9-2015

Properties of Catlin's reduced graphs and supereulerian graphs

Wei-Guo Chen

Zhi-Hong Chen
Butler University, chen@butler.edu

Mei Lu

Properties of Catlin's reduced graphs and supereulerian graphs

Wei-Guo, Chen, Guangdong Economic Information Center
Guangzhou, P. R. China
Zhi-Hong Chen*, Butler University
Indianapolis, IN 46208, USA.
Mei Lu, Tsinghua University
Beijing, P. R. China

Abstract

A graph G is called collapsible if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph H of G such that R is the set of vertices of odd degree in H. A graph is the reduction of G if it is obtained from G by contracting all the nontrivial collapsible subgraphs. A graph is reduced if it has no nontrivial collapsible subgraphs. In this paper, we first prove a few results on the properties of reduced graphs. As an application, for 3-edge-connected graphs G of order n with $d(u) + d(v) \geq 2(n/p - 1)$ for any $uv \in E(G)$ where $p > 0$ are given, we show how such graphs change if they have no spanning Eulerian subgraphs when p is increased from $p = 1$ to 10 then to 15.

1. Introduction

We shall use the notation of Bondy and Murty [4], except when otherwise stated. Graphs considered in this paper are finite and loopless, but multiple edges are allowed. The graph of order 2 and size 2 is called a 2-cycle and denoted by C_2. As in [4], $\kappa'(G)$ and $d_G(v)$ (or $d(v)$) denote the edge-connectivity of G and the degree of a vertex v in G, respectively. The size of a maximum matching in G is denoted by $\alpha'(G)$. A connected graph G is *Eulerian* if the degree of each vertex in G is even. An Eulerian subgraph H of G is called a *spanning Eulerian subgraph* if $V(G) = V(H)$ and is called a *dominating Eulerian subgraph* if $E(G - V(H)) = \emptyset$. A graph is *supereulerian* if it contains a spanning Eulerian subgraph. The family of supereulerian graphs is denoted by $\mathcal{SL}$.

*Email: chen@butler.edu

Let $O(G)$ be the set of vertices of odd degree in G. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph H_R of G with $O(H_R) = R$. $K_{3,3}-e$ and K_n $(n \geq 3)$ are collapsible [6]. K_1 is regarded as collapsible and supereulerian, and having $\kappa'(K_1) = \infty$. The family of collapsible graphs is denoted by $\mathcal{CL}$. Thus, $\mathcal{CL} \subset \mathcal{SL}$.

Throughout this paper, we use P for the Petersen graph and use P_{14} and P_{16} for the graphs defined in Figure 1.1.

(a) (b)

Figure 1.1

Like the study of many NP-complete problems in graph theory, various degree conditions for the existence of spanning and dominating Eulerian subgraphs in graphs have been derived (e.g, see [1, 5, 6, 8, 14, 15, 23, 22, 25]). For a graph G, we define

$\delta(G) = \min\{d(v) \mid v \in V(G)\}$;
$\sigma_2(G) = \min\{d(u) + d(v) \mid uv \notin E(G)\}$;
$\sigma_t(G) = \min\{\Sigma_{i=1}^{t} d(v_i) \mid \{v_1, v_2, \cdots, v_t\}$ is independent in G $(t \geq 2)\}$;
$\delta_F(G) = \min\{\max\{d(u), d(v)\} \mid \forall u, v \in V(G)$ with $dist(u, v) = 2\}$;
$\overline{\sigma}_2(G) = \min\{d(u) + d(v) \mid$ for every edge $uv \in E(G)\}$;
$\delta_L(G) = \min\{\max\{d_G(u), d_G(v)\} \mid$ for every edge $uv \in E(G)\}$.

These are all the degree parameters we know that have been studied by many for problems on spanning and dominating Eulerian subgraphs in graphs. In the following, we let

$$\Omega(G) = \{\delta(G), \sigma_2(G), \sigma_t(G), \delta_F(G), \overline{\sigma}_2(G), \delta_L(G)\}.$$

A powerful tool to work on spanning and dominating Eulerian subgraphs is Catlin's reduction method [6]. This reduction method has been applied to solve problems in Hamiltonian cycles in claw-free graphs [21], hamiltonian line graphs, a certain type of double cycle cover [9] and the total interval number of a graph [10], and others [11].

Catlin's reduction method

For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$ and use v_H for the vertex in G/H to which H is contracted. A contraction G/H is called a trivial contraction if $H = K_1$.

