some additional technical assumptions, they made significant progress towards a proof of this variant of the conjecture [41]. Here, we give a detailed description of the geometry of these symmetric Lagrangian spheres and the various ways in which the flow can be formulated in this setting. We then follow this with an exposition of the proofs of the long-time existence and convergence of the flow. We also find that an adjustment needs to be made to a technical assumption used in their proof. We provide a modified version of their result in Theorem 5.5.11.

We then restrict to the two-dimensional setting and study the flow of 2-spheres from a different perspective and provide a relatively short proof of the longtime existence of viscosity solutions under certain stability conditions, and their C^0 convergence to a smooth special Lagrangian 2-sphere. The main result here is Theorem 5.7.1.

Chapter 2: The degenerate special Lagrangian equation

2.1 Introduction

Let $f \in C^2([0,1] \times \mathbb{R}^n)$ and $\theta \in (-\pi, \pi]$. Then $f(t,x)$ satisfies the *degenerate special Lagrangian equation of phase* θ if

$$\operatorname{Im}\left(e^{-\sqrt{-1}\theta}\det(I_n + \sqrt{-1}\nabla^2 f)\right) = 0 \quad \text{and} \quad \operatorname{Re}\left(e^{-\sqrt{-1}\theta}\det(I + \sqrt{-1}\nabla_x^2 f)\right) > 0. \tag{2.1}$$

Here I_n denotes the diagonal $(n+1)\times(n+1)$ matrix with diagonal entries $(0,1,\ldots,1)$. The degenerate special Lagrangian equation (DSL) was introduced by Rubinstein–Solomon [29] in connection to geodesics in the space of positive Lagrangians of a Calabi–Yau manifold. It is a fully nonlinear, degenerate elliptic equation.

When the featured Calabi–Yau is $\mathbb{C}^n$, the geodesic endpoint problem in the space of positive graph Lagrangians corresponds to solving the Dirichlet problem for the DSL. In particular, the conditions in (2.1) capture, respectively, the notions of geodesic and positivity in this setting. Under appropriate boundary conditions, unique continuous solutions to the Dirichlet problem for the DSL exist. This was accomplished in [29] by finding a natural notion of subsolution to the DSL and then adapting the Dirichlet duality framework of Harvey–Lawson [18] for degenerate elliptic equations in Euclidean space.

Harvey–Lawson [19] have also developed a Dirichlet duality theory for equations on Riemanian manifolds. The starting point for this framework is an equation F in Euclidean space and a Riemannian manifold M. Assuming the topology on M is sufficiently mild and the symmetry of F is sufficiently high, one can define a global equation on M that is locally modelled on F. Thus, from this point of view, it is natural to consider the equation induced by the DSL on Riemannian manifolds, and that is the purpose of this note.

Geodesics in the space of positive Lagrangians play a crucial role in a program initiated by Solomon [37, 36] (see also [38]) to understand the existence and uniqueness of special Lagrangian submanifolds in Calabi–Yau manifolds. They also play a key role in a new approach to the Arnold conjecture put forth by Rubinstein–Solomon [29, Section 2.3] and in the development of a pluripotential theory for Lagrangian graphs initiated in [29] (see also [8]).

2.1.1 Organization

In the next section, we prove Proposition 1.2.1, showing that the DSL induces (in the sense of Harvey–Lawson) a global equation on every Riemannian manifold. We then geometrically motivate our study of the Riemannian DSL in Section 3 by proving Theorem 1.2.2 and Theorem 1.2.3. In Section 5, we extend the Dirichlet duality theory in the Riemannian setting to include domains with corners, proving a generalization of Theorem 1.2.4. In Section 6, we use these results to obtain unique continuous solutions to the Dirichlet problem for the DSL on Riemannian manifolds, and hence continuous geodesics. Finally, for ease of reference, we include an appendix with a brief summary of Dirichlet duality theory.

2.2 Geometry of the space of Lagrangians

The following section is based on the work of Solomon [37, 36] and briefly recalls the terminology concerning the geometry of the space of positive Lagrangians.

Let L be an n-dimensional real manifold and (X, J, ω, Ω) an almost Calabi–Yau manifold of complex dimension n. That is, (X, J, ω) is a Kähler manifold and Ω is a nowhere vanishing holomorphic n-form. Define

$$\mathcal{L} = \{\Gamma \subset X : \Gamma \text{ is an oriented Lagrangian submanifold diffeomorphic to } L\}.$$

For $\theta \in (-\pi, \pi]$, the space of θ-positive Lagrangians is defined as

$$\mathcal{L}_\theta^+ = \{\Gamma \in \mathcal{L} \mid \text{Re } (e^{-\sqrt{-1}\theta}\Omega)|_\Gamma > 0\}. \tag{2.2}$$

Denote by $\mathcal{O}_\theta \subset \mathcal{L}_\theta^+$ a connected component of the intersection of $\mathcal{L}_\theta^+$ with an orbit $\text{Ham}(X, \omega)$ acting on $\mathcal{L}$, where $\text{Ham}(X, \omega)$ is the group of compactly supported Hamiltonion diffeomorphisms of X.

When L is compact the tangent space to $\mathcal{O}_\theta$ can be identified with the space of smooth functions satisfying a normalization condition

$$T_\Gamma\mathcal{O}_\theta \equiv \{h \in C^\infty(\Gamma) \mid \int_\Gamma h\text{Re } \Omega = 0\}, \tag{2.3}$$

and a weak Riemannian metric on $\mathcal{O}_\theta$ is defined by

$$(h, k)_\theta|_\Gamma := \int_\Gamma hk\text{Re } (e^{-\sqrt{-1}\theta}\Omega|_\Gamma), \text{ for } h, k \in T_\Gamma\mathcal{O}_\theta. \tag{2.4}$$

When L is non-compact the normalization condition in (2.3) can be dropped and the tangent space at Γ is isomorphic to the space of compactly supported functions on Γ.

More specifically, given a path $\Lambda : [0,1] \to \mathcal{O}_\theta$ and a family of diffeomorphisms $g_t : L \to \Lambda_t$, let $h_t : \Lambda_t \to \mathbb{R}$ be the unique function satisfying

$$g_t^* \iota_{\frac{dg_t}{dt}} \omega = d(h_t \circ g_t), \tag{2.5}$$

and the normalization condition in (2.3). Then the velocity vector to Λ is defined as $\frac{d\Lambda_t}{dt} \equiv h_t$.

Given a vector field $q_t \in T_{\Lambda_t}\mathcal{O}_\theta$ along Λ, the Levi–Civita coavariant derivative of q_t in the direction of $\frac{d\Lambda_t}{dt}$ is defined by

$$\frac{Dq_t}{dt} = \left(\frac{\partial}{\partial t}(q_t \circ g_t) + g_t^* dq_t(\zeta_t) \right) \circ g_t^{-1}, \tag{2.6}$$

where ζ_t is the unique vector field on L such that

$$\iota_{\zeta_t} g_t^* \mathrm{Re}\,(e^{-i\theta}\Omega) = -g_t^* \iota_{\frac{dg_t}{dt}} \mathrm{Re}\,(e^{-i\theta}\Omega), \tag{2.7}$$

viewing $g_t : L \to \Gamma_t \subset X$ as a map from L to X.

The geodesic equation is then found by taking $q_t = h_t = \frac{d\Lambda_t}{dt}$:

$$\frac{Dh_t}{dt} = \left(\frac{\partial}{\partial t}(h_t \circ g_t) + g_t^* dh_t(\zeta_t) \right) \circ g_t^{-1} = 0. \tag{2.8}$$

2.3 The Riemannian DSL subequation

When $X = \mathbb{C}^n \cong \mathbb{R}^n \oplus \sqrt{-1}\mathbb{R}^n$, with the standard Calabi–Yau structure,

$$\omega = \frac{\sqrt{-1}}{2} \sum_j dz_j \wedge d\bar{z}_j \quad \text{and} \quad \Omega = dz_1 \wedge \ldots dz_n,$$

and L is identified with $\mathbb{R}^n \times \{0\} \subset \mathbb{C}^n$, the analysis in Section 2.2 leads to the degenerate special Lagrangian equation.

Theorem 2.3.1. *[29, Proposition 2.3] Let $\theta \in (-\pi, \pi]$ and let $k_i \in C^2(\mathbb{R}^n)$, $i = 0, 1$ be such that graph$(dk_i) \subset \mathbb{C}^n$ are elements of $\mathcal{O}_\theta$. Let $k \in C^2([0,1] \times \mathbb{R}^n)$ be such that graph$(d_x k(t, \cdot)) \subset \mathbb{C}^n$ is an element of $\mathcal{O}_\theta$ for each $t \in [0,1]$. Then $t \rightarrow$graph$(d_x k(t, \cdot))$ is a geodesic in $(\mathcal{O}_\theta, (\cdot, \cdot))$ with endpoints graph(dk_i), $i = 0, 1$, if and only if k satisfies*

$$\operatorname{Im}\left(e^{-\sqrt{-1}\theta} \det(I_n + \sqrt{-1}\nabla^2 k)\right) = 0 \quad \textit{and} \quad \operatorname{Re}\left(e^{-\sqrt{-1}\theta} \det(I + \sqrt{-1}\nabla_x^2 k)\right) > 0, \tag{2.9}$$

and $k(0, \cdot) = k_0 + c$, $k(1, \cdot) = k_1 + c$, for a constant $c \in \mathbb{R}$.

2.3.1 The DSL subequation

In order to obtain a subequation for the DSL, Rubinstein–Solomon associate to each $u \in C^2(\mathcal{D})$, where $\mathcal{D} = (0,1) \times D$, the circle valued function

$$\Theta_u(t, x) = \Theta(\nabla^2 u(t, x)) = \arg\det(I_n + \sqrt{-1}\nabla^2 u(t, x)) \in S^1,$$

defined where $\det(I_n + \sqrt{-1}\nabla^2 u(t, x)) \neq 0$. This angle Θ is called the *space-time Lagrangian angle* by analogy with the Lagrangian angle of Harvey–Lawson [17]. Accordingly, if $u \in C^2(\mathcal{D})$ solves the DSL of phase θ, then $\Theta_u \equiv \theta$.

For a complex matrix B, let spec(B) be its set of eigenvalues, and for $\lambda \in$ spec(B) denote by $m(\lambda)$ its multiplicity as a root of the characteristic polynomial. Let $\mathcal{S} \subset$Sym$^2(\mathbb{R}^{n+1})$ denote the set of symmetric matrices with all zeros in the first row and column, and for $A \in \mathrm{Sym}^2(\mathbb{R}^{n+1}) \setminus \mathcal{S}$, define

$$\widehat{\Theta}(A) = \sum_{\lambda \in \mathrm{spec}(I_n + \sqrt{-1}A)} m(\lambda) \arg(\lambda),$$

branch of arg with values in $(-\pi, \pi]$. Note that $\arg \det(I_n + \sqrt{-1}A) = \widehat{\Theta}(A) \bmod 2\pi$. Denote by $\widetilde{\Theta}$ the minimal upper semi-continuous extension of $\widehat{\Theta}$ to $Sym^2(\mathbb{R}^{n+1})$.

Theorem 2.3.2. *[29, Theorem 1.1] The function $\widehat{\Theta}$ is well-defined and differentiable on $\mathrm{Sym}^2(\mathbb{R}^{n+1} \setminus \mathcal{S})$, and for each $c \in (-(n+1)\pi/2, (n+1)\pi/2)$ such that $c \equiv \theta \bmod 2\pi$, the set*

$$\mathcal{F}_c = \{A \in \mathrm{Sym}^2(\mathbb{R}^{n+1}) : \widetilde{\Theta}(A) \geq c\}$$

is a subequation for the DSL of phase θ.

Remark 2.3.3. *The different choices of c for a given θ correspond to the different branches of the DSL. The DSL subequation is unique in the sense that it arises as the super-level set of an upper semi-continuous function and not a continuous one. $\widehat{\Theta}$ cannot be extended continuously to all of $\mathrm{Sym}^2(\mathbb{R}^{n+1})$. See [29, Section 3] for more details.*

The positivity condition defining the space of positive Lagrangians can also be phrased in terms of a subequation, namely the special Lagrangian subequation introduced by Harvey–Lawson,

$$F_c := \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathrm{tr} \tan^{-1}(A) \geq c\}.$$

Theorem 2.3.4. *[29, Corollary 5.6] Let $\theta \in (-\pi, \pi]$, let $D \subset \mathbb{R}^n$ be a domain and let $k \in C^2([0,1] \times D)$. Then k is a solution of the DSL if and only if for each*

$(t, x) \in [0,1] \times D$,

$$\nabla^2 k(t,x) \in \mathcal{F}_c \cap -\widetilde{\mathcal{F}}_c = \partial \mathcal{F}_c,$$

$$\nabla_x^2 k(t,x) \in \operatorname{int}\left(F_{c-\pi/2} \cap -F_{-c-\pi/2}\right),$$

for a fixed $c \in (-(n+1)\pi/2, (n+1)\pi/2)$ *satisfying* $c = \theta + 2\pi k$ *with* $k \in \mathbb{Z}$.

2.3.2 The Riemannian DSL subequation

In this section we prove the following.

Proposition 2.3.5. *For any Riemannian manifold* M*, the Riemannian manifold* $\mathbb{R} \times M$ *admits a global Riemannian subequation* $\mathcal{F}_c$ *locally modelled on the Euclidean degenerate special Lagrangian subequation* $\mathcal{F}_c$.

Proposition 1.2.1 is then an immediate consequence. To prove Proposition 2.3.5 we show that the $(n+1)$-dimensional manifold $\mathbb{R} \times M$ admits a topological O_n-structure and that $\mathcal{F}_c$ has compact invariance group O_n. This implies that $\mathcal{F}_c$ induces a global equation on $\mathbb{R} \times M$. See Section 3.1.5.

Proof of Proposition 2.3.5. To see that $\mathbb{R} \times M$ admits an O_n-structure (viewing $O_n \subset O_{n+1}$), observe that because $\mathbb{R} \times M$ is globally a product,

$$T(\mathbb{R} \times M) \cong T\mathbb{R} \oplus TM,$$

and $\mathbb{R}$, being parallelizable, admits a trivial structure. In terms of the metric, since O_n-structures are equivalent to Riemannian structures, this represents the fact that $\mathbb{R} \times M$ admits a global product metric.

Now we show that the compact invariance group of $\mathcal{F}_c$ contains O_n. Let S denote the elements $A \in \text{Sym}^2(\mathbb{R}^{n+1})$ of the form $A = \text{diag}(0, B)$, for some $B \in \text{Sym}^2(\mathbb{R}^n)$ and set $I_n = \text{diag}(0, I)$. From [29, Section 3.1] it follows that:

If $A \in S$,

$$\widetilde{\Theta}(A) = \frac{\pi}{2} + \text{ tr arg } (I + \sqrt{-1}B).$$

If $A \in \text{Sym}^2(\mathbb{R}^{n+1}) \setminus S$,

$$\widetilde{\Theta}(A) = \sum_{\lambda \in \text{spec}(I_n + \sqrt{-1}A)} m(\lambda)\text{arg}(\lambda).$$

Let $H = \text{diag}(1, h) \in O_{n+1}$, where $h \in O_n$.

When $A \in S$, $A = \text{diag}(0, B)$, so

$$\begin{aligned} \widetilde{\Theta}(HAH^t) =& \frac{\pi}{2} + \text{ tr } \arg(I + \sqrt{-1}hBh^t) \\ =& \frac{\pi}{2} + \text{ tr } \arg(I + \sqrt{-1}B). \\ =& \widetilde{\Theta}(A). \end{aligned}$$

When $A \notin S$, we have

$$I_n + \sqrt{-1}HAH^t = HI_nH^t + \sqrt{-1}HAH^t = H(I_n + \sqrt{-1}A)H^t,$$

and since $O_{n+1} \subset U_{n+1}$ the spectrum of

$$I_n + \sqrt{-1}A \quad \text{and} \quad I_n + \sqrt{-1}hAh^t$$

are the same. Thus, $\widetilde{\Theta}(HAH^t) = \widetilde{\Theta}(A)$. Therefore, the compact invariance group of $\mathcal{F}_c$ contains $O_n \subset O_{n+1}$. $\square$

The special Lagrangian subequation F_c has been studied by Harvey–Lawson. See [18, Section 10] and [19, Section 14]. Since F_c depends only on the eigenvalues of A it is O_n-invariant and caries over to a Riemannian subequation F_c on any n-dimensional Riemannian manifold.

2.3.3 The DSL on complex manifolds and higher corank

In unpublished notes [28], Rubinstein showed that the DSL subequation can also be defined in the complex setting, i.e., there is a well defined subequation for the equation

$$\operatorname{Im}\det(I_n + \sqrt{-1}\mathrm{Hess}_{\mathbb{C}}u) = 0, \tag{2.10}$$

where $\mathrm{Hess}_{\mathbb{C}}$ is the complex $(1,1)$ Hessian. Harvey–Lawson considered the non-degenerate case in [19, Section 15].

More specifically, let $k : \mathbb{C}^{n+1} \to \mathbb{C}$ so that

$$(\tau, z_1, \cdots, z_n) \mapsto k(\tau, z_1, \cdots, z_n) = k(t, z_1, \cdots, z_n),$$

where $\tau = t + \sqrt{-1}s$.

Then equation (2.10) is invariant under the unitary matrices $U_n \subset U_{n+1}$ in the sense that for any $U = \mathrm{diag}(1, V) \in U_{n+1}$, where $V \in U_n$,

$$\operatorname{Im}\det[I_n+\sqrt{-1}U\mathrm{Hess}_{\mathbb{C}}kU^*] = \operatorname{Im}\det[V(I_n+\sqrt{-1}\mathrm{Hess}_{\mathbb{C}}k)V^*] = \operatorname{Im}\det[I_n+\sqrt{-1}\mathrm{Hess}_{\mathbb{C}}k].$$

Any almost complex manifold X admits a topological U_n structure. Since $\mathbb{C}$ is paralellizable, $\mathbb{C} \times X$ also admits a topological U_n structure, viewing $U_n \subset U_{n+1}$. Thus, for any almost complex manifold X, there exists a global equation on $\mathbb{C} \times X$ locally modelled on Equation (2.10). Taking $D \subset X$ to be the domain

$$\mathcal{D} = \{(\tau, z) \in \mathbb{C} \times X : 0 \le t \le 1 \text{ and } z \in D\},$$

i.e., an infinite strip of width 1 in the complex plane times $D \subset X$, one can consider the Dirichlet problem for equation (2.10) on $\mathcal{D}$, with data depending only on the real part.

It was also shown by Rubinstein [28] that there are corresponding subequations for higher co-rank DSL equations on $\mathbb{R}^{n+k}$, with I_n replaced by the diagonal matrix $\mathrm{diag}(0, .., 0, 1, ...1)$ with k zeros and n ones. In an analogous manner, these equations will carry over to equations on $\mathbb{R}^k \times M$, for any n-dimensional Riemannian manifold M.

2.4 Geometry of the Riemannian DSL

In this section we prove Theorem 1.2.2 and Theorem 1.2.3.

2.4.1 Parallelizable manifolds

Recall that an n-dimensional manifold M is *parallelizable* if it admits a global frame field for the tangent bundle. In terms of its topological structure group (see Section 3.1.5), a manifold is parallelizable if it admits an I-structure, where I is the trivial subgroup in $\mathrm{GL}(n, \mathbb{R})$. Examples of parallelizable manifolds include all orientable 3-dimensional manifolds and all Lie groups [19, Section 5.2].

An *almost Calabi–Yau manifold* is an almost complex Hermitian manifold X with a global section of $\Lambda^{(n,0)}(T^*X)$ whose real part has comass 1 [19, Section 1]. This is equivalent to having topological structure group $SU(n)$. When M is parallelizable, we can explicitly construct an almost Calabi–Yau structure on T^*M

which respects the cotangent bundle fibration. The following construction is based on [19, Section 14].

Let (M, g) be a parallelizable Riemannian manifold. Taking a global orthonormal frame $v = (v_1, \ldots, v_n)$, we identify

$$TM = M \times \mathbb{R}^n.$$

Taking the global coframe $w = (w_1, \ldots, w_n)$ to v,

$$T^*M = M \times \mathbb{R}^n.$$

Thus, (v, w) forms a global frame for

$$T(T^*M) = T^*M \times \mathbb{R}^{2n},$$

where movement along M is captured by v and movement within the fibre by w. Let v^* and w^* denote the dual frames to v and w for the cotangent bundle of T^*M.

In terms of this framing, T^*M admits an almost Calabi–Yau structure.

Almost complex structure J:

$$Jv_i = w_i, \quad Jw_i = -v_i,$$

Non-vanishing $(n, 0)$-form Ω:

$$\Omega = (v_1^* + \sqrt{-1}w_1^*) \wedge \cdots \wedge (v_n^* + \sqrt{-1}w_n^*),$$

Non-degenerate 2-form ω:

$$\omega = \sum_i v_i^* \wedge w_i^*.$$

In general, this structure is not integrable. That is, J is not a (integrable) complex structure and ω and Ω are not closed. However, a certain degree of integrability is necessary for Solomon's geometry on the space of positive Lagrangians. For instance, if J is not integrable then Ω will not be closed and the connection on $\mathcal{O}$ may no longer be the Levi–Civita connection. To remedy this, we now consider a special class of parallelizable manifolds on which the above almost Calabi–Yau structure is a true Calabi–Yau structure, as defined in Section 2.2.

A manifold is called *integrably parallelizable* if it admits an atlas of charts such that the differentials of the transition maps are the identity. In terms of topological structure groups, this is equivalent to saying M admits an *integrable* I-structure.

Theorem 2.4.1. *[16, Section 1] Let M be connected and parallelizable. Then M is integrably paralellizable if and only if M is open (i.e., non-compact and without boundary) or diffeomorphic to the n-dimensional torus.*

Example 2.4.2. *[16] Examples of integrably parallelizable manifolds:*

i. *Open Lie groups;*

ii. *Punctured compact connected Lie groups;*

iii. *Open orientable* 3*-manifolds;*

iv. *Diffeomorphic images of the torus;*

v. *Punctured Stiefel manifolds.*

S^3 and $\mathbb{R}P^3$ are examples of parallelizable manifolds that are not integrably parallelizable. We now prove Theorem 1.2.2.

Proof of Theorem 1.2.2. We first construct a Calabi–Yau structure on T^*M. Since M is integrably parallelizable we have a covering of coordinate charts $\{U_\alpha\}_{\alpha\in A}$ such that the differential of the transition maps is the identity. Let x be coordinates on U_α, and consider the induced coordinate charts on T^*M:

$$V_\alpha = U_\alpha \times \mathbb{R}^n,$$

where $\mathbb{R}^n$ corresponds to the coordinates $\xi^1, \cdots, \xi^n$ for $dx^1, \cdots, dx^n$. It is easy to see that T^*M is also integrably parallelizable with this corresponding atlas. Let (v, w) to be the global frame for $T(T^*M)$ which on V_α takes the form $v_i = \frac{\partial}{\partial x^i}$ and $w_i = \frac{\partial}{\partial \xi^i}$, and take the almost Calabi–Yau structure on T^*M as defined above. This almost Calabi–Yau structure is integrable: since the complex structure is defined in terms of coordinate vector fields the Nijenhuis tensor vanishes, meaning the complex structure is integrable; it follows that Ω is holomorphic; and it is clear that ω is closed and compatible with J.

Next, we show that the Riemannian DSL coincides with the geodesic equation for gradient graphs. These computations closely follow [29, Section 2.4] so we mostly emphasize the differences.

Consider the path of Lagrangians in T^*M given by

$$\Lambda_t = \text{graph } (df_t),$$

where $f_t \in C^2(M)$ for $t \in [0, 1]$. Let $g_t : M \to \Lambda_t$, where

$$g_t(p) = (p, df_t|_p).$$

Then, in local coordinates,

$$\frac{dg_t}{dt} = \partial_t df_t = \sum_{i=1}^{n} \left(\frac{\partial^2 f_t}{\partial t \partial x_i} \right) \frac{\partial}{\partial \xi_i} \quad \text{and} \quad dg_t = \mathrm{I} \oplus \text{Hess } f_t \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right),$$

where Hess $f_t \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)$ denotes the Riemannian Hessian in local coordinates on M with respect to the (flat) metric on M induced by the Calabi–Yau structure on T^*M. It is obvious that dg_t is the identity in the horizontal direction. To see that it is the Riemannian Hessian in the vertical, we compute the image of $\frac{\partial}{\partial x_i}$ under dg_t. Let ∇^* denote the induced metric connection on the cotangent bundle, given in coordinates by

$$\nabla^*_{\frac{\partial}{\partial x^i}} dx^k = -\Gamma^k_{ij} dx^j.$$

Given $s : (-\epsilon, \epsilon) \to M$ satisfying $s(0) = p$ and $\frac{ds}{dt}(0) = \frac{\partial}{\partial x^i}$, the vertical component at $p \in M$ is given by

$$\begin{aligned}
\frac{d}{dt} \left[\sum_k \frac{\partial f_t}{\partial x^k}(s(t)) dx^k|_{s(t)} \right]_{t=0} &= \sum_k \frac{\partial^2 f_t}{\partial x^i \partial x^k}(p) dx^k|_p + \frac{\partial f_t}{\partial x^k} \frac{d}{dt}[dx^k|_{s(t)}]_{t=0} \\
&= \sum_k \frac{\partial^2 f_t}{\partial x^i \partial x^k}(p) dx^k|_p + \frac{\partial f_t}{\partial x^k} \nabla^*_{\frac{\partial}{\partial x^i}} dx^k|_p \\
&= \sum_k \frac{\partial^2 f_t}{\partial x^i \partial x^k}(p) dx^k|_p - \frac{\partial f_t}{\partial x^k} \Gamma^k_{ij} dx^j|_p \\
&= \left(\frac{\partial^2 f_t}{\partial x^i \partial x^j}(p) - \Gamma^k_{ij} \frac{\partial f_t}{\partial x^k} \right) dx^j|_p \\
&= \text{image under } i^{\text{th}} \text{ row of the matrix } \text{Hess}_p f_t \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right),
\end{aligned}$$

where the $(ij)^{\text{th}}$ entry of $\text{Hess}_p f_t \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)$ is $\text{Hess}_p f_t \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. Note that in this particular construction the $\Gamma^k_{ij} = 0$ as the metric on M is flat. This more general computation will be relevant in the next section where the metric on the base manifold is not flat.

Now expressing g_t with respect to the global frame (v, w),

$$g_t(p) = (p, df_t|_p(v)) = (p, df_t|_p(v_1), \cdots, df_t|_p(v_n)),$$

where $df_t|_p(v_i)$ is the coefficient for w_i,

$$\frac{dg_t}{dt} = \partial_t df_t|_p(v) = \sum_{i=1}^{n} \partial_t df_t(v_i) w_i, \qquad \text{and} \qquad dg_t = \mathrm{I} \oplus \mathrm{Hess}_p f_t(v, v).$$

Plugging into equation (5.25),

$$g_t^* \iota_{dg_t/dt} \omega(\cdot) = \omega\left(\frac{dg_t}{dt}, dg_t(\cdot)\right) = -\sum_j w_j^* \otimes v_j^* \left(\sum_{i=1}^{n} \partial_t df_t(v_i) w_i, dg_t(\cdot)\right) = -d\dot{f}_t(\cdot),$$

where $\dot{f}_t(p)$ denotes the derivative of $f_t(p)$ with respect to t, giving us

$$h_t \circ g(t, p) = -\dot{f}_t(p). \tag{2.11}$$

We then compute

$$\begin{aligned} g_t^* \iota_{dg_t/dt} \Omega &= \sum_{i=1}^{n} \Omega\left(dg_t/dt, dg_t(v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_n)\right) w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_n. \\ &= \sum_{i=1}^{n} \det B_i w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_n, \end{aligned}$$

where B_i, $i = 0, \ldots, n$, is the n-by-n matrix obtained by removing the $(i{+}1)$-th column from the n-by-$(n{+}1)$ matrix

$$B = \left[\sqrt{-1}\partial_t df_t|_x(v) \;\mid\; I + \sqrt{-1}\mathrm{Hess}_x f_t(v, v)\right].$$

Similarly,

$$\begin{aligned} g_t^* \Omega &= \det\left[\mathrm{I} \; + \sqrt{-1}\mathrm{Hess}_x f_t(v, v)\right] w_1 \wedge \cdots \wedge w_n \\ &= \det B_0 \, w^1 \wedge \cdots \wedge w_n \end{aligned}$$

From here, the analysis is the same as that in [29, Section 2.4]. Solving for the vector field ζ_t we get

$$\zeta_t = \sum_{i=1}^{n} a^i(t,p)v_i, \quad \text{where} \quad a^i(t,p) = -(-1)^i \frac{\text{Re } (e^{-\sqrt{-1}\theta} \det B_i)}{\text{Re } (e^{-\sqrt{-1}\theta} \det B_0)}.$$

Thus, the geodesic equation (2.8) becomes

$$\text{Im } e^{-\sqrt{-1}\theta} \det \left[I_n + \sqrt{-1}\text{Hess} f(\overline{v}, \overline{v}) \right] = 0, \tag{2.12}$$

where $\overline{v} = (\frac{\partial}{\partial t}, v)$ is a global frame on $[0,1] \times M$ and the Hessian of f is taken with respect to t and x. The positivity condition (2.2) on the Lagrangians implies that $\text{Re } (e^{-\sqrt{-1}\theta} \det B_0) > 0$, or

$$\text{Re } \det \left[\text{I } + \sqrt{-1}\text{Hess} f_t(v,v) \right] > 0. \tag{2.13}$$

$\square$

2.4.2 Calabi–Yau torus fibrations

In this section, inspired by a paper of Leung–Yau–Zaslow [24], we consider Calabi–Yau manifolds which admit a smooth torus fibration. That is, a Calabi–Yau manifold X which is actually a fibred manifold $\pi : X \to M$, where for any $p \in M$, $\pi^{-1}(p) = \mathbb{T}^n$. We show that the geodesic equation for positive Lagrangian sections corresponds to the Riemannian DSL on $[0,1] \times M$, proving Theorem 1.2.4. We begin by summarizing the calculations of [24, Section 3].

Let X be a Calabi–Yau n-fold admitting a smooth torus fibration over a base manifold M, possibly compact. And let ϕ be a $\mathbb{T}^n$-invariant Kähler potential on X. That is, $\phi(x^j, y^j) = \phi(x^j)$, where y are local coordinates on the fiber and x local coordinates on the base. The coordinates $z^j = x^j + iy^j$ are holomorphic on X, and the Kähler metric and form are given, respectively, by

$$h = \frac{\partial^2 \phi}{\partial x^i \partial x^j}(dx^i \otimes dx^j + dy^i \otimes dy^j) \quad \text{and} \quad \omega = \frac{\sqrt{-1}}{2} \frac{\partial^2 \phi}{\partial x^i \partial x^j}(dz^i \wedge dz^j).$$

By Calabi [?], X is Ricci-flat and $\Omega = dz^1 \wedge \cdots \wedge dz^n$ is covariant constant if and only if ϕ satisfies the real Monge–Ampère equation

$$\det \frac{\partial^2 \phi}{\partial x^i \partial x^j} = c, \tag{2.14}$$

for some constant c. Since ϕ satisfies (2.14), the Calabi–Yau condition

$$\frac{\omega^n}{n!} = c\,(-1)^{n(n-1)/2} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \overline{\Omega}$$

is satisfied.

Because of the semi-flatness of h, X is locally isometric to the tangent bundle TM with the metric induced by $g = \frac{\partial^2 \phi}{\partial x^i \partial x^j}(dx^i \otimes dx^j)$ on M. Moreover, if this metric on M is used to identify its tangent and cotangent bundles, then ω is the standard symplectic form on the cotangent bundle.

Consider a Lagrangian section C of this fibration, locally written as $y(x)$, in X. Using the identification with the cotangent bundle and the fact that C is Lagrangian with respect to ω if and only if it is closed and hence locally exact, it is shown [24] that locally

$$y^j = \phi^{jk} \frac{\partial f}{\partial x^k},$$

for some function f, and further computations show

$$dz^1 \wedge \cdots \wedge dz^n|_C = \det\left(I + \sqrt{-1} g^{-1} \text{Hess } f\right) dx^1 \wedge \cdots dx^n. \tag{2.15}$$

Proof of Theorem 1.2.3. Let $C_t : M \to X$ be a smooth path of Lagrangian sections, parametrized by $t \in [0,1]$. Take $g_t : M \to C_t$, where $g_t(p) = (p, C_t(p))$. Then, locally, by the above analysis,

$$\frac{dg_t}{dt} = \sum_{j=1}^{n} \phi^{jk} \frac{\partial^2 f_t}{\partial t \partial x^k} \frac{\partial}{\partial y^j}.$$

By calculations similar to those in Section 2.4.1 and [29, Section 2.4],

$$g_t^* \iota_{\frac{dg_t}{dt}} \omega = -d\dot{f}_t(x), \qquad g_t^* \iota_{\frac{dg_t}{dt}} \Omega = \sum_{i=1}^{n} \det B_i \; dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

and

$$\begin{aligned} g_t^* \Omega &= \det \left[\mathrm{I} \; + \sqrt{-1} g^{-1} \text{Hess } f_t \right] dx^1 \wedge \cdots \wedge dx^n \\ &= \det B_0 \, dx^1 \wedge \cdots \wedge dx^n, \end{aligned}$$

where B_i, $i = 0, \ldots, n$, is the n-by-n matrix obtained by removing the $(i+1)$-th column from the n-by-$(n+1)$ matrix

$$B = \left[\sqrt{-1} \partial_t df_t \;\mid\; I + \sqrt{-1} g^{-1} \text{Hess } f_t \right].$$

Taking $\mathfrak{g} = e \oplus g$, where e is the Euclidean metric on $\mathbb{R}$, we can express the positivity and geodesic conditions, respectively, as

$$\text{Re } e^{-i\theta} \det \left[\mathrm{I} + \sqrt{-1} g^{-1} \text{Hess } f_t \right] > 0,$$

and

$$\text{Im } e^{-\sqrt{-1}\theta} \det \left[\mathrm{I}_n + \sqrt{-1} \mathfrak{g}^{-1} \text{Hess } f \right] = 0,$$

where the second Hessian is taken with respect to $\mathfrak{g}$ and f is viewed as a function on $[0,1] \times M$. Finally, choosing a local orthonormal admissible frame $v = (v_1, \ldots, v_n)$

on M and extending this to $\mathbb{R} \times M$, by $\overline{v} = (\partial_t, v)$ we rewrite these conditions as

$$\text{Re } e^{-\sqrt{-1}\theta} \det \left[\mathrm{I} + \sqrt{-1}\text{Hess } f_t(v, v) \right] > 0,$$

and

$$\text{Im } e^{-\sqrt{-1}\theta} \det \left[\mathrm{I}_n + \sqrt{-1}\text{Hess } f(\overline{v}, \overline{v}) \right] = 0.$$

□

2.5 Solution of the Dirichlet problem for the Riemannian DSL

In this section we seek unique continuous solutions to the Dirichlet problem for the Riemannian DSL. Our set-up is the following. Let $D \subset M$ be a bounded domain with ∂D smooth and let $\mathcal{D} = (0, 1) \times D$, so that $\overline{\mathcal{D}}$ is a compact manifold with embedded corners in $\mathcal{M} = \mathbb{R} \times M$. We assume that both $F_{c-\pi/2}(D)$ and $\widetilde{F}_{c-\pi/2}(D)$ contain at least one C^2 function bounded below, where $F_{c-\pi/2}$ is the corresponding special Lagrangian subequation (see Section 3).

Theorem 2.5.1. *Suppose comparison holds for the Riemannian DSL subequation $\mathcal{F}_c$ on $\mathcal{M}$ and that ∂D is strictly $F_{c-\frac{\pi}{2}}$, $\widetilde{F}_{c+\frac{\pi}{2}}$ convex. Let $\phi \in C^0(\partial\mathcal{D})$ be consistent and affine in t when restricted to $[0, 1] \times \partial D \subset \partial\mathcal{D}$. Consider the following hypotheses:*

1. *$c > -\frac{\pi}{2}$ and for each $i \in \{0, 1\}$,*

$$\phi_i := \phi|_{\{i\}\times D} \in C^2(D) \cap F_{c-\frac{\pi}{2}}(D).$$

2. *For each $i \in \{0, 1\}$,*

$$\phi_i \in C^2(D) \cap F_{c-\frac{\pi}{2}}(D) \cap -F_{-c-\frac{\pi}{2}}(D).$$

If either (1) *or* (2) *holds, there exists a unique solution in* $C^0(\overline{\mathcal{D}})$ *to the* $\mathcal{F}_c$*-Dirichlet problem for* $(\mathcal{D}, \phi)$*.*

Remark 2.5.2. *a. The boundary assumptions hold for any* D *such that* ∂D *is strictly convex, in the sense that all of the eigenvalues of the second fundamental form* $II_{\partial D}$ *are strictly positive. See [19, Proposition 11.4 and Example 14.9].*

b. If M *(and hence* $\mathcal{M}$*) carries a strictly convex* C^2 *function, then by [19, Theorem 9.13] comparison holds for every pure second order subequation on* $\mathcal{M}$ *and thus for the Riemannian DSL. In particular, if* M *is a complete simply-connected Riemannian manifold with non-positive sectional curvature, then the square of the distance function from a fixed point is convex and thus comparison holds. Taking hypothesis 2., gives Theorem 1.2.4.*

c. It may happen that M *does not admit a* C^2 *convex function, but that some* $\overline{D} \subset M$ *do. In this case, [19, Theorem 9.13] implies that comparison (and the theorem) holds for the Riemannian DSL on* $\overline{D}$*.*

We prove Theorem 2.5.1 by showing that $\partial\mathcal{D}$ is appropriately convex and then applying Theorem 3.2.5, following as closely as possible the approach in [29, Section 8]. In Lemma 2.5.3 and Lemma 2.5.4, we construct subsolutions to the DSL that are maximal on various parts of the boundary. These are the analogues of [29, Lemma 8.3] and [29, Lemma 8.4], respectively.

