ABSTRACT

Title of dissertation: LOCALLY SYMMETRIC SPACES AND THE COHOMOLOGY OF THE WEIL REPRESENTATION

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We study generalized special cycles on Hermitian locally symmetric spaces $\Gamma\backslash D$ associated to the groups $G = \mathrm{U}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{O}^*(2n)$. These cycles are covered by symmetric spaces associated to subgroups of G which are of the same type. Using the oscillator representation and the thesis of Greg Anderson ([And]), we show that Poincaré duals of these generalized special cycles can be viewed as Fourier coefficients of a theta series. This gives new cases of theta lifts from the cohomology of Hermitian locally symmetric manifolds associated to G to vector-valued automorphic functions associated to the groups $G' = \mathrm{U}(m, m)$, $\mathrm{O}(m, m)$ or $\mathrm{Sp}(m, m)$ which are members of a dual pair with G in the sense of Howe. The above three groups are all the groups that show up in real reductive dual pairs of type I whose symmetric spaces are of Hermitian type with the exception of $\mathrm{O}(p, 2)$.

LOCALLY SYMMETRIC SPACES AND THE COHOMOLOGY OF THE WEIL REPRESENTATION

by

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Chapter 1: Introduction

1.1 Generalized special cycles

There are four classes of irreducible reductive dual pairs over $\mathbb{R}$ of type I in the sense of Howe, (see [Ad]):

1. $(\mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}))$
2. $(\mathrm{U}(p,q), \mathrm{U}(r,s))$
3. $(\mathrm{Sp}(p,q), \mathrm{O}^*(2n))$
4. $(\mathrm{O}(m,\mathbb{C}), \mathrm{Sp}(2n,\mathbb{C}))$

Each group belonging to any of the eight families of groups in the above table is the group that preserves either a non-degenerate Hermitian or skew-Hermitian form $(,)$ over a real,complex or quaternionic vector space V. Let Γ be a torsion free arithmetic subgroup of G such that $\Gamma\backslash G$ is compact. For this Introduction, we will assume that $G = \mathbb{G}(\mathbb{R})$ is the set of real points of a algebraic group $\mathbb{G}$ and Γ is a congruence subgroup of $\mathbb{G}(\mathbb{Z})$ (in general, we will need to assume $\mathbb{G}(\mathbb{R}) = G \times G_c$ where G_c is compact). Furthermore we assume we have chosen a lattice $\mathcal{L}$ in V which is invariant under $\mathbb{G}(\mathbb{Z})$. In each of the cases that we are interested in, the symmetric space $D = G/K$ associated to G where K is a fixed maximal compact subgroup has a realization as a subspace of a Grassmannian associated to V. In what follows let $M = \Gamma\backslash D$. Once we have chosen an orientation on D after passing to a (possibility deeper) congruence subgroup of Γ we may assume M is a compact oriented manifold.

We define and study cycles which are called "generalized special cycles", to be denoted $C_{\mathbf{x},z'}$ (see below for the explanation of the notation), in the locally symmetric spaces M. In this paper, we restrict our attention to the cases $G = \mathrm{U}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{O}^*(2n)$. In this case M is a compact Kahler manifold which is in fact a connected complex algebraic variety ([BB]), and the cycles $C_{\mathbf{x},z'}$ are algebraic cycles.

We now briefly introduce the definition of $C_{\mathbf{x},z'}$ when $G = \mathrm{U}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{O}^*(2n,\mathbb{R})$. We will give a self-contained definition of these cycles in Chapter 3. In what follows we will let $V = \mathbb{C}^{p+q}$ for the case $G = \mathrm{U}(p,q)$, $V = \mathbb{R}^{2n}$ for the case $G = \mathrm{Sp}(2n,\mathbb{R})$ and V be the rank n free right module over the Hamilton quaternions for the case $G = \mathrm{O}^*(2n)$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_m) \in V^m$. In the definition of the cycles $C_{\mathbf{x},z'}$ we will assume that the vectors $x_1, x_2, \cdots, x_m$ are linearly independent and the restriction of the form $(,)$ on $U = \mathrm{span}\{\mathbf{x}\}$ is non-degenerate. In particular, for $G = \mathrm{Sp}(2n,\mathbb{R})$ this implies that $m = 2r$ for some positive integer r. For $G = \mathrm{U}(p,q)$, let (r,s) be the signature of $(,)|_U$. This is an important invariant of the cycle $C_{\mathbf{x},z'}$. We then have the orthogonal splitting

$$V = U \oplus U^{\perp}.$$

For any non-degenerate subspace $W \subseteq V$, we denote by $G(W)$, resp. $K(W)$, resp. $\Gamma(W)$ the stabilizer of W in G, K or Γ respectively. Let $D(W) = G(W)/K(W)$ be the symmetric space of $G(W)$. Denote by $G_{\mathbf{x}}$ or $G(U^{\perp})$ the group that fixes each vector in $\mathbf{x}$, by $D_{\mathbf{x}}$ or $D(U^{\perp})$ its symmetric space and $\Gamma_{\mathbf{x}} = G_{\mathbf{x}} \cap \Gamma$.

There is no canonical embedding from $D_{\mathbf{x}}$ to D. However we can choose $z' \in D(U)$. Once this choice is made, there is an embedding

$$s_{\mathbf{x},z'} : D_{\mathbf{x}} \to D.$$

To be more precise, we can think of $D(U)$ as the set of (inner) Cartan involutions $\mathrm{Ad}(\sigma) \in \mathrm{Aut}(G(U))$, where Ad is the adjoint map and σ is a linear isometry of U satisfying a positivity condition. Similar descriptions hold for $D(U^{\perp})$ and D. Now suppose $z' = \mathrm{Ad}(\sigma') \in D(U)$ and $z = \mathrm{Ad}(\sigma) \in D(U^{\perp})$. Then

$$s_{\mathbf{x},z'} = \mathrm{Ad}(\sigma' \oplus \sigma) \in D.$$

We denote by $D_{\mathbf{x},z'}$ the image of $D_{\mathbf{x}}$ under $s_{\mathbf{x},z'}$.

We pass to the locally symmetric space level and denote (by abuse of notation) by $s_{\mathbf{x},z'}$ the map

$$s_{\mathbf{x},z'} : \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x}} \to M = \Gamma \backslash D$$

induced by $s_{\mathbf{x},z'}$ on the symmetric space level. Since $\Gamma_{\mathbf{x}} \subset \Gamma$ and $\Gamma_{\mathbf{x}}$ fixes z', the above map is well-defined.

$s_{\mathbf{x},z'}$ is an Riemannian immersion and is generically injective (see Lemma 3.2).

It is complex analytic hence algebraic by the GAGA principle since both $\Gamma_{\mathbf{x}}\backslash D_{\mathbf{x}}$ and M are projective varieties. We call the map $s_{\mathbf{x},z'} : \Gamma_{\mathbf{x}}\backslash D_{\mathbf{x}} \to \Gamma\backslash D$ a **generalized special cycle**. In general, $s_{\mathbf{x},z'}$ is not an embedding. However, for a given $\mathbf{x}$, we can pass to a finite index subgroup $\Lambda \subseteq \Gamma$ such that

$$s_{\mathbf{x},z'} : \Lambda_{\mathbf{x}}\backslash D_{\mathbf{x}} \to \Lambda\backslash D$$

is an embedding (see Lemma 3.1).

However, for general $\mathbf{x}$ (and U), we are forced to consider $C_{\mathbf{x},z'}$ to be a **singular** cycle in the sense of algebraic topology. Namely, we will consider the singular cycle given by the mapping $s_{\mathbf{x},z'} : C_{\mathbf{x}} \to M$. We will usually abuse notation and omit $s_{\mathbf{x},z'}$ and denote the resulting cycle by $C_{\mathbf{x},z'}$. In particular, if η is a differential form of degree equal to the dimension of $C_{\mathbf{x}}$, we will abbreviate $\int_{C_{\mathbf{x}}} s^*_{\mathbf{x},z'}\eta$ to $\int_{C_{\mathbf{x}}} \eta$.

We can also think of $C_{\mathbf{x},z'}$ as an element in the Chow group $\mathrm{Ch}^*(M)$ of M. We note that the image of $s_{\mathbf{x},z'}$ will often have self-intersections and sometimes will not be orientable (in which case the singular cycle is zero).

Remark 1.1. *It is an important fact that the homology class of $C_{\mathbf{x},z'}$* **does not** *depends on the choice z'. Hence when only the homology class is considered, for example when we take the period of a closed differential form η on $C_{\mathbf{x},z'}$, we often write $[C_{\mathbf{x}}]$ instead of $[C_{\mathbf{x},z'}]$.*

Remark 1.2. *When s or r is equal to 0 in cases (1),(2),(3) of the table, the corresponding cycle $C_{\mathbf{x},z'}$ is called special cycle by Kudla and Millson (see [KM1], [KM2],*

G	$G_{\mathbf{x}}$	G'
$\mathrm{U}(p,q)$	$\mathrm{U}(p-r,q-s),(1\le r\le p,1\le s\le q)$	$\mathrm{U}(r+s,r+s)$
$\mathrm{O}(p,q)$	$\mathrm{O}(p-r,q-s),(1\le r\le p,1\le s\le q)$	$\mathrm{Sp}(2(r+s),\mathbb{R})$
$\mathrm{Sp}(p,q)$	$\mathrm{Sp}(p-r,q-s),(1\le r\le p,1\le s\le q)$	$\mathrm{O}^*(2(r+s))$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{Sp}(2n-2r,\mathbb{R}),(1\le r\le n)$	$\mathrm{O}(2r,2r)$
$\mathrm{O}^*(2n)$	$\mathrm{O}^*(2n-2r),(1\le r\le n)$	$\mathrm{Sp}(r,r)$
$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{Sp}(2n-2r,\mathbb{C}),(1\le r\le n)$	$\mathrm{O}(4r,\mathbb{C})$
$\mathrm{O}(n,\mathbb{C})$	$\mathrm{O}(n-r,\mathbb{C}),(1\le r\le n)$	$\mathrm{Sp}(2r,\mathbb{C})$

Table 1.1: List of isometry groups of generalized special cycles

[KM3], [KM5]). In these cases $G(U)$ is compact and its symmetric space is a single point. In other words, the choice of z' in definition of $C_{\mathbf{x},z'}$ is not necessary.

The main goal of this paper is to put these generalized special cycles into a generating series and show that generating series is an automorphic form for another group G' such that (G,G') is a dual reductive pair in the sense of Howe (see [Ad]).

1.2 A special class in the Lie algebra cohomology with values in the oscillator representation

Let $\mathfrak{W}=\mathcal{S}(V^m)$, the Schwartz functions on V^m. There is a dual pair (in the sense of Roger Howe) (G,G') such that $G\times G'$ (in general a double cover of it, but we neglect this fact in the introduction) acts on $\mathfrak{W}$ by a unitary representation ω. In the three cases of our interest, we have

1. $G=\mathrm{U}(p,q)$, $G'=\mathrm{U}(m,m)$ with $m=r+s$, $0\le r\le p$, $0\le s\le q$,

2. $G=\mathrm{Sp}(2n,\mathbb{R})$, $G'=\mathrm{O}(m,m)$ with $m=2r$, $0\le r\le n$,

3. $G=\mathrm{O}^*(2n,\mathbb{R})$, $G'=\mathrm{Sp}(m,m)$.

In all cases, G' is the linear isometry group of $(W, <,>)$ where W is a $2m$ dimensional complex vector space, a $2m$ dimensional real vector space, or a $2m$ dimensional quaternionic vector space respectively, and $<,>$ is a skew-Hermitian, symmetric or Hermitian form respectively. Given an isotropic splitting $W = E + F$, we can view V^m as $V \otimes E$ and $\mathcal{S}(V^m)$ as $\mathcal{S}(V \otimes E)$. Let P' be the parabolic subgroup that preserves the subspace E. We call P' a Siegel parabolic. Then P' has a Langlands decomposition

$$P' = N'A'M',$$

where N' is the unipotent radical and M' is the Levi factor. We have

1. $A'M' \cong \mathrm{GL}_m(\mathbb{C})$,
2. $A'M' \cong \mathrm{GL}_m(\mathbb{R})$,
3. $A'M' \cong \mathrm{GL}_m(\mathbb{H})$,

respectively in the three cases. For $m' \in A'M'$, the action $\omega(m')$ on $\mathcal{S}(V^m)$ is simply

$$(\omega(m')f)(\mathbf{x}) = \det^n(m')f(\mathbf{x}m'), \tag{1.1}$$

where $f \in \mathcal{S}(V^m)$, $\mathbf{x} \in V^m$ and $n = \frac{p+q}{2}$ for the unitary group.

The action of G via ω is just the induced action on functions

$$(\omega(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x}).$$

for $f \in \mathfrak{W} = \mathcal{S}(V^m)$, $\mathbf{x} \in V^m$ viewed as a $(p+q)$ by m (resp. $2n$ by m in the other two cases) matrix.

We fix a point z_0 in the symmetric space D of G. In other words, we fix a maximal compact group K of G. Let

$$\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0$$

be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G (we drop the subscript 0 to indicate complexification). Let $\Omega^\bullet(D)$ be the space of smooth differential forms on D and $\Omega^\bullet(D, \mathfrak{W})$ be the space of differential forms with values in $\mathfrak{W}$.

Let $C^\bullet(\mathfrak{g}, K; \mathfrak{W}) = (\wedge^\bullet \mathfrak{p}^* \otimes \mathfrak{W})^K$. There is an isomorphism given by the evaluation at z_0

$$\Omega^\bullet(D, \mathfrak{W})^G \to (\wedge^\bullet \mathfrak{p} \otimes \mathfrak{W})^K.$$

Its inverse is given by

$$\psi \mapsto \tilde{\psi}(z, \mathbf{x}) := L^*_{g_z^{-1}}[\psi(g_z^{-1}\mathbf{x})],$$

where $\psi \in C^\bullet(\mathfrak{g}, K; \mathfrak{W})$ and $g_z z_0 = z$ and $L^*_{g_z^{-1}}$ is the pullback on differential forms induced by left translation of G on D. We can further extend the definition of $\tilde{\psi}$ to let it depend on G' by the oscillator representation ω,

$$\tilde{\psi}(z, g', \mathbf{x}) := L^*_{g_z^{-1}}[(\omega(g')\psi)(g_z^{-1}\mathbf{x})].$$

In his PhD thesis [And], Anderson constructed a cocycle

$$\varphi_+ \in C^\bullet(\mathfrak{g}, K; \mathfrak{W}) \cong \Omega^\bullet(D, \mathfrak{W})^G.$$

It is holomorphic (in particular, of Hodge type $(d, 0)$) and closed. Using φ_+ we can define a hodge type (d, d) class $\varphi \in C^{2d}(\mathfrak{g}, K; \mathcal{W})$ (see Chapter 7). In the unitary case, the construction of both φ_+ and φ depends on a pair of integers (r, s) ($0 \le r \le p$, $0 \le s \le q$) which will eventually match the signature of $(,)|_{\mathrm{span}\{\mathbf{x}\}}$. In this case we specify the signature by writing $\varphi_{r,s}$ instead of φ if necessary. We call φ a **special cocycle**. We will see that for large $\lambda \in \mathbb{R}$,

$$[\sum_{\mathbf{y} \in \Gamma \mathbf{x}} \tilde{\varphi}(\lambda \mathbf{y})]$$

is a constant multiple of the Poincaré dual of the cycle $C_{\mathbf{x},z'}$, where $[\cdot]$ means taking cohomology class in $H^\bullet(M)$. In the unitary case, the form $\varphi_{r,0}$ is studied by Kudla and Millson ([KM5]).

1.3 Theta series and statements of main theorems

Throughout this section we assume that φ is the special canonical cocycle in the last section. In the unitary case, we fix a signature (r, s) ($0 \le r \le p$, $0 \le s \le q$) and assume $\varphi = \varphi_{r,s}$.

Recall that $\mathcal{L}$ is a lattice in V fixed by Γ. By [Weil2], we can choose arithmetic subgroups $\Gamma \subset G$ and $\Gamma' \subset \tilde{G}'$ such that the distribution $\theta_{\mathcal{L}}$

$$\theta_{\mathcal{L},\psi} = \sum_{\mathbf{x}\in\mathcal{L}^m} \psi(\mathbf{x})$$

is $\Gamma \times \Gamma'$-invariant.

We now apply $\theta_{\mathcal{L}}$ to $\tilde{\varphi}$ to get

$$\theta_{\mathcal{L},\tilde{\varphi}} \in \Omega^{\bullet}(\Gamma\backslash D) \otimes C^{\infty}(\Gamma'\backslash G').$$

We also define

$$\theta_{\mathcal{L},\beta,\tilde{\varphi}}(z,g') = \sum_{\mathbf{x}\in\mathcal{L}^m,(\mathbf{x},\mathbf{x})=\beta} \tilde{\varphi}(z,g'). \tag{1.2}$$

for a matrix $\beta \in M_m(B)$. We have the following Fourier expansion of $\theta_{\mathcal{L},\tilde{\varphi}_\infty}$:

$$\begin{aligned}\theta_{\mathcal{L},\tilde{\varphi}}(z,g') =& \sum_{\beta} \sum_{\mathbf{x}\in\mathcal{L}^m,(\mathbf{x},\mathbf{x})=\beta} \tilde{\varphi}(z,g',\mathbf{x}) \\ =& \sum_{\beta} \theta_{\mathcal{L},\beta,\tilde{\varphi}}(z,g')\end{aligned}$$

where β runs over all possible inner product matrix $(\mathbf{x},\mathbf{x})$. We call $\theta_{\mathcal{L},\beta,\tilde{\varphi}}$ the β-th Fourier term of $\theta_{\mathcal{L},\tilde{\varphi}}$ as each $\theta_{\mathcal{L},\beta,\tilde{\varphi}}$ is a character function under the action of $\Gamma' \cap N'$. Recall that N' is the nilpotent radical of the Siegel parabolic P' and it is Abelian. See Chapter 5 for more details.

Now suppose $\mathcal{O} \subset V^m$ is a closed G orbit. Then by a theorem of Borel ([B], Theorem 9.11), $\mathcal{O} \cap L^m$ consists of a finite number of Γ-orbits. By Witt's theorem,

G acts transitively on the set

$$\{\mathbf{x} \in V^m | (\mathbf{x}, \mathbf{x}) = \beta\}$$

when β is non-degenerate.

Thus the set

$$\{\mathbf{x} \in \mathcal{L}^m | (\mathbf{x}, \mathbf{x}) = \beta\}$$

consists of finitely many Γ-orbits. We choose Γ-orbit representatives $\{\mathbf{x}_1, \ldots, \mathbf{x}_o\}$ and define

$$U_i = \mathrm{span}\mathbf{x}_i, 1 \leq i \leq o.$$

For each $1 \leq i \leq o$ choose a base point $z_i \in D(U_i)$. Let $C_{\mathbf{x}_i, z_i}$ be the generalized special cycle. Let

$$\mathbf{z} = \{z_1, z_2, \ldots, z_o\}.$$

Then define

$$C_{\beta,\mathbf{z}} = \sum_{i=1}^{o} C_{\mathbf{x}_i, z_i}.$$

$C_{\beta,\mathbf{z}}$ is a cycle in the Chow group of $\Gamma \backslash D$. By remark 1.1, the homology class $[C_{\beta,\mathbf{z}}]$ is independent of the choice of $\mathbf{z}$, so we simply denote by $[C_\beta]$ its homology class.

Our main theorem of the paper can be summarized as: if β is nondegenerate and in the unitary case has signature (r, s), the β-th Fourier term $\theta_{\mathcal{L},\beta,\tilde{\varphi}}$ of $\theta_{\mathcal{L},\tilde{\varphi}}$ will be a function of G' which is not constantly zero times the Poincare dual of $[C_\beta]$. In order to have precise statements, we use the notion of theta correspondence.

Definition 1.1. *Let η be any differential form on $\Gamma\backslash D$, define a smooth function $\theta_{\mathcal{L},\tilde{\varphi}}(\eta)$ on G' by*

$$\theta_{\mathcal{L},\tilde{\varphi}}(\eta) = \int_{\Gamma\backslash D} \eta \wedge \theta_{\mathcal{L},\tilde{\varphi}}(g').$$

We call the above map theta correspondence defined by $\tilde{\varphi}_\infty$. When η is closed, the above gives a map

$$\theta_{\mathcal{L},\tilde{\varphi}} : H^{d-d'}(\Gamma\backslash D, \mathbb{C}) \to C^\infty(\Gamma'\backslash G')$$

where $d = dimD$, d' is the degree of φ. $\theta_{\mathcal{L},\tilde{\varphi}}$ is called **(geometric) theta lifts**.

Also define $a_{\mathcal{L},\beta,\tilde{\varphi}}(\eta)$ to be the β-coefficient of $\theta_{\mathcal{L},\tilde{\varphi}}(\eta)$:

$$a_{\mathcal{L},\beta,\tilde{\varphi}}(\eta) = \int_{\Gamma\backslash D} \eta \wedge \sum_{\mathbf{x}\in\mathcal{L}^m, (\mathbf{x},\mathbf{x})=\beta} \tilde{\varphi}(z, g', \mathbf{x}) \tag{1.3}$$

Then we have the Fourier expansion:

$$\theta_{\mathcal{L},\tilde{\varphi}}(\eta) = \sum_\beta a_{\mathcal{L},\beta,\tilde{\varphi}}(\eta).$$

We prove the following theorem in Chapter 8.

Theorem 1.1. *Assuming that $G = \mathrm{U}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$ or $\mathrm{O}^*(2n,\mathbb{R})$, and β is a non-degenerate (nonsingular) Hermitian, skew symmetric or skew Hermitian matrix respectively in the three cases. Let m be the $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$-rank of β respectively in the three cases and $G' = \mathrm{U}(m,m)$, $\mathrm{O}(m,m)$ or $\mathrm{Sp}(m,m)$ respectively. Then we can find a canonical form $\varphi \in C^\bullet(\mathfrak{g}, K; \mathcal{S}(V^m))$ such that for any closed differential form η*

on M and $a_{\mathcal{L},\beta,\tilde{\varphi}}(\eta)$ defined in equation (1.3), the following is true

$$a_{\mathcal{L},\beta,\tilde{\varphi}}(\eta) = \kappa(g', \beta) \int_{C_\beta} \eta$$

where κ is an analytic function in G' that depends on β.

Remark 1.3. *The canonical form φ in the theorem depends on the signature of β. To be more specific, in the unitary case we have to assume $\varphi = \varphi_{r,s}$, where (r, s) is the signature of β. In the other two cases there is no notion of signature and there is no such dependence (on signature) as well.*

Let us also briefly recall Poincaré duality in terms of differential forms. For a closed submanifold C inside an oriented manifold M, we say that a closed form τ is a Poincaré dual form of C if it satisfies

$$\int_M \eta \wedge \tau = \int_C \eta$$

for any closed form η. Poincaré dual form is unique up to exact forms.

The above definition for a Poincaré dual form can be extended to a singular cycle. We say that a closed form τ is a Poincaré dual form of the singular cycle $f : C \to M$ if it satisfies

$$\int_M \eta \wedge \tau = \int_C f^* \eta$$

for any closed form η.

Remark 1.4. *If the image $f(C)$ has a stratification such that its open stratum $f(C)^0$*

is a submanifold of M, then we can replace the right-hand side of the above formula by $\int_{f(C)^0} \eta$. If $f(C)$ is an algebraic or a totally geodesic cycle (both are true here) then $f(C)$ has such a stratification.

With the above discussion in mind, Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2. *Keep the same assumptions on β and φ as in Theorem 1.1. Then*

$$[\theta_{\mathcal{L},\beta,\tilde{\varphi}}] = \mathrm{PD}([C_\beta])\kappa(g', \beta).$$

where $\mathrm{PD}([C_\beta]) \in H^*(\Gamma\backslash D)$ is the Poincaré dual of $[C_\beta]$ and κ is an analytic function in G' that depends on β.

If we can prove the function $\kappa(g', \beta)$ is not identically zero, then $\frac{1}{\kappa(g',\beta)}\theta_{\mathcal{L},\beta,\tilde{\varphi}}(z, g')$ is the Poincareé dual of $[C_\beta]$. We will prove this for generic g' in Chapter 10.

Theorem 1.3. *Let $\kappa(g', \beta)$ be the function defined in Theorem 1.1. Then $\kappa(g', \beta)$ is an analytic function on $\tilde{G}'$ that is not identically zero. To be more precise, there exists $m' \in M'$ such that for sufficiently large $\lambda \in \mathbb{R}$,*

$$\kappa(\lambda m', \beta) \neq 0.$$

In other words, for a generic g', $\frac{1}{\kappa(g',\beta)}\theta_{\mathcal{L},\beta,\tilde{\varphi}_\infty}(z, g')$ is a Poincaré dual of the cycle C_β.

Remark 1.5. *In a following paper [MS], Millson and the author of this paper will study generalized special cycles on the symmetric spaces associated to $G = \mathrm{O}(p, q)$,* Sp(p,q), Sp$(2n,\mathbb{C})$ *and* O$(n,\mathbb{C})$. *In other words, we can let G be any group that shows up in a irreducible reductive dual pair over $\mathbb{R}$ of type I in the sense of Howe other than the groups we study in this paper. We will prove the analogue of Theorem 1.1, Theorem 1.2 and Theorem 1.3 in those cases.*

In conclusion, $\theta_{\mathcal{L},\tilde{\varphi}}$ can be seen as a "generating" series of PD$([C_\beta])$. Of course, as for now we do not have an explanation for all the Fourier terms $\theta_{\mathcal{L},\beta,\tilde{\varphi}}$ as $\kappa(g',\beta)$PD$([C_\beta])$. Only for those $\theta_{\mathcal{L},\beta,\tilde{\varphi}}$ whose β is non-degenerate and in the unitary case has a fixed signature (r,s) (when $\varphi=\varphi_{r,s}$), do we have such an explanation.

We will show that the canonical special class φ transforms under an irreducible representation of a maximal compact group $K'\subset G'$. Moreover $\theta_{\mathcal{L},\tilde{\varphi}}$ can be viewed as matrix coefficients of a automorphic vector bundle on $\Gamma'\backslash G'/K'$.

1.4 History

The question proposed above has a long history of investigation. The modularity of intersection numbers of special cycles was first studied in [HZ] in case of Hillbert modular surfaces. Later in a series of work ([KM1], [KM2], [KM3], [KM5]), Kudla and Millson studied similar phenomenon for higher rank locally symmetric spaces associated to O(p,q) (resp. U(p,q) and Sp(p,q)). To be more specific they constructed via Weil representation differential forms that are Poincaré duals to $C_{\mathbf{x}}$ when $(,)|_{\mathbf{x}}$ is positive definite, namely, the special sub-cases of cases (1),(2),(3) in our table 1.1 when s has to be equal to zero. Sum over $\mathbf{x}$ one gets a theta series which is automorphic in the dual group Sp$(2r,\mathbb{R})$ (resp. U(r,r), O$^*(2r)$) of

$G = \mathrm{O}(p,q)$ (resp. $\mathrm{U}(p,q)$ and $\mathrm{Sp}(p,q)$). Moreover they prove these differential forms are holomorphic with respect to the dual group $\mathrm{Sp}(2r,\mathbb{R})$ (resp. $\mathrm{U}(r,r)$) on the cohomology level, thus give rise to holomorphic modular forms on Siegel upper half space of genus r by theta lifting. Using the results of Kudla and Millson, together with the classification of unitary representations with nonzero cohomology of Vogan-Zuckerman and endoscopic classification of automorphic representations of G', [BMM1] and [BMM2] are able to prove certain cases of Hodge Conjecture on arithmetic hyperbolic spaces and arithmetic quotients of complex balls.

In this paper, for the unitary group $\mathrm{U}(p,q)$ we remove the assumption of [KM1], [KM2], [KM3], [KM5] that $s = 0$. The corresponding cycles are no longer special cycles in the sense of Kudla and Millson. Moreover we give theta lifts from the cohomology of Hermtian locally symmetric manifolds associated to $G = \mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{O}^*(2n)$ to **vector valued** automorphic functions associated to the groups $G' = \mathrm{O}(m,m)$ or $\mathrm{Sp}(m,m)$.

In a following paper [MS], Millson and the author will deal with the rest cases when $G = \mathrm{O}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{O}(m,\mathbb{C})$ or $\mathrm{Sp}(2n,\mathbb{C})$. Thus for all groups G which show up in an irreducible real reductive dual pair of type I, we have constructed for all such cycles $C_{\beta,\mathbf{z}}$ (whenever β is non-degenerate) in the compact locally symmetric spaces of G, theta series that contains the Poincaré dual of $[C_\beta]$ as its Fourier coefficient.

We have to point out that there are many other cycles in locally symmetric spaces of G which correspond to subgroups of G. However we suspect that the generalized special cycle $C_{\mathbf{x}}$ we defined here may be the only class of cycles whose generating series is automorphic in a dual group. In the end, in order to apply Theta distributions to the Poincaré duals of the cycles, the cycles have to be determined by a tuple of vectors $\mathbf{x} \in V^m$ where V is the fundamental module V of G, which is exactly the case for (the homology classes of) these generalized special cycles.

1.5 Idea of Proof

We construct a fiber bundle

$$\pi : \Gamma_{\mathbf{x}} \backslash D \to \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z'},$$

whose fibers are (topologically) Euclidean spaces. We show that

$$L_{\mathbf{x}}(\varphi) = \sum_{\mathbf{y} \in \Gamma \mathbf{x}} \tilde{\varphi}(\mathbf{y})$$

is a (constant multiple of) the Thom form for the above fibration. To be more precise

$$\int_{\Gamma_{\mathbf{x}} \backslash D} \eta \wedge L_{\mathbf{x}}(\varphi) = \kappa(g', \beta) \int_{\Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z'}} \eta,$$

where

$$\kappa(g', \beta) = \int_{FD_{\mathbf{x},z'}} \tilde{\varphi}(z, g', \mathbf{x}),$$

and $FD_{\mathbf{x},z'}$ is any fiber of the fibration $\Gamma_{\mathbf{x}} \backslash D \to \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z'}$. In some cases, this integral can be computed explicitly (namely the Kudla-Millson cases in [KM1], [KM2], [KM3], [KM5]), but in general this integral is extremely hard to compute. So instead we show that

$$\int_{FD_{\mathbf{x},z'}} \tilde{\varphi}(z, g', \lambda\mathbf{x})$$

is nonzero for certain g' when $\lambda \to \infty$ using the method of Laplace. The method of Laplace is a very powerful tool and suits our situation very well, which will be used again in [MS]. After we have proved this, almost everything follows formally from the unfolding lemma 8.1.

1.6 Open problems

Many interesting questions follow from this paper. First, it is interesting to know the representation theory of φ in terms of $G \times G'$. It is possible that after taking its cohomology class, $[\varphi]$ is in a unique irreducible unitary representation $\pi \times \pi'$ of $G \times G'$. In fact, unitary representations π of G with nonzero cohomology are classified in [VZ]. We conjectured that $[\varphi]$ is in a unique representation π of G with nonzero cohomology and π' is the theta correspondence of π.

Secondly, the author would like to know more about $[\theta_{\mathcal{L},\beta,\tilde{\varphi}}]$ when β is degenerate or has the "wrong" signature in the unitary case. In the cases considered by Kudla and Millson it is shown that if β is not positive semi definite, then $[\theta_{\mathcal{L},\beta,\tilde{\varphi}_{m,0}}]$ is zero. The author wonders if similar phenomenon are true in the cases considered by this paper. To be more precise, the author would like to know if $[\theta_{\mathcal{L},\beta,\tilde{\varphi}_{r,s}}] = 0$ if β is not of signature (r, s).

Another possible direction of research will be the case when Γ is not co-compact. When $G = \mathrm{O}(p, q)$ and Γ is not co-compact, the boundary behavior of the special cohomology classes constructed by Kudla and Millson has been studied in [FM1], [FM2], [FM3] and [FM4]. It would be very interesting to study the boundary behaviour of φ constructed in this paper. In the case when G is $\mathrm{Sp}(2n, \mathbb{R})$, non-compact arithmetic quotients of D include Siegel modular varieties which are moduli spaces of principally polarized Abelian varieties. The objective along this line is to construct Poincaré dual of special cycles and theta lifts on various compactifications of these locally symmetric spaces.

1.7 Guidance for readers

This is a long paper so we provide a road map for readers. Chapter 2 constructs compact arithmetic quotients of D which are in fact complex projective varieties. Chapter 3 defines the generalized special cycle $C_{\mathbf{x}}$. In Chapter 4 we describe the symmetric spaces of $\mathrm{U}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{O}^*(2n, \mathbb{R})$, write down the local equations for the generalized special cycles and prove that they are in fact algebraic subvarieties. Chapter 5 and 6 reviews some fact about the oscillator representation and set up coordinate functions for later use. Chapter 7 reviews the construction of [And], constructs the special class φ, proves that it is closed and proves the important lemma, Lemma 7.7. In Chapter 8 we will prove Theorem 1.1 and Theorem 1.2 assuming rapid decreasing of $\tilde{\varphi}$ on the fiber $FD_{\mathbf{x},z'}$. Chapter 9.4 proves the rapid decreasing of $\tilde{\varphi}$ on the fiber $FD_{\mathbf{x},z'}$. Chapter 10 proves Theorem 1.3 using method of Laplace. Chapter 11 described the K'-type of φ in terms of highest weight theory. Readers who are familiar with arithmetic groups and the oscillator representation, can pick up the definition of special cycles in Chapter 3 and then proceed to Chapter 7 directly and go back to Chapters 4,5 and 6 if necessary. Chapter 8 consists of formal calculations which are similar to the counterparts of the previous work of Kudla and Millson ([KM1], [KM2], [KM3], [KM5]), but it include the case when the cycle C_β is singular which the author is not sure has been written down before. For Chapter 6, 7 and 10, one can focus only on the unitary group case for first reading as the other two cases are similar.

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Chapter 2: Compact quotients of Hermitian symmetric spaces

In this chapter we construct compact arithmetic quotients of Hermitian symmetric spaces associated to $\mathrm{U}(p,q)$, $\mathrm{Sp}(2n,R)$ or $\mathrm{O}^*(2n,\mathbb{R})$. By a Theorem of Baily and Borel ([BB]), these compact quotients are in fact projective algebraic varieties.

Let k be a totally real number field with ℓ distinct embeddings $\sigma_1,\ldots,\sigma_\ell$ into $\mathbb{R}$. Let $v_1,\ldots,v_\ell$ be the induced metrics and $k_{v_1},\ldots,k_{v_\ell}$ be the corresponding completions. Define $S_\infty=\{v_1,\ldots,v_\ell\}$. Let F be a CM field whose maximal real subfield is k. There are ℓ pairs of conjugate embeddings of F into $\mathbb{C}$. We choose one inside each pair, denote them by $\sigma_1,\ldots,\sigma_\ell$ by abusing notation. Notice that the abuse of notation is reasonable as $\sigma_i|_k$ is the σ_i defined for k.

Let (B,σ) be a k-algebra with involution of one of the following types:

$$(B,\sigma)=\begin{cases}(\mathrm{the\ CM\ field\ F, the\ generator\ of\ Gal(F/k)}) \\ \\ (\text{a quaternion algebra with center } k, \text{the standard involution})\end{cases} \tag{2.1}$$

Let V be a right B vector space and $(,)$ a non-degenerate σ-skew Hermitian form (or σ-Hermitian form respectively) on V satisfying

$$(vb,\tilde{v}\tilde{b})=b^\sigma(v,\tilde{v})\tilde{b} \tag{2.2}$$

for $v, \tilde{v} \in V$ and $b, \tilde{b} \in B$. Let G be the unitary group of V such that

$$G = \{g \in \mathrm{GL}_B(V) | (gv, g\tilde{v}) = (v, \tilde{v}), \forall v, \tilde{v} \in V\} \tag{2.3}$$

Define $V_v = V \otimes_k k_v$ and $G_v = G(k_v)$, where v is an Archimedean place (metric) v of k. Extend $(,)$ to V_v and denote the new form by $(,)_v$. Also define

$$V_\infty = \prod_{v \in S_\infty} V_v,$$

$$G_\infty = \prod_{v \in S_\infty} G_v.$$

We want to choose the form $(,)$ to be anisotropic, which is to say that there is no vector $v \in V$ such that $(v, v) = 0$.

Let $\mathcal{O}_k$ be the ring of integers of k and $B(\mathcal{O}_k)$ be the integral closure of $\mathcal{O}_k$ in B. Choose an $B(\mathcal{O}_k)$ lattice $\mathcal{L} \subset V$ and define

$$G(\mathcal{O}_k) = \{g \in G | g\mathcal{L} = \mathcal{L}\}.$$

Also define $G(I)$ to be the congruence subgroup

$$G(I) = \{g \in G(\mathcal{O}_k) | g \equiv Id(modI)\} \tag{2.4}$$

for an ideal I in $B(\mathcal{O}_k)$. We want to choose I big enough, namely containing some fixed ideal J, such that $G(I)$ acts simply on the symmetric space D of G_∞. Let

$\Gamma = G(I)$. It is a theorem of Borel (see [B]) that Γ is a co-compact subgroup in G_∞. Moreover by a Theorem of Baily and Borel ([BB]), $\Gamma\backslash D$ is a complex projective variety.

We need a lemma.

Lemma 2.1. *There is a $c \in k$ such that $\sqrt{c} \notin k$ and $\sigma_i(c) \in U_i$ for $1 \le i \le \ell$, where U_i is any open subset of $\mathbb{R}$.*

Proof. Choose a non-Archimedean place $\mathfrak{p}$ of k such at $\mathcal{O}_k/\mathfrak{p}$ is not a field of characteristic two. As taking square is a two to one map on $\mathcal{O}_k/\mathfrak{p}$, there exists a $b \in \mathcal{O}_k$ such that $x^2 \equiv b\ mod(\mathfrak{p})$ has no solution in $\mathcal{O}_k/\mathfrak{p}$. Thus $x^2 = b$ has no solution in $\mathcal{O}_k$ and k. Now choose ϵ small enough such that $x^2 = a$ has a solution when $|a-1|_\mathfrak{p} \le \epsilon$. By weak appoximation theorem, there exists a $c \in k$ such that

1. $\sigma_i(c) \in U_i$ $(1 \le i \le \ell)$

2. $|c-b|_\mathfrak{p} < |b|_\mathfrak{p} \cdot \epsilon$

Then c satisfies the assumption of the lemma. □

Detailed construction of (B,σ) and $(V,(,))$ will be given separately in three cases.

2.1 The U(p,q) case

Choose $d_1,\dots,d_p,d_{p+1},\dots d_{p+q}$ be $p+q$ purely imaginary numbers of F satisfying

1. $Im(\sigma_1(d_\alpha)) < 0$, $Im(\sigma_1(d_\mu)) > 0$ for $1 \le \alpha \le p$ and $p+1 \le \mu \le p+q$

2. $Im(\sigma_i)(d_j) < 0$ for $2 \leq i \leq \ell$ and $1 \leq j \leq p+q$.

This is possible because of weak approximation theorem. Let V be a $p+q$ dimensional right F vector space and $(,)$ be the skew Hermitian form defined by the diagonal matrix with diagonal entries $d_1, \ldots, d_{p+q}$. If G is defined by equation 2.3, we have

1. $G_{v_1} \cong \mathrm{U}(p,q)$

2. $G_{v_i} \cong \mathrm{U}(p+q)$ for $2 \leq i \leq \ell$,

where $G_v = G(k_v)$ for an archimedean place (metric) v of k. In particular since $(,)_{v_i}$ is definite for $2 \leq i \leq \ell$, there is no nonzero isotropic vector for $(,)$ in V.

2.2 The $\mathrm{Sp}(2n, \mathbb{R})$ case

By lemma 2.1, we can choose c_1, c_2 such that $\sigma_1(c_j) > 0$, $\sigma_i(c_j) < 0$ and $\sqrt{c_j} \notin k$ for $j = 1, 2$ and $2 \leq i \leq \ell$. Let $B = \mathbb{H}_k(c_1, c_2)$ be the quarternion algebra over k generated by ϵ_1, ϵ_2 with relations

$$\epsilon_1^2 = c_1, \epsilon_2^2 = c_2, \epsilon_1\epsilon_2 = -\epsilon_2\epsilon_1.$$

We put $\epsilon_3 = \epsilon_1\epsilon_2$. Then an element $\xi \in B$ can be written as $\xi = \xi_0 + \xi_1\epsilon_1 + \xi_2\epsilon_2 + \xi_3\epsilon_3$, where $\xi_j \in k$ for $0 \leq j \leq 3$. We define an anti-involution σ on B by

$$\sigma(\xi) = \xi_0 - \xi_1\epsilon_1 - \xi_2\epsilon_2 - \xi_3\epsilon_3.$$

With the given assumption we know that

$$B \otimes_k k_{v_1} \cong M_2(\mathbb{R})$$

$$B \otimes_k k_{v_i} \cong \mathbb{H}$$

for $2 \leq i \leq \ell$, where $\mathbb{H}$ is the classical Hamiltonian quarternions.