Catlin [6] showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$ such that $V(G) = \cup_{i=1}^{c} V(H_i)$. The contraction of G obtained from G by contracting each H_i into a single vertex v_i $(1 \le i \le c)$ is called the *reduction* of G and denoted by G'. For a vertex $v \in V(G')$, there is a unique maximal collapsible subgraph in G, denoted by $H(v)$, such that v is the contraction image of $H(v)$. We call $H(v)$ the *preimage* of v. A graph G is *reduced* if $G = G'$. By the definition of contraction, we have $\kappa'(G') \ge \kappa'(G)$. If the reduction of a graph G_A is a graph G_B, we said that graph G_A can be reduced to graph G_B.

The main theorem of Catlin's reduction method is the following:
Theorem A (Catlin [6]). Let G be a graph, and let G' be the reduction of G. Let H be a collapsible subgraph of G. Then each of the following holds:

(a) $G \in \mathcal{CL}$ if and only if $G/H \in \mathcal{CL}$. In particular, $G \in \mathcal{CL}$ if and only if $G' = K_1$.

(b) $G \in \mathcal{SL}$ if and only if $G/H \in \mathcal{SL}$. In particular, $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.

With Theorem A, we can see that to determine if a graph is supereulerian can be reduced to a problem of the reduction of the graph. For instance, by combining the prior results in [8, 14, 15, 19] and the results proved recently in [17, 18], we have:
Theorem B. Let G be a 3-edge-connected graph of order n. Let $p > 1$ and ϵ be given numbers. Let $D(G) \in \Omega(G)$. If $D(G) \ge \frac{n}{p} - \epsilon$, then when n is large, either $G \in \mathcal{SL}$ or G' has order at most cp where c is a constant.

To be more specific, let $D(G) = \delta_F(G)$, we have
Theorem C (W. Chen and Z. Chen [17]). Let G be a 3-edge-connected graph of order n with girth $g \in \{3, 4\}$. Let G' be the reduction of G. If $\delta_F(G) > \frac{n}{(g-2)p} - \epsilon$ where $p \ge 2$ and $\epsilon > 0$ are fixed and n is large, then either $G \in \mathcal{SL}$ or $G' \ne K_1$ has order at most $5(p-2)$.

For $D(G) = \overline{\sigma}_2(G)$, we have
Theorem D (Chen and Lai [14, 19]). Let $p > 0$ be an integer. Let G be a 3-edge-connected simple graph of order n. Let G' be the reduction of G. If $n \ge 12p(p-1)$ and $\overline{\sigma}_2(G) \ge \frac{2n}{p} - 2$, then either $G \in \mathcal{SL}$ or $G' \ne K_1$ with $\alpha'(G') \le p/2$ and $|V(G')| \le 3p/2 - 4$. $\square$

With Theorems B, C and D, the problem to determine if a graph G with $D(G) \ge \frac{n}{p} - \epsilon$ is in $\mathcal{SL}$ can be reduced to the problem of a finite number of reduced graphs. The main challenge to solve such problems become solving the problems of reduced graphs.

In this paper, we first prove some results on the properties and structures of reduced graphs. Then as an application, we prove a result on $\overline{\sigma}_2(G) \geq \frac{2n}{p} - 2$ conditions for 3-edge-connected graphs. Combining prior results on $\overline{\sigma}_2(G)$ conditions, it reveals how such graphs are change from supereulerian to graphs that can be reduced to the Petersen graph and then to graphs that can be reduced to P_{14} when p is increased from 1 to 10 then to 15.

2. Prior theorems on Catlin's reduction and π-reduction methods

For a graph G, let $F(G)$ be the minimum number of extra edges that must be added to G, to obtain a spanning supergraph having two edge-disjoint spanning trees.

Theorem E. Let G be a connected reduced graph. Then each of the following holds:

(a) [6] G is simple and K_3-free with $\delta(G) \leq 3$. Any subgraph H of G is reduced.

(b) [7] $F(G) = 2|V(G)| - |E(G)| - 2$.

(c) [12] If $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t}(t \geq 1)\}$.

(d) [19] If $\delta(G) \geq 3$, then $\alpha'(G) \geq (|V(G)| + 4)/3$.

For a graph G, define $D_i(G) = \{v \in V(G) \mid d(v) = i\}$.