The proof of Lemma 2.5.3 is essentially identical to that of [29, Lemma 8.3] (note that our initial data is C^2). However, the proof of [29, Lemma 8.4] does not carry over to a proof of Lemma 2.5.4. This is due to the absence of appropriate global defining functions in the Riemannian setting. Instead, Lemma 2.5.4 is proved by combining the techniques used to prove [29, Lemma 8.4] and [19, Proposition F].

Lemma 2.5.5 then uses both Lemma 2.5.3 and Lemma 2.5.4 to to show that $\partial\mathcal{D}$ is both $(\mathcal{F}_c, \phi)$ and $(\widetilde{\mathcal{F}}_c, -\phi)$ strictly convex. We omit its proof as it is identical to [29, Lemma 8.5].

2.5.1 Proof of Theorem 2.5.1

Given $\phi_i \in C^2(D)$ (as above), define $v_i \in C^0(\overline{\mathcal{D}})$, by

$$v_0 = \phi_0 - Ct, \quad v_1 = \phi_1 - C(1-t), \tag{2.16}$$

where t is the coordinate on $\mathbb{R}$.

Lemma 2.5.3. *Suppose $\phi_i \in C^2(D) \cap F_{c-\frac{\pi}{2}}(D)$. For each $i \in \{0,1\}$, the function v_i is of type $\mathcal{F}_c$.*

Proof. Suppose v_0 is not of type $\mathcal{F}_c$. Then, by the definition of $\mathcal{F}_c$ (see Section 2.3.1), there is a point $(t,x) \in \mathcal{D}$ such that

$$\widetilde{\Theta}\left(\mathrm{Hess}_{(t,x)} v_0(\overline{e}, \overline{e})\right) = \widetilde{\Theta}\left(\mathrm{diag}[0, \mathrm{Hess}_x \phi_0(e,e)]\right) < c,$$

where $\overline{e} = (\partial_t, e)$ is an admissible frame near (t,x). Thus, by the definition of $\widetilde{\Theta}$,

$$\mathrm{tr} \tan^{-1} \mathrm{Hess}_x \phi_0(e,e) < c - \pi/2.$$

However, ϕ_0 is of type $F_{c-\pi/2}$, so this is a contradiction. The same argument holds for v_1.

$\square$

Lemma 2.5.4. *Let $\phi \in C^0(\partial\mathcal{D})$ be consistent and affine in t when restricted to $[0,1] \times \partial D$. Let $\delta > 0$ and let $(t_0, x_0) \in [0,1] \times \partial D$. If ∂D is $F_{c-\pi/2}$ strictly convex, then there exists a subsolution to the $\mathcal{F}_c$ Dirichlet problem for $(\mathcal{D}, \phi)$ that is δ-maximal at (t_0, x_0).*

Proof. Since the boundary of D is strictly $F_{c-\pi/2}$ convex at x_0, by Theorem 3.1.7 there exists a local defining function ρ for ∂D near x_0 which defines a barrier for $F_{c-\pi/2}$ at x_0. That is, there exists $C_0 > 0$, $\epsilon > 0$, and $r > 0$ such that in local coordinates the functions

$$\beta_i(x) = \phi_i(x_0) - \delta + C\left(\rho(x) - \epsilon\frac{|x - x_0|^2}{2}\right)$$

are strictly $F_{c-\pi/2}$ subharmonic on $B(x_0, r)$ for all $C \geq C_0$. Here we have written ϕ_i to mean $\phi|_{\{i\}\times D}$, for $i = 0, 1$.

By the continuity of ϕ, we can shrink $r > 0$ so that

$$\phi_i(x_0) - \delta < \phi_i(x) \text{ on } \partial D \cap B(x_0, r).$$

Let $\psi \in F_{c-\pi/2}(D)$ be bounded below, and pick $N > \sup_{\partial D} |\phi_i| + \sup_{\overline{D}} \psi$ so that

$$\psi - N < \phi_i - \delta \text{ on } \partial D.$$

Choose C sufficiently large so that on $(B(x_0, r) \setminus B(x_0, r/2)) \cap \overline{D}$

$$\beta_i < \psi - N$$

and on $B(x_0, r/2) \cap \overline{D}$

$$\beta_i < \phi_i(x).$$

Note that since ρ is a boundary defining function it is negative inside D, where defined. As ϕ is affine in t along the boundary of $\mathcal{D}$, it follows that on $[0,1] \times (B(x_0, r/2) \cap \partial D)$

$$\beta(t,x) = \phi(t,x_0) - \delta + C\left(\rho(x) - \epsilon\frac{|x-x_0|^2}{2}\right) \le \phi(t,x),$$

and on $\partial\mathcal{D}$

$$(\psi - N)(t,x) = (\psi - N)(x) \le \phi(t,x).$$

Now set $w(t,x) := \max\{\beta, (\psi - N)\}$. Then, for every t, $w(t,x)$ is equal to $\beta(t,x)$ near x_0 and equal to $\psi - N$ outside $B(x_0, r/2)$. Since

$$\text{Hess } \beta(t,x) = \text{diag}(0, \text{Hess } \beta(x)) \quad \text{and} \quad \text{Hess } (\psi - N)(t,x) = \text{diag}(0, \text{Hess } (\psi - N)(x)),$$

it follows that $\beta(t,x)$ and $(\psi - N)(t,x)$ are $\mathcal{F}_c$-subharmonic. Thus, $w(t,x)$, the max of two $\mathcal{F}_c$-subharmonic functions, is also of type $\mathcal{F}_c$ by Theorem 3.1.2.

Since $w(t,x)$ is equal to $\beta(t,x)$ near x_0 it is immediate that $w(t_0,x_0) = \phi(t_0,x_0) - \delta$. □

Lemma 2.5.5. *Let $\mathcal{D}$ and ϕ be as in Theorem 2.5.1. Then $\partial\mathcal{D}$ is $(\mathcal{F}_c, \phi)$ strictly convex and $(\widetilde{\mathcal{F}}_c, -\phi)$ strictly convex.*

Proof. The proof of this is essentially identical to the proof [29, Lemma 8.5]. □

Proof of Theorem 2.5.1. Combine Lemma 2.5.5 and Theorem 3.2.5. □

2.6 Fourier-Mukai transform

2.6.1 Background and Motivation

Let M be an n-dimensional Calabi-Yau, and assume M admits a smooth torus fibration over a base B:

$$\pi : M \to B, \quad \pi^{-1}(b) = \mathbb{T}^m.$$

Let $z^j = x^j + iy^j$ be local holomorphic coordinates on M, where x^j and y^j are coordinates on the base B and fibre $\mathbb{T}^m$, respectively.

Let $\phi = \phi(x^j, y^j) = \phi(x^j)$ (semi-flatness condition) be the Kähler potential, where $\det \phi_{ij} = \text{const}$. Then in these coordinates the Calabi-Yau structure is

$$g = \phi_{ij}(dx^i dx^j + dy^i dy^j), \quad \omega = \frac{\sqrt{-1}}{2}\phi_{ij} dz^i \wedge dz^j, \quad \Omega = dz^1 \wedge \cdots \wedge dz^n.$$

Leung–Yau–Zaslow [24] showed that the special Lagrangian equation, $\operatorname{Im}\Omega = 0$ transforms via the Fourier-Mukai transform into the deformed Hermitian-Yang-Mills equation, $\operatorname{Im}(\tilde{\omega} + F_A)^n = 0$, where $\tilde{\omega}$ is the Kähler form on the mirror manifold W and F_A is the curvature form of the corresponding connection (more details below). Here we take the Fourier-Mukai transform of the degenerate Special Lagrangain equation.

When the Lagrangian is a section of the fibration, and thus locally a (gradient) graph over the base, the special Lagrangian equation becomes an equation for a function $f : B \to \mathbb{R}$,

$$\operatorname{Im}\det(g + \sqrt{-1}\operatorname{Hess}(f)) = 0.$$

Similarly, when a parametrized path of Lagranians are assumed to be sections of the fibration, the degenerate Special Lagrangain equation on $\mathbb{R} \times B$ takes the form

$$\operatorname{Im}\det(g_0 + \sqrt{-1}\text{Hess}(f)) = 0,$$

where $g_0 = 0 \oplus g$ and the Hessian of f is taken with respect to x and t variables. Taking the Fourier-Mukai transform gives:

$$\operatorname{Im}(\tilde{\omega}_0 + F_{\dot{A}})^{n+1} = 0,$$

where $\tilde{\omega}_0 = 0 \oplus \omega$ and $F_{\dot{A}}$ is the curvature form of the corresponding connection.

2.6.2 Lagrangian Sections

Locally a section is given by a graph $y(x)$ over the base. A section C is Lagrangian with respect to ω if an only if

$$y^j(x) = \phi^{jk}(x)\frac{\partial f}{\partial x^k}(x).$$

Then

$$dy^j = \phi^{jl}\left(\frac{\partial^2 f}{\partial x^l \partial x^k} - \phi^{pq}\phi_{lkp}\frac{\partial f}{\partial x^q}\right)dx^k,$$

which is the product of the inverse of the metric times and the Riemannian Hessian. Thus, since $dz^j = dx^j + \sqrt{-1}dy^j$

$$\operatorname{Im}\Omega|_C = dz^1 \ldots dz^n|_C = \det(I + \sqrt{-1}g^{-1}\text{Hess}(f))dx^1 \ldots dx^n.$$

Setting equal to zero and multiplying through by g gives the special Lagrangian equation above. Similar calculations give the degenerate special Lagrangina equation above.

Leung-Yau-Zaslow showed that from this SLag data one can construct a connection over the mirror manifold W which satisfies the deformed HYM equation. Our goal is to follow this construction as closely as possible with the DSLag data.

2.6.3 Mirror Manifold

The dual manifold W to M is constructed by replacing each torus fibre T in M by the dual torus $\tilde{T}$ =Hom(T, S^1). This leads to the following Calabi-Yau structure on W:

$$\tilde{g} = \phi^{ij}(d\tilde{x}_i d\tilde{x}_j + d\tilde{y}_i d\tilde{y}_j), \quad \tilde{\omega} = \frac{\sqrt{-1}}{2}\phi^{ij} d\tilde{z}_i \wedge d\tilde{z}_j, \quad \tilde{\Omega} = d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_n,$$

where $\tilde{y}_j$ and $\tilde{x}_j$ are dual coordinates to y_j and x_j, with

$$\tilde{x}_j = \tilde{x}_j(x) \text{ such that } \frac{\partial \tilde{x}_j}{\partial x^k} = \phi_{jk}$$

and holomorphic coordinates $\tilde{z}_j = \tilde{x}_j + \sqrt{-1}\tilde{y}_j$.

2.6.4 Fourier-Mukai Transform

On each torus fibre there is the canonical isomorphism $T = \mathrm{Hom}(\tilde{T}, S^1) = \mathrm{Hom}(\pi_1(\tilde{T}), S^1)$, where each point $y = (y^1, \ldots, y^n) \in T$ defines a flat connection D_y on its dual $\tilde{T}$. This is the real Fourier-Mukai transform. More specifically,

$$g_y : \tilde{T} \to \sqrt{-1}(\mathbb{R}/\mathbb{Z}) = S^1, \quad \tilde{y} \mapsto \sqrt{-1}\sum y^j \tilde{y}_j,$$

and

$$D_y = d + A = d + \sqrt{-1} dg_y = d + \sqrt{-1}\sum y^j d\tilde{y}_j.$$

As y varies as a function of x, we get a torus family of 1-forms over the base, i.e., a a connection on W.

The curvature form of this connection is

$$F_A = dA = \sum_{k,j} \sqrt{-1}\frac{\partial y^j}{\partial x_k} d\tilde{x}_k \wedge d\tilde{y}_j.$$

In particular,

$$F_A^{2,0} = \frac{1}{2}\sum_{j,k}\left(\frac{\partial y^k}{\partial \tilde{x}_j} - \frac{\partial y^j}{\partial \tilde{x}_k}\right) d\tilde{z}_j \wedge d\tilde{z}_k,$$

so D_A being integrable is equivalent to there existing a function $f = f(x)$ such that $y^j = \frac{\partial f}{\partial \tilde{x}_j} = \phi^{jk}\frac{\partial f}{\partial x^k}$, which is equivalent to the section C, locally $y(x)$, being Lagrangian.

Since $\frac{\partial y^j}{\partial \tilde{x}_k} = \frac{\partial^2 f}{\partial \tilde{x}_j \partial \tilde{x}_k}$ is equal to the Hessian of f in the x variables, the section C being special Lagrangian is equivalent to Im $(\tilde{\omega} + F_A)^m = 0$.

2.6.5 Transforming DSL

For the degenerate special Lagrangian equation we start with an equation on $\mathbb{R} \times B$, so to make the transformation more natural we add a torus above this $\mathbb{R}$ factor to get $\dot{M} = M \times \mathbb{R} \times S^1$, with

$$\pi : \dot{M} \to B \times \mathbb{R}.$$

Then take the product Kähler metric, with the flat metric on the $\mathbb{R} \times S^1$ factor (let t be the coordinate on $\mathbb{R}$, s the coordinate on S^1, and $v = t + is$):

$$\dot{g} = \phi_{ij}(dx^i dx^j + dy^i dy^j) + (dt^2 + ds^2),$$

$$\dot{\omega} = \frac{\sqrt{-1}}{2}\sum_{i,j}\phi_{ij}(dz^i \wedge d\overline{z}_j) + \frac{\sqrt{-1}}{2}(dv \wedge d\overline{v}).$$

Then

$$D_{\dot{A}} = d + \sqrt{-1}\sum_{j=1}^{n} d\tilde{y}_j + \sqrt{-1}sd\tilde{s},$$

and

$$F_{\dot{A}} = d\dot{A} = \sum_{k,j=1}^{n} \sqrt{-1}\frac{\partial y^j}{\partial \tilde{x}_k}(d\tilde{x}_k \wedge d\tilde{y}_j) + \sum_{k=1}^{n} \sqrt{-1}\frac{\partial s}{\partial \tilde{x}_k}(d\tilde{x}_k \wedge d\tilde{s})$$
$$+ \sum_{j=1}^{n} \sqrt{-1}\frac{\partial y^j}{\partial \tilde{t}}(d\tilde{t} \wedge d\tilde{y}_j) + \sqrt{-1}\frac{\partial s}{\partial \tilde{t}}(d\tilde{t} \wedge d\tilde{s}).$$

Integrability is then equivalent to the existence of a function $f = f(\tilde{t}, \tilde{x})$, where

$$y^j = \frac{\partial f}{\partial \tilde{x}_j} = \phi^{jk}\frac{\partial f}{\partial x^k} \quad \text{and} \quad s = \frac{\partial f}{\partial \tilde{t}} = \frac{\partial f}{\partial t},$$

and so

$$\frac{\partial y^j}{\partial \tilde{x}_k} = \frac{\partial^2 f}{\partial \tilde{x}_j \partial \tilde{x}_k}, \quad \frac{\partial y^j}{\partial \tilde{t}} = \frac{\partial^2 f}{\partial \tilde{x}_j \partial t}, \quad \frac{\partial s}{\partial \tilde{x}_k} = \frac{\partial^2 f}{\partial \tilde{x}_j \partial \tilde{x}_k}, \quad \frac{\partial s}{\partial \tilde{t}} = \frac{\partial^2 f}{\partial t^2},$$

which in terms of the (t, x)-variables is the Riemannian Hessian of $f(t, x)$ on $\mathbb{R} \times B$ (flat in the t-direction). Thus, the degenerate special Lagrangian equation is equivalent to

$$\mathrm{Im}(\tilde{\omega}_0 + F_{\dot{A}})^{n+1} = 0,$$

where $\tilde{\omega}_0 = 0 \oplus \omega$.

2.6.6 Transforming the positivity condition

The degenerate special Lagrangian equation governs geodesics in the space of positive Lagrangians of a Calabi-Yau. In this setting the positivity conditions translates to

$$\mathrm{Re}\det(g + \sqrt{-1}\mathrm{Hess}_x(f) > 0, \text{ for all } t,$$

where $\mathrm{Hess}_x(f)$ denotes the Riemannian Hessian of $f(t,x)$ with respect to just the x-variables. Under the Fourier-Mukai transform this positivity condition becomes

$$\mathrm{Re}(\tilde{\omega}+F_A)^n > 0, \text{ for all } t.$$

2.7 Analytic solutions to the Cauchy problem for the DSL

2.7.1 Preliminaries

Let $k \in C^2(\mathbb{R}\times\mathbb{R}^n)$. Then $k(t,x)$ satisfies the degenerate special Lagrangian equation if

$$\mathrm{Im}\left(\det(I_n+\sqrt{-1}\nabla^2 f)\right) = 0 \quad \text{and} \quad \mathrm{Re}\left(\det(I+\sqrt{-1}\nabla_x^2 f)\right) > 0. \tag{2.17}$$

Here we consider the following Cauchy problem for the degenerate special Lagrangian equation:

$$\begin{aligned} &\mathrm{Im}\left(\det(I_n+\sqrt{-1}\nabla^2 u)\right) = 0, \\ &\partial_t u(0,x) = \phi_0, \quad u(0,x) = \phi_0, \end{aligned} \tag{2.18}$$

where ϕ_0 and ϕ_1 are analytic.

Note that (2.18) is (almost) a special case of the general Cauchy problem:

$$F\left(x,(\partial^\alpha u)_{|\alpha|\le k}\right) = 0, \qquad \partial_\nu^j u = \phi_j \text{ on } S \qquad (0 \le j \le k), \tag{2.19}$$

where $x \in \mathbb{R}^{n+1}$ and $S \subset \mathbb{R}^{n+1}$ is

Theorem 2.7.1 ([14]). *[Cauchy-Kovalevskya] If G and ϕ_j are analytic functions near* 0*, then the non-linear Cauchy problem*

$$\partial_t^k u = G\left(x,t,\partial_t^j\partial_x^\alpha u\right), \tag{2.20}$$

where $j < k$ and $|\alpha| + j \leq k$, with initial conditions

$$\partial_t^j u(x,0) = \phi_j(x), \quad 0 \leq j < k, \tag{2.21}$$

has a unique analytic solution near 0.

2.7.2 The two-dimensional case

Let $u \in C^2(\mathbb{R} \times \mathbb{R}^2)$. Then the DSL becomes

$$u_{tt} = \det(\nabla^2 u) \quad \text{and} \quad \det(\nabla_x^2 u) < 1. \tag{2.22}$$

Rewriting

$$\det(\nabla^2 u) = u_{tt}\det(\nabla_x^2 u) - u_{tx}\left(u_{xt}u_{yy} - u_{yt}u_{xy}\right) + u_{ty}\left(u_{xt}u_{yx} - u_{yt}u_{xx}\right),$$

the first part of (2.22) becomes

$$u_{tt} = \frac{-u_{tx}\left(u_{xt}u_{yy} - u_{yt}u_{xy}\right) + u_{ty}\left(u_{xt}u_{yx} - u_{yt}u_{xx}\right)}{1 - \det(\nabla_x^2 u)}. \tag{2.23}$$

Since the denominator is not equal to zero, this is analytic as a function of lower t and x derivatives of u.

2.7.3 General case

Theorem 2.7.2. *Given analytic initial data, the DSL admits analytic solution in neighborhood of initial Lagrangian.*

Proof. Expanding $\det(I_n + \sqrt{-1}\nabla^2 k)$ in terms of cofactors:

$$\det(I_n + \sqrt{-1}\nabla^2 k) = \sqrt{-1}\left(k_{tt}C_{tt} + k_{t1}C_{t1} + \cdots + k_{t1}C_{t1}\right), \tag{2.24}$$

where $C_{ij} = (-1)^{i+j} M_{ij}$, with M_{ij} denoting the (i, j)-minor of $(I_n + \sqrt{-1}\nabla^2 k)$, i.e., the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting the i-th row and the j-th column of B. Since $C_{tt} = \det(I + \sqrt{-1}\nabla_x^2 k)$,

$$\det(I_n + \sqrt{-1}\nabla^2 k) = \sqrt{-1}\left(k_{tt}\det(I + \sqrt{-1}\nabla_x^2 k) + k_{t1}C_{t1} + \cdots + k_{t1}C_{t1}\right).$$

Then the first part of (2.17) becomes

$$\operatorname{Im}\left[\sqrt{-1}\left(k_{tt}\det(I + \sqrt{-1}\nabla_x^2 k) + k_{t1}C_{t1} + \cdots + k_{t1}C_{t1}\right)\right] = 0,$$

or

$$\operatorname{Re}\left[k_{tt}\det(I + \sqrt{-1}\nabla_x^2 k) + k_{t1}C_{t1} + \cdots + k_{t1}C_{t1}\right] = 0.$$

Since k_{tt} is real,

$$k_{tt}\operatorname{Re}\left[\det(I + \sqrt{-1}\nabla_x^2 k)\right) = -\operatorname{Re}\left[k_{t1}C_{t1} + \cdots + k_{t1}C_{t1}\right],$$

which can be rewritten

$$k_{tt} = \frac{-\operatorname{Re}\left[k_{t1}C_{t1} + \cdots + k_{t1}C_{t1}\right]}{\operatorname{Re}\left[\det(I + \sqrt{-1}\nabla_x^2 k)\right]}.$$

By the second part of (2.17) the denominator is never equal to zero. As the C_{ti} do not contain double t derivatives for $i \geq 1$, this is an analytic expression for k_{tt} in terms of lower t and x derivatives of k. $\square$

Chapter 3: Dirichlet duality

3.1 Overview

This chapter provides a summary of Harvey–Lawson's Dirichlet duality theory on Riemannian manifolds [19]. For the Euclidean formulation, see [18]. See also the related work of Slodkowski [31, 33, 32].

3.1.1 The second-order jet bundle

Let X be a smooth n-dimensional manifold. The *second-order jet bundle* $J^2(X) \to X$ is the bundle whose fibre at a point $x \in X$ is the quotient $J^2_x = C^\infty_x / C^\infty_{x,3}$, where C^∞_x denotes the germs of smooth functions at x and $C^\infty_{x,3}$ the subspace of germs which vanish to order 3 at x.

If X carries a Riemannian metric then the Riemannian Hessian, defined for any C^2 function u and vector fields V and W on X by $(\text{Hess}\ u)(V, W) := V(Wu) - (\nabla_V W)(u)$, is a section of $\text{Sym}^2(T^*X)$. The following is a well-known result concerning the Riemannian Hessian [19, Section 4].

Theorem 3.1.1 (The canonical splitting)**.** *The Riemannian Hessian provides a* *bundle isomorphism*

$$J^2(X) \to \mathbb{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X) \quad \text{by mapping} \quad J^2_x u \to (u(x), (du)_x, \mathrm{Hess}_x u)$$

for a C^2 function u at x.

3.1.2 Subequations

Let $P = \{A \in \mathrm{Sym}^2(T^*_x X) : A \geq 0\}$. A subset $F \subset J^2(X)$ satisfies the *Positivity Condition* (P) if

$$F + P \subset F.$$

Take the canonical splitting $J^2(X) = \mathbb{R} \oplus J^2_{\mathrm{red}}(X)$, where $\mathbb{R}$ denotes the 2-jets of locally constant functions and $J^2_{\mathrm{red}}(X)_x \equiv \{J^2_x u : u(x) = 0\}$ is the space of *reduced 2-jets* at x, and define $N \subset \mathbb{R} \subset J^2(X)$ to have fibres $N_x = \mathbb{R}^- = \{c \in \mathbb{R} : c \leq 0\}$. A subset $F \subset J^2(X)$ satisfies the *Negativity Condition* (N) if

$$F + N \subset F.$$

A subset $F \subset J^2(X)$ satisfies the *Topological Condition* (T) if

$$(i)\ \ F = \overline{\mathrm{Int}\ F}, \qquad (ii)\ \ F_x = \overline{\mathrm{Int}\ F_x}, \qquad (iii)\ \ \mathrm{Int}\ F_x = (\mathrm{Int}\ F)_x.$$

The main existence and uniqueness results for Dirichlet duality assume that F satisfies (P), (T), and (N), so this is formalized as follows. A *subequation F on a manifold X* is a subset $F \subset J^2(X)$ satisfying conditions (P), (T), and (N).

3.1.3 F-subharmonic functions

Let $F \subset J^2(X)$ be closed. The function $u \in C^2(X)$ is F*-subharmonic* if its 2-jet satisfies $J^2_x u \in F_x$, for all $x \in X$, and *strictly* F*-subharmonic* if its 2-jet satisfies $J^2_x u \in (\mathrm{Int}F)_x$, for all $x \in X$. This definition extends to the larger class of upper semi-continuous functions on X taking values in $[-\infty, \infty)$, USC(X), in a viscosity-like way: $u \in \mathrm{USC}(X)$ is said to be $F-$*subharmonic* if for each $x \in X$ and each function ϕ which is C^2 near x, one has that

$$\{u \leq \phi \text{ near } x_0 \text{ and } u(x_0) = \phi(x_0)\} \implies J^2_x \phi \in F_x.$$

The set of all such functions is denoted by $F(X)$.

Theorem 3.1.2 (Remarkable Properties of F-Subharmonic Functions)**.** *Let F be an arbitrary subequation.*

(Maximums) If $u, v \in F(X)$, then $w = \ max\{u, v\} \in F(X)$.

(Coherence) If $u \in F(X)$ is twice differentiable at $x \in X$, then $D^2_x u \in F_x$.

(Decreasing Sequences) If $\{u_j\}$ is decreasing sequence of functions in $F(X)$ then limit is of type $\mathcal{F}$.

(Uniform Limits) If $\{u_j\}$ is a sequence of functions in $F(X)$ that converges uniformly on compact sets then the limit is if type F.

(Families Locally Bounded Above) If $\mathcal{F} \subset F(X)$ is a family which is locally uniformly bounded above. Then the USC regularization v^* of the upper envelope $v(x) = \sup_{f \in \mathcal{F}} f(x)$ belongs to $F(X)$.

Given a subset $F \subset J^2(X)$ the *Dirichlet dual* $\widetilde{F}$ of F is defined by $\widetilde{F} = \sim (-\mathrm{Int}F) = -(\sim \mathrm{Int}F)$, and a function u is F-*harmonic* if $u \in F(X)$ and $-u \in \widetilde{F}(X)$.

3.1.4 Local trivialization

When $X = \mathbb{R}^n$ the 2-jet bundle is canonically trivialized by $J^2_x u = (u(x), D_x u, D^2_x u)$, where

$$D_x u = \left(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_n}(x) \right) \text{ and } D^2_x u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \right).$$

Thus, for any open subset $X \subset \mathbb{R}^n$ there is a canonical trivialization

$$J^2(X) = X \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n), \quad \text{with fibre} \quad \mathbf{J}^2 = \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n).$$

The notation $J = (r, p, A) \in \mathbf{J}^2$ will be used for the coordinates on $\mathbf{J}^2$. Any subset $\mathbf{F} \subset \mathbf{J}^2$ which satisfies conditions (P), (N), and (T) determines a *Euclidean subequation* on any open subset $X \subset \mathbb{R}^n$ by setting $F = X \times \mathbf{F} \subset J^2(X)$. This subequation is often referred to as just F.

Let $e = (e_1, ..., e_n)$ be a choice of local framing of the tangent bundle TX on some neighborhood $U \subset X$. With this framing the canonical splitting determines a trivialization of $J^2(U)$ given at $x \in U$ by

$$\Phi^e : J^2_x(U) \to \mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n), \quad \text{defined by} \quad \Phi^e(J^2_x(u)) \equiv (u, e(u), (\mathrm{Hess}\ u)(e, e)),$$

where $e(u) = (e_1 u, ..., e_n u)$ and $(\mathrm{Hess}\ u)(e, e)$ is the $n \times n-$matrix with entries $(\mathrm{Hess}\ u)(e_i, e_j)$.

3.1.5 Riemannian G-manifolds and Riemannian G-subequations

The general linear group $\mathrm{GL}_n(\mathbb{R})$ has a natural action on the fibre J^2 given by

$$h(r,p,A) = (r, hp, hAh^t) \qquad \text{for } h \in \mathrm{GL}_n(\mathbb{R}).$$

For each Euclidean subequation $F \subset J^2$ this action determines a *compact invariance group*

$$G(\mathbf{F}) = \{h \in O_n : h(\mathbf{F}) = \mathbf{F}\}.$$

Fix a subgroup $G \subset O_n$. A *topological G-structure on X* is a family of smooth local trivializations of TX over open sets in a covering $\{U_\alpha\}$ of X with G-valued transition functions. A *Riemannian G-manifold* is a Riemannian manifold equipped with a topological G-structure.

Lemma 3.1.3. *[19, Lemma 5.2] Suppose* $\mathbf{F}$ *is a Euclidean subequation with compact invariance group G and X is a Riemannian G-manifold. For $x \in X$, the condition on a 2-jet $J \equiv J_x^2 u$ that*

$$\Phi^e(J) \equiv (u(x), e_x(u), (\mathrm{Hess}\,_x u)(e,e)) \in \boldsymbol{F}$$

is independent of the choice of G-frame e at x. Hence there is a well-defined subset $F \subset J^2(X)$ given by

$$J \in F_x \iff \Phi^e(J)(x) \in \boldsymbol{F}.$$

This subset $F \subset J^2(X)$ is a subequation on X and will be called the *Riemannian G-subequation on X with Euclidean model F.*

Now in local coordinates $x = (x_1, ..., x_n)$ on X, the Riemannian Hessian takes the following form

$$(\text{Hess } u)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^k(x) \frac{\partial u}{\partial x_k},$$

where Γ_{ij}^k denote the Christoffel symbols of the Levi–Civita connection. In shorthand,

$$(\text{Hess } u)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = D^2 u - \Gamma_x(Du).$$

Proposition 3.1.4. *[19, Proposition 5.5] Let F be a Riemannian G-subequation on X with Euclidean model $\boldsymbol{F}$ on a Riemannian G-manifold X. Suppose $x = (x_1, ..., x_m)$ is a local coordinate system on U and that $e_1, ..., e_n$ is an admissible G-frame on U. Let h denote the GL_n-valued function on U defined by $e = h\frac{\partial}{\partial x}$. Then a C^2-function u is F-subharmonic on U if and only if*

$$(u, hDu, h(D^2u - \Gamma(Du))h^t) \in \mathbf{F} \text{ on } U.$$

3.1.6 Comparison and approximation

There is a comparison and approximation theory for subequations, which addresses when the sum of an F-subharmonic function and an $\widetilde{F}$-subharmonic function satisfy the maximum principle and when an arbitrary F-subharmonic function can be uniformly approximated with strictly F-subharmonic functions. We briefly introduce the relevant terminology and an important result.

Comparison holds for the subequation F on X if for all compact sets $K \subset X$, whenever

$$u \in F(K) \quad \text{and } v \in \widetilde{F}(K),$$

the Zero Maximum Principle holds for $u+v$ on K, that is,

$$u+v \le 0 \text{ on } \partial K \quad \Longrightarrow \quad u+v \le 0 \text{ on } K.$$

Strict approximation holds for F on X if for each compact set $K \subset X$, each function $u \in F(X)$ can be uniformly approximated by strictly F-subharmonic functions on K. A function $u \in C^2(X)$ is said to be *strictly F-subharmonic* on X if $J_x^2 u \in \operatorname{Int} F$ for all $x \in X$. This notion extends to upper semicontinuous functions (see [19, Definition 7.4]). Let $F_{\text{strict}}(X)$ denote the set of all upper semicontinuous strictly F-subharmonic functions.

A subset $M \subset J^2(X)$ is a *convex monotonicity cone for F* if M is a convex cone with vertex at the origin and $F + M \subset F$.

Theorem 3.1.5. *[19, Theorem 10.3] Suppose F is a Riemannian G-subequation on a manifold X. If X supports a C^2 strictly M-subharmonic function, where M is a monotonicity cone for F, then comparison holds for F on X.*

3.1.7 Boundary convexity and barriers

Recall the canonical decomposition

$$J^2(X) = \mathbb{R} \oplus J^2_{\text{red}}(X)$$

with fibre coordinates $J \equiv (r, J_0)$. A subequation of the form $\mathbb{R} \oplus F$ with $F \subset J^2_{\text{red}}(X)$ is referred to as a *reduced subequation* or a *subequation independent of the r variable.*

Given a subequation $F \subset J^2_{\text{red}}(X)$ independent of the r-variable, the *asymptotic interior $\vec{F}$ of F* is the set of all $J \in J^2_{\text{red}}(X)$ for which there exists a neighborhood $\mathcal{N}(J)$ in the total space of $J^2_{\mathrm{red}}(X)$ and a number $t_0 > 0$ such that

$$t \cdot \mathcal{N}(J) \subset F \quad \text{for all } t \geq t_0.$$

Let Ω be a domain in X with smooth boundary $\partial\Omega$. A *defining function for* $\partial\Omega$ is a smooth function ρ defined on a neighborhood of $\partial\Omega$ such that

$$\partial\Omega = \{x : \rho(x) = 0\}, \quad d\rho \neq 0 \text{ on } \partial\Omega, \quad \text{and } \rho < 0 \text{ on } \Omega.$$

For $x \in \partial\Omega$, $J^2_x\rho = \{0\} \times J^2_{\mathrm{red},x}\rho$, so we use the notation $J^2_x\rho = J^2_{\mathrm{red},x}\rho$

Given a reduced subequation F on X with asymptotic interior $\vec{F}$ and $\Omega \subset X$ a smoothly bounded domain, the $\partial\Omega$ is called *strictly F-convex at $x \in \Omega$* if there exists a local defining function ρ for $\partial\Omega$ near x such that $J^2_x\rho \in \vec{F}_x$. If this holds at every point $x \in \Omega$ then boundary $\partial\Omega$ is *strictly F-convex.*

For general subequations boundary convexity is defined as follows. Given any subequation $F \subset J^2(X)$ there is a family of reduced subequations $F_\lambda \subset J^2_{\mathrm{red}}(X)$, $\lambda \in \mathbb{R}$ defined by

$$\{\lambda\} \times F_\lambda = F \cap \left\{\{\lambda\} \times J^2_{\mathrm{red}}(X)\right\}.$$

Definition 3.1.6. *[19, Definition 11.10] Given a general subequation $F \subset J^2(X)$ and a domain $\Omega \subset X$ with smooth boundary, we say that $\partial\Omega$ is strictly F-convex at a point x if $\partial\Omega$ is strictly F_λ-convex at x for each $\lambda \in \mathbb{R}$. The boundary $\partial\Omega$ is called globally F-convex if it is F-convex at every $x \in \partial\Omega$.*

The importance of boundary convexity is that it implies the existence of barrier functions at boundary points.

Let $\Omega \subset X$ be a smooth domain and let ρ be a local defining function for $\partial\Omega$ near $x_0 \in \partial\Omega$. Then given any $\lambda \in \mathbb{R}$, ρ *defines a* λ*-barrier for* F *at* $x_0 \in \partial\Omega$ if there exists $C_0 > 0$, $\epsilon > 0$, and $r_0 > 0$ such that the function

$$\beta(x) = \lambda + C\left(\rho(x) - \epsilon\frac{|x - x_0|^2}{2}\right) \tag{3.1}$$

is strictly F-subharmonic on $B(x_0, r_0)$ for all $C \geq C_0$. If F is a reduced subequation, then we say that ρ *defines a barrier for* F *at* x_0, since the same ρ works for all $\lambda \in \mathbb{R}$. The following result [19, Theorem 11.12] connects boundary convexity to the existence of barriers.

Theorem 3.1.7 (Existence of Barriers)**.** *Suppose* $\Omega \subset X$ *is a domain with smooth boundary* $\partial\Omega$ *which is strictly* F*-convex at* $x_0 \in \partial\Omega$*. Then for each* $\lambda \in \mathbb{R}$ *there exists a local defining defining function* ρ *for* $\partial\Omega$ *near* x_0 *which defines a* λ*-barrier for* F *at* x_0*.*

3.1.8 Solution of the Dirichlet problem

Let $\Omega \subset\subset X$. Then $g : \overline{\Omega} \to \mathbb{R}$ is said to *solve the* F*-Dirichlet problem on* $\overline{\Omega}$ *for boundary values* ϕ if:

$$(a)\quad g \in C(\overline{\Omega}), \quad (b)\quad g \text{ is } F \text{ harmonic on } \Omega, \quad (c)\quad g = \phi \text{ on } \partial\Omega.$$

Given $\phi \in C(\partial\Omega)$, define the *Perron family*

$$F(\phi) \equiv \{u \in \mathrm{UCS}(\overline{\Omega}) : u|_\Omega \in F(\Omega) \text{ and } u|_{\partial\Omega} \leq \phi\}$$

and the *Perron function* $u_\phi(x) \equiv \sup\{u(x) : u \in F(\phi)\}$. Assuming that both

$F_{\text{strict}}(\overline{\Omega})$ and $\widetilde{F}_{\text{strict}}(\overline{\Omega})$ contain at least one function bounded below (this assumption is minor - see [19, Section 12]), Harvey–Lawson prove the following.

