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Fan-Type Conditions for Collapsible Graphs

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ABSTRACT. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph of G whose set of odd degree vertices is R. A graph is *supereulerian* if it contains a spanning closed trail. It is known that every collapsible graph is supereulerian. A graph G of order n is said to satisfy a Fan-type condition if $\max\{d(u), d(v)\} \geq n/(g-2)p - \epsilon$ for each pair of vertices u, v at distance two, where $g \in \{3, 4\}$ is the girth of G, and $p \geq 2$ and $\epsilon \geq 0$ are fixed numbers. In this paper, we study the Fan-type conditions for collapsible graphs and supereulerian graphs.

1 Introduction

We follow the notation of Bondy and Murty [2], except that graphs have no loops. Let G be a graph. A cycle of order n is denoted by C_n. The distance, denoted $dist(u,v)$, between two vertices u and v of a connected graph is the minimum length of all paths joining u and v. For a graph G, let u be a vertex in G. Define $N_G(u) = \{v \in V(G) \mid uv \in E(G)\}$. A graph G is called hamiltonian if G has a cycle containing every vertex of G. Let $\kappa'(G)$ denote the edge-connectivity of G, and let $O(G)$ denote the set of vertices of odd degree in G. A graph G is eulerian if it is connected with $O(G) = \emptyset$. A graph G is called supereulerian if it has a spanning eulerian subgraph. A graph G is called collapsible if for every even set $X \subseteq V(G)$ there is a spanning connected subgraph H_X of G, such that $O(H_X) = X$. Examples of collapsible graphs include K_n with $n \geq 3$, C_2, and C_3, but not C_t with $t \geq 4$. It is known that all collapsible graphs are supereulerian (see [3],[4]). The trivial graph K_1 is both supereulerian and collapsible, and is

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regarded as having infinite edge-connectivity. The line graph of G, denoted by $L(G)$, has $E(G)$ as its set of vertices, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. Harary and Nash-Williams [13] showed that if G has at least 3 edges, then $L(G)$ is hamiltonian if and only if G has an eulerian subgraph that contains at least one end of every edge in G.

Various sufficient degree conditions for the existence of spanning eulerian subgraphs and hamiltonian line graphs have been derived.

Theorem A. (Lesniak–Foster and Williamson [15]). *Let G be a graph of order $n \geq 6$. If $\delta(G) \geq 2$ and if any pair u, v of non-adjacent vertices of G, $d(u) + d(v) \geq n - 1$ then G is supereulerian.* □

Theorem B. (Catlin [5]). *Let G be a simple graph of order n with edge-connectivity $\kappa'(G) = k \in \{2, 3\}$. If n is sufficiently large and if any pair u, v of non-a vertices of G, $d(u)+d(v) > \frac{2n}{(k-1)5} - 2$, then G is supereulerian.* □

Theorem C. (Chen, Lai [10], and Veldman [16]). *Let G be a 3-edge-connected simple graph of order n. If n is sufficiently large and if for every edge $uv \in E(G)$, $d(u) + d(v) > \frac{n}{5} - 2$, then G is supereulerian.* □

Theorem D. (Lai [14]). *Let G be a 2-edge-connected simple graph of order n. If for every edge $uv \in E(G)$, $\max\{d(u), d(v)\} \geq n/5 - 1$, then for n large, $L(G)$ is hamiltonian except for a class of well characterized graphs.* □

In the study of hamiltonian graphs, Fan [11] proved the following:

Theorem E. *If G is 2-connected simple graph of order n and $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v with $dist(u, v) = 2$ in G, then G is hamiltonian.* □

A simple graph G of order n is said to satisfy a Fan-type condition if for every pair of vertices u, v with $dist(u, v) = 2$ in G

$$\max\{d(u), d(v)\} \geq \frac{n}{(g-2)p} - \epsilon, \tag{1}$$

where g is the girth of G, $p \geq 2$ and $\epsilon > 0$ are fixed numbers.

Note that it is easy to show that for a simple graph G of order n, if $\max\{d(u), d(v)\} \geq n/m$ for every pair of distance-2 vertices u, v in G, where m is a fixed number, and if n is large, then G has girth at most 4. Thus, we only consider $g \in \{3, 4\}$ in (1).

In this paper, we shall use Catlin's reduction technique [3] to study the Fan-type conditions for collapsible graphs and supereulerian graphs.

Catlin's reduction technique. In section 3, we study the structures of the reduction graphs of graphs satisfying (1). Our main results are presented in section 4.