Theorem F (Chen [13, 16]). Let G be a connected simple graph of order n with $\delta(G) \geq 2$. Let G' be the reduction of G. Then each of the following holds:

(a) [13] If $n \leq 7$, $\delta(G) \geq 2$ and $|D_2(G)| \leq 2$, then G is not reduced and $G' \in \{K_1, K_2\}$.

(b) [16] If $\kappa'(G) \geq 3$ and $n \leq 14$, then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$.

(c) [16] If $\kappa'(G) \geq 3$, $n = 15$, $G \notin \mathcal{SL}$ and $G' \notin \{P, P_{14}\}$, then $G = G'$ has girth at least 5 and $V(G) = D_3(G) \cup D_4(G)$ where $D_4(G)$ is an independent set with $|D_4(G)| = 3$.

Catlin's π-reduction method [7]: Let G be a graph containing an induced 4-cycle $uvzwu$ and let $E = \{uv, vz, zw, wu\}$. Denote by G/π the graph obtained from $G - E$ by identifying u and z to form a vertex x, and by identifying v and w to form a vertex y, and by adding an edge $e_\pi = xy$. The way to obtain G/π from G is called *π-reduction method* (Catlin [7]).

Theorem G (Catlin [7]). Let G be a connected graph and let G/π be the graph defined above, then each of the following holds:

(a) If $G/\pi \in \mathcal{CL}$, then $G \in \mathcal{CL}$;

(b) If $G/\pi \in \mathcal{SL}$ then $G \in \mathcal{SL}$. $\square$

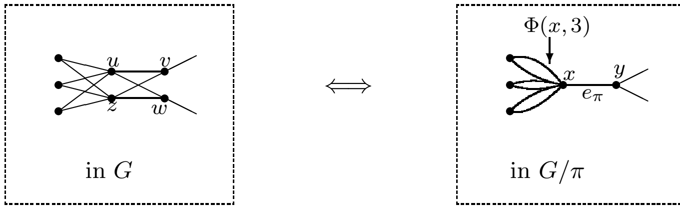


Figure 2.2

Let $\Phi(v,t)$ be the graph obtained from $K_{1,t}$ with center at v by replacing each edge in $K_{1,t}$ by a C_2. Thus, $\Phi(v,t)$ is a graph formed by t C_2s with all the edges incident with v and $|V(\Phi(v,t))| = t+1$ and $|E(\Phi(v,t))| = 2t$. (See $\Phi(x,3)$ in Figure 2.2).

Lemma 2.1. Let G be a connected reduced graph with $\delta(G) \geq 3$. Let $H = uvzwu$ be a 4-cycle in G. Let G/π be the graph defined by π-reduction on G with $e_\pi = xy$. Then G/π has at most two nontrivial collapsible subgraphs. Furthermore, if H_0 is a nontrivial maximal collapsible subgraph of G/π, then $|V(H_0) \cap \{x,y\}| = 1$ and either $H_0 = \Phi(v,t)$ for some $t \geq 1$ ($v \in \{x,y\}$) and $2|V(H_0)| - |E(H_0)| = 2$, or $3 \leq 2|V(H_0)| - |E(H_0)|$. Hence,

$$2 \leq 2|V(H_0)| - |E(H_0)|.$$

Proof. Since G is reduced with $\delta(G) \geq 3$, by Theorem A and Theorem G, $G \neq K_1$ and $(G/\pi)' \neq K_1$. If G/π is not reduced, let H_0 be a nontrivial maximal collapsible subgraph of G/π. If $V(H_0) \cap \{x,y\} = \emptyset$, then H_0 is a nontrivial collapsible subgraph of G, contrary to that G is reduced. If $\{x,y\} \subseteq V(H_0)$, then by Theorem G, $G[E(H) \cup \{uv, vz, zw, wu\}]$ is a nontrivial collapsible subgraph of G, a contradiction again. Thus, any nontrivial maximal collapsible subgraph of G/π must contain one and only one vertex in $\{x,y\}$.

We may assume $x \in V(H_0)$. Then G has a subgraph H_1 with $V(H_1) = (V(H_0) - \{x\}) \cup \{u,z\}$ and $E(H_1) = E(H_0)$. If $H_0 \neq \Phi(x,t)$ ($t \geq 1$), then $H_1 \neq K_{2,t}$. Since H_0 is nontrivial, $H_1 \neq K_2$. By Theorem E(c), $F(H_1) \geq 3$. Then

$$\begin{aligned} 3 \leq F(H_1) &= 2|V(H_1)| - |E(H_1)| - 2 \\ &= 2(|V(H_0)| + 1) - |E(H_0)| - 2 = 2|V(H_0)| - |E(H_0)|. \end{aligned}$$

Lemma 2.1 is proved. □

3. Properties of Catlin's reduced graphs

Catlin had the following conjectures on reduced graphs:

Conjecture A (Conjecture 4 of [9]). A 3-edge-connected nontrivial reduced graph G with $F(G) = 3$ must be the Petersen graph P.