Theorem 3.1.8. *[19, Theorem 13.3] Assume comparison holds for the subequation F on X and the domain $\Omega \subset\subset X$ has smooth boundary. If $\partial\Omega$ is both F and $\widetilde{F}$ strictly convex, then for each $\phi \in C(\partial\Omega)$ the Perron function u_ϕ uniquely solves the Dirichlet problem on Ω for boundary values ϕ.*

3.2 Extension to domains with corners

Let F be a Riemannian subequation on a manifold $\mathcal{M}$. In this section we extend Dirichlet duality theory to include certain domains $U \subset \mathcal{M}$ with corners.

Following the conventions of Joyce [21], given a manifold with corners U, the boundary ∂U is itself a manifold with corners, equipped with a map

$$i_U : \partial U \to U,$$

which may not be injective. The manifold U is said to be a *manifold with embedded corners* if ∂U can be written as the disjoint union of a finite number of open and closed subsets on each of which i_U is injective. A function ϕ on ∂U is called *consistent* if it is constant on the fibres of i_U.

In the case that $\mathcal{M} = \mathbb{R}^n$, Rubinstein–Solomon extended Dirichlet duality to include such domains.

Theorem 3.2.1. *[29, Theorem 7.8] Let F be a subequation in $Sym^2(\mathbb{R}^n)$, and let U be a bounded domain in $\mathbb{R}^n$ such that $\overline{U}$ is a manifold with embedded corners. Let ϕ*

be a consistent continuous function on ∂U. Assume ∂U is strictly (F, ϕ)-convex and strictly $(\widetilde{F}, -\phi)$-convex. Then the F-Dirichlet problem for (U, ϕ) admits a unique solution in $C^0(\overline{U})$.

Theorem 3.2.1 was then used to obtain continuous solutions to the Dirichlet problem for the DSL, which, as previously mentioned, is naturally posed on a domain with corners. Our goal here is to achieve a similar extension in the setting of Riemannian manifolds (Theorem 3.2.5) and use it to obtain solutions to the Dirichlet problem for the Riemmanian DSL.

We briefly outline our approach and provide context for it in relation to [29, 18, 19]. In Section 3.2.1, we extend the notion of boundary convexity (in a weakened sense) to domains with corners. This is accomplished by decomposing the boundary into a part that is convex (in the original sense) and a part where given subsolutions are well-behaved. Our definitions come straight from [29, Section 7.2].

Section 4.2 is then devoted to proving Theorem 3.2.5. Because of the local nature of the arguments used in Dirichlet duality in the Riemannian setting, the proofs in [19, Section 12] carry over almost exactly to this setting. This can be contrasted to the Euclidean setting, where the use of global defining functions to construct barriers [18, Theorem 5.12] makes this extension more difficult. See [29, Proposition 7.3].

3.2.1 Weak boundary convexity

Let $U \subset \mathcal{M}$ be a bounded domain, and let ∂U denote the boundary of $\overline{U}$ considered as a manifold with corners.

Definition 3.2.2. *The boundary component ∂U_i is called strictly F-convex if for each $x \in \partial U_i$, ∂U_i is strictly F-convex at x in the sense of Definition 3.1.6.*

Definition 3.2.3. *Let $\phi \in C^0(\partial U)$ be consistent. A subsolution of the F-Dirichlet problem for (U, ϕ) is a function $u \in F(U) \cap USC(\overline{U})$ such that $u|_{\partial U} \leq \phi$. A subsolution u for (U, ϕ) is called δ-maximal at $p \in \partial U$ if $u(p) \geq \phi(p) - \delta$, and maximal at p if $u(p) = \phi(p)$.*

Definition 3.2.4. *We say ∂U is strictly (F, ϕ)-convex if we can decompose ∂U as the disjoint union $A \cup B$, where A and B are unions of components and satisfy the following:*

1. *For each $p \in A$ and $\delta > 0$ there exists a $C^0(\overline{U})$ subsolution of the F-Dirichlet problem for (U, ϕ) that is δ-maximal at p.*
2. *B is strictly F-convex.*

3.2.2 Solution of the Dirichlet problem

The main result of this section is the following extension of Theorem 3.1.8 and analogue of Theorem 3.2.1. Here we make the same minor technical assumption that is made for Theorem 3.1.8 - that is, we assume $F(U)$ and $\widetilde{F}(U)$ both contain at least one function bounded from below.

Theorem 3.2.5. *Suppose F is a subequation on $\mathcal{M}$ for which comparison holds. Let $U \subset \mathcal{M}$ be a bounded domain such that $\overline{U}$ is a manifold with embedded corners, and let ϕ be a consistent function on ∂U. If ∂U is strictly (F, ϕ)-convex and strictly $(\widetilde{F}, -\phi)$-convex, then the F-Dirichlet problem for (U, ϕ) admits a unique solution in $C^0(\overline{U})$.*

Remark 3.2.6. *By Theorem 3.1.5, the existence of certain subequation-specific C^2 functions on $\mathcal{M}$ implies that comparison holds for that subequation. For instance, if $\mathcal{M}$ carries a strictly convex C^2 function, then by [19, Theorem 9.13] comparison holds for every pure second order subequation on $\mathcal{M}$. Thus, in particular, when $\mathcal{M} = \mathbb{R}^n$ comparison holds for all pure second order subequations.*

The proof of Theorem 3.2.5 is divided into a series of smaller steps, following almost exactly [19, Section 12].

Definition 3.2.7. *Given a consistent continuous function ϕ on ∂U, consider the Perron family*

$$F(\phi) \equiv \{u \in USC(\overline{U}) : u|_U \in F(U) \text{ and } u|_{\partial U} \leq \phi\}$$

and define the Perron function

$$u_\phi(x) \equiv \sup\{u(x) : u \in F(\phi)\}$$

to be the upper envelope of the Perron family.

Proposition 3.2.8 (F)**.** *Let ϕ be a consistent continuous function on ∂U and suppose ∂U is strictly (F, ϕ)-convex at $x_0 \in \partial U$. Then for each $\delta > 0$ small, there exists $w \in F(\phi)$ such that* *i.* w *is continuous at* x_0

ii. $w(x_0) \geq \phi(x_0) - \delta$

iii. $w \in F(\overline{U})$.

Lemma 3.2.9. *Let* ϕ *be a consistent continuous function on* ∂U*. Let* x_0 *be a point of a boundary component* $\partial U_i \subset \partial U$ *that is strictly* F*-convex. Then for each* $\delta > 0$ *small, there exists* $w \in F(\phi)$ *such that*

i. w *is continuous at* x_0

ii. $w(x_0) = \phi(x_0) - \delta$

iii. $w \in F(\overline{X})$.

Proof. The proof of this is identical to that of [19, Proposition F], as the existence of barriers (Theorem 3.1.7) is a purely local condition. □

Clearly, an analogous result holds for strictly $\widetilde{F}$-convex boundary components, providing an element in $\widetilde{F}(-\phi)$ with the corresponding properties.

Proof of Proposition 3.2.8. This follows either by assumption or Lemma 3.2.9. □

Proposition 3.2.10 ($\widetilde{F}$)**.** *Let* ϕ *be a consistent continuous function on* ∂U *and suppose* ∂U *is strictly* $(\widetilde{F}, \phi)$*-convex at* $x_0 \in \partial U$*. Then for each* $\delta > 0$ *small, there exists* $w' \in \widetilde{F}(-\phi)$ *such that*

i. w' *is continuous at* x_0

ii. $w'(x_0) \geq -\phi(x_0) - \delta$

iii. $w' \in \widetilde{F}(\overline{U})$.

Proof. Same as Proposition 3.2.8 with an exchange of roles. □

Given a function f, let usc f denote its *upper semicontinuous regularization*

$$\text{usc } f := \lim_{\delta \to 0} \sup\{f(y) : y \in U \text{ and } d(x,y) < \delta\},$$

and let lsc f denote its lower semicontinuous regularization, defined analogously.

Lemma 3.2.11 (F)**.** usc $u_\phi|_U \in F(U)$

Proof. The proof of this is identical to that of [19, Lemma F]. □

Lemma 3.2.12 ($\widetilde{F}$)**.** $-$lsc $u_\phi|_U \in \widetilde{F}(U)$

Proof. The proof of this is identical to that of [19, Lemma $\widetilde{F}$]. □

Corollary 3.2.13 (F)**.** $\phi(x_0) \leq$ lsc $u_\phi(x_0)$

Proof. This follows from Proposition 3.2.8, and is essentially identical to the proof of [19, Corollary F]. Since $w \in F(\phi)$, we have $w \leq u_\phi$ and thus lsc $w \leq$ lsc u_ϕ. Because w is continuous at x_0 and $w(x_0) \geq \phi(x_0) - \delta$,

$$\phi(x_0) - \delta \leq \text{lsc } u_\phi(x_0) \quad \forall \delta > 0 \text{ small.}$$

□

Corollary 3.2.14 ($\widetilde{F}$)**.** usc $u_\phi(x_0) \leq \phi(x_0)$

Proof. The proof of this is essentially identical to that of [19, Corollary $\widetilde{F}$]. Take $u \in F(\phi)$, arbitrary. Since $w' \leq -\phi$ on ∂U, this implies

$$u + w' \leq 0 \quad \text{on } \partial U.$$

Since $w' \in \widetilde{F}(\overline{U})$, by comparison

$$u + w' \le 0 \quad \text{on } \overline{U}.$$

Thus, $u_\phi + w' \le 0$ on $\overline{U}$. By the continuity of w' at x_0, and the fact that $w'(x_0) \ge -\phi(x_0) - \delta$, we have

$$\text{usc } u_\phi(x_0) \le -w'(x_0) \le \phi(x_0) + \delta \quad \forall \delta > 0 \text{ small.}$$

□

From this series of results we can draw the following conclusions.

1. By Corollary 3.2.13 and Corollary 3.2.14, we have lsc $u_\phi = u_\phi =$ usc $u_\phi = \phi$ on ∂U. Thus, u_ϕ is continuous on ∂U.

2. By Corollary 3.2.14 and Lemma 3.2.11, it follows that usc $u_\phi \in F(\phi)$.

3. And since usc $u_\phi \in F(\phi)$, this means usc $u_\phi \le u_\phi$ on ∂U. Thus, $u_\phi =$ usc u_ϕ.

We can now prove Theorem 3.2.5. This proof is identical to that of [19, Theorem 12.4].

Proof of Theorem 3.2.5. It only remains to show that u_ϕ is F-harmonic. By Corollary 3.2.13 and Lemma 3.2.12,

$$-\text{lsc } u_\phi \in \widetilde{F}(-\phi).$$

By conclusion (1) above,

$$-\text{lsc } u_\phi = -u_\phi \quad \text{on } \partial U.$$

Since $u_\phi|_U \in F(U)$, $-\text{lsc } u_\phi|_U \in \widetilde{F}(U)$, and $u_\phi - \text{lsc } u_\phi \le 0$ on ∂U, comparison implies

$$u_\phi - \text{lsc } u_\phi \le 0 \quad \text{ on } \overline{U}.$$

Thus, lsc $u_\phi = u_\phi$, and so u_ϕ is F-harmonic. $\square$

Chapter 4: Theorem of Slodkowski

4.1 Introduction

In this chapter we explore a theorem of Slodkowski regarding the "largest eigenvalue" of a convex function. This result plays a key role in proving the uniquenss of solutions to the Dirichlet problem in the Euclidean formulation of Harvey–Lawson's Dirichlet duality theory [18].

4.1.1 Motivation

It is known that a convex function u on $\mathbb{R}^n$ is differentiable almost everywhere and has distributional second-order partial derivatives. It is also known that a convex function is twice differentiable almost everywhere in the sense that for a.e. $x \in \mathbb{R}^n$, there exists a symmetric positive semi-definite matrix $D^2 f(x)$ such that

$$f(x+h) = f(x) + \langle \nabla f(x), h \rangle + \frac{1}{2}\langle D^2 f(x)h, h \rangle + o(||h||^2).$$

The operator $D^2 f$ is called the *second-order Peano derivative.* Note that its existence does not imply the existence of ∇f in a neighbourhood, so it should not be considered the second derivative of f in the usual sense.

In [34], Slodkowski studies uniqueness for a generalized Dirichlet problem in the class of $q-$plurisubharmonic (q–psh) functions (for C^2 functions on $\mathbb{C}^n$ this is equivalent to the complex Hessian having $n-q$ nonnegative eigenvalues at every point). The problem of uniqueness reduces to showing that the difference of two such functions is $n-1$–psh, which implies that it satisfies a maximum principle, from which uniqueness then follows. Functions of this q–psh class can be approximated by a subclass which are convex up to a quadratic polynomial. Because of this it is sufficient to study this smaller class, which given their quasi-convexity, retain some of the nice properties of convex functions. In particular, quasi-convex functions are a.e. twice differentiable, in the above sense. Thus, the second-order behavior of these functions and their difference is known a.e. However, to show that the difference is a member of the above mentioned class, they must satisfy this eigenvalue property everywhere. To this end, Slodkowski introduces a generalized second-order derivative, which for C^2 functions is simply the largest eigenvalue of the Hessian, and proves that if this quantity is bounded below almost everywhere in some domain, it is bounded below everywhere in that domain. Using this, he shows that the difference is contained in the desired $n-1$–psh class.

Following Slodkowski [34, Section 3], we define the largest "eigenvalue" of a convex function.

Definition 4.1.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$. If $\nabla u(x_0)$ exists, $K(u, x_0)$ is defined by the formula*

$$K(u, x_0) = \limsup_{\epsilon \to 0} 2\epsilon^{-2} \max\{u(x_0 + \epsilon h) - u(x_0) - \epsilon \langle \nabla u(x_0), h \rangle : h \in S^{n-1}\}$$

otherwise $K(u, x)$ is defined as $+\infty$.

This is the generalized second-order derivative that Slodkowski defines. For the sake of context, note that this quantity is a modification to the *second-order upper Peano derivative* of u in the direction of h, which is defined as

$$\limsup_{\epsilon \to 0^+} 2\epsilon^{-2}(u(x_0 + \epsilon h) - u(x_0) - \epsilon \langle \nabla u(x_0), h \rangle).$$

Being maximal, this second-order derivative is of particular interest because it corresponds to the largest eigenvalue of the Hessian when defined (which it does, in the above sense, almost everywhere for convex functions), and gives a useful quantity to work with otherwise, especially in the context of Slodkowski's $C^{1,1}$ estimates.

Regarding this quantity $K(u, x)$, Slodkowski [34] shows the following.

Theorem 4.1.2. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a locally convex function in $U \subset \mathbb{R}^n$, such that $K(u, x) \geq M$ for almost every $x \in U$. Then $K(u, x) \geq M$ for all $x \in U$.*

The recent work of Harvey and Lawson [18] on the Dirichlet problem was one of our motivations for studying this quantity $K(u, x)$ and Slodkowski's proof of the above result. They study fully non-linear degenerate elliptic equations of the form

$$F(\mathrm{Hess}(u)) = 0 \text{ on } \Omega \tag{4.1}$$

$$u = \phi \text{ on } \partial\Omega. \tag{4.2}$$

Given certain convexity assumptions on the boundary, they establish the existence and uniqueness of continuous solutions using their new Dirichlet duality theory. The work of Slodkowski [34] was "an inspiration" for that paper, and in particular Theorem 4.1.2 is the "deepest ingredient" of their proof of uniqueness of viscosity solutions of (4.1). These existence and uniqueness results apply to many important problems including all branches of the homogeneous Monge-Ampère equation, all branches of the special Lagrangian potential equation, and equations appearing naturally in Lagrangian and calibrated geometry.

Given the usefulness of this generalized derivative and the above result to recent progress on important problems, it makes sense to better understand both the derivative and the proof of the theorem. The proof is technical and very geometric so here an illustrated exposition is provided. The quantity $K(u, x)$ is then studied further for convex u. In particular, the Legendre–Fenchel transform is applied to give a simple alternative characterization of $K(u, x)$ in terms of the convexity of the dual function u^* to u. This allows for an alternative proof to a key proposition needed to prove Slodkowski's theorem. Altogether, there are now three ways to view this generalized derivative $K(u, x)$: analytic (Definition 4.1.1), geometric (Proposition 4.1.6), and dual-analytic (Theorem 4.1.9).

4.1.2 Summary

Theorem 4.1.2 follows immediately from the following theorem [34, Theorem 3.2], the proof of which is the focus of the first part of this paper.

Theorem 4.1.3. *Let u be convex near $x_0 \in \mathbb{R}^n$. Assume that $K(u, x_0) = k_0$ is finite. Then for every $k > k_0$ the set $\{x : K(u, x) < k\}$ is Borel and its lower density at x_0 is not less than $\left(\frac{k-k_0}{2k}\right)^n$.*

Lower density is defined as follows.

Definition 4.1.4. *The lower density of a Lebesgue measurable set $Z \subset \mathbb{R}^n$ at $x_0 \in$*

$\mathbb{R}^n$ is the number

$$\liminf_{\varepsilon \to 0} \frac{m_n\left(Z \cap B(x_0, \varepsilon)\right)}{m_n\left(B(x_0, \varepsilon)\right)},$$

where m_n denotes the n-dimensional Lebesgue measure.

Slodkowski's proof of Theorem 4.1.3 divides naturally into two parts. First, an equivalent geometric characterization of a bound on $K(u, x)$ is given in terms of spheres tangent to the graph of u. This is the content of the following definition and proposition [34, Proposition 3.3].

For $c = (c_1, \ldots, c_{n+1}) \in \mathbb{R}^n$, let $S(c, r)$ denote the n-sphere with center c and radius r, and $B(c, r)$ denote the open $n + 1$-disk of radius r centered at c.

Definition 4.1.5. *The sphere $S(c, r)$ is a sphere of support from above at $y = (x_0, u(x_0))$ if $y \in S(c, r)$, $B(c, r) \cap \text{graph}(u) = \emptyset$ and $c_{n+1} > u(P(c))$, where P denotes the orthogonal projection of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$.*

Thus, $S(c, r)$ can be visualized as a ball resting on a "surface" that is the graph of u, and such that $(x_0, u(x_0))$ is one of its resting points.

Proposition 4.1.6. *Let $U \subset \mathbb{R}^n$ be open and $u : U \to \mathbb{R}$ be convex. Assume that u has gradient at x.*

(i) If u has second-order Peano derivatives at x, then $K(u, x)$ is equal to the norm (i.e. the largest eigenvalue) of the real Hessian of u at x.

(ii) If $K(u, x)$ is finite, then for every $K > K(u, x)$ there is $\varepsilon > 0$ such that

$$u(x + h) - u(x) - \langle \nabla u(x), h \rangle \ \leq \frac{1}{2} K |h|^2.$$

(iii) If there is a sphere $S(c, r), r > 0$ which supports the graph of u from the above at $(x, u(x))$, then

$$K(u, x) \leq \frac{(1 + |\nabla u(x)|^2)^{\frac{3}{2}}}{r}. \tag{4.3}$$

Parts (ii) and (iii) give the above mentioned equivalence between a bound on $K(u, x)$ and a sphere of support to the graph of a corresponding radius at $(x, u(x))$. See Section 4.2.2 for a more detailed explanation.

The second part of the proof then uses this alternative characterization of $K(u, x)$ to obtain a density result, which is essentially the statement of the theorem in terms of spheres of support as opposed to $K(u, x)$. This is the content of the following lemma [34, Lemma 3.4].

Lemma 4.1.7. *Let u be a non-negative convex function in $B(0, d) \subset \mathbb{R}^n$, $d > 0$, such that $u(0) = 0$ and $\nabla u(0) = 0$. Let $R > 0$ and assume that the closed ball $\bar{B}(c, R)$, $c = (0, ..., 0, R) \in \mathbb{R}^{n+1}$, intersects the graph of u only at $0 \in \mathbb{R}^{n+1}$. Let X_r, $0 < r < R$ denote the set of all $x \in B(0, d) \subset \mathbb{R}^n$ such that there exists a sphere of radius r supporting the graph of u from above at $(x, u(x))$. Then the lower density of X_r at 0 is not less than $((R - r)/2R)^n$.*

As will be seen in more detail in Section 4.2, there is an inverse relationship between the bound on $K(u, x)$ and the radius of the sphere of support to the graph of u at $(x, u(x))$. This will explain the similarity between the lower bound on density given in the lemma and the one in the theorem.

The geometric characterization of $K(u, x)$ is key to proving Theorem 4.1.3 and helpful in understanding what quality this generalized derivative captures about the function u and its graph. Since the results here concern functions that are at least locally convex, it is natural to study them via the Legendre–Fenchel transform, the classical transform of convex analysis. By definition, the set of points above the graph of a convex function (epigraph) is a convex set. Any convex set in $\mathbb{R}^n$ can be defined entirely by a family of supporting hyperplanes. Thus, since the epigraph of u completely determines the graph of u, which in turn completely determines u, this family of hyperplanes can be considered an alternative description or parametrization of u. This is essentially how the transform of u (or dual function to u) u^* is defined. Each point $p \in \mathbb{R}^n$ defines a collection of hyperplanes (via gradient), and u^* specifies a point $u^*(p) \in \mathbb{R}$, such that $(0, ..., 0, -u^*(p)) \in \mathbb{R}^n$ lies on the one hyperplane of this collection which supports the epigraph (or graph) of u.

Interestingly, under the Legendre–Fenchel transform, differentiability properties of u correspond to convexity properties of u^*. Two classic examples of this are the following.

Proposition 4.1.8. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then*

(i) f is strictly convex if and only if f^* is differentiable.

(ii) f is strongly convex with modulus c if and only if f^* is differentiable and ∇f^* is Lipschitz continuous with constant $\frac{1}{c}$.

Given that $K(u, x)$ is a (local) differentiability property of u, it seems there should be an appropriate (local) convexity property corresponding to u^*. In section 3 we prove the following result.

Theorem 4.1.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. If $K(f, x_0) = k_0 < k$ then f^* is quadratically convex at $y_0 = \nabla f(x_0)$ with modulus $\frac{1}{k}$. Conversely, if u^* is quadratically convex with modulus $\frac{1}{k}$, then $K(f, x_0) = k_0 \leq k$.*

Quadratically convex at y_0, which is defined in section 3, is a more local form of convexity than the two types of convexity referred to in Proposition 4.1.8. This dual characterization of $K(u, x)$ allows for an alternative proof of Proposition 4.1.6. Using quadratics to define different types of convexity is standard (e.g. quasi-convexity, strong convexity). See section 3 for definitions of all these terms and a more detailed discussion.

In Slodkowski's proof, quadratics arise naturally via the definition of $K(u, x)$, and from this, spheres. The geometric properties of spheres make certain arguments very clear (see proof of Lemma 4.1.7), however some manipulations and calculations are simpler with quadratics, given their constant second-order behavior. For example, in [**?**] Harvey and Lawson provide an alternative proof of Slodkoski's lemma (as well as Alexandrov's theorem stated above) via a generalization by using quadratics instead of spheres. Their proof is modelled off of Slodkowski's, and they obtain their result for the larger class of quasi-convex functions. Instead of spheres of support, they use the notion of upper contact jets, where given $p \in \mathbb{R}^n$, and A a real symmetric $n \times n$ matrix, (p, A) is an *upper contact jet for u at x* if there exists a neighbourhood of x such that

$$u(y) \leq u(x) + \langle p, y - x \rangle + \frac{1}{2}\langle A(y - x), y - x \rangle.$$

Slodkowski's result then corresponds to $A = \lambda I$.

4.1.3 Organization

Section 4.2 contains the exposition of Slodkowski's proof of Theorem 4.1.3: Section 4.2.1 gives an overview of the proof, Section 4.2.2 a slight variation of Slodkowski's proof to Proposition 4.1.6 (the generalized $C^{1,1}$ estimate), Section 4.2.3 an expanded and illustrated version of Slodkowski's proof to Lemma 4.1.7, and Section 4.2.4 combines these for the proof of the theorem.

Section 4.3 studies $K(u, x)$ from the dual perspective: Section 4.3.1 recalls some basic convex analysis, including Legendre–Fenchel duality, Section 4.3.2 provides an equivalent interpretation of $K(u, x)$ in terms of the dual function to u, and uses this for an alternative proof of the $C^{1,1}$ estimate.

The appendix considers Lipschitz continuity of the gradient and the geometric interpretation of $K(u, x)$: Section 4.4.1 demonstrates $K(u, x)$ is bounded by the Lipschitz constant when u is $C^{1,1}$, Section 4.4.2 gives an example of a function with a sphere of support that is not $C^{1,1}$ on any neighbourhood, Section 4.4.3 compares $K(u, x)$ to the classical notion of an osculating circle to a plane curve and gives an extension of this to higher dimensions, Section 4.4.4 relates the radius of a sphere of support to a function to that of the radius of a supporting sphere to its dual.

4.2 Exposition of Slodkowski's proof

4.2.1 Overview

Theorem 4.1.3 is concerned with the set of points (near x_0) such that $K(u, x) < k$, for some fixed $k > k_0 = K(u, x_0)$. However this set may be difficult to study directly given that the only information available about u is that it is continuous (bounded and convex) on some neighbourhood of x_0 and $K(u, x_0) = k_0 < \infty$. In particular, knowing the value of $K(u, x)$ at a given point does not immediately suggest anything about its value nearby. Thus, the first step towards a better understanding of this set of points is an alternative characterization of what it means for $K(u, x)$ to bounded at some point.

If at the point x, $K(u, x_0) < \infty$ this is equivalent to a (local) sphere of support from above to the graph of u at $(x, u(x))$. This is precisely what Proposition 4.1.6 (ii) and (iii) states. (ii) implies the existence (locally) of a quadratic function tangent to the graph of u at $(x, u(x))$ which majorizes u on some neighbourhood, and this in turn implies the (local) existence of a sphere of support to the graph of u at $(x, u(x))$. The content of (iii) is clear.

With this alternative geometric characterization in hand, Lemma 4.1.7 then proves the theorem in terms of these spheres of support. To accomplish this another change in perspective is needed, which takes further advantage of this more geometric interpretation of $K(u, x)$. Instead of looking at points x in the domain of u such that there exists a sphere of support to the graph of u at $(x, u(x))$, it is better to consider for each point x in domain of u an $n-$sphere (of fixed radius) in $\mathbb{R}^{n+1}$ above the graph of u with center $c \in \mathbb{R}^{n+1}$ such that $P(c) = x$, where $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map. If we lower this sphere down towards x it will of course eventually intersect the graph of u. Since u is continuous, it is not difficult to show that on a small enough neighbourhood these spheres will come down on a closed part of the graph of u and thus there will be an initial point of contact. This sphere is by definition a sphere of support to the graph of u at that point. The next step is to show that for every ϵ neighbourhood of 0 ($x_0 = 0$ for Lemma 4.1.7) there is a corresponding $\delta = \delta(\epsilon)$ such that the spheres above the points in $B(0, \delta)$ are spheres of support to the graph at points $(x, u(x))$, where $x \in B(0, \epsilon)$. Now $B(0, \delta)$ is a much nicer set to work with then $X_r \cap B(0, \epsilon)$, and these two sets can be related by a few simple Lipschitz maps. Since Lipschitz maps behave nicely with respect to measures, this allows us to place a lower bound on the measure $m(X_r \cap B(0, \epsilon))$ for each epsilon. A limiting argument is then used to obtain the lower bound on the lower density at 0.

Proposition 4.1.6 and Lemma 4.1.7 can then be combined to give Theorem 4.1.3. A sketch of the proof is as follows. Start with a point x_0 where $K(u, x_0)$ is finite (hypothesis of Theorem 4.1.3), and choose any $k > K(u, x_0)$. Note it can be assumed without loss of generality that $x_0 = 0$, $u(0) = 0$, and $\nabla u(0) = 0$ (see Section 4.2.3 for details). Then apply Proposition 4.1.6 (ii), which locally gives a sphere of support of radius $1/k$ at $(x_0, u(x_0))$. Now, apply Lemma 4.1.7 to get a lower bound on the density of X_r, $r < 1/k$, at x_0. Next, apply Proposition 4.1.6

(iii) to convert this into a statement about the density of X'_k, where

$$X'_k \equiv \{x \in \mathrm{dom}(u) | K(u,x) < k\}.$$

This last step is accomplished by using the continuity of the gradient to show that in a small enough neighbourhood $X_r \subset X'_k$. More explicitly, $x \in X_r$ implies $K(u,x) \leq r^{-1}(1 + |\nabla u(x)|^2)^{3/2}$ and $\nabla u(x_0) = 0$, so by continuity of the gradient of convex functions and since $k > 1/r$, $\nabla u(x)$ will eventually be small enough so that $r^{-1}(1 + |\nabla u(x)|^2)^{3/2} < k$. Thus, for $x \in X_r$, $K(u,x) < 1/k$. This gives the theorem by choosing R arbitrarily close to $1/k_0$ and r arbitrarily close to $1/k$ (see Section 4.2.4 for a detailed proof).

4.2.2 The generalized $C^{1,1}$ estimate

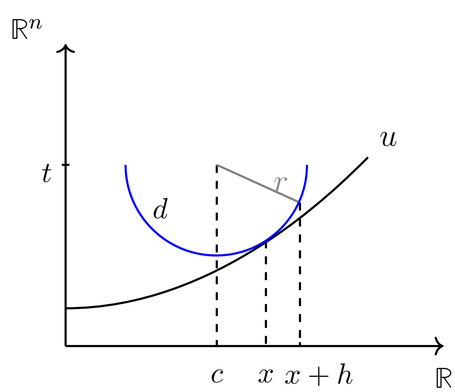

Figure 4.1: Lower hemishpere function $d : B(c,r) \to \mathbb{R}$

In this subsection we provide an alternative proof to Proposition 4.1.6 (iii). The main idea is as follows: given a sphere of support of radius r to the graph of u at the point $(x, u(x))$, the lower hemisphere of this sphere defines the graph of a smooth convex function that agrees up to first order with u at x and majorizes u

elsewhere. Denote this function by d. It immediately follows that $K(u,x) \leq K(d,x)$, and the rest of the proof consists in computing $K(d,x)$, which is equal to the largest eigenvalue of d because d is smooth.

Proof of Proposition 4.1.6 (iii). Assume that the sphere $S((c,t),r)$, $c \in \mathbb{R}^n$ supports the graph of u from the above at $(x, u(x_0))$ and that u is differentiable at x_0. Define $d : B(c,r) \to \mathbb{R}$ to be the function whose graph is the lower open hemisphere of $S((c,t),r)$. Recall the definition for $K(f,x_0)$:

$$K(u,x_0) := \limsup_{\epsilon\to 0} 2\epsilon^{-2} \max \{u(x_0+\epsilon h) - u(x_0) - \epsilon\langle \nabla u(x_0), h\rangle : |h| = 1\}.$$

Clearly, since $d(x_0) = u(x_0)$ and $\nabla d(x_0) = \nabla u(x_0)$,

$$K(u,x_0) \leq K(d,x_0).$$

Since d is smooth,

$$\begin{aligned}
K(d,x_0) &= \limsup_{\epsilon\to 0} 2\epsilon^{-2} \max \{d(x_0+\epsilon h) - d(x_0) - \epsilon\langle \nabla d(x_0), h\rangle : |h| = 1\} \\
&= \limsup_{\epsilon\to 0} 2\epsilon^{-2} \max \{\frac{1}{2}\langle \nabla^2 d(x_0 + \gamma_{\epsilon,h}\epsilon h)\epsilon h, \epsilon h\rangle : |h| = 1\}, \quad 0 < \gamma_{\epsilon,h} < 1 \\
&= \limsup_{\epsilon\to 0} \max \{\langle \nabla^2 d(x_0 + \gamma_{\epsilon,h}\epsilon h) h, h\rangle : |h| = 1\}, \quad 0 < \gamma_{\epsilon,h} < 1 \\
&= \max \{\langle \nabla^2 d(x_0) h, h\rangle : |h| = 1\} \text{ by continuity and compactness.} \\
&= \lambda_{\max}, \quad \text{maximum eigenvalue of } \nabla^2 d(x_0)
\end{aligned}$$

Thus, now we show that

$$\lambda_{\max} = \frac{(1 + (\nabla u(x_0))^2)^{\frac{3}{2}}}{r}.$$

The equation for d, the sphere of radius r centered at (c,t), where $c \in \mathbb{R}^n$ and $t \in \mathbb{R}$, is

$$d(x) = t - \sqrt{r^2 - |c-x|^2}.$$

Without loss of generality we may assume that the sphere of support is centered at the origin and x_0 has just first component non-zero, as otherwise we could always shift and then rotate without affecting the second-order behavior. In other words, assume $(c,t) = 0 \in \mathbb{R}^{n+1}$ and $x_0 = (s_1, ..., s_n) = (s, 0, ..., 0) \in \mathbb{R}^n$. Then $d(x) = -\sqrt{r^2 - |x|^2}$.

Let

$$w(x) := \frac{1}{r^2 - s^2}\left(|x - x_0|^2 + 2\langle x - x_0, x_0\rangle\right).$$

Since

$$|x|^2 = \langle x, x\rangle = \langle (x - x_0) + x_0, (x - x_0) + x_0\rangle = |x - x_0|^2 + 2\langle x_0, x - x_0\rangle + |x_0|^2$$

and $|x_0|^2 = s^2$, we can write $d(x)$ as

$$d(x) = -\sqrt{r^2 - s^2}\sqrt{1 - w(x)}.$$

Now expanding $\sqrt{1 - w(x)}$ as a series and dropping the terms of order higher than two (as they will have 0 Hessian at x_0),

$$d(x) \approx -\sqrt{r^2 - s^2}\left(1 - \frac{w(x)}{2} - \frac{w(x)^2}{8}\right).$$

This can be further reduced to

$$d(x) \approx -\sqrt{r^2 - s^2}\left(1 - \frac{w(x)}{2} - \frac{1}{8}\left(\frac{2\langle x - x_0, x_0\rangle}{r^2 - s^2}\right)^2\right),$$

since we are only concerned with the expression for d, modulo powers higher than two.

Thus, $d(x)$ has been replaced by a diagonal quadratic form and straightforward computations give

$$\nabla d(x_0) = \frac{x_0}{\sqrt{r^2 - s^2}},$$

and

$$\nabla^2 d(x_0) = \frac{1}{\sqrt{r^2 - s^2}} I + \frac{s}{(r^2 - s^2)^{3/2}} A,$$

where I is the $n \times n$ identity matrix and A is the $n \times n$ matrix with first row $x_0 = (s, 0, ..., 0)$ and zeros elsewhere. It follows immediately that

$$\lambda_{max} = \frac{1}{\sqrt{r^2 - s^2}} + \frac{s^2}{(r^2 - s^2)^{3/2}}.$$

Since $|\nabla u(x_0)|^2 = |\nabla d(x_0)|^2 = \frac{s^2}{r^2 - s^2}$,

$$\lambda_{max} = \frac{1 + |\nabla u(x_0)|^2}{\sqrt{r^2 - s^2}}.$$

Furthermore, the vector $(x_0, u(x_0))$ is of length r, proportional to the upward pointing unit normal to the graph of u at $(x_0, u(x_0))$, which is equal to

$$(1 + |\nabla u(x_0)|^2)^{-\frac{1}{2}}(-\nabla u(x_0), 1).$$

Scaling by r, we obtain

$$x_0 = -r(1 + |\nabla u(x_0)|^2))^{-\frac{1}{2}} \nabla u(x_0).$$

Giving

$$x_0 = \frac{r \nabla u(x_0)}{\sqrt{1 + |\nabla u(x_0)|^2}}, \quad s^2 = |x_0|^2 = \frac{r^2 |\nabla u(x_0^2)}{1 + |\nabla u(x_0)|^2}.$$

Therefore,

$$\lambda_{max} = \frac{(1 + |\nabla u(x_0)|^2)^{\frac{3}{2}}}{r}.$$

$\square$

4.2.3 The density lemma

If at the point $x_0 = 0$ there is a sphere of support of radius R, Lemma 4.1.7 provides a lower bound on the lower density of the set X_r of points with sphere of support of a radius $r < R$. Note that without loss of generality it may be assumed that $x_0 = 0$, $u(0) = 0$, and $\nabla u(0) = 0$, since any convex function $\tilde{u}$ can always be adjusted by a constant and linear term so that this is true without affecting the 2nd-order behaviour of $\tilde{u}$.