2 Catlin's reduction technique

Let G be a graph, and let H be a connected subgraph of G. The contraction G/H is the graph obtained from G by contracting all edges of H, and by deleting any resulting loops. In [3], Catlin showed that every graph G has a unique collection of vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$ such that $\bigcup_{i=1}^{c} V(H_i) = V(G)$. The reduction of G, denoted by G', is the graph obtained from G by successively contracting $H_1, H_2, \cdots, H_c$. Since $V(G) = \bigcup_{i=1}^{c} V(H_i)$, $|V(G')| = c$. Let v be the vertex in G' that is the contraction image of a subgraph $H(v)$. Then $H(v)$ is called the preimage of v. If $H(v) = K_1$, we say $H(v)$ is the trivial preimage of v and we call v a trivial vertex. A graph G is reduced if $G = G'$. Note that a reduced graph is a simple and K_3-free graph, and every subgraph of a reduced graph is also reduced [3]. By the definition of contraction and $\kappa'(K_1) = \infty$, we have $\kappa'(G') \geq \kappa'(G)$. It is known that the smallest 2-edge-connected reduced non-supereulerian graph is $K_{2,3}$, and the smallest 3-edge-connected reduced non-supereulerian graph is the Petersen graph [9].

Throughout this paper, we let $d(v)$ and $d'(v)$ denote the degree of v in G and G', respectively. Let $E(G')$ be the edge set of G'. We regard $E(G')$ as a subset of $E(G)$. Note that $E(G') = E(G) - \bigcup_1^c E(H_i)$. For a vertex $v \in V(G)$, we define

$$I(v) = \{u \in V(G') |\, uv \in E(G')\}.$$

Define $i(v) = |I(v)|$, which is the number of edges in G' incident with v in G. Note that since each edge $uv \in E(G')$ is also an edge in $E(G)$, we also view as a subset of $V(G)$. We define $N_{G-I(v)}(v) = N_G(v) - I(v)$, where $v \in V(G)$. Thus, for any $v \in V(G)$,

$$d(v) = i(v) + |N_{G-I(v)}(v)|. \tag{2}$$

As an example, consider a graph G_0 of order $n \geq 15$ obtained from $K_{2,3}$ by replacing each vertex of $K_{2,3}$ by a complete graph of order $n/5$ in the way as shown in Figure 1 below (where each circle in G_0 represent a $K_{n/5}$ subgraph). Then G_0 has five vertex-disjoint maximal collapsible subgraphs $H_i = K_{n/5}$ $(1 \leq i \leq 5)$, and so G_0', the reduction of G_0, is $K_{2,3}$, and $E(G_0') = \{a, b, c, d, e, f\} = E(G_0) - \bigcup_1^5 E(H_i)$. For a vertex $v \in G_0$, if v is not incident with any edges in $\{a, b, c, d, e, f\}$, then $I(v) = \emptyset$. For instance, from the figure below, we can see that $i(v_0) = 0$, but $i(v_1) = i(v_2) = 1$, and $i(v_3) = 3$.

$G_0 \qquad\qquad G_0' = K_{2,3}$

Figure 1

We shall make use of the following theorem:

Theorem F. (Catlin [3],[6]). *Let G be a connected graph and let G' be the reduction of G.*

(a) *Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if $G' = K_1$.*

(b) *G is supereulerian if and only if G' is supereulerian.*

(c) *G' is a simple graph with $\delta(G') \leq 3$, and G' contains no K_3 or $K_{3,3}-e$ as a subgraph, and either $G' \in \{K_1, K_2\}$ or*

$$|E(G')| \leq 2|V(G')| - 4.$$

□

3 The Fan-Type Conditions and the Reduction of a Graph

Let G be a graph satisfying Fan-type condition (1). Let G' be the reduction of G. Define

$$\Pi(G') = \{v' \in V(G') | \text{there is a } x \in V(H(v')) \text{ such that } d(x) \geq \frac{n}{(g-2)p} - \epsilon\},$$

and define $\Gamma(G') = V(G') - \Pi(G')$.

Lemma 1. *Let G be a k-edge-connected simple graph of order n satisfying (1) with girth $g \in \{3, 4\}$ and $k \in \{2, 3\}$. Let G' be the reduction of G. Let $V(G') = \{v_1', v_2', \cdots, v_c'\}$. Let $H(v_i')$ be the preimage of v_i' in G ($1 \leq i \leq c$).*

(a). *If $v_i' \in \Pi(G')$, then there is a $u_i \in V(H(v_i'))$ such that*

$$\frac{n}{(g-2)p} - \epsilon \leq d(u_i) \leq i(u_i) + \frac{|V(H(v_i'))| + g - 4}{g-2}.$$

(b). *If $v_i' \in \Gamma(G')$ then there is a $u_i \in V(H(v_i'))$ such that*

$$k \le d(u_i) \le i(u_i) + \frac{|V(H(v_i'))| + g - 4}{g-2}.$$

Proof: The lemma is trivial if $g = 3$. Thus, we may assume $g = 4$. If $|V(H(v_i'))| = 1$ then (a) and (b) are trivially true. In the following, we assume that $|V(H(v_i'))| > 1$. Then since $H(v_i')$ is K_3-free and collapsible, $|V(H(v_i'))| \ge 6$ and $|E(H(v_i'))| > 1$.