Let E be any field that contains $\sqrt{c_1}$. We define an anti-involution σ' on $M_2(E)$ by

$$\sigma'(x) = J^t x J^{-1},$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We can embed B into $M_2(E)$ as follows

$$i(\xi_0 + \xi_1\epsilon_1 + \xi_2\epsilon_2 + \xi_3\epsilon_3) = \begin{pmatrix} \xi_0 + \xi_1\sqrt{c_1} & c_2(\xi_2 + \xi_3\sqrt{c_1}) \\ \xi_2 - \xi_3\sqrt{c_1} & \xi_0 - \xi_1\sqrt{c_1} \end{pmatrix}.$$

It is easy to check that $\sigma' \circ i = i \circ \sigma$, so from now on we abuse notation and denote both involutions by σ. Now let V be a n-dimensional right B vector space and $(,)$ be a Hermitian form on V satisfying 2.2. G be the group defined as in 2.3.

Let e_{ij} be the matrix with the (i,j)-th entry 1 and all the other entries zero. Let $e = e_{11}$. As $B \otimes_k E \cong M_2(E)$, we get a decomposition

$$V_E = V_E e + V_E e^\sigma$$

as a E vector space, where $V_E = V \otimes_k E$. Let S_E be the E-bilinear form on $V_E e$ defined by

$$S_E(xe, ye)e_{21} = (xe, ye)$$

Following a result of section 2 of [LM] we see that S_E is skew symmetric and

$$G_E \cong \mathrm{Sp}((V_E, S_E)) \cong \mathrm{Sp}(2n, E).$$

In particular when $E = k_{v_1} \cong \mathbb{R}$, we know that $G_{v_1} \cong \mathrm{Sp}((V_{v_1}, S_{v_1})) \cong \mathrm{Sp}(2n, \mathbb{R})$.

And we also have

$$G_{v_i} \cong \mathrm{Sp}(p_{v_i}, q_{v_i})$$

for $2 \leq i \leq \ell$, where $p_{v_i} + q_{v_i} = n$ as $B \otimes_k k_{v_i} \cong \mathbb{H}$. In particular we can choose the form $(,)$ to be defined by a diagonal matrix with diagonal entries $d_1, \ldots, d_n \in k$ satisfying

$$\sigma_i(d_j) > 0$$

for $2 \leq i \leq m$ and $1 \leq j \leq n$. We then have

$$G_{v_i} \cong \mathrm{Sp}(n)$$

for $2 \leq i \leq n$. With this choice G will be anisotropic as $\mathrm{Sp}(n)$ is. We have

$$G_\infty \cong \mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(n)^{\ell-1}.$$

2.3 The $\mathrm{O}^*(2n, \mathbb{R})$ case

The construction of G in this case is very similar to that of $\mathrm{Sp}(2n, \mathbb{R})$. Let k be the same totally real number field. By lemma 2.1, we can choose c_1, c_2 such that

$$\sigma_1(c_1) < 0, \sigma_1(c_2) < 0$$

$$\sigma_i(c_1) > 0, \sigma_i(c_2) < 0$$

$$\sqrt{c_1}, \sqrt{c_2} \notin k$$

for $2 \leq i \leq \ell$. $B = \mathbb{H}_k(c_1, c_2)$ as before. This time we have

$$B \otimes_k k_{v_1} \cong \mathbb{H}$$

$$B \otimes_k k_{v_i} \cong M_2(\mathbb{R})$$

for $2 \leq i \leq \ell$, where $\mathbb{H}$ is the Hamiltonian quarternions. Now let V be a n-dimensional right B vector space and $(,)$ be a σ-skew Hermitian form on V satisfying 2.2. G be the group defined as in 2.3. Then we see that

$$G_{v_1} \cong \mathrm{O}^*(2n, \mathbb{R})$$

as $D \otimes_k k_{v_1} \cong \mathbb{H}$. Following a result of section 2 of [LM], we also have

$$G_{v_i} \cong \mathrm{O}(p_{v_i}, q_{v_i})$$

for $2 \leq i \leq \ell$, where $p_{v_i} + q_{v_i} = 2n$ as $B \otimes_k k_{v_i} \cong M_2(\mathbb{R})$. In particular we can choose $d_1, \ldots, d_n \in k$ such that $\sigma_i(d_j) > 0$ for all $1 \leq i \leq \ell$, $1 \leq j \leq n$, and let $(,)$ be the form defined by the diagonal matrix with diagonal entries $d_1\epsilon_2, \ldots, d_n\epsilon_2$. By lemma 2.1 of [LM], we know that G_{v_i} $(2 \leq i \leq \ell)$ is defined by a block diagonal matrix S_{v_i} with 2 by 2 diagonal blocks $-J(d_1\epsilon_2), \ldots, -J(d_n\epsilon_2)$. Since

$$-J(d_j\epsilon_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & c_2 d_j \\ d_j & 0 \end{pmatrix} = \begin{pmatrix} d_j & 0 \\ 0 & -c_2 d_j \end{pmatrix},$$

by our assumption S_{v_i} will be positive definite for $2 \leq i \leq \ell$. Thus $G_{v_i} \cong \mathrm{O}(2n, \mathbb{R})$ for $2 \leq i \leq \ell$. This implies that G is anisotropic and

$$G_\infty \cong \mathrm{O}^*(2n, \mathbb{R}) \times \mathrm{O}(2n, \mathbb{R})^{\ell-1}.$$

Chapter 3: Hermitian symmetric spaces and generalized special cycles

Notations and assumptions are as in the previous chapter. In this chapter we define generalized special cycles in $\Gamma \backslash D$, where D is the symmetric space associated to $G_\infty = \prod_{v \in S_\infty} G_v$. As a set, it is the set of maximal compact subgroups of G_∞ or equivalently the set of Cartan involutions. Recall that $G_{v_1} \cong \mathrm{U}(p,q), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}^*(2n,\mathbb{R})$ respectively and all the other factors of G_∞ are compact. The symmetric space of G_∞ is the set of maximal compact subgroups of G_∞ or equivalently the set of Cartan involutions of G_∞ and in our case there will be a canonical one-to-one corresponence with those of G_{v_1}. We will accordingly identify the two sets.

Recall that V is a right B vector space and $(,)$ a non-degenerate σ-skew Hermitian or σ-Hermitian form. Let

$$\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{L}^m.$$

We will require that $U = \mathrm{span}_B(\mathbf{x})$ to be non-degenerate with respect to $(,)$. We then have the decomposition $V = U + U^{\perp}$. We define

$$G(U^{\perp}) = \{g \in G | gv = v, \forall v \in U\} \tag{3.1}$$

and

$$G(U) = \{g \in G | gv = v, \forall v \in U^{\perp}\}.$$

Let $D(U^{\perp})$ be the symmetric space associated to $G(U^{\perp})_{v_1}$ and $D(U)$ be the symmetric space associated to $G(U)_{v_1}$. Thus we are thinking of $G(\bullet)$ and $D(\bullet)$ as functors on the categories of real or complex vector spaces equipped with nondegenerate forms.

Remark 3.1. *In order to be consistent with the notation of the work of Kudla and Millson we will also use the symbol G_U to denote the group $G(U^{\perp})$ defined above with similar alternative notations for $D(U)$ etc. Hence we have*

$$G_U = G(U^{\perp}),\ G_{U^{\perp}} = G(U),\ D_U = D(U^{\perp}),\ D_{U^{\perp}} = D(U). \tag{3.2}$$

Define $\Gamma_U = \Gamma \cap G_U$. By our construction $\Gamma_U \backslash D_U$ is a compact locally symmetric manifold. We want to map $\Gamma_U \backslash D_U$ to $\Gamma \backslash D$ to get an algebraic cycle. This requires the choice of a point in $D(U)$ as we will now see.

Let $r_U = Id|_U \oplus (-Id)|_{U^{\perp}}$, so $r_U \in G$ and let $\sigma_U = Ad(r_U) \in \mathrm{Aut}(G)$. Any $z' \in D(U)$ corresponds to a Cartan involution $\sigma(z')$ of $G(U)$. We need to extend $\sigma(z')$ to all of D. To do this let $Y \subset U$ be the orthogonal complement of z' thought of as a linear subspace of U (in our usual realization of $D(U)$ as an open subset of a Grassmannian of U). Let r_Y be reflection in Y. As an involution of D we have $\sigma(z') = Ad(r_Y)$. Now consider Y as a subspace of V and define our extension of $\sigma(z')$ to be reflection through Y in V.

For any $\sigma \in \mathrm{Aut}(G)$, define

$$H^\sigma = \{g \in H | \ \sigma(g) = g\}$$

for any subgroup H of G carried into itself by the action of σ_i. Since $\sigma_U\sigma(z') = \sigma(z')\sigma_U$, we know that G^{σ_U} is carried into itself by the action of $\sigma(z')$. Then we define

$$G^{\sigma_U,\sigma(z')} \triangleq (G^{\sigma_U})^{\sigma(z')} = (G^{\sigma(z')})^{\sigma_U}$$

It is easy to see that

$$G^{\sigma_U} = G(U) \times G(U^\perp).$$

Consequently the symmetric space D^{σ_U} of $G^{\sigma_U}_{v_1}$ is the product

$$D(U, U^\perp) := D^{\sigma_U} = D(U) \times D(U^\perp)$$

A point $z \in D(U, U^\perp)$ is a pair of Cartan involutions (τ_1, τ_2) where τ_1 (τ_2 resp.) is a Cartan involution of $G(U)_{v_1}$ ($G(U^\perp)_{v_1}$ resp.). Equivalently we can view z as a subspace of V, then

$$z \in D(U, U^\perp) \Leftrightarrow z = z \cap U \oplus z \cap U^\perp.$$

Definition 3.1. *Let U be a non-degenerate subspace of V and $z \in D$. We say (U, z) (resp. $(\mathbf{x}, z)$) is a* **compatible pair** *if $z \in D(U, U^{\perp})$.*

There is a natural embedding π defined by

$$\rho : D(U, U^{\perp}) \hookrightarrow D : (\tau_1, \tau_2) \mapsto \tau_1 \oplus \tau_2.$$

When there is no ambiguity we also denote the image of ρ by $D(U, U^{\perp})$.

Since $G^{\sigma_U} = G(U) \times G(U^{\perp})$, we know that

$$G^{\sigma_U, \sigma(z')} = G(U)^{\sigma(z')} \times G(U^{\perp}).$$

Notice that $G(U)_{v_1}^{\sigma(z')}$ is the maximal subgroup of $G(U)_{v_1}$ fixed by $\sigma(z')$. Define $\Gamma_{U,\sigma(z')} = \Gamma \cap G^{\sigma_U, \sigma(z')}$. By definition

$$\Gamma_{U,\sigma(z')} = (\Gamma \cap G(U)^{\sigma(z')}) \times \Gamma_U,$$

where $\Gamma \cap G(U)^{\sigma(z')}$ is discrete in the compact group $G(U)_{v_1}^{\sigma(z')}$ hence finite. Moreover $\Gamma \cap G(U)^{\sigma(z')}$ acts trivially on $D(U^{\perp})$, so $\Gamma_U \backslash G(U^{\perp}) = \Gamma_{U,\sigma(z')} \backslash G(U^{\perp})$.

For a Cartan involution σ of $G(U^{\perp})_{v_1}$, we define an embedding $i_{z'}$:

$$i_{z'} : D(U^{\perp}) \hookrightarrow D(U, U^{\perp}) : \sigma \mapsto (z', \sigma).$$

Let $\rho_{z'}$ be the composite map $\rho \circ i_{z'}$.

Definition 3.2. *Denote by $D_{U,z'}$ (or $D_{\mathbf{x},z'}$) the image of D_U under the map $\rho_{z'}$. We* *call it a generalized special sub symmetric space of D.*

Remark 3.2. *Note that the fixed point set D_Y of Y is the cover of a special cycle in the sense of Kudla and Millson and*

$$D_{U,z'} = D(U, U^{\perp}) \cap D_Y$$

so the universal cover of our generalized special cycle is the simultaneous fixed point set of a pair of commuting involutions.

$i_{z'}$, ρ and $\rho_{z'}$ induce maps (still denoted as $i_{z'}$, ρ and $\rho_{z'}$) of locally symmetric spaces

$$\Gamma_U \backslash D(U^{\perp}) \to \Gamma^{\sigma_U} \backslash D(U, U^{\perp}), D(G^{\sigma_U}) \to \Gamma \backslash D, \Gamma_U \backslash D(U^{\perp}) \to \Gamma \backslash D$$

respectively. Since both σ_U is rational, the image of the above maps are closed. The induced map $i_{z'}$ is always injective(hence an embedding). But in general ρ (hence $\rho_{z'}$) won't be injective and the image won't be a manifold. The following two lemmas "resolves the singularities" of the image in two different ways.

Lemma 3.1. *There is an arithmetic subgroup $\Gamma' \subseteq \Gamma$ of finite index such that the following diagram commutes, and ρ' is an embedding.*

$$\begin{array}{ccc} (\Gamma')^{\sigma_U} \backslash D(U, U^{\perp}) & \xrightarrow{\rho'} & \Gamma' \backslash D \\ \downarrow & & \downarrow \\ \Gamma^{\sigma_U} \backslash D(U, U^{\perp}) & \xrightarrow{\rho} & \Gamma \backslash D \end{array} .$$

This implies that $\rho'_{z'} : \Gamma'_U \backslash D_U \to \Gamma' \backslash D$ *is an embedding.*

Proof. $U \oplus U^\perp$ is a rational eigenspace decomposition of V under r_U. Let

$$\Gamma' = \Gamma \cap r_U \Gamma r_U.$$

Then Γ' is fixed by $Ad(r_U)$ as $r_U^2 = Id$. We claim that $\Gamma(N) \subseteq \Gamma'$ for some ideal $N \subset \mathcal{O}_k$. r_U is represented by a matrix m for a basis of the lattice L. Let J be the fractional ideal generated by matrix entries of m and choose N such that $NJ^2 \subset \mathcal{O}_k$. Then any $g \in \Gamma(N)$ can be written as

$$g = Id + g_1, \ g_1 \equiv 0 (mod N).$$

Hence all entries of mgm are in $\mathcal{O}_k$. That is to say $r_U \Gamma(N) r_U \subseteq \Gamma$. Hence $\Gamma(N) \subseteq \Gamma'$. In particular Γ' is of finite index in Γ.

Suppose that $x_1, x_2 \in \Gamma_U \backslash D(U^\perp)$ and $\gamma \in \Gamma'$ such that $\gamma x_1 = x_2$. Let $\mu = \gamma r_U \gamma^{-1} r_U$. Then since $r_U \gamma^{-1} r_U \in \Gamma'$, $\mu \in \Gamma'$. But $\mu x_2 = x_2$ hence $\mu = 1$ as Γ acts simply. It follows that $r_U \gamma^{-1} r_U = \gamma^{-1}$ hence $\gamma \in G^{\sigma_U}$. So $\gamma \in (\Gamma')^{\sigma_U}$. So the map $\rho' : (\Gamma')^{\sigma_U} \backslash D(U, U^\perp) \to \Gamma' \backslash D$ is injective. Since locally it is an analytic immersion (see Chapter 4), the lemma is proved. □

Lemma 3.2. *The map* $\Gamma_U \backslash D(U^\perp) \xrightarrow{\rho_{z'}} \Gamma \backslash D$ *is a finite birational morphism onto its image.*

Proof. $\forall \gamma \in \Gamma - \Gamma_U$, define

$$D_\gamma = D(U^\perp) \cap \gamma D(U^\perp).$$

Then D_γ is an analytic subset of $D(U^\perp)$. We claim that it is a proper subset. Otherwise γ is in the stablizer of $D(U^\perp)$ which is $G(U^\perp)$. Hence $\gamma \in \Gamma \cap G(U^\perp) = \Gamma_U$, a contradiction.

The image V_γ of D_γ under the natural quotient map $D(U^\perp) \to \Gamma_U \backslash D(U^\perp)$ is a proper analytic sub variety of $\Gamma_U \backslash D(U^\perp)$. Define

$$V = \bigcup_{\gamma \in \Gamma - \Gamma_U} V_\gamma.$$

Then $\rho_{z'} : \Gamma_U \backslash D(U^\perp) \to \Gamma \backslash D$ is injective outside V.

By the commuting diagram

$$\begin{array}{ccc} \Gamma'_U \backslash D(U^\perp) & \xrightarrow{\rho'_{z'}} & \Gamma' \backslash D \\ \downarrow & & \downarrow \\ \Gamma_U \backslash D(U^\perp) & \xrightarrow{\rho_{z'}} & \Gamma \backslash D \end{array}$$

where Γ' is as in Lemma 3.1, the map $\rho_{z'}$ is quasi-finite. $\rho_{z'}$ is a regular map between projective varieties (see Theorem 4.1). It is a projective morphism by Lemma 28.41.15 of Stack Project, hence is a finite morphism. By the argument in the previous paragraph, it is injective outside a set of measure 0 with respect to the measure defined by the Kahler metric on $D(U^\perp)$. Hence the degree of the finite morphism $\rho_{z'}$ must be 1. It must be a birational morphism. □

We denote the image of $\rho_{z'}$ by $C_{U,z'}$. We sometimes also use $C_{\mathbf{x},z'}$ to denote $C_{U,z'}$ for convenience but it really just depends on U. We will prove in Theorem 4.1 that $C_{U,z'}$ is a subvariety of $\Gamma\backslash D$.

Definition 3.3. *We call $C_{U,z'}$ (or $C_{\mathbf{x},z'}$) a generalized special cycle.*

Definition 3.4. *Denote by M_U the image $\rho(D(U,U^\perp))$. We call M_U a mixed cycle associate to U.*

Remark 3.3. *In the unitary group case, when U_{v_1} is not positive or negative definite, M_U is called a mixed (special) cycle and denoted D_U by [KM5]. In the current paper, our notation is different.*

The cycle $C_{U,z'}$ depends on the choice of $z' \in D(U)$. However, its homology class does not. In fact, for any two $z', z'' \in D(U)$, there is a continuous path

$$c : [0,1] \to D(U)$$

such that $c(0) = z'$, $c(1) = z''$. Thus we can define a map $D(U^\perp) \times [0,1] \to D$ by

$$(\sigma, t) \mapsto c(t)|_U \oplus \sigma|_{U^\perp}.$$

Since Γ_U fixes U, this map defines a map $\Gamma_U\backslash D(U^\perp) \times [0,1] \hookrightarrow \Gamma\backslash D$ which is a homotopy equivalence between two different embeddings of $\Gamma_U\backslash D(U^\perp)$. From now on we specify the choice of embedding if necessary, otherwise we use the notation $[C_U]$ or $[C_{\mathbf{x}}]$ to refer to an equivalent class of cycles.

We illustrate the above abstract construction in case of the unitary group. In the next chapter, we will write down in explicit coordinates generalized special cycles in the symmetric spaces associated to the real groups $\mathrm{U}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{O}^*(2n,\mathbb{R})$. The abstract constructions in this chapter will then become concrete.

3.1 Example: the unitary group case

Recall that we assume that $G_{v_1} \cong \mathrm{U}(p,q)$ and $G_v \cong \mathrm{U}(p+q)$ for the rest Archimedean places v. It is well-known that the symmetric space D can be identified with the set of negative q-planes in V_{v_1}

$$D = \{z \in \mathrm{Gr}_q(V_{v_1}) |\ (,)|_z < 0\}$$

Let U be a F-subspace of V. If $(,)|_{U_{v_1}}$ has signature (r,s), then we have $G(U^\perp)_{v_1} \cong \mathrm{U}(p-r, q-s)$. And

$$D(U^\perp) = \{z \in \mathrm{Gr}_{q-s}(U^\perp_{v_1}) |\ (,)|_z < 0\}$$

Choose an orthogonal decomposition of F-vector space $U = U^+ + U^-$ with respect to $(,)$, such that $(,)_{v_1}|_{U^+_{v_1}}$ is positive definite and $(,)_{v_1}|_{U^-_{v_1}}$ is negative definite. Equivalently choose a rational point $z'_0 = U^-$ in the symmetric space $D(U)$. This is always possible by weak approximation theorem. We can now define an embedding $D_U \hookrightarrow D$ by

$$z \mapsto z \oplus U^-_{v_1}.$$

Under this embedding we identify $D(U^{\perp})$ with its image D_{U,z'_0} in D:

$$D_{U,z'_0} = \{z \in D|\ U^-_{v_1} \subseteq z \subseteq (U^+_{v_1})^{\perp}\}. \tag{3.3}$$

After passing to arithmetic quotient, $C_{U,z'_0} = \rho_{z'_0}(\Gamma_U \backslash D(U^{\perp}))$ becomes an algebraic cycle of $\Gamma\backslash D$ denoted as C_{U,z'_0}.

Proposition 3.1. *When U_{v_1} is positive or negative definite, the above embedding is canonically defined. In other words, the choice of z'_0 is unnecessary. $D_{U,z'_0} = D(U, U^{\perp})$ Moreover $D_{U,z'_0} = D_{U^+} \cap D_{U_-}$.*

Proof. When U_{v_1} is positive (negative resp.) definite, the group $G(U)$ is compact and the symmetric space $D(U)$ consists of one point $z'_0 = \{0\}$ ($z'_0 = \{U\}$ resp.). This proves the first statement.

$D(U, U^{\perp}) = D(U) \times D(U^{\perp}) = D(U) \times D_U$ but the first factor is trivial, the second statement is proved. By equation (3.3)

$$D_{U^-} = \{z \in D|U^-_{v_1} \subseteq z\}, D_{U^+} = \{z \in D|z \subseteq (U^+_{v_1})^{\perp}\}.$$

The third statement follows from equation (3.3) again. □

Remark 3.4. *When U_{v_1} is positive (resp. negative) definite, we simply denote C_{U,z'_0} by C_U. This is the situation in [KM1] [KM2], [KM3] and [KM5]. C_U is called a special cycle there. When U_{v_1} is not necessarily definite, the name special cycle is reserved for $\Gamma^{\sigma_U}\backslash D(U, U^{\perp})$.*

3.2 Fibration $\pi : D \to D_{U,z'}$

We construct a fibering $\pi : D \to D_{U,z'_0}$. Let $N(D_{U,z'_0})$ be the normal bundle of $D_{U,z'_0} \subset D$. And denote

$$N_z D_{U,z'_0} = \{v \in T_z D \mid v \perp T_z D_{U,z'}\}$$

for each $z \in D_{U,z'_0}$. Then the Riemannian exponential map induces a map $F : N(D_{U,z'_0}) \to D$ by the formula

$$F(z, v) = \exp_z(v).$$

The image of the line through v in the normal fiber $N_z(D_{U,z'_0})$ over z under the exponential map is the unique geodesic through z orthogonal to D_{U,z'_0}. Theorem 14.6 of [He] then states that since D_{U,z'_0} is totally geodesic in D the tube D is the disjoint union of the geodesics in D which are perpendicular to D_{U,z'_0}. Hence we obtain

Lemma 3.3. *The Riemannian exponential map* $\exp : N(D_{U,z'_0}) \to D$ *is a diffeomorphism.*

Let $\pi : D \to D_{U,z'_0}$ be the map whose fibers are geodesics that are perpendicular to D_{U,z'_0}. By Lemma 3.3, $\pi : D \to D_{U,z'_0}$ is isomorphic to $N(D_{U,z'_0}) \to D_{U,z'_0}$ as a fiber bundle. We denote by

$$F_z D_{U,z'} := \pi^{-1}(z) \tag{3.4}$$

the fiber of π for any $z \in D_{U,z'}$.

Since

$$\exp(\mathrm{Ad}(g)x) = \mathrm{Ad}(g)\exp(x),$$

π is $G_U = G(U^\perp)$-equivariant. Thus it induces a fibration which we still denote by π:

$$\pi : \Gamma_U \backslash D \to \Gamma_U \backslash D_{U,z'}.$$

Chapter 4: Coordinates of Hermtian symmetric spaces

The notations and assumptions in this chapter are as in the last chapter except that we work in the category of real groups. We assume $G = \mathrm{U}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$ or $\mathrm{O}^*(2n,\mathbb{R})$ and $(V, (,))$ be its fundamental module. Let $\mathbf{x} \in V^m$ and $U = \mathrm{span}\{\mathbf{x}\}$. We write down explicitly coordinates of the symmetric spaces of G. We then write down the generalized special cycles in these local coordinates and prove that they are algebraic subvarieties of the ambient locally symmetric spaces. We then construct embeddings of the symmetric spaces of $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{O}^*(2n,\mathbb{R})$ into the symmetric space of $\mathrm{U}(n,n)$ and explain the relations of the corresponding generalized special cycles.

4.1 The $\mathrm{U}(p,q)$ case

We start with the case $G = \mathrm{U}(p,q)$. We use an explicit inhomogeneous coordinate of the bounded symmetric domain model of D. Recall that D is the set of negative q-planes. Let (U, z_0) be a compatible pair (Definition 3.1). We can choose an orthonormal basis $\{v_1, \ldots, v_p, v_{p+1}, v_{p+q}\}$ such that

$$(v_\alpha, v_\alpha) = 1 \text{ if } 1 \leq \alpha \leq p,$$

$$(v_\mu, v_\mu) = -1 \text{ if } p+1 \leq \mu \leq p+q,$$

$$(v_i, v_j) = 0 \text{ if } i \neq j,$$

and

$$U = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r, v_{p+1}, \ldots, v_{p+s}\}, \quad z_0 = \mathrm{span}_{\mathbb{C}}\{v_{p+1}, \ldots, v_{p+q}\}$$

Define

$$U^+ = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r\}, \quad U^- = \mathrm{span}_{\mathbb{C}}\{v_{p+1}, \ldots, v_{p+s}\},$$

and

$$z_0' = z_0 \cap U = U^- = \mathrm{span}_{\mathbb{C}}\{v_{p+1}, \ldots, v_{p+s}\}.$$

U has signature (r, s) w.r.t. the Hermitian form $(,)$. Then by Proposition 3.1, we know that

$$D_{U,z_0'} = \{z \in D | U^- \subseteq z, z \perp U^+\}.$$

The Cartan involution θ_0 corresponding to z_0 is conjugation by the $g_0 = Id_{V_+} + (-Id_{V_-})$.

For a negative q-plane z we can find a unique ordered basis $\{u_1, \ldots, u_q\}$ of z such that

$$(u_1, \ldots, u_q) = (v_1, \cdots, v_{p+q}) \begin{pmatrix} A \\ I_q \end{pmatrix} \tag{4.1}$$

where $A = (a_{ij})_{1 \le i,j \le p}$ is a p by p matrix such that

$$A^*A < Id_q. \tag{4.2}$$

Here $<$ is the usual partial order on the space of q by q Hermitian matrices (defined later). Conversely given A satisfying the inequalities (4.2), the above equation defines a negative q-frame. We call the matrix A the inhomogeneous coordinates of the point z and the representation of z as the $p+q$ by q matrix $\begin{pmatrix} A \\ I_q \end{pmatrix}$ as the homogeneous coordinates of z.

The action of $\mathrm{U}(p,q)$ on the inhomogeneous coordinates of a point z is as follows.

Lemma 4.1. *For $g \in \mathrm{U}(p,q)$, the action of $\mathrm{U}(p,q)$ on D represented by the inhomogeneous coordinate is*

$$g \cdot A = BC^{-1}$$

if $g \begin{pmatrix} A \\ I_q \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix}$ *where B is a p by q matrix and C is a q by q matrix.*

Proof. First compute matrix multiplication $g \begin{pmatrix} A \\ I_q \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix}$, then multiply on the right by C^{-1} to get $\begin{pmatrix} BC^{-1} \\ I_q \end{pmatrix}$. The first step takes $\{u_1, \dots, u_q\}$ to the negative q-frame $\{gu_1, \dots, gu_q\}$. Since the latter spans a negative q-plane as well, by our coordinate assumption, ${}^*BB - {}^*CC$ is negative definite, hence C is invertible.

Then multiplying on the right by C^{-1} is equivalent to choosing another basis inside span$\{gu_1, \ldots, gu_q\}$ so that we get back to the inhomogeneous coordinate. By definition $g \cdot A = BC^{-1}$. $\square$

Suppose z is represented by A in inhomogeneous coordinate, the above condition says the first r rows and first s columns of A are zero, i.e.

$$D_{U,z_0'} = \{A \in M_{p\times q}(\mathbb{C}) | A = \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}, A_4 \in M_{(p-r)\times(q-s)}(\mathbb{C}), {}^*AA < I_q\}. \quad (4.3)$$

4.2 The $\mathrm{Sp}(2n, \mathbb{R})$ case

In this section let D_1 be the symmetric space of $\mathrm{Sp}(2n, \mathbb{R})$ and D be the symmetric space of $\mathrm{U}(n, n)$. Let V_0 be a $2n$ dimensional real vector space with a non-degenerate skew symmetric form $(,)_1$. Let $V = V_0 \otimes \mathbb{C}$. For $v \in V$ let $\overline{v}$ denote the conjugation of v relative to the real subspace V_0. We extend $(,)_1$ from V_0 to V anti-linearly in the first variable and linearly in the second variable. Denote the resulting form by $(,)$. It can be easily checked that $(,)$ is skew Hermitian. By a calculation using basis of V_0 (which we will see soon), one can show that the Hermitian form $\frac{1}{i}(,)$ has signature (n, n). We can also extend $(,)_1$ linearly in both variables and still denote the resulting skew-symmetric form by $(,)_1$. Hence we have

$$(v, w) = (\overline{v}, w)_1, \text{ for } v, w \in V.$$

We then have three linear isometry groups: $G(V_0, (,)_1) \cong \mathrm{Sp}(2n, \mathbb{R})$, $G(V, (,)) \cong$

$\mathrm{U}(n,n)$ and $G(V,(,)_1) \cong \mathrm{Sp}(2n,\mathbb{C})$. Denote by σ the complex conjugation on $\mathrm{GL}(V)$ induced by the above conjugation on V, so from real vector subspace V_0 inside V.

Lemma 4.2. *$(\bar{v},\bar{w}) = \overline{(v,w)}$, $(\bar{v},\bar{v}) = -(v,v)$ for all $v,w \in V$. $\sigma(\mathrm{U}(n,n)) = \mathrm{U}(n,n)$, $\sigma(\mathrm{Sp}(2n,\mathbb{C})) = \mathrm{Sp}(2n,\mathbb{C})$.*

Proof. Let $\bar{v}$ be the complex conjugate of $v \in V$.

$$(\bar{v},\bar{w}) = (v,\bar{w})_1 = -(\bar{w},v)_1 = -(w,v) = \overline{(v,w)}.$$

Since (v,v) is purely imaginary $(\bar{v},\bar{v}) = -(v,v)$. For any $g \in \mathrm{U}(n,n)$

$$\begin{aligned}(\sigma(g)x,\sigma(g)y) =&(\overline{g\bar{x}},\overline{g\bar{y}}) = \overline{(g\bar{x},g\bar{y})} \\ =&\overline{(\bar{x},\bar{y})} = (x,y)\end{aligned}$$

This shows that $\sigma(\mathrm{U}(n,n)) = \mathrm{U}(n,n)$. Similarly, $\sigma(\mathrm{Sp}(2n,\mathbb{C})) = \mathrm{Sp}(2n,\mathbb{C})$. $\square$

The following is easy to verify

$$\mathrm{Sp}(2n,\mathbb{R}) = \mathrm{U}(n,n)^\sigma = \mathrm{Sp}(2n,\mathbb{C})^\sigma = \mathrm{U}(n,n) \cap \mathrm{Sp}(2n,\mathbb{C}).$$

D_1 is the set of n dimensional $\mathbb{C}$-subspace z in V such that

1. $(,)_1$ restricted on z is zero (z is Lagragian for $(,)_1$).

2. $\frac{1}{i}(,)$ is negative definite on z.

Since D is the set of negative n-planes of $(V,\frac{1}{i}(,))$, D_1 naturally injects into D, this injection is in fact an embedding. The involution σ of $\mathrm{U}(n,n)$ induces an involution of D. To be more precise, for any $z \in D$, let $\bar{z}$ be the complex conjugate of z with respect to the real structure defined by V_0. By Lemma 4.2, $\bar{z}$ is positive definite. Define

$$\sigma(z) = \bar{z}^{\perp}.$$

with respest to the Hermitian form $\frac{1}{i}(,)$. One sees immediately that

Lemma 4.3.

$$\sigma(g \cdot z) = \sigma(g) \cdot \sigma(z), \forall g \in \mathrm{U}(n,n)$$

and $D_1 = D^{\sigma}$.

Proof.

$$\sigma(g \cdot z) = (\overline{g \cdot z})^{\perp} = (\sigma(g) \cdot \bar{z})^{\perp} = \sigma(g)(\bar{z})^{\perp}$$

where the second equation follows from the definition of $\sigma(g)$ and the third equality follows from the fact that σ is an anti-isometry of $(,)$ (see the proof of the previous lemma). Hence the first statement is proved. Let $z \in D$, then

$$\begin{aligned} z \in D^{\sigma} &\Leftrightarrow z = (\bar{z})^{\perp} \\ &\Leftrightarrow (\bar{x}, y) = 0, \forall x, y \in z \\ &\Leftrightarrow (x, y)_1 = 0, \forall x, y \in z \\ &\Leftrightarrow z \in D_1. \end{aligned}$$

□

Now let $U_0 \subset V_0$ be a non-degenerate subspace with respect to $(,)_1$ and z_0 be a point in $D(U_0)$. We can find a symplectic basis $\{E_1, \ldots, E_n, F_1, \ldots, F_n\}$ such that

1. $(E_k, F_j)_1 = \delta_{kj}$, $(E_k, E_j)_1 = (F_k, F_j)_1 = 0$.

2. $U_0 = \mathrm{span}\{E_1, F_1, \ldots, E_r, F_r\}$ and $z_0 = \mathrm{span}_{\mathbb{C}}\{E_1 - F_1 i, \ldots, E_r - F_r i\}$.

According to definition 3.2, we can define a special subsymmetric space $D_{U_0,z_0} \subset D_1$. Let $U = U_0 \otimes \mathbb{C}$, then $U \subset V$ is of signature (r, r) and we have the special subsymmetric space $D_{U,z_0} \subset D$. The natural embedding $D_1 \to D$ maps D_{U_0,z_0} into D_{U,z_0}.

Define an orthonormal basis of V for the form $(,)$ by

$$v_1 = \frac{1}{\sqrt{2}}(E_1+F_1 i), \ldots, v_n = \frac{1}{\sqrt{2}}(E_n+F_n i), v_{n+1} = \frac{1}{\sqrt{2}}(E_1-F_1 i), \ldots, v_{2n} = \frac{1}{\sqrt{2}}(E_n-F_n i).$$

Clearly, $\overline{v_\alpha} = v_{\alpha+n}$ and $\overline{v_{\alpha+n}} = v_\alpha$ for $1 \le \alpha \le n$. Hence, if $\begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{C}^{2n}$ is the column vector of coordinates of the vector $v \in V$ relative to the basis $\{v_1, \cdots, v_{2n}\}$ then we have

$$\overline{v} = \begin{pmatrix} \overline{w} \\ v \end{pmatrix} \tag{4.4}$$

in the same coordinates. For any $z \in D$, we express z as a matrix A as in Equation (4.1) of the last section. By Equation (4.4), if z is the span of the column vectors in the matrix $\begin{pmatrix} A \\ I_n \end{pmatrix}$ then $\bar{z}$ is the span of the column vectors in the matrix $\begin{pmatrix} I_n \\ \bar{A} \end{pmatrix}$.

The column vectors in the matrix $\begin{pmatrix} {}^tA \\ I_n \end{pmatrix}$ are perpendicular to the column vectors in the matrix $\begin{pmatrix} I_n \\ \bar{A} \end{pmatrix}$ with respect to $\frac{1}{i}(,)$ which has the matrix form $\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Hence we conclude that in the inhomogeneous coordinate above

$$\sigma(A) = {}^tA.$$

From this description we see that σ is holomorphic and $D_1 = D^\sigma$ in the inhomogeneous coordinate is the set of $n \times n$ symmetric matrix with operator norm less than 1. Recall that in the inhomogeneous coordinate D_{U,z_0} is represented by the matrices

$$\{A \in M_{n\times n}(\mathbb{C}) | A = \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}, A_4 \in M_{(n-r)\times(n-r)}(\mathbb{C}), {}^*A_4A_4 < I_{n-r}\}. \tag{4.5}$$

It is easy to see that D_{U_0,z_0} is the subset of D_{U,z_0} with the additional requirement that ${}^tA = A$. Hence

$$(D_{U,z_0})^\sigma = D_{U_0,z_0}.$$

4.3 The $\mathrm{O}^*(2n,\mathbb{R})$ case

In this section let D_2 be the symmetric space of $\mathrm{O}^*(2n,\mathbb{R})$ and D be the symmetric space of $\mathrm{U}(n,n)$. Let V_0 be a n dimensional right $\mathbb{H}$-vector space with non-degenerate skew Hermitian form $(,)_2$ that satisfies

$$(vh, \tilde{v}\tilde{h})_2 = \bar{h}(v, \tilde{v})\tilde{h}.$$

Let V be the underlying complex vector space of V_0. Define $(,)$ on V by

$$(v, \tilde{v}) = a + bi \text{ if } (v, \tilde{v})_2 = a + bi + cj + dk.$$

Then $(,)$ is s skew Hermitian form and $\frac{1}{i}(,)$ is of signature (n, n). Also define $S(,)$ on V by

$$S(v, \tilde{v}) = (vj, \tilde{v}).$$

It is easy to see that $S(,)$ is bi-linear and symmetric. We have three linear isometry groups $G(V_0, (,)_2) \cong \mathrm{O}^*(2n, \mathbb{R})$, $G(V, (,)) \cong \mathrm{U}(n, n)$ and $G(V, S(,)) \cong \mathrm{O}(2n, \mathbb{C})$. Denote by σ the involution on $\mathrm{GL}_{\mathbb{C}}(V) \cong \mathrm{GL}(2n, \mathbb{C})$ defined by conjugating by the element $j \in \mathbb{H}$.

Lemma 4.4. *$(vj, wj) = \overline{v, w}$, $(vj, vj) = -(v, v)$ for all $v, w \in V$. Moreover $\sigma(\mathrm{U}(n, n)) = \mathrm{U}(n, n)$, $\sigma(\mathrm{O}(2n, \mathbb{C})) = \mathrm{O}(2n, \mathbb{C})$.*

Proof.

$$(vj, wj) = -j(v, w)j = \overline{(v, w)}j \cdot j = \overline{(v, w)}.$$

Since $(,)$ is skew Hermitian (v, v) is purely imaginary. It follows that $(vj, vj) = -(v, v)$. Define $J(v) = v \cdot j$ for $v \in V_0$. Then for $g \in \mathrm{U}(n, n)$,

$$\begin{aligned}(\sigma(g)x, \sigma(g)y) =&(JgJ^{-1}x, JgJ^{-1}y) = \overline{(gJ^{-1}x, gJ^{-1}y)} \\ =&\overline{(J^{-1}x, J^{-1}y)} = (x, y).\end{aligned}$$

Hence $\sigma(g) \in \mathrm{U}(n, n)$ as well. Similarly $\sigma(\mathrm{O}(2n, \mathbb{C})) = \mathrm{O}(2, \mathbb{C})$. □

The following can be verified

$$\mathrm{O}^*(2n, \mathbb{R}) = \mathrm{U}(n, n)^\sigma = \mathrm{O}(2n, \mathbb{C})^\sigma = \mathrm{U}(n, n) \cap \mathrm{O}(2n, \mathbb{C}).$$

D_2 is the set of n dimensional $\mathbb{C}$-subspace z of V such that

1. $S(,)$ restricted on z is zero.

2. $\frac{1}{i}(,)$ is negative definite on z.

Since D is the set of negative n-planes of $(V, \frac{1}{i}(,))$, D_2 naturally injects into D, this injection is in fact an embedding. The involution σ of $\mathrm{U}(n, n)$ induces an involution of D. To be more precise, for any $z \in D$, let $z \cdot j$ be the complex vector space $\{vj | v \in z\}$. By Lemma 4.4, $z \cdot j$ is positive definite with respect to $\frac{1}{i}(,)$. Define

$$\sigma(z) = (z \cdot j)^\perp$$

with respest to the Hermitian form $\frac{1}{i}(,)$.