As mentioned in section 2.1, Lemma 4.1.7 is proved by looking not directly at X_r but at small neighbourhoods of 0 that are the projection of the set of centers of spheres of support to the graph of u on shrinking neighbourhoods. For each $\epsilon > 0$ a $\delta = \delta(\epsilon)$ is needed so that $B(0, \delta)$ is contained in the projection onto $\mathbb{R}^n$ of the set of centers of spheres of support to the graph of u restricted to an epsilon neighbourhood. Since the only information about u is that there is a sphere of support at 0, this is what is used to construct ϵ and δ. More specifically, the appropriate ϵ's and δ's are found by constructing a family of convex functions that are identical to u on a neighbourhood of 0, but greater and simpler outside this neighbourhood. This allows one to fully utilize the only initial information given. Using this family of simple functions and basic geometry, three key set inclusions are obtained, which essentially relate $B(0,\delta(\epsilon))$ to $X_r \cap B(0,\epsilon)$. Then using Lipschitz maps to relate these sets and by applying properties of Lipschitz functions on measure, the lower density bound is shown. This whole construction is crucial because it provides a much simpler approach to studying the possibly very complex set X_r. The following is the proof given by Slodkowski.

Proof of Lemma 4.1.7. The number $r \in (0,R)$ will be kept fixed so let $X \equiv X_r$. Define

$$Z = \{(x,u(x)) \in \mathbb{R}^{n+1} : x \in X\}.$$

It is clear that $Z \cap (\bar{\mathrm{B}}(0,d') \times \mathbb{R})$ is compact for every $d' < d$, thus $X \cap (\bar{\mathrm{B}}(0,d') \times \mathbb{R})$ is also compact, as it is the orthogonal projection $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ of Z. Since compact sets are Lebesgue measurable, the notion of lower density is applicable to both X and Z.

It is more convenient to first estimate the density of Z at 0 with respect to Hausdorff measure, and then use the properties of Lipschitz functions on measure to obtain bounds on the density of X. To accomplish this a family of convex functions, built from the initial sphere of support of radius R at 0, which modify u outside a small neighbourhood of 0 will be constructed. As mentioned above, these functions will be identical to u on a neighbourhood of 0 and very simple outside this neighbourhood. These functions will enable us to find a corresponding $\delta = \delta(\epsilon)$ neighbourhood for each ϵ so that $x \in B(0,\delta)$ implies that $x = P(c)$, where $c \in \mathbb{R}^n$ is the center of a sphere of support to $(x', u(x'))$, for some $x' \in B(0,\epsilon) \cap X_r$.

Step One. A family of convex functions is constructed which will let us find an appropriate $\delta(\epsilon)$, as explained above. For each α such that $0 < \alpha < \frac{1}{2}\arcsin(\frac{d}{R})$, define the function

$$v_\alpha : B(0, R) \to [0, \infty),$$

as follows. First, define

$$Y = \{y \in \mathbb{R}^{n+1} : |y - c| = R, (y - c, 0 - c) = 2\alpha\}, \tag{4.4}$$

where $c = (0, ..., 0, R) \in \mathbb{R}^{n+1}$ is the center of the sphere of support to u at $(0, u(0))$. Y forms a “ring ”on $S(c, R)$, and clearly the projection of Y, $P(Y)$, onto $\mathbb{R}^n$ is the

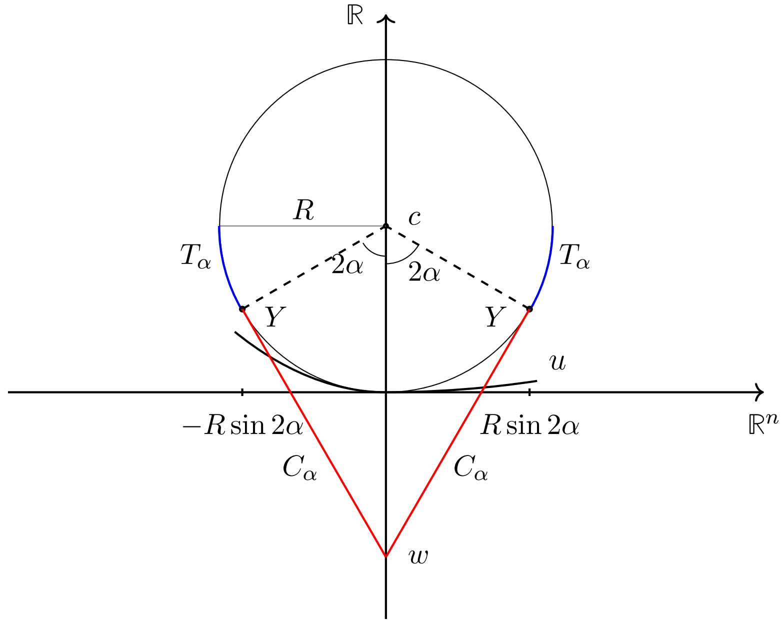

Figure 4.2: Construction of Auxiliary Convex Functions

$n - 1$ sphere of radius $R \sin 2\alpha$, centered at 0. Next, let C_α denote the union of all closed segments $\overline{\mathrm{wy}}$ with one endpoint w on the axis $0 \times \mathbb{R} \subset \mathbb{R}^{n+1}$ and tangent to the sphere $S(c, R)$ at the other endpoint y, where $y \in Y$. Note that w is independent of which $y \in Y$ that is being used. C_α is simply a finite cone with vertex w and base Y, tangent to $S(c, R)$ along Y.

Define now

$$T_\alpha = \{y \in S(c, R) : R(1 - \cos 2\alpha) \leq y_{n+1} < R\}. \tag{4.5}$$

T_α can be visualized as a "strip" of $S(c, r)$, and note that $T_\alpha \cap C_\alpha = Y$ and that $T_\alpha \cup C_\alpha$ defines a convex function $k_\alpha : B(0, R) \to R$.

For $0 < \alpha < \frac{1}{2} \arcsin(\frac{d}{R})$, define

$$v_\alpha = \begin{cases} \max(u(x), k_\alpha(x)), & |x| < R \sin 2\alpha \\ k_\alpha(x), & R \sin 2\alpha \leq |x| < R. \end{cases}$$

Note that u is only defined on $B(0, d)$ and $R \sin 2\alpha < d < R$, so that is why v_α is defined this way. It is clear that

$$v_\alpha(x) \geq u(x), \text{ for } |x| < d. \tag{4.6}$$

Observe that v_α is locally convex on the set $|x| \neq R \sin 2\alpha$ since for $|x| > R \sin 2\alpha, v_\alpha = k_\alpha(x)$, which is convex, and for $|x| < R \sin 2\alpha, v_\alpha$ is the maximum of two convex functions which is convex. If $|x| = R \sin 2\alpha$, then $(x, v_\alpha(x)) \in Y \subset S(c, r)$ Since $S(c, r)$ lies above the graph of u, so $k_\alpha|_Y > u|_Y$. Thus near $Y, v_\alpha \equiv k_\alpha$, and so v_α is locally convex in $B(0, R)$, which implies that v_α is convex.

Step Two. For any convex function the following Lipschitz map can be constructed. This will let us relate the possibly complex set, X, to the disk $B(0, \delta(\epsilon))$. Given a convex function $v : B(0, R) \to \mathbb{R}$. Let $E(v) = \{(x, t) \in \mathbb{R}^{n+1} : t > v(x)\}$ denote the strict epigraph of v, and define Z^v as the set of all $y = (x, v(x))$, where $|x| < R$, and such that for some $c' \in \mathbb{R}^{n+1}$, $B(c', r) \subset E(v)$ and $y \in S(c', r)$, where $r < R$, as defined earlier.

Note that if $y = (x, v(x)) \in Z^v$, then the graph(v) has a unique supporting hyperplane at y (since any such hyperplane is tangent to $S(c', r)$), and thus c' is uniquely determined by y.

Now consider the map $\gamma^v : Z^v \to \mathbb{R}^{n+1}$, where $\gamma^v(y) = c'$. This map is Lipschitz with constant one. To see this, let $y_1, y_2 \in Z^v$ and $c_i' = \gamma^v(y_i), i = 1, 2$. The set $E(v)$ is convex (by definition since v is convex), and so it contains $W := \mathrm{co}(B(c_1, r) \cup B(c_1, r))$, where co() denotes the convex hull. In particular, $W\cap$ graph(v)=$\emptyset$. Since $y_i \in S(c_i, r) \cap \mathrm{graph}(v)$, $y_i \in S(c_i', r) \setminus W, i = 1, 2$. Thus, y_1 and y_2 do not belong to, and are separated by, the open region between two hyperplanes which are orthogonal to the segment $\overline{c_1' c_2'}$ and pass through its ends. Therefore $|c_1' - c_2'| \le |y_1 - y_2|$. The importance of this map will be seen below, where combined with u and the projection map P it allows the set of interest in $\mathbb{R}^n$ to be related to a small disk.

Step Three. Three key set inclusions are established. Along with step two this will allow on small neighborhoods the measure of X to be bounded from below by the volume of small $n-$ balls. Using the notation above, let Z^α and γ^α denote the set Z^v and map γ^v, respectively, for $v = v_\alpha$, where $0 < \alpha < \frac{1}{2}\arcsin(\frac{d}{R})$.

Consider the set

$$U_\alpha = \mathrm{graph}(v_\alpha) \setminus (C_\alpha \cup T_\alpha). \tag{4.7}$$

Note that this is a subset of the graph of u. For $\alpha \in (0, \frac{1}{2}\arcsin(\frac{d}{R}))$, we have the following three inclusions:

$$P(U_\alpha) \subset B(0, R\sin 2\alpha) \tag{4.8}$$

$$Z^\alpha \cap U_\alpha \subset Z \cap U_\alpha \tag{4.9}$$

$$B_N(0,\delta) \subset P\gamma^\alpha(Z^\alpha \cap U_\alpha), \text{where } \delta = (R-r)\tan\alpha. \tag{4.10}$$

The first inclusion follows directly from the definition of U_α: $|x| \geq R\sin 2\alpha \Rightarrow v_\alpha(x) \in T_\alpha$.

By (5), $Z^\alpha \cap \operatorname{graph}(u) \subset Z$. To see this, let $z \in Z^\alpha$. Thus we have a $c' \in \mathbb{R}^{n+1}$ such that $B(c', r) \subset E(v_\alpha)$ and $z \in S(c', r)$. So there is a sphere of radius r supporting the graph of v_α from above at z. If $z \in \operatorname{graph}(u)$, then we must have $z \in Z$: $B(c', r) \subset E(v_\alpha)$ and $v_\alpha(x) \geq u(x)$ give us that $B(c', r) \cap \operatorname{graph}(u) = \emptyset$ and $c'_{n+1} > u(Pc')$, which together with $z \in S(c', r)$ imply that $z \in Z$, by definition. Since $U_\alpha \subset \operatorname{graph}(u), Z^\alpha \cap U_\alpha \subset Z^\alpha \cap \operatorname{graph}(u) \subset Z$. And of course $Z^\alpha \cap U_\alpha \subset U_\alpha$, so together we have $Z^\alpha \cap U_\alpha \subset Z \cap U_\alpha$, which gives us the second inclusion.

The third inclusion is the critical aforementioned relation between the set of points with spheres of support and a disk in $\mathbb{R}^n$. (Below we will take $\epsilon = R\sin\alpha$ and $\delta = (R-r)\tan\alpha$). To obtain this inclusion we proceed as follows. Let $x \in \mathbb{R}^n$, be such that $|x| < R - r$, and consider the set

$$\{c' \in \{x\} \times \mathbb{R} : B(c', r) \subset E(v_\alpha)\}. \tag{4.11}$$

This set is a non-empty, closed half-line. To see this, consider lowering the sphere $S((x, c'_{n+1}), r)$ in $\mathbb{R}^{n+1}$ onto the graph of v_α, by continuously decreasing the last coordinate. Because the radius of this sphere is r and $|x| < R-r$, this sphere comes

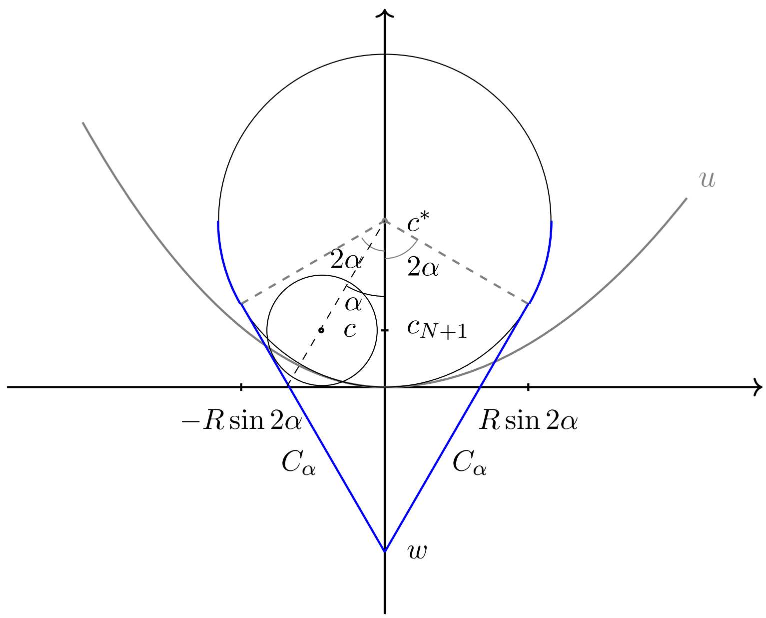

Figure 4.3: Closest Supporting Spheres to Origin

down on a closed subset of the graph of v_α. Once contact is made with the graph of v_α we stop, and the corresponding value of (x, c'_{n+1}) is our closed endpoint. Let $c' \in \mathbb{R}^{n+1}$ be this endpoint and $y \in S(c', r) \cap \mathrm{graph}(v_\alpha)$ (note that y may not be unique). Then $c' = \gamma^\alpha(y)$ and $x = P\gamma^\alpha(y)$, and so

$$B_N(0, R-r) \subset P\gamma^\alpha(Z^\alpha). \tag{4.12}$$

Now $Z^\alpha \backslash (C_\alpha \cup T_\alpha) \subset \mathrm{graph}(v_\alpha) \backslash (C_\alpha \cup T_\alpha) = U_\alpha$, so clearly $Z^\alpha \backslash (C_\alpha \cup T_\alpha) \subset Z^\alpha \cap U_\alpha$. Therefore,

$$P\gamma^\alpha(Z^\alpha) \setminus P\gamma^\alpha(Z^\alpha \cap (C_\alpha \cup T_\alpha)) \subset P\gamma^\alpha(Z^\alpha \cap U_\alpha). \tag{4.13}$$

This relation and (12) will give us our third inclusion (4.10), once we show that

$$P\gamma^\alpha(Z^\alpha \cap (C_\alpha \cup T_\alpha)) \cap B_N(0, \delta) = \emptyset. \tag{4.14}$$

Consider the family of all spheres $S(c', r)$ which support $C_\alpha \backslash Y$ from above and are contained in the upper half space $y_{n+1} \geq 0$. Clearly the smallest value of $|P(c')|$

is attained when the sphere $S(c', r)$ is tangent to both C_α and $\{y_{n+1} = 0\}$ (see Fig. 3). It is not difficult to see that in this case $(c' - c, 0 - c) = \alpha$, where c here is the center of the initial sphere of support. This gives us $|P(c')| = (|c| - c'_{N+1}) \tan \alpha = (R - r) \tan \alpha = \delta$, which implies

$$P\gamma^\alpha(Z^\alpha \cap C_\alpha) \cap B_N(0, \delta) = \emptyset. \tag{4.15}$$

Now when $S(c', r)$ supports $T_\alpha \setminus Y$ from the above at some point y, the segment $\overline{c', y}$ is normal to $S(c, R)$ and $y_{N+1} \geq R(1 - \cos 2\alpha) \geq \delta$. Thus $(c' - c, 0 - c) \geq 2\alpha$ and, as above, $|P(c)| \geq (R - r) \tan 2\alpha \geq \delta$ (note $0 \leq \alpha \leq \frac{\pi}{4}$). This gives

$$P\gamma^\alpha(Z^\alpha \cap T_\alpha) \cap B_N(0, \delta) = \emptyset. \tag{4.16}$$

Combining (4.15) and (4.16) we have (4.14), which gives the third inclusion.

Step Four. Estimate of the density of X. The above inclusions and the effect of Lipschitz maps on measure, will be enough to estimate the density of $X = P(Z)$. Recall that $Z = \{(x, u(x)) \in \mathbb{R}^{N+1} | x \in X\}$, where X is the set of points in $B(0, d) \subset \mathbb{R}^N$ such that there exists a sphere of radius r supporting the graph of u from above at $(x, u(x))$.

Using a few theorems from Rockafellar [27], it can be shown that the map $\varphi : P(U_\alpha) \to U_\alpha$, where $\varphi(x) = (x, u(x))$ is Lipschitz with constant $(1 + g_\alpha^2)^{\frac{1}{2}}$, where $g_\alpha = \sup\{|\nabla u| : |x| < R \sin 2\alpha\}$. More specifically, by [27, Theorem 10.4], u is Lipschitz, and by [27, Theorems 24.7, 25.5, and 25.6] g_α is a Lipschitz bound for $u|_{B(0, R \sin 2\alpha)})$. A simple Pythagorean argument then shows $(1 + g_\alpha^2)^{\frac{1}{2}}$ is a Lipschitz bound for φ. Notice that φ maps $X \cap P(U_\alpha) = P(Z \cap U_\alpha)$ onto $Z \cap U_\alpha$.

A basic theorem regarding the effect of Lipschitz maps on Hausdorff measures (see Rogers [**?**, Theorem 2.29]), along with our first inclusion from above (7), leads to:

$$\begin{aligned} H^n(Z \cap U_\alpha) \leq & (1+g_\alpha^2)^{\frac{n}{2}} m_n(X \cap P(U_\alpha)) \\ \leq & (1+g_\alpha^2)^{\frac{n}{2}} m_n(X \cap B(0,\varepsilon)), \quad \varepsilon = R\sin 2\alpha, \end{aligned}$$

where again H^n and m_n denote the Hausdorff and Lebesgues measure on $\mathbb{R}^n$, respectively. Furthermore

$$\begin{aligned} m_n(B(0,\delta)) \leq & m_n(P\gamma^\alpha(Z^\alpha \cap U_\alpha)) && \text{by (9)} \\ \leq & H^n(Z^\alpha \cap U_\alpha) && P\gamma^\alpha \text{ is Lipshitz with constant } \leq 1 \\ \leq & H^n(Z \cap U_\alpha) && \text{by (8).} \end{aligned}$$

Finally, combining these inequalities one obtains

$$\begin{aligned} \frac{m_n(X \cap B(0,\varepsilon))}{m_n(B(0,\varepsilon))} &\geq (1+g_\alpha^2)^{\frac{-n}{2}} \frac{m_n(B(0,\delta))}{m_n(B(0,\varepsilon))} \\ &= (1+g_\alpha^2)^{\frac{-n}{2}} \left(\frac{(R-r)\tan\alpha}{R\sin 2\alpha}\right)^n \\ &= (1+g_\alpha^2)^{\frac{-n}{2}} \left(\frac{R-r}{2R}\right)^n \cos^{-2n}\alpha, \end{aligned}$$

where the volume of an n-ball of radius r is $\dfrac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}$ in the first equality, and Γ denotes the gamma function. Thus,

$$\liminf_{\varepsilon\to 0} \frac{m_n(X \cap B(0,\varepsilon))}{m_n(B(0,\varepsilon))} \geq \liminf_{\varepsilon\to 0}(1+g_\alpha^2)^{\frac{-n}{2}} \left(\frac{R-r}{2R}\right)^n \cos^{-2n}\alpha.$$

Now since $\varepsilon = R\sin 2\alpha$ and $0 < \alpha < \frac{\pi}{4}$, as $\varepsilon \to 0$, $\alpha \to 0$. And as the gradient of a convex function is continuous (Theorem 25.5, [1]), $g_\alpha \to 0$ as well since $\nabla u(0) = 0$. Therefore the lower density of X at 0 is not less than $\left(\dfrac{R-r}{2R}\right)^N$. □

4.2.4 Proof of Theorem 4.1.3

Lemma 4.1.7 and Proposition 4.1.6 now combine nicely to give us Theorem 4.1.3.

Proof of Theorem 4.1.3. Let $X'_k = \{x \in \text{dom}\ (u) : K(u,x) < k\}$. Note that $K(u,x)$ is of first Baire class, as it is the pointwise limit of a sequence of continuous functions. Since the inverse image of an open set under a Baire function is Borel [8, 4.1], we have that X'_k is Borel.

Now without loss of generality, let $x_0 = 0, u(x_0) = 0, \nabla u(x_0) = 0$. Note that by the convexity of u this implies $u \geq 0$. Set $k_0 = K(u,x_0) = K(u,0)$, and let $k > k_0$ be fixed and take K such that $k > K > k_0$.

Set $R = \dfrac{1}{K}$ and note that $R - (R^2 - |x|)^{\frac{1}{2}} \geq \dfrac{1}{2R}|x|^2 = \dfrac{K}{2}|x|^2$, $\forall x$ such that $|x| < R$. This follows immediately by contradiction. The left-hand side of this inequality is the last component of the point $(x,t) \in \mathbb{R}^n$, where $x \in \mathbb{R}^n$, on the $(n+1)$-dimensional sphere of radius R centered $(0, ..., 0, R) \in \mathbb{R}^{n+1}$ (i.e the value of $d(x)$, where d is the lower hemisphere function defined in the proof of the proposition, see Figure **??**).

Since $K > K(u,0)$, by Proposition 4.1.6 (ii) there exists $d > 0$ such that

$$u(0+h) - u(0) - \langle \nabla u(0), h\rangle \ \leq \frac{1}{2}K|h|^2 \text{ for every } |h| < d.$$

So

$$u(h) \leq \frac{1}{2}K|h|^2 \text{ for every } |h| < d.$$

Thus the sphere $S(c,R)$, where $c = (0, ...0, R) \in \mathbb{R}^{n+1}$, supports the graph of $u|_{B(0,d)}$ form above at $0 \in \mathbb{R}^{n+1}$, and Lemma 4.1.7 can be applied to the function $u|_{B(0,d)}$.

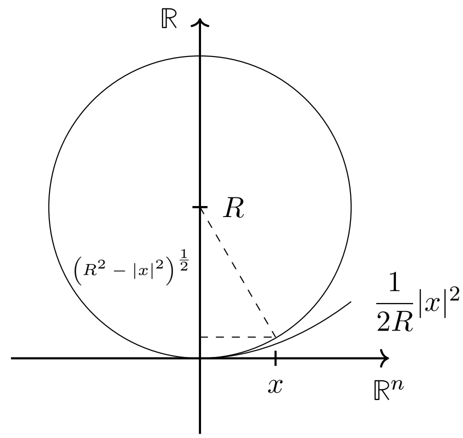

Figure 4.4: Tangent sphere

Let r, such that $\frac{1}{k} < r < R$, be arbitrary, and let $X = X_r$ and $Z = Z_r$ be defined as in Lemma 4.1.7. By Proposition 4.1.6 (iii), $\forall x \in X$

$$K(u,x) \leq \frac{(1+g^2)^{\frac{3}{2}}}{r}, \qquad \text{where } g = |\nabla u(x)|.$$

Set

$$g_\varepsilon = \sup\{|\nabla u(x)| : |x| < \varepsilon\}.$$

Then clearly

$$K(u,x) \leq \frac{(1+g_\varepsilon^2)^{\frac{3}{2}}}{r} \quad \forall x \in X \cap B(0,\varepsilon).$$

By the continuity of the gradient function, $\lim_{\varepsilon\to 0} g_\varepsilon = |\nabla u(0)| = 0$. Thus since $\dfrac{1}{r} < k$, there exists ε', where $0 < \varepsilon' < d$, such that

$$\frac{(1+g_\varepsilon^2)^{\frac{3}{2}}}{r} < k, \qquad \text{for } 0 < \varepsilon < \varepsilon',$$

and so

$$(B(0,\varepsilon) \cap X) \subset (B(0,\varepsilon) \cap X_k'), \quad \text{for } 0 < \varepsilon < \varepsilon'.$$

If $x \in X$ then there exists a supporting sphere of radius r at $(x, u(x))$, and if $x \in B(0, \varepsilon)$, where $\varepsilon < \varepsilon'$, then $K(u, x) < k$.

It follows by Lemma 4.1.7 that

$$\begin{aligned}\liminf_{\varepsilon \to 0} \frac{m_n(X'_k \cap B(0,\varepsilon))}{m_n(B(0,\varepsilon))} &\geq \liminf_{\varepsilon \to 0} \frac{m_n(X \cap B(0,\varepsilon))}{m_n(B(0,\varepsilon))} \\ &\geq \left(\frac{R-r}{2R}\right)^n.\end{aligned}$$

Now recall that $R = \dfrac{1}{K}$ was chosen arbitrarily so that it satisfied the inequality $\dfrac{1}{k} < \dfrac{1}{K} < \dfrac{1}{k_0}$, where k and k_0 are fixed. Similarly, r was chosen arbitrarily so that $\dfrac{1}{k} < r < \dfrac{1}{K}$. Thus we can choose $R = \dfrac{1}{K}$ and r arbitrarily close to $\dfrac{1}{k_0}$ and $\dfrac{1}{k}$, respectively, giving us the desired bound $\left(\dfrac{k-k_0}{2k}\right)^n$. □

4.3 Dual Perspective

4.3.1 Background

Since u is convex near x_0, it is natural to study this quantity $K(u, x_0)$ from the dual perspective as well. Let $Cvx(\mathbb{R}^n)$ denote the space of convex, lower semi-continuous functions on $\mathbb{R}^n$. Given a function $u \in Cvx(\mathbb{R}^n)$, one can apply the Legendre–Fenchel transform $L : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ of u to obtain its conjugate or dual function u^*, where

$$u^* \equiv Lu(s) = \sup_x (\langle s, x\rangle - u(x)).$$

L is an order-reversing, involutive transform on $Cvx(\mathbb{R}^n)$, and for sufficiently nice convex functions (differentiable, strictly convex, and 1-coercive), u^* is given by

$$u^*(s) = \langle s, (\nabla u)^{-1}(s)\rangle - u((\nabla u)^{-1}(s)).$$

The conjugate function u^* can be viewed as a reparametrization of the original function u in terms of its tangents using the duality between points and hyperplanes. More specifically, given a vector in $\mathbb{R}^n$, there is an associated family of hyperplanes with that gradient. u^* distinguishes the one that supports the epigraph of u by specifying a point on that plane.

For convex functions defined only in a neighbourhood it is standard to extend the function to all of $\mathbb{R}^n$ by setting it equal $+\infty$ outside that neighbourhood. In our case, we are given u convex near x_0, so we extend it in this manner, if necessary. Clearly this does not affect $K(u, x_0)$, which is a purely local property. Recall the following basic definitions:

Definition 4.3.1. *The differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, x' \in$ $\mathbb{R}^n$*

$$f(x') \geq f(x) + \langle \nabla f(x), (x' - x) \rangle,$$

and strictly convex if the inequality is strict for $x \neq x'$.

Definition 4.3.2. *The differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex with modulus c if and only if for all $(x, x') \in \mathbb{R}^n \times \mathbb{R}^n$,*

$$f(x') \geq f(x) + \langle \nabla f(x), (x' - x) \rangle + \frac{1}{2} c |x' - x|^2.$$

When f is not differentiable a lot of analysis can still be done using the calculus of subdifferentials.

Definition 4.3.3. *3.3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. The subdifferential of f, denoted ∂f, is a set function, where $\partial f(x) = \{ s \in \mathbb{R}^n : f(y) \geq f(x) + \langle s, y - x \rangle \ \forall y \in \mathbb{R}^n \}$.*

Under the Legendre transform, differentiability of u corresponds to convexity or monotonicity of u^*. Recall from Proposition 1.8, two properties that transform especially well are (i) $u \in C^1$ if and only if u^* is strictly convex, and (ii) $u \in C^{1,1}$, where ∇u has Lipschitz constant c if and only if u is strongly convex with modulus $\frac{1}{c}$.

4.3.2 Quadratic convexity

In this section we look at how a bound on $K(u, x_0)$ or equivalently a sphere of support to the graph of u at $(x_0, u(x_0))$ transforms to a property of u^*. More specifically, since K or a sphere of support is a bound on a generalized second-order derivative of u, how does this translate to information about the convexity of u^*? We should expect a more localized property then in Proposition 1.8, as we only have information at x_0. Further, we are not assuming any regularity beyond differentiable at x_0.

Now, strong convexity may also defined in terms of quadratic functions: u is strongly convex with modulus m if $u - \frac{1}{2}m|x|^2$ is convex. Similarly, quasi-convexity, is defined via quadratics: u is λ- quasi-convex if $u + \frac{1}{2}\lambda|x|^2$ is convex.

Let $u : \mathbb{R}^n \to \mathbb{R}$ be convex with $K(u, x_0) = k_0 < \infty$. By the definition of $K(u, x)$, for any $k > k_0$ there exists $\epsilon > 0$ such that

$$u(x_0 + h) - u(x_0) - \langle \nabla u(x_0), h \rangle \leq \frac{1}{2}k|h|^2, \text{ for all } |h| < \epsilon.$$

This motivates the following definition.

Definition 4.3.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Then f is quadratically (resp. sub-quadratically) convex at x_0 with modulus $m > 0$ if there exists $\epsilon > 0$ and a quadratic function $Q : \mathbb{R}^n \to \mathbb{R}$ with $\nabla^2 Q = mI$ such that*

$$f(x_0) = Q(x_0) \text{ and } f(x) \geq Q(x), \qquad \forall x \in B(x_0, \epsilon)$$

resp.

$$f(x_0) = Q(x_0) \text{ and } f(x) \leq Q(x), \qquad \forall x \in B(x_0, \epsilon).$$

Example 4.3.5. *$f(x) = |x|^{4/3}$ is quadratically convex at 0, but not sub-quadratically convex at 0. Note also that $K(f, 0) = +\infty$ and it does not have a sphere of support at 0.*

Example 4.3.6. *More generally, consider any function of the form $f(x) = A|x|^k$, at $x = 0$. If $0 < k < 1$, f is not convex. If $k = 1$, f is quadratically convex at 0, but not sub-quadratically convex. If $1 < k < 2$ then f is strictly convex and quadratically convex but not sub-quadratically convex. If $k = 2$, f is both quadratically convex and sub-quadratically convex. If $k > 2$, f is sub-quadratically convex but not quadratically convex.*

If f is of the form $f = \frac{|x|^k}{k}$, then $f^* = \frac{|y|^q}{q}$, where $\frac{1}{k} + \frac{1}{q} = 1$. So, in general, given that the Legendre-Fenchel transform is order-reversing and quadratics are transformed into quadratics, it follows that if f is quadratically convex, f^* is sub-quadratically convex. For a convex C^2 function f, if $\nabla^2 f(x_0)$ is positive definite then f is both quadratically and sub-quadratically convex at x_0.

Proof of Theorem 4.1.9. Suppose $K(u, x_0) = k_0 < \infty$. As stated above, by definition of $K(u,x_0)$, for any $k > k_0$, there exists $\epsilon > 0$ such that u satisfies

$$u(x) - u(x_0) - \langle \nabla u(x_0), x - x_0 \rangle \leq \frac{1}{2}k|x - x_0|^2,$$

for all $x \in B(x_0, \epsilon)$. Thus, on this neighbourhood of x_0

$$u(x) \leq u(x_0) + \langle \nabla u(x_0), x - x_0 \rangle + \frac{1}{2}k|x - x_0|^2.$$

By assumption u is convex, and $k > k_0 \geq 0$, so the right-hand side is also convex. Taking the Legendre transform gives

$$u^*(y) \geq \langle \nabla u(x_0), x_0 \rangle - u(x_0) + \langle x_0, y - \nabla u(x_0) \rangle + \frac{1}{2}k\left|\frac{y - \nabla u(x_0)}{k}\right|^2.$$

Now u^* may not be differentiable at $\nabla u(x_0)$, however $\nabla u(x_0) \in \partial u(x_0)$ if and only if $x_0 \in \partial u^*(\nabla u(x_0))$, which is equivalent to $u^*(\nabla u(x_0)) = \langle \nabla u(x_0), x_0 \rangle - u(x_0)$. So the above inequality simplifies to

$$u^*(y) \geq u^*(\nabla u(x_0)) + \langle x_0, y - \nabla u(x_0) \rangle + \frac{1}{2k}|y - \nabla u(x_0)|^2.$$

Note that there is equality at $y_0 = \nabla u(x_0)$ and the Hessian of the right-hand side is $\frac{1}{k}I$ so u^* is quadratically convex with modulus $\frac{1}{k}$.

On the other hand, if u^* is quadratically convex at $y_0 = \nabla u(x_0)$ with modulus $\frac{1}{k}$ then u will be sub-quadratically convex with modulus k at x_0, and it follows that $K(u, x_0) \leq k$. □

In the above proof we do not need to worry about $\partial u(B(x_0, \epsilon))$ being degenerate (for example if u is locally a hyperplane at x_0) because in that case $u^*(y)$ will then be $+\infty$ away from $\nabla u(x_0)$ so clearly the inequality will hold on some neighbourhood.

Our goal now is to obtain the nice bound on $K(u, x)$ in Proposition 4.1.6 using the dual function, given a sphere of support to the graph of u at $(x, u(x))$. The following elementary lemma, which we state without proof, will enable us to reduce arguments on $\mathbb{R}^n$ to ones on $\mathbb{R}$.

Lemma 4.3.7. *Let S_r be an n-sphere with radius r in $\mathbb{R}^{n+1}$, centered at $(0, ..., 0, r)$, and let $d : \mathbb{R}^n \to \mathbb{R}$ be the function defined by the lower hemisphere, i.e., for $z \in B_n(0, r)$, $d(z) = r - \sqrt{r^2 - |z|^2}$. Then for any $x \in B_n(0, r)$ and $v \in \mathbb{R}^n$, $|v| = 1$, the graph of $\psi : I \subset \mathbb{R} \to \mathbb{R}^{n+1}$ defined by $\psi(t) = d(x + tv)$ is a lower semi-circle in $\mathbb{R}^{n+1}$ of radius $\leq r$, where $I = (-\epsilon, \epsilon')$ is of maximal length.*

Proposition 4.3.8. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2 and convex and suppose there exists a sphere of support to the graph of f at $(x_0, f(x_0))$ of radius r. Then*

$$K(f, x_0) \leq \frac{(1 + \nabla f|_{x_0}^2)^{\frac{3}{2}}}{r}.$$

Proof. In this case $K(f, x_0)$ is the largest eigenvalue λ_{max} of $\nabla^2 f(x_0)$. If λ_{max}=0 then the bound on $K(f, x_0)$ is trivial, so let $\lambda_{max} > 0$. Since f is convex $\nabla^2 f(x_0)$ is symmetric positive semi-definite, so there exists an orthonormal basis of eigenvectors. By Lemma 4.3.7, along each of these vectors the sphere of support can be considered simply a circle of support, the second derivative the corresponding eigenvalue, and the first derivative the inner product with $\nabla f(x_0)$. The largest eigenvalue will be achieved in the direction of the gradient (or equivalently away from the center) since this places x_0 as far as possible from the center of a semi-circle, where the gradient grows fastest. So without loss of generality assume that f is a function on $\mathbb{R}$.

Let $S((c,t),r)$ be a sphere of support of radius r, to the graph of f at x_0. Let d be the function defined by the lower hemisphere of this sphere, i.e.

$$\begin{aligned} d(x) =& t - \sqrt{r^2 - (x-c)^2}, \;\; x \in [x-c, x+c] \\ d(x) =& \infty, \;\text{ else.} \end{aligned}$$

Clearly d is convex and $d \geq f$, by definition of a supporting sphere. By properties of convex functions and their conjugates the following relations hold:

$$f(x_0) = d(x_0) \qquad \nabla f(x_0) = \nabla d(x_0) \qquad f^* \geq d^* \qquad \nabla f^*(x_0) = \nabla d^*(x_0) = y_0.$$

Now d^* can be computed directly using the Legendre transforms of common functions. First rewriting d:

$$\begin{aligned} d(x) =& t - \sqrt{r^2 - (x-c)^2} \\ =& t - r\sqrt{1 - \left(\frac{x}{r} - \frac{c}{r}\right)^2}, \end{aligned}$$

and then applying the following well-known conjugate pairs:

$$\begin{aligned} h(x) &= -\sqrt{1-x^2} & h^*(y) &= \sqrt{1+y^2} \\ g(x) &= \alpha + \beta x + \gamma u(\lambda x + \delta) & g^*(x) &= -\alpha - \delta \frac{y-\beta}{\lambda} + \gamma u^*(\frac{y-\beta}{\gamma\lambda}). \end{aligned}$$

This gives

$$\begin{aligned} d^*(y) =& -t + cy + r\sqrt{1+y^2} \\ \nabla d^*(y) =& c + \frac{ry}{\sqrt{1+y^2}} \\ \nabla^2 d^*(y) =& \frac{r}{(1+y^2)^{\frac{3}{2}}} \end{aligned}$$

The following relationship exists between the Hessians of dual functions

$$\nabla^2 f^*(y_0) = \nabla^2 f(x_0)^{-1} \qquad \text{where } y_0 = \nabla f(x_0).$$

Since $\nabla^2 f(x_0) > 0$ and f is C^2 there exists an open interval I on which $\nabla^2 f > 0$, and a corresponding interval I^* on which $\nabla^2 f^* > 0$. And since $f^* \geq d^*$, $f^*(x_0) = d^*(x_0)$, $\nabla f^*(y_0) = \nabla d^*(y_0)$, by basic calculus

$$\begin{aligned}\nabla^2 f^*(y_0) \geq \nabla^2 d^*(y_0) =& \frac{r}{(1+y_0^2)^{\frac{3}{2}}}\\ =& \frac{r}{(1+\nabla f(x_0)^2)^{\frac{3}{2}}}\end{aligned}$$

Therefore,

$$K(f, x_0) = \nabla^2 f(x_0) = \frac{1}{\nabla^2 f^*(y_0)} \leq \frac{(1+\nabla f(x_0)^2)^{\frac{3}{2}}}{r}.$$

□

The more general case, where f is not assumed to be C^2, will use Proposition 4.3.8 and quadratic convexity of the dual.