Case 1. $v_i' \in \Pi(G')$.

Let x be a vertex in $V(H(v_i'))$ such that

$$\frac{n}{(g-2)p} - \epsilon \le d(x) = |N_G(x)|. \tag{3}$$

Since $\kappa'(G) \ge 2$ and $|V(H(v_i'))| \ge 6$, $|N_{G-I(x)}(x)| \ge 2$. Let y and z be two vertices in $N_{G-I(x)}(x)$ with $d(y) \ge d(z)$. Since G is K_3-free, yxz is a length-2 path in G. Therefore, by (1),

$$\frac{n}{(g-2)p} - \epsilon \le d(y). \tag{4}$$

Since G is K_3-free, $N_{G-I(x)}(x) \cap N_{G-I(y)}(y) = \emptyset$. Therefore,

$$|N_{G-I(x)}(x)| + |N_{G-I(y)}(y)| \le |V(H(v_i'))|, \tag{5}$$

$$\min\{|N_{G-I(x)}(x)|, |N_{G-I(y)}(y)|\} \le \frac{|V(H(v_i'))|}{2} = \frac{|V(H(v_i'))| + g - 4}{g-2}.$$

Hence, Lemma 1(a) follows from (3), (4), and (5).

Case 2. $v_i' \in \Gamma(G')$.

Since $|E(H(v_i'))| \ge 1$, $H(v_i')$ contains an edge (say $e = xy$). Since G is K_3-free, $N_{G-I(x)}(x) \cap N_{G-I(y)}(y) = \emptyset$. Then (5) still holds in this case. Since G is k-edge-connected, $d(x) \ge k$ and $d(y) \ge k$, and so by (2) and (5), Lemma 1(b) holds. □

Lemma 2. *Let G be a graph satisfying (1). Let G' be the reduction of G. Let $v' \in \Gamma(G')$. Then all vertices in $N_{G'}(v')$ except for at most one are in $\Pi(G')$.*

Proof: Let $r = d'(v')$. Let $N_{G'}(v') = \{x_1', x_2', \cdots, x_r'\}$. Let $H(v')$ be the preimage of v' in G. Let $H(x_i')$ be the preimages of x_i' in G. Let $v_i x_i$ be the corresponding edge of $v' x_i'$ in G.

Case 1. $|V(H(v'))| = 1$.

In this case, $v_i = v_j$ $(1 \le i, j \le r)$, and so $V(H(v')) = \{v'\}$. Then $x_i v' x_j$ $(i \ne j)$ is a length-2 path in G. Suppose that $d(x_1) < n/(g-2)p - \epsilon$. By (1),

$d(x_i) \geq n/(g-2)p - \epsilon$ for all $2 \leq i \leq r$. This shows that $\{x'_2, x'_3, \cdots, x'_r\} \subseteq \Pi(G')$. Thus, the lemma holds.

Case 2. $|V(H(v'))| > 1$.

Since $|V(H(v'))| > 1$ and $V(H(v')) \cap V(H(x_i)) = \emptyset$, for each $v_i x_i$, there exists a vertex y_i in $V(H(v'))$ such that $y_i v_i x_i$ is a length-2 path in G. Since $v' \notin \Pi(G')$, $d(y_i) < n/(g-2)p - \epsilon$. By (1), $d(x_i) \geq n/(g-2)p - \epsilon$ for all $1 \leq i \leq r$. This shows that $\{x'_1, x'_2, \cdots, x'_r\} \subseteq \Pi(G')$. The proof is complete. □

Lemma 3. *Let G be a 2-edge-connected simple graph. For any $v \in V(G)$, $i(v) \leq |\Pi(G')|$.*

Proof: Let $r = i(v)$. Let $I(v) = \{x'_1, x'_2, \cdots, x'_r\}$. Let H be the maximal collapsible subgraph of G containing v, and let v' be the contraction image of H. Then $I(v) \subseteq N_{G'}(v')$. Let $H(x'_i)$ be the preimage of x'_i. For each x'_i, there is a vertex x_i in $H(x'_i)$ such that vx_i is the corresponding edge of $v'x'_i$ in G. Since $V(H(x_i)) \cap V(H(x_j)) = \emptyset$ $(i \neq j)$, $x_i x_j \notin E(G)$, and so $x_i v x_j$ is a length-2 path in G. By (1), all vertices in $I(v)$ except for at most one are in $\Pi(G')$. If one of the vertices in $I(v)$ is not in $\Pi(G')$, say x'_1, then since $\kappa'(G') \geq \kappa'(G) \geq 2$, $|N_{G'}(x'_1) - \{v'\}| > 0$. Let y' be a vertex in $N_{G'}(x'_1) - \{v'\}$. Since G' is K_3-free, $y' \notin I(v)$. Let y be the vertex in $H(y')$ such that $x_1 y$ is the corresponding edge of $x'_1 y'$ in G. Then vx_1y is a length-2 path in G. By (1), either $v' \in \Pi(G')$ or $y' \in \Pi(G')$. We may assume that $y' \in \Pi(G')$. Summing up above, we have

$$\text{either } I(v) \subseteq \Pi(G') \text{ or } (I(v) - \{x'_1\}) \cup \{y'\} \subseteq \Pi(G').$$

Thus, $i(v) \leq |\Pi(G')|$. □

In the following, we use an abbreviated notation $n >> p$ to say that n is sufficiently large relative to p.