Lemma 4.5.

$$\sigma(g \cdot z) = \sigma(g) \cdot \sigma(z), \forall g \in \mathrm{U}(n,n)$$

and $D_2 = D^\sigma$.

Proof. Define $J(v) = v \cdot j$ for $v \in V_0$.Then

$$(gv) \cdot j = J(gv) = JgJ^{-1}Jv = \sigma(g)(v \cdot j).$$

For any $g \in \mathrm{U}(n,n)$

$$((gz) \cdot j)^\perp = (\sigma(g)(z \cdot j))^\perp = \sigma(g)(z \cdot j)^\perp.$$

The second equality follows from the fact that $\sigma(g) \in \mathrm{U}(n,n)$(Lemma 4.4). This proves the first statement. Let $z \in D$, then

$$\begin{aligned} z \in D^\sigma \Leftrightarrow & z = (z \cdot j)^\perp \\ \Leftrightarrow & (x \cdot j, y) = 0, \forall x, y \in z \\ \Leftrightarrow & S(x,y) = 0, \forall x, y \in z \\ \Leftrightarrow & z \in D_2. \end{aligned}$$

□

Now let $U_0 \subset V_0$ be a non-degenerate $\mathbb{H}$-subspace with respect to $(,)_2$ and z_0 be a point in $D(U_0)$. We can find a basis $\{v_1, \ldots, v_n\}$ such that

1. $(v_\alpha, v_\beta)_2 = i\delta_{\alpha\beta}$ for $1 \le \alpha, \beta \le n$.

2. $U_0 = \mathrm{span}_{\mathbb{H}}\{v_1, \ldots, v_r\}$ and $z_0 = \mathrm{span}_{\mathbb{C}}\{v_1 j, \ldots, v_r j\}$.

According to definition 3.2, we can define a special subsymmetric space $D_{U_0,z_0} \subset D_2$. Let U be the underlying complex vector space of U_0, then $U \subset V$ is of signature (r, r) and we have the special subsymmetric space $D_{U,z_0} \subset D$. The natural embedding $D_2 \to D$ maps D_{U_0,z_0} into D_{U,z_0}.

Define an orthonormal basis of V:

$$v_1, \ldots, v_n, v_{n+1} = v_1 j, \ldots, v_{2n} = v_n j.$$

Apparently $v_\alpha j = v_{\alpha+n}$ and $v_{\alpha+n} j = -v_\alpha$ for $1 \le \alpha \le n$. For any $z \in D$, we express z as a matrix A as in the last section. z is the span of the column vectors in the matrix $\begin{pmatrix} A \\ I_n \end{pmatrix}$ and $z \cdot j$ is the span of the column vectors in the matrix $\begin{pmatrix} -I_n \\ \bar{A} \end{pmatrix}$. The column vectors in the matrix $\begin{pmatrix} -{}^t A \\ I_n \end{pmatrix}$ are perpendicular to the column vectors in the matrix $\begin{pmatrix} -I_n \\ \bar{A} \end{pmatrix}$ with respect to $\frac{1}{i}(,)$ which has the matrix form $\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Hence we conclude that in the inhomogeneous coordinate above

$$\sigma(A) = -{}^t A.$$

From this description we see that σ is holomorphic and $D_2 = D^\sigma$ in the inhomogeneous coordinate is the set of $n \times n$ anti-symmetric matrix with operator norm less than 1. Recall that in the inhomogeneous coordinate D_{U,z_0} is represented by the matrices

$$\{A \in M_{n\times n}(\mathbb{C}) | A = \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}, A_4 \in M_{(n-r)\times(n-r)}(\mathbb{C}), {}^*A_4 A_4 < I_{n-r}\}.$$

It is easy to see that D_{U_0,z_0} is the subset of D_{U,z_0} with the additional requirement that ${}^tA = -A$. Hence

$$(D_{U,z_0})^\sigma = D_{U_0,z_0}.$$

4.4 Generalized special cycles are subvarieties

Theorem 4.1. *The generalized special cycle $C_{U,z'}$ defined in 3.3 is an algebraic subvariety in $M = \Gamma\backslash D$.*

Proof. First let us assume that the map $\rho_{z'} : \Gamma_U\backslash D_U \to \Gamma\backslash D$ is an embedding. By equation (4.3) and its analogues for $G = \mathrm{Sp}(2n, \mathbb{R})$ and $G = \mathrm{O}^*(2n, \mathbb{R})$, the subsymmetric space $D_{U,z'}$ is a complex analytic subvariety of D. Hence $\Gamma_U\backslash D_U$ is a complex analytic subvariety of D. By the main theorem of [Chow], $\Gamma_U\backslash D_U$ is a complex algebraic subvariety of $\Gamma\backslash D$.

In the general case, we apply Lemma 3.1. By the previous argument we see that (in the notation of Lemma 3.1) $(\Gamma')_U\backslash D_U$ is a complex algebraic subvariety of $\Gamma'\backslash D$. Since the map $f : \Gamma'\backslash D \to \Gamma\backslash D$ is an analytic covering between complex projective varieties, it is automatically a regular map of complex projective algebriac varieties by [Serre]. Hence f is projective by Lemma 28.41.15 of Stack Project. Being a finite covering map, f is automatically quasi-finite, hence a finite morphism. Hence f is proper and in particular closed. Then the image $C_{U,z'} = f((\Gamma')_U \backslash D_U)$ is a closed subvariety of $\Gamma \backslash D$. $\square$

Chapter 5: The oscillator representation

5.1 Dual reductive pairs

Let (B, σ) be defined as in 2.1. Let ϵ be -1 or 1. We define a non-degenerate ϵ-Hermitian form $(,)$ on a right B vector space V and a $-\epsilon$-Hermitian form $<,>$ on a left B vector space W which satisfy

$$(v_1 b_1, v_2 b_2) = b_1^\sigma (v_1, v_2) b_2$$

$$(v_1, v_2) = \epsilon (v_2, v_1)^\sigma$$

$$< b_1 w_1, b_2 w_2 >= b_1 < w_1, w_2 > b_2^\sigma$$

$$< w_1, w_2 >= -\epsilon < w_2, w_1 >^\sigma$$

for $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $b_1, b_2 \in B$. Let G be defined as in 2.3 and

$$G' = \{g' \in \mathrm{GL}_B(W) | < w_1 g', w_2 g' >=< w_1, w_2 > \forall w_1, w_2 \in W\}.$$

We view G and G' as algebraic groups over k. Let

$$n = dim_B V,$$

and

$$2n' = dim_B W.$$

Also let

$$\mathbb{W} = V \otimes_B W$$

and

$$\langle\langle , \rangle\rangle = tr_{B/k}((,) \otimes <, >^\sigma) \tag{5.1}$$

So that $\mathbb{W}$ is a k-vector space with a non-degenerate alternating form $\langle\langle , \rangle\rangle$. Then (G, G') is a dual reductive pair in the sense of [Ho2].

5.2 the Schrodinger model

Now we assume that $(W, <, >)$ is split over B. i.e. there is a decomposition

$$W = W' + W''$$

with B subspaces W' and W'' which are isotropic for $<, >$. We fix a decomposition of W and choose B-bases $e_1, \ldots, e'_n$ for W' and $e'_1, \ldots e'_n$ for W'' such that

$$< e_i, e'_j >= \delta_{ij}.$$

This choice of basis gives rise to an isomorphism

$$G' \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_{2n}(B) | ad^* - bc^* = 1, ab^* = ba^*, cd^* = dc^* \right\},$$

where $a^* = {}^\top a^\sigma$ etc and to isomorphisms

$$\mathbb{W} \cong V^{2n'}$$

$$\mathbb{W}' := V \otimes_B W' \cong V^n$$

$$\mathbb{W}'' := V \otimes_B W'' \cong V^n.$$

Under these isomorhpism

$$\langle\langle [x_1, y_1], [x_2, y_2] \rangle\rangle = tr_{B/k}((x_1, y_2) - (y_1, x_2)),$$

where we think of $[x_1, y_1], [x_2, y_2]$ as elements in W for $x_1, y_1, x_2, y_2 \in V^n$ and

$$(x, y) = ((x_i, y_j)) \in M_n(B)$$

if $x, y \in V^n$. Also note that the parabolic subgroup $P' \subset G'$ which stabilizes W'' then has the form

$$P' = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G' \right\}$$

and hs unipotent radical

$$N' = \left\{ n'(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} | b \in M_n(B) \text{ with } {}^{\top}b^{\sigma} = \epsilon b \right\} \tag{5.2}$$

and Levi factor

$$M' = \left\{ m'(a) = \begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix} | a \in \mathrm{GL}_n(B) \text{ and } \hat{a} = ({}^{\top}a^{-1})^{\sigma} \right\}. \tag{5.3}$$

Fix a non-trivial additive chracter ψ of $k_{\mathbb{A}}$ trivial on k and let

$$(\omega, L^2(V(\mathbb{A})^{n'}))$$

be the Schródinger model of the global oscillator representation of $\widetilde{\mathrm{Sp}(\mathbb{W}(\mathbb{A}))}$, the two fold metaplectic cover of $\mathrm{Sp}(\mathbb{W}(\mathbb{A}))$, corresponding to ψ and the polarization (1.7) in [Weil2] and [Ho2]. As usual , we identify $\mathrm{Sp}(\mathbb{W}(k))$ with its image in $\widetilde{\mathrm{Sp}(\mathbb{W}(\mathbb{A}))}$ under the canonical splitting, and we have the $\mathrm{Sp}(\mathbb{W}(k))$-invariant distribution θ defined by

$$\theta_{\varphi} = \sum_{\mathbf{x} \in V(k)^n} \varphi(\mathbf{x}),$$

where $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$, the Bruhat Schwartz functions on $V(\mathbb{A})^n$. Let $\tilde{G}'(\mathbb{A})$ denote the inverse image of $G'(\mathbb{A})$ in $\widetilde{\mathrm{Sp}(\mathbb{W}(\mathbb{A}))}$. Then the action of $G'(\mathbb{A})$ in $L^2(V(\mathbb{A})^n)$ defined by the restriction of ω to $G'(\mathbb{A})$ commutes with the natural action of $G(\mathbb{A})$

defined by

$$\omega(g)\varphi(\mathbf{x}) = \varphi(g^{-1}\mathbf{x}),$$

where $g \in G(\mathbb{A}), \varphi \in L^2(V(\mathbb{A})^n)$.

To describe the action of the parabolic subgroup $\tilde{P}'(\mathbb{A})$ of $\tilde{G}'(\mathbb{A})$ we fix a section of the covering $\widetilde{\mathrm{Sp}(\mathbb{W}(\mathbb{A}))} \to \mathrm{Sp}(\mathbb{W}(\mathbb{A}))$ and hence an identification:

$$\widetilde{\mathrm{Sp}(\mathbb{W}(\mathbb{A}))} \cong \mathrm{Sp}(\mathbb{W}(\mathbb{A})) \times \mu_2$$

as in [R]. Then

$$\omega(m'(a), \zeta)\varphi(\mathbf{x}) = \zeta\chi_V(a)\varphi(\mathbf{x}a) \tag{5.4}$$

where

$$\chi_V(a) = \begin{cases} \epsilon_V(a)|a|^{\frac{m}{2}} \text{ case 1} \\ \\ |a|^{\frac{m}{2}} \text{ cases 2 and 3,} \end{cases}$$

where $|a|$ is the modulus of multiplication by a on $B^n_{\mathbb{A}}$ and $\epsilon_v(a)$ is as in the notation of [R]. Also

$$\omega(n'(b), \zeta)\varphi(\mathbf{x}) = \zeta\psi(\frac{1}{2}tr(b(X, X)))\varphi(\mathbf{x}). \tag{5.5}$$

Define $\mathcal{L} = \{\beta \in M_n(B)|\ {}^{\top}b^{\sigma} = \epsilon b\}$ and we have the Fourier expansion according to the action(character) of $\widetilde{N}'(\mathbb{A})$:

$$\theta_{\varphi} = \sum_{\beta \in \mathcal{L}} \sum_{\mathbf{x} \in V(k)^n, (\mathbf{x}, \mathbf{x}) = \beta} \varphi(X).$$

Notice that this is just a way to rewrite the summation. Finally we define

$$\theta_\varphi(g, g') = \sum_{\mathbf{x} \in V(k)^n} \omega(g)\omega(g')\varphi(\mathbf{x}).$$

5.3 Classical theta distribution

Let $\mathbb{L}$ be an open subgroup of $V(\mathbb{A}_f)$. Let $\varphi_{\mathbb{L}}$ be the characteristic function of an open subgroup $\mathbb{L}^n \subset V(\mathbb{A}_f)$. For any $\varphi \in \mathcal{S}(V_\infty^n)$, define $\varphi_\infty \in \mathcal{S}(V(\mathbb{A})^n)$ by

$$\varphi_\infty = \varphi \otimes \varphi_{\mathbb{L}}.$$

We now apply the $\mathrm{Sp}(\mathbb{W}(k))$-invariant theta distribution to φ_∞:

$$\theta_{\varphi_\infty}(g, g') = \sum_{\mathbf{x} \in V(k)^n} \varphi_\infty(g, g', \mathbf{x}).$$

Define

$$L = \mathbb{L} \cap V(k).$$

Via the map $V(k) \to V_\infty$, L is a lattice in V_∞. There exist arithmetic subgroups $\Gamma \subset G(k)$ (actually $\Gamma = G(k) \cap \mathrm{GL}(\mathbb{L})$) and $\Gamma' \subset G'(k)$ such that for any $g \in \Gamma, g' \in \Gamma'$, we have

$$\sum_{\mathbf{x} \in V(k)^m} \omega(g, g')(\varphi \otimes \varphi_{\mathbb{L}})(\mathbf{x}) = \sum_{\mathbf{x} \in L^n} \omega(g, g')(\varphi \otimes 1)(\mathbf{x}).$$

In other words, each open subgroup $\mathbb{L} \subset V(\mathbb{A}_f)$ induces a V_∞ lattice L, groups $\Gamma \subset G(k)$, $\Gamma' \subset G'(k)$, and a $\Gamma \times \Gamma'$-invariant distribution θ_L on $\mathcal{S}(V_\infty^n)$ defined by

$$\theta_{\mathcal{L},\varphi_\infty} = \sum_{\mathbf{x} \in L^n} \varphi_\infty(\mathbf{x}).$$

This is the classical theta distribution that we will use later.

Chapter 6: Real reductive dual pairs

In this chapter, we study real dual reductive pairs. We construct the division algebra (B, σ), the B vector space V(resp. W), the form $(,)$(resp. $<,>$) and the group G(resp. G') as in section 5.1 except that now we are interested in the case $k = \mathbb{R}$ instead of a number field (so B=$\mathbb{R}$,$\mathbb{C}$ or $\mathbb{H}$).

6.1 The infinitesimal Fock model

The discussion in this section follows that of section 6 in [KM5] closely. But we correct a sign error in that paper. Let $\mathbb{W}$ be a be a vector space over $\mathbb{R}$ with a non-degenerate skew-symmetric form $\langle\langle,\rangle\rangle$ and J_0 be a positive definite complex structure (i.e. the form $\langle\langle J_0,\rangle\rangle$ is positive definite) on $\mathbb{W}$. We may decompose $\mathbb{W}\otimes\mathbb{C}$ according to

$$\mathbb{W} \otimes \mathbb{C} = \mathbb{W}' + \mathbb{W}'',$$

where $\mathbb{W}'$ is the $+i$ eigenspace of J_0 and $\mathbb{W}''$ is the $-i$ eigenspace of J_0. Notice that both $\mathbb{W}'$ and $\mathbb{W}''$ are isotropic for $\langle\langle,\rangle\rangle$.

We now construct a one-parameter family of representations ω_λ of $\mathfrak{sp}(\mathbb{W}\otimes\mathbb{C})$ on $\mathrm{Sym}(\mathbb{W}')^*$, the symmetric algebra of the dual of $\mathbb{W}'$. We observe that $\mathbb{W}'$ and

$\mathbb{W}''$ are dually paired by $\langle\langle , \rangle\rangle$ and it suffices to construct a one-parameter family of representations on $\mathrm{Sym}(\mathbb{W}'')$.

Define $\mathcal{W}_\lambda$ to be the quotient of the tensor algebra $T^\bullet(\mathbb{W}\otimes\mathbb{C})$ of the complexification of $\mathbb{W}$ by the ideal generated by the elements $x\otimes y - y\otimes x - \lambda\langle\langle x, y\rangle\rangle 1$ where $x, y \in \mathbb{W}$. Let $p : T^\bullet(\mathbb{W}\otimes\mathbb{C}) \to \mathcal{W}_\lambda$ be the quotient map. Clearly $p(T^\bullet(\mathbb{W}')) = \mathrm{Sym}^\bullet(\mathbb{W}')$ and $p(T^\bullet(\mathbb{W}'')) = \mathrm{Sym}^\bullet(\mathbb{W}'')$. $\mathcal{W}_\lambda$ has a filtration $F^\bullet$ inherited from the grading of $T^\bullet(\mathbb{W}\otimes\mathbb{C})$ and

$$[F^p\mathcal{W}_\lambda, F^q\mathcal{W}_\lambda] \subset F^{p+q-2}\mathcal{W}_\lambda.$$

Thus $F^2\mathcal{W}_\lambda$ is a Lie algebra and we have a split short exact sequence of Lie algebras

$$0 \to F^1\mathcal{W}_\lambda \to F^2\mathcal{W}_\lambda \to \mathfrak{sp}(\mathbb{W}\otimes\mathbb{C}) \to 0$$

where the splitting $j : \mathfrak{sp}(\mathbb{W}\otimes\mathbb{C}) \cong \mathrm{Sym}^2(\mathbb{W}\otimes\mathbb{C}) \to F^2(\mathcal{W}_\lambda)$ is given by

$$j(x\circ y) = \frac{1}{2\lambda}[p(x)p(y) + p(y)p(x)].$$

Now let $\mathcal{I}$ be the left ideal in $\mathcal{W}_\lambda$ generated by $\mathbb{W}'$. Then $\mathfrak{W}_\lambda = \mathcal{W}_\lambda/\mathcal{I}$ is a $\mathcal{W}_\lambda$-module and a fortiori an $\mathfrak{sp}(\mathbb{W}\otimes\mathbb{C})$ module via the splitting j. The projection $p : \mathrm{Sym}^\bullet(\mathbb{W}'') \to \mathcal{W}_\lambda$ induces an isomorphism onto $\mathcal{W}_\lambda/\mathcal{I}$ and we obtain an action of $\mathcal{W}_\lambda$ and $\mathfrak{sp}(\mathbb{W}\otimes\mathbb{C})$ by left multiplication.

Explicitly let $\{e_1, \ldots, e_N, f_1, \ldots f_N\}$ be a symplectic basis for $\mathbb{W}$ such that

$$J_0 e_j = f_j \text{ and } J_0 f_j = -f_j \tag{6.1}$$

for $1 \leq j \leq N$. Define

$$w_j' = e_j - f_j i \text{ and } w_j'' = e_j + f_j i$$

for $1 \leq j \leq N$. Then $\{w_1', \ldots, w_N'\}$ (resp. $\{w_1'', \ldots, w_N''\}$) is a basis for $\mathbb{W}'$ (resp. $\mathbb{W}''$). Let z_j be the linear functional given by

$$z_j(w') = \langle\langle w', w_j'' \rangle\rangle.$$

Then $\mathrm{Sym}^\bullet(\mathbb{W}'')$ can be identified with $\mathrm{Pol}(\mathbb{W}') \cong \mathrm{Pol}(\mathbb{C}^N) = \mathbb{C}[u_1, \ldots, u_N]$. Denote by ρ_λ the action of $\mathcal{W}_\lambda$ on $\mathrm{Pol}(\mathbb{C}^N)$. We have (Lemma 6.1 of [KM5])

Lemma 6.1.

1. $\rho_\lambda(w_j'') = u_j$
2. $\rho_\lambda(w_j') = 2i\lambda \frac{\partial}{\partial u_j}$.

From now on we specialize to the case $\lambda = 2\pi i$ and let $\mathfrak{W} = \mathfrak{W}_{2\pi i}$. Let ω denote the action of $\mathfrak{sp}(\mathbb{W} \otimes \mathbb{C})$ on $\mathfrak{W}$. We call $(\mathfrak{W}, \omega)$ the infinitesimal Fock model of the oscillator representation of $\mathfrak{sp}(\mathbb{W} \otimes \mathbb{C})$.

6.2 The Schrodinger Model

If we decompose $\mathbb{W}$ as

$$\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are Lagrangian subspaces of $\mathbb{W}$. The Schrodinger model can be viewed as the Schwartz functions $\mathcal{X}$. Explicitly if we assume

$$\mathbb{X} = \mathrm{span}\{e_1, \ldots, e_N\}$$

$$\mathbb{Y} = \mathrm{span}\{f_1, \ldots, f_N\}$$

Then we have

$$\rho_\lambda(e_j) = \frac{\partial}{\partial x_j}$$

$$\rho_\lambda(f_j) = \lambda x_j$$

where $\{x_1, \ldots, x_N\}$ are coordinate functions with respect to the basis $\{e_1, \ldots, e_N\}$. We define

$$\varphi_0 = \exp(-\pi \sum_{i=1}^{N} x_i^2).$$

φ_0 is the unique vector in $\mathcal{S}(\mathbb{X})$ that is annihilated by $\rho(w'_j)$ for all $1 \leq j \leq N$. The oscillator representation ω is a representation of the metaplectic group $\mathrm{Mp}(2N, \mathbb{R})$ and $\widetilde{\mathrm{U}(N)}$ is a maximal compact subgroup of $\mathrm{Mp}(2N, \mathbb{R})$. We then have a unique $\mathcal{W}_\lambda$-intertwining (thus $\mathfrak{sp}(\mathbb{W})$-intertwining) operator $\iota : \mathfrak{W}_\lambda \to \mathcal{S}(\mathbb{X})$. ι maps the infinitesimal Fock model $\mathfrak{W}$ onto the $\widetilde{\mathrm{U}(N)}$-finite vectors in the Schrodinger model which consists of functions on $\mathbb{X}$ of the form $p(\mathbf{z})\varphi_0(\mathbf{z})$ where $p(\mathbf{z})$ is a polynomial function on $\mathbb{X}$. More specifically, it maps $1 \in \mathfrak{W}$ to φ_0 and it is easy to show that (Lemma 6.3 of [KM5])

Lemma 6.2.

$$\iota(z_j) = (\frac{\partial}{\partial x_j} - 2\pi x_j)$$

As in previous chapters we are going to give detailed construction of Fock and Schrodinger model case by case. We will see later that the Fock (resp. Schrodinger) models of the dual pairs $(\mathrm{U}(n,n), \mathrm{U}(m,m'))$, $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(2m,2m'))$ and $(\mathrm{O}^*(2n), \mathrm{Sp}(m,m'))$ are isomorphic since they come from isomorphic $(\mathbb{W}, \langle\langle, \rangle\rangle)$. The common Fock (Schrodinger) model of these dual pairs can be used to study the seesaw dual pairs:

$$\begin{array}{ccc} \mathrm{U}(n,n) & & \mathrm{O}(2r,2s) \\ \uparrow & \nwarrow\!\!\!\!\nearrow & \uparrow \\ \mathrm{Sp}(2n,\mathbb{R}) & & \mathrm{U}(r,s) \end{array} \tag{6.2}$$

and

$$\begin{array}{ccc} \mathrm{U}(n,n) & & \mathrm{Sp}(r,s)\ . \\ \uparrow & \nwarrow\!\!\!\!\nearrow & \uparrow \\ \mathrm{O}^*(2n) & & \mathrm{U}(r,s) \end{array} \tag{6.3}$$

6.3 The $(\mathrm{U}(p,q), \mathrm{U}(m,m'))$ case

Let V be a $p+q$ dimensional right vector space over the complex numbers. $(,)$ be a non-degenerate skew-Hermitian form on V with signature (p,q) satisfying

$$(vh, v'h') = \overline{h}(v, v')h'$$

for $h, h' \in \mathbb{C}$ and $v, v' \in V$. Choose a basis $\{v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}\}$ of V such that

1. $(v_\alpha, v_\alpha) = -i$

2. $(v_\mu, v_\mu) = i$

for $1 \leq \alpha \leq p, p+1 \leq \mu \leq p+q$ (in this section we keep this convention of index) and $(v_j, v_k) = 0$ if $j \neq k$.

Let W be a $m+m'$ dimensional vector space over the complex numbers with a Hermitian form $<,>$ of signature (m,m') satisfying

$$< hw, h'w' >= h < w_1, w_2 > \overline{h'}$$

for $h, h' \in \mathbb{C}$ and $w, w' \in W$. Choose a basis $\{w_1, \ldots, w_m, w_{m+1}, \ldots, w_{m+m'}\}$ of W such that

1. $< w_a, w_a >= 1$

2. $< w_k, w_k >= -1$

for $1 \leq a \leq m, m+1 \leq k \leq m+m'$(in this section we keep this convention of index) and $< w_j, w_k >= 0$ if $j \neq k$.

Define $\mathbb{W} = V \otimes_{\mathbb{C}} W$ and $\langle\langle , \rangle\rangle$ on $\mathbb{W}$ by

$$\langle\langle v \otimes w, \tilde{v} \otimes \tilde{w} \rangle\rangle = (v, \tilde{v}) < \tilde{w}, w > .$$

One checks easily that $\langle\langle , \rangle\rangle$ is a skew Hermitian form that is anti-linear in the first variable and linear in the second variable. Define $\langle\langle , \rangle\rangle_{\mathbb{R}} = Re\langle\langle , \rangle\rangle$, then $\langle\langle , \rangle\rangle_{\mathbb{R}}$ is a symplectic form on the underlying real vector space of $\mathbb{W}$.

Define $J_0 = iI_{p,q} \otimes I_{m,m'}$, where $I_{a,b}$ is the matrix

$$\begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}.$$

Then J_0 is a positive definite complex structure for the symplectic form $\langle\langle , \rangle\rangle_{\mathbb{R}}$.

Now define $\mathbb{W}_{\mathbb{C}} = \mathbb{W} \otimes_{\mathbb{R}} \mathbb{C} \cong V \otimes_{\mathbb{C}} (W \otimes_{\mathbb{R}} \mathbb{C})$. Denote the new complex structure by right multiplication by i. Define

$$w'_a = w_a - iw_a i \tag{6.4}$$

$$w''_a = w_a + iw_a i$$

$$w'_k = w_k + iw_k i$$

$$w''_k = w_k - iw_k i.$$

$\mathbb{W}_{\mathbb{C}} = \mathbb{W}' \oplus \mathbb{W}''$ where $\mathbb{W}'(W''$ resp.) is the $+i(-i$ resp.) eigenspace of J_0. Then we have

Lemma 6.3.

$$\mathbb{W}' = span_{\mathbb{C}}\{v_\alpha \otimes w'_a, v_\mu \otimes w''_a, v_\alpha \otimes w'_k, v_\mu \otimes w''_k\}$$

$$\mathbb{W}'' = span_{\mathbb{C}}\{v_\alpha \otimes w''_a, v_\mu \otimes w'_a, v_\alpha \otimes w''_k, v_\mu \otimes w'_k\}$$

Define linear functionals

1. $u_{\alpha a}(v \otimes w) = \langle\langle v_\alpha \otimes w''_a, v \otimes w\rangle\rangle_{\mathbb{R}}$
2. $u_{\mu a}(v \otimes w) = \langle\langle v_\mu \otimes w'_a, v \otimes w\rangle\rangle_{\mathbb{R}}$
3. $u_{\alpha k}(v \otimes w) = \langle\langle v_\alpha \otimes w''_k, v \otimes w\rangle\rangle_{\mathbb{R}}$
4. $u_{\mu k}(v \otimes w) = \langle\langle v_\mu \otimes w'_k, v \otimes w\rangle\rangle_{\mathbb{R}}$

We can now identify $\mathrm{Sym}^\bullet(\mathbb{W}'') \cong \mathrm{Pol}(\mathbb{W}')$ with the space of polynomials in complex variables $\{u_{ij}, 1 \leq i \leq p+q, 1 \leq j \leq m+m'\}$ and this will be the Fock model $\mathfrak{W}$.

Now we assume $m = m'$. In this case, the Schrodinger model of the oscillator representation is given by the space of Schwartz functions on $\mathcal{S}(V \otimes_{\mathbb{C}} E) \cong \mathcal{S}(V^m)$ on V^m where E is a given Lagrangian subspace of W. We use complex coordinates

$$\mathbf{z} = (z_1, \ldots, z_m)$$

with

$$z_j = {}^t(z_{1,j}, \ldots, z_{p+q,j})$$

$$z_{j,k} = x_{j,k} + iy_{j,k}.$$

with respect to the basis $\{v_1, \ldots, v_{p+q}\}$. If we fix the parameter $\lambda = 2\pi i$, then we have

$$\varphi_0(\mathbf{z}) = \exp(-\pi \sum_{k=1}^{p+q} \sum_{a=1}^{m} |z_{ka}|^2).$$

The Weil representation action of $\mathfrak{sp}(\mathbb{W}, \langle\langle , \rangle\rangle)$ now arises by the following action of quantum algebra $\mathcal{W}_\lambda$:

$$\begin{aligned}
\rho_\lambda(v_\alpha \otimes w'_a) &= \frac{1}{\sqrt{2}}(-\lambda i \bar{z}_{\alpha a} + 2\frac{\partial}{\partial z_{\alpha a}}), & \rho_\lambda(v_\mu \otimes w''_a) &= \frac{1}{\sqrt{2}}(-\lambda i z_{\mu a} + 2\frac{\partial}{\partial \bar{z}_{\mu a}}),\\
\rho_\lambda(v_\alpha \otimes w'_{a+n}) &= \frac{1}{\sqrt{2}}(\lambda i z_{\alpha a} - 2\frac{\partial}{\partial \bar{z}_{\alpha a}}), & \rho_\lambda(v_\mu \otimes w''_{a+n}) &= \frac{1}{\sqrt{2}}(\lambda i \bar{z}_{\mu a} - 2\frac{\partial}{\partial z_{\mu a}}),\\
\rho_\lambda(v_\alpha \otimes w''_a) &= \frac{1}{\sqrt{2}}(\lambda i z_{\alpha a} + 2\frac{\partial}{\partial \bar{z}_{\alpha a}}), & \rho_\lambda(v_\mu \otimes w'_a) &= \frac{1}{\sqrt{2}}(\lambda i \bar{z}_{\mu a} + 2\frac{\partial}{\partial z_{\mu a}}),\\
\rho_\lambda(v_\alpha \otimes w''_{a+n}) &= \frac{1}{\sqrt{2}}(-\lambda i \bar{z}_{\alpha a} - 2\frac{\partial}{\partial z_{\alpha a}}), & \rho_\lambda(v_\mu \otimes w'_{a+n}) &= \frac{1}{\sqrt{2}}(-\lambda i z_{\mu a} - 2\frac{\partial}{\partial \bar{z}_{\mu a}})
\end{aligned}$$

where $1 \leq \alpha \leq p, p+1 \leq \mu \leq p+q$, $1 \leq a \leq m$ and

$$\frac{\partial}{\partial z_{jk}} = \frac{1}{2}(\frac{\partial}{\partial x_{jk}} - \frac{\partial}{\partial y_{jk}} i)$$

$$\frac{\partial}{\partial \bar{z}_{jk}} = \frac{1}{2}(\frac{\partial}{\partial x_{jk}} + \frac{\partial}{\partial y_{jk}} i)$$

$\iota : \mathfrak{W} \to \mathcal{S}(V^m)$ maps $1 \in \mathfrak{W}$ to φ_0 and

Lemma 6.4.

$$\iota(u_{\alpha a}) = \frac{1}{\sqrt{2}}(\lambda i z_{\alpha a} + 2\frac{\partial}{\partial \bar{z}_{\alpha a}}), \qquad \iota(u_{\mu a}) = \frac{1}{\sqrt{2}}(\lambda i \bar{z}_{\mu a} + 2\frac{\partial}{\partial z_{\mu a}}),$$
$$\iota(u_{\alpha k}) = \frac{1}{\sqrt{2}}(-\lambda i \bar{z}_{\alpha a} - 2\frac{\partial}{\partial z_{\alpha a}}), \qquad \iota(u_{\mu k}) = \frac{1}{\sqrt{2}}(-\lambda i z_{\mu a} - 2\frac{\partial}{\partial \bar{z}_{\mu a}})$$

for $1 \le \alpha \le p, p+1 \le \mu \le p+q,\ 1 \le a \le m,\ k = a+m$

Let p be a monomial of degree d of the variables $\{u_{\alpha a}, u_{\mu a}, u_{\alpha k}, u_{\mu k} | \le \alpha \le p, p+1 \le \mu \le p+q, 1 \le a \le m, m+1 \le k \le 2m\}$. Suppose

$$p = \prod_{\alpha=1}^{p} \prod_{\mu=p+1}^{p+q} \prod_{a=1}^{m} \prod_{k=m+1}^{2m} u_{\alpha a}^{d_{\alpha a}} u_{\mu a}^{d_{\mu a}} u_{\alpha k}^{d_{\alpha k}} u_{\mu k}^{d_{\mu k}}.$$

Then we have

$$\iota(p) = \prod_{\alpha=1}^{p} \prod_{\mu=p+1}^{p+q} \prod_{a=1}^{m} \prod_{k=m+1}^{2m} (\sqrt{2})^{d_{\alpha a}+d_{\mu a}} (-\sqrt{2})^{d_{\alpha k}+d_{\mu k}}$$
$$\cdot (\frac{\partial}{\partial \bar{z}_{\alpha a}} - \pi z_{\alpha a})^{d_{\alpha a}} (\frac{\partial}{\partial z_{\mu a}} - \pi \bar{z}_{\mu a})^{d_{\mu a}} (\frac{\partial}{\partial z_{\alpha a}} - \pi \bar{z}_{\alpha a})^{d_{\alpha k}} (\frac{\partial}{\partial \bar{z}_{\mu a}} - \pi z_{\mu a})^{d_{\mu a}} \varphi_0.$$

We will need the following lemma later

Lemma 6.5.

$$\iota(p) = \tilde{p}\varphi_0.$$

where $\tilde{p}$ *is a polynomials of the variables* $\{z_{\alpha a}, z_{\mu a}, z_{\alpha k}, z_{\mu k} | 1 \le \alpha \le p, p+1 \le \mu \le$

$p+q, 1 \leq a \leq m, m+1 \leq k \leq 2m\}$ *whose highest order term (degree d term) is*

$$\prod_{\alpha=1}^{p} \prod_{\mu=p+1}^{p+q} \prod_{a=1}^{m} \prod_{k=m+1}^{2m} (-2\sqrt{2}\pi z_{\alpha a})^{d_{\alpha a}} (-2\sqrt{2}\pi \bar{z}_{\mu a})^{d_{\mu a}} (2\sqrt{2}\pi \bar{z}_{\alpha a})^{d_{\alpha k}} (2\sqrt{2}\pi z_{\mu a})^{d_{\mu k}}.$$

Proof. Since

$$\varphi_0 = \prod_{\alpha=1}^{p+q} \prod_{a=1}^{m} \exp(-\pi |z_{\alpha a}|^2)$$

and the operators $\{\iota(u_{\alpha a}), \iota(u_{\mu a}), \iota(u_{\alpha k}), \iota(u_{\mu k}) | \leq \alpha \leq p, p+1 \leq \mu \leq p+q, 1 \leq a \leq m, m+1 \leq k \leq 2m\}$ commute with each other, it suffices to prove the lemma for one variable case. That is it suffices to prove the case when

$$\varphi_0 = \exp(-\pi |z|^2)$$

and

$$\iota(u) = \sqrt{2}(\frac{\partial}{\partial \bar{z}} - \pi z)$$

$$\overline{\iota(u)} = \sqrt{2}(\frac{\partial}{\partial z} - \pi \bar{z}).$$

This is an easy induction on the degree d. When $d = 0$, there is nothing to prove. Suppose

$$\iota(u)^{d_1} \overline{\iota(u)}^{d_2} \varphi_0 = p_{d_1,d_2} \varphi_0$$

where p_{d_1,d_2} is a degree d_1+d_2 polynomial of $z, \bar{z}$ with highest order term $(-2\sqrt{2}\pi z)^{d_1}(-2\sqrt{2}\pi \bar{z})^{d_2}$. Then

$$\frac{\partial}{\partial z} \varphi_0 = -\pi \bar{z} \varphi_0$$

$$\frac{\partial}{\partial \bar{z}}\varphi_0 = -\pi z\varphi_0.$$

Both $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ reduce the degree of p_{d_1,d_2}. So

$$\iota(u)^{d_1+1}\overline{\iota(u)}^{d_2}\varphi_0 = p_{d_1+1,d_2}\varphi_0$$

where p_{d_1+1,d_2} is a polynomial of with highest order term $(-2\sqrt{2}\pi z)^{d_1+1}(-2\sqrt{2}\pi\bar{z})^{d_2}$. The lemma is proved. □

6.4 The seesaw dual pairs 6.2

Let V_0 be a $2n$ dimensional real vector space with a non-degenerate skew symmetric form $(,)_0$. Let $V = V_0 \otimes \mathbb{C}$ and we extend $(,)_0$ from V_0 to V anti-linearly in the first variable and linearly in the second variable. Denote the resulting skew Hermitian form by $(,)$.

Let W be a $r+s$ dimensional complex vector space with a Hermitian form $<,>$ of signature (r,s) which is linear in the first variable and anti-linearly in the second variable. We denote the underlying real vector space of W as $W_{\mathbb{R}}$. And define $<,>_{\mathbb{R}}= \mathrm{Re} <,>$. Then $<,>_{\mathbb{R}}$ is a symmetric form of signature $(2r,2s)$. We have

$$V \otimes_{\mathbb{C}} W = (V_0 \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} W \cong V_0 \otimes_{\mathbb{R}} W_{\mathbb{R}}.$$

Define $\mathbb{W} = V \otimes_{\mathbb{C}} W$ and $\langle\langle,\rangle\rangle$ on $\mathbb{W}$ by

$$\langle\langle v \otimes w, \tilde{v} \otimes \tilde{w}\rangle\rangle = (v,\tilde{v}) < \tilde{w}, w > .$$

Also define a symplectic form $\langle\langle , \rangle\rangle_{\mathbb{R}}$ on $\mathbb{W}$ (regarded as $V_0 \otimes W_{\mathbb{R}}$) by

$$\langle\langle v \otimes w, \tilde{v} \otimes \tilde{w} \rangle\rangle_{\mathbb{R}} = (v, \tilde{v})_0 < \tilde{w}, w >_{\mathbb{R}} .$$

It is easy to check directly that

Lemma 6.6.

$$\mathrm{Re}\langle\langle , \rangle\rangle = \langle\langle , \rangle\rangle_{\mathbb{R}}.$$

Remark 6.1. *The above lemma shows that the seesaw dual pairs in 6.2 share the same underlying symplectic module* $(\mathbb{W}, \langle\langle , \rangle\rangle_{\mathbb{R}})$*. Thus they give rise to the same oscillator representation. Thus the Fock and Schrodinger model of* $(\mathrm{U}(n, n), \mathrm{U}(r, s))$ *can serve as the Fock and Schrodinger model of* $(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(2r, 2s))$ *as well.*

Now suppose $r = s$. We want to write down the coordinate functions in the Schrodinger model $\mathcal{S}(V_0^{2r}) \cong \mathcal{S}(V^r)$. Let $E_1, \ldots, E_n, F_1, \ldots, F_n$ be a symplectic basis of $(V_0, (,)_0)$. Then

1. $v_\alpha = \frac{1}{\sqrt{2}}(E_\alpha - iF_\alpha)$ for $1 \leq \alpha \leq n$
2. $v_\mu = \frac{1}{\sqrt{2}}(E_{\mu-n} + iF_{\mu-n})$ for $n+1 \leq \mu \leq 2n$.