Proposition 4.3.9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex with a sphere of support at x_0 of radius r. Then $K(f, x_0) \leq \dfrac{(1+\nabla f|_{x_0}^2)^{\frac{3}{2}}}{r}$.*

Proof. Let d be the lower hemisphere function. Then $d(x_0) = f(x_0)$, and

$$d \geq f \Rightarrow f^* \geq d^*.$$

If $y_0 = \nabla f(x_0)$ (which exists since there is a sphere of support) then

$$d^*(y_0) = f^*(y_0) \text{ and } \nabla d^*(y_0) \in \partial f^*(y_0).$$

From Proposition 4.3.8 the smallest eigenvalue of $\nabla^2 d^*(y_0)$ is equal to $\frac{r}{(1+|y_0|^2)^{\frac{3}{2}}}$, so for any $m < \frac{r}{(1+|y_0|^2)^{\frac{3}{2}}}$ there exists a neighbourhood U of x_0 such that

$$f^*(y) \geq d^*(y) \geq d^*(y_0) + \langle \nabla d^*(y_0), y - y_0 \rangle + \frac{1}{2} m |y - y_0|^2.$$

Thus, f^* is quadratically convex with modulus m.

It follows that $f = (f^*)^*$ is sub-quadratically convex at x_0 with modulus $\frac{1}{m}$. Let Q_m be a satisfying quadratic. This implies that

$$K(f, x_0) \leq K(Q_m, x_0) = \frac{1}{m},$$

and since this holds for any $m < \frac{r}{(1+|y_0|^2)^{\frac{3}{2}}}$,

$$K(f, x_0) \leq \frac{(1 + |y_0|^2)^{\frac{3}{2}}}{r} = \frac{(1 + |\nabla f(x_0)|^2)^{\frac{3}{2}}}{r}.$$

$\square$

4.4 Appendix

4.4.1 Lipschitz gradient

Here we show that the generalized derivative $K(f, x)$ retains the following standard property regarding the derivative of a Lipschitz continuous function.

Proposition 4.4.1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $C^{1,1}$ (i.e f is differentiable and has Lipschitz gradient), with Lipschitz constant L. Then $K(f, x) \leq L$ for all x.*

Proof. Let $x_0 \in \mathbb{R}^n$.

$$K(f, x_0) := \limsup_{\epsilon \to 0} 2\epsilon^{-2} \max \{ f(x_0 + \epsilon h) - f(x_0) - \epsilon \langle \nabla f(x_0), h \rangle : |h| = 1 \},$$

which can be can written as

$$K(f,x_0) = \limsup_{\epsilon\to 0}\ \max\ \left\{2\frac{f(x_0+\epsilon h)-f(x_0)-\epsilon\langle\nabla f(x_0),h\rangle}{\epsilon^2} : |h|=1\right\}.$$

Differentiability lets us use the Cauchy mean value theorem. Let $\phi_1(\epsilon) = f(x_0 + \epsilon h) - \epsilon\langle\nabla f(x_0), h\rangle$, and $\phi_2(\epsilon) = \epsilon^2$. Note that

$$2\frac{f(x_0+\epsilon h)-f(x_0)-\epsilon\langle\nabla f(x_0),h\rangle}{\epsilon^2} = 2\frac{\phi_1(\epsilon)-\phi_1(0)}{\phi_2(\epsilon)-\phi_2(0)}.$$

Thus, there exists $\gamma \in (0,\epsilon)$ such that

$$\begin{aligned}2\frac{\phi_1(\epsilon)-\phi_1(0)}{\phi_2(\epsilon)-\phi_2(0)} = 2\frac{\phi_1'(\gamma)}{\phi_2'(\gamma)} &= \frac{\langle\nabla f(x_0+\gamma h),h\rangle - \langle\nabla f(x_0),h\rangle}{\gamma}\\ &= \frac{\langle\nabla f(x_0+\gamma h)-\nabla f(x_0),h\rangle}{\gamma}\\ &\leq \frac{|\nabla f(x_0+\gamma h)-\nabla f(x_0)|}{\gamma} \leq L\end{aligned}$$

Therefore $K(f,x_0) \leq L$, and thus $\frac{1}{K(f,x_0)}$ bounds the modulus of convexity of f^*, for any x_0. □

4.4.2 Example of a non $C^{1,1}$ function with a sphere of support

Example 4.4.2. *It may seem that since a bound on $K(u,x)$ implies a sphere of support to the graph of u at $(x,u(x))$, that this in turn implies some kind Lipschitz continuity of the gradient in a small neighbourhood of x. Here we construct an example of a strictly convex function f that is C^1 and twice differentiable with $K(f,0) < \infty$, but with gradient not Lipschitz in any neighbourhood of 0, to show this is not the case. Let $f : [-1,1] \to \mathbb{R}$ be given by $f(0) = 0$, and for $x \geq 0$*

$$f'(x) = \int_0^x \gamma(t)dt, \quad \textit{where } \gamma(t) := n+4 \textit{ on } I_n \textit{ and } 0 \textit{ otherwise,}$$

with $I_n = \frac{1}{(n+4)^2}[1 - \frac{1}{(n+4)^2},\ 1]$. *Define* $f'(-x) := -f'(x)$.

Then f' *is clearly increasing and so* f *is convex. And for* $x_n = \frac{1}{(n+4)^2}$,

$$f'(x_n) = \int_0^{x_1} \gamma(t)\, dt = \sum_{k\geq n} \frac{1}{(k+4)^3} \leq \int_{n+3}^{\infty} \frac{dt}{t^3} = \frac{1}{2(n+3)^2} < \frac{1}{(n+4)^2} = x_n.$$

So we have $f'(x) \leq x$ *for all* $x \in [0,1]$ *and* $f'(x) \geq x$ *for all* $x \in [-1,0]$. *Since* $d'(x) \geq x$ *for all* $x \in [0,1]$ *and* $d'(x) \leq x$ *for all* $x \in [-1,0]$, *it follows that the graph of* d, *and thus the unit circle centered at* $(0,1)$, *is always at or above the graph of* f, *with* $f(0) = d(0)$. *Therefore,* f *has a sphere of support at* $x_0 = 0$.

However, there exist sequences $\{x_i\}, \{x_j\}$ *such that*

$$\frac{f'(x_i) - f'(x_j)}{x_i - x_j}$$

blows up: Taking x_i *and* x_j *as the endpoints of* I_n,

$$\frac{f'(x_i) - f'(x_j)}{x_i - x_j} = \frac{1}{x_i - x_j}\left(\int_0^{x_i} \gamma(t)dt - \int_0^{x_j} \gamma(t)dt\right) = (n+4)^4 \int_{x_j}^{x_i} n+4dt = n+4.$$

We can make f *strictly convex by adding an* x^m *term, which does not affect any of the above analysis. The above example can be adjusted to show that* f' *is not* α*-Holder continuous for any* α.

4.4.3 Osculating and locally supporting spheres

Here we extend the concept of an osculating circle to a plane curve to that of an "osculating sphere" to the graph of a function in higher dimensions. The bound on the "largest eigenvalue" $K(u, x)$ can be seen as a generalization of the relationship between the second derivative of a C^2 plane curve u and the radius of its osculating circle:

Let $u : \mathbb{R} \to \mathbb{R}$ be C^2. Provided $u'' \neq 0$, the radius of curvature at x is defined as

$$r_{u,x} := \frac{1}{\kappa} = \frac{(1+u'^2)^{\frac{3}{2}}}{u''},$$

where κ is the curvature of u at x, and the right-hand side is the standard formula for computing the curvature of a planar curve [2, §8]. Thus,

$$u'' = \frac{(1+u'^2)^{3/2}}{r}.$$

Definition 4.4.3. *The osculating circle, or circle of curvature, to a planar curve C at p is the circle that touches C (on the concave side) at p and whose radius is the radius of curvature of C at p.*

We extend this to the graphs of C^2 convex functions in higher dimensions by

Definition 4.4.4. *For a convex function $u : \mathbb{R}^n \to \mathbb{R}$ let the osculating sphere to the graph of u at x be the $n-$sphere tangent to the graph of u at x the with radius equal to that of $\frac{1}{\lambda_{max}}$.*

It is easy to show that any tangent sphere at $(x, u(x))$ with radius less than the osculating sphere at that point is a (local) sphere of support. And any tangent sphere at $(x, u(x))$ with radius greater than the osculating sphere cannot be a (local) sphere of support.

4.4.4 Spheres of support to a function and its dual

Given a convex function u with a sphere of support at $(x_0, u(x_0))$, the conjugate function u^* will not necessarily have a sphere of support at the corresponding point $(\nabla u(x_0), u^*(\nabla u(x_0)))$. For example take $u = \frac{1}{4}|x|^4$ and $u^* = \frac{3}{4}|x|^{\frac{4}{3}}$. However, for more regular and sufficiently convex functions (e.g. C^2 and locally strongly convex), we will have a sphere of support (locally) to both graphs at corresponding points, and the order-reversing property of L provide a simple inequality relating the radii of these spheres. We state this without proof.

Proposition 4.4.5. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be strongly convex and C^2 near x_0, and suppose u has a sphere of support of radius r_{x_0}. If r_{y_0} is the radius of a sphere of support to u^* at $y_0 = \nabla u(x_0)$, then*

$$r_{y_0} \leq \frac{(1 + |x|^2)^{\frac{3}{2}} \, (1 + |\nabla u(x_0)|^2)^{\frac{3}{2}}}{r_{x_0}}.$$

Chapter 5: The almost Lagrangian mean curvature flow of symmetric spheres in Milnor fibers

5.1 Introduction

In Sections 5.2 and 5.3, we introduce the Lagrangian mean curvature flow, highlighting its special features in the Calabi–Yau setting, and state the Thomas–Yau conjecture.

In Section 5.4, we then discuss a class of algebraic varieties, called Milnor fibers. These spaces admit an almost Calabi–Yau structure and contain a particularly natural family of Lagrangians spheres, which will be the primary focus of this chapter. Sections 5.5 and 5.6 provide an exposition of the work of Thomas–Yau [41], which includes a detailed study of the (almost) Lagrangian mean curvature flow of these spheres, and a proof of a modified version of their conjecture in this setting, assuming some key technical assumptions. The content of these Sections 5.4, 5.5, and 5.6 is not new. However, in revisiting the work of Thomas–Yau, we have concluded their work requires a few technical assumptions that were not originally stated. In many places we have clarified notation, added background, expanded arguments, and, in a few places, made some corrections.

In the penultimate Section 5.7, we provide a new, alternative proof to a Thomas–Yau type conjecture in two-dimensional Milnor fibers. Here, we show the existence of viscosity solutions to the (almost) Lagrangian mean curvature flow and their convergence in C^0 to a smooth special Lagrangian.

Finally, in Section 5.8 we collect a number of important results that are used in the preceding sections.

5.2 Lagrangian mean curvature flow

Some of the notation here has been introduced in previous sections but we take a fresh start here for the sake of clarity.

Let (X, g) be a Riemannian manifold of dimension m, L a smooth embedded submanifold of dimension $n < m$, and

$$\iota : L \to X \tag{5.1}$$

the inclusion mapping. The metric g on X restricts to a metric g_L on L, given by

$$g_L := \iota^* g. \tag{5.2}$$

Definition 5.2.1. *At each $p \in L \subset X$, the tangent space T_pX splits as an orthogonal direct sum*

$$T_pX = \iota_* T_pL \oplus N_pL,$$

where $N_pL := (\iota_* T_pL)^\perp$ is the normal space at p with respect to g on T_pX. The normal bundle of L in X is defined as

$$NL := \bigcup_{p \in L} N_pL. \tag{5.3}$$

Definition 5.2.2. *The second fundamental form of L, denoted II_L, is a quadratic form on the tangent bundle of L with values in the normal bundle of L:*

$$\mathrm{II}_L : TL \times TL \to NL,$$

such that, given $V, W \in TL$,

$$\mathrm{II}_L(V, W) := (\nabla_{\iota_* V} \iota_* W)^{\perp}, \tag{5.4}$$

where $\perp$ denotes projection onto the normal bundle (5.3) and ∇ is covariant differentiation on (X, g).

Proposition 5.2.3. *The value of the second fundamental form at $p \in L$ depends only on the values of V and W at p.*

Proof. Let $x_1, \ldots, x_n, \ldots, x_m$ be local coordinates on X about $p \in L$, so that $x_1, \ldots, x_n$ form local coordinates for the embedded submanifold L. Let $V, W \in TL$. In this coordinate frame we can write

$$\iota_* V = V^1 \partial_1 + \ldots V^n \partial_n, \quad \iota_* W = W^1 \partial_1 + \ldots W^n \partial_n,$$

where $\partial_i := \frac{\partial}{\partial x_i}$. The second fundamental form (5.4) is then

$$\begin{aligned} \mathrm{II}_L(V, W) := (\nabla_{\iota_* V} \iota_* W)^{\perp} &= \sum_{k=1}^{m} \left(\left(V^j W^i \Gamma^k_{ij} + V^j \frac{\partial W^k}{\partial x_j} \right) \partial_k \right)^{\perp} \\ &= \sum_{k=n+1}^{m} \left(V^j W^i \Gamma^k_{ij} + V^j \frac{\partial W^k}{\partial x_j} \right) \partial_k \\ &= \sum_{k=n+1}^{m} \left(V^j W^i \Gamma^k_{ij} \right) \partial_k, \end{aligned}$$

since $W^k = 0$ for $k > n$. $\square$

The mean curvature vector is then defined as the trace with respect to g_L of the second fundamental form:

Definition 5.2.4. *Let $p \in L$, and let $\{V_1, \ldots, V_n\}$ be a basis for T_pL. Then the mean curvature vector at p is defined as the trace of the second fundamental form at p:*

$$\mathrm{tr}_{g_L} \mathrm{II}_L = g_L^{ij} \mathrm{II}_L(V_i, V_j), \tag{5.5}$$

where $g_L^{ij} = g_L(V_i, V_j)$.

Note that the mean curvature vector is well-defined, i.e., independent of the choice of basis $\{V_1, \ldots, V_n\}$, since at every point in L the second fundamental form is a linear map and the trace of a linear map is basis-independent.

Definition 5.2.5. *Let $L(t)$, for $t \in [0, T)$, be a family of Lagrangians in X. We say that the $L(t)$ evolves by the Lagrangian mean curvature flow if*

$$\frac{d}{dt} L(t) = \mathrm{tr}_{g_{L(t)}} \mathrm{II}_{L(t)}, \quad L(0) = L. \tag{5.6}$$

More precisely, if $\iota_t : L \to L(t)$ is a family of embeddings, such that $\iota_t(L) = L(t)$ and ι_0 is the identity map on L, then $\frac{d}{dt} L(t) := \frac{d}{dt} \iota_t(L)$.

We now consider the case where X is Calabi–Yau and $L \subset X$ is Lagrangian. Recall from Section 1.1.1 the definition of a Calabi–Yau manifold (X, J, ω, Ω) of complex dimension n, i.e., (X, J, ω) is a Kähler manifold, with complex structure J and Kähler form ω, and Ω is a compatible (1.1) nowhere vanishing holomorphic $(n, 0)$-form. Recall also from Section 1.1.1 the definition of a Lagrangian submanifold, i.e., $\dim_{\mathbb{R}} L = n$ and $\iota^* \omega = 0$.

The Riemannian metric on X is given by

$$g = \omega(\cdot, J\cdot).$$

As in (5.2), we define g_L as the restriction of this metric to L, and denote by

$$dV_{g_L}$$

the corresponding Riemannian volume on L. Since Ω is a nowhere vanishing $(n,0)$-form, it restricts to a nowhere vanishing complex-valued n-form on L, denoted by $\iota^*\Omega$. These two n-forms on L are related by the Harvey–Lawson formula [17]:

$$\iota^*\Omega = e^{\sqrt{-1}\theta_L} dV_{g_L}, \tag{5.7}$$

where θ_L is an S^1-valued function on L, referred to as the *Lagrangian angle*.

Definition 5.2.6. *A Lagrangian $L \subset X$ is called a special Lagrangian if the function $\theta_L : L \to S^1$ is constant.*

The Lagrangian angle, θ_L, defines a homology class $[d\theta_L] \in H^1(L, \mathbb{Z})$, called the Maslov class of L. If the Maslov class vanishes, then θ_L lifts to a real-valued function.

Definition 5.2.7. *A grading of L, denoted by $\theta : L \to \mathbb{R}$, is the choice of a smooth "lift" of the Lagrangian angle, θ_L, in the sense that for any $p \in L$*

$$\theta(p) = \theta_L(p) \mod 2\pi. \tag{5.8}$$

Note that a grading is not unique.

From here on, we will only concern ourselves with compact, graded Lagrangians.

A special feature of Lagrangian mean curvature flow in the Calabi–Yau setting is that the mean curvature vector can be expressed in terms of the gradient of the Lagrangian angle. See [17, p. 96], [41, p. 1077], [26].

Proposition 5.2.8. *Let X be a Calabi–Yau manifold and $L \subset X$ a graded Lagrangian submanifold. Then the mean curvature vector on L is given by $J\iota_*\nabla^{g_L}\theta$.*

Remark 5.2.9. *In most of the literature this vector is denoted by $J\nabla\theta$, however this is rather abusive notation in a number of ways.*

Proof. First, observe that if $\{E_i\}_{i=1}^n$ is a local orthornormal frame for TL, then $\{e_i = \iota_* E_i\}_{i=1}^n$ is a local orthonormal frame for $\iota_* TL$, and the second fundamental form (Definition 5.2.2) is given by:

$$\mathrm{II}_L(E_i, E_j) = (\nabla_{e_i} e_j)^\perp.$$

Since L is Lagrangian, the complex structure, J, on X provides an isomorphism $J : TL \to NL$, and thus the component of $\mathrm{II}_L(E_i, E_j)$ in the Je_k direction is simply $g(\nabla_{e_i} e_j, Je_k)$. Therefore,

$$\mathrm{II}_L(E_i, E_j) = \sum_k g(\nabla_{e_i} e_j, Je_k) Je_k.$$

The mean curvature vector then becomes

$$\mathrm{tr}_{g_L} \mathrm{II}_L = \sum_k g\left(\sum_i \nabla_{e_i} e_i, Je_k\right) Je_k. \tag{5.9}$$

Now, let $p \in L$ and take any vector $V \in T_pL$. Since

$$V\theta = d\theta(V) = g_L(\nabla^{g_L}\theta, V) = \iota^* g(\nabla^{g_L}\theta, V) = g(\iota_*\nabla^{g_L}\theta, \iota_* V) = g(J\iota_*\nabla^{g_L}\theta, J\iota_* V),$$

we want to show that

$$V\theta = g(\mathrm{tr}_{g_L}\mathrm{II}_L, J\iota_* V). \tag{5.10}$$

We first simplify the right-hand side. Writing $\iota_* V = \sum_l v_l e_l$ in the orthonormal frame and using equation (5.9) we can expand the right-hand side of (5.10) as:

$$\begin{aligned} g(\mathrm{tr}_{g_L}\mathrm{II}_L, J\iota_* V) =& g\left(\sum_k g(\sum_i \nabla_{e_i} e_i, Je_k)Je_k, \sum_l v_l J e_l\right) \\ =& \sum_k g\left(g(\sum_i \nabla_{e_i} e_i, Je_k)Je_k, v_k J e_k\right) \\ =& g\left(\sum_i \nabla_{e_i} e_i, JV\right) \\ =& g\left(\sum_i \nabla_{e_i} Je_i, V\right). \end{aligned}$$

Since $g(Je_i, \iota_* V) = 0$ for all i,

$$0 = \nabla_{e_i}\left(g(Je_i, \iota_* V)\right) = g(\nabla_{e_i} Je_i, \iota_* V) + g(Je_i, \nabla_{e_i}\iota_* V),$$

this implies

$$\sum_i g(\nabla_{e_i} Je_i, \iota_* V) = -\sum g(Je_i, \nabla_{e_i}\iota_* V) = -\sum_i g(e_i, J\nabla_{\iota_* V} e_i), \tag{5.11}$$

where in the last equality we applied J to both vectors and used the fact that J is parallel, i.e., $\nabla \iota_* J = 0$, and that we can extend $\iota_* V$ so that $[e_i, \iota_* V] = 0$.

Now, we rewrite the left-hand side of (5.10). Note that $\{e_i, Je_i\}_{i=1}^n$ is an orthonormal frame for TX near p. Letting $\{\alpha_i, \beta_i\}_{i=1}^n$, where $\beta_i = -\alpha_i \circ J$, be the dual basis of 1-forms for T^*X near p, we can express Ω as

$$\Omega = e^{\sqrt{-1}\theta} \bigwedge_j (\alpha_j + \sqrt{-1}\beta_j),$$

where θ is the Lagrangian angle on L. Since Ω is parallel,

$$
\begin{aligned}
0 = \nabla_{\iota_* V}\Omega &= \nabla_{\iota_* V}\left(e^{\sqrt{-1}\theta}\bigwedge_j(\alpha_j+\sqrt{-1}\beta_j)\right)\\
&= e^{\sqrt{-1}\theta}\sqrt{-1}\iota_* V\theta\bigwedge_j(\alpha_j+\sqrt{-1}\beta_j)+e^{\sqrt{-1}\theta}\nabla_{\iota_* V}\left(\bigwedge_j(\alpha_j+\sqrt{-1}\beta_j)\right).
\end{aligned}
$$

Thus,

$$
\sqrt{-1}\iota_* V(\theta)\bigwedge_j(\alpha_j+\sqrt{-1}\beta_j) = -\sum_k(\alpha_1+\sqrt{-1}\beta_1)\wedge\cdots\wedge\nabla_{\iota_* V}(\alpha_k+\sqrt{-1}\beta_k)\wedge\ldots(\alpha_n+\sqrt{-1}\beta_n).
$$

Evaluating this against the $(n,0)$-vector $\bigwedge_j\left(\frac{1}{2}(e_j-\sqrt{-1}Je_j)\right)$ gives

$$
-\sum_k\left[\nabla_{\iota_* V}(\alpha_k+\sqrt{-1}\beta_k)[\frac{1}{2}(e_k-\sqrt{-1}Je_k)]\right]\bigwedge_j(\alpha_j+\sqrt{-1}\beta_j). \tag{5.12}
$$

Thus, comparing the right-hand side (5.11) and left-hand side (5.12), we are left with showing

$$
\sum_k\left[\nabla_{\iota_* V}(\alpha_k+\sqrt{-1}\beta_k)[\frac{1}{2}(e_k-\sqrt{-1}Je_k)]\right] = \sum_i g(e_i, J\nabla_{\iota_* V}e_i),
$$

or equivalently

$$
\nabla_{\iota_* V}(\alpha_k+\sqrt{-1}\beta_k)[(e_k-\sqrt{-1}Je_k)] = 2\sqrt{-1}g(e_k, J\nabla_{\iota_* V}e_k). \tag{5.13}
$$

Since

$$
\begin{aligned}
0 =&\nabla_{\iota_* V}\left[(\alpha_k+\sqrt{-1}\beta_k)(e_k-\sqrt{-1}Je_k)\right]\\
=&\nabla_{\iota_* V}(\alpha_k+\sqrt{-1}\beta_k)[e_k+\sqrt{-1}Je_k]+(\alpha_k+\sqrt{-1}\beta_k)\nabla_{\iota_* V}(e_k-\sqrt{-1}Je_k),
\end{aligned}
$$

we can write

$$
\begin{aligned}
\nabla_{\iota_* V}(\alpha_k + \sqrt{-1}\beta_k)[e_k - \sqrt{-1}Je_k] &= -(\alpha_k + \sqrt{-1}\beta_k)\nabla_{\iota_* V}(e_k - \sqrt{-1}Je_k) \\
&= -\alpha_k[\nabla_{\iota_* V}e_k] + \sqrt{-1}\alpha_k[\nabla_{\iota_* V}(Je_k)] \\
&\qquad - \sqrt{-1}\beta_k[\nabla_{\iota_* V}e_k] - \beta_k[\nabla_{\iota_* V}(Je_k)] \\
&= \sqrt{-1}\alpha_k[\nabla_{\iota_* V}(Je_k)] - \sqrt{-1}\beta_k[\nabla_{\iota_* V}e_k] \\
&= \sqrt{-1}\left(\alpha_k[\nabla_{\iota_* V}(Je_k)] + \alpha_k[\nabla_{\iota_* V}(Je_k)]\right. \\
&= 2\sqrt{-1}\alpha_k[\nabla_{\iota_* V}(Je_k)] \\
&= 2\sqrt{-1}g(e_k, J\nabla_{\iota_* V}e_k),
\end{aligned}
$$

which verifies (5.13). $\square$

Corollary 5.2.10. *If $L \subset X$ is a graded embedded Lagrangian submanifold, then the mean curvature flow is a Hamiltonian deformation and thus preserves the Hamiltonian deformation class of L.*

Using the identity in Proposition 5.2.8, Smoczyk [35, Theorem 1.9] showed that the Lagrangian mean curvature flow in the Calabi–Yau setting preserves the Lagrangian condition, and that along the flow θ evolves by the following equations:

Proposition 5.2.11. *Under mean curvature flow, the phase θ and the Riemannian volume form satisfy:*

$$
\dot{\theta} = \Delta\theta \qquad \text{and} \qquad \frac{d}{dt}dV_{g_L} = -|d\theta|^2_{g_L}dV_{g_L}. \tag{5.14}
$$

Proof. We will prove a more general result in Proposition 5.5.7. $\square$

5.3 The Thomas–Yau conjecture

The Thomas–Yau conjecture suggests conditions under which the Lagrangian mean curvature flow exists for all time and converges. These conditions are formulated in terms of the variation of the grading θ on L and involve a notion of stability, formulated in terms of Lagrangian connect sums, which we discuss below.

5.3.1 Lagrangian connect sums

In this section we summarize the discussion on connect sums in [41, Section 3.1].

Let X be an n-dimensional Calabi–Yau and $L_1, L_2 \subset X$ be two Lagrangians in X that intersect transversally at the point p. About this point one can chooses Darboux coordinates $(x_i, y_i)_{i=1}^n$, in which $\omega = \sum_i dx_i \wedge dy_i$, and such that L_2 is represented by the $(x_1 \ldots x_n)$-plane:

$$L_2 = \{y_1 = \cdots = y_n = 0\},$$

and

$$L_1 = \{y_i = \tan(\alpha)x_i, i = 1, \ldots, n\},$$

for some $\alpha \in (0, \pi)$. Identifying this coordinate patch with $\mathbb{C}^n$, via $z_i = x_i + \sqrt{-1}y_i$, we can express the L_i as:

$$L(\alpha) = e^{\sqrt{-1}\alpha}\mathbb{R}_{>0}.S^{n-1}(1) := \{z = re^{\sqrt{-1}\alpha}a : r \in \mathbb{R}_{>0}, a = (a_j)_{j=1}^n \in S^{n-1}(1) \subset \mathbb{R}^n \subset \mathbb{C}^n\},$$

where $L_2 = L(0)$ and $L_1 = L(\alpha)$. (Note that this complex structure may not coincide with the complex structure on X.)

More generally, given a curve $\gamma \subset \mathbb{C}$, one can define a Lagrangian

$$L_\gamma = \gamma.S^{n-1} = \{z_j = \gamma a_j : a = (a_j)_{j=1}^n \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n\}.$$

In this notation, L_1 is represented by $\gamma_1 = e^{\sqrt{-1}\alpha}[0,\infty) \subset \mathbb{C}$, L_2 by $\gamma_2 = [0,\infty) \subset \mathbb{C}$, and $L_1 \cup L_2$ by the union of these two curves $\gamma_1 \cup \gamma_2 \subset \mathbb{C}$.

Definition 5.3.1. *The Lagrangian connect sum of L_1 and L_2, denoted $L_1 \# L_2$, is represented by any smoothing of $\gamma_1 \cup \gamma_2$ staying inside the cone $\{re^{\sqrt{-1}\beta} : r > 0, \beta \in [0,\alpha]\}$ and coinciding with $\gamma_1 \cup \gamma_2$ outside a compact set. We denote any such smoothing by $\gamma_1 \# \gamma_2$.*

5.3.2 A conjecture for the LMCF

Let

$$\arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi] \tag{5.15}$$

be the principal branch of the argument function defined on the non-zero complex numbers. Note its discontinuity along the negative x-axis.

Definition 5.3.2. *Let L be a compact, graded (as in Definition 5.2.7), Lagrangian submanifold of a Calabi–Yau manifold X. Then the phase of L is defined as*

$$\phi(L) := \arg\left(\int_L \iota^*\Omega\right) = \arg\left(\int_L e^{\sqrt{-1}\theta} dV_{g_L}\right),$$

and depends only on the homology class of L.

The following conjecture is due to Thomas–Yau [41, p. 1101]:

Conjecture 5.3.3. *Let L be a compact, graded, Lagrangian submanifold of a Calabi–Yau manifold X, and let θ (as in Definition 5.2.7) and ϕ (as in Definition 5.3.2) be the grading and phase of L, respectively. If L satisfies*

$$[\phi(L_1), \phi(L_2)] \not\subset (\inf_L \theta, \sup_L \theta), \tag{5.16}$$

for all graded connect sums $[L]_{Ham} = [L_1 \# L_2]_{Ham}$, then the mean curvature flow of L (5.2.5) exists for all time and converges to a special Lagrangian in its Hamiltonian deformation class.

We emphasize that we will not study this conjecture but rather a modification of this conjecture, see Conjecture 5.5.8.

5.4 Milnor fibers

5.4.1 Introduction

In this section we introduce an important class of almost Calabi–Yau manifolds, called Milnor fibers [25, 22], following the exposition by [41, Section 6]. As mentioned earlier, in general, there are very few explicitly known Calabi–Yau structures. The next best thing is then an explicit almost Calabi–Yau structure, which, as we will see (Proposition 5.4.3), Milnor fibers admit. Moreover, due to their high degree of symmetry, Milnor fibers can be represented by 1-dimensional objects, which further simplifies their analysis.

Definition 5.4.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial with only simple roots.*

Then the Milnor fiber of degree n is the complex submanifold of $\mathbb{C}^{n+1}$, defined by

$$X = X_f^n := \{(w, z_1, \dots, z_n) \in \mathbb{C}^{n+1} : z_1^2 + \dots z_n^2 = f(w)\}.$$

We claim that X is indeed a complex submanifold. To see this, set

$$F(w, z_1, \dots, z_n) := z_1^2 + \dots z_n^2 - f(w)$$

we can write

$$X = \{F(w, z_1, \dots, z_n) = 0\}.$$

Thus, the complex Jacobian matrix of F at a point $(w, z_1, \dots, z_n)$ is given by

$$\left[-\partial_w f, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right] = \left[-\partial_w f(w), 2z_1, \dots, 2z_n\right],$$

which is surjective at every point on X. In particular, $z_1 = z_2 = \dots = z_n = 0$ only at a root w_0 of f, but by the simpleness of the roots $\partial_w f(w_0) \neq 0$. Therefore, one can apply the holomorphic implicit function theorem [20, Theorem 2.1.2], and X is a complex submanifold of $\mathbb{C}^{n+1}$ of dimension n.

As a complex submanifold of $(\mathbb{C}^{n+1}, \omega)$, where ω is the standard Euclidean Kähler form, X inherits a Kähler structure,

$$\omega_X := \iota_n^* \omega, \tag{5.17}$$

where $\iota_n : X \to \mathbb{C}^{n+1}$ is the inclusion map. To give X an almost Calabi–Yau structure, it needs to be equipped with a non-vanishing holomorphic $(n, 0)$-form (Section 1.1.1). This is given by the Poincaré residue of $dw \wedge dz_1 \wedge \dots \wedge dz_n$, which we briefly discuss in the next section.

5.4.2 Poincaré residue

Let

$$\iota_V : V \to \mathbb{C}^m$$

be an analytic hypersurface, and let

$$\psi$$

be a meromorphic m-form on $\mathbb{C}^m$ with a single pole along V. The Poincaré residue is a higher-dimensional generalization of the residue of a meromorphic function in complex analysis.

If V is defined by $\phi = 0$, where

$$\phi : \mathbb{C}^m \to \mathbb{C}$$

is holomorphic and $d\phi$ is nowhere vanishing, then the 1-form

$$\frac{d\phi}{\phi},$$

is meromorphic on $\mathbb{C}^m$, with a single pole along V.

Definition 5.4.2. *The Poincaré residue of ψ, which we denote by ψ_V, is defined to be the unique holomorphic $(m-1, 0)$-form on V for which there exists a holomorphic $(m, 0)$-form β on $\mathbb{C}^m$ such that $\psi_V = \iota_V^* \beta$, and*

$$\psi = \frac{d\phi}{\phi} \wedge \beta.$$

More explicitly, in coordinates $z_1, \ldots, z_m$ on $\mathbb{C}^m$, if

$$\psi = \frac{h(z) dz_1 \wedge \cdots \wedge dz_m}{\phi(z)},$$

where $h(z)$ is holomorphic, then β corresponds to the unique holomorphic $(m-1,0)$-form satisfying

$$\psi = \frac{d\phi}{\phi} \wedge \beta,$$

i.e.,

$$\frac{h(z)dz_1 \wedge \cdots \wedge dz_m}{\phi(z)} = \left(\frac{1}{\phi(z)} \sum \frac{\partial \phi}{\partial z_i}(z) dz_i \right) \wedge \beta.$$

Thus, on the set in $\mathbb{C}^m$ on which $\frac{\partial \phi}{\partial z_i} \neq 0$, we can take

$$\beta = (-1)^{i-1} \frac{h(z)dz_1 \wedge \cdots \wedge \hat{dz_i} \wedge \cdots \wedge dz_m}{\frac{\partial \phi}{\partial z_i}}.$$

Clearly, β is a holomorphic $(m-1,0)$-form. Note that β is defined on all of $\mathbb{C}^m$ since we assumed $d\phi$ is nowhere vanishing. Therefore,

$$\psi_V = (-1)^{i-1} \frac{h(z)dz_1 \wedge \cdots \wedge \hat{dz_i} \wedge \cdots \wedge dz_m}{\frac{\partial \phi}{\partial z_i}} \Bigg|_V . \tag{5.18}$$

See [15, Chapter 1] for additional discussion on the Poincaré residue.

5.4.3 Almost Calabi–Yau structure

The Milnor fiber X is an analytic hypersurface in $\mathbb{C}^{n+1}$. Consider the meromorphic $(n+1,0)$-form on $\mathbb{C}^{n+1}$ given by

$$\psi = \frac{dw \wedge dz_1 \wedge \cdots \wedge dz^n}{f(w) - \sum z_i^2}.$$

Then, by (5.18), in a chart where $z_i \neq 0$, the Poincaré residue of ψ can be expressed as

$$\Omega_X = (-1)^i \, \frac{dw \wedge dz_1 \wedge \ldots \hat{dz_i} \cdots \wedge dz^n}{2z_i} \Bigg|_X , \tag{5.19}$$

and, in a chart where $\partial_w f(w) \neq 0$, it can be expressed as

$$\Omega_X = \left.\frac{dz_1 \wedge \cdots \wedge dz^n}{\partial_w f(w)}\right|_X. \tag{5.20}$$

Note that these charts cover X by assumption on the simpleness of the roots of f.

Proposition 5.4.3. *With ω_X and Ω_X as defined in (5.17), (5.19)–(5.20), respectively, $(X, J, \omega_X, \Omega_X)$ is an almost Calabi–Yau n-fold.*

Proof. As we saw above (Section 5.4.1), (X, J, ω_X) is Kähler, where J and ω_X are the induced complex structure and Kähler form from $\mathbb{C}^{n+1}$. Taking Ω_X, as defined in (5.19)–(5.20), equips X with a nowhere-vanishing holomorphic n-form. □

Remark 5.4.4. Note that X is only an *almost* Calabi–Yau manifold in the sense that the metric is not Ricci-flat, i.e., the condition

$$\frac{\omega_X^n}{n!} = (-1)^{n(n-1)/2} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega_X \wedge \overline{\Omega_X}, \tag{5.21}$$

does *not* hold. To see this, consider the point $(w, z_1, \ldots, z_n) = (w_0, 0, \ldots, 0)$ on X, where w_0 is a root of f. At this point:

$$\Omega_X = \frac{1}{\partial_w f(w_0)} dz_1 \wedge \cdots \wedge dz_n \qquad \text{and} \qquad \omega_X = dz_1 \wedge \overline{dz}_1 + \cdots + dz_n \wedge \overline{dz}_n,$$

which does not satisfy condition (5.21). The advantage of these forms is that they are the restrictions of explicit Euclidean forms.