Lemma 4. *Let G be a k-edge-connected simple noncollapsible graph of order n with girth $g \in \{3, 4\}$ and $k \in \{2, 3\}$. Let G' be the reduction of G. If G satisfies (1) and if $n >> p \geq 2$ and $n >> \epsilon$, then*

$$2 \leq |\Pi(G')| \leq p. \tag{6}$$

Proof: The lower bound is trivial. In the following, we shall prove $|\Pi(G')| \leq p$ only. Let $c = |V(G')|$ and $t = |\Pi(G')|$. Then $|\Gamma(G')| = c - t$. By Lemma 1, for each vertex $u' \in \Pi(G')$, there is a vertex u in $V(H(u'))$ such that

$$\frac{n}{(g-2)p} - \epsilon \leq d(u) \leq i(u) + \frac{|V(H(u'))| + g - 4}{(g-2)},$$

and so

$$\frac{n}{(g-2)p} + \frac{4-g}{(g-2)} - \epsilon \leq i(u) + \frac{|V(H(u'))|}{g-2}. \tag{7}$$

Then by (7) and $t = |\Pi(G')|$,

$$t\{\frac{n}{(g-2)p} + \frac{(4-g)}{(g-2)p} - \epsilon\} \le \sum_{u' \in \Pi(G')} i(u) + \frac{\sum_{u' \in \Pi(G')} |V(H(u'))|}{(g-2)}. \quad (8)$$

For each $v' \in \Gamma(G')$, by Lemma 1, there is a vertex v in $V(H(v'))$ such that

$$k \le d(v) \le i(v) + \frac{|V(H(v'))| + g - 4}{g-2},$$

and so

$$k + \frac{4-g}{g-2} \le i(v) + \frac{|V(H(v'))|}{g-2}.$$

Then since $|\Gamma(G')| = c - t$,

$$\{k + \frac{4-g}{g-2}\}(c-t) \le \sum_{v' \in \Gamma(G')} i(v) + \frac{\sum_{v' \in \Gamma(G')} |V(H(v'))|}{(g-2)}. \quad (9)$$

By (8), and (9), and $V(G') = \Pi(G') \cup \Gamma(G')$,

$$\{k + \frac{4-g}{g-2}\}(c-t) + t\{\frac{n}{(g-2)p} + \frac{4-g}{g-2} - \epsilon\}$$
$$\le \sum_{v' \in V(G')} i(v) + \frac{\sum_{v' \in V(G')} |V(H(v'))|}{g-2}.$$
$$\{k + \frac{4-g}{g-2}\}c + t\{\frac{n}{(g-2)p} - (k+\epsilon)\}$$
$$\le \sum_{v' \in V(G')} i(v) + \frac{\sum_{v' \in V(G')} |V(H(v'))|}{g-2}. \quad (10)$$

Note that $\sum_{v' \in V(G')} i(v) \le \sum_{v' \in V(G')} d'(v') = 2|E(G')|$ and $\sum_{v' \in V(G')} |V(H(v'))| = n$. By Theorem F(c), $|E(G')| \le 2c - 4$. By (10),

$$\{(g-2)k + 4 - g\}c + t\{\frac{n}{p} - (g-2)(k+\epsilon)\} \le (g-2)(4c-8) + n,$$
$$t\{\frac{n}{p} - (g-2)(k+\epsilon)\} \le \{(g-2)(4-k) - 4 + g\}c - 8(g-2) + n. \quad (11)$$

Since $c \le n$, by (11)

$$t\{\frac{n}{p} - (g-2)(k+\epsilon)\} \le \{(g-2)(4-k) - 3 + g\}n - 8(g-2). \quad (12)$$

Since $g \in \{3,4\}$ and $k \in \{2,3\}$,

$$(g-2)(4-k) - 3 + g \leq 5. \tag{13}$$

By (12) and (13),

$$t\{\frac{n}{p} - (g-2)(k+\epsilon)\} \leq 5n - 8(g-2),$$
$$t \leq 5p + \frac{5p^2(g-2)(k+\epsilon) - 8(g-2)p}{n-(g-2)(k+\epsilon)p}.$$