Then

1. $(v_\alpha, v_\alpha) = -i$
2. $(v_\mu, v_\mu) = i$

Choose a split basis $\{w_1, \ldots, w_{2r}\}$ of $(W, <, >)$ such that

1. $< w_a, w_{b+r} >= \delta_{ab}$

2. $< w_a, w_b >=< w_{a+r}, w_{b+r} >= 0$

for $1 \leq a, b \leq r$. Then $\{w_1, iw_1, \dots w_r, iw_r, w_{r+1}, w_{r+1}, \dots, w_{2r}, iw_{2r}\}$ is a split basis of $(W_{\mathbb{R}}, <, >_{\mathbb{R}})$. Define

$$E = \mathrm{span}_{\mathbb{C}}\{w_1, \dots, w_r\}.$$

Then the Schrodinger model of the seesaw dual pair (6.2) is $\mathcal{S}(V \otimes_{\mathbb{C}} E) \cong \mathcal{S}(V_0 \otimes E_{\mathbb{R}}) \cong \mathcal{S}(V^r) \cong \mathcal{S}(V_0^{2r})$. We use complex coordinates

$$\mathbf{z} = (z_1, \dots, z_r)$$

with

$$z_j = {}^t(z_{1,j}, \dots, z_{2n,j})$$

$$z_{j,k} = x_{j,k} + iy_{j,k}.$$

with respect to the basis $\{v_1, \dots, v_{2n}\}$ and $\{w_1, \dots, w_r\}$. Then it is easy to see that

$$\begin{aligned} x_{\alpha,a}(E_\alpha \otimes w_a) &= \frac{1}{\sqrt{2}}, & x_{\alpha+n,a}(E_\alpha \otimes w_a) &= \frac{1}{\sqrt{2}}, \\ y_{\alpha,a}(E_\alpha \otimes iw_a) &= \frac{1}{\sqrt{2}}, & y_{\alpha+n,a}(E_\alpha \otimes iw_a) &= \frac{1}{\sqrt{2}}, \\ y_{\alpha,a}(F_\alpha \otimes w_a) &= \frac{1}{\sqrt{2}}, & y_{\alpha+n,a}(F_\alpha \otimes w_a) &= -\frac{1}{\sqrt{2}}, \\ x_{\alpha,a}(F_\alpha \otimes iw_a) &= -\frac{1}{\sqrt{2}}, & x_{\alpha+n,a}(F_\alpha \otimes iw_a) &= \frac{1}{\sqrt{2}}. \end{aligned}$$

for $1 \leq \alpha \leq n$, $1 \leq a \leq r$. All other pairings between $\{x_{j,k}, y_{j,k} | 1 \leq j \leq n, 1 \leq k \leq$

$r\}$ and $\{E_\alpha \otimes w_a, F_\alpha \otimes w_a, E_\alpha \otimes iw_a, F_\alpha \otimes iw_a | 1 \leq \alpha \leq n, 1 \leq a \leq r\}$ are zero.

6.5 The seesaw dual pair 6.3

Let V be a n-dimensional right $\mathbb{H}$ vector space with skew Hermitian form $(,)$ satisfying

$$(vh, \tilde{v}\tilde{h}) = \overline{h}(v, \tilde{v})\tilde{h},$$

where $v, \tilde{v} \in V$ and $h, \tilde{h} \in \mathbb{H}$. $V_{\mathbb{C}}$ is the underlying complex vector space of V. Define a skew Hermitian form $H(,)$ on $V_{\mathbb{C}}$ by

$$H(v, \tilde{v}) = a + bi$$

if $(v, \tilde{v}) = a + bi + cj + dk$. Define $S(,)$ on $V_{\mathbb{C}}$ by

$$S(v, \tilde{v}) = H(vj, \tilde{v}).$$

S is symmetric and complex linear. Let W be a $r+s$ dimensional left $\mathbb{C}$ vector spae with non-degenerate Hermitian form $<,>$ of signature (r, s) that is comlex linear in the first variable and anti-linear in the second variable. Define $W_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{C}} W$, extend $<,>$ to a form on $W_{\mathbb{H}}$ denoted as $<,>_{\mathbb{H}}$ satisfying

$$< hv, \tilde{h}\tilde{v} >_{\mathbb{H}} = h < v, \tilde{v} >_{\mathbb{H}} \overline{\tilde{h}},$$

for $h, \tilde{h} \in \mathbb{H}$. Then we have a canonical isomorphism

$$V \otimes_{\mathbb{H}} W_{\mathbb{H}} \cong V_{\mathbb{C}} \otimes_{\mathbb{C}} W$$

Let $\mathbb{W} = V \otimes_{\mathbb{H}} W_{\mathbb{H}}$. Define $\langle\langle, \rangle\rangle_{\mathbb{R}}$ on $\mathbb{W}$ by

$$\langle\langle v \otimes w, \tilde{v} \otimes \tilde{w} \rangle\rangle = Re[(v, \tilde{v}) < \tilde{w}, w >_{\mathbb{H}}].$$

$\langle\langle, \rangle\rangle_{\mathbb{R}}$ is well-defined on $\mathbb{W}$ and is a symplectic form. Also define $\langle\langle, \rangle\rangle$ on $\mathbb{W}$ (regarded as $V_{\mathbb{C}} \otimes_{\mathbb{C}} W$)by

$$\langle\langle v \otimes w, \tilde{v} \otimes \tilde{w} \rangle\rangle = H(v, \tilde{v}) < \tilde{w}, w > .$$

Then we have

Lemma 6.7.

$$\langle\langle, \rangle\rangle_{\mathbb{R}} = \mathrm{Re}\langle\langle, \rangle\rangle.$$

Remark 6.2. *The above lemma shows that the seesaw dual pairs in 6.2 share the same underlying symplectic module* $(\mathbb{W}, \langle\langle, \rangle\rangle_{\mathbb{R}})$. *Thus they give rise to the same oscillator representation. Thus the Fock and Schrodinger model of* $(\mathrm{U}(n,n), \mathrm{U}(r,s))$ *can serve as the Fock and Schrodinger model of* $(\mathrm{O}^*(2n), \mathrm{Sp}(r,s))$ *as well.*

Now suppose $r = s$. We want to write down the coordinate functions in the

Schrodinger model $\mathcal{S}(V^r) \cong \mathcal{S}(V_{\mathbb{C}}^r)$. Choose a basis $v_1, \ldots, v_n$ of V. such that

$$(v_\alpha, v_\beta) = -i\delta_{\alpha\beta}.$$

Let $v_{\alpha+n} = v_\alpha j$. Then $\{v_1, \ldots, v_n, v_{n+1}, v_{2n}\}$ is a basis of $V_{\mathbb{C}}$. Choose a split basis $\{w_1, \ldots, w_{2r}\}$ of $(W, <, >)$ such that

1. $< w_a, w_{b+r} >= \delta_{ab}$
2. $< w_a, w_b >=< w_{a+r}, w_{b+r} >= 0$

for $1 \leq a, b \leq r$.

Define

$$E = \text{span}_{\mathbb{C}}\{w_1, \ldots, w_r\}.$$

Then the Schrodinger model of the seesaw dual pair (6.2) is $\mathcal{S}(V \otimes_{\mathbb{C}} E) \cong \mathcal{S}(V^r)$. We use complex coordinates

$$\mathbf{z} = (z_1, \ldots, z_r)$$

with

$$z_j = {}^t(z_{1,j}, \ldots, z_{2n,j})$$

$$z_{j,k} = x_{j,k} + iy_{j,k}.$$

with respect to the basis $\{v_1, \ldots, v_n, v_1 j, \ldots, v_n j\}$ and $\{w_1, \ldots, w_r\}$.

Chapter 7: Special Schwartz classes in the relative Lie Algebra cohomology of the Weil representation

In this section we review the construction of holomorphic differential forms in [And]. We use this result to construct the special canonical class φ as in Theorem 1.1. We prove that φ is closed.

Recall that

$$C^\bullet(\mathfrak{g}, K; \mathcal{M}) = \mathrm{Hom}_K(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{M}) \cong \mathrm{Hom}_K(\wedge^\bullet \mathfrak{p}, \mathcal{M}) \cong (\wedge^\bullet \mathfrak{p}^* \otimes \mathcal{M})^K$$

is a cochain complex and gives rise to the relative Lie algebra cohomology $H^\bullet(\mathfrak{g}, K; \mathcal{M})$ (see [BW]) for the $(\mathfrak{g}, K)$ module $\mathcal{M}$.

Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}$ be its complexification. Fix a maximal compact subgroup $z_0 = K$ of G and the corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. Identify $T_{z_0}D$ with $\mathfrak{p}_0$ where $D = G/K$. Note that D is Hermitian symmetric. Decompose $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$ into holomorphic and anti-holomorphic tangent vectors

$$\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-.$$

In his Phd thesis, [And], G. Anderson constructed cochains φ_+ in $\mathrm{Hom}_K(\wedge^\bullet \mathfrak{p}_+, \mathfrak{W}^{\mathfrak{p}_-})$

where $\mathfrak{W}$ is the Fock model of the oscillator representation. Here, the notation $\mathfrak{W}^{\mathfrak{p}_-}$ denotes the subspace of $\mathfrak{W}$ annihilated by $\mathfrak{p}_-$. By left-translating using elements in G, the cochains φ_+ give rise to G-invariant holomorphic differential forms on D with values in $\mathfrak{W}$. We now give this construction case by case.

7.1 The $\mathrm{U}(p,q)$ case

We follow the assumptions and notations of section 6.3. Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Given a Hermitian form $(,)$ on V, there is a map $V \to V^*$ given by

$$v \mapsto (v, \cdot).$$

Hence we can view the underlying Abelian group of V^* as the same with that of V, and we denote by v^* the element $v \in V$ if we think of v as an element in V^* instead. The scalar multiplication in V^* is defined by

$$hv^* = v\overline{h}, \forall h \in \mathbb{C}.$$

where the left hand side is scalar multiplication in V^* while the right hand side is scalar multiplication in V.

One can also identify $V \otimes_{\mathbb{C}} V^*$ with $\mathrm{Hom}_{\mathbb{C}}(V, V)$ by the map

$$v \otimes \tilde{v}^* \mapsto v(\tilde{v}, \cdot).$$

Let $\mathrm{Sym}(V \otimes V^*)$ be the symmetric tensor inside $V \otimes_{\mathbb{C}} V^*$ (this makes sense since V and V^* have the same underlying Abelian group). By the above identification, $\mathrm{Sym}(V \otimes V^*)$ acts on V by

$$(v \circ \tilde{v})(x) = v(\tilde{v}, x) + \tilde{v}(v, x)$$

where $v \circ \tilde{v} = v \otimes \tilde{v}^* + \tilde{v}^* \otimes v$. One can check that this action satisfies

$$((v \circ \tilde{v})(x), y) + (x, (v \circ \tilde{v})(y)) = 0$$

Moreover we have

Lemma 7.1.

$$\mathrm{Sym}(V \otimes V^*) \cong \mathfrak{u}(V, (,)) = \mathfrak{u}(p, q)$$

Define $V^+ = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_p\}$ and $V^- = \mathrm{span}_{\mathbb{C}}\{v_{p+1}, \ldots, v_{p+q}\}$. The splitting $V = V^+ + V^-$ corresponds to a point in the symmetric space D of G and gives a Cartan Decomposition of $\mathfrak{g}_0 = \mathfrak{u}(V, (,))$.

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0,$$

where

$$\mathfrak{k}_0 = (\mathrm{Hom}(V^-, V^-) \oplus \mathrm{Hom}(V^+, V^+)) \cap \mathfrak{g}_0$$

and

$$\mathfrak{p}_0 = (\mathrm{Hom}(V^-, V^+) \oplus \mathrm{Hom}(V^+, V^-)) \cap \mathfrak{g}_0.$$

More explicitly define

$$E_{mn} = v_m \circ v_n \text{ and } F_{mn} = iv_m \circ v_n \tag{7.1}$$

for $1 \leq m, n \leq p + q$ then we have

1. $\mathfrak{k}_0 = \text{span}_{\mathbb{R}}\{E_{\alpha\beta}, F_{\alpha\beta}, E_{\mu\nu}, F_{\mu\nu}\}$

2. $\mathfrak{p}_0 = \text{span}_{\mathbb{R}}\{E_{\alpha\mu}, F_{\alpha\mu}\}$

where $1 \leq \alpha, \beta \leq p$ and $p+1 \leq \mu, \nu \leq p+q$(in this section we keep this convention of index). In terms of matrices we have

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} 0 & A \\ {}^*A & 0 \end{pmatrix} |A \in M_{p\times q}(\mathbb{C}) \right\}.$$

We now describe an $Ad(K)$-invariant almost complex structure $J_{\mathfrak{p}}$ acting on $\mathfrak{p}$ that induces the structure of Hermitian symmetric domain on $\mathrm{U}(p,q)/(\mathrm{U}(p)\times\mathrm{U}(q))$. Let $\zeta = e^{\frac{\pi i}{4}}$. Define $a(\zeta)$ by

$$a(\zeta)(v_\alpha) = v_\alpha\zeta \text{ and } a(\zeta)v_\mu = v_\mu\zeta^{-1}. \tag{7.2}$$

Now we define

$$J_{\mathfrak{p}} = Ad(a(\zeta)).$$

It is easy to check under the identification of $V \otimes_{\mathbb{C}} V^* \cong \mathrm{Hom}_{\mathbb{C}}(V, V)$ we have

$$Ad(a(\zeta))(v \otimes \tilde{v}^*) = (a(\zeta)v) \otimes (a(\zeta)\tilde{v}^*).$$

This implies that

$$J_{\mathfrak{p}}(E_{\alpha\mu}) = F_{\alpha\mu},\ J_{\mathfrak{p}}(F_{\alpha\mu}) = -E_{\alpha\mu}.$$

Define

$$X_{\alpha\mu} = E_{\alpha\mu} - iF_{\alpha\mu} = 2v_\alpha \otimes v_\mu^*,$$

$$Y_{\alpha\mu} = E_{\alpha\mu} + iF_{\alpha\mu} = 2v_\mu \otimes v_\alpha^*.$$

Let $\mathfrak{p}_+$(resp. $\mathfrak{p}_-$) be the $+i$ (resp. $-i$) eigenspace of $J_{\mathfrak{p}}$. We then have

$$\mathfrak{p}_+ = \mathrm{span}_{\mathbb{C}}\{X_{\alpha\mu}\},\ \mathfrak{p}_- = \mathrm{span}_{\mathbb{C}}\{Y_{\alpha\mu}\}.$$

In matrix form we have

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} | A \in M_{p\times q}(\mathbb{C}) \right\}$$

$$\mathfrak{p}_- = \left\{ \begin{pmatrix} 0 & 0 \\ {}^tA & 0 \end{pmatrix} | A \in M_{p\times q}(\mathbb{C}) \right\}.$$

We also let $\{\xi'_{\alpha\mu} | 1 \le \alpha \le p, p+1 \le \mu \le p+q\}$ (resp. $\{\xi''_{\alpha\mu} | 1 \le \alpha \le p, p+1 \le \mu \le p+q\}$)be the basis of $\mathfrak{p}_+^*$ (resp. $\mathfrak{p}_-^*$).

Now fix $1 \leq r \leq p, 1 \leq s \leq q$. Define

$$U^+ = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r\},\ U^- = \mathrm{span}\{v_{p+1}, \ldots, v_{p+s}\}$$

and

$$U = U^+ + U^-.$$

Also define

$$U = \mathrm{span}_{\mathbb{C}}\{v_1, v_2, \ldots, v_r, v_{p+1}, \ldots, v_{p+s}\}$$

Recall that the symmetric space D of $G = \mathrm{U}(p, q)$ is the set of negative q-planes. We fix $z_0' = U^-$. In Chapter 3 we define a sub symmetric space of D

$$D_{U,z_0'} = \{z \in D | U^- \subset z \subset U^+\}.$$

Fix a base point $z_0 = \mathrm{span}\{v_{p+1}, \ldots, v_{p+q}\} \in D_{U,z_0'}$. Now it is obvious that

$$T_{z_0} D_{U,z_0'} = \mathrm{span}_{\mathbb{R}}\{E_{\alpha\mu}, F_{\alpha\mu} | r + 1 \leq \alpha \leq p, p + s + 1 \leq \mu \leq p + q\}$$

where $T_{z_0} D_{U,z_0'}$ is the tangent space of D_U at z_0. And the holomorphic tangent space of D_U at z_0 is

$$T^+_{z_0} D_{U,z_0'} = \mathrm{span}_{\mathbb{C}}\{X_{\alpha\mu} | r + 1 \leq \alpha \leq p, p + s + 1 \leq \mu \leq p + q\}$$

Recall that we define a fiber bundle $\pi : D \to D_{U,z_0'}$ in section 3.2 as follows. At each point $z \in D_{U,z_0'}$, the fiber is the union of all geodesics that are perpendicular to $D_{U,z_0'}$ at z. The tangent space of the fiber $F_{z_0}D_{U,z_0'} = \pi^{-1}(z_0)$ at z_0 can be described as

$$N_{z_0}D_{U,z_0'} = \text{span}_{\mathbb{R}}\{E_{\alpha\mu}, F_{\alpha\mu}|(\alpha,\mu) \in I\} \tag{7.3}$$

where I is the index set

$$I = \{(\alpha,\mu)|1 \le \alpha \le r, p+1 \le \mu \le p+q\} \cup \{(\alpha,\mu)|r+1 \le \alpha \le p, p+1 \le \mu \le p+s\}. \tag{7.4}$$

$N_{z_0}D_{U,z_0'} \perp T_{z_0}D_{U,z_0'}$ with respect to the Killing form of $\mathfrak{g}$. We also have

$$N^+_{z_0}D_{U,z_0'} = \{X_{\alpha\mu}|(\alpha,\mu) \in I\}.$$

Next let $\mathfrak{W}_-$ be the infinitesimal Fock model defined in section 6 for the dual pair $(\mathrm{U}(p,q), \mathrm{U}(0, r+s))$. We now define polynomials $f_+, f_- \in \mathfrak{W}_-$ by

Definition 7.1.

$$f_+ = det\begin{pmatrix} u_{11} & u_{12} & \dots & u_{1r} \\ \dots & \dots & \dots & \dots \\ u_{r1} & u_{r2} & \dots & u_{rr} \end{pmatrix}$$

$$f_- = det\begin{pmatrix} u_{p+1\ r+1} & u_{p+1\ r+2} & \dots & u_{p+1\ r+s} \\ \dots & \dots & \dots & \dots \\ u_{p+s\ r+1} & u_{p+s\ r+2} & \dots & u_{p+s\ r+s} \end{pmatrix}$$

These polynomials are special cases of the polynomials studied in [KV]. We define an element of $(\wedge^\bullet \mathfrak{p}_+ \otimes \mathfrak{W}_-^{\mathfrak{p}_-})^{\mathrm{SU}(p,q)}$ following the construction of [And]. To be more precise, let $K = \mathrm{U}(p) \times \mathrm{U}(q)$ and $\tilde{K}$ be its two fold cover which is the preimage of K under the map $\mathrm{Mp}(\mathbb{W}) \to \mathrm{Sp}(\mathbb{W})$ and $\tilde{K}^0$ be the identity component of $\tilde{K}$. Then it is a fact known to experts that $\tilde{K}$ is the $det^{-\frac{r+s}{2}}$-cover of K (see for example [P]):

$$\tilde{K} \cong \{(g,z) \in K \times \mathbb{C}^\times \mid z^2 = det(g)^{-\frac{r+s}{2}}\}.$$

Define

$$e_{D_{U,z_0'}} = \bigwedge_{(\alpha,\mu)\in I} X_{\alpha\mu}.$$

Also define

$$f_{D_{U,z_0'}} = f_+^{q-s} f_-^{p-r}$$

It can be shown that $\mathfrak{k}$ acts on $\mathfrak{W}_-$ by

$$\omega(v_\alpha \otimes v_\beta^*) = \sum_{k=1}^{r+s} z_{\alpha k} \frac{\partial}{\partial z_{\beta k}} + \frac{1}{2}\delta_{\alpha\beta}(r+s)$$

$$\omega(v_\mu \otimes v_\nu^*) = -\sum_{k=1}^{r+s} z_{\nu k} \frac{\partial}{\partial z_{\mu k}} - \frac{1}{2}\delta_{\mu\nu}(r+s).$$

The adjoint action of $\mathfrak{k}$ on $\mathfrak{p}_+$ induces an action on $\wedge^\bullet \mathfrak{p}_+$. Define

$$\mathfrak{b} = \mathrm{span}_{\mathbb{C}}\{v_\alpha \otimes v_\beta^* | 1 \le \alpha \le \beta \le p\} \oplus \mathrm{span}_{\mathbb{C}}\{v_\mu \otimes v_\nu^* | p+1 \le \nu \le \mu \le p+q\}.$$

$$\mathfrak{t} = \mathrm{span}_{\mathbb{C}}\{v_\alpha \otimes v_\alpha^* | 1 \le \alpha \le p\} \oplus \mathrm{span}_{\mathbb{C}}\{v_\mu \otimes v_\mu^* | p+1 \le \mu \le p+q\}.$$

$$\mathfrak{n}_- = \mathrm{span}_{\mathbb{C}}\{v_\alpha \otimes v_\beta^* | 1 \le \beta < \alpha \le p\} \oplus \mathrm{span}_{\mathbb{C}}\{v_\mu \otimes v_\nu^* | p+1 \le \mu < \nu \le p+q\}.$$

Then $\mathfrak{b}$ is a Borel sub-algebra of $\mathfrak{k}$. One can verify that both $e_{D_{U,z'_0}}$ and $f_{D_{U,z'_0}}$ are highest weight vectors with respect to $\mathfrak{b}$. The weight of $e_{D_{U,z'_0}}$ with respect to $\mathfrak{b}$ is

$$(\underbrace{q,\ldots,q}_{r},\underbrace{s,\ldots,s}_{p-r},\underbrace{-p,\ldots,-p}_{s},\underbrace{-r,\ldots,-r}_{q-s}).$$

The weight of $f_{D_{U,z'_0}}$ with respect to $\mathfrak{b}$ is

$$\Big(\underbrace{q+\frac{1}{2}(r-s),.,q+\frac{1}{2}(r-s)}_{r},\underbrace{\frac{1}{2}(r+s),.,\frac{1}{2}(r+s)}_{p-r},$$
$$\underbrace{\frac{1}{2}(r-s)-p,\frac{1}{2}(r-s)-p}_{s},\underbrace{-\frac{1}{2}(r+s),.,-\frac{1}{2}(r+s)}_{q-s}\Big).$$

It is easy to observe that these two weights differ by

$$-(\underbrace{\frac{1}{2}(r-s),\ldots,\frac{1}{2}(r-s)}_{p+q}).$$

Now denote the irreducible representation of $\tilde{K}^0$ generated by $e_{D_{U,z'_0}}$ as $V(U)$, the irreducible representation of $\tilde{K}^0$ generated by $f_{D_{U,z'_0}}$ as $A(U)$ where $\tilde{K}^0$ is the identity component $\tilde{K}$. By the theory of highest weight we have a $\tilde{K}^0$-equivariant map $\psi_+ : V(U) \to A(U) \otimes det^{-\frac{1}{2}(r-s)}$ such that

$$\psi_+(e_{D_{U,z'_0}}) = f_{D_{U,z'_0}} \otimes 1.$$

When r and s have different parities, $\tilde{K} = \tilde{K}^0$. When r and s have the same parity, we know that $\tilde{K} = K \times \{\pm 1\}$. The main result of [And] in this case can be rephrased as

Theorem 7.1.

$$\psi_+ \in \mathrm{Hom}_{\tilde{K}^0}(\wedge^{rq+ps-rs}\mathfrak{p}_+, \mathfrak{W}_-^{\mathfrak{p}_-} \otimes det^{-\frac{1}{2}(r-s)}).$$

Let $\{\epsilon_1, \ldots, \epsilon_d\}$ be a basis of $V(U) \subset \wedge^\bullet \mathfrak{p}_+$ such that each ϵ_i is a weight vector of $\mathfrak{t}$. Extend $\{\epsilon_1, \ldots, \epsilon_d\}$ to a basis of $\wedge^{rq+ps-rs}\mathfrak{p}_+$, take the dual basis inside $\wedge^{rq+ps-rs}\mathfrak{p}_+^*$ and denote the first d basis vectors by $\Omega_1, \ldots, \Omega_d$. We have an isomorphism

$$\mathrm{Hom}_{\tilde{K}^0}(\wedge^\bullet \mathfrak{p}_+, \mathfrak{W}_-^{\mathfrak{p}_-} \otimes det^{-\frac{1}{2}(r-s)}) \cong (\wedge^\bullet \mathfrak{p}_+^* \otimes \mathfrak{W}_-^{\mathfrak{p}_-} \otimes det^{-\frac{1}{2}(r-s)})^{\tilde{K}^0}.$$

Under this isomorphism ψ_+ maps to an element $\phi_+ \in (\wedge^{rq+ps-rs}\mathfrak{p}_+^* \otimes \mathfrak{W}_-^{\mathfrak{p}_-} \otimes det^{-\frac{1}{2}(r-s)})^{\tilde{K}^0}$:

$$\phi_+ = \sum_{i=1}^{d} \psi_+(\epsilon_i)\Omega_i. \tag{7.5}$$

The element thus defined is independent of the choice of the basis $\{\epsilon_1, \ldots, \epsilon_d\}$ and is actually in $(\wedge^\bullet \mathfrak{p}_+ \otimes \mathfrak{W}_-^{\mathfrak{p}_-})^{\mathrm{SU}(p,q)}$.

We suppose

$$\mathrm{span}\{\mathbf{x}\} = \{v_1, \ldots, v_r, v_{p+1}, \ldots, v_{p+s}\} \in V^{r+s}.$$

Let

$$i : F_{z_0} D_{U,z_0'} \to D$$

be the natural embedding (see equation (3.4) for the definition of $F_{z_0}D_{U,z_0'}$). We have the following crucial lemma which states that when restricted to the fiber $F_{z_0}D_{U,z_0'}$ at z_0 there is only one term left in ϕ_+.

Lemma 7.2.

$$i^*(\phi_+(\mathbf{x}))|_{z_0} = f_+^{q-s} f_-^{p-r} \bigwedge_{(\alpha,\mu)\in I} \xi'_{\alpha,\mu}$$

Proof. Recall that $\{X_{\alpha\mu} \mid (\alpha,\mu) \in I\}$ (I is defined in Equation (7.4)) span the holomorphic tangent space $N^+_{z_0}D_{U,z_0'}$ of $F_{z_0}D_{U,z_0'}$ at z_0, and $X_{\alpha\mu}$ is perpendicular to $N^+_{z_0}D_{U,z_0'}$ if $(\alpha,\mu) \notin I$. Similarly $\{\xi'_{\alpha\mu} \mid (\alpha,\mu) \in I\}$ span the holomorphic cotangent space of $F_{z_0}D_{U,z_0'}$ at z_0, and $i^*(\xi'_{\alpha\mu})|_{z_0} = 0$ if $(\alpha,\mu) \notin I$.

Notice that $\bigwedge_{(\alpha,\mu)\in I} \xi'_{\alpha\mu}$ is the unique lowest weight vector in the irreducible $\mathrm{U}(p) \times \mathrm{U}(q)$-representation generated by itself. All the other weight vectors in the representation are obtained by applying highering operators to $\bigwedge_{(\alpha,\mu)\in I} \xi'_{\alpha\mu}$ and hence are of the form

$$X = \sum_{j_1,\ldots,j_d} \xi'_{j_1} \wedge \ldots \wedge \xi'_{j_d},$$

where $\{j_1,\ldots,j_d\} \in \{(\alpha,\mu) \mid 1 \le \alpha \le p, p+1 \le \mu \le p+q\}$ and for each d-tuple of index $\{j_1,\ldots,j_d\}$ that appears in the above summation, there is an index $j \in \{j_1,\ldots,j_d\}$ such that $j \notin I$. Hence

$$i^*(X)|_{z_0} = 0.$$

By the definition of ϕ_+, the lemma follows.

□

Similarly let $\mathfrak{W}_+$ be the Fock model for the dual pair $(\mathrm{U}(p,q), \mathrm{U}(r+s,0))$. One can define an element

$$\psi_- \in \mathrm{Hom}_{\tilde{K}^0}(\wedge^{rq+ps-rs}\mathfrak{p}_-, \mathfrak{W}_+^{\mathfrak{p}_+} \otimes det^{\frac{1}{2}(r-s)})$$

and an element

$$\phi_- \in (\wedge^{rq+ps-rs}\mathfrak{p}_-^* \otimes \mathfrak{W}_+^{\mathfrak{p}_+} \otimes det^{\frac{1}{2}(r-s)})^{\tilde{K}^0}$$

We omit the construction of ψ_- and ϕ_- because it is completely analogous to the construction of ϕ_+. Both ϕ_+ and ϕ_- are closed by Theorem 7.4 which we will prove later.

Now let $\mathfrak{W}$ be the infinitesimal Fock model for the dual pair $(\mathrm{U}(p,q), \mathrm{U}(r+s, r+s))$. We have

$$\mathfrak{W} = \mathfrak{W}_- \otimes \mathfrak{W}_+ \cong \mathfrak{W}_- \otimes det^{-\frac{1}{2}(r-s)} \otimes \mathfrak{W}_+ \otimes det^{\frac{1}{2}(r-s)}.$$

We define

$$\phi = \phi_+ \wedge \phi_-.$$

It is immediate that

$$\phi \in (\wedge^{2rq+2ps-2rs}\mathfrak{p}^* \otimes \mathfrak{W})^{\tilde{K}^0}.$$

Moreover since the differential operator d for the chain complex $C^\bullet(g, \tilde{K}^0; \mathfrak{W})$ satisfies

$$d = d_- \otimes 1 + 1 \otimes d_+$$

where d_- (resp. d_+) is the differential operator for the chain complex $C^\bullet(g, \tilde{K}^0; \mathfrak{W}_- \otimes det^{-\frac{1}{2}(r-s)})$ (resp. $C^\bullet(g, \tilde{K}^0; \mathfrak{W}_+ \otimes det^{\frac{1}{2}(r-s)})$), we have

$$d\phi = 0.$$

7.2 The $\mathrm{Sp}(2n, \mathbb{R})$ case

We use the fact that $G = \mathrm{Sp}(2n, \mathbb{R}) \cong \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(n, n)$. More specifically let V be a $2n$-dimensional complex vector space with a skew Hermitian form $H(,)$ and a skew symmetric form $S(,)$. We further assume that we can choose a basis $\{v_1, \dots v_{2n}\}$ such that

1. $H(v_\alpha, v_\alpha) = -i$,
2. $H(v_\mu, v_\mu) = i$,
3. $< v_j, v_k >= 0$ if $j \neq k$,

for $1 \leq \alpha \leq n, n+1 \leq \mu \leq 2n$(in this section we keep this convention of index) and

1. $S(v_\alpha, v_{\alpha+n}) = i$,
2. $S(v_j, v_k) = 0$ if $j \neq k + n$ and $k \neq j + n$.

Given these assumptions, it can be shown that $\mathrm{U}(V, H(,)) \cap \mathrm{Sp}(V, S(,)) \cong \mathrm{Sp}(2n, \mathbb{R})$. We have seen in the last section that

$$\mathfrak{u}(V, H(,)) \cong \mathrm{Sym}(V \otimes V^*).$$

It can also be shown that

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C} \cong \mathfrak{sp}(V, S(,)) \cong \mathrm{Sym}^2_{\mathbb{C}}(V)$$

where $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$ and $\mathrm{Sym}^2_{\mathbb{C}}(V)$ acts on V by

$$v \circ_{\mathbb{C}} \tilde{v}(x) = vS(\tilde{v}, x) + \tilde{v}S(v, x).$$

We will denote a product element in $\mathrm{Sym}^2_{\mathbb{C}}(V)$ as $x \circ_{\mathbb{C}} y$ and a product element in $\mathrm{Sym}(V \otimes V^*)$ as $x \circ y$ for $x, y \in V$. The linear transformation $a(\zeta)$ introduced in equation (7.2) sits inside $\mathrm{Sp}(V, S(,)) \cap \mathrm{U}(V, H(,))$. Thus the almost complex structure

$$J_{\mathfrak{p}} = Ad(a(\zeta))$$

introduced in the last section stablize $\mathfrak{g}_0$ and we have a Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

and

$$\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$$

where $\mathfrak{k}$ is the 0 eigenspace of $J_{\mathfrak{p}}$ and $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) is the $+i$ (resp. $-i$) eigenspace of $J_{\mathfrak{p}}$. If we define $V^+ = \mathrm{span}_{\mathbb{C}}\{v_1, \dots, v_n\}$ and $V^- = \mathrm{span}_{\mathbb{C}}\{v_{n+1}, \dots, v_{2n}\}$ as in the last section, we have the identifications

$$\mathfrak{k} = \mathrm{span}_{\mathbb{C}}\{v \circ_{\mathbb{C}} \tilde{v} | v \in V^+, \tilde{v} \in V^-\}$$

$$\mathfrak{p}_+ = \mathrm{span}_{\mathbb{C}}\{v \circ_{\mathbb{C}} \tilde{v} | v, \tilde{v} \in V^+\}$$

$$\mathfrak{p}_- = \mathrm{span}_{\mathbb{C}}\{v \circ_{\mathbb{C}} \tilde{v} | v, \tilde{v} \in V^-\}.$$

More explicitly define

$$X_{\alpha\beta} = v_\alpha \circ_{\mathbb{C}} v_\beta$$

$$Y_{\mu\nu} = v_\mu \circ_{\mathbb{C}} v_\nu$$

for $1 \leq \alpha \leq \beta \leq n$ and $n+1 \leq \mu \leq \nu \leq 2n$. Then

$$\mathfrak{p}_+ = \mathrm{span}_{\mathbb{C}}\{X_{\alpha\beta}\}$$

$$\mathfrak{p}_- = \mathrm{span}_{\mathbb{C}}\{Y_{\mu\nu}\}.$$

in terms of matrices we have

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} 0 & A \\ {}^*A & 0 \end{pmatrix} | A \in M_n(\mathbb{C}), {}^tA = A \right\} \tag{7.6}$$

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} | A \in M_n(\mathbb{C}), {}^tA = A \right\}$$

$$\mathfrak{p}_- = \left\{ \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} | A \in M_n(\mathbb{C}), {}^tA = A \right\}.$$

We also let $\{\xi'_{\alpha\beta} | 1 \leq \alpha \leq \beta \leq n\}$ (resp. $\{\xi''_{\mu\nu} | n+1 \leq \mu \leq \nu \leq 2n\}$) be the basis of $\mathfrak{p}_+^*$ (resp. $\mathfrak{p}_-^*$).

Now fix $1 \leq r \leq n$. Define

$$U = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r, v_{n+1}, \ldots, v_{n+r}\}.$$

Recall that the symmetric space D of $G = \mathrm{Sp}(2n, \mathbb{R})$ is the set of positive definite almost complex structures with respect to $S(,)$. Define the complex structure J_U on U to be

1. $J_U(v_\alpha) = iv_\alpha$ for $1 \leq \alpha \leq r$

2. $J_U(v_{\alpha+n}) = -iv_{\alpha+n}$ for $1 \leq \alpha \leq r$.

In Chapter 3 we define a sub symmetric space D_U of D:

$$D_{U,J_U} = \{J \in D | J|_U = J_U\}.$$

Fix the base point $z_0 = a(\zeta)^2 \in D_{U,J_U}$. We have

$$T_{z_0} D_{U,J_U} \otimes \mathbb{C} = T^+_{z_0} D_{U,J_U} \oplus T^-_{z_0} D_{U,J_U}$$

where $T_{z_0} D_{U,J_U}$ is the tangent space of D_{U,J_U} at z_0, and $T^+_{z_0} D_{U,J_U}$ (resp. $T^-_{z_0} D_{U,J_U}$)

is the holomorphic (resp. anti-holomorphic) part of it. And we have

$$T^+_{z_0} D_{U,J_U} = \text{span}_{\mathbb{C}}\{X_{\alpha\beta} | r+1 \leq \alpha \leq \beta \leq n\}$$

$$T^-_{z_0} D_{U,J_U} = \text{span}_{\mathbb{C}}\{Y_{\mu\nu} | n+r+1 \leq \mu \leq \nu \leq 2n\}$$

The holomorphic tangent space of the fiber $F_{z_0} D_{U,z'_0}$ (see Section 3.2) at z_0 is

$$N^+_{z_0} D_{U,J_U} = \text{span}_{\mathbb{C}}\{X_{\alpha\beta} | (\alpha, \beta) \in I\} \tag{7.7}$$

where I is the index set

$$I = \{(\alpha, \beta) | 1 \leq \alpha \leq r, \alpha \leq \beta \leq n\}. \tag{7.8}$$

Define $e_{D_{U,J_U}} \in \wedge^{\bullet} \mathfrak{p}_+$ by

$$e_{D_{U,J_U}} = \bigwedge_{(\alpha,\beta)\in I} X_{\alpha\beta}.$$

Let $\mathfrak{W}_-$ be the infinitesimal Fock model for the dual pair $\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(0, 2r)$ defined in Chapter 6. Define $f_+ \in \mathfrak{W}_-$ by

$$f_+ = det \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1r} \\ \dots & \dots & \dots & \dots \\ u_{r1} & u_{r2} & \dots & u_{rr} \end{pmatrix}$$

Also define

$$f_{D_{U,J_U}} = f_+^{n-r+1}.$$

It can be shown that $\mathfrak{k}$ acts on $\mathcal{W}_-$ by

$$\omega(v_\alpha \circ_{\mathbb{C}} v_{\beta+n}) = -i\sum_{k=1}^{r}(u_{\alpha+n,k}\frac{\partial}{\partial u_{\beta+n,k}} + u_{\alpha,k}\frac{\partial}{\partial u_{\beta,k}}) - ir\delta_{\alpha\beta}.$$

The adjoint action of $\mathfrak{k}$ on $\mathfrak{p}_+$ induces an action on $\wedge^\bullet\mathfrak{p}_+$. Define

$$\mathfrak{b} = \mathrm{span}_{\mathbb{C}}\{v_\alpha \circ_{\mathbb{C}} v_{\beta+n} | 1 \le \alpha \le \beta \le n\}.$$

$$\mathfrak{t} = \mathrm{span}_{\mathbb{C}}\{v_\alpha \circ_{\mathbb{C}} v_{\alpha+n} | 1 \le \alpha \le n\}.$$

$$\mathfrak{n}_- = \mathrm{span}_{\mathbb{C}}\{v_\alpha \circ_{\mathbb{C}} v_{\beta+n} | 1 \le \beta < \alpha \le n\}.$$

$\mathfrak{b}$ is a Borel sub-algebra of $\mathfrak{k}$. Both $e_{D_{U,J_U}}$ and $f_{D_{U,J_U}}$ are eigenvectors under the action of $\mathfrak{b}$. Moreover they have the same weight

$$(\underbrace{n+1,\ldots,n+1}_{r},\underbrace{r,\ldots,r}_{n-r}).$$

Let $K = \mathrm{U}(n)$ and $\tilde{K}$ be its two cover which is the preimage of K under the map $\mathrm{Mp}(\mathbb{W}) \to \mathrm{Sp}(\mathbb{W})$. Using the seesaw pair

$$\begin{array}{ccc} \mathrm{U}(n,n) & & \mathrm{O}(0,2r+2s) \\ \uparrow & \nwarrow\!\!\!\!\nearrow & \uparrow \\ \mathrm{Sp}(2n,\mathbb{R}) & & \mathrm{U}(0,r+s) \end{array} \tag{7.9}$$

and fact about $\widetilde{\mathrm{U}(r+s)}$ (see section 7.5 or [P]), we can see that

$$\tilde{K} = K \times \{\pm 1\}.$$

Now denote the irreducible representation of K generated by $e_{D_{U,J_U}}$ as $V(U)$, the irreducible representation of K generated by $f_{D_{U,J_U}}$ as $A(U)$. By the theory of highest weight we have a K-equivariant map $\psi_+ : V(U) \to A(U)$ such that

$$\psi_+(e_{D_{U,J_U}}) = f_{D_{U,J_U}}.$$

The main result of [And] in this case is

Theorem 7.2.

$$\psi_+ \in \mathrm{Hom}_K(\wedge^{\frac{1}{2}n(n+1)-\frac{1}{2}r(r+1)}\mathfrak{p}_+, \mathfrak{W}_-^{\mathfrak{p}_-}).$$

Let $\{\epsilon_1, \ldots, \epsilon_d\}$ be a basis of $V(U) \subset \wedge^\bullet \mathfrak{p}_+$ such that each ϵ_i is a weight vector of $\mathfrak{t}$. Extend $\{\epsilon_1, \ldots, \epsilon_d\}$ to a basis of $\wedge^{\frac{1}{2}n(n+1)-\frac{1}{2}r(r+1)}\mathfrak{p}_+$, take the dual basis inside $\wedge^{\frac{1}{2}n(n+1)-\frac{1}{2}r(r+1)}\mathfrak{p}_+^*$ and denote the first d basis vectors by $\Omega_1, \ldots, \Omega_d$. We have an isomorphism

$$\mathrm{Hom}_K(\wedge^\bullet \mathfrak{p}_+, \mathfrak{W}_-^{\mathfrak{p}_-}) \cong (\wedge^\bullet \mathfrak{p}_+^* \otimes \mathfrak{W}_-^{\mathfrak{p}_-})^K.$$

Under this isomorphism ψ_+ maps to an element $\phi_+ \in (\wedge^{\frac{1}{2}n(n+1)-\frac{1}{2}r(r+1)}\mathfrak{p}_+^* \otimes \mathfrak{W}_-^{\mathfrak{p}_-})^K$:

$$\phi_+ = \sum_{i=1}^{d} \psi_+(\epsilon_i)\Omega_i.$$

The element thus defined is independent of the choice of the basis $\{\epsilon_1, \ldots, \epsilon_d\}$.