Remark 5.4.5. There are not many known explicit Calabi–Yau metrics. Although there are several known in the non-compact setting (e.g., Stenzel [39] and Eguchi-Hanson [12] metrics), there are currently no closed form expressions for a Ricci-flat metric on any nontrivial compact Calabi–Yau [11].

Since X is a complex submanifold of $\mathbb{C}^{n+1}$, with Kähler form the restriction of the Euclidean Kähler form on $\mathbb{C}^{n+1}$, it is computationally advantageous to express tangent vectors to X as vectors in $\mathbb{C}^{n+1}$. Throughout the rest of this chapter we make use the following notation:

Notation 5.4.6. *(i) Let $(w, z_1, \ldots, z_n)$ be coordinates on $\mathbb{C}^{n+1}$, with real and imaginary parts*

$$w = u + \sqrt{-1}v, \quad z_j = x_j + \sqrt{-1}y_j.$$

We represent a vector

$$V = (v_0, \ldots, v_n) \in \mathbb{C}^{n+1}$$

by

$$V = v_0\partial_u + \sum v_j\partial_{x_j},$$

and identify

$$\sqrt{-1}\partial_u = \partial_v, \quad \sqrt{-1}\partial_{x_j} = \partial_{y_j}.$$

(ii) Let

$$\pi_n : \mathbb{C}^{n+1} \to \mathbb{C}$$

denote projection onto the first coordinate, i.e.,

$$\pi_n(w, z_1, \ldots z_n) \to w.$$

(iii) The map $\gamma : [0,1] \to \mathbb{C}$, where $a \mapsto \gamma(a)$, will always denote a simple, regular path in $\mathbb{C}$, i.e., γ does not cross itself and

$$\gamma'(a) \neq 0$$

for all $a \in [0,1]$. *Here,*

$$\gamma' := \frac{d\gamma}{da} \in \mathbb{C},$$

which, using the above (i) notation, we could express as $\frac{d\gamma}{da}\partial_u$.

(iv) Let

$$g_{\mathbb{C}^k} := |dz_1|^2 + \cdots + |dz_k|^2$$

denote the Euclidean metric on $\mathbb{C}^k$.

(v) Let $\iota_n : X^n \hookrightarrow \mathbb{C}^{n+1}$ *denote the inclusion mapping. We denote the metric on* X, *as in (5.2), by*

$$g_{X^n} := \iota_n^*(g_{\mathbb{C}^{n+1}}).$$

5.4.4 Symmetric Lagrangian spheres

We focus on a specific type of Lagrangian inside of X. These are the symmetric n-spheres, with an S^{n-1} fibration that respects the fibration structure of X, which we will discuss below. In this section, we explain how these spheres are constructed and show that they are Lagrangian.

Let $w \in \mathbb{C}$. If $f(w) \neq 0$, then the smooth fiber in X over w has a Lagrangian S^{n-1} 'real' slice:

$$S_w^{n-1} := \left\{ (w, z_1, \ldots, z_n) \in X : \frac{z_i}{\sqrt{f(w)}} \in \mathbb{R} \right\}. \tag{5.22}$$

We can identify S_w^{n-1} with a real $(n-1)$-sphere in $\mathbb{C}^n$ via

$$\{\sqrt{f(w)}(r_1, \ldots, r_n) : r \in S^{n-1}(1) \subset \mathbb{R}^n\} \subset \mathbb{C}^n, \tag{5.23}$$

where $S^{n-1}(1) := \{r = (r_1, \dots, r_n) \in \mathbb{R}^n : |r|^2 = 1\}$ denotes the unit $(n-1)$-sphere in $\mathbb{R}^n$. The tangent space to S_w^{n-1} (thought of as a real subspace of $\mathbb{C}^n = \mathbb{R}^{2n}$) is then contained in the image of $\mathbb{R}^n$ under multiplication by $\sqrt{f(w)}$. Thus, taking

$$A = \operatorname{diag}\left(\frac{\sqrt{f(w)}}{|\sqrt{f(w)}|}, \dots, \frac{\sqrt{f(w)}}{|\sqrt{f(w)}|}\right) \in U(n),$$

this can be expressed as

$$T_p S_w^{n-1} \subset A(\mathbb{R}^n \times \{0\}) \subset \mathbb{C}^n.$$

Let $\Lambda(n)$ denote the set of Lagrangian n-planes in $\mathbb{C}^n$, i.e., the real n-planes in $(\mathbb{C}^n, \omega)$ on which ω vanishes. Harvey–Lawson [17, p.87] showed that the unitary group $U(n)$ acts transitively on $\Lambda(n)$, and that the isotropy subgroup at the point $\mathbb{R}^n \in \Lambda(n)$ is $SO(n)$ acting diagonally on $\mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$. Thus, symbolically we can write

$$\Lambda(n) = U(n)/SO(n).$$

Therefore, the Kähler form on the fiber, which is the restriction of the Euclidean form

$$\frac{\sqrt{-1}}{2} \sum_j dz_j \wedge \overline{dz_j}$$

on $\mathbb{C}^n$, vanishes. In summary:

Lemma 5.4.7. *The $(n-1)$-sphere S_w^{n-1} (5.23) is a Lagrangian submanifold of $\mathbb{C}^n$.*

These Lagrangian $(n-1)$-spheres are invariant under $O(n)$ acting on X (on the z coordinates), and from them we can construct an $O(n)$-invariant Lagrangian n-sphere in X as follows.

Definition 5.4.8. *Let $\gamma : [0,1] \to \mathbb{C}$ be any regular, simple path between two distinct roots of f and not containing any roots in its interior. Then γ can be lifted to*

$$\Gamma = \Gamma_\gamma := \overline{\bigcup_{a \in (0,1)} S^{n-1}_{\gamma(a)}}, \tag{5.24}$$

which is an n-sphere in X, S^{n-1}-fibered over γ, except at the end points where it closes up. See Figure 5.1.

According to Proposition 5.8.4, Γ is actually a smooth submanifold of X, diffeomorphic to the n-sphere. The following proposition is due to Thomas–Yau [41, Section 6]:

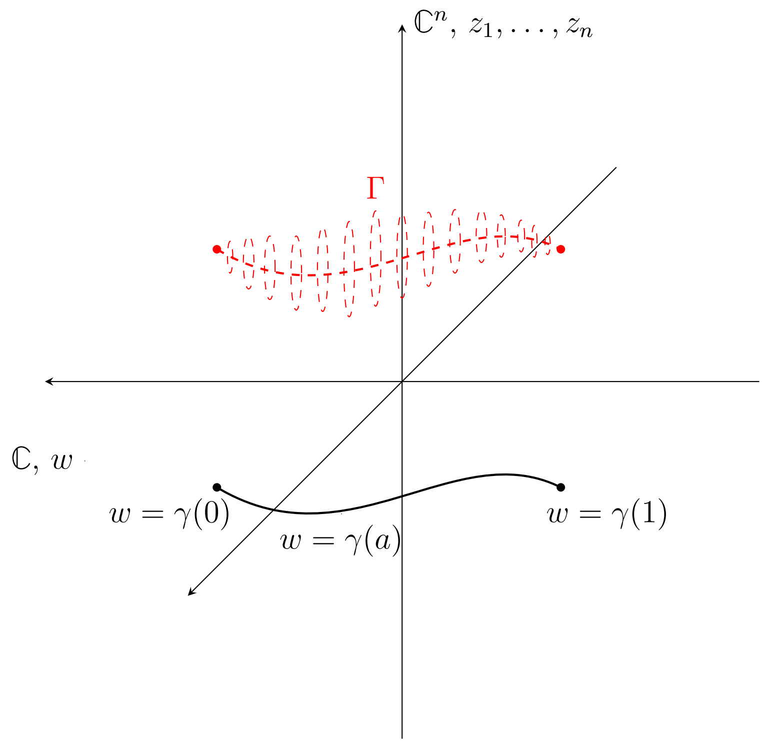

Figure 5.1: Lift of γ to the Lagrangian sphere, Γ

Proposition 5.4.9. Γ *is Lagrangian.*

Proof. Since Γ is an S^{n-1} fibration over the curve γ. The tangent space at any point can be decomposed as the tangent space to the S^{n-1} fiber plus the tangent space along γ. The Kähler form on X is the restriction of the Euclidean Kähler form $\omega = \frac{\sqrt{-1}}{2}\left(dw \wedge \overline{dw} + \sum_j dz_j \wedge \overline{dz_j}\right)$. So for Γ to be Lagrangian, we check that $\omega|_\Gamma = 0$.

We saw above (Lemma 5.4.7) that the restriction of $\frac{\sqrt{-1}}{2}\sum_j dz_j \wedge \overline{dz_j}$ is zero on the S^{n-1} component of the tangent space. Thus, we now need to check that ω vanishes on tangent vectors to Γ, along the path γ.

Lemma 5.4.10. *Let $\gamma : [0,1] \to \mathbb{C}$ be a simple, regular path between two distinct roots of f and not containing any roots in its interior. For any $a_0 \in (0,1)$, the vector*

$$\gamma'(a_0)\left(\partial_u + \frac{\partial_w f(\gamma(a_0))}{2f(\gamma(a_0))}\sum z_i \partial_{x_i}\right), \tag{5.25}$$

is tangent to Γ at the point $(\gamma(a_0), z_1, \ldots, z_n)$, and projects to the vector $\gamma'(a_0)\partial_u \in \mathbb{C}$ tangent to γ, i.e.

$$\pi_*\left(\gamma'(a_0)\left(\partial_u + \frac{\partial_w f(\gamma(a_0))}{2f(\gamma(a_0))}\sum z_i \partial_{x_i}\right)\right) = \gamma'(a_0)\partial_u,$$

where $\pi : \mathbb{C}^n \to \mathbb{C}$ is the projection map and we are using the notation from Notation 5.4.6.

Proof. Fix a point $(\gamma(a_0), z_1, \ldots, z_n) \in \Gamma$ in the fiber above $t = \gamma(a_0)$. Using the identification (5.23), we can write this point as:

$$(\gamma(a_0), z_1, \ldots, z_n) = \left(\gamma(a_0), \sqrt{f(\gamma(a_0))}r_1, \ldots, \sqrt{f(\gamma(a_0))}r_n\right),$$

where $(r_1, \ldots r_n) \in S^{n-1}$ is fixed.

Now consider the curve $C_\gamma : (0,1) \to \Gamma$, defined by

$$C_\gamma(a) = \left(\gamma(a), \sqrt{f(\gamma(a))}r_1, \ldots . \sqrt{f(\gamma(a))}r_n\right) \in \mathbb{C}^{n+1},$$

This curve passes through our fixed point at $a = a_0$. Thus, the derivative at $a = a_0$

$$\left.\frac{dC_\gamma}{da}\right|_{a_0} = \left(\gamma'(a_0), \gamma'(a_0)\frac{\partial_w f(\gamma(a_0))}{2\sqrt{f(\gamma(a_0)}}r_1, \ldots, \gamma'(a_0)\frac{\partial_w f(\gamma(a_0))}{2\sqrt{f(\gamma(a_0)}}r_n\right) \in \mathbb{C}^{n+1}$$

is tangent to Γ at this point. Rewriting this in terms of the $z_i = \sqrt{f(\gamma(a_0))}r_i$, we get

$$\left(\gamma'(a_0), \gamma'(a_0)\frac{\partial_w f(\gamma(a_0))}{2f(\gamma(a_0))}z_1, \ldots, \gamma'(a_0)\frac{\partial_w f(\gamma(a_0))}{2f(\gamma(a_0))}z_n\right),$$

which we express, using Notation 5.4.6, as

$$\gamma'(a_0)\left(\partial_u + \frac{\partial_w f(\gamma(a_0))}{2f(\gamma(a_0))}\sum z_i \partial_{x_i}\right).$$

$\square$

Notice that since Γ is $O(n)$-invariant, the lift of γ' to another point on the same fiber of $S^{n-1}_{\gamma(a_0)}$, i.e., some other $(r_1, \ldots, r_n)$, is the image under

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in O(n+1) \subset U(n+1), \text{ for } A \in O(n).$$

Thus, since ω is invariant under $U(n+1)$ it can be calculated at any point in the fiber. For simplicity, choose $z_1 = \sqrt{f(w)} = \sqrt{f(\gamma(a_0))}$, and $z_i = 0$, for $i = 2, \ldots, n$. By (5.25), the following vectors form a basis for the tangent space at $p = (\gamma(a_0), \sqrt{f(\gamma(a_0))}, 0, \ldots, 0)$:

$$\left\{\gamma'(a_0)\left(\partial_u + \frac{\partial_w f(\gamma(a_0))}{2\sqrt{f(\gamma(a_0))}}\partial_{x_1}\right), \sqrt{f(\gamma(a_0))}\partial_{x_2}, \ldots, \sqrt{f(\gamma(a_0))}\partial_{x_n}\right\}. \qquad (5.26)$$

Notice that at this point we are at one of the poles of S^{n-1}, so the fiber component of the tangent space is spanned by the last $(n-2)$ vectors in (5.26).

Finally, we see that

$$\omega = \frac{\sqrt{-1}}{2}\left(dw \wedge \overline{dw} + \sum_j dz_j \wedge \overline{dz_j}\right) = ds \wedge dq + \sum_j dx_j \wedge dy_j$$

evaluates to zero on (5.26):

$$\omega|_p\left(\sqrt{f(\gamma(a_0))}\partial_{x_i}, \sqrt{f(\gamma(a_0))}\partial_{x_j}\right) = 0 \qquad \forall i, j = 2, \ldots, n$$

and

$$\omega|_p\left(\sqrt{f(\gamma(a_0))}\partial_{x_i}, \gamma'(a_0)(\partial_u + \frac{\partial_w f(\gamma(a_0))}{2\sqrt{f(\gamma(a_0))}}\partial_{x_1})\right) = 0 \qquad \forall i = 2, \ldots n.$$

By continuity and the fact that Γ is smooth (Proposition 5.8.4), $\omega|_\Gamma$ must also vanish at the endpoints of γ. Therefore, $\omega|_\Gamma = 0$ and so Γ is Lagrangian. $\square$

Lemma 5.4.11. *Let $\Gamma \subset$ be as in Definition (5.24). Then the Lagrangian angle, $\theta_\Gamma : \Gamma \to S^1$, is an $O(n)$-invariant function, given by*

$$\theta_\Gamma(a) = \arg(\gamma'(a)) + (n/2 - 1)\arg(f(\gamma(a))) \mod 2\pi, \tag{5.27}$$

where $\arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ denotes principal branch of the argument function defined on non-zero complex numbers, as in Definition 5.15.

Proof. Since $\Omega_X = \frac{1}{\partial_w f(w)} dz_1 \wedge \ldots dz_n$ is unchanged by an $O(n)$ action on the z_i, the phase function θ_Γ is also $O(n)$-invariant and thus a function of $u \in [0, 1]$, where $u \mapsto \gamma(u) \in \mathbb{C}$. We can calculate θ_Γ at the point $z_1 = \sqrt{f(w)}$, $z_i = 0$, for $i > 1$. Take the basis (5.26) for the tangent space to Γ at this point:

$$\gamma'\left(\partial_u + \frac{\partial_w f(\gamma)}{2\sqrt{f(\gamma)}}\partial_{x_1}\right), \sqrt{f(\gamma)}\partial_{x_2}, \ldots, \sqrt{f(\gamma)}\partial_{x_n}. \tag{5.28}$$

Plugging these vectors (5.28) into Ω_X evaluates to

$$\gamma' \frac{\partial_w f(\gamma)}{2\sqrt{f(\gamma)}} \frac{(\sqrt{f(\gamma)})^{n-1}}{\partial_w f(\gamma)} = \frac{1}{2}\gamma'(f(\gamma))^{n/2-1}.$$

Thus, the phase function (5.7) on Γ is given by

$$\theta_\Gamma(a) = \arg(\gamma'(a)) + (n/2-1)\arg(f(\gamma(a))).$$

$\square$

Definition 5.4.12. *Given any regular curve $c : [0,1] \to \mathbb{C}$, such that $c(a) \neq 0, \forall a \in (0,1)$, we define a lift of* $\arg$ *(as defined in 5.15) along c, denoted by $\widetilde{\arg}$, as follows:*

1. *$\widetilde{\arg}(c(a))$ is continuous in $a \in [0,1]$,*

2. *$\widetilde{\arg}(c(a)) = \arg(c(a)) \mod 2\pi$.*

3. $$\widetilde{\arg}(c(0)) := \begin{cases} \arg(c(0)), & c(0) \neq 0 \\ \arg(\frac{d}{da}|_{a=0^+} c(a)), & c(0) = 0 \end{cases}.$$

Note that this defines $\widetilde{\arg}(c(1))$ by continuity.

Remark 5.4.13. *This works regardless if c is a simple curve.*

Corollary 5.4.14. *Γ is a graded Lagrangian, and*

$$\theta(a) = \widetilde{\arg}(\gamma'(a)) + (n/2-1)\widetilde{\arg}(f(\gamma(a))), \tag{5.29}$$

is a grading for Γ, where $\widetilde{\arg}$ is given by Definition 5.4.12.

Proof. This follows immediately from the proof of Lemma 5.4.11, with arg replaced with $\widetilde{\arg}$, since $\gamma' f^{n/2-1} \neq 0$ except at the endpoints of γ. $\square$

Remark 5.4.15. $\widetilde{\arg}(\gamma'(a)(f(\gamma(a)))^{(n/2-1)})$ *is also a grading, possibly different from (5.29).*

5.5 The almost mean curvature vector in Milnor fibers

Let X be an almost Calabi–Yau manifold, and let $L \subset X$ a Lagrangian submanifold. In general, Proposition 5.2.8 will no longer hold. However, $J\iota_*\nabla\theta$ is still a normal vector field along L and the corresponding flow represents an interesting geometric process [41, Section 6]. In this more general setting, we refer to this vector as follows:

Definition 5.5.1. *Let X be an almost Calabi–Yau manifold, and let $L \subset X$ be a Lagrangian submanifold with grading $\theta : L \to \mathbb{R}$. Then we refer to $J\iota_*\nabla\theta$ as the almost mean curvature vector on L.*

Let $\Gamma \subset X$ be the $O(n)$-invariant Lagrangian n-sphere, fibered over the simple, regular curve

$$\gamma : [0,1] \to \mathbb{C}, \tag{5.30}$$

between two distinct roots of f, and let

$$\pi_n : \mathbb{C}^{n+1} \to \mathbb{C}$$

be projection onto the w-coordinate, as described in Notation 5.4.6. The following is shown in [41, Section 6].

Proposition 5.5.2. *Let $\theta : \Gamma \to \mathbb{R}$ be the grading of Γ, as in (5.29). Then, where π_n is defined as in Notation 5.4.6,*

$$\pi_{n*}J\iota_*\nabla\theta = \frac{1}{1+\frac{|\partial_w f(\gamma)|^2}{4|f(\gamma)|}}\big(\kappa + (1-n/2)N(\log|f(\gamma)|)\big)N, \tag{5.31}$$

where κ is the curvature of γ and N is the unit (upward-pointing) normal to γ. At the endpoints γ, $\pi_{n*}J\iota_*\nabla\theta$ vanishes.

Proof. Using (5.29), we can write

$$d\theta = \left(\frac{d}{da}\widetilde{\arg}(\gamma'(a)) + (n/2-1)\frac{d}{da}\widetilde{\arg}(f(\gamma(a))\right)\pi_n^*\gamma_* da. \tag{5.32}$$

The basis (5.26) for the tangent space to Γ is orthogonal with respect to the metric g on X, Dividing these basis vectors by their norms gives an orthonormal basis for the tangent space to Γ:

$$\frac{\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})}{|\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})|_g}, \frac{\sqrt{f}\partial_{x_2}}{|\sqrt{f}\partial_{x_2}|_g}, \ldots, \frac{\sqrt{f}\partial_{x_n}}{|\sqrt{f}\partial_{x_n}|_g}, \tag{5.33}$$

since $g = \iota_n^* g_{\mathbb{C}^{n+1}}$ and these vectors are orthogonal in $\mathbb{C}^{n+1}$.

We then see that the one-forms:

$$g\left(\frac{\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})}{|\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})|_g^2}, \quad\cdot\quad\right) \qquad \text{and} \qquad \pi_n^*\gamma_* da\,(\cdot) \tag{5.34}$$

both evaluate to 1 on the vector $\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})$. This follows from (5.33) and the computation:

$$\pi_n^*\gamma_* da\left(\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})\right) = (\gamma_* da)(\gamma'\partial_u) = 1,$$

where $\widetilde{\gamma_*\partial_u}$ denotes the lift (5.25) of $\gamma_*\partial_u$ to Γ. Therefore,

$$(\pi_n^*\gamma_* da)^{\#g} = \frac{\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})}{|\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})|_g^2}.$$

Using (5.32) and the fact that the projection π_n is holomorphic, we have

$$\pi_{n*}J = i\pi_{n*},$$

as in Notation 5.4.6. We can then calculate that $J\iota_*\nabla\theta$ is the lift (5.25) of

$$\left(\frac{d}{da}\widetilde{\arg}(\gamma') + (n/2-1)\frac{d}{da}\widetilde{\arg}(f)\right)\pi_{n*}\left[J\frac{\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})}{|\gamma'(\partial_u + \frac{\partial_w f}{2f}z_1\partial_{x_1})|_g^2}\right]$$

$$= \frac{\frac{d}{da}\left(\widetilde{\arg}(\gamma') + (\frac{n}{2}-1)\widetilde{\arg}(f)\right)}{|\gamma'|\left(1+|\partial_w f|^2/4|f|\right)}\sqrt{-1}\frac{\gamma'}{|\gamma'|}\partial_u. \tag{5.35}$$

Let

$$T := \gamma'\partial_u/|\gamma'| \quad \text{and} \quad N := \sqrt{-1}T = \gamma'\partial_v/|\gamma'| \tag{5.36}$$

denote the unit tangent and normal vectors to γ at $\gamma(a)$ using the notation from Notation 5.4.6. The vector (5.35) can then be written as

$$\frac{T\left[\widetilde{\arg}(\gamma') + (n/2-1)\widetilde{\arg}(f)\right]}{1+|\partial_w f|^2/4|f|}N. \tag{5.37}$$

The first term in the numerator is the curvature vector of γ in the Euclidean metric on $\mathbb{C}$, i.e., the rate at which the unit tangent vector is rotating, which we will denote by κ (see Definition 5.8.5).

For the second term, consider the holomorphic function

$$\log f = \log|f| + \sqrt{-1}\widetilde{\arg}(f).$$

By the Cauchy–Riemann equations,

$$\partial_u\widetilde{arg}(f) = -\partial_v \log|f|,$$

in the notation of Notation 5.4.6. Therefore, (5.36) gives

$$T\widetilde{\arg}(f) = -N\log|f|.$$

Rewriting (5.37), we see that $\pi_{n*}J\iota_*\nabla\theta$ is

$$\frac{1}{1+|\partial_w f|^2/4|f|}\big(\kappa + (1-n/2)N(\log|f|)\big)N. \tag{5.38}$$

To see that $\pi_{n*}J\iota_*\nabla\theta$ vanishes at the endpoints, we can look directly at $\theta : \Gamma \to \mathbb{R}$. Since θ is $O(n)$-invariant, the directional derivative of θ at the point on Γ corresponding to the endpoint of γ must be the same in all directions. However, this is only possible if $\nabla\theta = 0$. Therefore, $\pi_{n*}J\iota_*\nabla\theta = 0$ at the endpoints of γ. $\square$

Recall, from Definition 5.4.1, that $X^1 = X^1_f$ is defined as

$$X^1_f := \left\{(w, z) \in \mathbb{C}^2 : z = \pm\sqrt{f(w)}\right\},$$

and thus forms a double cover of $\mathbb{C}$, branched over the roots of f. Let $Z_f \subset \mathbb{C}$ denote the discrete finite set of roots of f, i.e.,

$$Z_f := \{w \in \mathbb{C} : f(w) = 0\}, \tag{5.39}$$

and let B_f denote the set of branch points of X^1, i.e.,

$$B_f := \{(w, 0) \in X^1 : w \in Z_f\}. \tag{5.40}$$

Define the "upper" layer of X^1 by

$$X^1_{up} := X^1_{f,up} := \left\{(w, z) \in \mathbb{C}^2 : z = \sqrt{f(w)}\right\}, \tag{5.41}$$

where $\sqrt{f(w)}$ denotes a consistent choice of a branch of the square root function which makes X^1_{up} a smooth submanifold. Observe that away from B_f, X^1_{up} is the image of $\mathbb{C} \setminus Z_f$ under the smooth map

$$\Psi : \mathbb{C} \setminus Z_f \to X^1_{up} \setminus B_f, \tag{5.42}$$

$$w \mapsto (w, \sqrt{f(w)}).$$

Proposition 5.5.3. *Away from its branch points, X^1 is conformally equivalent to $\mathbb{C}$. More precisely, $(X^1_{up} \setminus B_f, g_{X^1})$ is isometric to $(\mathbb{C} \setminus Z_f, hg_{\mathbb{C}})$, where*

$$h = \left(1 + \frac{|\partial_w f|^2}{4|f|}\right) \qquad \text{and} \qquad g_{\mathbb{C}} = |dw|^2,$$

as in Notation 5.4.6.

Proof. Recall from Section 5.4.3 that since X^1 is a smooth submanifold of $\mathbb{C}^2$, the Euclidean metric on $\mathbb{C}^2$, $g_{\mathbb{C}^2} = |dw|^2 + |dz|^2$, induces a metric

$$g_{X^1} = \iota_1^* g_{\mathbb{C}^2}, \tag{5.43}$$

on X^1, where $\iota : X^1 \hookrightarrow \mathbb{C}^2$, is the inclusion map.

Using (5.42) and (5.43), we can pull the metric on $X^1_{up} \setminus B_f$ back to $\mathbb{C} \setminus Z_f$:

$$\Psi^* \iota_1^* g_{\mathbb{C}^2} = \Psi^*(|dw|^2 + |dz|^2) = \left(1 + \frac{|\partial_w f|^2}{4|f|}\right)|dw|^2 = \left(1 + \frac{|\partial_w f|^2}{4|f|}\right) g_{\mathbb{C}}.$$

$\square$

$$\begin{array}{c} \Gamma^1 \subset \left(X^1_{up} \setminus B_f, g_{X^1}\right) \\ \Big\uparrow{\scriptstyle \Psi} \\ \gamma \subset (\mathbb{C} \setminus Z_f, hg_{\mathbb{C}}) \end{array}$$

Figure 5.2: Local isometry between X^1 and $\mathbb{C}$

Therefore, one can view the map

$$\Psi : \mathbb{C} \setminus Z_f \to X^1_{up} \setminus B_f,$$

as coordinates on X^1_{up}, which we can use to endow $\mathbb{C} \setminus Z_f$ with the pull-back metric $\Psi^*(g_{X^1}) = hg_{\mathbb{C}}$, where

$$h = \left(1 + \frac{|\partial_w f|^2}{4|f|}\right). \tag{5.44}$$

Equivalently, Ψ can be viewed as an isometry between the Riemannian manifolds $(\mathbb{C} \setminus Z_f, hg_{\mathbb{C}})$ and $\left(X^1_{up} \setminus B_f, g_{X^1}\right)$.

Lemma 5.5.4. *Let (M, g_M) and (N, g_N) be Riemannian manifolds, and $\phi : M \to N$ an isometry, i.e., ϕ is a diffeomorphism and*

$$\phi^*(g_N) = g_M.$$

Let $S_M \subset M$ and $S_N \subset N$ be submanifolds of M and N, such that $\phi(S_M) = S_N$. Then the the mean curvature of S_M is equal to the pull-back of the mean curvature of S_N, i.e.,

$$\phi^* \mathrm{tr}_{g_{S_N}} \mathrm{II}_{S_N} = \mathrm{tr}_{g_{S_M}} \mathrm{II}_{S_M}, \tag{5.45}$$

where g_{S_N} denotes the metric induced on S_N from g_N, and g_{S_M} denotes the metric induced on S_M from g_M, as in (5.2).

Proof. The mean curvature vector (5.5) of S_N is given by

$$\mathrm{tr}_{g_{S_N}} \mathrm{II}_{S_N}.$$

Since ϕ is an isometry, $\phi^* g_{S_n} = g_{S_M}$. Thus, it is sufficient to show that

$$\phi_* \mathrm{II}_{S_M} = \mathrm{II}_{S_N}.$$

Given $V, W \in TS_M$, the second fundamental form (5.4), is given by

$$\mathrm{II}_{S_M}(V, W) := (\nabla^M_V W)^\perp,$$

where ∇^M is the Levi-CIvita connection on M. To prove that ϕ preserves the second fundamental form, it is then sufficient to show that

$$\phi_*(\nabla^M_V W) = \nabla^N_{\phi_* V}\phi_* W. \tag{5.46}$$

To see this, we use the uniqueness of the Levi-Civita connection. Define a connection $\overline{\nabla}$ on N by (5.46),i.e.,

$$\overline{\nabla}_{\phi_* V}\phi_* W := \phi_*(\nabla^M_V W).$$

We show that $\overline{\nabla}$ is the Levi-Civita connection on N and therefore, by uniqueness, $\overline{\nabla} = \nabla^N$.

- (Metric)

$$g_N(\overline{\nabla}_{\phi_* V}\phi_* W, \phi_* U) + g_N(\phi_* W, \overline{\nabla}_{\phi_* V}\phi_* U)$$

$$\begin{aligned}
&= g_N(\phi_*(\nabla^M_V W), \phi_*(U)) + g_N(\phi_*(W), \phi_*(\nabla^M_V U)) \\
&= g_M(\nabla^M_V W, U) + g_M(W, \nabla^M_V U) \\
&= V g_M(W, U) \\
&= \phi_* V g_N(\phi_* W, \phi_* U),
\end{aligned}$$

  where in the last equality we used the fact that if $m(t)$ is a curve in M such that $\frac{d}{dt}m(t) = V$ and $m(0) = m$, then

$$\phi_*|_m V g_N(\phi_*|_m W, \phi_*|_m U) = \frac{d}{dt} g_N(\phi_*|_{m(t)} W, \phi_*|_{m(t)} U) = \frac{d}{dt} g_M(W, U) = V g_M(W, U).$$

- (Symmetry)

$$\begin{aligned}\overline{\nabla}_{\phi_* V}\phi_* W - \overline{\nabla}_{\phi_* W}\phi_* V &= \phi_*(\nabla^M_V W) - \phi_*(\nabla^M_W V)\\ &= \phi_*(\nabla^M_V W - \nabla^M_W V)\\ &= \phi_*([V, W])\\ &= [\phi_* V, \phi_* W],\end{aligned}$$

since Lie bracket is preserved by diffeomorphism.

□

Therefore, since $\Psi(\gamma \setminus Z_f) = \Gamma^1_{up} \setminus B_f$, the mean curvature of $\Gamma^1 \subset (X^1, g_{X^1})$ away from the branch points of X^1 is equal to the pull-back of the mean curvautre of $\gamma \subset (\mathbb{C}, hg_{\mathbb{C}})$ away from the roots of f. We denote this quantity by κ^1. By Lemma 5.8.7, the mean curvature of $\gamma \subset (\mathbb{C}, hg_{\mathbb{C}})$ is

$$\kappa^1 = \frac{1}{h}\left(\kappa - \frac{1}{2}N(\log h)\right), \tag{5.47}$$

where κ is curvature of γ and N is the unit normal to γ in $(\mathbb{C}, g_{\mathbb{C}})$.

Proposition 5.5.5. *Let $\pi_{n*}J\iota_*\nabla\theta$ be as in (5.31), and let $\Psi : \mathbb{C} \setminus Z_f \to X^1 \setminus B_f$ be as in (5.42). Then,*

$$\Psi_*\pi_{n*}J\iota_*\nabla\theta$$

$$= \left(\kappa^1 - \frac{1}{2}(n-1)N^1(\log|f(\gamma)|) + \frac{1}{2}N^1\left(\log(|f(\gamma)| + |\partial_w f(\gamma)|^2/4)\right)\right)N^1, \tag{5.48}$$

where κ^1 and N^1 are the curvature and unit normal vector to $\Gamma^1 \subset X^1$. Since $\pi_{n*}J\iota_*\nabla\theta = 0$ at the endpoints of γ, we extend this vector field to all of Γ^1 by setting it equal to 0 at the branch points. See Figure 5.3.

Proof. By (5.31), we want to show that (5.48) is equal to

$$\Psi_*\left(\frac{1}{1+\frac{|\partial_w f(\gamma)|^2}{4|f(\gamma)|}}\big(\kappa+(1-n/2)N(\log|f(\gamma)|)\big)N\right). \tag{5.49}$$

Note that we are using Lemma 5.5.4 to identify, away from the branch points, the curvature and normal of $\Gamma^1 \subset (X^1, g_X)$ with the curvature and normal of $\gamma \subset (\mathbb{C}, hg_{\mathbb{C}})$, where, as in (5.44),

$$h = 1+\frac{|\partial_w f(\gamma)|^2}{4|f(\gamma)|}.$$

Then, the normal vector to $(\mathbb{C}, hg_{\mathbb{C}})$ is given by

$$N^1 = \frac{1}{h^{1/2}}N,$$

and (5.49) becomes

$$\Psi_*\left(\frac{1}{h}\big(\kappa+(1-n/2)N(\log|f(\gamma)|)\big)N\right).$$

Motivated by formula (5.47), we express this as

$$\begin{aligned}
&\Psi_*\left(\frac{1}{h}\big(\kappa+(1-n/2)N(\log|f(\gamma)|)\big)N\right)\\
&\qquad=\Psi_*\left(\frac{1}{h}\big(\kappa-\frac{1}{2}N(\log h)+\frac{1}{2}N(\log h)+(1-n/2)N(\log|f(\gamma)|)\big)N\right)\\
&\qquad=\kappa^1N^1+\Psi_*\left(\frac{1}{h}\left(\frac{1}{2}N(\log h)+(1-n/2)N(\log|f(\gamma)|)\right)N\right).
\end{aligned} \tag{5.50}$$

Expanding the $N(\log h)$ term:

$$N(\log h) = N\log\left(\frac{|f(\gamma)|+|\partial_w f(\gamma)|^2/4}{|f(\gamma)|}\right) = N\left(\log(|f(\gamma)|+|\partial_w f(\gamma)|^2/4)\right)-N(\log|f(\gamma)|).$$

Thus,

$$\frac{1}{h}\left(\frac{1}{2}N(\log h)+(1-n/2)N(\log|f(\gamma)|)\right)N$$

$$= \frac{1}{h}\left(\frac{1}{2}N\left(\log(|f(\gamma)| + |\partial_w f(\gamma)|^2/4)\right) - \frac{1}{2}N(\log|f(\gamma)|) + (1-n/2)N(\log|f(\gamma)|)\right)N$$
$$= \frac{1}{h}\left(-\frac{1}{2}(n-1)N(\log|f(\gamma)|) + \frac{1}{2}N\left(\log(|f(\gamma)| + |\partial_w f(\gamma)|^2/4)\right)\right)N.$$

In sum,

$$\Psi_*\left(\frac{1}{h}\left(\frac{1}{2}N(\log h) + (1-n/2)N(\log|f(\gamma)|)\right)N\right)$$
$$= \left(-\frac{1}{2}(n-1)N^1(\log|f(\gamma)|) + \frac{1}{2}N^1\left(\log(|f(\gamma)| + |\partial_w f(\gamma)|^2/4)\right)\right)N^1.$$

Plugging into (5.50), gives (5.49). $\square$

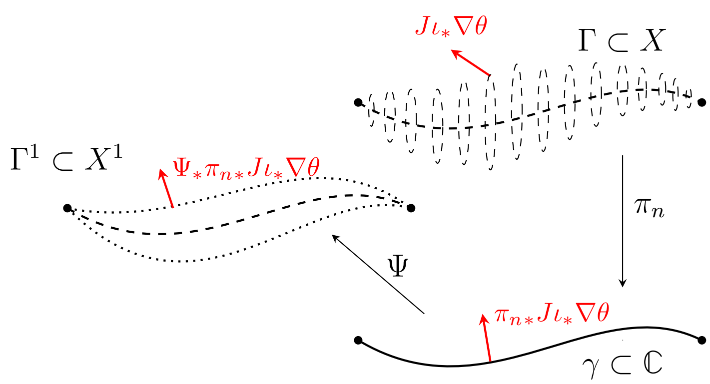

Figure 5.3: Projected and lifted flow

5.5.1 Statement of the modified Thomas–Yau conjecture

Definition 5.5.6. *Let $L(t)$, for $t \in [0, T)$, be a family of Lagrangians in the almost Calabi–Yau manifold, X. We say that the $L(t)$ evolves by the almost Lagrangian mean curvature flow if*

$$\frac{d}{dt}L(t) = J\iota_*\nabla\theta, \quad L(0) = L. \tag{5.51}$$

More precisely, if $\iota_t : L \to L(t)$ is a family of embeddings, such that $\iota_t(L) = L(t)$, then, $\frac{d}{dt}L(t) := \frac{d}{dt}\iota_t(L)$.