Therefore, when $n >> p$ and $n >> \epsilon$,

$$t \leq 5p. \tag{14}$$

By Lemma 3, $i(v) \leq |\Pi(G')| = t \leq 5p$. By (8), (14) and $\sum_{v\in\Pi(G')} |V(H(v))| \leq n$,

$$t(n + (4-g)p - (g-2)p\epsilon) \leq (g-2)p \sum_{v\in\Pi(G')} i(v) + np,$$
$$t(n + (4-g)p - (g-2)p\epsilon) \leq (g-2)5p^2 t + np,$$
$$t\{n + (4-g)p - (g-2)p\epsilon - 5(g-2)p^2\} \leq np,$$
$$t \leq p + \frac{p\{(g-2)p(\epsilon+5p) - (4-g)p\}}{n - \{(g-2)p\epsilon + 5(g-2)p^2 - (4-g)p\}}.$$

Therefore, $t \leq p$ for $n >> p$ and $n >> \epsilon$. The proof is complete. □

Remark. One can check that Lemma 4 holds if n satisfies the following:

$$n > \begin{cases} (3+\epsilon)(p+1)p - 8p & \text{if } k=3 \text{ and } g=3, \\ \max\{2(3+\epsilon)p(3p+1) - 16p, 2p(p+1)(\epsilon+3p)\} & \text{if } k=3 \text{ and } g=4, \\ \max\{(2+\epsilon)p(2p+1) - 8p, p(p+1)(2p+\epsilon-1)\} & \text{if } k=2 \text{ and } g=3, \\ \max\{2(2+\epsilon)p(5p+1) - 16p, 2p(p+1)(5p+\epsilon)\} & \text{if } k=2 \text{ and } g=4. \end{cases}$$

However, these bounds are not the best possible.

Corollary 1. *Let G be a graph satisfying the conditions of Lemma 4. Let $v \in \Pi(G')$. Let $H(v)$ be the preimage of v in G. Then*

$$|V(H(v))| \geq \frac{n}{p} + (4-g) - (g-2)(\epsilon + p). \tag{15}$$

Furthermore, if $|\Pi(G')| = p$ then

$$|\Gamma(G')| \leq p\{(g-2)(\epsilon+p) + g - 4\}. \tag{16}$$

Proof: The inequality (15) follows from Lemmas 3 and 4, and (7) in the proof of Lemma 4. For the inequality (16), if $|\Pi(G')| = p$, then by (15)

$$\begin{aligned} n = \sum_{v\in V(G')} |V(H(v))| &= \sum_{v\in\Gamma(G')} |V(H(v))| + \sum_{v\in\Pi(G')} |V(H(v))| \\ &\geq |\Gamma(G')| + p\{\frac{n}{p} + (4-g) - (g-2)(\epsilon+p)\} \end{aligned}$$

Thus, $|\Gamma(G')| \leq p\{(g-2)(\epsilon+p) + g - 4\}$. □

Lemma 5. *Let G be a graph satisfying the conditions in Lemma 4 Let G' be the reduction of G. Suppose that $|\Pi(G')| = p$ and $n >> p$ and $n >> e$. Let $v' \in \Pi(G')$.*

(a). *If $g = 3$ and $N_{G'}(v') \cap \Gamma(G') \neq \emptyset$, then*

$$|V(H(v'))| \geq \frac{n}{p} + 1 - \epsilon. \tag{17}$$

(b). *If $g = 4$, then*

$$|V(H(v'))| \geq \frac{n}{p} - 2\epsilon. \tag{18}$$

Furthermore, if $|V(H(v'))| = \frac{n}{p} - 2\epsilon$, then $H(v')$ is a bipartite graph with bipartition $V(H(v')) = X \cup Y$ such that $|X| = |Y| = \frac{n}{2p} - \epsilon$.

Proof: Since $|\Pi(G')| = p$, by Corollary 1,

$$|V(G')| = |\Pi(G')| + |\Gamma(G')| \leq p + p\{(g-2)(\epsilon+p) + g - 4\}, \tag{19}$$

and so

$$d'(v') \leq |V(G')| - 1 \leq p + p\{(g-2)(\epsilon+p) + g - 4\} - 1. \tag{20}$$

There are two cases. Let us consider the case when $g = 3$ first.

Case 1. $g = 3$.

Let $x' \in N_{G'}(v') \cap \Gamma(G')$. Let xv be the edge in G corresponding to $x'v'$ in $E(G')$. Then $d(x) < n/p - \epsilon$ since $x' \notin \Pi(G')$. Let $\mathcal{A} = N_{G-I(v)}(v)$. Then $\mathcal{A} \subseteq V(H(v')) - I(v)$. Note that since G' is simple and K_3-free, for any $z \in \mathcal{A}$, $zx \notin E(G)$, and so zvx is a length-2 path in G.

Subase 1: There is a vertex y in $\mathcal{A}$ such that $i(y) = 0$.