We suppose

$$\mathrm{span}_{\mathbb{C}}\{\mathbf{x}\} = \mathrm{span}\{v_1, \ldots, v_r\}.$$

Let

$$i : F_{z_0} D_{U,J_U} \to D$$

be the natural embedding (see equation (3.4) for the definition of $F_{z_0} D_{U,J_U}$). We can prove the following lemma similarly as Lemma 7.2:

Lemma 7.3.

$$i^*(\phi_+(\mathbf{x}))|_{z_0} = f_+^{n-r+1} \bigwedge_{(\alpha,\mu)\in I} \xi'_{\alpha,\mu}$$

Let $\mathfrak{W}_+$ be the infinitesimal Fock model for the dual pair $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(2r,0))$. One can analogously define an element

$$\psi_- \in \mathrm{Hom}_K(\wedge^{\frac{1}{2}n(n+1)-\frac{1}{2}r(r+1)}\mathfrak{p}_-, \mathfrak{W}_+^{\mathfrak{p}_+})$$

and an element

$$\phi_- \in (\wedge^{\frac{1}{2}n(n+1)-\frac{1}{2}r(r+1)}\mathfrak{p}_-^* \otimes \mathfrak{W}_+^{\mathfrak{p}_+})^K$$

Now let $\mathfrak{W}$ be the infinitesimal Fock model for the dual pair $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(2r,2r))$. We have

$$\mathfrak{W} = \mathfrak{W}_- \otimes \mathfrak{W}_+.$$

We regard both ϕ_+ and ϕ_- as elements in $C^\bullet(g, K; \mathfrak{W})$ and define

$$\phi = \phi_+ \wedge \phi_-.$$

It is immediate that

$$\phi \in (\wedge^{n(n+1)-r(r+1)}\mathfrak{p}^*, \mathfrak{W})^K.$$

$$d\phi_+ = d\phi_- = d\phi = 0$$

The proof of this fact is similar to the one in the previous section which is a consequence of Theorem 7.4.

7.3 The $\mathrm{O}^*(2n)$ case

Let V be a n-dimensional right $\mathbb{H}$ vector space with skew Hermitian form $(,)$ satisfying

$$(vh, \tilde{v}\tilde{h}) = \overline{h}(v, \tilde{v})\tilde{h},$$

where $v, \tilde{v} \in V$ and $h, \tilde{h} \in \mathbb{H}$. $V_{\mathbb{C}}$ is the underlying complex vector space of V. Define V^* the dual vector space of V as follows. The underlying Abelian group of V^* is the same with that of V and we denote by v^* the element $v \in V$ if we think of v as an element in V^* instead. The scalar multiplication in V^* is defined by

$$hv^* = v\overline{h}, \forall h \in \mathbb{H}.$$

where the left hand side is scalar multiplication in V^* while the right hand side is scalar multiplication in V. There is an isomorphism $V^* \to \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ given by the form $(,)$

$$v \mapsto (v, \cdot).$$

One can also identify $V \otimes_{\mathbb{C}} V^*$ with $\mathrm{Hom}_{\mathbb{C}}(V, V)$ by the map

$$v \otimes \tilde{v}^* \mapsto v(\tilde{v}, \cdot).$$

Let $\mathrm{Sym}(V \otimes V^*)$ be the symmetric tensor inside $V \otimes_{\mathbb{C}} V^*$ (this makes sense since V and V^* have the same underlying Abelian group). By the above identification, $\mathrm{Sym}(V \otimes V^*)$ acts on V by

$$(v \circ_{\mathbb{H}} \tilde{v})(x) = v(\tilde{v}, x) + \tilde{v}(v, x)$$

where $v \circ_{\mathbb{H}} \tilde{v} = v \otimes \tilde{v}^* + \tilde{v}^* \otimes v$. One can check that this action is $\mathbb{H}$-linear and satisfies

$$((v \circ_{\mathbb{H}} \tilde{v})(x), y) + (x, (v \circ_{\mathbb{H}} \tilde{v})(y)) = 0$$

Moreover we have

Lemma 7.4.

$$\mathrm{Sym}(V \otimes V^*) \cong \mathfrak{o}^*(V, (,)) = \mathfrak{o}^*(2n)$$

Define $H(,)$ on $V_{\mathbb{C}}$ by

$$H(v,\tilde{v}) = a + bi \text{ if } (v,\tilde{v}) = a + bi + cj + dk.$$

Then $H(,)$ is (complex) skew Hermitian of signature (n,n). We also define $S(,)$ on $V_{\mathbb{C}}$ by

$$S(v,\tilde{v}) = H(vj,\tilde{v}).$$

One can check that $S(,)$ is symmetric complex bilinear. We have the following fact

$$\mathrm{O}^*(V,(,)) = \mathrm{U}(V_{\mathbb{C}}, H(,)) \cap \mathrm{O}(V_{\mathbb{C}}, S(,))$$

In other words

$$\mathrm{O}^*(2n) = \mathrm{U}(n,n) \cap \mathrm{O}(2n,\mathbb{C}).$$

It can also be shown that

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C} \cong \mathfrak{o}(V, S(,)) \cong \wedge^2_{\mathbb{C}}(V)$$

where $\mathfrak{g}_0 = \mathfrak{o}^*(V,(,))$ and $\wedge^2_{\mathbb{C}}(V)$ acts on V by

$$v \wedge \tilde{v}(x) = -vS(\tilde{v},x) + \tilde{v}S(v,x).$$

Explicitly choose an orthogonal basis $\{v_1, \ldots, v_n\}$ of V such that

$$(v_\alpha, v_\beta) = -i\delta_{\alpha\beta}$$

for $1 \leq \alpha, \beta \leq n$. Then $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ is a basis of $V_{\mathbb{C}}$ where

$$v_\mu = v_{\mu-n}j$$

for $n+1 \leq \mu \leq 2n$. The $\mathbb{C}$-linear transformation $a(\zeta)$ introduced in equation (7.2) sits inside $\mathrm{O}(V, S(,)) \cap \mathrm{U}(V, H(,))$. Thus the almost complex structure

$$J_{\mathfrak{p}} = Ad(a(\zeta))$$

stablize $\mathfrak{g}_0$ and we have a Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

and

$$\mathfrak{p} = \mathfrak{p}_+ + \mathfrak{p}_-$$

where $\mathfrak{k}$ is the 0 eigenspace of $J_{\mathfrak{p}}$ and $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) is the $+i$ (resp. $-i$) eigenspace of $J_{\mathfrak{p}}$.

If we define $V^+ = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_n\}$ and $V^- = \mathrm{span}_{\mathbb{C}}\{v_{n+1}, \ldots, v_{2n}\}$ as in the last section, we have the identifications

$$\mathfrak{k} = \mathrm{span}_{\mathbb{C}}\{v \wedge \tilde{v} | v \in V^+, \tilde{v} \in V^-\}$$

$$\mathfrak{p}_+ = \mathrm{span}_{\mathbb{C}}\{v \wedge \tilde{v} | v, \tilde{v} \in V^+\}$$

$$\mathfrak{p}_- = \mathrm{span}_{\mathbb{C}}\{v \wedge \tilde{v} | v, \tilde{v} \in V^-\}.$$

More explicitly define

$$X_{\alpha\beta} = v_\alpha \wedge v_\beta$$

$$Y_{\mu\nu} = v_\mu \wedge v_\nu$$

for $1 \leq \alpha < \beta \leq n$ and $n+1 \leq \mu < \nu \leq 2n$. Then

$$\mathfrak{p}_+ = \mathrm{span}_{\mathbb{C}}\{X_{\alpha\beta}\}$$

$$\mathfrak{p}_- = \mathrm{span}_{\mathbb{C}}\{Y_{\mu\nu}\}.$$

in terms of matrices we have

$$\mathfrak{p}_0 = \Big\{ \begin{pmatrix} 0 & A \\ {}^*A & 0 \end{pmatrix} | A \in M_n(\mathbb{C}), {}^tA = -A \Big\} \tag{7.10}$$

$$\mathfrak{p}_+ = \Big\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} | A \in M_n(\mathbb{C}), {}^tA = -A \Big\}$$

$$\mathfrak{p}_- = \Big\{ \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} | A \in M_n(\mathbb{C}), {}^tA = -A \Big\}.$$

We also let $\{\xi'_{\alpha\beta} | 1 \leq \alpha < \beta \leq n\}$ (resp. $\{\xi''_{\mu\nu} | n+1 \leq \mu < \nu \leq 2n\}$) be the basis of $\mathfrak{p}_+^*$ (resp. $\mathfrak{p}_-^*$).

Now fix $1 \leq r \leq n$. Define

$$U = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r, v_{n+1}, \ldots, v_{n+r}\}.$$

Recall that the symmetric space D of $G = \mathrm{O}^*(2n)$ is the set of n-dimensional subspace z of $V_{\mathbb{C}}$ such that

1. $S(,)|_z$ is zero

2. $\frac{1}{-i}H(,)|_z$ is positive definite.

Define $z'_0 \subset U$ by

$$z'_0 = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r\}.$$

In Chapter 3 we define a sub symmetric space D_{U,z'_0} of D:

$$D_{U,z'_0} = \{z \in D | z_0 \in z\}.$$

Fix the base point $z_0 = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_n\} \in D$. We have

$$T_{z_0} D_{U,z'_0} \otimes \mathbb{C} = T^+_{z_0} D_{U,z'_0} \oplus T^-_{z_0} D_{U,z'_0}$$

where $T_{z_0}D_{U,z_0'}$ is the tangent space of $D_{U,z_0'}$ at z_0, and $T^+_{z_0}D_{U,z_0'}$ (resp. $T^-_{z_0}D_{U,z_0'}$) is the holomorphic (resp. anti-holomorphic) part of it. And we have

$$T^+_{z_0}D_{U,z_0'} = \mathrm{span}_{\mathbb{C}}\{X_{\alpha\beta}|r+1 \le \alpha < \beta \le n\}$$

$$T^-_{z_0}D_{U,z_0'} = \mathrm{span}_{\mathbb{C}}\{Y_{\mu\nu}|n+r+1 \le \mu < \nu \le 2n\}$$

The holomorphic tangent space of the fiber $F_{z_0}D_{U,z_0'}$ (see Section 3.2) at z_0 is

$$N^+_{z_0}D_{U,z_0'} = \mathrm{span}_{\mathbb{C}}\{X_{\alpha\beta}|(\alpha,\beta) \in I\} \tag{7.11}$$

where I is the index set

$$I = \{(\alpha,\beta)|1 \le \alpha \le r, \alpha < \beta \le n\}$$

Define $e_{D_{U,z_0'}} \in \wedge^\bullet \mathfrak{p}_+$ by

$$e_{D_{U,z_0'}} = \bigwedge_{(\alpha,\beta)\in I} X_{\alpha\beta}.$$

Let $\mathfrak{W}_-$ be the Fock model defined in the last section for the dual pair $(\mathrm{O}^*(2n), \mathrm{Sp}(0,r))$. Define $f_+ \in \mathfrak{W}_-$ by

$$f_+ = det\begin{pmatrix} u_{11} & u_{12} & \dots & u_{1r} \\ \dots & \dots & \dots & \dots \\ u_{r1} & u_{r2} & \dots & u_{rr} \end{pmatrix}$$

Also define $f_{D_{U,z_0'}} \in \mathcal{W}_-$ by

$$f_{D_{U,z_0'}} = f_+^{n-r-1}.$$

It can be shown that $\mathfrak{k}$ acts on $\mathcal{W}_-$ by

$$\omega(v_\alpha \wedge v_{\beta+n}) = -i\sum_{k=1}^{r}(u_{\alpha+n,k}\frac{\partial}{\partial u_{\beta+n,k}} + u_{\alpha,k}\frac{\partial}{\partial u_{\beta,k}}) - ir\delta_{\alpha\beta}.$$

The adjoint action of $\mathfrak{k}$ on $\mathfrak{p}_+$ induces an action on $\wedge^\bullet\mathfrak{p}_+$. Define

$$\mathfrak{b} = \mathrm{span}_{\mathbb{C}}\{v_\alpha \wedge v_{\beta+n} | 1 \le \alpha \le \beta \le n\}.$$

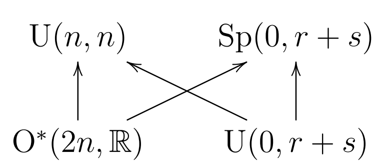
$$\mathfrak{t} = \mathrm{span}_{\mathbb{C}}\{v_\alpha \wedge v_{\alpha+n} | 1 \le \alpha \le n\}.$$

$$\mathfrak{n}_- = \mathrm{span}_{\mathbb{C}}\{v_\alpha \wedge v_{\beta+n} | 1 \le \beta < \alpha \le n\}.$$

$\mathfrak{b}$ is a Borel sub-algebra of $\mathfrak{k}$. Both $e_{D_{U,z_0'}}$ and $f_{D_{U,z_0'}}$ are eigenvectors under the action of $\mathfrak{b}$. Moreover they have the same weight

$$(\underbrace{n-1,\dots,n-1}_{r},\underbrace{r,\dots,r}_{n-r}).$$

Let $K = \mathrm{U}(n)$ and $\tilde{K}$ be its two cover which is the preimage of K under the map $\mathrm{Mp}(\mathbb{W}) \to \mathrm{Sp}(\mathbb{W})$. Using the seesaw pair



(7.12)

and fact about $\widetilde{\mathrm{U}(r+s)}$ (see Section 7.5 or [P]), we can see that

$$\tilde{K} = K \times \{\pm 1\}.$$

Now denote the irreducible representation of K generated by $e_{D_{U,z_0'}}$ as $V(U)$, the irreducible representation of K generated by $f_{D_{U,z_0'}}$ as $A(U)$. By the theory of highest weight we have a K-equivariant map $\psi_+ : V(U) \to A(U)$ such that

$$\psi_+(e_{D_{U,z_0'}}) = f_{D_{U,z_0'}}.$$

The main result of [And] in this case is

Theorem 7.3.

$$\psi_+ \in \mathrm{Hom}_K(\wedge^\bullet \mathfrak{p}_+, \mathfrak{W}_-^{\mathfrak{p}_-}).$$

Let $\{\epsilon_1, \ldots, \epsilon_d\}$ be a basis of $V(U) \subset \wedge^\bullet \mathfrak{p}_+$ such that each ϵ_i is a weight vector of $\mathfrak{t}$. Extend $\{\epsilon_1, \ldots, \epsilon_d\}$ to a basis of $\wedge^{\frac{1}{2}n(n-1)-\frac{1}{2}r(r-1)}\mathfrak{p}_+$, take the dual basis inside $\wedge^{\frac{1}{2}n(n-1)-\frac{1}{2}r(r-1)}\mathfrak{p}_+^*$ and denote the first d basis vectors by $\Omega_1, \ldots, \Omega_d$. We have an isomorphism

$$\mathrm{Hom}_K(\wedge^\bullet \mathfrak{p}_+, \mathfrak{W}_-^{\mathfrak{p}_-}) \cong (\wedge^\bullet \mathfrak{p}_+^* \otimes \mathfrak{W}_-^{\mathfrak{p}_-})^K.$$

Under this isomorphism ψ_+ maps to an element $\phi_+ \in (\wedge^{\frac{1}{2}n(n-1)-\frac{1}{2}r(r-1)}\mathfrak{p}_+^* \otimes \mathfrak{W}_-^{\mathfrak{p}_-})^K$:

$$\phi_+ = \sum_{i=1}^{d} \psi_+(\epsilon_i)\Omega_i.$$

We suppose

$$\mathrm{span}\{\mathbf{x}\} = \mathrm{span}\{v_1, \ldots, v_r\}.$$

Let

$$i : F_{z_0} D_{U,z_0'} \to D$$

be the natural embedding (see equation (3.4) for the definition of $F_{z_0} D_{U,z_0'}$). We can prove the following lemma similarly as Lemma 7.2:

Lemma 7.5.

$$i^*(\phi_+(\mathbf{x}))|_{z_0} = f_+^{n-r-1} \bigwedge_{(\alpha,\mu)\in I} \xi'_{\alpha,\mu}$$

Let $\mathfrak{W}_+$ be the infinitesimal Fock model for the dual pair $(\mathrm{O}^*(2n), \mathrm{Sp}(r, 0))$. One can analogously define an element

$$\psi_- \in \mathrm{Hom}_K(\wedge^{\frac{1}{2}n(n-1)-\frac{1}{2}r(r-1)}\mathfrak{p}_-, \mathfrak{W}_+^{\mathfrak{p}_+})$$

and an element

$$\phi_- \in (\wedge^{\frac{1}{2}n(n-1)-\frac{1}{2}r(r-1)}\mathfrak{p}_-^* \otimes \mathfrak{W}_+^{\mathfrak{p}_+})^K$$

Now let $\mathfrak{W}$ be the infinitesimal Fock model for the dual pair $(\mathrm{O}^*(2n), \mathrm{Sp}(r, r))$. We have

$$\mathfrak{W} = \mathfrak{W}_- \otimes \mathfrak{W}_+.$$

We regard both ϕ_+ and ϕ_- as elements in $(\wedge^{n(n-1)-r(r-1)}\mathfrak{p}^* \otimes \mathfrak{W})^K$ and define

$$\phi = \phi_+ \wedge \phi_-.$$

It is immediate that

$$\phi \in (\wedge^{n(n-1)-r(r-1)}\mathfrak{p}^* \otimes \mathfrak{W})^K.$$

$$d\phi_+ = d\phi_- = d\phi = 0$$

The proof of this fact is similar to the one in the unitary case which is a consequence of Theorem 7.4..

7.4 Closedness of holomorphic differentials

In this subsection, we prove that the Anderson cochains φ_+ are closed hence cocycles. First we recall the following well-known fact

Lemma 7.6. *A holomorphic form on a compact Kähler manifold M is closed.*

Proof. On compact Kähler manifold we have the following equality of Laplacians

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

A holomorphic form φ is $\bar{\partial}$-closed. It is also $\bar{\partial}^*$-closed as well because it has Hodge-type $(p, 0)$. Hence φ is $\Delta_{\bar{\partial}}$-harmonic, hence Δ_d-harmonic. Hence φ is closed. □

Theorem 7.4. *Let $G = \mathrm{SU}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{SO}^*(2n, \mathbb{R})$. Then the ϕ_+ (ϕ_- resp.) constructed in [And] is closed as an element of $C^\bullet(g, K; \mathfrak{W}_-)$ ($C^\bullet(g, K; \mathfrak{W}_+)$ resp).*

Proof. We prove the holomorphic case, the anti-holomorphic case is similar.

Recall that ϕ_+ is generated by the special harmonic polynomial f ($f = f_+^{q-s} f_-^{p-r}$, f_+^{n-r+1}, f_-^{n-r-1} respectively for our three cases) considered in [KV]. By

[KV] f is in a representation $\mathcal{A} \boxtimes \theta(\mathcal{A})$ of $G \times G'$ where $\mathcal{A}$ is the K-finite vectors of an irreducible unitary representation and $\theta(\mathcal{A})$ is a finite dimensional representation of the compact group G'. Hence the Anderson cocycle ϕ_+ is a holomorphic cocycle in $H^{(p,0)}(\mathfrak{g}, K; \mathcal{A})$.

By the proof of Proposition 2.3 in [And], there is a cocompact lattice Γ of G and a $(\mathfrak{g}, K)$-map

$$I : \mathfrak{W}_- \to C^\infty(\Gamma \backslash G),$$

such that $I(f) \neq 0$. Since $f \in \mathcal{A}$ and $\mathcal{A}$ is irreducible, we know that I is injective restricted on $\mathcal{A}$. Hence the map on the cochains:

$$I_* C^\bullet(\mathfrak{g}, K, \mathcal{A}) \to C^\bullet(\mathfrak{g}, K, C^\infty(\Gamma \backslash G))$$

is also injective.

Now $I_*(\varphi_+)$ is a holomorphic form on a compact Kähler manifold. So it is closed by Lemma 7.6. Since I_* is a map of chain complexes we know that

$$I_*(d\varphi_+) = dI_*(\varphi_+) = 0.$$

Because I_* is injective on $C^\bullet(\mathfrak{g}, K, \mathcal{A})$, we know that

$$d\varphi_+ = 0.$$

□

7.5 φ in the Schrodinger Model

Let K be $\mathrm{SU}(p,q)$, $\mathrm{U}(n)$ (regarded as the maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{R})$) or $\mathrm{U}(n)$ (regarded as the maximal compact subgroup of $\mathrm{O}^*(2n)$). In all three cases we have constructed elements in the relative Lie algebra cohomology $\phi_+, \phi_-, \phi \in (\wedge^\bullet \mathfrak{p}^* \otimes \mathfrak{W})^K$. Recall that we have an defined a map $\iota : \mathfrak{W} \to \mathcal{S}(V^{r+s})$. Since ι is an isomorphism between $\mathfrak{W}$ and its image in $\mathcal{S}(V^m)$, we define

$$\varphi_+ = (\iota^{-1})^*(\phi_+), \quad \varphi_- = (\iota^{-1})^*(\phi_-), \quad \varphi = (\iota^{-1})^*(\phi)$$

in $(\wedge^\bullet \mathfrak{p}^* \otimes \mathcal{S}(V^m))^K$. More explicitly we have in the Schrodinger Model (see equation (7.5)):

$$\varphi = \sum_{i,j=1}^{d} [(\iota \circ \psi_+(\epsilon_i))(\iota \circ \psi_-(\bar{\epsilon}_j))\varphi_0]\Omega_i \wedge \bar{\Omega}_j.$$

By our construction of ψ and lemma 6.4, $\iota \circ \psi_+(\epsilon_i)$ and $\iota \circ \psi_-(\bar{\epsilon}_j)$ are polynomials of the operators that show up on the right hand side of lemma 6.4. Hence we have

$$(\iota \circ \psi_+(\epsilon_i))(\iota \circ \psi_-(\bar{\epsilon}_j))\varphi_0 = p_{ij}\varphi_0,$$

and

$$\varphi = \varphi_0 \sum_{i,j=1}^{d} p_{ij}\Omega_i \wedge \bar{\Omega}_j \tag{7.13}$$

where p_{ij} is a polynomial in the variables $\{z_{\alpha a}, \bar{z}_{\alpha a}, z_{\mu a}, \bar{z}_{\mu a} \mid 1 \leq \alpha \leq p, p+1 \leq \mu \leq p+q, 1 \leq a \leq m\}$ (in the $\mathrm{Sp}(2n,\mathbb{R})$ or $\mathrm{O}^*(2n)$ case, $p = q = n, m = r$). Define

$$f'_+ = (-2\sqrt{2}\pi)^r \det \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1r} \\ \dots & \dots & \dots & \dots \\ z_{r1} & z_{r2} & \dots & z_{rr} \end{pmatrix}$$

$$f'_- = (-2\sqrt{2}\pi)^s \det \begin{pmatrix} \bar{z}_{p+1\ r+1} & \bar{z}_{p+1\ r+2} & \dots & \bar{z}_{p+1\ r+s} \\ \dots & \dots & \dots & \dots \\ \bar{z}_{p+s\ r+1} & \bar{z}_{p+s\ r+2} & \dots & \bar{z}_{p+s\ r+s} \end{pmatrix}$$

Recall that $i : F_{z_0}D_{U,z'_0} \to D$ is the natural embedding where $U = \mathrm{span}\{\mathbf{x}\}$ and z'_0 are defined as in each subsection.

Lemma 7.7. *The highest term (in terms of the degree of the polynomial in front of φ_0) of $i^*(\varphi(\mathbf{x}))|_{z_0}$ is*

$$(f'_+\bar{f}'_+)^{q-s}(f'_-\bar{f}_-)^{p-r}\mathrm{dVol}(F_{z_0}D_{U,z'_0})|_{z_0},$$

$$(f'_+\bar{f}'_+)^{n-r+1}\mathrm{dVol}(F_{z_0}D_{U,z'_0})|_{z_0},$$

$$(f'_+\bar{f}'_+)^{n-r-1}\mathrm{dVol}(F_{z_0}D_{U,z'_0})|_{z_0},$$

respectively for $G = \mathrm{U}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{O}^*(2n,\mathbb{R})$, where $\mathrm{dVol}(F_{z_0}D_{U,z'_0})$ is the volume form of the Fiber $F_{z_0}D_{U,z'_0}$.

Proof. It follows from Lemma 7.2 (resp. Lemma 7.3 or Lemma 7.5), the analogous expression of $i^*(\varphi_-)$ and Lemma 6.5. □

Chapter 8: The Thom form for the fiber bundle $\Gamma\backslash D \to \Gamma_{\mathbf{x}}\backslash D_{\mathbf{x},z'}$

In this chapter we start to prove the main theorems of the paper. We give the symmetric space $D = G_\infty/K_\infty$ the G_∞-invariant Riemannian metric τ induced by the trace form on $\mathfrak{p}_0$. D is then a negatively curved symmetric Kahler manifold whose sectional curvatures are automatically bounded as it is homogeneous. We suppose the sectional curvature of D is bounded below by $-\rho^2$. Let $\mathbf{x} \in V^m$ viewed as a m-tuple of vectors in V.

Let $\mathbf{x} \in V^m$ $(1 \leq m \leq n)$ satisfy the following assumptions in the three cases of our interests respectively:

1. The Hermitian form $i(,)_{v_1}$ restricted to $\mathrm{span}_B \mathbf{x} \otimes k_{v_1}$ is non-degenerate and has signature (r, s). In particular $1 \leq r \leq p, 1 \leq s \leq q$ and $r + s = m$.
2. The Hermitian form $(,)$ restricted to $\mathrm{span}_B \mathbf{x}$ is non-degenerate. $m = r$.
3. The skew-Hermitian form $(,)$ restricted to $\mathrm{span}_B \mathbf{x}$ is non-degenerate. $m = r$.

In each case we define a $2m$ dimensional B-vector space W, a form $<,>$ on W that is non-degenerate and split. We assume that it is Hermitian, skew Hermitian and Hermtian respectively in case (1), (2) and (3). Let G' be the group of B-linear transformations on W preserving $<,>$. Define G'_v and G'_∞ for G' the same way for

G. We have

1. $G'_\infty = \mathrm{U}(r+s, r+s)^\ell$

2. $G'_\infty = \mathrm{O}(2r, 2r)^\ell$

3. $G'_\infty = \mathrm{Sp}(r, r)^\ell$

for case (1), (2) and (3) respectively.

We fix point z_0 in the symmetric space D of G_∞. That is to say we fix a maximal compact group K_∞ of G_∞. By our assumptions, $K_\infty = K_{v_1} \times \prod_{v \in S_\infty, v \neq v_1} G_v$. Let

$$\mathfrak{g}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0$$

be the corresponding Cartan decomposition on the Lie algebra $\mathfrak{g}_0$ of G_∞ (we drop the subscript 0 to indicate complexification). Let $\Omega^\bullet(D)$ be the space of smooth differential forms on D.

We think of $(\wedge^\bullet \mathfrak{p}^* \otimes \mathcal{S}(V_\infty^m))^{K_\infty}$ as a cotangent vector with values in $\mathcal{S}(V_\infty^m)$ at the base point z_0 where $\mathcal{S}$ stands for Schwartz functions. There is an isomorphism given by the restriction map

$$(\Omega^\bullet(D) \otimes \mathcal{S}(V_\infty^m))^{G_\infty} \to (\wedge^\bullet \mathfrak{p} \otimes \mathcal{S}(V_\infty^m))^{K_\infty}$$

Let $\tilde{G}'_{v_1}$ be the metaplectic cover of G'_{v_1} which is the preimage of G'_{v_1} under the map $\mathrm{Mp}((V^m)_{v_1}) \to \mathrm{Sp}((V^m)_{v_1})$. And let

$$\tilde{G}'_\infty = \tilde{G}'_{v_1} \times \prod_{v \neq v_1} G'_v.$$

$\tilde{G}'$ acts on $\mathcal{S}(V^n_\infty)$ by the oscillator(Weil) representation ω and the action commutes with that of G. Recall that the action of G_∞ via ω is just the induced action on functions

$$(\omega(g)f)(\mathbf{x}) = f(g^{-1}\mathbf{x}).$$

for $f \in \mathcal{S}(V^m_\infty)$, $\mathbf{x} \in V^m_\infty$ viewd as $(p+q)$ by m matrix.

Any form $\psi \in (\wedge^\bullet \mathfrak{p}^* \otimes \mathcal{S}(V^m_\infty))^K$ give rise to form $\tilde{\psi}$

$$\tilde{\psi}(g, g', \mathbf{x}) = L^*_{g^{-1}}(\omega(g, g')\psi(\mathbf{x}))$$

where $g \in G_\infty$, $g' \in \tilde{G}'_\infty$ and L_g denotes the left translation map by g on G_∞ and L^*_g is its induced pullback map on $\Omega^\bullet(D)$. Then we have

$$\tilde{\psi} \in (\Omega^\bullet(D) \otimes \mathcal{S}(V^m_\infty))^{G_\infty \times \tilde{G}'_\infty}.$$

We will often write $\tilde{\psi}(z, g', \mathbf{x})$ for $z \in D$ as $\tilde{\psi}$ only depends on g *mod* K_∞, and

$$\tilde{\psi}(z, g', \mathbf{x}) = L^*_{g_z{}^{-1}}(\omega(g_z, g')\psi(\mathbf{x}))$$

where g_z is any element in G that maps our base point z_0 to z.

Let φ_v be the vacuum vector (Gaussian function) of $\mathcal{S}(V_v)$ for any Archimedean place $v \neq v_1$. Define

$$\varphi_\infty = \varphi \otimes \prod_{v \neq v_1} \varphi_v \in (\wedge^\bullet \mathfrak{p} \otimes \mathcal{S}(V_\infty^m))^{K_\infty}$$

where φ is the form we constructed in the last chapter. Apply the above construction to φ_∞ to get a form $\tilde{\varphi}_\infty \in (\Omega^\bullet(D) \otimes \mathcal{S}(V_\infty^m))^{G_\infty \times \tilde{G}'_\infty}$.

Choose an open subgroup $\mathbb{L}$ of $V(\mathbb{A}_f)$. Recall from chapter 5 that we can choose arithmetic subgroups $\Gamma \subset G(k)$ and $\Gamma' \subset \tilde{G}'(k)$ and a $\Gamma \times \Gamma'$-invariant distribution θ_L

$$\theta_{\mathcal{L},\psi} = \sum_{\mathbf{x} \in L^m} \psi(X)$$

where $\mathcal{L} = \mathbb{L} \cap V(k)$ is a lattice in V_∞.

We now apply $\theta_{\mathcal{L}}$ to $\tilde{\varphi}_\infty$ to get

$$\theta_{\mathcal{L},\tilde{\varphi}_\infty} \in \Omega^\bullet(\Gamma \backslash D) \otimes C^\infty(\Gamma' \backslash \tilde{G}'_\infty).$$

We also define

$$\theta_{\mathcal{L},\beta,\tilde{\varphi}_\infty}(z, g') = \sum_{\mathbf{x} \in L^m, (\mathbf{x},\mathbf{x}) = \beta} \tilde{\varphi}_\infty(z, g', \mathbf{x})) \tag{8.1}$$

for a matrix $\beta \in M_m(B)$. We have the following Fourier expansion of $\theta_{\mathcal{L},\tilde{\varphi}_\infty}$:

$$\begin{aligned}\theta_{\mathcal{L},\tilde{\varphi}_\infty}(z,g') &= \sum_{\beta} \sum_{\mathbf{x}\in L^m,(\mathbf{x},\mathbf{x})=\beta} \tilde{\varphi}_\infty(z,g',\mathbf{x})) \\ &= \sum_{\beta} \theta_{\mathcal{L},\beta,\tilde{\varphi}_\infty}(z,g')\end{aligned}$$

where β runs over all possible inner product matrix $(\mathbf{x},\mathbf{x})$. We call $\theta_{\mathcal{L},\beta,\tilde{\varphi}_\infty}$ the β-th Fourier coefficient of $\theta_{\mathcal{L},\tilde{\varphi}_\infty}$ as it's a character function under the action of $\Gamma' \cap N'$ for certain Nilpotent subgroup $N' \subset \tilde{G}'$. See chapter 5 for more details.

Now suppose $\mathcal{O} \subset V_\infty^m$ is a closed G_∞ orbit. Then by a theorem of Borel ([B], Theorem 9.11), $\mathcal{O} \cap L^m$ consists of a finite number of Γ-orbits. By Witt's theorem, G_∞ acts transitively on the set

$$\{\mathbf{x} \in V_\infty^m | (\mathbf{x},\mathbf{x}) = \beta\}$$

when β is non-degenerate.

Thus the set

$$\{\mathbf{x} \in \mathcal{L}^m | (\mathbf{x},\mathbf{x}) = \beta\}$$

consists of finitely many Γ-orbits. We choose Γ-orbit representatives $\{\mathbf{x}_1, \ldots, \mathbf{x}_o\}$ and define

$$U_i = \text{span}\mathbf{x}_i, 1 \le i \le o.$$

For each $1 \le i \le o$ choose a base point $z_i \in D(U_i)$. Let $C_{\mathbf{x}_i,z_i}$ be the generalized special cycle. Let

$$\mathbf{z} = \{z_1, z_2, \ldots, z_o\}.$$

Then define

$$C_{\beta,\mathbf{z}} = \sum_{i=1}^{o} C_{\mathbf{x}_i, z_i}.$$

$C_{\beta,\mathbf{z}}$ is a cycle in the Chow group of $\Gamma\backslash D$. By remark 1.1, the homology class $[C_{\beta,\mathbf{z}}]$ is independent of the choice of $\mathbf{z}$, so we simply denote by $[C_\beta]$ its homology class. In this chapter whenever we take the period of a closed differential form η on $C_{\beta,\mathbf{z}}$ (resp. $C_{\mathbf{x},z'}$) we write $\int_{C_\beta} \eta$ (resp. $\int_{C_\mathbf{x}} \eta$).

Let η be any differential form on $\Gamma\backslash D$, define a smooth function $\theta_{\mathcal{L},\tilde{\varphi}_\infty}(\eta)$ on $\tilde{G}'_\infty$ by

$$\theta_{\mathcal{L},\tilde{\varphi}_\infty}(\eta) = \int_{\Gamma\backslash D} \eta \wedge \theta_{\mathcal{L},\tilde{\varphi}_\infty}(g').$$

We call the above map theta lifts defined by $\tilde{\varphi}_\infty$. When η is closed, the above gives a map

$$H^{d-d'}(\Gamma\backslash D, \mathbb{C}) \to \mathcal{A}(\Gamma'\backslash\tilde{G}'_\infty)$$

where $d = dimD$, d' is the degree of φ and $\mathcal{A}(\Gamma'\backslash\tilde{G}'_\infty)$ is the space of automorphic functions on $\Gamma'\backslash\tilde{G}'_\infty$. Also define $a_{\mathcal{L},\beta(\eta),\tilde{\varphi}_\infty}$ to be the β-coefficient of $\theta_{\mathcal{L},\tilde{\varphi}_\infty}(\eta)$:

$$a_{\mathcal{L},\beta(\eta),\tilde{\varphi}_\infty}(\eta) = \int_{\Gamma\backslash D} \eta \wedge \theta_{\mathcal{L},\beta,\tilde{\varphi}_\infty}(z, g') = \int_{\Gamma\backslash D} \eta \wedge \sum_{\mathbf{x}\in L^m, (\mathbf{x},\mathbf{x})=\beta} \tilde{\varphi}_\infty(z, g', \mathbf{x}). \quad (8.2)$$

Then we have the Fourier expansion:

$$\theta_{\mathcal{L},\tilde{\varphi}_\infty}(\eta) = \sum_\beta a_{\mathcal{L},\beta,\tilde{\varphi}_\infty}(\eta).$$

Theorem 8.1. *Assuming that β is non-degenerate (a nonsingular matrix), and in case (1) also assume that $\sigma_1(\beta)$ has signature (r,s). Then*

$$a_{\mathcal{L},\beta(\eta),\tilde{\varphi}_\infty}(\eta) = \kappa(g',\beta)\int_{C_\beta}\eta$$

where κ is an analytic function in g' that depends on β.

Let us also briefly recall Poincaré duality in terms of differential forms. For a closed submanifold C inside an oriented manifold M, we say that τ is a Poincaré dual form of C if it is a closed form such that

$$\int_M \eta\wedge\tau = \int_C \eta$$

for any closed form η. Poincaré dual form is unique up to exact forms.

The above definition for Poincaré dual form can be extended to a closed cycle C (a member of $H_*(M)$) given that C is almost a submanifold. To be more precise, if C has a stratification such that its open stratum is a submanifold of M, then the above definition works.

With the above theory of Poincaré duality in mind, theorem 8.1 then is equivalent to

Theorem 8.2. *Assuming that β is non-degenerate (a nonsingular matrix), and in case (1) also assume that $\sigma_1(\beta)$ has signature (r, s). Then*

$$[\theta_{\mathcal{L},\beta,\tilde{\varphi}_\infty}(z, g')] = \mathrm{PD}([C_\beta])\kappa(g', \beta).$$

where $[\theta]$ is the value of θ in $H^*(\Gamma\backslash D)$ and $\mathrm{PD}([C_\beta]) \in H^*(\Gamma\backslash D)$ is the Poincaré dual of C_β.

When the function $\kappa(g', \beta)$ is nonzero, $\frac{1}{\kappa(g',\beta)}[\theta_{\mathcal{L},\beta,\tilde{\varphi}_\infty}(z, g')]$(see equation (8.1)) is the Poincaré dual of C_β. We will prove this for generic g' in Chapter 10.

Theorem 8.3. *$\kappa(g', \beta)$ is an analytic function on $\tilde{G}'$ that is not identically zero. In particular, $\kappa(g', \beta)$ is nonzero for generic g'. To be more precise, there exists $m' \in M'$ such that for sufficiently large $\lambda \in \mathbb{R}$,*

$$\kappa(\lambda m', \beta) \neq 0.$$

In other words, for a generic g', $\frac{1}{\kappa(g',\beta)}[\theta_{\mathcal{L},\beta,\tilde{\varphi}_\infty}(z, g')]$ is the Poincareé dual of C_β.

These theorems mean that $[\theta_{\mathcal{L},\tilde{\varphi}_\infty}]$ can be seen as a "generating" series of $\mathrm{PD}([C_\beta])$. Of course, as for now we do not have an explanation for all the "Fourier coefficients" as Poincaré duals of cycles. Only for those satisfying the condition of theorem 8.2 do we have such an explanation.

First we need the following "unfolding" lemma:

Lemma 8.1.

$$\int_{\Gamma\backslash D}\eta\wedge\sum_{\mathbf{x}\in\mathcal{L}^m,(\mathbf{x},\mathbf{x})=\beta}\tilde{\varphi}_\infty(z,g',\mathbf{x})=\sum_{j=1}^{o}\int_{\Gamma_{\mathbf{x}_j}\backslash D}\eta\wedge\tilde{\varphi}_\infty(z,g',\mathbf{x}_j)$$

Proof. First we can switch the order of integration and summation on the left hand side of the equation as η is bounded and $\tilde{\varphi}_\infty(z,g',\mathbf{x})$ is uniformly fast decreasing in $\mathbf{x}$:

$$\begin{aligned}\int_{\Gamma\backslash D}\eta\wedge\sum_{\mathbf{x}\in\mathcal{L}^m,(\mathbf{x},\mathbf{x})=\beta}\tilde{\varphi}_\infty(z,g',\mathbf{x}) &= \sum_{\mathbf{x}\in\mathcal{L}^m,(\mathbf{x},\mathbf{x})=\beta}\int_{\Gamma\backslash D}\eta\wedge\tilde{\varphi}_\infty(z,g',\mathbf{x})\\ &=\sum_{j=1}^{o}\sum_{\mathbf{y}\in\Gamma\mathbf{x}_j}\int_{\Gamma\backslash D}\eta\wedge\tilde{\varphi}_\infty(z,g',\mathbf{y})\end{aligned}$$

We have to justify the exchange of summation and integration in the first equality above. Let $B(r)$ be the ball of radius r in V_∞. Since $\{\mathbf{x}\in\mathcal{L}\cap B^m(r)\}$ has finite cardinality, $\{\mathbf{x}\in\mathcal{L}^m,(\mathbf{x},\mathbf{x})\}\cap B^m(r)$ also has finite cardinality. As $||\tilde{\varphi}_\infty||$ is a Schwartz function in V_∞^m, the exchanges of summation and integration follows from dominated convergence theorem.