For an almost Calabi–Yau manifold (X, Ω, ω, J) and graded Lagrangian $L \subset X$

$$\Omega|_L = \rho^{n/2} e^{\sqrt{-1}\theta} dV_{g_L},$$

where

$$\rho : L \to \mathbb{R}_{>0}$$

is defined by

$$\rho^n \frac{\omega^n}{n!} = (-1)^{n(n-1)/2} (\sqrt{-1}/2)^n \Omega \wedge \overline{\Omega}, \tag{5.52}$$

and measures the deviation of the metric $g = \omega(\cdot, J\cdot)$ from being a Calabi–Yau metric. When $\rho = 1$, g is Ricci-flat. The following proposition [41, Section 7] generalizes Proposition 5.2.11.

Proposition 5.5.7. *Let X be almost Calabi–Yau, and set $\phi := \rho^{n/2}$, where ρ is defined by (5.52). Then, under the flow of a Lagrangian by $J\iota_*\nabla\theta$, the function θ satisfies*

$$\frac{d}{dt}\theta = \Delta\theta + \frac{\langle d\theta, d\phi\rangle}{\phi} \quad \text{and} \quad \frac{d}{dt}\left(\phi\, dV_{g_L}\right) = -|d\theta|^2 \left(\phi\, dV_{g_L}\right). \tag{5.53}$$

Proof. Expanding out the derivative

$$\frac{d}{dt}\left(e^{\sqrt{-1}\theta}\phi\, dV_{g_L}\right) = \sqrt{-1}e^{\sqrt{-1}\theta}\frac{d\theta}{dt}\phi\, dV_{g_L} + e^{\sqrt{-1}\theta}\frac{d}{dt}\left(\phi\, dV_{g_L}\right). \tag{5.54}$$

By Lie derivative and Cartan's formula

$$\frac{d}{dt}\left(e^{\sqrt{-1}\theta}\phi\, dV_{g_L}\right) = \mathcal{L}_{J\iota_*\nabla\theta}\Omega|_L = d\left(\iota_{J\iota_*\nabla\theta}\,\Omega\right)|_L. \tag{5.55}$$

Writing out the right-hand side of (5.55),

$$\begin{aligned}\sqrt{-1}d\left(\phi e^{\sqrt{-1}\theta}\iota_{\nabla\theta}\, dV_{g_L}\right) &= \left(-\phi e^{\sqrt{-1}\theta}d\theta + \sqrt{-1}e^{\sqrt{-1}\theta}d\phi\right)\wedge(\iota_{\nabla\theta}\, dV_{g_L}) \\ &\quad - \sqrt{-1}\phi e^{\sqrt{-1}\theta}d^*d\theta\, dV_{g_L} \\ &= \left(-\phi e^{\sqrt{-1}\theta}d\theta\wedge\iota_{\nabla\theta}\, dV_{g_L} + \sqrt{-1}e^{\sqrt{-1}\theta}d\phi\wedge\iota_{\nabla\theta}\, dV_{g_L}\right) \\ &\quad - \sqrt{-1}\phi e^{\sqrt{-1}\theta}d^*d\theta\, dV_{g_L}.\end{aligned}$$

Using the identity $\alpha\wedge\iota_{\widetilde{\beta}}\, dV_{g_L} = \langle\alpha,\beta\rangle\, dV_{g_L}$, for 1-forms α and β, where $\langle\cdot,\cdot\rangle$ is the metric-induced inner product on forms, and $d^*d = \Delta$,

$$\frac{d}{dt}\left(e^{\sqrt{-1}\theta}\phi\, dV_{g_L}\right) = -\phi e^{\sqrt{-1}\theta}|d\theta|^2\, dV_{g_L} + \sqrt{-1}e^{\sqrt{-1}\theta}\langle d\theta, d\phi\rangle\, dV_{g_L} - \sqrt{-1}e^{\sqrt{-1}\theta}\phi\Delta\theta\, dV_{g_L}.$$

Equating with (5.54) proves the proposition. □

In this setting of almost Calabi–Yau manifolds, we refer to the following as the modified Thomas–Yau conjecture:

Conjecture 5.5.8. *Let L be a compact, graded, Lagrangian submanifold of an almost Calabi–Yau manifold X. If L satisfies*

$$[\phi(L_1), \phi(L_2)] \not\subset (\inf_L \theta, \sup_L \theta), \tag{5.56}$$

for all graded connect sums $[L]_{Ham} = [L_1 \# L_2]_{Ham}$, then the almost Lagrangian mean curvature flow (Definition 5.5.6) of L exists for all time and converges to a special Lagrangian in its Hamiltonian deformation class.

Thomas–Yau stated the following result which would resolve Conjecture 5.5.8, modulo a technical assumption (5.57), for $O(n)$-invariant Lagrangian spheres in Milnor fibers [41, Theorem 7.6]:

Theorem 5.5.9. *Let $\Gamma \subset X^n$ be an $O(n)$-invariant Lagrangian sphere. Suppose the phase, θ, of Γ satisfies condition (5.56), and also*

$$\sup_\Gamma \theta - \inf_\Gamma \theta < \frac{2\pi}{3}. \tag{5.57}$$

Then the almost Lagrangian mean curvature flow (5.51) of Γ exists for all time and converges in C^∞ to a smooth special Lagrangian.

Remark 5.5.10. *As noted by Thomas–Yau, it is enough to assume (5.56) is satisfied for just $O(n)$-invariant Lagrangian spheres L_1, L_2.*

It seems to us that their proof in fact requires a slightly different technical assumption: see equation (5.59) below which we use to replace equation (5.57) above. Moreover, we need to assume that:

There exists C^1 solutions, γ_c, to the one-dimensional

initial value problem for the special Lagrangian equation:

$$\begin{gathered} \widetilde{\arg}(\gamma_c') + (n/2-1)\widetilde{\arg}(f(\gamma_c)) = c, \qquad \text{for } c = \inf_\Gamma \theta, \sup_\Gamma \theta, \\ \gamma_c(0) = \text{ any of the two roots of } f \text{ that are the endpoints of } \gamma \text{ (5.30).} \end{gathered} \tag{5.58}$$

Theorem 5.5.11. *Let $\Gamma \subset X^n_f$ be an $O(n)$-invariant Lagrangian sphere. Suppose the grading, θ, of Γ satisfies condition (5.56), that (5.58) holds, and that*

$$\sup_\Gamma \theta - \inf_\Gamma \theta < \begin{cases} \pi, & \text{if } n = 1, \\ \frac{\pi n}{2(n-1)}, & \text{if } n > 1. \end{cases} \tag{5.59}$$

Then the almost Lagrangian mean curvature flow (5.51) of Γ exists for all time and converges in C^∞ to a smooth special Lagrangian.

5.5.2 Formulations of the almost Lagrangian mean curvature flow in Milnor fibers

The Milnor fiber X is an almost Calabi–Yau manifold (Proposition 5.4.3). Let $\Gamma_{initial}$ be the $O(n)$-invariant Lagrangian sphere in X, with grading $\theta_{initial} : \Gamma \to \mathbb{R}$ fibered over the regular, simple curve $\gamma_{initial} : [0,1] \to \mathbb{C}$. In Section 5.5, we saw that due to the high degree of symmetry of Γ there are several ways of representing the vector $J\iota_* \nabla\theta$ (see Propositions 5.5.2 and 5.5.5). These different representations allow for different formulations of the almost Lagrangian mean curvature flow (Definition 5.5.6).

(i) *The flow of the curve γ in $(\mathbb{C}, g_{\mathbb{C}})$, with fixed endpoints.*

Let $\gamma = \gamma(s,t)$, then

$$\gamma_t = \frac{1}{1 + |\partial_w f(\gamma)|^2/4|f(\gamma)|}\big(\gamma_{ss} + (1 - n/2)N(\log|f(\gamma)|)\big)N, \tag{5.60}$$

$$\gamma(s,0) = \gamma_{initial}(s), \qquad \gamma(s_i,t) = \gamma_{initial}(s_i), \quad i = 0,1, \quad \forall t$$

where t is time, s is arclength, N is the upward-pointing unit normal to $\gamma(s,t)$, and $s_0 = 0$, $s_1 = s_1(t)$ are the pre-images of the endpoints of $\gamma(\cdot,t)$. See Proposition 5.5.2.

(ii) *The flow of the closed curve $\Gamma^1 \cong S^1$ in (X^1, g_{X^1}).*

Let $\Gamma^1 = \Gamma^1(s,t)$, then

$$\Gamma^1_t = \left(\kappa^1 - \frac{1}{2}(n-1)N^1(\log|f(\pi_1(\Gamma^1))|)\right)N^1 \tag{5.61}$$

$$+\frac{1}{2}N^1(\log(|f(\pi_1(\Gamma^1))| + |\partial_w f(\pi_1(\Gamma^1))|^2/4))N^1,$$

$$\pi_1(\Gamma^1(s,0)) = \gamma_{initial} \qquad .$$

where κ^1 is the curvature and N^1 is the upward-pointing unit normal to $\Gamma^1 \subset (X^1, g_{X^1})$. At the branch points Γ^1 is fixed, i.e. $\Gamma^1(0,t) = \Gamma^1(0,0)$ and $\Gamma^1(s_1(t),t) = \Gamma^1(s_1(0),0)$. See Proposition 5.5.5.

Note: This is the lift of the flow (5.60), *not* the $J\iota_*\nabla\theta$ flow on Γ^1. See Remark 5.5.12.

(iii) *The almost Lagrangian mean curvature flow of* $\Gamma \subset (X, g_X)$.

Let $\Gamma = \Gamma(t)$, then

$$\frac{d}{dt}\Gamma(t) = J\iota_{t*}\nabla^{g_{\Gamma(t)}}\theta, \tag{5.62}$$

where $\iota_t : \Gamma(t) \to X$ is the inclusion map and where θ is a grading of the Lagrangian angle $\theta_{\Gamma(t)}$. See Definition 5.5.6.

(iv) A consequence of flow (iii) is the following n-dimensional evolution equation for θ along the flow. See Proposition 5.5.7.

Let $\theta = \theta(p,t)$, then

$$\theta_t = \Delta\theta + \frac{\langle d\theta, d\phi.\rangle}{\phi} \tag{5.63}$$

$$\theta(p,0) = \theta_{initial},$$

where $\theta_{initial}$ is a grading of θ_Γ

Remark 5.5.12. One can think of flow (ii) in the following way. The $J\iota_*\nabla\theta$ flow (iii) of $\Gamma \subset X$ corresponds to the flow (i) of $\gamma \subset$ with fixed endpoints (Propositions

5.5.2). Given a flow of γ we can look at the corresponding family of lifted curves $\Gamma^1 \subset X^1$ over these curves in $\mathbb{C}$ (Definition 5.24). Since the flow of $\gamma \subset \mathbb{C}$ keeps the endpoints fixed, the family of lifted closed curves $\Gamma^1 \subset X^1$ is fixed at the branch points. One can look at how this family Γ^1 evolves in coordinates (5.42) and this evolution equation is (ii). Roughly speaking, looking at the flow in $\mathbb{C}$ (i) is not as natural as looking at the flow in X^1 (ii) because the projection map π_n collapses the geometry of Γ at its ends (where it closes up) to a point. This is why the flow in $\mathbb{C}$ degenerates at the endpoints (in the sense that the coefficients vanish at the endpoints). This corresponds to the fact that the flow vector $J\iota_*\nabla\theta$ vanishes as the ends of Γ. However, the $J\iota_*\nabla\theta$ is a uniformly parabolic equation (the curvature itself vanishes at the ends, but the coefficient does not vanish). The remedy to this is to lift the flow of $\gamma \in \mathbb{C}$ to Γ^1. Here, the flow is uniformly parabolic (the coefficient of curvature term no longer degenerates). The reason for this, as suggested in [41, Section 6], is that X^1 is canonically embedded in X and thus maintains the geometry at the endpoints.

5.6 Proof of Theorem 5.5.9

In this section we provide an exposition of the proof of Theorem 5.5.9 [41, Section 7] with some additional details and corrections.

5.6.1 Long-time existence

Using formulation (5.61), we show that the flow exists as long as the curvature stays bounded [41, Lemma 7.7].

Lemma 5.6.1. *Suppose that the flow (5.61) does not exists up to time T. Then*

$$\sup_{t\in[0,T)} \sup_{a\in[0,1]} |\kappa^1(\Gamma^1(a,t))|_{g_{X^1}} = \infty.$$

Proof. Suppose, for the sake of contradiction, that the curvature $|\kappa^1(\Gamma^1(a,t)|_{g_{X^1}}$ is uniformly bounded for $t < T$.

1. (Short-time existence) Given any initial regular, $C^{2,\alpha}$ curve $\gamma_{initial}$ two zeros of f, short-time existence for the flow is given by [1, Theorem 3.1]. (See Theorem 5.8.8 and note that the function V for our flow (5.61) satisfies $\frac{\partial V}{\partial k} = 1$.)

2. (Bounded flow vector) If κ^1 is uniformly bounded for $t < T$, then the flow vector, i.e., the right-hand side of (5.61):

$$\left(\kappa^1 - \frac{1}{2}(n-1)N^1(\log|f|) + \frac{1}{2}N^1(\log(|f| + |\partial_w f|^2/4))\right) N^1$$

will also remain bounded. The third term is the derivative of a smooth bounded function (by the simpleness of the roots of f), and therefore bounded. And the second term, recalling (Section 5.8.2) that $N^1 = h^{-1/2}N$, where $h = 1 + \frac{|\partial_w f|^2}{4|f|}$ and N is the unit normal to γ in $\mathbb{C}$, can be written as

$$N^1(\log|f(\gamma)|)N^1 = h^{-1}N(\log|f(\gamma)|)N = \frac{1}{1+\frac{|\partial_w f(\gamma)|^2}{4|f(\gamma)|}} \frac{1}{|f(\gamma)|} N(|f(\gamma)|)N.$$

This simplifies to

$$\frac{1}{|f(\gamma)| + |\partial_w f(\gamma)|^2/4} N(|f(\gamma)|) N < \max\left\{\frac{4}{|\partial_w f(\gamma)|}, \frac{1}{|f(\gamma)|}\right\},$$

which is bounded by the simpleness of the roots of f.

3. (C^1 limit curve) Since the flow vector is uniformly bounded for $t < T$, the flow will converge point-wise to a limit curve, which we denote by Γ^1_T. Recall (Section 5.8.2) that the curvature of a curve on a surface is given by the covariant derivative of the unit tangent vector, i.e., if $\Gamma^1(s,t)$ is parametrized with respect to arclength,

$$\kappa^1 = \left|\frac{D}{ds}\frac{d\Gamma^1(s,t)}{ds}\right|_{g_{X^1}}.$$

A bound on the curvature then implies a bound on the first and second derivatives with respect to arclength of the curves $\Gamma^1(s,t)$, for $t < T$. Therefore, by Arzela-Ascoli, there exists a subsequence which converges in C^1 to a C^1 limit curve. By the Lebesgue dominated convergence theorem, this limit curve has bounded, possibly weak, curvature, i.e., the second derivative may only be in L^1. By the uniqueness of the point-wise limit, this limit curve must be Γ^1_T.

4. ($C^{2,\alpha}$ limit curve) The goal now is to show that the flow can be continued beyond time T. To accomplish this, they use the classical theory of parabolic equations [23] to show that Γ^1_T is sufficiently smooth with Hölder continuous curvature, i.e., $\Gamma^1_T \in C^{2,\alpha}$ for some $\alpha > 0$. One can then use a result of Angenent [1, Theorem 3.1] (see Theorem 5.8.8) on the short-time existence of $C^{2,\alpha}$ curves on surfaces. Indeed, the key assumption in that theorem is

satisfied as we now check. In the notation of Theorem 5.8.8, the flow (5.61) corresponds to a function V of the form

$$V(E,k) := k - W(E),$$

so that that $\frac{dV}{dk} = 1 > 0$.

This is accomplished as follows. Since the phase θ on Γ is $O(n)$-invariant, i.e., only depends on arclength, bounds on the derivative of the phase of Γ^1 give bounds on the derivative of the phase of Γ via (5.29):

$$\theta = \widetilde{\arg}(\gamma') + \left(\frac{n}{2} - 1\right)\widetilde{\arg}(f(\gamma)).$$

Thus, the phase function θ on Γ is C^0 convergent to the phase of Γ_T.

Recall that along the flow the phase function satisfies the following uniformly parabolic n-dimensional equation (5.53):

$$\frac{d\theta}{dt} = \Delta\theta + \frac{\langle d|\Omega|, d\theta\rangle}{|\Omega|}.$$

Differentiating this equation with respect to arclength and expressing it in local coordinates gives a uniformly parabolic equation with bounded coefficients and a bounded solution, θ_s, for $t \in [0,T]$. By parabolic regularity [23, Section III, Theorem 10.1], θ_s is α-Hölder continuous for some $\alpha > 0$ and so Γ^1_T is $C^{2,\alpha}$. By the short-time existence for curves with Hölder continuous initial data [1, Theorem 3.1], the flow exists for some time $t > T$. □

Lemma 5.6.2. *While the flow (5.60) exists γ cannot form a 180° kink. More precisely, suppose the flow (5.60) exists for $t \in [0,T]$. Then, the quantity:*

$$\limsup_{|s-s'|\to 0} |\widetilde{\arg}(\gamma'(s,t)) - \widetilde{\arg}(\gamma'(s',t))| < \pi, \tag{5.64}$$

for every fixed $t \in [0, T]$*, where* s *is the arclength along* $\gamma(\cdot, t)$*.*

Proof. First, we show that γ must stay at a bounded distance from the other roots of f. To do this we will use the maximum principle and the stability condition. We begin by giving a simple proof of this for the $n = 2$ case, and then give a more general argument for arbitrary n.

For $n = 2$, the stability condition becomes:

$$[\phi(\Gamma_1), \phi(\Gamma_2)] \not\subset (\inf_{\Gamma} \theta, \sup_{\Gamma} \theta) = (\inf_{\gamma} \widetilde{\arg}(\gamma'), \sup_{\gamma} \widetilde{\arg}(\gamma')), \tag{5.65}$$

for all graded connect sums $[\Gamma_1 \# \Gamma_2]_{Ham} = [L]_{Ham}$.

Let w_1 and w_2 be the roots of f corresponding to the endpoints of γ, and let w_3 be another root of f. By stability, w_3 cannot be "under" the graph of γ in the sense that if we concatenate with γ the straight line path from w_2 to w_1, this closed loop does not contain w_3. Taking Γ_1 to be the SLag (corresponding to the straight line path γ_1 from w_3 to w_2) and Γ_2 to be the SLag (corresponding to the straight line path γ_2 from w_1 to w_3), this follows immediately from the mean value theorem. See Figure 5.4.

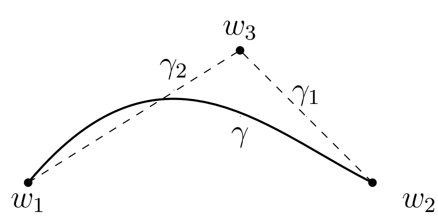

Figure 5.4: Stable curve

Therefore, γ is "under" w_3, in the above sense, and we want to show that it is precluded from approaching w_3. Observe, by (5.65), that either

$$\sup_\gamma \widetilde{\arg}(\gamma') \leq \gamma_2' \quad \text{or} \quad \inf_\gamma \widetilde{\arg}(\gamma') \geq \gamma_1'.$$

Without loss of generality, assume $\inf_\gamma \widetilde{\arg}(\gamma') \geq \gamma_1'$. Recall that $\theta = \widetilde{\arg}(\gamma')$ satisfies the uniformly parabolic equation (5.63) on the compact manifold Γ. Thus, by the maximum principle, unless θ is constant, the infimum of θ will increase, i.e., at any later time, t_1, (for which the flow exists) there exists $\epsilon > 0$, such that

$$\inf_\Gamma \theta(t_1) = \inf_\gamma \widetilde{\arg}(\gamma'(\cdot, t_1)) > \gamma_1' + \epsilon, \tag{5.66}$$

and this remains true for all $t > t_1$ by the maximum principle.

Now, suppose, for the sake of contradiction, that γ approached w_3. Then, the average slope of γ between the points where γ approached w_3 and the endpoint of γ at w_2 approaches γ_1'. Therefore, by the mean value theorem, there exists a a point on γ where

$$\gamma' < \gamma_1' - \epsilon.$$

However, this contradicts (5.66).

For $n \neq 2$, the grading θ does not have such a simple form, so we give a more general argument using the cohomological phase. Recall (Definition 5.3.2) that the phase of a graded Lagrangian L is defined as

$$\phi(L) := \arg\left(\int_L \iota^*\Omega\right) = \arg\left(\int_L e^{\sqrt{-1}\theta} dV_{g_L}\right),$$

and depends only on the homology class of L. For a Lagrangian sphere, Γ, fibered over γ, this becomes

$$\phi(\Gamma) = \arg\left(\int_\gamma e^{\sqrt{-1}\theta} da\right). \tag{5.67}$$

As above, Γ satisfies the stability condition

$$[\phi(\Gamma_1), \phi(\Gamma_2)] \not\subset (\inf_\Gamma \theta, \sup_\Gamma \theta),$$

for all graded connect sums $[\Gamma_1 \# \Gamma_2]_{Ham} = [\Gamma]_{Ham}$. So assume, without loss of generality, that

$$\inf_\Gamma \theta \geq \phi(\Gamma_1). \tag{5.68}$$

As above, unless θ is constant, the maximum principle, for a uniformly parabolic equation on a compact manifold, implies that for any later time, $t_1 > 0$, (for which the flow exists), there exists $\epsilon > 0$ such that

$$\inf \theta(t) > \phi(\Gamma_1) + \epsilon, \tag{5.69}$$

for all $t \geq t_1$.

Now, suppose that γ approaches another root of f at $w_3 \in \mathbb{C}$. Then we can choose $(a_i, t_i)_{i=1}^\infty$ be such that

$$\lim_{i\to\infty} \gamma(a_i, t_i) = w_3.$$

Define

$$\gamma_{a_i,t_i} := \gamma(\cdot, t_i) : [a_i, 1] \to \mathbb{C},$$

and let $\overline{\gamma_{a_i,t_i}}$ denote any smooth continuation of γ_{a_i,t_i} to w_3. Note, by construction,

$$\lim_{i\to\infty} \text{length}(\overline{\gamma_{a_i,t_i}}) = 0. \tag{5.70}$$

Denote by Γ_{a_i,t_i} the Lagrangian fibered over the curve

$$\gamma_{a_i,t_i} \cup \overline{\gamma_{a_i,t_i}},$$

which connects w_3 to w_2. Now, since Γ_{a_i,t_i} is homologous to Γ_1, we have that

$$\phi(\Gamma_1) = \phi(\Gamma_{a_i,t_i}).$$

We can expand

$$\phi(\Gamma_{a_i,t_i}) = \arg\left(\int_{\overline{\gamma_{a_i,t_i}}} e^{\sqrt{-1}\theta} + \int_{\gamma_{a_i,t_i}} e^{\sqrt{-1}\theta} da\right).$$

By (5.70), the contribution of $\overline{\gamma_{a_i,t_i}}$ to the integral becomes negligible. Giving us

$$\lim_{i\to\infty} \arg\left(\int_{\gamma_{a_i,t_i}} e^{\sqrt{-1}\theta} da\right) = \phi(\Gamma_1),$$

which contradicts (5.68).

We note that this same argument works to eliminate the possibility of γ intersecting one of its own endpoints at some time during the flow. Thus, knowing that γ cannot approach another root of f (or its endpoints), the proof of (5.64) can be divided into two cases:

- (Case 1) Outside a fixed neighborhood of the endpoints of γ, i.e., $s \in (s_1, s_2)$.

  When $n > 1$, recall that, by assumption, the initial curve γ, i.e., at $t = 0$, satisfies:

  $$\sup_{\Gamma} \theta - \inf_{\Gamma} \theta < \frac{\pi n}{2(n-1)}. \tag{5.71}$$

  Along the flow θ evolves by the parabolic equation (5.63), and thus satisfies the maximum principle, i.e., it achieves its maximum and minimum on the parabolic boundary. Since Γ is a closed manifold this means these both occur at time $t = 0$, and thus the initial bound on θ in (5.71) is preserved by the flow.

Outside of a fixed neighborhood of the endpoints of γ, the variation in $\widetilde{\arg}(f(\gamma))$ approaches 0 as $|s-s'| \to 0$, by continuity. Thus, since (5.71) is preserved by the flow, plugging into (5.29) gives

$$\limsup_{|s-s'|\longrightarrow 0} |\widetilde{\arg}(\gamma'(s,t)) - \widetilde{\arg}(\gamma'(s',t))| < \frac{\pi n}{2(n-1)} < \pi.$$

When $n = 1$, the same exact argument goes through with $\frac{\pi n}{2(n-1)}$ replaced by π in (5.71).

- (Case 2) Inside an arbitrarily small neighborhood of the endpoints of γ, i.e., $s \in (0, \epsilon)$.

In this case, one no longer has control over the variation of $\widetilde{\arg}(f(\gamma))$ since we are arbitrarily close to a root of f, so the desired bound is obtained less directly in the following way. Consider the curves in $\mathbb{C}$ representing special Lagrangians of phase $\sup_\gamma \theta_\Gamma$ and $\inf_\gamma \theta_\Gamma$ emanating from a root of f, i.e., the curves γ_S, γ_I in $\mathbb{C}$ solving

$$\widetilde{\arg}(\gamma_S') + (n/2-1)\widetilde{\arg}(f(\gamma_S)) = \sup_\gamma \theta_\Gamma,$$

and

$$\widetilde{\arg}(\gamma_I') + (n/2-1)\widetilde{\arg}(f(\gamma_I)) = \inf_\gamma \theta_\Gamma.$$

These ODEs can be solved with solutions emanating from either of the roots of f by assumption (5.58) (see also [30, Section 5] for some heuristics).

We claim that in the tangent space to the root of f this gives a cone of angle

$$\frac{2}{n}\left(\sup_\gamma \theta_\Gamma - \inf_\gamma \theta_\Gamma\right) < \pi/(n-1),$$

which γ lies inside and cannot cross at any time during the flow (5.60). To see this, note that, because the roots of f are simple, we can approximate $f(\gamma(s))$ near the endpoint ($s = 0$) using Taylor's theorem as follows:

$$f(\gamma(s)) = \partial_w f(\gamma(0))\gamma'(0)s + o(s).$$

Thus, since the $\widetilde{\arg}$ function is smooth (by Cauchy–Riemann equations), we can approximate $\widetilde{\arg}(f(\gamma))$ near the endpoint by

$$\widetilde{\arg}(f(\gamma(s))) = \widetilde{\arg}(\partial_w f(\gamma(0)) + \widetilde{\arg}(\gamma'(0)) + o(s). \tag{5.72}$$

Therefore, near a root, the above equations become, up to second-order,

$$\widetilde{\arg}(\gamma'_S) + (n/2 - 1)[\widetilde{\arg}(\partial_w f) + \widetilde{\arg}(\gamma'_S)] = \sup_\gamma \theta_\Gamma,$$

$$\widetilde{\arg}(\gamma'_I) + (n/2 - 1)[\widetilde{\arg}(\partial_w f) + \widetilde{\arg}(\gamma'_I)] = \inf_\gamma \theta_\Gamma.$$

Subtracting these from one another gives

$$(n/2)[\widetilde{\arg}(\gamma'_S) - \widetilde{\arg}(\gamma'_I)] = \sup_\gamma \theta_\Gamma - \inf_\gamma \theta_\Gamma < \frac{\pi n}{2(n-1)},$$

which gives a cone of angle

$$\widetilde{\arg}(\gamma'_S) - \widetilde{\arg}(\gamma'_I) < \pi/(n-1).$$

Thus, in a sufficiently small neighborhood of this root, γ cannot leave this cone because its phase, which we can approximate using (5.72), would then either surpass its maximum, $\sup_\gamma \theta_\Gamma$, or drop below its minimum, $\inf_\gamma \theta_\Gamma$. Therefore,

$$\widetilde{\arg}(\gamma(s,t)) < \pi/(n-1), \quad \forall s < \epsilon, \quad \forall t. \tag{5.73}$$

We can now bound the variation of $\widetilde{\arg}(f(\gamma))$ inside a similar cone near the root of f. Using Taylor's theorem again, for $w \in \mathbb{C}$ near $\gamma(0)$

$$f(w) = f(\gamma(0)) + \partial_w f(\gamma(0))w + o(w).$$

Thus,

$$\widetilde{\arg}(f(\gamma)) = \widetilde{\arg}(\partial_w f(\gamma(0))) + \widetilde{\arg}(\gamma) + o(\gamma)$$

which, using (5.73) gives a cone

$$\widetilde{\arg}(f(\gamma(s,t))) < \pi/(n-1), \quad \forall s < \epsilon, \quad \forall t. \tag{5.74}$$

Finally, to obtain the desired bound, we can write the formula for the phase (5.29) as:

$$-\widetilde{\arg}(\gamma') = -\theta + (n/2-1)\widetilde{\arg}(f(\gamma)).$$

Then, using the cone (5.74) and the fact that the bound $\sup_\gamma \theta_\Gamma - \inf_\gamma \theta_\Gamma = \frac{\pi n}{2(n-1)}$ on the variation θ is preserved by the maximum principle, we get, by the triangle inequality,

$$|\widetilde{\arg}(\gamma'(s,t)) - \widetilde{\arg}(\gamma'(s',t))| < \frac{\pi n}{2(n-1)} + (n/2-1)\frac{\pi}{(n-1)} = \pi.$$

□

By Lemma 5.6.1, the flow exists, unless the curvature of Γ^1 blows up. If the curvature were to blow-up at time $T \in \mathbb{R}_{>0} \cup +\{\infty\}$, then one can choose

$$\{s_i\}, \quad \{t_i\}, \quad i = 1, 2, \ldots,$$

such that t_i converges to T, and the curvature,

$$\kappa_i^1 := \kappa^1(\Gamma^1(s_i, t_i)), \tag{5.75}$$

is maximal over the curvatures of $\Gamma^1(s,t)$ for all s and all $t \leq t_i$.

Definition 5.6.3. *Suppose the curvature of $\Gamma^1(s,t)$ blows up at time T, and let $\{s_i\}, \{t_i\}, i = 1, 2, \ldots$ be chosen as above (5.75). The curvature is said to blow-up at the branch point $\Gamma^1(0,T)$ if*

$$|s_i| = O\left(\frac{1}{|\kappa_i^1|_{g_{X^1}}}\right). \tag{5.76}$$

The curvature is said to blow-up in the interior if

$$|s_i| >> \frac{1}{|\kappa_i^1|_{g_{X^1}}}. \tag{5.77}$$

Here, $|\cdot|$ denotes the Euclidean norm (note that s denotes arclength), and $|\cdot|_{g_{X^1}}$ denotes the norm with respect to g_{X^1}.

Lemma 5.6.4. *If the curvature κ^1 blows-up, it must blows-up at one of the branch points of Γ^1.*

More precisely, suppose

$$\sup_{t\in[0,T)} \sup_{a\in[0,1]} |\kappa^1(\Gamma^1(a,t))|_{g_{X^1}} = \infty,$$

and let $\{s_i\}, \{t_i\}, i = 1, 2, \ldots$ be chosen as above (5.75). Then,

$$|s_i| = O\left(\frac{1}{|\kappa_i^1|_{g_{X^1}}}\right).$$

Proof. For the sake of contradiction, suppose the curvature blows-up and that this blow-up occurs in the interior, as in Definition 5.6.3. Then there are two cases to consider:

- (Case 1) The blow-up occurs at a finite distance $|s_i| > \epsilon$ from either branch point of Γ^1.

Here, the flow will be a finite perturbation of mean curvature flow, satisfying the conditions of [1, Theorem 9.1] (see Theorem 5.8.9), and thus a 180° kink must occur. However, this contradicts Lemma 5.6.2.

- (Case 2) The blow-up tends to one of the branch points such that $s_i \to 0$, while $r_i := |s_i||\kappa_i^1|_{g_{X^1}} \to \infty$.

  We divide the proof that this cannot occur into the following steps:

  1. (Parabolic rescaling) For each i, following [1, Section 9], we can rescale the variables to zoom in around the blow-up point. This gives a new metric on X^1 for each i:

     $$s \mapsto |\kappa_i^1|_{g_{X^1}} s, \quad g_{X^1} \mapsto |\kappa_i^1|_{g_{X^1}} g_{X^1} \quad t \mapsto \kappa_i^{1/2}(t - t_i), \tag{5.78}$$

     where g_{X_1} is the original metric on X^1.

     The flow vector (5.61) for each i becomes:

     $$(\gamma_i)_t = \left(\kappa^1 - \frac{1}{2}(n-1)N_i^1(\log|f|_i) + \frac{1}{2}N_i^1(\log(|f|_i + |\partial_w f|_i^2/4))\right) N_i^1, \tag{5.79}$$

     where $|\cdot|_i$ is the norm and N_i^1 is the unit normal in the i^{th} metric $|\kappa_i^1| g_{X^1}$. Note that gradients and the curvature get scaled by $1/|\kappa_i^1|_{g_{X^1}}$. Thus, the curvature has a maximum over $t \leq 0$ of 1 at y_i at time $t = 0$.

  2. (Non-curvature terms vanish) We want to show that the second and third terms on the right-hand side of (5.79) converge to 0 as $i \to \infty$.

     In the i^{th} metric on X^1, take a geodesic disc of radius $r_i/2$ about y_i (where $r_i := |\kappa_i^1|_{g_{X^1}} s_i \to \infty$ as $i \to \infty$, and $y_i = \Gamma^1(s_i, t_i) \in X^1$ were chosen above).

Since $s_i \longrightarrow 0$, this is within an arbitrarily small neighborhood of z (root of f) in the original metric. Thus, by the proof of Lemma 5.6.2, the angle of γ' is less than π, which implies that $\Gamma^1_s(s,t)$ varies within an angle $\pi/2$ cone on X^1. i.e., no spiraling around the root

In the rescaled variable ($s \mapsto |\kappa^1_i|_{g_{X^1}} s$), $y_i \in \Gamma^1_i$ is at an arclength of $s = q_i = |\kappa^1_i|_{g_{X^1}} s_i \geq r_i$ from z, the root of f, at $s = 0$. Note that $r_i = s_i |\kappa^1_i|_{g_{X^1}}$, is the same as above and not rescaled, so it approaches infinity and gives us the radius of neighborhood in X^1_i.

Thus, all of the points in the geodesic disc of radius $r_i/2$ are at a distance $\geq cr_i/2$ from the root of f (for some constant $c > 0$ for all $i >> 1$) in the new metric. It follows that for i sufficiently large:

$$|f|_i^{1/2} \geq C(1/|\kappa^1_i|_{g_{X^1}})(cr_i/2),$$

where C is a constant just less than the norm of the derivative of $f^{1/2}$ at the root z in the original metric, and the $1/|\kappa^1_i|_{g_{X^1}}$ factor comes from scaling the derivative to the new metric.

We can now bound the non-curvature terms by:

$$|(\gamma^1_i)_t - \kappa^1|_{g_{X^1}} \leq (n-1)\frac{|\kappa^1_i|^{-1}_{g_{X^1}} \sup |d(f^{1/2})|_{g_{X^1}}}{C|\kappa^1_i|_{g_{X^1}}(cr_i/2)}$$
$$+\frac{1}{2}|\kappa^1_i|^{-1}_{g_{X^1}} \sup |d\log(|f| + |\partial_w f|^2/4)|_{g_{X^1}},$$

where both sups are taken over small neighborhoods of z in the original metric on X^1 (because this includes all of the blown-up neighborhoods).

In these two terms the derivative (d) is taken on X^1 with respect to the original metric and then scaled by $|\kappa_i^1|_{g_{X^1}}$ when pulled back. Both of these supremums are finite:

$$|d(f^{1/2})|_{g_{X^1}} = \left(\frac{4|f|}{4|f| + |\partial_w f|^2}\right)^{1/2} \frac{\partial_w f}{f^{1/2}} = \frac{2\partial_w f}{(4|f| + |\partial_w f|^2)^{1/2}},$$

and

$$\begin{aligned} |d \log(|f|+|\partial_w f|^2/4)_{g_{X^1}} \\ &= \left(\frac{4|f|}{4|f| + |\partial_w f|^2}\right)^{1/2} \frac{4}{4|f| + |\partial_w f|^2} \left(\frac{f}{|f|} + d(|\partial_w f|^2/4)\right) \\ &\leq \frac{8}{(4|f| + |\partial_w f|^2)^{3/2}} \left(|f|^{1/2} + d(|\partial_w f|^2/4)\right). \end{aligned}$$

As $i \longrightarrow \infty$, $|\kappa_i^1|_{g_{X^1}}, r_i \longrightarrow \infty$, so the above bound tends to root, and the radius of the disc we are working on $r_i/2 \longrightarrow \infty$. Thus, in the limit the flow is the mean curvature flow of a curve inside an infinite flat disc $\mathbb{R}^2$.