Since yvx is a length-2 path in G and since $d(x) < n/p - \epsilon$, by (1), $i(y) = 0$ and $y \in \mathcal{A} \subseteq V(H(v'))$,

$$\frac{n}{p} - \epsilon \leq d(y) \leq |V(H(v'))| - 1.$$

Inequality (17) follows in this case.

Subcase 2. Every vertex y in $\mathcal{A}$ has $i(y) \neq 0$.

By the definition of reduction, each vertex in $N_{G'}(v')$ can only be adjacent to one vertex in $V(H(v'))$, and so in this case $|\mathcal{A}| \leq |N_{G'}(v')| = d'(v')$. By Lemmas 3 and 4, $i(v) \leq |\Pi(G')| \leq p$. Thus, by (2) and (20) with $g = 3$, and $n >> p$ and $n >> \epsilon$,

$$d(v) = i(v) + |N_{G-I(v)}(v)| = i(v) + |\mathcal{A}| \leq |\Pi(G')| + d'(v') < \frac{n}{p} - \epsilon. \qquad (21)$$

Note that for any $z \in \mathcal{A}$, zvx is a length-2 path in G. By (1), since $d(x) < n/p - \epsilon$,

$$|N_G(z)| = d(z) \geq \frac{n}{p} - \epsilon.$$

By (19) and $g = 3$, $|V(G')| \leq p(\epsilon+p)$. When n is large, $|N_G(z)| \geq n/p - \epsilon > |V(G')| > d'(v) \geq |\mathcal{A}|$, and so there exists a vertex u in $N_G(z) - \mathcal{A}$ such that $i(u) = 0$. Since $uz \in E(G)$, $z \in V(H(v'))$, and $i(u) = 0$, $u \in V(H(v'))$. Therefore, since $u \notin \mathcal{A}$ and $i(u) = 0$, $uv \notin E(G)$, and so vzu is a length-2 path in G. Thus, (21), (1) with $g = 3$, and $i(u) = 0$ implies that

$$\frac{n}{p} - \epsilon \leq d(u) \leq i(u) + |V(H(v'))| - 1 = |V(H(v'))| - 1.$$

This proves the lemma for $g = 3$. Next we consider the case when $g = 4$.

Case 2. $g = 4$.

Since $v' \in \Pi(G')$, there is a vertex (say y) in $V(H(v'))$ such that

$$|N_G(y)| = d(y) \geq \frac{n}{2p} - \epsilon.$$

If $i(y) = 0$, then $|N_{G-I(y)}(y)| = |N_G(y)| \geq n/2p - \epsilon > d'(v') + p$. If $i(y) \geq 1$, then since $i(y) \leq p$, $|N_{G-I(y)}(y)| = |N_G(y)| - i(y) \geq n/2p - \epsilon - p > d'(v')$. Note that each vertex in $N_{G'}(v')$ can be adjacent to only one vertex in $H(v')$. This shows that not matter whether y is adjacent to a vertex in $N_{G'}(v')$ or not, there are two vertices (say x_1, x_2) in $N_{G-I(y)}(y)$ such that $i(x_1) = i(x_2) = 0$. Since G is K_3-free, $x_1x_2 \notin E(G)$, and so x_1yx_2 is a length-2 path in G. We may assume that $d(x_1) \geq d(x_2)$. By (1) and $i(x_1) = 0$,

$$|N_G(x_1)| = d(x_1) \geq \frac{n}{2p} - \epsilon, \qquad (22)$$

$$N_G(x_1) \subseteq V(H(v')). \qquad (23)$$

By (22) and (23), similar to the argument above, there are two vertices (say z_1, z_2) in $N_G(x_1)$ such that $i(z_1) = i(z_2) = 0$ and $d(z_1) \geq d(z_2)$. Thus, by (1) and $i(z_1) = 0$,

$$|N_G(z_1)| = d(z_1) \geq \frac{n}{2p} - \epsilon, \tag{24}$$

$$N_G(z_1) \subseteq V(H(v')). \tag{25}$$

Since G is K_3-free, and $x_1z_1 \in E(G)$, $N_G(x_1) \cap N_G(z_1) = \emptyset$ Therefore, by (22), (23), (24) and (25),

$$|V(H(v'))| \geq |N_G(x_1)| + |N_G(z_1)| \geq 2(\frac{n}{2p} - \epsilon) = \frac{n}{p} - 2\epsilon. \tag{26}$$

Furthermore, if $|V(H(v'))| = n/p - 2\epsilon$, then (26) holds with equality, and so $V(H(v')) = N_G(x_1) \cup N_G(z_1)$ with $|N_G(x_1)| = |N_G(z_1)| = n/2p - \epsilon$. The proof is complete. □

4 The Fan-Type Conditions for Collapsible Graphs

We are now ready to prove our main results.