We do the summation in each Γ-orbit:

$$\sum_{\mathbf{y}\in\Gamma\mathbf{x}}\int_{\Gamma\backslash D}\eta\wedge\tilde{\varphi}_\infty(z,g',\mathbf{y})=\sum_{\gamma'\in\Gamma_{\mathbf{x}}\backslash\Gamma}\int_{\Gamma\backslash D}\eta\wedge\tilde{\varphi}_\infty(z,g',\gamma'^{-1}\mathbf{x})$$

By definition of $\tilde{\varphi}_\infty(z, g', \mathbf{x})$ and the fact that L_g^* is contravariant in g we have

$$\begin{aligned}\tilde{\varphi}_\infty(z, g', \gamma^{-1}\mathbf{x}) =& L^*_{g_z^{-1}}(\varpi(g_z, g')\varphi_\infty(\gamma^{-1}\mathbf{x}))\\ =& L^*_{g_z^{-1}}(\varpi(g')\varphi_\infty(g_z^{-1}\gamma^{-1}\mathbf{x}))\\ =& L^*_\gamma \circ L^*_{(\gamma g)_z^{-1}}(\varpi(g')\varphi_\infty((\gamma g_z)^{-1}\mathbf{x}))\\ =& L^*_\gamma(\tilde{\varphi}_\infty(\gamma z, g', \mathbf{x})).\end{aligned}$$

And

$$\begin{aligned}\sum_{\gamma\in\Gamma_{\mathbf{x}}\backslash\Gamma}\int_{\Gamma\backslash D}\eta\wedge\tilde{\varphi}_\infty(z, g', \gamma^{-1}\mathbf{x}) &= \sum_{\gamma\in\Gamma_{\mathbf{x}}\backslash\Gamma}\int_{\Gamma\backslash D}\eta\wedge L^*_\gamma(\tilde{\varphi}_\infty(\gamma z, g', \mathbf{x}))\\ &= \sum_{\gamma\in\Gamma_{\mathbf{x}}\backslash\Gamma}\int_{\Gamma\backslash D} L^*_\gamma(\eta\wedge\tilde{\varphi}_\infty(\gamma z, g', \mathbf{x}))(\text{as } \eta \text{ is } \Gamma\text{-invariant})\end{aligned}$$

Now if F is a fundamental domain of the Γ action on D, then $\cup_{\gamma\in\Gamma_{\mathbf{x}}\backslash\Gamma}(\gamma\cdot F)$ is a fundamental domain of the $\Gamma_{\mathbf{x}}$ action on D. So we have

$$\begin{aligned}\sum_{\gamma\in\Gamma_{\mathbf{x}}\backslash\Gamma}\int_{\Gamma\backslash D} L^*_\gamma(\eta\wedge\tilde{\varphi}_\infty(\gamma z, g', \mathbf{x})) &= \sum_{\gamma\in\Gamma_{\mathbf{x}}\backslash\Gamma}\int_F L^*_\gamma(\eta\wedge\tilde{\varphi}_\infty(\gamma z, g', \mathbf{x}))\\ &= \sum_{\gamma\in\Gamma_{\mathbf{x}}\backslash\Gamma}\int_{\gamma(F)}\eta\wedge\tilde{\varphi}_\infty(z, g', \mathbf{x})\\ &= \int_{\Gamma_{\mathbf{x}}\backslash D}\eta\wedge\tilde{\varphi}_\infty(z, g', \mathbf{x}).\end{aligned}$$

The conclusion of our lemma follows. □

Define

$$U_{\mathbf{x}} = \mathrm{span}\mathbf{x}.$$

Let $G_{\mathbf{x}}$ be the stabilizer in G_∞ of $\mathbf{x}$ and $D_{U_{\mathbf{x}}}$ be the symmetric space associated to $G_{\mathbf{x}}$ (i.e. the symmetric space associated to $U_{\mathbf{x}}^{\perp}$). Also let $\Gamma_{\mathbf{x}} = \Gamma \cap G_{\mathbf{x}}$. Recall in Chapter 3, we have an embedding $D_{U_{\mathbf{x}}} \to D$ by choosing a point z_0' in the symmetric space associated to $(U_{\mathbf{x}}, (,))$. The image of the embedding $D_{U_{\mathbf{x}},z_0'}$ is totally geodesic. After mod out Γ, we have the cycle $C_{U_{\mathbf{x}},z_0'} = \pi_{z_0'}(\Gamma_{\mathbf{x}} \backslash D_{U_{\mathbf{x}}})$ of $\Gamma \backslash G_\infty$. In this chapter, for simplicity we denote $D_{U_{\mathbf{x}},z_0'}$ by $D_{\mathbf{x},z_0'}$ and $C_{U_{\mathbf{x}},z_0'}$ by $C_{\mathbf{x},z_0'}$.

The critical topological observation is that $\Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z_0'}$ is a totally geodesic sub manifold of the space $E_{\mathbf{x}} = \Gamma_{\mathbf{x}} \backslash D$ and $E_{\mathbf{x}}$ is in a natural way (topologically) a vector bundle over $\Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z_0'}$. There is a fibering $\pi : E_{\mathbf{x}} \to \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z_0'}$ defined in section 3.2. We denote the fiber at a point $z \in \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z_0'}$ by $F_z D_{\mathbf{x},z_0'}$. For later use, we also define $N_z D_{\mathbf{x},z_0'}$ by

$$N_z D_{\mathbf{x},z_0'} = \{\vec{v} \in T_z D | v \perp T_z D_{\mathbf{x},z_0'}\}$$

where $T_z D$ is the tangent space of D at z. $N_z D_{\mathbf{x},z_0'}$ is the tangent space of $F_z D_{\mathbf{x},z_0'}$ at z.

As before suppose $d = dim(D)$,$d' = dim(D_{\mathbf{x},z_0'})$. The following theorem is a special case of theorem 2.1 of [KM4].

Theorem 8.4. *Let Φ be a differential d'-form on $E_{\mathbf{x}}$ satisfying*

1. *Φ is closed.*

2. *$||\Phi|| \leq \exp(-d\rho r)p(r)$ for some polynomial p where r is the geodesic distance to $\Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z_0'}$.*

If η is a closed bounded $(d-d')$-form on $E_{\mathbf{x}}$ we have

$$\int_{E_{\mathbf{x}}} \eta \wedge \Phi = \kappa \int_{\Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z_0'}} \eta$$

where

$$\kappa = \int_{F_z D_{\mathbf{x},z_0'}} \Phi.$$

for any $z \in \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z_0'}$.

Remark 8.1. *When $\kappa = 1$, Φ is a Thom form of the fiber product $E_{\mathbf{x}} \to \Gamma_{\mathbf{x}} \backslash D_{\mathbf{x},z_0'}$.*

Proof of Theorem 8.1 assuming rapid decreasing of $\tilde{\varphi}_\infty$ on the fiber $F_z D_{\mathbf{x},z_0'}$: Assume that $\tilde{\varphi}_\infty$ satisfies the condition of Theorem 8.4.

By the unfolding lemma 8.1, we know that

$$a_\beta(\eta) = \sum_{j=1}^{o} \int_{\Gamma_{\mathbf{x}_j} \backslash D} \eta \wedge \tilde{\varphi}_\infty(z, g', \mathbf{X_j}).$$

It suffices to show that

$$\int_{\Gamma_{\mathbf{x}} \backslash D} \eta \wedge \tilde{\varphi}_\infty(z, g', \mathbf{x}) = \kappa(g', \beta) \int_{C_{\mathbf{x}}} \eta$$

for each $\mathbf{x} = \mathbf{x}_j$.

We apply theorem 8.4 to $\Phi = \tilde{\varphi}_\infty(z, g', \mathbf{x})$. We already know that $\tilde{\varphi}_\infty$ is closed. We will prove in Theorem 9.3 that it is fast-decaying. So it satisfies both conditions of the above theorem. Define

$$\kappa(g', \mathbf{x}, z_0') = \int_{F_{z_0} D_{\mathbf{x}, z_0'}} \tilde{\varphi}_\infty(z, g', \mathbf{x})$$

for any $z_0 \in C_{\mathbf{x}, z_0'}$. By theorem 8.4, we know that

$$\int_{\Gamma_\mathbf{x} \backslash D} \eta \wedge \tilde{\varphi}_\infty(z, g', \mathbf{x}) = \kappa(g', \mathbf{x}, z_0') \int_{\Gamma_\mathbf{x} \backslash D_{\mathbf{x}, z_0'}} \eta.$$

When the map $\pi_{z_0'} : \Gamma_\mathbf{x} \backslash D_{U_\mathbf{x}} \to \Gamma \backslash D$ is an embedding, in other words, we can identify $\Gamma_\mathbf{x} \backslash D_{U_\mathbf{x}}$ with $C_{\mathbf{x}, z_0'}$. One can immediately conclude that

$$\int_{\Gamma_\mathbf{x} \backslash D} \eta \wedge \tilde{\varphi}_\infty(z, g', \mathbf{x}) = \kappa(g', \mathbf{x}, z_0') \int_{C_{\mathbf{x}, z_0'}} \eta. \tag{8.3}$$

In general one use lemma 3.2 to conclude that the map

$$\pi_{\sigma_2} : \Gamma_\mathbf{x} \backslash D_{\mathbf{x}, z_0'} \to C_{\mathbf{x}, z_0'}$$

is generically injective, so

$$\int_{C_{\mathbf{x}, z_0'}} \eta = \int_{\Gamma_\mathbf{x} \backslash D_{\mathbf{x}, z_0'}} \eta,$$

and equation (8.3) holds again. This finishes the proof of theorem 8.1 except that we have to show $\kappa(g', \mathbf{x}, z_0')$ only depends on g' and β.

The fiber $F_{z_0} D_{\mathbf{x}, z_0'}$ does not depend on the group $\Gamma_\mathbf{x}$, it only depends on $D_{\mathbf{x}, z_0'}$ ($D \to D_{\mathbf{x}, z_0'}$ and $\Gamma_\mathbf{x} \backslash D \to C_{\mathbf{x}, z_0'}$ have the same fiber). We want to show that

Lemma 8.2. *$\kappa(g', \mathbf{x}, z_0')$ is independent of the choice of z_0'. Moreover it only depends on $\beta = (\mathbf{x}, \mathbf{x})$ when β is non-degenerate.*

Let η be a Γ-invariant form on D such that $\int_{C_{\mathbf{x},z_0'}} \eta \neq 0$. We have seen in the proof of lemma 8.1 that

$$\int_{\Gamma \backslash D} \eta \wedge \sum_{g \in \Gamma} \tilde{\varphi}_\infty(z, g', g\mathbf{x}) = \int_{\Gamma_{\mathbf{x}} \backslash D} \eta \wedge \tilde{\varphi}_\infty(z, g', \mathbf{x}).$$

Theorem 8.4 tells us that

$$\int_{\Gamma_{\mathbf{x}} \backslash D} \eta \wedge \tilde{\varphi}_\infty(z, g', \mathbf{x}) = \kappa(g', \mathbf{x}, z_0') \int_{C_{\mathbf{x},z_0'}} \eta.$$

For different choices of z_0', $C_{\mathbf{x},z_0'}$ are homologous. Thus $\int_{C_{\mathbf{x},z_0'}} \eta$ is independent of z_0'. By the nondegeneracy of the pairing between $H^*(M)$ and $H_*(M)$, we can choose η such that

$$\int_{C_{\mathbf{x},z_0'}} \eta \neq 0.$$

So $\int_{\Gamma \backslash D} \eta \wedge \sum_{g \in \Gamma} \tilde{\varphi}_\infty(z, g', g\mathbf{x})$ is independent of z_0'. This shows that $\kappa(g', \mathbf{x}, z_0')$ is independent of the choice of λ, from now on we simiply denote it by $\kappa(g', \mathbf{x})$.

Let $\tilde{\mathbf{x}} \in V_\infty^m$ and $\tilde{\lambda}$ be a point in the symmetric space associated to $(U_{\tilde{\mathbf{x}}}, (,))$, $z_0 \in D_{\tilde{\mathbf{X}},\tilde{\lambda}}$. Suppose there is a $g \in G_\infty$ such that $g\mathbf{x} = \tilde{\mathbf{x}}$ then as in the proof of 8.1, we have

$$\begin{aligned}\int_{F_{z_0} D_{\tilde{\mathbf{x}},\tilde{\lambda}}} \tilde{\varphi}_\infty(x, g', \tilde{\mathbf{x}}) &= \int_{F_{z_0} D_{g\mathbf{x},\tilde{\lambda}}} \tilde{\varphi}_\infty(z, g', g\mathbf{x}) \\ &= \int_{F_{z_0} D_{g\mathbf{x},\tilde{\lambda}}} L^*_{g^{-1}}(\tilde{\varphi}_\infty(g^{-1}z, g', \mathbf{x})) \\ &= \int_{L_{g^{-1}}(F_{z_0} D_{g\mathbf{x},\tilde{\lambda}})} \tilde{\varphi}_\infty(z, g', \mathbf{x}) \\ &= \int_{F_{g^{-1}z_0} D_{\mathbf{x},g^{-1}\tilde{\lambda}}} \tilde{\varphi}_\infty(z, g', \mathbf{x})\end{aligned}$$

This proves that $\kappa(g', \mathbf{x})$ only depends on the G_∞-orbit of $\mathbf{x}$. In particular, when β is non-degenerate and has signature (r, s) in the unitary case, we can define

$$\kappa(g', \beta) = \kappa(g', \mathbf{x})$$

for any $\mathbf{x}$ such that $(\mathbf{x}, \mathbf{x}) = \beta$.

Chapter 9: Rapid decrease of $\tilde{\varphi}$ on the Riemannian normal fiber

In this chapter, we prove that $\tilde{\varphi}_\infty$ defined in Chapter 8 satisfies the condition of Theorem 8.4.

Throughout the chapter we assume V is a complex vector space with a non-degenerate skew Hermitian form $(,)$ of signature (p, q). We fix an orthnormal basis $\{v_1, \dots, v_{p+q}\}$ as before. For the latter two groups we assume $p = q = n$ in addition.

For $G = \mathrm{U}(p, q)$, $n = p + q$, $m = r + s$, $\mathbf{x} = (\vec{x}_1, \dots, \vec{x}_{r+s})$ and

$$U := \mathrm{span}\{\mathbf{x}\}.$$

Let $z_0 \in D$ such that (U, z) is a compatible pair (Definition 3.1). Notice that U together with z_0 gives rise to a sub-symmetric space $D_{U,z_0'}$ where $z_0' = z_0 \cap U$.

Recall from Section 4.2 that V_0 is a $2n$ dimensional real vector space with a skew symmetric form $(,)_1$ and $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$. $\mathrm{Sp}(2n, \mathbb{R}) = G(V_0, (,)_1)$ is a subgroup of $\mathrm{U}(n, n)$ and its symmetric space D_1 embedds D, the symmetric of $\mathrm{U}(n, n)$. In this case let $m = r$ and assume $\mathbf{x}_0 = (\vec{x}_1, \dots, \vec{x}_r, \vec{y}_1, \dots, \vec{y}_r) \in V_0^{2r}$. Denote by U_0 and U

$$U_0 = \mathrm{span}_{\mathbb{R}}\{\vec{x}_1, \dots, \vec{x}_r, \vec{y}_1, \dots, \vec{y}_r\} \subset V_0, \quad U = U_0 \otimes_{\mathbb{R}} \mathbb{C} \subset V$$

respectively. Let $z_0 \in D_1$ such that (U, z_0) is a compatible pair. Notice that U_0 together with z_0 gives rise to a sub-symmetric space $D_{U_0,z_0'} \in D_1$ where $z_0' = z_0 \cap U$. Recall that we have the isomorphisms $V_0^{2r} \cong V_0 \otimes_{\mathbb{R}} E \cong V \otimes_{\mathbb{C}} E \cong V^r$ (as in Subsection 6.4). Under this isomorphism $\mathbf{x}_0$ is mapped to $\mathbf{x}$ in V^r, more explicitly

$$\mathbf{x} = \frac{1}{\sqrt{2}}(\vec{x}_1 - i\vec{y}_1, \ldots, \vec{x}_r - i\vec{y}_r) \in V^m.$$

Recall from Section 4.3 that $\mathrm{O}^*(2n, \mathbb{R})$ is a subgroup of $\mathrm{U}(n, n)$ and its symmetric space D_2 embedds D, the symmetric of $\mathrm{U}(n, n)$. In this case let $m = r$ and assume $\mathbf{x} = (\vec{x}_1, \ldots, \vec{x}_r) \in V^r$ and

$$U = \mathrm{span}_{\mathbb{H}}\{\mathbf{x}\}.$$

We can also view $\mathbf{x}$ as in both $V_{\mathbb{H}}^r$ and V^r. Let $z_0 \in D_2$ such that (U, z_0) is a compatible pair. U together with z_0 gives rise to a sub-symmetric space $D_{U,z_0'} \in D_2$ where $z_0' = z_0 \cap U$.

In all three cases, a point $z_0 \in D$ (in the latter two cases view z_0 as in D instead of D_1 or D_2) gives rise to a Cartan involution $\theta_z : V \to V$. Define a positive definite Hermitian form $(,)_{z_0}$ on V by

$$(x, y)_{z_0} = i(x, \theta_{z_0} y).$$

We call this form the **majorant** of $(,)$ with respect to z_0. We denote $(v, v)_{z_0}$ as $\|v\|_{z_0}^2$.

In all the above cases, for any $z_0 \in D$, define $M_{z_0} : D \times V^m \to \mathbb{R}$ to be the function

$$M_{z_0}(z, \mathbf{x}) = \sum_{\ell=1}^{m} ||g_z^{-1}\vec{x}_\ell||_{z_0}^2, \tag{9.1}$$

where $g_z \in G$ is any element such that $g_z z_0 = z$. Since the isotropic group of z_0 is K and for any $k \in K$, $\theta_{z_0}k = k\theta_{z_0}$, we know that

$$\begin{aligned}(kx, ky)_{z_0} =& i(kx, \theta_{z_0}ky)\\ =& i(kx, k\theta_{z_0}y)\\ =& i(x, \theta_{z_0}y) = (x, y)_{z_0}.\end{aligned}$$

Hence the function $M_{z_0}(z, \mathbf{x})$ is well-defined.

Lemma 9.1.

$$M_{hz_0}(hz, h\mathbf{x}) = M_{z_0}(z, \mathbf{x}).$$

Proof. Choose a g such that $gz_0 = z$, then $hgh^{-1}hz_0 = hz$. Hence

$$\begin{aligned}M_{hz_0}(hz, h\mathbf{x}) =& \sum_{\ell=1}^{m} ||hg^{-1}h^{-1}h\vec{x}_\ell||_{hz_0}^2\\ =& \sum_{\ell=1}^{m}(hg^{-1}h^{-1} \cdot h\vec{x}_\ell, h\theta_{z_0}h^{-1} \cdot (hg^{-1}h^{-1} \cdot h\vec{x}_\ell))\\ =& \sum_{\ell=1}^{m}(hg^{-1}\vec{x}_\ell, h\theta_{z_0}g^{-1}\vec{x}_\ell)\\ =& \sum_{\ell=1}^{m}(g^{-1}\vec{x}_\ell, \theta_{z_0}g^{-1}\vec{x}_\ell)\\ =& M_{z_0}(z, \mathbf{x}).\end{aligned}$$

□

Recall that the Riemannian distance $d(D', z)$ between the totally geodesic submanifold D' and $z \in D$ is the length of the shortest geodesic joining z to a point of D'. This geodesic is necessarily normal to D'. Choose a base point $z_0 \in D'$. If $z = \exp_{z_0}(tu)$ for $u \in N_{z_0}D'$ where $\exp_{z_0}$ denotes the exponential map at the base point z_0, then

$$d(D', z) = t.$$

Our first goal of this chapter is to prove the following estimate on M for all three cases.

Theorem 9.1. *Let* $(\mathbf{x}, z_0)$ *be a compatible pair (Definition 3.1) and* $z_0' = z_0 \cap U$*. There is a positive constant* c *depending on* $\mathbf{x}$ *and* z_0' *but not on* z_0 *such that*

$$M_{z_0}(z, \mathbf{x}) \geq c \exp(2c \cdot d(D_{U,z_0'}, z)). \tag{9.2}$$

It is easy to see that

$$d(D_{U,z_0'}, z) = d(D_{hU,hz_0'}, hz). \tag{9.3}$$

Equation (9.3) together with Lemma 9.1 implies that in order to prove Theorem 9.1 it suffices to assume (by replacing $(\mathbf{x}, z_0)$ by $(h\mathbf{x}, hz_0)$ for some $h \in G$)

1. For $G = \mathrm{U}(p, q)$, $\mathrm{span}\{\mathbf{x}\} = \mathrm{span}_{\mathbb{C}}\{v_1, \dots, v_r, v_{p+1}, \dots, v_{p+s}\}$ and $z_0 = \mathrm{span}_{\mathbb{C}}\{v_{p+1}, \dots, v_{p+q}\}$.

2. For $G = \mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{span}_{\mathbb{R}}\{\mathbf{x}_0\} = \mathrm{span}_{\mathbb{R}}\{E_1, \dots, E_r, F_1, \dots, F_r\}$ and $z_0 =$

$\text{span}_{\mathbb{C}}\{E_1 + iF_1, \ldots, E_n + iF_n\}$.

3. For $G = \mathrm{O}^*(2n, \mathbb{R})$, $\text{span}\{\mathbf{x}\}_{\mathbb{H}} = \text{span}_{\mathbb{H}}\{v_1, \ldots, v_r\}$ and $z_0 = \text{span}_{\mathbb{C}}\{v_1 j, \ldots, v_n j\}$.

Recall that we have a $G_{\mathbf{x}}$-equivariant fibration $\pi : D \to D_{U,z_0'}$ with the fiber $F_{z_0} D_{U,z_0'}$ over z_0. By the definition of $M_{z_0}(z, \mathbf{x})$ (Equation (9.1)), we see that

$$M_{z_0}(gz, \mathbf{x}) = M_{z_0}(z, \mathbf{x}), \forall g \in G_{\mathbf{x}}.$$

Hence we can translate z by elements in $G_{\mathbf{x}}$ and assume that $z \in F_{z_0} D_{U,z_0'}$. Hence we can assume that

$$z = \exp_{z_0}(X),$$

for some $X \in N_{z_0} D_{U,z_0'}$ where $\exp_{z_0}$ is the exponential map of the symmetric space starting from z_0. It is well-known that (see for example Section 3 of Chapter IV of [He])

$$\exp_{z_0}(X) = \exp(X) z_0,$$

where we identify $N_{z_0} D_{U,z_0'}$ as a subspace of $\mathfrak{p}$ and exp is the exponential map of the group G. From now on we assume that $z = \exp(X) z_0$ with $X \in N_{z_0} D_{U,z_0'} \subset \mathfrak{p}$. We also denote for simplicity $(,)_0 = (,)_{z_0}$.

9.1 Theorem 9.1 for the unitary case

The theorem will be a consequence of Lemmas 9.2 through 9.6.

Recall that we have the Cartan Decomposition

$$\mathfrak{u}(p,q) = \mathfrak{k}_0 + \mathfrak{p}_0$$

where $\mathfrak{p}_0$ are Hermitian matrices of the form

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} 0 & A \\ {}^*A & 0 \end{pmatrix} | A \in M_{p\times q}(\mathbb{C}) \right\}.$$

Lemma 9.2. *Let $A \in \mathrm{Herm}_n$ and $\epsilon > 0$ be given. Then there exists δ depending on ϵ and A such that for any $B \in \mathrm{Herm}_n$ with $\|A - B\| < \delta$ there exist $R, S \in \mathrm{U}(n)$ such that RAR^{-1} and SBS^{-1} are diagonal with $\|RAR^{-1} - SBS^{-1}\| < \epsilon$ and $\|R - S\| < \epsilon$.*

Proof. For $U \in \mathrm{U}(n)$, the statement in the lemma is true for A if and only it is true for UAU^{-1} (with the same ϵ and δ). Hence without lost of generality we can assume we can assume A is diagonal of the form

$$A = \begin{pmatrix} \lambda_1 I_{r_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 I_{r_2} & 0 & \cdots & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \lambda_k I_{r_k} \end{pmatrix} \tag{9.4}$$

where $\lambda_1, \lambda_2, \ldots \lambda_r$ are distinct eigenvalues of A and $r_1 + r_2 + \cdots + r_k = n$.

First we assume that $\lambda_1, \lambda_2, \ldots \lambda_r$ are all nonzero.

Define a Lie subalgebra $\mathfrak{u}_A \subseteq \mathfrak{u}(n)$ to be the set of matrix of the form

$$X = \begin{pmatrix} 0_{r_1} & X_{12} & \cdots & X_{1k} \\ -X_{12}^* & 0_{r_2} & \cdots & X_{2k} \\ & & & \\ -X_{1k}^* & -X_{2k}^* & \cdots & 0_{r_k} \end{pmatrix}. \tag{9.5}$$

We define a map

$$\phi : \mathfrak{u}_A \times \mathrm{Herm}_{r_1} \times \cdots \times \mathrm{Herm}_{r_k} \to \mathrm{Herm}_n,$$

given by

$$\phi(X, m_1, \cdots, m_k) = \exp(X) A \begin{pmatrix} \exp(m_1) & 0 & \cdots & 0 \\ 0 & \exp(m_2) & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & \exp(m_k) \end{pmatrix} \exp(-X) \tag{9.6}$$

Then the differential of ϕ at $(0, 0, \ldots, 0)$ is given by

$$d\phi_0(X, m_1, \cdots, m_k) = A \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & m_k \end{pmatrix} + [X, A],$$

which in turn is equal to

$$\begin{pmatrix} \lambda_1 m_1 & (\lambda_1-\lambda_2)X_{12} & \cdots & (\lambda_1-\lambda_k)X_{1k} \\ (\lambda_1-\lambda_2)X_{12}^* & \lambda_2 m_2 & \cdots & (\lambda_2-\lambda_k)X_{2k} \\ & & & \\ (\lambda_1-\lambda_k)X_{1k}^* & (\lambda_2-\lambda_k)X_{2k}^* & \cdots & \lambda_k m_k \end{pmatrix}$$

Because we assume that $\lambda_1, \lambda_2, \ldots \lambda_r$ are distinct and nonzero, $d\phi_0$ is an isomorphism from $\mathfrak{u}_A \times \mathrm{Herm}_{r_1} \times \cdots \times \mathrm{Herm}_{r_k}$ to Herm_n. Hence by inverse function theorem, ϕ is a diffeomorphism from the product of a ball $B(0,\eta)$ of radius η around the origin in $\mathfrak{u}_A$ with a ball $B(0,\eta')$ of radius η' around the origin of $\mathrm{Herm}_{r_1} \times \cdots \times \mathrm{Herm}_{r_k}$ to a neighborhood $U(\eta,\eta')$ of A in Herm_n.

For a given ϵ, shrink the size of η and η' if necessary such that

$$X \in \mathfrak{u}_A \text{ and } ||X|| < \eta \Rightarrow ||I - \exp(X)|| < \epsilon \tag{9.7}$$

and

$$Y \in \mathrm{Herm}_{r_1} \times \cdots \times \mathrm{Herm}_{r_k} \text{ and } \|Y\| \leq \eta' \Rightarrow \|A - A\exp(Y)\| < \epsilon. \tag{9.8}$$

Choose δ such that $B(A,\delta) \subset U(\eta,\eta')$. Suppose $B \in B(A,\delta)$. Since $B \in U(\eta,\eta')$ we have a unique expression

$$B = \exp(X)A\exp(Y)\exp(-X) \text{ with } X \in B(0,\eta) \text{ and } Y \in B(0,\eta').$$

Put $R_1 = I$ and $S_1 = \exp(-X)$ so $R_1AR_1^{-1} = A$ is diagonal and $S_1BS_1^{-1} = A\exp(Y)$ is block diagonal of the form

$$S_1BS_1^{-1} = \begin{pmatrix} B_{11} & 0 & \cdots & 0 \\ 0 & B_{22} & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & B_{kk} \end{pmatrix}, \tag{9.9}$$

where B_{ii} is of size $r_i \times r_i$.

By the above choices of η (equation (9.7)), η' (equation (9.8)) and δ, it is clear that we have

$$\|A - B\| < \delta \Rightarrow \|R_1 - S_1\| < \epsilon \text{ and } \|R_1AR_1^{-1} - S_1BS_1^{-1}\| < \epsilon. \tag{9.10}$$

Now there is a block diagonal unitary matrix

$$R = \begin{pmatrix} R_{11} & 0 & \cdots & 0 \\ 0 & R_{22} & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & R_{kk} \end{pmatrix},$$

where $R_{ii} \in \mathrm{U}(r_i)$ such that $RS_1BS_1^{-1}R^{-1}$ is diagonal. Notice that $RA = AR$, hence $RAR^{-1} = A$ is also diagonal.

Let $S = RS_1$. Since R is unitary, by equation 9.10 we have

$$||R - S|| = ||R(R_1 - S_1)|| < \epsilon,$$

and

$$||RAR^{-1} - SBS^{-1}|| = ||R(A - S_1 B S_1^{-1})R^{-1}|| < \epsilon.$$

The lemma is now proved for A a block diagonal matrix of the form (9.4) and $\lambda_1, \ldots, \lambda_k$ are nonzero.

In general we can choose a λ such that $A' = A + \lambda I_n$ does not have zero eigenvalue. By the previous argument, there are $B' \in \mathrm{Herm}_n$ and $R, S \in \mathrm{U}(n)$ such that $||R - S|| < \epsilon$ and $||RA'R^{-1} - SB'S^{-1}|| < \epsilon$. Now let $B = B' - \lambda I_n$. The lemma is now proved. □

Recall that the tangent space $N_{z_0} D_{U,z_0'}$ of the fiber $F_{z_0} D_{U,z_0'}$ at z_0 can be identified with a subspace of $\mathfrak{p}_0$ given by equation (7.3). Our goal is to prove

The Proposition will be a consequence of the following discussion and lemma. Let $X \in N_{z_0} D_{U,z_0'}$ so in particular $X \in \mathfrak{p}$ and is Hermitian with respect to $(,)_0$. Let $\tilde{v}_1, \cdots \tilde{v}_n$ be an orthonormal basis for $(\ ,\ )_0$ of V consisting of eigenvectors of X. Then

$$X(\tilde{v}_k) = \lambda_k \tilde{v}_k, 1 \leq k \leq n.$$

Suppose

$$\vec{x}_j = \sum_{k=1}^{n} a_j^k \tilde{v}_k.$$

for $1 \leq k \leq n$. Then

$$\|\vec{x}_j\|_0^2 = \sum_{\lambda_k} \|p_{\lambda_k}(\vec{x}_j)\|^2 = \sum_{k=1}^{n} |a_j^k|^2$$

where p_{λ_k} is the orthogonal projection using the metric $(\ , \ )_0$ onto the eigenspace corresponding to λ_k.

Remark 9.1. *When it is necessary to distinguish to which $X \in N_{z_0}D_{U,z_0'}$ the numbers a_j^k and λ_k belong we will write $a_j^k(X)$ and $\lambda_k(X)$.*

Lemma 9.3. *We have*

$$(\exp(-tX)\vec{x}_j, \exp(-tX)\vec{x}_j)_0 = \sum_{k=1}^{n} \|p_{\lambda_k}(\vec{x}_j)\|^2 \exp(-2\lambda_k t)$$

where the sum is over all eigenvalues of X.

Proof. Since $\{\tilde{v}_k : 1 \leq k \leq n\}$ is an orthonormal basis for V we have

$$\begin{aligned}(\exp(-tX)\vec{x}_j, \exp(-tX)\vec{x}_j)_0 &= \sum_{i=1}^{n} |(\tilde{v}_i, (\exp -(tX)\vec{x}_j)_0|^2 = \sum_{i=1}^{n} |\big(\tilde{v}_i, \exp(-tX)(\sum_{k=1}^{n} a_j^k \tilde{v}_k)\big)|^2 \\ &= \sum_{i=1}^{n} |(\tilde{v}_i, \sum_{k=1}^{n} a_j^k \exp(-\lambda_k t)\tilde{v}_k)_0|^2 = \sum_{i=1}^{n} |a_j^i|^2 \exp(-2\lambda_i t)\end{aligned}$$

□

We now define

$$f(X) = -\sum_{j=1}^{r+s} \sum_{\lambda_i(X)<0} |a_j^i|^2(X)\lambda_i(X) = -\sum_{j=1}^{r+s} \sum_{\lambda_i(X)<0} \|p_{\lambda_i(X)}(\vec{x}_j)\|_0^2 \lambda_i(X). \quad (9.11)$$

Since all the terms in the sum defining $f(X)$ are nonnegative it follows that $f(X) = 0$ if and only all the term in the sum are zero. By Lemma 9.2, we can prove the following.

Lemma 9.4. *$f(X)$ is continuous on $\mathfrak{p}_0 \cong T_{z_0}D$.*

Proof. Let $X_1 \in \mathfrak{p}_0$. Then for any $X \in \mathfrak{p}_0$ we have

$$|f(X) - f(X_1)| = |\sum_{j=1}^{r+s} \sum_{\lambda_i(X)<0} |a_j^i|^2(X)\lambda_i(X) - \sum_{j=1}^{r+s} \sum_{\lambda_i(X_1)<0} |a_j^i|^2(X_1)\lambda_i(X_1)| \quad (9.12)$$

Let $\epsilon > 0$ be given. Apply Lemma 9.2 with $A = X_1$ to find δ such that whenever $\|X - X_1\| < \delta$, there exist unitary matrices R, S such that RX_1R^{-1} and SXS^{-1} are diagonal and

$$\|R - S\| < \epsilon \text{ and } \|RAR^{-1} - SBS^{-1}\| < \epsilon.$$

But $\|R - S\| < \epsilon$ implies that suitably chosen eigenvectors of A and B are close. More precisely, if $\tilde{v}_i(X)$, resp. $\tilde{v}_i(X_1), 1 \leq i \leq n$ is the eigenvector of X (resp. X_1), corresponding to the eigenvalue $\lambda_i(X), resp.\lambda_i(X_1)$ which is the i-th row of R, resp. i-th row of S we have

$$\epsilon^2 > \|R - S\|^2 = \sum_{i=1}^{n} \|\tilde{v}_i(X) - \tilde{v}_i(X_1)^2\|_0^2 \Rightarrow \|\tilde{v}_i(X) - \tilde{v}_i(X_1)\|_0^2 < \epsilon^2, 1 \leq i \leq m.$$

Hence, for all $i, j, 1 \leq i \leq n$ and $1 \leq j \leq n$ we have

$$|a_j^i(X) - a_j^i(X_1)|^2 = |(\vec{x}_j, \tilde{v}_i(X) - v_i(X_1))_0|^2 \leq \|\vec{x}_j\|_0^2 \ \|\tilde{v}_i(X) - \tilde{v}_i(X_1)\|_0^2 < \|\vec{x}_j\|_0^2 \ \epsilon^2.$$

Also

$$\|RAR^{-1} - SBS^{-1}\|^2 = \sum_{i=1}^{m} (\lambda_i(X) - \lambda_i(X_1))^2$$

and consequently

$$\|RAR^{-1} - SBS^{-1}\|^2 < \epsilon^2 \Rightarrow \sum_{i=1}^{n} (\lambda_i(X) - \lambda_i(X_1))^2 < \epsilon \Rightarrow (\lambda_i(X) - \lambda_i(X_1))^2 < \epsilon^2, 1 \leq i \leq n.$$

Hence, for all $i, 1 \leq i \leq n$ we have

$$|\lambda_i(X) - \lambda_i(X_1)| < \epsilon.$$

Since X is fixed, we assume that

$$\lambda_i(X) \leq M, \lambda_i(X_1) \leq M$$

for $1 \leq i \leq n$.

Now using the identity $|ab - a'b'| \leq |b||a - a'| + |a'||b - b'|$ we obtain

$$||a_j^i|^2(X)\lambda_i(X) - |a_j^i|^2(X_1)\lambda_i(X_1)| \leq |\lambda_i(X)||a_j^i|^2(X) - |a_j^i|^2(X_1)| + |a_j^i|^2(X_1)||\lambda_i(X) - \lambda_i(X_1)|.$$

Since $|\lambda_i(X)| \leq M$ and $|a_j^i|^2(X_1) \leq \|\vec{x}_j\|_0^2 \epsilon^2$ we have

$$
\begin{aligned}
||a_j^i|^2(X)\lambda_i(X) - |a_j^i|^2(X_1)\lambda_i(X_1)| \leq & M||a_j^i|^2(X) - |a_j^i|^2(X_1)| + \|\vec{x}_j\|_0^2|\lambda_i(X) - \lambda_i(X_1)| \\
\leq & M||\vec{x}_j||_0^2\ \epsilon^2 + ||\vec{x}_j||_0^2\epsilon = ||\vec{x}_j||_0^2(M\epsilon^2 + \epsilon).
\end{aligned}
$$

Suppose that the strictly negative eigenvalues of X are $\lambda_1(X), \cdots, \lambda_k(X)$ and the strictly negative eigenvalues of X_1 are $\lambda_1(X_1), \cdots, \lambda_\ell(X_1)$. We assume $k > \ell$. The case $k = \ell$ is easier (in this case, we have only the first sum in Eqation (9.13) below) and the case $k < \ell$ can be treated in a manner symmetrical to that of the case $k > \ell$.

We have

$$
|f(X)-f(X_1)| = |\sum_{j=1}^{r+s}\sum_{i=1}^{\ell}\Big(|a_j^i|^2(X)\lambda_i(X)-|a_j^i|^2(X_1)\lambda_i(X_1)\Big)-\sum_{j=1}^{r+s}\sum_{i=\ell+1}^{k}|a_j^i|^2(X)\lambda_i(X)| \tag{9.13}
$$

The first sum is clearly majorized by $\ell \sum_{j=1}^{r+s} \|\vec{x}_j\|_0^2(2M\epsilon^2 + \epsilon)$ using the inequality immediately above. To majorize the second sum we note that

$$
\ell < i \leq k \Rightarrow \lambda_i(X) < 0 \text{ and } \lambda_i(X_1) \geq 0.
$$

Hence $|\lambda_i(X) - \lambda_i(X_1)| = -\lambda_i(X) + \lambda_i(X_1)$. Note that each of the two terms is positive. But

$$
|\lambda_i(X) - \lambda_i(X_1)| < \epsilon \Rightarrow -\lambda_i(X) + \lambda_1(X_1) < \epsilon \Rightarrow -\lambda_i(X) < \epsilon.
$$

Hence the second summand is majorized by $(k-\ell)\sum_{j=1}^{r+s}\|\vec{x}_j\|_0^2\epsilon$.

Lemma 9.4 follows.

$\square$

Let $S(N_{z_0}D_{U,z_0'})$ be the unit sphere of $N_{z_0}D_{U,z_0'}$, then we have

Lemma 9.5. *$f(Y)$ does not take the value zero on $S(N_{z_0}D_{U,z_0'})$. As $S(N_{z_0}D_{U,z_0'})$ is compact and $f(Y)\geq 0$, there exists $C>0$ so that*

$$f(Y)\geq C,\ Y\in S(N_{z_0}D_{U,z_0'}).$$

Proof. Assume $f(Y)=0$. Suppose v is an eigenvector of Y corresponding to a strictly negative eigenvalue so

$$Y(v)=\lambda v,\lambda<0.$$

Then

$$f(Y)=0\Rightarrow\|\mathfrak{p}_\lambda(\vec{x}_j)\|^2=0\Rightarrow(\vec{x}_j,v)_0=0.$$

for $1\leq j\leq r$ or $p+1\leq j\leq p+s$.