3. (Apply theorem of Angenent) By Theorem 5.8.9 and Remark 5.8.10, for the curvature to blow-up, a $180°$ kink must occur in the curve $\Gamma^1(s,t)$. However, this is a contradiction to Lemma 5.6.2.

□

Lemma 5.6.5. *The curvature κ^1 of Γ^1 does not blow-up in finite time.*

Proof. In Lemma 5.6.4 it was shown that the curvature does not blow up in the interior (in the sense of Definition 5.6.3). Thus, if it can be shown that the curvature does not blow-up at the endpoints (the only alternative), then the curvature will not blow-up in finite time.

More precisely, if the curvatures κ_i^1, at points $y_i = \Gamma^1(s_i, t_i)$, blow up, then by Lemma 5.6.2 there exists some constant A such that for all i:

$$|s_i||\kappa_i^1|_{g_{X^1}} < A.$$

Assume for the sake of contradiction that such a blow-up occurs. We first rescale the variables as above (5.78):

$$s \mapsto |\kappa_i|^1_{g_{X^1}} s, \quad g_{X^1} \mapsto |\kappa_i^1|_{g_{X^1}} g_{X^1} \quad t \mapsto |\kappa_i^1|^2_{g_{X^1}} (t - t_i), \tag{5.80}$$

where g_{X^1} is the original metric on X^1. We will work on an interval of length $|\kappa_i^1|^{1/2}_{g_{X^1}} \longrightarrow \infty$ (in the new metric) on Γ^1, centered at the root of f $(s = 0)$. Notice that this is contained inside the ball of radius $|\kappa_i^1|^{-1/2}_{g_{X^1}} \to 0$ about z in X^1 in the original metric, so for i sufficiently large we can assume $f(w) - Cw$ $(w \in \mathbb{C})$ is arbitrarily small in any C^k-norm. We then proceed as follows:

1. Obtain bounds on the polar angle of the curve and its tangent vector (recall that the derivative of the angle of tangent vector with respect to arclength is the curvature). This will involve using polar coordinates and applying Lemma 5.6.2.

   Taking i sufficiently large so that the metric on the radius $\kappa_i^{1/2}$ disc about z in X^1 is close to being flat, define geodesic polar coordinates on X^1 by

   $$r^1 := |\Gamma^1|_i \quad \text{and} \quad \theta^1 := \theta(\Gamma^1),$$

   where $|\cdot|$ and θ are length and angle on X^1, with respect to the i-th metric. So $\theta(\Gamma^1) \approx \theta(\gamma)/2$, up to a constant. Thomas–Yau then claim that θ_s^1 is arbitrarily C^1 close to

$$\frac{1}{r}\sin\left(\theta(\Gamma^1_s) - \theta^1\right), \tag{5.81}$$

which is the exact formula for θ^1_s in the flat metric and polar coordinates. Thus,

$$|r\theta^1_s| \leq \sin\left(\theta(\Gamma^1_s) - \theta^1\right).$$

By construction, the curvature of Γ^1 is bounded by 1, i.e., $|\left(\theta(\Gamma^1_s)\right)_s| \leq 1$. Thus, we can bound

$$|\theta(\Gamma^1_s)| \leq s.$$

Substituting $f = \theta(\Gamma^1_s) - \theta^1$ into (5.81), we get (in the flat setting)

$$f_s = \kappa_i - \frac{\sin f}{r},$$

for $f(0) = 0$ and $|\kappa_i| \leq 1$. This implies that $|f(s)| \leq |s|$, so for i sufficiently large that our polar coordinates are sufficiently flat one gets a bound on $|\theta(\Gamma^1_s) - \theta^1|$, which gives a bound on θ^1/s and, subsequently (by uniform comparison bounds of r and s of Lemma 5.6.4), a bound on θ^1/r.

2. Analyze equation (4) for θ_Γ on Γ in the new metrics, as $i \longrightarrow \infty$:

$$\frac{d\theta_\Gamma}{dt} = \Delta\theta_\Gamma + \frac{\langle d|\Omega|_i, d\theta_\Gamma\rangle}{|\Omega|_i}. \tag{5.82}$$

Here $|\Omega|_i$ is the pullback of $|\Omega|$ to Γ^1 with the ith metric. *They use the above polar angle estimates to show that the second term goes to zero as $i \longrightarrow \infty$. First, observe that

$$|d\theta_\Gamma| = |(\theta_\Gamma)_s| = \left|\partial_u[\theta(\Gamma^1_s)/2 + (n/2-1)\theta(f(\Gamma^1))]\right|,$$

since $\theta_\Gamma(s) := \theta(\gamma') + (n/2-1)\theta(f(\gamma))$, pulling back to Γ^1 and using the fact that $\theta(\gamma_s) \approx \theta(\Gamma^1_s)/2$. For i sufficiently large, this is then bounded by the estimates above, as $\theta(f(\Gamma^1))$ is C^1 close to $\theta^1/2$ in the disc in which we are working. i.e., we have bounds on the first term (curvature) and bounds on the second term $\partial_u\theta^1 \approx \theta^1/s$. Therefore, the second term (5.82) can be bounded by

$$\kappa_i^{-1}\frac{\sup|d\Omega|}{\inf|\Omega|},$$

where the sup and inf are taken over a small neighborhood (in the original metric) of the point $(0,\ldots,0,z) \in X^1$, where z is the endpoint of γ, root of f. This clearly goes to 0 as $i \longrightarrow \infty$.

Computing the Laplacian on the space Γ, using a radial coordinate s and rotational symmetry about the origin $s=0$, makes (5.82)

$$(\theta_\Gamma)_t = (\theta_\Gamma)_{ss} + (n-1)\frac{R^i_r}{R^i}(\theta_\Gamma)_s + O(\kappa_i^{-1}),$$

where $R^i = R^i(r)$ is the radius of the sphere S^{n-1} at r in the new metric.

We use the arclength s on $\Gamma^1 \subset X^1$, the radial coordinate $r = \kappa_i s$ on X^1, and a radial coordinate ρ on $\mathbb{C}$. Then, for i sufficiently large, and for $s \le \kappa_i^{1/2}$ in the new metric, f can be approximated linearly, i.e., $f \approx \partial_w f(z)\rho$. So we can assume that R^i is as close as we like to $\kappa_i\sqrt{|\partial_w f(z)|\rho}$ in C^2, as lengths in the new metric gets scaled by the curvature. Thus,

$$R^i_r = \frac{R_\rho}{r_\rho} = \frac{R^i_\rho}{\sqrt{\kappa_i^2 + (R^i_\rho)^2}} = \frac{1}{\sqrt{1+\frac{\kappa_i^2}{(R^i_\rho)^2}}}$$

By obtaining bounds on the derivatives of R^i as $i \longrightarrow \infty$ one can show that it converges in C^1 to some R with $R_r(0) = 1$. Moreover, the limit curve γ^1_∞ satisfies a limiting equation for θ^∞:

$$\theta^\infty_t = \theta^\infty_{ss} + (n-1)\frac{R_s}{R}\theta^\infty_s.$$

The solution to this is $\theta^\infty = \text{constant}$.

3. However, by construction $\max \theta_s = 1$ for all i, and θ_s is Hölder continuous with uniform (in i) bounds, by [23, Section III, Theorem 10.1] and the above bounds. Therefore, one can show that $\max \theta^\infty_s = 1$, which is a contradiction. Thus, the curvature cannot blow-up in this way.

□

5.6.2 Smooth convergence to a special Lagrangian

Lemma 5.6.6. *The flow (5.31) converges in C^∞ to a special Lagrangian.*

Proof. The convergence of the flow to a special Lagrangian can be seen as follows:

1. (Bounded curvature) The curvature of Γ^1 is bounded for all time. This follows from the same argument used in the proof of Lemma 5.6.5 to show that the curvature does not blow up in finite time. This implies, by the O_n-symmetry, that the curvature of Γ is uniformly bounded We also have a C^1 bound on θ (with respect to arclength and which is constant in the fibers).

2. (Convergence of subsequence in C^∞) It follows that the coefficients in

$$\frac{d\theta}{dt} = \Delta\theta + \frac{\langle d|\Omega|, d\theta\rangle}{|\Omega|}$$

have uniform C^1 bounds. (The coefficients involve only first derivatives of the metric). This implies, by parabolic Schauder estimates, [23, III. Theorem 12.1], that θ actually has uniform C^3 bounds, which then gives, again by symmetry as described above, uniform C^2 bounds on the curvature (curvature is first derivative of θ). This can be continued to give C^∞ bounds and so gives a subsequence converging in C^∞ to a special Lagrangian submanifold.

3. (Unique limit) To see that every convergent subsequence converges to the same limit, we show that θ converges in a (weighted) L^2-norm. To do this, they first rewrite the evolution equation for θ as:

$$\dot{\theta} = -\Delta^\Omega \theta := \frac{1}{|\Omega|} * d(|\Omega| * d\theta),$$

similar to how the classical Laplacian can be written as $\Delta = *d * d$, where $*$ is the Hodge star.

Then, setting $\overline{\theta}$ to be the average angle over Γ:

$$\overline{\theta} = \frac{1}{\int_\Gamma |\Omega| dV_{g_\Gamma}} \int_\Gamma \theta |\Omega| dV_{g_\Gamma},$$

which is constant on Γ, but not in time, we compute

$$\begin{aligned} \frac{d}{dt} \int_\Gamma (\theta - \overline{\theta})^2 |\Omega| dV_{g_\Gamma} &= \int_\Gamma \left[2(\theta - \overline{\theta})(-\Delta^\Omega \theta - \frac{d}{dt}\overline{\theta}) - (\theta - \overline{\theta})^2 |d\theta|^2 \right] |\Omega| dV_{g_\Gamma} \\ &\leq -2 \int_\Gamma (\theta - \overline{\theta}) \Delta^\Omega (\theta - \overline{\theta}) |\Omega| dV_{g_\Gamma}. \end{aligned} \tag{5.83}$$

Recall that on a compact manifold, the classical Laplacian can also be written as $\Delta = d^* d$, where d^* is the adjoint of d with respect to the L^2-norm, i.e.,

$$\int_\Gamma h \Delta k dV_{g_\Gamma} = \int_\Gamma h d^*(dk) dV_{g_\Gamma} = \int_\Gamma \langle dh, dk \rangle dV_{g_\Gamma}.$$

Similarly, the operator Δ^Ω can be expressed as $\Delta^\Omega = d^{*\Omega} d$, where $d^{*\Omega}$ is the adjoint of d with respect to the weighted L^2-norm $|\Omega| dV_{g_\Gamma}$ we have been using here, i.e.,

$$\int_\Gamma h\Delta^\Omega k|\Omega|dV_{g_\Gamma} = \int_\Gamma hd^{*\Omega}(dk)|\Omega|dV_{g_\Gamma} = \int_\Gamma \langle dh, dk\rangle|\Omega|dV_{g_\Gamma}.$$

Therefore, the kernel contains just the constant functions. And the uniform C^∞ bounds on the metric and $|\Omega|$ (for all t) give a uniform lower bound $\lambda > 0$ for the smallest eigenvalue of Δ^Ω.

This gives a bound of the form

$$\int_\Gamma (f\Delta^\Omega f)|\Omega|dV_{g_\Gamma} \geq \lambda \int_\Gamma f^2|\Omega|dV_{g_\Gamma}.$$

To see this, write $f = a_1 f_1 + a_2 f_2 + \ldots$, where the f_i are an orthornormal basis of eigenfunctions, corresponding to the eigenvalues $\lambda_1 < \lambda_2 < \ldots$ of Δ^Ω. Then,

$$\int_\Gamma f\Delta^\Omega f|\Omega|dV_{g_\Gamma} = \int_\Gamma (a_1 f_1 + a_2 f_2 + \ldots)(\lambda_1 a_1 f_1 + \lambda_2 a_2 f_2 + \ldots)|\Omega|dV_{g_\Gamma}.$$

By the orthonormality,

$$\begin{aligned} &= \int_\Gamma (\lambda_1 a_1^2 f_1^2 + \lambda_2 a_2^2 f_2^2 + \ldots)|\Omega|dV_{g_\Gamma} \\ &\geq \int_\Gamma \lambda_1 (a_1^2 f_1^2 + a_2^2 f_2^2 + \ldots)|\Omega|dV_{g_\Gamma} \\ &= \lambda_1 \int_\Gamma f^2|\Omega|dV_{g_\Gamma}. \end{aligned}$$

Taking $f = \theta - \overline{\theta}$ and combining with (5.83), we get

$$\frac{d}{dt}\int_\Gamma (\theta - \overline{\theta})^2|\Omega|dV_{g_\Gamma} \leq -2\lambda \int_\Gamma (\theta - \overline{\theta})^2|\Omega|dV_{g_\Gamma}.$$

Thus, since this quantity decreases monotonically in time, the convergent subsequence implies that it tends to 0 as $t \to \infty$, and we have convergence to constant phase.

$\square$

5.7 A viscosity approach

5.7.1 Introduction

In this section we study the flow (5.31) from a slightly different perspective, in that we work directly with the degenerate parabolic equation for a curve with fixed endpoint in $\mathbb{C}$ evolving by (5.31). Our main result is the following theorem:

Theorem 5.7.1. *Let Γ be a stable (5.65), $O(2)$-invariant, Lagrangian 2-sphere in the Milnor fiber X. Then there exists viscosity solutions to the almost Lagrangian mean curvature flow (5.5.6) of Γ for all time, and this flow converges in C^0 to a special Lagrangian 2-sphere.*

Although our graphical assumption is more restrictive, the constraint on the variation of the grading is weakened to π, which coincides with the modification in (5.59).

5.7.2 The equation

As above, let z_0 and z_1 be two roots of the polynomial f in $\mathbb{C}$, and let $\gamma : [0,1] \to \mathbb{C}$ be a smooth path over which L_0 is fiberd, such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$. The flow of γ then corresponds to the evolution of $u(t,x)$ by:

$$u_t = A(x,u)\frac{u_{xx}}{1+u_x^2}, \tag{5.84}$$

where

$$A(x,u) = \frac{4|f(x+iu)|}{4|f(x+iu)| + |f'(x+iu)|^2}.$$

Thus, the existence of the flow of γ corresponds to solving equation (5.84) for u on the domain

$$\Omega_T = [0,T) \times (0,1), \tag{5.85}$$

with initial and boundary conditions:

$$u(0,x) = \text{Im } \gamma(x), \quad \text{for } x \in [0,1], \tag{5.86}$$

$$u(t,0) = \text{Im } \gamma(0) \text{ and } u(t,1) = \text{Im } \gamma(1) \text{ for } t \in [0,T). \tag{5.87}$$

Observations:

1. $A(x,u)$ is well-defined because the roots of f are simple, and therefore f and f' are never zero at the same place and time.

2. Since f stays at a bounded distance from its other roots (see proof of Lemma 5.6.2), it follows that:

   For $0 < x < 1$,

   $$0 < A(x,u) \leq 1.$$

   For $x = 0$ and $x = 1$,

   $$A(x,u) = 0.$$

Thus, the equation is strictly parabolic on the interior but degenerates at the endpoints.

3. The equation is quasilinear, but because it degenerates at the boundary, the standard parabolic theory of linear and quasilinear equations [23] is not directly applicable.

5.7.3 Viscosity solutions

In this section we provide a short proof of the existence of viscosity solutions for equation (5.84).

Lemma 5.7.2. *For any $T > 0$, equation (5.84) admits viscosity solutions.*

Proof. Without loss of generality, assume that $z_0 = 0$ and $z_1 = x_1$, i.e., z_0 and z_1 lie on the x-axis, so $u(0) = u(1) = 0$. Now consider the following perturbed boundary conditions:

$$u(0,x) = \operatorname{Im}\,\gamma(x) + \epsilon, \qquad u(t,0) = u(t,1) = \epsilon \text{ for all } t \in [0,\infty). \tag{5.88}$$

Writing equation (5.84) in the more standard PDE form, we have an equation of the form:

$$u_t = a(u_x, u, x, t)u_{xx}, \tag{5.89}$$

where,

$$a(u_x, u, x, t) = \frac{A(x,u)}{1+u_x^2} = \frac{4|f(x+iu)|}{\left(4|f(x+iu)| + |f'(x+iu)|^2\right)(1+u_x^2)}.$$

With the perturbed boundary conditions (5.88) this equation is now uniformly parabolic, i.e., for all bounded $\mathcal{K} \subset \mathbb{R} \times \mathbb{R} \times \Omega \times [0,T]$ we can find positive $\lambda_{\mathcal{K}}$

and $\Lambda_{\mathcal{K}}$ such that

$$\lambda_{\mathcal{K}} \leq a(p,q,x,t) \leq \Lambda_{\mathcal{K}}, \text{ when } (p,q,x,t) \in \mathcal{K}.$$

We would like to use the Leray-Schauder theory for quasilinear parabolic equation [23] to obtain a unique solution $u^{\epsilon}(t,x) \in C^{2+1}_{x+t}(\Omega_T) \cap C(\overline{\Omega_T})$, for each $\epsilon > 0$. To prove existence one follows the standard approach: a bound on $\sup |u|$; a bound on $\sup |u_x|$; a Hölder gradient bound $|u_x|_\alpha$; and finally the application of a fixed point theorem. We will carry these steps out for each ϵ to give a solution u^ϵ. Uniqueness follows from the maximum principle.

We are ultimately seeking a solution to the limit $\epsilon \to 0$. However, taking the limit of these u^ϵ would require the above bounds to be ϵ-independent, which they are not - the Hölder gradient bound depends on the uniformly parabolic bounds, which, essentially by construction, depend on ϵ, as the original problem degenerates on the boundary. However, we do have ϵ-independence for the sup bound on u and its spatial gradient, as well as an ϵ-independent Hölder norm in time. Combining these gives us a family of solutions with a uniform Hölder bound. Hence, by Arzela-Ascoli, we can take a uniformly convergent subsequence of these solutions. Viewing the u^ϵ solutions as simply viscosity solutions and then using the fact that the uniform limit of viscosity solutions on a compact set is a viscosity solution tells us that $u^0 := \lim_{\epsilon \to 0} u^\epsilon$ is also a viscosity solution.

More precisely, for each ϵ we have the following bounds on u^ϵ:

i. **Sup bound**. By the comparison principle, [7, Lemma 4.2]:

$$\sup_{\Omega \times [0,T]} |u^\epsilon(t,x)| \leq \sup |u_0^\epsilon|,$$

where $u_0^\epsilon = u^\epsilon|_{\mathcal{P}(\Omega_T)}$ is our initial data on the parabolic boundary.

ii. **Boundary gradient estimate**. Since u_0^ϵ is time-independent, by [7, Lemma 4.3]:

$$\sup_{\substack{(x,t)\in\partial\Omega\times[0,T]\\(y,s)\in\Omega\times[0,t]}} \frac{|u(x,t)-u(y,s)|}{|(x,t)-(y,s)|} \leq L,$$

where L depends only on osc u_0^ϵ, $|u_0^\epsilon|_{1+\beta,\beta/2}$.

iii. **Global gradient estimate**. Using the above boundary estimate and the fact that the oscillation of u^ϵ is bounded, by [7, Lemma 4.5]:

$$\sup_{\Omega\times[0,T]} |u_x^\epsilon| \leq 2L.$$

iv. **Hölder bound in time**. Since $a(p,q,x,t) \leq 1$ and u^ϵ admits a global spatial gradient bound, by [7, Corollary 4.9]:

$$|u^\epsilon(x,t) - u^\epsilon(x,s)| \leq C_0|t-s|^{1/2},$$

where $C = C(L)$

Therefore, being C^1 in space and $C^{0,1/2}$ in time, u^ϵ is $C^{0,1/2}(\Omega_T)$. Since the above bounds are independent of ϵ, this gives us a family of uniformly bounded functions in $C^{0,1/2}(\Omega_T)$. Thus, by Arzelà-Ascoli there is a uniformly convergent subsequence. Let $u^0 \in C^{0,1/2}(\Omega_T)$ denote this limit.

As the uniform limit of viscosity solutions to an equation is itself a viscosity solution to that same equation, u^0 is a viscosity solution to equation (5.84). Moreover, by continuity, u^0 satisfies the appropriate boundary conditions:

$$u^0(0,x) = \operatorname{Im}\,\gamma(x) \quad \text{and} \quad u^0(t,0) = u^0(t,1) = 0.$$

$\square$

5.7.4 Convergence to special Lagrangians

Recall from the previous section that the Lagrangian angle, θ_Γ, of an $O(n)$-invariant sphere is given by:

$$\theta_\Gamma = \arg(\gamma') + \left(\frac{n}{2} - 1\right) \arg(f(\gamma)) \mod 2\pi,$$

where $\arg(z)$ denotes the the principal branch of the argument function, as in Notation 5.4.6 (iii).

In two-dimensional Milnor fibers the curves in $\mathbb{C}$ corresponding to special Lagrangian spheres are the straight lines:

$$\theta_\Gamma = \arg(\gamma') = \text{constant}.$$

Therefore, we want to show that the flow of γ, formulated in terms of the potential function u (5.84 - 5.87) converges to a straight line as $t \to \infty$.

Lemma 5.7.3. *Let $u^0 : \Omega_T \to \mathbb{R}$ be a solution to the flow (5.84), as constructed above, with initial curve γ. Then as $t \to \infty$ the solution converges to the straight line connecting $\gamma(0)$ and $\gamma(1)$.*

Proof. First consider the ϵ-perturbed problem and its solution

$$u^\epsilon(t, x) \in C^{2+1}([0, \infty) \times [0, 1]).$$

The length of the curve $u^\epsilon(t, \cdot) : [0, 1] \to \mathbb{R}$ at time t is given by

$$L(t) = \int_0^1 ds_t.$$

Along the flow (5.84) the length changes over time according to

$$\frac{d}{dt}L(t) = -\int_0^1 A_\epsilon \kappa^2 ds.$$

Since $A_\epsilon > 0$ and u^ϵ admits a global gradient bound it follows that u^ϵ converges to a straight line as $t \to \infty$.

To see that u^0 also converges to a straight line as $t \to \infty$, suppose for the sake of contradiction that it does not. Without loss of generality, assume $\gamma(0) = \gamma(1) = 0$, so that $\lim_{t\to\infty} u_\epsilon(t,x) = \epsilon$. Then there exists $\{x_n\}_{n=1}^\infty \subset [0,1]$, $t_n \to \infty$, and $\delta > 0$ such that

$$|u^0(t_n, x_n)| > \delta \quad \text{ for all } n.$$

However, this contradicts the fact that u^ϵ is a decreasing sequence of functions and $u^\epsilon \to u^0$ uniformly on $[0,t_n] \times [0,1]$, so that

$$|u^0(t_n, x_n) - u^\epsilon(t_n, x_n)| \longrightarrow 0.$$

$\square$

5.8 Appendix

In this appendix, we collect important results that are used in the earlier sections of Chapter 5.

5.8.1 Smooth Lagrangian spheres

In this section we give the proof of Solomon-Yuval [38, Proposition 3.7] that $\Gamma = \Gamma_\gamma$ (see 5.24) is in fact smooth and diffeomorphic to an n-sphere. We first introduce their notation and terminology:

Throughout, we think of S^1 as the additive quotient group $\mathbb{R}/2\pi\mathbb{Z}$, with inversion given by $-$, i.e., $-\pi = \pi$.

Definition 5.8.1. *A symmetric circle is an embedding*

$$\gamma : S^1 \to X^1 \tag{5.90}$$

$$u \mapsto (z(u), t(u)), \tag{5.91}$$

satisfying $(z(-u), t(-u)) = (-z(u), t(u))$*, for all* $u \in S^1$*, and for all* $u \neq 0, \pi$*,* $z(u) \neq 0$.

Lemma 5.8.2. *Let* $g : (-\epsilon, \epsilon) \to \mathbb{R}$ *be smooth and satisfy* $g(-x) = g(x)$ *for all* x*. Then, the composition* $h = g \circ \sqrt{} : [0, \epsilon^2) \to \mathbb{R}$ *is infinitely differentiable from the right at* 0 *and satisfies* $h'(0) = g''(0)/2$.

Lemma 5.8.3. *Let* $\alpha : (-\epsilon, \epsilon) \to X^1$*,* $x \mapsto (z(x), t(x))$*, be a smooth embedding satisfying* $z(0) = 0$*,* $z(x) \neq 0 \quad \forall x \neq 0$*, and*

$$(z(-x), t(x)) = (-z(x), t(x)) \forall x.$$

Let $c = \{t(x) : x \in (-\epsilon, \epsilon)\} \subset \mathbb{C}$*. Then the map* $\nu : [0, \epsilon^2) \to c$*,* $x \mapsto t(\sqrt{x})$ *is a diffeomorphism.*

Proof. At any $(t, z) \in X^1$, where $z \neq 0$, the projection $(t, z) \mapsto t$ is a local diffeomorphism. Thus, we need only show the regularity of ν at 0. By assumption, t is even, and so ν is infinitely differentiable from the right at 0 by the previous lemma. We need to verify that $\nu'(0) \neq 0$, or equivalently $t''(0) \neq 0$. Differentiating

$(z(x))^2 = f(t(x))$ yields

$$2z(x)z'(x) = f'(t(x))t'(x),$$

Differentiating again yields

$$2(z'(x))^2 + 2z(x)z''(x) = f''(t(x))(t'(x))^2 + f'(t(x))t''(x).$$

Since $z(0) = t'(0) = 0$, substituting $x = 0$ yields

$$2(z'(0))^2 = f'(t(0))t''(0).$$

And since α is an embedding and $t'(0) = 0$, we must have $z'(0) \neq 0$. $\square$

Proposition 5.8.4. *Let γ be a symmetric circle. Then $\Gamma = \Gamma(\gamma)$ is diffeomorphic to S^n.*

Proof. More specifically, we set

$$\phi : S^1 \times S^{n-1} \to \Gamma, \quad \text{where } (u, x) \mapsto (z(u) \cdot x, t(x))$$

and

$$\chi : S^1 \times S^{n-1} \to S^n, \quad \text{where } (u, x) \mapsto (\sin(u) \cdot x, \cos(u)).$$

There is a unique map $\Psi : S^n \to \Gamma$, which satisfies $\Psi \circ \chi = \phi$ and is a diffeomorphism.

The existence, uniqueness, and bijectivity of Ψ follow from the universal property of quotients. Both ϕ and χ are local diffeomorphisms at any (u, x) where $u \neq 0, \pi$. Hence Ψ is smooth and regular anywhere away from the two poles. Let D^n be the open n-dimensional unit ball, and parametrize the north n-hemisphere by

$$Y : D^n \to S^n, y \mapsto (y, 1 - \sqrt{1 - |y|^2}).$$

We show that $\Psi \circ Y$ is smooth and regular at $y = 0$. By assumption on γ, there is a smooth even nonvanishing function $r : S^1 \to \mathbb{C}$ such that for all $u \in S^1$,

$$z(u) = r(u) \sin(u).$$

Let $U \subset S^1$ be a small neighborhood of 0. If $y \in D^n$ and $(u, x) \in U \times S^{n-1}$ satisfy $Y(y) = \chi(u, x)$, then $u = \pm \arcsin |y|$, and it follows that

$$\Psi \circ Y(y) = (r(\arcsin |y|) \cdot y, t(\arcsin |y|)).$$

The function $y \mapsto r(\arcsin |y|)$ can written as $y \mapsto r \circ \arcsin \circ \sqrt{|y|^2}$, which is smooth by Lemma 5.8.2 since $r \circ \arcsin$ is smooth and even. The function $y \mapsto t(\arcsin |y|)$ is smooth by a similar argument. Hence $\Psi \circ Y$ is smooth, and it is regular at $y = 0$ since r is non-vanishing. The other pole can be treated similarly. $\square$

5.8.2 Curvature of planar curves

In this section we briefly recall some definitions and basic results concerning curves in the plane. For curves in the plane there is only one notion of curvature, so given a curve $\gamma \subset \mathbb{C}$ we simply refer to the curvature of γ.

Definition 5.8.5. *Let $\gamma : [0, 1] \to \mathbb{C}$ be a C^2 curve, parametrized by arclength, i.e., $|\gamma'(s)| = 1$, and let $N(s) = i\gamma'(s)$ be the upward-pointing unit normal to γ. Then the curvature of γ, denoted by κ, is defined by*

$$\kappa(s)N(s) = \gamma''(s). \tag{5.92}$$

Remark 5.8.6 (Alternative definition)**.** *An equivalent and more geometric definition of the curvature of a plane curve can be formulated in terms of the angle of the* *tangent vector. Let $\gamma : [0,1] \to \mathbb{C}$ be a C^2 curve, parametrized by arclength. Since $|\gamma'(s)| = 1$ for all s, only the direction or angle of the tangent vector is changing along the curve. Thus, $\gamma''(s)$ is a measure of how fast the angle of the tangent vector is changing:*

$$\kappa(s) = \frac{d}{ds} \arg(\gamma'(s)), \tag{5.93}$$

where $\arg(\gamma'(s))$ *is the phase of $\gamma'(s) \in \mathbb{C}$.*

As mentioned earlier, the flow of $\Gamma \subset X^1$ by the vector $J\iota_* \nabla\theta_\Gamma$ is equivalent to the flow of $\gamma \subset \mathbb{C}$ (where γ is the curve used to construct Γ). The flow of γ can be formulated as a perturbation of the mean curvature flow in $\mathbb{C}$ with a different metric. In order to show this, we need the following lemma [41, Lemma 6.6]:

Lemma 5.8.7. *Let $\langle \cdot, \cdot \rangle$ be the Euclidean metric on $\mathbb{C}$, and h a positive real-valued function on $\mathbb{C}$. Then, with respect to the metric $h\langle \cdot, \cdot \rangle$, the curvature of a curve $\gamma \subset \mathbb{C}$ is,*

$$\frac{1}{h}\left(\kappa - \frac{1}{2}N(\log h)\right),$$

where κ is curvature of and N is the unit normal to γ in $\mathbb{C}$ with the Euclidean metric $\langle \cdot, \cdot \rangle$.

Proof. Let A be the endomorphism-valued 1-form on $\mathbb{C}$ defined by

$$A_X Y = \frac{1}{2h}\left((Xh)Y + (Yh)X - \langle X, Y\rangle \nabla h\right).$$

For vector fields X, Y on $\mathbb{C}$, A is symmetric and so defines a torsion-free connection on $\mathbb{C}$ by $D + A$, where D is the usual connection on $\mathbb{C}$:

$$(D + A)_{fX} Y = f(D + A)_X Y$$

and

$$(D+A)_X fY = D_X(fY) + A_X(fY) = (df(X)Y + fD_XY) + fA_XY = f(D+A)_XY.$$

Moreover, $\overline{D} = D + A$ is the Levi–Civita connection on $\mathbb{C}$ with respect to $h\langle\cdot,\cdot\rangle$, i.e., for vector fields X, Y, Z,

$$\partial_X h\langle Y, Z\rangle = h\langle (D+A)_X Y, Z\rangle + h\langle Y, (D+A)_X\rangle \tag{5.94}$$

To see this, observe that the left-hand side of (5.94) is:

$$dh(X)\langle Y, Z\rangle + h\langle D_XY, Z\rangle + h\langle Y, D_XZ\rangle,$$

and the right-hand side of (5.94) is:

$$h\langle D_XY, Z\rangle + h\langle Y, D_XZ\rangle + h\langle A_XY, Z\rangle + h\langle Y, A_XZ\rangle.$$

Thus, we need to show

$$dh(X)\langle Y, Z\rangle = h\langle A_XY, Z\rangle + h\langle Y, A_XZ\rangle. \tag{5.95}$$

Expanding this out

$$h\langle A_XY, Z\rangle = h\langle \frac{1}{2h}\left((Xh)Y + (Yh)X - \langle X, Y\rangle Dh\right), Z\rangle,$$

so the right-hand side of (5.95) is:

$$(Xh)\langle Z, Y\rangle + \frac{1}{2}\left(\langle (Yh)X, Z\rangle + \langle (Zh)X, Y\rangle - \langle\langle X, Y\rangle Dh, Z\rangle - \langle\langle X, Z\rangle Dh, Z\rangle\right).$$

Note that $\langle\langle X, Y\rangle Dh, Z\rangle = \langle X, Y\rangle dh(Z) = \langle X, Y\rangle (Zh)$ by the definition of Dh, so the right-hand side of (5.95) reduces to just $dh(X)\langle Y, Z\rangle$.

Now let N be the unit normal to γ in the standard metric. Then $h^{-1/2}N$ is the unit normal in the metric $h\langle\cdot,\cdot\rangle$. If T is the unit tangent vector $\frac{\gamma'}{||\gamma'||}$, where $||\cdot||$ is the norm in the metric $h\langle\cdot,\cdot\rangle$, then the curvature vector in $\mathbb{C}$ with respect to the connection $\overline{D} = D + A$ is:

$$\kappa N := (\overline{D}_T T)^{\perp} = \frac{1}{||\gamma'||^2}(\overline{D}_{\gamma'}\gamma')^{\perp}.$$

Expanding $\overline{D} = D + A$ and taking the normal component, we get:

$$\frac{h\langle (D+A)_{\gamma'}\gamma', h^{-1/2}N\rangle}{h|\gamma'|^2} h^{-1/2}N = \frac{1}{h}\left(\frac{\langle \gamma'', N\rangle}{|\gamma'|^2} + \frac{\langle A_{\gamma'}\gamma', N\rangle}{|\gamma'|^2}\right) N. \tag{5.96}$$

The first term on the right-hand side of (5.96) is the curvature in the standard metric. For the second term, expanding and using orthogonality of γ' and N:

$$\frac{\langle A_{\gamma'}\gamma', N\rangle}{|\gamma'|^2} = \frac{\langle (\gamma' h)\gamma' + (\gamma' h)\gamma' - |\gamma'|^2 Dh, N\rangle}{2h|\gamma'|^2}$$

$$= -\frac{1}{2h}\langle Dh, N\rangle = -\frac{1}{2h}dh(N) = -\frac{1}{2}N(\log h).$$

Therefore, the curvature of γ in the new metric is

$$\frac{1}{h}\left(\kappa - \frac{1}{2}N(\log h)\right).$$

□

5.8.3 Theorems of Angenent

Let (M, g) be a smooth 2-dimensional Riemannian manifold, and denote its unit tangent bundle by

$$T^1M := \{E \in TM : g(E, E) = 1\}.$$

Let S^1 denote the unit circle. Given a curve,

$$\gamma : S^1 \to M,$$

we denote its unit tangent vector by

$$T := \gamma_s(s),$$

its unit upward-pointing normal by N, and its curvature by $\kappa_\gamma(s)$.

The following result is due to Angenent [1, Theorem 3.1].

Theorem 5.8.8. *Assume $V : T^1(M) \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,1}$ function which satisfies*

$$\frac{\partial V}{\partial k} > 0 \quad \text{for all } (E, k) \in T^1(M) \times \mathbb{R}.$$

Let $\gamma_0 : S^1 \to M$ be a $C^{2,\alpha}$ curve in M. Then there exists a maximal time, $t_{max} > 0$, for which the equation

$$\gamma_t(s,t) = V(\gamma_s(s,t), \kappa_\gamma(s,t))N$$

admits a unique solution $\gamma(s,t)$, with $\gamma(\cdot, t) \in C^{3,\alpha}$ for all $0 < t < t_{max}$.

The unit tangent bundle is a smooth submanifold of the tangent bundle and thus carries a natural Riemannian metric. Let

$$TT^1M$$

denote the tangent bundle to T^1M, and ∇ the connection on T^1M. Taking an orthogonal splitting of T^1M into its horizontal and vertical subbundles,

$$TT^1M = HT^1M \oplus VT^1M,$$

one can decompose ∇ into its horizontal and vertical components

$$\nabla = \nabla^H \oplus \nabla^V.$$

The following result is from [1, Theorem 9.1]:

Theorem 5.8.9. *Assume $V : T^1(M) \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,1}$ function which satisfies*

$$\frac{\partial V}{\partial k} > 0 \quad \textit{for all } (E,k) \in T^1(M) \times \mathbb{R},$$

and $|\nabla^H V| + |k||\nabla^V V| \leq C(1 + k^2)$ for almost every $(E,k) \in T^1M \times \mathbb{R}$, for some $C > 0$. Let $\gamma_0 : S^1 \to M$ be a $C^{2,\alpha}$ curve in M, and let $\gamma(s,t)$ be a maximal solution to

$$\gamma_t(s,t) = V(\gamma_s(s,t), \kappa_\gamma(s,t))N,$$

for $t \in [0, t_{max})$, where $t_{max} < \infty$. Then, for any $\epsilon > 0$

$$\limsup_{t \to t_{max}} \sup_{|s_1 - s_0| < \epsilon} |\widetilde{\arg}(\gamma_s(s_1,t)) - \widetilde{\arg}(\gamma_s(s_0,t))| \geq \pi.$$

Remark 5.8.10. *Here one compares angle, by parallel transporting $\gamma_s(s_1,t) \in T_{\gamma(s_1,t)}M$ to $T_{\gamma(s_0,t)}M$. In Euclidean space, this is of course not necessary.*

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