Theorem 1. *Let G be a simple 2-edge-connected graph of order n satisfying (1) with girth $g = 3$, $p = 2$, and $\epsilon = 2$. If n is sufficiently large ($n > 30$) then either G is collapsible, or G can be contracted to C_4 (4-cycle) in such a way that the preimages of two vertices in C_4 are K_r, or $K_r - e$, where $r = n/2 - 1$.*

Proof: Let G' be the reduction of G. Suppose that G is not collapsible. Then $c = |V(G')| \geq 4$. By Lemma 4, $t = |\Pi(G')| = 2$. Let $\Pi(G') = \{v'_1, v'_2\}$. Let $H(v'_i)$ be the preimage of v'_i in G, $(1 \leq i \leq 2)$. Since $d'(v'_i) \geq 2$ and $|\Pi(G')| = 2$, $N_{G'}(v'_i) \cap \Gamma(G') \neq \emptyset$. Therefore, by Lemma 5(a) with $g = 3$, $p = \epsilon = 2$,

$$|V(H(v'_i))| \geq \frac{n}{2} - 1.$$

Therefore,

$$n \geq |\Gamma(G')| + \sum_{i=1}^{2} |V(H(v'_i))| \geq (c-2) + 2(\frac{n}{2} - 1) = c - 4 + n. \tag{27}$$

Since $c \geq 4$, (27) holds with equality, and so $c = |V(G')| = 4$ and $|V(H(v'_i))| = n/2 - 1$ $(1 \leq i \leq 2)$. This shows that $G' = C_4$. By (1) with $p = \epsilon = 2$ and $g = 3$, $H(v'_i) \in \{K_{n/2-1}, K_{n/2-1} - e\}$. The proof is complete. □

Theorem 2. *Let G be a simple 2-edge-connected and K_3-free ($g=4$) graph of order n satisfying (1) with $p=2$ and $\epsilon=1/2$. If n is sufficiently large ($n>126$), then either G is collapsible, or G can be contracted to C_4 in such a way that the preimages of two vertices in C_4 are $K_{s,s}$ or $K_{s,s}-e$, where $s=(n-2)/4$.*

Proof: Since G is K_3-free, G has girth $g=4$. Let G' be the reduction of G. Suppose that G is not collapsible. Then $c=|V(G')|\geq 4$. By Lemma 4, $|\Pi(G')|=2$. Let $\Pi(G')=\{v_1',v_2'\}$. Let $H(v_i')$ be the preimage of v_i' in G, $(1\leq i\leq 2)$. Since $d'(v_i')\geq 2$ and $|\Pi(G')|=2$, $N_{G'}(v_i')\cap\Gamma(G')\neq\emptyset$. By Lemma 5(b) with $g=4$, $p=2$ and $\epsilon=1/2$,

$$|V(H(v_i'))|\geq\frac{n}{2}-1. \tag{28}$$

Therefore,

$$n\geq|\Gamma(G')|+\sum_{i=1}^{2}|V(H(v_i'))|\geq c-2+2(\frac{n}{2}-1)=c-4+n. \tag{29}$$

Hence, $c=|V(G')|=4$, and so $G'=C_4$. Then (28) and (29) hold with equality. By Lemma 5(b), $H(v_i')$ is a bipartite graph with bipartition $V(H(v_i'))=X\cup Y$ and $|X|=|Y|=(n-2)/4$. By (1) with $p=2$ and $\epsilon=1/2$, this forces that $H(v_i')\in\{K_{s,s},K_{s,s}-e\}$, where $s=(n-2)/4$. The proof is complete. □

Since collapsible graphs and C_4 are supereulerian, by Theorem F(b), we have

Corollary 2. *A graph satisfying Theorem 1 or Theorem 2 is supereulerian.* □

Remark. Let $V(K_{2,3})=\{x,y,v_1,v_2,v_3\}$, where $d(x)=d(y)=3$ and $d(v_i)=2$. Let G be a graph obtained from $K_{2,3}$ by replacing vertices x and y by $H(x)=K_{(n-3)/2}$ and $H(y)=K_{(n-3)/2}$ (or $K_{s,s}$ where $s=(n-3)/4$), respectively, and replacing each path xv_iy by a path $x_iv_iy_i$ such that $x_i\neq x_j$ and $y_i\neq y_j$ ($i\neq j$), where $x_i\in V(H(x))$ and $y_i\in V(H(y))$. Then G is a 2-edge-connected simple (or K_3-free) graph of order n such that $\max\{d(u),d(v)\}\geq(n-5)/2$ (or $\max\{d(u),d(v)\}\geq(n-3)/4$) for each pair of distance-2 vertices u, v in G. The reduction of G is the non-supereulerian graph $K_{2,3}$. Thus, Theorem 1, Theorem 2, and Corollary 2 are best possible.