Let $U_+=\text{span}\{v_1,\ldots,v_r\}$ and $U_-=\text{span}\{v_{p+1},\ldots,v_{p+s}\}$. Let $U_+^\perp=\text{span}\{v_{r+1},\ldots,v_p\}$ and $U_-^\perp=\text{span}\{v_{p+s+1},\ldots,v_{p+q}\}$. Then $v\perp U_+\oplus U_-$ as $\text{span}\{\vec{x}_1,\ldots,\vec{x}_{r+s}\}=U_+\oplus U_-$. Let $u\in V$. Using the orthogonal decomposition $V=U_++U_+^\perp+U_-+U_-^\perp$ we may write $u=(v_1,w_1,v_2,w_2)$ with $v_1\in U_+,w_1\in U_+^\perp,v_2\in U_-,w_2\in U_-^\perp$. Then in this representation we have

$$v = (0, w_1, 0, w_2) \text{ with } w_1 \neq 0 \text{ or } w_2 \neq 0$$

and

$$Y(v) = \lambda v = (0, \lambda w_1, 0, \lambda w_2).$$

But since $Y \in N_{z_0} D_{\mathbf{x}, z_0'}$ we have (see equation (7.3))

$$Y = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & 0 \\ {}^*a & {}^*c & 0 & 0 \\ {}^*b & 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$Y(v) = (bw_2, 0, {}^*cw_1, 0) = \lambda v = (0, \lambda w_1, 0, \lambda w_2).$$

Since $\lambda < 0$ the equation immediately above implies w_1 and $w_2 = 0$, a contradiction.

□

Assume now we have ordered the eigenvalues of X so that the first n eigenvalues are negative and the rest nonnegative.

Lemma 9.6. *There exists strictly positive numbers b, c such that for some i, j, with $1 \leq i \leq n$, we have*

$$|\lambda_i| = -\lambda_i \geq b \text{ and } |a^i_j|^2 \geq c$$

Proof. Since

$$f(X) = -\sum_{j=1}^{r+s} \sum_{\substack{i=1 \\ \lambda_i(X)<0}}^{n} |a_j^i|^2(X)\lambda_i(X)$$

is bounded below by C, at least one of terms in the sum is bounded below by $c = \frac{C}{N}$ where N is the number of terms in the sum. Suppose this term is $-|a_j^i|^2\lambda_i$. Hence

$$|a_j^i|^2|\lambda_i| = -|a_j^i|^2\lambda_i \geq c \text{ for some } i, j.$$

But since $\sum_{i=1}^n |a_j^i|^2 = \|\vec{x}_j\|_0^2$ it follows that

$$|a_j^i|^2 \leq \|\vec{x}_j\|_0^2, 1 \leq i \leq n, 1 \leq j \leq r+s.$$

Hence

$$|\lambda_i| \geq \frac{c}{\|\vec{x}_j\|_0^2}.$$

We put

$$b = \frac{c}{\|\vec{x}_j\|_0^2}.$$

But since $\|X\| = \sum_{i=1}^n \lambda_i^2 = 1$, it follows that

$$|\lambda_i| \leq 1$$

and hence

$$|a_j^i|^2 \geq c.$$

$\square$

Proof of Theorem 9.1 for the unitary case: We assume $X \in S(N_{z_0}D_{\mathbf{x},z_0'})$, $z = exp(Xt)z_0$ and $g = exp(Xt)$. Then

$$M_{z_0}(z,\mathbf{x}) = \sum_{j=1}^{r+s}(\exp(-tX)\vec{x}_j, \exp(-tX)\vec{x}_j)_0 = \sum_{j=1}^{r+s}\sum_{i=1}^{n}\|p_{\lambda_i}(\vec{x}_j)\|_0^2\exp(-2\lambda_i t).$$

But all the terms in the sum immediately above are nonnegative and we have proved in Lemma 9.6 that one of them is minorized by $c\exp(2bt)$. Hence the entire sum is also minorized by $c\exp(2bt)$ and we obtain

$$M_{z_0}(z,\mathbf{x}) \geq c\exp(2bt).$$

Since

$$d(D_{\mathbf{x},z_0'}, z) = t$$

Theorem 9.1 is proved.

9.2 Proof of Theorem 9.1 for $G = \mathrm{Sp}(2n,\mathbb{R})$

We know that $\mathrm{Sp}(2n,\mathbb{R}) \in \mathrm{U}(n,n)$ and the symmetric space $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$ embedds into $\mathrm{U}(n,n)/(\mathrm{U}(n)\times\mathrm{U}(n))$. In this section we denote $D = \mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$, $D' = \mathrm{U}(n,n)/(\mathrm{U}(n)\times\mathrm{U}(n))$. Let

$$z_0 = \mathrm{span}\{v_{n+1},\ldots,v_{2n}\}.$$

Then z_0 is a negative n-plane of V. At the same time z_0 is a Lagrangian subspace of $(,)_1$. Here $(,)_1$ is the symplectic form on V_0 and we extend it complex linearly to V. So z_0 can be naturally viewed as a point in both D and D'.

z_0 corresponds to a Cartan decomposition of $\mathfrak{g}_0$

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0.$$

With respect to the basis $\{v_1, \ldots, v_{2n}\}$ $\mathfrak{p}_0$ is given by equation (7.6). Now we need to study a Schrödinger model of the following seesaw dual pairs

$$\begin{array}{ccc} \mathrm{U}(n,n) & & \mathrm{O}(2r,2r) \\ \uparrow & \nwarrow\!\!\!\!\nearrow & \uparrow \\ \mathrm{Sp}(2n,\mathbb{R}) & & \mathrm{U}(r,r) \end{array} \tag{9.14}$$

as in Section 6.3 and Section 6.4. To be more precise, recall that W is $2r$ dimensional complex vector space with a Hermitian form of signature (r,r). Let E be a Lagrangian subspace of W. Then the Schrödinger model of the above seesaw dual pair is $\mathcal{S}(V \otimes_{\mathbb{C}} E) \cong \mathcal{S}(V^r)$. Also recall that we have a symplectic basis $\{E_1, \ldots, E_n, F_1, \ldots, F_n\}$ of V_0 and an orthonormal basis $\{v_1, \ldots v_{2n}\}$ of V with relations given in Section 6.4.

Moreover there is an isomorphism $V^r \cong V \otimes_{\mathbb{C}} E \cong V_0 \otimes_{\mathbb{R}} E \cong V_0^{2r}$. Under this isomorphism, $\mathbf{x}_0 = (\vec{x}_1, \ldots, \vec{x}_r, \vec{y}_1, \ldots, \vec{y}_r) \in V_0^{2r}$ is mapped to

$$\mathbf{x}' = (\vec{x}_1 - i\vec{y}_1, \ldots, \vec{x}_r - i\vec{y}_r) \in V^m.$$

Let $\mathbf{x}'' = (\vec{x}_1 + i\vec{y}_1, \ldots, \vec{x}_r + i\vec{y}_r) \in V^m$. We have assumed that $\mathrm{span}_{\mathbb{R}}\{\mathbf{x}_0\} = \mathrm{span}_{\mathbb{R}}\{E_1, \ldots, E_r, F_1, \ldots, F_r\}$. Hence $\mathrm{span}_{\mathbb{C}}\{\mathbf{x}', \mathbf{x}''\} = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r, v_{n+1}, \ldots, v_{n+r}\}$. Let $U = \mathrm{span}_{\mathbb{R}}\{\mathbf{x}_0\}$ and $U' = \mathrm{span}_{\mathbb{C}}\{\mathbf{x}', \mathbf{x}''\}$.

We know that

$$\begin{aligned} M_{z_0}(z, \mathbf{x}_0) =& \sum_{j=1}^{r} ((g_z^{-1}x_j, g_z^{-1}x_j)_0 + (g_z^{-1}y_j, g_z^{-1}y_j)_0) \\ =& \sum_{j=1}^{r} (g_z^{-1}(\vec{x}_j - i\vec{y}_j), g_z^{-1}(\vec{x}_j - i\vec{y}_j))_0 \\ =& M_{z_0}(z, \mathbf{x}'). \end{aligned}$$

where $(,)_0$ is the Hermtian form on V with $\{v_1, \ldots, v_{2n}\}$ an orthonormal basis and $g_z \in \mathrm{Sp}(2n, \mathbb{R})$ such that $g_z z_0 = z$. Since g_z is real and $(,)_0$ is invariant under complex conjugation, it is also true that

$$M_{z_0}(z, \mathbf{x}_0) = M_{z_0}(z, \mathbf{x}'') = \sum_{j=1}^{r} (g_z^{-1}(\vec{x}_j + i\vec{y}_j), g_z^{-1}(\vec{x}_j + i\vec{y}_j))_0.$$

Hence

$$M_{z_0}(z, \mathbf{x}_0) = \frac{1}{2} \sum_{j=1}^{r} ((g_z^{-1}(\vec{x}_j + i\vec{y}_j), g_z^{-1}(\vec{x}_j + i\vec{y}_j))_0 + (g_z^{-1}(\vec{x}_j - i\vec{y}_j), g_z^{-1}(\vec{x}_j - i\vec{y}_j))_0). \tag{9.15}$$

For $X \in T_{z_0}D \subset \mathfrak{p}_0$, let $z = \exp(tX)z_0$, then as in Lemma 9.3, we have

$$\begin{aligned} M_{z_0}(z, \mathbf{x}) =& \frac{1}{2}\sum_{j=1}^{r}(\exp(-tX)(\vec{x}_j + i\vec{y}_j), \exp(-tX)(\vec{x}_j + i\vec{y}_j))_0 + \\ & (\exp(-tX)(\vec{x}_j - i\vec{y}_j), \exp(-tX)(\vec{x}_j - i\vec{y}_j))_0) \\ =& \frac{1}{2}\sum_{j=1}^{r}\sum_{\lambda(X)}\{\|p_{\lambda(X)}(\vec{x}_j - i\vec{y}_j)\|_0^2 \exp(-\lambda(X)t) + \|p_{\lambda(X)}(\vec{x}_j + i\vec{y}_j)\|_0^2 \exp(-\lambda(X)t)\}, \end{aligned}$$

where the summation runs over the eigenvalues of X and $p_{\lambda(X)}$ is the projection onto the eigenspace of X with eigenvalue $\lambda(X)$.

Since $T_{z_0}D \subset T_{z_0}D'$, we can define $f(X)$ as in equation (9.11)

$$f(X) = -\sum_{j=1}^{r}\sum_{\lambda(X)<0}\{\|p_{\lambda(X)}(\vec{x}_j - i\vec{y}_j)\|_0^2\lambda(X) + \|p_{\lambda(X)}(\vec{x}_j + i\vec{y}_j)\|_0^2\lambda(X)\},$$

where the summation runs over the negative eigenvalues of X. By lemma 9.4, we know f is continuous on $T_{z_0}D'$. As before, we have a generalized special cycle $D_{U,z_0'} \in D$ where $z_0' = z_0 \cap U$. Let $N_{z_0}D_{U,z_0'}$ be the normal vectors to $D_{U,z_0'}$ at z_0 and $S(N_{z_0}D_{U,z_0'})$ be its unit sphere.

Lemma 9.7. *$f(X)$ does not take zero value on $S(N_{z_0}D_{U,z_0'})$. As $S(N_{z_0}D_{U,z_0'})$ is compact there exists $C > 0$ such that*

$$f(X) \geq C, X \in S(N_{z_0}D_{U,z_0'}).$$

Remark 9.2. *This is the analogue of lemma 9.5. Notice that the symmetric space D here is different from that of lemma 9.5.*

Proof. Assume $f(X) = 0$. Suppose $v \in V_0$ is an eigenvector of X corresponding to a strictly negative eigenvalue so

$$X(v) = \lambda v, \lambda < 0.$$

Then

$$f(X) = 0 \Rightarrow \|\mathfrak{p}_\lambda(\vec{x}_j - i\vec{y}_j)\|^2 = \|\mathfrak{p}_\lambda(\vec{x}_j + i\vec{y}_j)\|^2 = 0$$

Since $\mathrm{span}_{\mathbb{C}}\{\vec{x}_1 - i\vec{y}_1, \ldots, \vec{x}_r - i\vec{y}_r, \vec{x}_1 + i\vec{y}_r, \ldots, \vec{x}_r + i\vec{y}_r\} = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r, v_{n+1}, \ldots, v_{n+r}\}$, we know that

$$(v, v_j)_0 = 0$$

for $1 \leq j \leq r$ and $n + 1 \leq j \leq n + r$.

Recall that for $X \in N_{z_0} D_{\mathbf{x}, z_0'}$ we have (see equation (7.7) and equation (7.6))

$$X = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & {}^t b & 0 \\ {}^* a & \bar{b} & 0 & 0 \\ {}^* b & 0 & 0 & 0 \end{pmatrix}.$$

where $a = {}^t a$. The rest of the proof is exactly the same with that of lemma 9.5. $\square$

With lemma 9.7, the conclusion of lemma 9.6 holds for $X \in N_{z_0} D_{\mathbf{x}, z_0}$ as well, so for $z \in F_{z_0} D_{U, z_0'}$ Theorem 9.1 can be proved similarly as in the unitary case. The general case can be derived from this using the argument right after Theorem 9.1.

9.3 Proof of Theorem 9.1 for $G = \mathrm{O}^*(2n)$

In this section we use notations from Section 6.5. We denote

$$D = \mathrm{O}^*(2n)/\mathrm{U}(n),\ D' = \mathrm{U}(n,n)/(\mathrm{U}(n) \times \mathrm{U}(n)).$$

Then $D \subset D'$. Let

$$z_0 = \mathrm{span}\{v_{n+1}, \ldots, v_{2n}\}.$$

Then z_0 is a negative n-plane of $(V_{\mathbb{C}}, H(,))$. At the same time z_0 is a Lagrangian subspace of $(V_{\mathbb{C}}, S(,))$. So z_0 can be naturally viewed as a point in both D and D'. z_0 corresponds to a Cartan decomposition of $\mathfrak{g}_0$

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0.$$

With respect to the basis $\{v_1, \ldots, v_{2n}\}$ $\mathfrak{p}_0$ is given by equation (7.10). Now we need to study a Schrödinger model of the following seesaw dual pairs

$$\begin{array}{ccc} \mathrm{U}(n,n) & & \mathrm{Sp}(r,r) \\ \uparrow & \times & \uparrow \\ \mathrm{O}^*(2n,\mathbb{R}) & & \mathrm{U}(r,r) \end{array} \tag{9.16}$$

as in Section 6.3 and Section 6.5. To be more precise, recall that W is $2r$ dimensional complex vector space with a Hermitian form of signature (r,r). Let E be a Lagrangian subspace of W. Then the Schrödinger model of the above seesaw dual pair is $\mathcal{S}(V \otimes_{\mathbb{C}} E) \cong \mathcal{S}(V^r)$. Also recall that we have an orthonormal $\mathbb{H}$-basis $\{v_1, \ldots v_n\}$ of V with relations given in Section 6.5.

Recall that $W_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{C}} W$, and we can extend the Hermitian form on W to $W_{\mathbb{H}}$. Let E be a Lagrangian subspace of W, and $E_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{C}} E$. We have an isomorphism:

$$V^r \cong V \otimes_{\mathbb{C}} E \cong V \otimes_{\mathbb{H}} E_{\mathbb{H}} \cong V^r.$$

Let $\mathbf{x} = (\vec{x}_1, \ldots, \vec{x}_r) \in V^r$, we have assumed that

$$U = \operatorname{span}_{\mathbb{H}}\{\mathbf{x}\} = \operatorname{span}_{\mathbb{H}}\{v_1, \ldots, v_r\}.$$

Hence

$$\operatorname{span}_{\mathbb{C}}\{\mathbf{x}, \mathbf{x}j\} = \operatorname{span}_{\mathbb{C}}\{v_1, \ldots, v_r, v_{n+1}, \ldots, v_{n+r}\}.$$

We know that

$$M_{z_0}(z, \mathbf{x}) = \sum_{\alpha=1}^{r} (g_z^{-1}\vec{x}_\alpha, g_z^{-1}\vec{x}_\alpha)_0.$$

where $(,)_0$ is the Hermtian form on V with $\{v_1, \ldots, v_{2n}\}$ an orthonormal basis and $g_z \in \mathrm{O}^*(2n, \mathbb{R})$ such that $g_z z_0 = z$.

For $X \in T_{z_0}D \subset \mathfrak{p}_0$, let $z = \exp(tX)z_0$, then as in Lemma 9.3, we have

$$\begin{aligned} M_{z_0}(z, \mathbf{x}) &= \sum_{\alpha=1}^{r} (\exp(-tX)\vec{x}_\alpha, \exp(-tX)\vec{x}_\alpha)_0 \\ &= \sum_{\alpha=1}^{r} \sum_{\lambda(X)} \|p_{\lambda(X)}(\vec{x}_\alpha)\|_0^2 \exp(-\lambda(X)t), \end{aligned}$$

where the summation runs over the eigenvalues of X and $p_{\lambda(X)}$ is the projection onto the eigenspace of X with eigenvalue $\lambda(X)$.

Since $T_{z_0}D \subset T_{z_0}D'$, we can define $f(X)$ as in equation (9.11)

$$f(X) = -\sum_{\alpha=1}^{r} \sum_{\lambda(X)<0} \|p_{\lambda(X)}(\vec{x}_\alpha)\|_0^2 \lambda(X),$$

where the summation runs over the negative eigenvalues of X. By lemma 9.4, we know f is continuous on $T_{z_0}D'$. As before, we have a generalized special cycle $D_{U,z'_0} \in D$ where $z'_0 = z_0 \cap U$. Let $N_{z_0}D_{U,z'_0}$ be the normal vectors to D_{U,z'_0} at z_0 and $S(N_{z_0}D_{U,z'_0})$ be its unit sphere.

Lemma 9.8. *$f(X)$ does not take zero value on $S(N_{z_0}D_{\mathbf{x},z_0})$. As $S(N_{z_0}D_{\mathbf{x},z_0})$ is compact and $f(X) \geq 0$ there exists $C > 0$ such that*

$$f(X) \geq C, X \in S(N_{z_0}D_{\mathbf{x},z_0}).$$

Proof. Since $X \in \mathfrak{o}^*(2n)$, it commutes with right multiplication by j on V which is a complex anti-linear map on $V_{\mathbb{C}}$. In particular if λ is an eigenvalue of X (λ must be real as X is Hermitian) and V_λ is the λ-eigenspace of X, V_λ will be preserved by right multiplication by j.

Suppose V_λ is the λ-eigenspace of X corresponding to a strictly negative eigenvalue λ. Assume $f(X) = 0$, then

$$f(X) = 0 \Rightarrow \|\mathfrak{p}_\lambda(\vec{x}_\alpha)\|_0^2 = 0 \Rightarrow v_\alpha \perp V_\lambda,$$

for $1 \leq \alpha \leq r$. As V_λ is preserved by right multiplication by j and $(,)_0$ is preserved by right multiplication by j, the above implies $\vec{x}_\alpha j \perp V_\lambda$ for $1 \leq \alpha \leq r$. Since $\mathrm{span}_{\mathbb{C}}\{\mathbf{x}, \mathbf{x}j\} = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_r, v_{n+1}, \ldots, v_{n+r}\}$, we have

$$v_\alpha \perp V_\lambda,$$

for $1 \leq \alpha \leq r$ and $n+1 \leq \alpha \leq n+r$.

Recall that for $X \in N_{z_0} D_{\mathbf{x}, z_0'}$ we have (see equation (7.11) and equation (7.10))

$$X = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -{}^t b & 0 \\ {}^* a & -\bar{b} & 0 & 0 \\ {}^* b & 0 & 0 & 0 \end{pmatrix}.$$

where $a = -{}^t a$. The rest of the proof is exactly the same with that of lemma 9.5. □

With lemma 9.8, the conclusion of lemma 9.6 holds for $X \in N_{z_0} D_{\mathbf{x}, z_0}$ as well, so for $z \in F_{z_0} D_{U, z_0'}$, Theorem 9.1 can be proved similarly as in the unitary case. The general case can be derived from this using the argument right after Theorem 9.1.

9.4 Rapid decrease of the cocycles over the fiber $F_{z_0} D_{\mathbf{x}, z_0'}$

In this section we want to prove that for any $\psi \in C^\bullet(\mathfrak{g}, K; \mathcal{S}(V^m))$, $\tilde{\psi}$ satisfies the decreasing condition of Theorem 8.4.

Theorem 9.2. *All the assumptions are as in Theorem 9.1. For any Schwartz func-*

tion $\psi \in \mathcal{S}(V^m)$ and any constant $\rho > 0$, there is a constant C_ρ such that

$$\psi(g^{-1}\mathbf{x}) \le C_\rho \exp(-\rho\, d(z, g, \mathbf{x})), \tag{9.17}$$

where $d(z, g)$ is defined in Theorem 9.1.

Proof. Since ψ is a Schwartz function, for any positive integer N, there is a positive constant C_N such that

$$\psi(\mathbf{x}) \le \frac{C_N}{(||\mathbf{x}||_0^2)^N}.$$

By Theorem 9.1, we know that

$$\psi(g^{-1}\mathbf{x}) \le \frac{C_N}{c} \exp(-2Nc\, d(z, g, \mathbf{x})).$$

We fix a $N > \frac{\rho}{2c}$ and let $C_\rho = \frac{C_N}{c}$, the theorem is proved. □

Theorem 9.3. *Fix a nondegenerate $\mathbf{x}$, a point $z \in D$ such that $(\mathbf{x}, z)$ is a compatible pair, $g' \in G'$ and $\psi \in C^\bullet(\mathfrak{g}, K; \mathcal{S}(V^m))$. For any $\rho > 0$, there is a positive constant C'_ρ such that*

$$||\tilde{\psi}(z, g', \mathbf{x})|| \le C'_\rho \exp(-\rho d(z, g, \mathbf{x})),$$

where the norm is taken with respect to the Riemannian metric τ and $d(z, g, \mathbf{x})$ is as in Theorem 9.1. In particular, $\tilde{\varphi}_\infty(z, g', \mathbf{x})$ satisfies the condition of Theorem 8.4 and is integrable on $F_{z_0} D_{\mathbf{x}, z'_0}$.

Proof. Let $g = g_z = \exp(Xt)$ with $\|X\| = 1$. So $d(D_{\mathbf{x},z_0'}, z) = t$. Recall that

$$\tilde{\psi} = (L_{g^{-1}})^*(\psi).$$

where

$$\psi = \sum_I^d \psi_I \Omega_I,$$

where $\psi_I \in \mathcal{S}(V^m)$ are polynomial and $\Omega_I \in \wedge^{\bullet}\mathfrak{p}^*$. Hence

$$\tilde{\psi}(z, g', \mathbf{x}) = \sum_I (\omega(g')\psi_I)(g^{-1}\mathbf{x}) L_{g^{-1}}^*(\Omega_I).$$

Weil representation preserves the space of Schwartz functions, hence $\omega(g')\psi_I \in \mathcal{S}(V^m)$.

By Theorem 9.2, we know that for any I and $\rho' > 0$, there is a constant $C_\rho^I > 0$ such that

$$(\omega(g')\psi_I)(g^{-1}\mathbf{x}) \leq C_\rho^I \exp(-\rho\, d(z, g, \mathbf{x})).$$

Since the left action of G on D is isometric, we know that

$$||L_{g^{-1}}^*(\Omega_I)|| = ||\Omega_I||.$$

Hence define

$$C_\rho' = \sqrt{\sum_I (C_\rho^I)^2 ||\Omega_I||^2}.$$

We know that $||\tilde{\psi}_\infty(z, g', \mathbf{x})|| \leq C_\rho' \exp(-\rho d(z, g))$.

□

Chapter 10: Asymptotic evaluations of fiber integrals

We want to compute the fiber integral $\kappa(g',\beta)$ defined by

$$\kappa(g',\beta) = \int_{F_{z_0} D_{\mathbf{x},z_0'}} \tilde{\varphi}_\infty(z,g',\mathbf{x}),$$

where $(\mathbf{x},\mathbf{x}) = \beta$ and $(\mathbf{x},z_0)$ is a compatible pair (Definition 3.1) and $z_0' = z_0 \cap \mathrm{span}\{\mathbf{x}\}$. Our goal is to prove theorem 8.3. This only depends on the symmetric space G_∞/K_∞ so we only need to work over real groups. Recall that $\varphi_\infty = \varphi \otimes \prod_{v\neq v_1} \varphi_v$ where φ is the cocycle defined in Chapter 7, φ_v is the Gaussian function of V_v^m and v is an Archimedean place for the number field k. Throughout the chapter we assume that $g' = (g_1', Id, \ldots, Id) \in G'_\infty = \prod_v G'_v$. Hence $\varphi_v(g,g')$ is a nonzero constant for $v \neq v_1$. So in order to prove the integral $\kappa(g',\beta)$ is nonzero, it suffices to compute the following rescaled integral

$$\int_{F_{z_0} D_{\mathbf{x},z_0'}} \tilde{\varphi}(z,g',\mathbf{x})$$

with $g' \in G'_{v_1}$. So from now on in this chapter, we change our notation and let $G = G_{v_1}$, $G' = G'_{v_1}$ and $\kappa(g',\beta) = \int_{F_{z_0} D_{\mathbf{x},z_0'}} \tilde{\varphi}_\infty(z,g',\mathbf{x})$. Recall that (G,G') can be the following three dual pairs

1. $(\mathrm{U}(p,q), \mathrm{U}(r,r))$

2. $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(2r,2r))$

3. $(\mathrm{O}^*(2n), \mathrm{Sp}(r,r))$

Recall that the group $M' \subset G'$ is

$$M' = \left\{ m'(a) = \begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix} \mid a \in \mathrm{GL}_m(B) \text{ and } \hat{a} = ({}^\top a^{-1})^\sigma \right\}.$$

An element $(m'(a), \zeta)$ in its double cover acts by (see (Chapter 5))

$$\omega(m'(a), \zeta)\varphi(\mathbf{x}) = \zeta\chi_V(a)\varphi(\mathbf{x}a).$$

If

$$(\mathbf{x}, \mathbf{x}) = \beta.$$

Then β is an ℓ-tuple of $r \times r$ $\mathbb{C}$-skew Hermitian (resp. $2r$ by $2r$ real-valued skew-symmetric or r by r $\mathbb{H}$-skew Hermitian) matrix. We have

$$(\mathbf{x}a, \mathbf{x}a) = {}^\top a^\sigma \beta a.$$

It follows that

$$\kappa(g'(m'(a), \zeta), \beta) = \zeta\chi_V(a)\kappa(g', {}^\top a^\sigma \beta a).$$

Suppose β satisfies the condition of theorem 8.1. By the above formula and Gram-Schmidt process, we can choose $m = m'(a)$ such that ${}^\top a^\sigma \beta a$ is of the following form:

1. For the group $\mathrm{U}(p,q)$ we assume ${}^\top a^\sigma \beta a$ is an $r+s$ by $r+s$ diagonal matrix with diagonal entries $\{\underbrace{-i,\ldots,-i}_{r},\underbrace{i,\ldots,i}_{s}\}$.

2. For the group $\mathrm{Sp}(2n,\mathbb{R})$ we assume ${}^\top a^\sigma \beta a$ is the $2r$ by $2r$ matrix $\begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}$

3. For the group $\mathrm{O}^*(2n)$ we assume ${}^\top a^\sigma \beta a$ is an r by r $\mathbb{H}$-valued diagonal matrix with diagonal entries $\{\underbrace{-i,\ldots,-i}_{r}\}$.

So from now on we assume β is of the above form. By translating by appropriate $g \in G$, we can choose any $\mathbf{x}$ such that $(\mathbf{x},\mathbf{x}) = \beta$ and compute $\kappa(g',\beta)$ as

$$\kappa(g',\beta) = \int_{F_{z_0} D_{\mathbf{x},z_0'}} \tilde{\varphi}_\infty(z,g',\mathbf{x}).$$

Hence we can assume

1. $\mathbf{x} = (v_1,\ldots,v_r,v_{p+1},\ldots,v_{p+s}),\ z_0 = \mathrm{span}_{\mathbb{C}}\{v_{p+1},\ldots,v_{p+q}\}$,

2. $\mathbf{x} = (E_1,\ldots,E_r,F_1,\ldots,F_r),\ z_0 = \mathrm{span}_{\mathbb{C}}\{v_n,\ldots,v_{2n}\}$,

3. $\mathbf{x} = (v_1,\ldots,v_r),\ z_0 = \mathrm{span}_{\mathbb{C}}\{v_n,\ldots,v_{2n}\}$,

in the three cases respectively (the choice of basis of the vector spaces is as in Chapter 7). Let $a(t) \in \mathrm{GL}_m(B)$ be scalar matrix $t \cdot Id$. The exact value of $\kappa(g',\beta)$ is hard to compute in general, instead we approximate $\kappa(g',\beta)$ for $g' = (m'(a(t)),1)$ as $t \to \infty$. We need the following theorem.

Theorem 10.1. *Let $f(x), h(x)$ be smooth functions on $\mathbb{R}^n$. And let $J(v)$ be the integral*

$$J(t) = \int_{\mathbb{R}^n} f(x) e^{-th(x)} dx.$$

And we assume that

1. *The integral $J(t)$ converges absolutely for all $t > 0$.*

2. *For every $\epsilon > 0$, $\rho(\epsilon) > 0$, where*

$$\rho(\epsilon) = \inf\{h(x) - h(0) : x \in \mathbb{R}^n, |x - 0| \geq \epsilon\}$$

3. *the Hessian matrix*

$$A = (\frac{\partial^2 h}{\partial x_i \partial x_j})|_{x=0}$$

is positive definite.

Then we have

$$J(t) \sim (\frac{2\pi}{t})^{\frac{n}{2}} f(0) det(A)^{-\frac{1}{2}} exp[-th(0)],$$

as $t \to \infty$.

The above theorem is one special case of the so-called method of Laplace. The proof can be found in Section 3 of Chapter IX in [Wong]. To apply Theorem 10.1, we choose a base point $z_0 \in D_{\mathbf{x},z_0'}$ Recall that we identify $T_{z_0}D \cong \mathfrak{g}_0/\mathfrak{k}_0$ with $\mathfrak{p}_0$. One half of the trace form defines a $Ad(K)$-invariant metric τ on $\mathfrak{p}_0$ (thus τ extends t a G-invariant metric on D). If exp_{z_0} is the exponential map of the Riemanian manifold (D, τ) and exp is the exponential map in G, then we have the following commutative diagram([He] Chapter VI, section 3):

$$\begin{array}{ccc} \mathfrak{p}_0 & \xrightarrow{exp} & G \\ \downarrow \cong & & \downarrow \\ T_{z_0}D & \xrightarrow{exp_{z_0}} & G/K \end{array}$$

The metric τ also makes $T_{z_0}D$ a (Euclidean) Riemannian manifold in an obvious way, we abuse notation and denote this metric also as τ. If $X \in \mathfrak{p}$ and $g_z = exp(X), z = exp_{z_0}(X)$ then we have $d(z, z_0) = ||X||$, where both $d(z, z_0)$ and $||X||$ are defined with respect to the metric τ.

Using the exponential map $\exp_{z_0}$ we can identify the normal vector space $N_{z_0}D_{\mathbf{x},z_0'}$ with the fiber $F_{z_0}D_{\mathbf{x},z_0'}$ for a suitable choice of base point z_0. Then we apply theorem 10.1 to

$$J(t) = \kappa(g', \mathbf{x}) = \int_{F_{z_0}D_{\mathbf{x},z_0'}} \tilde{\varphi}(z, g', \mathbf{x}) = \int_{N_{z_0}D_{\mathbf{x},z_0'}} exp^*_{z_0}(\tilde{\varphi}(Y, g', \mathbf{x}))$$

for $Y \in N_{z_0}D_{\mathbf{x},z_0'}$ and $g' = (m'(a(t)), 1)$.

We verify the assumptions of theorem 10.1 one by one. The first assumption is already proved in Theorem 9.3. Hence we will verify the remaining two in this section. We will emphasize on the case $G = \mathrm{U}(p, q)$ and mention the other two cases briefly. As the reader will see, the case when $G = \mathrm{Sp}(2n, \mathbb{R})$ or $G = \mathrm{O}^*(2n)$ follows easily from the case $G = \mathrm{U}(n, n)$.

Proof of theorem 8.3: In theorems 10.2, 10.3 and 10.4 we will compute the asymptotic value of $\kappa(g', \beta)$ in each cases and show it's nonzero. It is easy to show that the action of $\tilde{G}'$ is analytic on the polynomial Fock space. By the definition of $\kappa(g', \beta)$ and the fact that $\tilde{\varphi}_\infty$ is fast decaying on $F_{z_0} D_{\mathbf{x},\lambda}$, $\kappa(g', \beta)$ is analytic on $\tilde{G}'$, and the theorem is proved.

10.1 The method of Laplace for the $\mathrm{U}(p, q)$ Case

Our goal is to apply theorem 10.1. To apply the theorem there are three conditions to check. (1) is checked in Chapter 9.4. Now we check (2) and (3). We have assumed

$$z_0 = \mathrm{span}\{v_{p+1}, \ldots, v_{p+q}\} \in D, z_0' = \mathrm{span}\{v_{p+1}, \ldots, v_{p+s}\},$$

$$\mathbf{x} = (v_1, \ldots, v_r, v_{p+1}, \ldots, v_{p+q}) \in V^m.$$

Let

$$U_j = \mathrm{span}_{\mathbb{C}}\{v_j\}, 1 \leq j \leq p + q$$

$$D_j = D_{U_j}$$

Recall that the definition of D_j does not require a base point (see remark 3.4). Our key observation is that

Lemma 10.1.

$$D_{\mathbf{x},z_0'} = (\bigcap_{j=1}^{r} D_j) \bigcap (\bigcap_{j=p+1}^{p+s} D_j).$$

Proof. By definition for $1 \leq j \leq p$, we have

$$D_j = \{z \in D | z \subset U_j^{\perp}\}.$$

For $p+1 \leq j \leq p+q$ we have

$$D_j = \{z \in D | U_j \subset z\}.$$

Since (see remark 3.3)

$$D_{\mathbf{x},z_0'} = \{z \in D | \oplus_{j=p+1}^{p+s} U_j \subset z \subset (\oplus_{j=1}^{r} U_j)^{\perp}\},$$

the lemma follows. $\square$

We define functions $M^j(X) : D \to \mathbb{R}$ by

$$M^j(z) = ||g_z^{-1} v_j||_0^2,$$

where $z \in D$ and $g_z \cdot z_0 = z$. It is well defined as $|| \cdot ||_0$ is K invariant.

Lemma 10.2. *For any element $z \in D$, we have*

$$h^j(z) = cosh^2(t) + sinh^2(t)$$

where $t = d(z, D_j)$. *In particular,*

$$M^j(z) \geq M^j(z_0).$$

Equality holds if and only if $z \in D_{e_j}$.

Proof. Let us assume $1 \leq j \leq p$. The case $p+1 \leq j \leq p+q$ is similar. Without loss of generality we can assume $j = 1$. It is easy to see that

$$exp(-tE_{\alpha\mu})v_1 = \begin{cases} cosh(t) \cdot v_1 - isinh(t) \cdot v_\mu & \text{if } \alpha = 1 \\ v_1 & \text{otherwise} \end{cases}.$$

$$exp(-tF_{\alpha\mu})v_1 = \begin{cases} cosh(t) \cdot v_1 + sinh(t) \cdot v_\mu & \text{if } \alpha = 1 \\ v_1 & \text{otherwise} \end{cases}.$$

Recall that the group G_{U_1} fixes v_1. Hence

$$M^1(z) = M^1(gz), \forall g \in G_{U_1}.$$

We have a G_{U_1}-equvariant fibration $\pi_1 : D \to D_1$ (see Section 3.2). By translating z using an element in G_{U_1}, we can assume that $\pi_1(z) = z_0$ and is of the form $z = exp_{z_0}(X)$ where $X \in N_{z_0}D_1$ and (recall the definition of $E_{\alpha\mu}, F_{\alpha\mu}$ in eqaution (7.1))

$$N_{z_0}D_1 = \text{span}\{E_{1\mu}, F_{1\mu} | p+1 \leq \mu \leq p+q\}.$$

We assume that

$$X = \sum_{\mu=p+1}^{p+q} x_{1\mu} E_{1\mu} - \sum_{\mu=p+1}^{p+q} y_{1\mu} F_{1\mu} = \sum_{\mu=p+1}^{p+q} v_1 \wedge (x_{1\mu} + i y_{1\mu}) v_\mu.$$

We define

$$t = \sqrt{\sum_{\mu=p+1}^{p+q} (|x_{1\mu}|^2 + |y_{1\mu}|^2)}.$$

and

$$v = \frac{1}{t} \sum_{\mu=p+1}^{p+q} (x_{1\mu} + i y_{1\mu}) v_\mu.$$

Then $(v, v) = i$ and $(v_1, v) = 0$. We have

$$\exp(-X)(v_1) = cosh(t) \cdot v_1 - isinh(t) \cdot v.$$

So

$$M^1(z) = \|cosh(t) \cdot v_1 - isinh(t) \cdot v\|_0^2 = cosh^2(t) + sinh^2(t).$$

Since $z = \exp(tv_1 \circ v)z_0$ we know that $t = d(z, z_0) = d(z, D_1)$. The claim of the lemma is proved. $\square$

Define $M : D \to \mathbb{R}$ by

$$M(z) = \sum_{j=1}^{r} h^j(z) + \sum_{j=p+1}^{p+s} h^j(z).$$

Define $h : N_{z_0}D_{\mathbf{x},z_0'} \to \mathbb{R}$ (h^j resp.) by

$$h = M \circ \exp_{z_0} \circ i \ (h^j = M^j \circ \exp_{z_0} \circ i \text{ resp .}), \tag{10.1}$$

where i is the injection $i : N_{z_0}D_{\mathbf{x},z_0'} \hookrightarrow T_{z_0}D$. M is the function defined in Equation (9.1).

Proposition 10.1. *The function h satisfies condition* (2) *and* (3) *of theorem 10.1.*

Proof. By lemma 10.2 and lemma 10.1, $M(z)$ obtains its minimal value at z if and only if $z \in D_{\mathbf{x},z_0'}$. In particular, z_0 is the unique point with minimal h value on $F_{z_0}D_{\mathbf{x},z_0'}$, hence a critical point of h. Now we suppose $X \in N_{z_0}D_{\mathbf{x},z_0'}$ with $||X|| = 1$. We define $h_X : \mathbb{R} \to \mathbb{R}$ by

$$h_X(t) = h(\exp_{z_0}(-Xt)).$$

Then $t = 0$ is a global minimum for $h_X(t)$, thus we have

$$\frac{d}{dt}h_X(t)|_{t=0} = 0. \tag{10.2}$$

By lemma 9.3, we have

$$(\exp(-tX)v_j, \exp(-tX)v_j)_0 = \sum_{k=1}^{m} |a_j^k|^2 \exp(-2\lambda_k t).$$

From this we know

$$\frac{d^2}{dt^2}h_{\tau,X}(t) = \sum_{k=1}^{m}\{\sum_{j=1}^{r} 4|a_j^k|^2\lambda_k^2 \exp{(-2\lambda_k t)} + \sum_{j=r+1}^{r+s} 4|a_j^k|^2\lambda_k^2 \exp{(-2\lambda_k t)}\}.$$

With Lemma 9.6 in mind, we have a uniform lower bound of $\frac{d^2}{dt^2}h_{\tau,X}(t)$ for all $\{||X|| = 1|X \in T_{z_0}F_{\mathbf{x},\mathbf{y}}\}$ and $t \geq 0$. And a similar argument works for $t < 0$. So $h_X(t)$, we can assume that $h''_X(t) \geq C$ for a positive constant C. It follows that

$$h(X) \geq h(0) + \frac{1}{2}C||X||^2.$$

Hence h satisfies condition (2) of theorem 10.1. Condition (3) is also satisfied because we know from the above that the Hessian matrix of h is positive definite with the smallest eigenvalue bigger or equal to C. □

Any $X \in N_{z_0}D_{\mathbf{x},z'_0}$ can be written as

$$X = \sum_{(\alpha,\mu)\in I} x_{\alpha\mu}E_{\alpha\mu} + \sum_{(\alpha,\mu)\in I} y_{\alpha\mu}F_{\alpha\mu}$$

where I is the index set defined in equation (7.4). We denote the function $h^j \circ \exp_{z_0} \circ i$ by $\tilde{h}^j$. We think of these as functions in variables $\{x_\alpha, y_{\alpha\mu}|(\alpha,\mu) \in I\}$. We need

Corollary 10.1. *Suppose* $(\alpha,\mu),(\beta,\nu) \in I$. *For* $1 \leq j \leq p$, *we have*

$$\frac{\partial^2 h^j}{\partial x_{\alpha\mu}\partial x_{\beta\nu}} = 2\delta_{\alpha\beta}\delta_{\alpha j}\delta_{\mu\nu}$$

$$\frac{\partial^2 h^j}{\partial y_{\alpha\mu}\partial y_{\beta\nu}} = 2\delta_{\alpha\beta}\delta_{\alpha j}\delta_{\mu\nu}$$

$$\frac{\partial^2 h^j}{\partial x_{\alpha\mu}\partial y_{\beta\nu}} = 0.$$

For $p+1 \leq j \leq p+q$, we have

$$\frac{\partial^2 h^j}{\partial x_{\alpha\mu}\partial x_{\beta\nu}} = 2\delta_{\alpha\beta}\delta_{\mu j}\delta_{\mu\nu}$$

$$\frac{\partial^2 h^j}{\partial y_{\alpha\mu}\partial y_{\beta\nu}} = 2\delta_{\alpha\beta}\delta_{\mu j}\delta_{\mu\nu}$$

$$\frac{\partial^2 h^j}{\partial x_{\alpha\mu}\partial y_{\beta\nu}} = 0.$$

Proof. We need to compute the following

$$\frac{\partial^2}{\partial s\partial t}[(\exp{(sX+tY)}v, \exp{(sX+tY)}v)_0]|_{s=t=0}.$$

Using the second order approximation of the exponential map

$$\exp{(sX+tY)} = I+(sX+tY)+\frac{1}{2}(s^2X^2+t^2Y^2+stXY+stYX)+\text{higher order terms}$$

One can check that

$$\begin{aligned}&\frac{\partial^2}{\partial s\partial t}[(\exp{(sX+tY)}v, \exp{(sX+tY)}v)_0]|_{s=t=0}\\ =&\frac{1}{2}((XY+YX)v, v)_0 + \frac{1}{2}(v, (XY+YX)v)_0 + (Yv, Xv)_0 + (Xv, Yv)_0.\end{aligned}$$

We also have

$$E_{\alpha\mu}(v_\alpha) = -iv_\mu,\ E_{\alpha\mu}(v_\mu) = iv_\alpha$$

$$F_{\alpha\mu}(v_\alpha) = -v_\mu,\ F_{\alpha\mu}(v_\mu) = -v_\alpha$$

for $1 \leq \alpha \leq p, p+1 \leq \mu \leq p+q$. Also recall that $(,)_0$ is a positive definite Hermitian form with orthonormal basis $\{v_1, \ldots, v_{p+q}\}$. With these preparations, the formulas in the lemma follow from routine calculations. □

The Hessian matrix A of h at 0 is a $2(rq+ps-rs)$ by $2(rq+ps-rs)$ matrix.

Corollary 10.2. *A is diagonal with diagonal entries*

$$\frac{\partial^2 h}{\partial^2 x_{\alpha\mu}} = \frac{\partial^2 h}{\partial^2 y_{\alpha\mu}} = 4$$

for $1 \leq \alpha \leq r, p+1 \leq \mu \leq p+s$. And

$$\frac{\partial^2 h}{\partial^2 x_{\alpha\mu}} = \frac{\partial^2 h}{\partial^2 y_{\alpha\mu}} = 2$$

for $1 \leq \alpha \leq r, p+s+1 \leq \mu \leq p+s$ or $r+1 \leq \alpha \leq p, p+1 \leq \mu \leq p+s$. In particular it is positive definite with determinant

$$det(A) = 4^{rq+ps}.$$

Proof. Since

$$h = \sum_{j=1}^{r} h^j + \sum_{j=p+1}^{p+s} h^j.$$

The corollary follows from 10.1 and simple algebra. □

With all the preparations we are ready to prove our main theorem of this section. Recall that

$$J(t) = \kappa(g', \mathbf{x}) = \int_{F_{z_0} D_{\mathbf{x}, z_0'}} \tilde{\varphi}(z, g', \mathbf{x}) = \int_{N_{z_0} D_{\mathbf{x}, z_0'}} i^* \circ exp_{z_0}^* (\tilde{\varphi}(Y, g', \mathbf{x}))$$

for $Y \in N_{z_0} D_{\mathbf{x}, z_0'}$ and $g' = (m'(a(t)), 1)$.

Theorem 10.2.

$$J(t) \sim \zeta 2^{3rq+3ps-7rs} \pi^{3rq+3ps-5rs} t^{\frac{1}{2}(p+q)-2rs} \exp(-(r+s)t^2),$$

where ζ is an eighth root of unity.

Proof. Think of $N_{z_0} D_{\mathbf{x}, z_0'}$ as $\mathbb{R}^{2(rq+ps-rs)}$. Recall that

$$\tilde{\varphi}_\infty = (L_{g^{-1}})^*(\varphi) = (L_{g^{-1}})^*(\varphi).$$

where $gz_0 = z$. By equation (7.13), we have

$$\varphi = \varphi_0 \sum_{i,j=1}^{d} p_{ij} \Omega_i \wedge \bar{\Omega}_j.$$

where p_{ij} are polynomial functions on V^m and $\Omega_i \in \wedge^\bullet \mathfrak{p}_+^*$ and

$$\varphi_0(\mathbf{x}) = \exp(-\pi Tr(\mathbf{x}, \mathbf{x})_0) = \exp(-\pi \sum_{i=1}^{m} ||\vec{x}_m||^2),$$

with $\mathbf{x} = (v_1, \ldots, v_r, v_{p+1}, \ldots, v_{p+s})$. So we have

$$(i^* \circ \exp_{z_0}^*)(\tilde{\varphi})(Y, Id, \mathbf{x}) = \varphi_0(\exp(-Y)\mathbf{x}) \sum_{i,j} p_{ij}(\exp(-Y)\mathbf{x}) i^*((L_{\exp(-Y)})^*(\Omega_i) \wedge \overline{(L_{\exp(-Y)})^*(\Omega_j)}), \tag{10.3}$$

with

$$\varphi_0(\exp(-Y)\mathbf{x}) = \exp(-h(Y)),$$

for $Y \in N_{z_0} D_{\mathbf{x}, z_0'}$. Since $\tilde{\varphi}$ and $N_{z_0} D_{\mathbf{x}, z_0'}$ have the same degree.

$$(i^* \circ \exp_{z_0}^* \tilde{\varphi})(Y, Id, \mathbf{x}) = f(Y) \exp(-h(Y)) dvol$$

for a function f in Y, where $dvol$ is the volume form of the Euclidean space $(N_{z_0} D_{\mathbf{x}, z_0'}, \tau)$.

Recall equation 5.4, so we have

$$\omega(m'(a(t)), 1)\varphi(\mathbf{x}) = \zeta t^{\frac{1}{2}(p+q)} \varphi(\mathbf{x}t).$$

So for a polynomial function p of degree d on V^m, we know

$$\begin{aligned}\omega(m'(a(t)), 1)(p\varphi_0)(\mathbf{x}) = \zeta t^{\frac{1}{2}(p+q)} \sum_{i=1}^{d} t^i p_i(\mathbf{x}) \varphi_0(\mathbf{x}t) \\ = \zeta t^{\frac{1}{2}(p+q)} \sum_{i=1}^{d} t^i p_i(\mathbf{x}) \exp(-h(Y)t^2)\end{aligned}$$

where p_i is the degree i homogeneous part of p_i. After applying $g' = omega(m'(a(t)), 1)$ to each term in the summation of Equation (10.3) and combining terms according to t-degree, we have

$$(i^* \circ \exp_{z_0}^* \tilde{\varphi})(Y, g', \mathbf{x}) = \zeta t^{\frac{p+q}{2}} \sum_{i=1}^{2(rq+ps-rs)} t^i f_i(Y) \exp(-h(Y)t^2) dvol. \tag{10.4}$$

Theorem 10.1 can be applied to $(i^* \circ \exp_{z_0}^* \tilde{\varphi})(Y, g', \mathbf{x})$. Condition (1) of 10.1 is checked in theorem 9.3 as each term of Equation (10.4) is in $C^\bullet(D, \mathcal{S}(V^m))^G$. Condition (2) and (3) are checked in Proposition 10.1, and the determinant of the Hession of h is checked in corollary 10.2.

As we are interested in asymptotic value when $t \to \infty$, only the highest degree (which is $2(rq + ps - rs)$) term of t in Equation (10.4) matters. By Lemma 7.7, we know that the highest degree term of $(i^* \circ \exp_{z_0}^* \tilde{\varphi})(0, g', \mathbf{x})$ evaluated at

$$\mathbf{x} = (v_1, \ldots, v_r, v_{p+1}, \ldots, v_{p+s})$$

is

$$(2\sqrt{2}\pi t)^{2(rq+ps-rs)} \bigwedge_{(\alpha,\mu)\in I} \xi'_{\alpha\mu} \wedge \bigwedge_{(\alpha,\mu)\in I} \xi''_{\alpha\mu}$$

where

$$I = \{(\alpha, \mu) | 1 \le \alpha \le r, p+1 \le \mu \le p+q\} \cup \{(\alpha, \mu) | r+1 \le \alpha \le p, p+1 \le \mu \le p+s\}.$$

Since

$$(i^* \circ \exp_{z_0}^*)|_0 (\bigwedge_{(\alpha,\mu)\in I} \xi'_{\alpha\mu} \wedge \bigwedge_{(\alpha,\mu)\in I} \xi''_{\alpha\mu}) = dvol$$

By 10.1 we know

$$
\begin{aligned}
J(t) \sim & (2\sqrt{2}\pi t)^{2(rq+ps-2rs)} \cdot t^{\frac{1}{2}(p+q)} \cdot (\frac{2\pi}{t^2})^{ps+rq-rs} 2^{-ps-rq} \exp(-(r+s)t^2) \\
\sim & 2^{3rq+3ps-7rs} \pi^{3rq+3ps-5rs} t^{\frac{1}{2}(p+q)-2rs} \exp(-(r+s)t^2).
\end{aligned}
$$

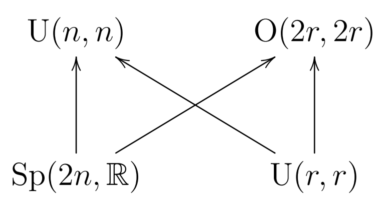
The theorem is proved. □

10.2 The method of Laplace for the $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(2r,2r))$ Case

We want to apply theorem 10.1 to compute

$$
J(t) = \int_{N_{z_0}, D_{\mathbf{x}, z_0'}} \exp^*_{z_0}(\tilde{\varphi}_\infty(Y, g', \mathbf{x}))
$$

in case of the dual pair $(\mathrm{Sp}(2n,\mathbb{R}), \mathrm{O}(2r,2r))$. We need to check conditions (1),(2),(3) of theorem 10.1 again. Fortunately, we don't need to start from scratch. Our strategy is to use the seesaw dual pair



(10.5)

to reduce a substantial part of the problem to the unitary dual pair case. We proceed quickly by omitting the proofs that are similar to the unitary group case.

Recall from Section 6.4 that we have a $\mathbb{R}$ vector space V_0 and $V = V_0 \otimes \mathbb{C}$. Let

$$D = \mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n), \quad \tilde{D} = \mathrm{U}(r, r)/(\mathrm{U}(r) \times \mathrm{U}(r).$$

We have assumed that

$$\mathbf{x} = (E_1, \ldots, E_r, F_1, \ldots, F_r), \; z_0 = \mathrm{span}_{\mathbb{C}}\{v_n, \ldots, v_{n+1}\}.$$

Let $U_0 = \mathrm{span}_{\mathbb{R}}\{\mathbf{x}\}$, $U = \mathrm{span}_{\mathbb{C}}\{\mathbf{x}\}$ and $z_0' = U \cap z_0 = \mathrm{span}_{\mathbb{C}}\{v_n, \ldots, v_{n+r}\}$. Then we can define generalized special cycles $D_{U,z_0'} \in D$ and $\tilde{D}_{U,z_0'} \in \tilde{D}$.

We have studied the function

$$M(z) = M_{z_0}(z, \mathbf{x}) = \sum_{\alpha=1}^{r}(||g_z^{-1}E_\alpha||_0^2 + ||g_z^{-1}F_\alpha||_0^2)$$

in Chapter 9, where $g_z z_0 = z$. By equation (9.15), we know that

$$M(z) = \sum_{\alpha=1}^{r}(||g_z^{-1}v_\alpha||_0^2 + ||g_z^{-1}v_{\alpha+n}||_0^2)$$

Recall that we have defined function $h : \tilde{D} \to \mathbb{R}$ by $h = M \circ \exp_{z_0} \circ i$ where $i : N_{z_0}\tilde{D}_{U,z_0'} \to T_{z_0}\tilde{D}$. Let j be the injection $D \to \tilde{D}$ and $j_* : N_{z_0}D_{U,z_0'} \to N_{z_0}\tilde{D}_{U,z_0'}$ be the induced map. Then we know that

$$\exp_{\tilde{D},z_0}(j_*(X)) = j(\exp_{D,z_0}(X)) = \exp(X)z_0, \forall X \in N_{z_0}D_{U,z_0'},$$

where $\exp_{\tilde{D},z_0}$ and $\exp_{D,z_0}$ are exponential map in the corresponding symmetric spaces and exp is the exponential map on the group $\mathrm{U}(n,n)$. We then have

Lemma 10.3. *The function $h \circ j_*$ satisfies condition (2) of theorem 10.1.*

Proof. By Proposition 10.1, $h(z)$ obtains its minimal value when $z = z_0$ for $z \in \tilde{D}$. As $D \subset \tilde{D}$. $h|_D$ also obtains its minimal value at z_0. If we suppose $X \in N_{z_0}D_{\mathbf{x},z_0}$ with $||X|| = 1$. We define $h_X : \mathbb{R} \to \mathbb{R}$ by

$$h_X(t) = h(j_*(Xt)).$$

Then as in the proof of Lemma 10.1, we know that $\frac{d^2}{dt^2}h_{\tau,X}(t)$ is bounded below uniformly by Lemma 9.7. This suggest that z_0 is actually the unique minimal point of $h \circ j_*$ on $F_{z_0}D_{\mathbf{x},z_0'}$. It also implies that $h \circ j_*$ satisfies condition (2) of theorem 10.1 as in the proof of Proposition 10.1. □

It is then easy to see that the Hession matrix A of $h \circ exp_{z_0} \circ i$ at $0 \in N_{z_0}D_{\mathbf{x},z_0}$ is a $2nr - r^2 + r$ by $2nr - r^2 + r$ positive definite diagonal matrix with determinant

$$det(A) = 4^{2nr - r^2}.$$

Recall that

$$J(t) = \kappa(g', \mathbf{x}) = \int_{F_{z_0}D_{\mathbf{x},z_0'}} \tilde{\varphi}(z, g', \mathbf{x}) = \int_{N_{z_0}D_{\mathbf{x},z_0'}} i^* \circ exp_{z_0}^*(\tilde{\varphi}(Y, g', \mathbf{x}))$$

for $Y \in N_{z_0}D_{\mathbf{x},z_0'}$ and $g' = (m'(a(t)), 1)$.

Theorem 10.3.

$$J(t) \sim 2^{2nr-\frac{5}{2}r^2+\frac{7}{2}r}\pi^{3nr-\frac{5}{2}r^2+\frac{5}{2}r}t^{n-r^2+r}\exp(-rt^2).$$

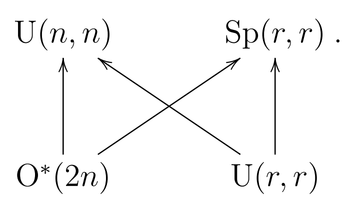
Proof. Apply lemma 6.5, lemma 7.3 and theorem 10.1. The details are similar to that of theorem 10.2. □

10.3 The method of Laplace for the $(\mathrm{O}^*(2n), \mathrm{Sp}(r,r))$ Case

We want to apply theorem 10.1 to compute

$$J(t) = \int_{N_{z_0}, D_{\mathbf{x}, z_0'}} \exp^*_{z_0}(\tilde{\varphi}_\infty(Y, g', \mathbf{x}))$$

in case of the dual pair $(\mathrm{O}^*(2n), \mathrm{Sp}(r,r))$. We need to check conditions (1),(2),(3) of theorem 10.1. Again we make use of the seesaw dual pair



(10.6)

to reduce a substantial part of the problem to the unitary dual pair case.. We use the assumptions and notations of Section 6.5.

Let

$$D = \mathrm{O}^*(2n, \mathbb{R})/\mathrm{U}(n), \quad \tilde{D} = \mathrm{U}(r,r)/(\mathrm{U}(r) \times \mathrm{U}(r)$$

We assume that

$$\mathbf{x} = (v_1, \ldots, v_r),\ z_0 = \mathrm{span}_{\mathbb{C}}\{v_1, \ldots, v_n\}.$$

Recall that we have defined function $M : \tilde{D} \to \mathbb{R}$ by

$$M(z) = M_{z_0}(z, \mathbf{x}) = \sum_{j=1}^{r} \|g_z^{-1} v_j\|_{z_0}^2.$$

Let $i : N_{z_0} \to T_{z_0} D$. We then define $h = M \circ exp_{z_0} \circ i$.

Lemma 10.4. *The function h satisfies condition (2) of theorem 10.1.*

Proof. Similar to that of Lemma 10.3 or Proposition 10.1. □

It is then easy to see that the Hession matrix A of $h \circ exp_{z_0} \circ i$ at $0 \in N_{z_0} D_{\mathbf{x}, z_0}$ is a $2nr - r^2 - r$ by $2nr - r^2 - r$ positive definite diagonal matrix with determinant

$$det(A) = 4^{2nr - r^2 - r}.$$

In this case we have

Theorem 10.4.

$$J(t) \sim 2^{2nr - \frac{5}{2}r^2 - \frac{5}{2}r} \pi^{3nr - \frac{5}{2}r^2 - \frac{5}{2}r} t^{n - r^2 - r} \exp(-rt^2).$$

Proof. Apply lemma 6.5, lemma 7.5 and theorem 10.1. The details are similar to that of theorem 10.2. □

Chapter 11: The associated vector bundle $\mathcal{E}$ on $\Gamma'\backslash D'$

Recall that $\tilde{G}'_{v_1}$ ($\tilde{K}'_{v_1}$ resp.)is the metaplectic cover of G'_{v_1} (K'_{v_1} resp.) which is the preimage of G'_{v_1} (K'_{v_1} resp.) under the map $\mathrm{Mp}((V^m)_{v_1}) \to \mathrm{Sp}((V^m)_{v_1})$, and

$$\tilde{G}'_\infty = \tilde{G}'_{v_1} \times \prod_{v \neq v_1} G'_v, \ \tilde{K}'_\infty = \tilde{K}'_{v_1} \times \prod_{v \neq v_1} K'_v.$$

$\tilde{G}'$ acts on $\mathcal{S}(V^n_\infty)$ by the oscillator(Weil) representation ω and the action commutes with that of G. In this appendix, we show that $\theta_{\mathcal{L},\tilde{\varphi}}$ is a matrix coefficient of an automorphic vector bundle

$$\mathcal{E} \to \Gamma'\backslash\tilde{G}'_\infty/\tilde{K}'_\infty,$$

where $\tilde{K}'_\infty$ is the maximal compact subgroup of $\tilde{G}'_\infty$ that fixes the vacuum vector $\prod_v \varphi_v$ where φ_v is the Gaussian function of $\mathcal{S}(V_v)$. Let K'_∞ be the image of $\tilde{K}'_\infty$ in G'_∞ under natural projection. Recall that we have

1. $G'_\infty = \mathrm{U}(r+s, r+s)^\ell$
2. $G'_\infty = \mathrm{O}(2r, 2r)^\ell$
3. $G'_\infty = \mathrm{Sp}(r, r)^\ell$

for case (1), (2) and (3) respectively.

We will compute the $\tilde{K}'_\infty$ action on

$$\varphi_\infty = \varphi \otimes \prod_{v \neq v_1} \varphi_v$$

where φ is the special class constructed in Chapter 7. Since $\tilde{K}'_\infty = \prod_v \tilde{K}'_v$ and $\tilde{K}'_v$ acts on φ_v trivially for $v \neq v_1$, it suffices to compute the $\tilde{K}'_{v_1}$ action on φ. It turns out that often times $\tilde{K}'_{v_1}$ is the trivial two-fold cover of K'_{v_1} and the action descends to K'_{v_1}. If this is the case, we actually compute the action of K'_{v_1}. The following general argument applies to both $\tilde{K}'_{v_1}$ and K'_{v_1} representation so we just deal with the $\tilde{K}'_{v_1}$ case for brevity.

We will show that φ is a highest weight vector of an irreducible representation of $\tilde{K}'_{v_1}$. We denote this representation by σ. To be more precise there is an irreducible representation (E_σ, σ) of $\tilde{K}'_\infty$ inside $\mathcal{W}$ such that

$$\omega(k')\varphi = \sigma(k')\varphi, \forall k' \in \tilde{K}'_\infty$$

Hence

$$\omega(g'k')\phi = \omega(g')(\sigma(k')\varphi), \tag{11.1}$$

for all $g' \in \tilde{G}'_\infty$, $k' \in \tilde{K}'_\infty$ and $\phi \in E$.

Let $E^*_\sigma = \mathrm{Hom}(E_\sigma, \mathbb{C})$ be the dual representation of E. There is a canonical element $\Phi \in E_\sigma \otimes E^*_\sigma$ which corresponds to the identity element in $E_\sigma \otimes E^*_\sigma \cong \mathrm{Hom}(E_\sigma, E_\sigma)$. Explicitly we choose a basis $\{\varphi_1, \ldots, \varphi_d\}$ of E_σ and we assume $\varphi_1 = \varphi$, the special class defined in Theorem 1.1. Let $\{e_1, \ldots, e_d\}$ be the corresponding dual basis of E_σ^*. Then we have

$$\Phi = \sum_{i=1}^{d} \varphi_i \otimes e_i.$$

By definition the diagonal action of $\tilde{G}'_\infty$ on Φ leaves it invariant:

$$(\sigma \otimes \sigma^*)(k')\Phi = \Phi,\ k' \in \tilde{K}'_\infty \tag{11.2}$$

Equivalently,

$$(\sigma \otimes Id)(k')\Phi = (Id \otimes \sigma^*)((k')^{-1})\Phi,\ k' \in \tilde{K}'_\infty \tag{11.3}$$

where Id stands for the trivial action. For each $\mathbf{x} \in \mathcal{L}^n$,

$$\Phi(g', \mathbf{x}) = \omega(g')(\Phi)(\mathbf{x})$$

is a function on $\tilde{G}'_\infty$ with values in E_σ^*.

Any $\phi \in \mathcal{W}$ defines a function on $\tilde{G}'_\infty$ with values in $\mathcal{W}$ by associating $g' \in \tilde{G}'_\infty$ to $\omega(g')\phi$. We use $R(g')$ to denote the right translation action of $g' \in G'$ on the functions with values in a vector space. Equation (11.1) and equation (11.3) together imply that

$$R(k')\Phi = (Id \otimes \sigma^*)((k')^{-1})\Phi. \tag{11.4}$$

In other words, Φ is a section of the associated vector bundle $\mathcal{E} \to \tilde{G}'_\infty/\tilde{K}'_\infty$ to the representation (E_σ^*, σ^*) of $\tilde{K}'_\infty$.

We now apply θ-distribution to get

$$\theta_{\mathcal{L},\Phi}(g') = \sum_{\mathbf{x}\in\mathcal{L}^n} (\omega(g') \otimes Id)(\Phi)(\mathbf{x}).$$

There is a subgroup $\Gamma' \in G'$ such that its pullback in $\tilde{G}'$ is the trivial double cover and

$$\theta_{\mathcal{L},\Phi}(\gamma' g') = \theta_{\mathcal{L},\Phi}(g'), \gamma' \in \Gamma'.$$

By equation (11.4) and the Γ'-invariance of $\theta_{\mathcal{L},\Phi}$, $\theta_{\mathcal{L},\Phi}$ is a section of the bundle $\Gamma'\backslash\mathcal{E} \to \Gamma'\backslash\tilde{G}'_\infty/\tilde{K}'_\infty$. The cocycle $\theta_{\mathcal{L},\varphi}$ is then a matrix coefficient of $\theta_{\mathcal{L},\Phi}$:

$$\theta_{\mathcal{L},\varphi}(g') = \langle \theta_{\mathcal{L},\Phi}(g'), \varphi \rangle,$$

where $\langle , \rangle$ is an $\tilde{K}'_\infty$-invariant bilinear pairing between E_σ and E^*_σ.

The group $\tilde{G}'_\infty$ can be decomposed as $G' = P'\tilde{K}'_\infty$. And recall that $P' = N'A'M'$ (Langlands decomposition). We can define

$$\kappa(\Phi, g', \beta) = \int_{F_{\mathbf{x},z'}} \Phi(g', \mathbf{x}),$$

it is a section of $\mathcal{E} \to G'/\tilde{K}'_\infty$. Then we have

$$\kappa(g', \beta) = \langle \kappa(\Phi, g', \beta), \varphi \rangle.$$

Now we proceed in cases. In case (1) we compute the $(\tilde{K}'_{v_1})^0$ (the identity component of $\tilde{K}'_{v_1}$) action by computing highest weights. In case (2) and (3) $\tilde{K}'_{v_1}$ is the trivial two-fold cover of K'_{v_1} and the action descends to K'_{v_1}, so we compute the K'_{v_1} action. The results are essentially the same with those of [KV]. We proceed by cases.

11.1 Case (1)

In this case $G'_{v_1} \cong \mathrm{U}(m,m)$ where $m = r+s$. Recall in Section 6.3 we choose a basis $\{w_1, \ldots, w_m, w_{m+1}, \ldots, w_{m+m'}\}$ of W_{v_1} with the Hermitian form $<,>$ such that

1. $< w_a, w_a >= 1$

2. $< w_k, w_k >= -1$

for $1 \leq a \leq m, m+1 \leq k \leq m+m'$ and $< w_j, w_k >= 0$ if $j \neq k$. Under this basis $\mathrm{U}(m) \times \mathrm{U}(m)$ is the block diagonal matrices

$$\Big\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} |{}^*AA = Id, {}^*BB = Id \Big\}.$$

For simplicity we denote by $\mathrm{U}(m,0)$ the subgroup that is identity on the lower right block and by $\mathrm{U}(0,m)$ the subgroup that is identity on the upper left block.

Let $\mathfrak{k}'$ be the Lie algebra of $\mathrm{U}(m,m)$. Then $\mathfrak{k}' \otimes \mathbb{C}$ acts via the oscillator representation by

$$\omega(w_a \otimes w_b^*) = -\sum_{\alpha=1}^{p} u_{\alpha,b}\frac{\partial}{\partial u_{\alpha,a}} + \sum_{\mu=p+1}^{p+q} u_{\mu,a}\frac{\partial}{\partial u_{\mu,b}} - \frac{1}{2}\delta_{ab}(p-q)$$

for $1 \leq a, b \leq m$ and

$$\omega(w_k \otimes w_\ell^*) = \sum_{\alpha=1}^{p} u_{\alpha,k}\frac{\partial}{\partial u_{\alpha,\ell}} - \sum_{\mu=p+1}^{p+q} u_{\mu,\ell}\frac{\partial}{\partial u_{\mu,k}} + \frac{1}{2}\delta_{ab}(p-q) \tag{11.5}$$

for $m+1 \leq k, \ell \leq 2m$.

Let $\mathfrak{t}$ be the diagonal torus of $\mathfrak{u}(0,m)$, $\mathfrak{n}$ be the strictly upper triangular Lie algebra of $\mathfrak{u}(0,m)$:

$$\mathfrak{n} = \{w_k \otimes w_\ell^* | m+1 \leq k < \ell \leq 2m\}$$

Recall that in Section 7 we define the element φ_+ using the special harmonic $f_+^{q-s} f_-^{p-r}$. Here f_+ and f_- are as in definition 7.1 except that we shift b to $b+m$ where b is the second index of the variable u_{ab}. This is because in definition 7.1 our assumption is that W is negative definite (hence $K' = \mathrm{U}(0,m)$). So now $\mathrm{U}(m,0)$ acts trivially on $f_+^{q-s} f_-^{p-r}$.

Using equation (11.5), it is easy to see that $f_+^{q-s} f_-^{p-r}$ has weight

$$(\underbrace{q-s+\frac{1}{2}(p-q),\ldots,q-s+\frac{1}{2}(p-q)}_{r},\underbrace{-(p-r)+\frac{1}{2}(p-q),\ldots,-(p-r)+\frac{1}{2}(p-q)}_{s})$$

$$= (\underbrace{-s+\frac{1}{2}(p+q),\ldots,-s+\frac{1}{2}(p+q)}_{r},\underbrace{r-\frac{1}{2}(p+q),\ldots,r-\frac{1}{2}(p+q)}_{s})$$

under $\mathfrak{k}$. Moreover we can show that $f_+^{q-s} f_-^{p-r}$ is killed by $\mathfrak{n}$. We have three cases

1. $m+1 \leq k < \ell \leq m+r$

2. $m+1 \leq k \leq m+r$,$m+r+1 \leq \ell \leq m+r+s$

3. $m+r+1 \leq k < \ell \leq m+r+s$.

In case (1), $w_k \otimes w_\ell^*$ replaces a column of f_+ by an existing column and acts trivially on all the variables in f_-. In case (2), $w_k \otimes w_\ell^*$ acts trivially on all the variables in f_+ and f_-. In case (3), $w_k \otimes w_\ell^*$ acts trivially in all the variables in f_+ and replaces a column of f_- by an existing column. In any case $w_k \otimes w_\ell^*$ kills $f_+^{q-s} f_-^{p-r}$.

The conclusion is that $f_+^{q-s} f_-^{p-r}$ hence φ_+ is a highest weight vector of $\mathfrak{u}(0, m)$. Similarly φ_- is a highest weight vector of $\mathfrak{u}(m, 0)$ with weight

$$= (\underbrace{s - \frac{1}{2}(p+q), \ldots, s - \frac{1}{2}(p+q)}_{r}, \underbrace{-r + \frac{1}{2}(p+q), \ldots, -r + \frac{1}{2}(p+q)}_{s})$$

Hence $\varphi = \varphi_+ \wedge \varphi_-$ is a highest weight vector of weight

$$(\underbrace{s - \frac{1}{2}(p+q), \ldots,}_{r} \underbrace{-r + \frac{1}{2}(p+q), \ldots,}_{s} \underbrace{-s + \frac{1}{2}(p+q), \ldots,}_{r} \underbrace{r - \frac{1}{2}(p+q), \ldots}_{s})$$

under the action of the connected component of $\mathrm{U}\widetilde{(m, m)}$.

11.2 Case (2)

In this case $G'_{v_1} = \mathrm{O}(2r, 2r)$.

Let us use the notation of Section 6.4. Recall that W is a complex vector space with a Hermitian form $<,>$ of signature (r,r). We denote by $W_{\mathbb{R}}$ the underlying real vector space of W and let $<,>_{\mathbb{R}}= \mathrm{Re} <,>$. G'_{v_1} is the linear isometry group of $(W_{\mathbb{R}}, <,>_{\mathbb{R}})$. We can choose an orthonormal basis $\{w_a, w_k | 1 \leq a \leq r, r+1 \leq k \leq 2r\}$ of W such that

$$< w_a, w_a >= 1, < w_k, w_k >= -1$$

for $1 \leq a \leq r, r+1 \leq k \leq 2r$. Let $\mathrm{O}(2r,0)$ be the linear isometry group of the subspace W_+ spanned by the first r basis vectors, and $\mathrm{O}(0,2r)$ be the linear isometry group of the subspace W_- spanned by the last r basis vectors. Then $K'_{v_1} = \mathrm{O}(0,2r) \times \mathrm{O}(2r,0)$ is a maximal compact subgroup of G'_{v_1}. Let $\mathfrak{k}'$ be its Lie algebra. Then

$$\mathfrak{k}' \cong \wedge^2(W_+) \oplus \wedge^2(W_-) = \mathfrak{sp}(r,0) \oplus \mathfrak{sp}(0,r).$$

First we focus on $\mathrm{O}(0,2r)$. Define

$$w'_k = w_k + i w_k i$$

$$w''_k = w_k - i w_k i,$$

where $r+1 \leq k \leq 2r$. Notice that if we extend the form $(,)_{\mathbb{R}}$ complex linearly to a symmetric form on $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Then

$$(w'_k, w''_\ell)_{\mathbb{R}} = -2\delta_{k\ell}$$

and the inner product between any other vectors among these are zero. Then we have a split torus $\mathfrak{t}$ of $\mathfrak{o}(0, 2r) \otimes \mathbb{C}$ spanned by

$$\{w'_k \wedge w''_k = w'_k \otimes (w'_k)^* - w''_k \otimes (w''_k)^* | r+1 \leq k \leq 2r\}.$$

We also have a nilpotent algebra $\mathfrak{n}$ of $\mathfrak{o}(0, 2r) \otimes \mathbb{C}$:

$$\mathfrak{n} = \operatorname{span}\{w''_k \wedge w''_\ell | r+1 \leq k, \ell \leq 2r\} \oplus \operatorname{span}\{w'_k \wedge w''_\ell | r+1 \leq k < \ell \leq 2r\}$$

The Lie algebra $\mathfrak{o}(0, 2r) \times \mathbb{C}$ acts by the Weil representation in the following way:

$$\omega(w'_k \wedge w'_\ell) = 2 \sum_{\alpha=1}^{n} (u_{\alpha+n,k} \frac{\partial}{\partial u_{\alpha,\ell}} - u_{\alpha+n,\ell} \frac{\partial}{\partial u_{\alpha,k}}),$$

$$\omega(w''_k \wedge w''_\ell) = 2 \sum_{\alpha=1}^{n} (u_{\alpha,k} \frac{\partial}{\partial u_{\alpha+n,\ell}} - u_{\alpha,\ell} \frac{\partial}{\partial u_{\alpha+n,k}}),$$

$$\omega(2w'_k \otimes (w'_\ell)^* - 2w''_\ell \otimes (w''_k)^*) = \omega(w'_k \wedge w''_\ell) = 2 \sum_{\alpha=1}^{n} (u_{\alpha+n,k} \frac{\partial}{\partial u_{\alpha+n,\ell}} - u_{\alpha,\ell} \frac{\partial}{\partial u_{\alpha,k}}),$$

Recall that in Chapter 7 we define the element φ_+ using the special harmonic f_+^{n-r+1}. Here f_+ is as in definition 7.1 except that we shift b to $b+r$ where b is the second index of the variable u_{ab}. This is because in definition 7.1 our assumption is that W is negative definite (hence $K' = \mathrm{O}(0, 2r)$). So now $\mathrm{O}(2r, 0)$ acts trivially on f_+^{n-r+1}.

Moreover it is easy to see that f_+^{n-r+1} is killed by $\mathfrak{n}$ and has weight

$$(\underbrace{-(n-r+1), \ldots, -(n-r+1)}_{r})$$

under $\mathfrak{t}$. So f_+^{n-r+1} hence φ_+ is a lowest weight vector of $\mathfrak{so}(0, 2r)$. The corresponding highest weight is

$$(\underbrace{(n-r+1), \ldots, (n-r+1)}_{r}).$$

Under the group $\mathrm{O}(0, 2r)$, f_+^{n-r+1} generates an irreducible representation that splits into two irreducible representations of $\mathfrak{so}(0, 2r)$ with highest weights

$$(\underbrace{(n-r+1), \ldots, (n-r+1)}_{r})$$

and

$$(\underbrace{(n-r+1), \ldots, (n-r+1)}_{r-1}, -(n-r+1)).$$

Let us denote this representation by E

Similarly φ_- is a highest weight vector of $\mathfrak{o}(2r, 0)$ with weight

$$(\underbrace{(n-r+1), \ldots, (n-r+1)}_{r})$$

and it generates the same representation E as before under the group action of $\mathrm{O}(2r, 0)$.

11.3 Case (3)

In this case $G'_{v_1} = \mathrm{Sp}(r,r)$.

Let us use the notation of Section 6.4. Recall that $W_{\mathbb{H}}$ is a $\mathbb{H}$-vector space with a Hermitian form $<,>_{\mathbb{H}}$ of signature (r,r). $G'_{v_1} \cong$ is the linear isometry group of $(W_{\mathbb{H}}, <,>_{\mathbb{H}})$. We can choose an orthonormal basis $\{w_a, w_k | 1 \leq a \leq r, r+1 \leq k \leq 2r\}$ of $W_{\mathbb{H}}$ such that

$$< w_a, w_a >_{\mathbb{H}} = 1, < w_k, w_k >_{\mathbb{H}} = -1$$

for $1 \leq a \leq r, r+1 \leq k \leq 2r$. Let $\mathrm{Sp}(r,0)$ be the linear isometry group of the subspace W_+ spanned by the first r basis vectors, and $\mathrm{Sp}(0,r)$ be the linear isometry group of the subspace W_- spanned by the last r basis vectors. Then $K'_{v_1} = \mathrm{Sp}(0,r) \times \mathrm{Sp}(r,0)$ is a maximal compact subgroup of G'_{v_1}. Let $\mathfrak{k}'$ be its Lie algebra. Then

$$\mathfrak{k}' \cong \wedge^2(W_+) \oplus \wedge^2(W_-) = \mathfrak{sp}(r,0) \oplus \mathfrak{sp}(0,r).$$

First we focus on $\mathrm{Sp}(0,r)$. We have a split torus $\mathfrak{t}$ of $\mathfrak{sp}(0,r) \otimes \mathbb{C}$ spanned by

$$\{w_k \wedge w_k - i w_k \wedge w_k i = 2 w_k \otimes w_k^* | r+1 \leq k \leq 2r\}.$$

We also have a nilpotent algebra $\mathfrak{n}$ of $\mathfrak{o}(0,2r) \otimes \mathbb{C}$:

$$\mathfrak{n} = \mathrm{span}\{w_k \wedge_{\mathbb{H}} w_\ell - i w_k \wedge_{\mathbb{H}} w_\ell i | r+1 \leq k < \ell \leq 2r\} \oplus \mathrm{span}\{w_k \wedge_{\mathbb{H}} j w_\ell - i w_k \wedge_{\mathbb{H}} j w_\ell i | r+1 \leq k, \ell \leq 2r\}$$

The Lie algebra $\mathfrak{sp}(0,r)$ acts by the Weil representation in the following way:

$$\omega(w_k \wedge_{\mathbb{H}} w_\ell - iw_k \wedge_{\mathbb{H}} w_\ell i) = 2\sum_{\alpha=1}^{n}(u_{\alpha,k}\frac{\partial}{\partial u_{\alpha,\ell}} - u_{\alpha+n,\ell}\frac{\partial}{\partial u_{\alpha+n,k}}),$$

$$\omega(w_k \wedge_{\mathbb{H}} jw_\ell - iw_k \wedge_{\mathbb{H}} jv_\ell i) = 2\sum_{\alpha=1}^{n}(u_{\alpha,k}\frac{\partial}{\partial u_{\alpha+n,\ell}} + u_{\alpha,\ell}\frac{\partial}{\partial u_{\alpha+n,k}}),$$

$$\omega(w_k \wedge_{\mathbb{H}} jw_\ell + iw_k \wedge_{\mathbb{H}} jw_\ell i) = -2\sum_{\alpha=1}^{n}(u_{\alpha+n,\ell}\frac{\partial}{\partial u_{\alpha,k}} + u_{\alpha+n,k}\frac{\partial}{\partial u_{\alpha,\ell}}).$$

Recall that in Chapter 7 we define the element φ_+ using the special harmonic f_+^{n-r-1}. Here f_+ is as in definition 7.1 except that we shift b to $b+r$ where b is the second index of the variable u_{ab}. This is because in definition 7.1 our assumption is that W is negative definite (hence $K' = \mathrm{Sp}(0,r)$). So now $\mathrm{Sp}(r,0)$ acts trivially on f_+^{n-r-1}.

Moreover it is easy to see that f_+^{n-r-1} is killed by $\mathfrak{n}$ and has weight

$$(\underbrace{-(n-r-1),\ldots,-(n-r-1)}_{r})$$

under $\mathfrak{t}$. So f_+^{n-r-1} hence φ_+ is a lowest weight vector of $\mathfrak{sp}(0,r)$. And the corresponding highest weight is

$$(\underbrace{(n-r-1),\ldots,(n-r-1)}_{r}).$$

Similarly φ_- is a highest weight vector of $\mathfrak{sp}(r,0)$ with weight

$$(\underbrace{(n-r-1),\ldots,(n-r-1)}_{r}).$$

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