For 3-edge-connected graphs, we have the following:

Theorem 3. *Let G be a 3-edge-connected simple graph of order n satisfying (1) with $p=4$ and $\epsilon=9/4$. If n is sufficiently large, then G is collapsible.*

To prove Theorem 3, the following lemma is needed.

Lemma 6. (Chen [9]). *Let G be a 3-edge-connected simple graph on $n \leq 11$ vertices. Then either G is collapsible or G is the Petersen graph.* □

Proof of Theorem 3: Let G' be the reduction of G. Suppose that G is not collapsible. Then by Lemma 4, $t = |\Pi(G')| \leq 4$, and by Lemma 6,

$$c = |V(G')| \geq 10. \tag{30}$$

Claim 1. $t = |\Pi(G')| = 4$.

By contradiction, suppose that $t \leq 3$. Then by (30) $|\Gamma(G')| = c - t \geq 7$. Note that G is 3-edge-connected, and so is G'. Hence, $d'(u) \geq 3$ for every $u \in V(G')$. By Lemma 2, for every $u \in \Gamma(G')$, all neighbors of u except for most one are in $\Pi(G')$. Since $|\Gamma(G')| \geq 7$, and $d'(u) \geq 3$ for every $u \in \Gamma(G')$, and $V(G') = \Pi(G') \cup \Gamma(G')$, by observation, one can find that G' contains either a K_3 or a $K_{3,3} - e$ as a subgraph, and so by Theorem F(c) G' is not a reduced graph, a contradiction. Claim 1 is proved.

Claim 2. Let $v' \in \Pi(G')$. Then $N_{G'}(v') \cap \Gamma(G') \neq \emptyset$.

By contradiction, suppose that $N_{G'}(v') \subseteq \Pi(G') - \{v'\}$. Then by Claim 1, $|N_{G'}(v')| \leq t - 1 = 3$. Since G' is 3-edge-connected, $|N_{G'}(v')| = 3$. Let $S = V(G') - (N_{G'}(v') \cup \{v'\})$. Since $|\Pi(G')| = 4$, $S = \Gamma(G')$ and $|S| = c - 4 \geq 6$. Note that for any $u \in S = \Gamma(G')$, $uv' \notin E(G')$. Therefore, by Lemma 2, all neighbors of u except for at most one are in $\Pi(G') - \{v'\} = N_{G'}(v')$. Since $|N_{G'}(v')| = 3$ and $|S| \geq 6$, G' contains either a K_3 or a $K_{3,3} - e$ as a subgraph, contrary to that G' is a reduced graph. Claim 2 is proved.

By Claims 1 and 2, Lemma 5(a) can be applied, and so $|V(H(v))| \geq (n-5)/4$ for any $v \in \Pi(G')$. Therefore,

$$n \geq c - t + \sum_{v \in \Pi(G')} |V(H(v))| \geq c - 4 + 4(\frac{n-5}{4}) = c - 9 + n,$$

and so $c \leq 9$. This is contrary to (30). The proof is complete. □

Theorem 4. *Let G be a 3-edge-connected simple and K_3-free ($g = 4$) graph of order n satisfying (1) with $p = 4$ and $\epsilon = 5/8$. Then G is collapsible.*

Proof: By way of contradiction, suppose that G is not collapsible. Then by Lemma 6, $c = |V(G')| \geq 10$. Similar to the proof in Theorem 3, $t = |\Pi(G')| = 4$ still holds.

By Lemma 5(b), $|V(H(v))| \geq (n-5)/4$ for every $v \in \Pi(G')$. Therefore, since $|\Pi(G')| = 4$,

$$n \geq c - t + \sum_{v \in \Pi(G')} |V(H(v))| \geq c - 4 + 4(\frac{n-5}{4}) = c - 9 + n,$$

and so $c \leq 9$. This is contrary to $c \geq 10$. The proof is complete. $\square$

Remark. Let G be a graph obtained from the Petersen graph P by replacing four vertices of P by $H_i = K_r$ $(1 \leq i \leq 4)$ where $r = (n-6)/4$ (or $H_i = K_{s,s}$ if $g = 4$ where $s = (n-6)/8$) in an appropriate way (see Figure 2). Therefore, G is a 3-edge-connected simple (K_3-free if $g = 4$) graph of order n such that $\max\{d(u), d(v)\} \geq (n-10)/4$ (or $\max\{d(u), d(v)\} \geq (n-6)/8$ if $g = 4$) for every pair of distance-2 vertices u, v in G. However, the reduction of G is the Petersen graph. This shows that Theorem 3 and Theorem 4 are best possible.

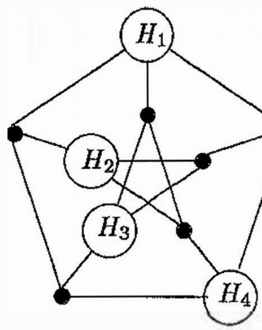

Figure 2

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