Conjecture B ([10]). A 3-edge-connected simple graph G of order at most 17 is either in $\mathcal{SL}$ or its reduction is in $\{P, P_{14}, P_{16}\}$. Thus, either $G \in \mathcal{SL}$ or G can be contracted to P.

Theorem F(b) indicates that these conjectures are valid for graphs with at most 14 vertices. In this section, we prove some results on certain structure properties of reduced graphs that are related to these conjectures and that will be needed in section 4.

For convenience, for a connected graph G, we define

$$f(G) = 2|V(G)| - |E(G)| - 2.$$

By Theorem E(b), if G is reduced, then $F(G) = f(G)$.

Theorem 3.1. Let G be a connected reduced graph with $F(G) = 3$ and $\delta(G) \geq 3$. If $G \notin \mathcal{SL}$, then G has no 4-cycles.

Proof. By way of contradiction, suppose that G has a 4-cycle $H_0 = uvzwu$. Using π-reduction method, we have G/π from G with $e_\pi = xy$ and

$$|V(G/\pi)| \quad = \quad |V(G)| - 2 \text{ and } |E(G/\pi)| = |E(G)| - 3. \qquad (1)$$

By Theorem G and the definition of G/π, since $G \notin \mathcal{SL}$ with $\delta(G) \geq 3$, $G/\pi \notin \mathcal{SL}$ with $\delta(G/\pi) \geq 3$. By (1) and $F(G) = 3$,

$$\begin{aligned} f(G/\pi) &= 2|V(G/\pi)| - |E(G/\pi)| - 2 \\ &= 2(|V(G)| - 2) - (|E(G)| - 3) - 2 \\ &= 2|V(G)| - |E(G)| - 2 - 1 = F(G) - 1 = 2. \end{aligned} \qquad (2)$$

If G/π is reduced, then by Theorem E(b) and (2), $F(G/\pi) = f(G/\pi) = 2$. By Theorem E(c), $G/\pi \in \{K_1, K_2, K_{2,t}\}$, contrary to $\delta(G/\pi) \geq 3$. Thus, G/π is not reduced.

By Lemma 2.1, we may assume G/π has a maximal collapsible subgraph H_x with $x \in V(H_x)$. By Lemma 2.1,

$$2 \leq 2|V(H_x)| - |E(H_x)|. \qquad (3)$$

Let $G_x = (G/\pi)/H_x$. Since $G/\pi \notin \mathcal{SL}$, by Theorem A, $G_x \neq K_1$. Let v_x be the vertex in G_x obtained from G/π by contracting H_x. Since $\delta(G/\pi) \geq 3$, all the vertices in G_x have degree at least 3 except v_x as the result of contracting $H_x = C_2$. By (2) and (3),

$$\begin{aligned} f(G_x) &= 2|V(G_x)| - |E(G_x)| - 2 \\ &= 2(|V(G/\pi)| - |V(H_x)| + 1) - (|E(G/\pi)| - |E(H_x)|) - 2 \\ &= f(G/\pi) + 2 - (2|V(H_x)| - |E(H_x)|) \leq f(G/\pi) = 2. \end{aligned}$$

If G_x is reduced, then by Theorem E(c) $G_x \in \{K_1, K_2, K_{2,t}\}$, contrary to that all the vertices in G_x except at most one vertex have degree at least 3. Then G_x cannot be reduced.

Let H_y be the another nontrivial maximal collapsible subgraph of G/π. By Lemma 2.1, G/π has at most two nontrivial maximal collapsible subgraphs. Then $G_{xy} = G_x/H_y = ((G/\pi)/H_x)/H_y$ is reduced. Similar to the way of finding $f(G_x) \leq 2$, we have $f(G_{xy}) \leq f(G_x) \leq 2$ and so $F(G_{xy}) = f(G_{xy}) \leq 2$. By Theorem E(c), $G_{xy} \in \{K_1, K_2, K_{2,t}\}$ $(t \geq 1)$.

If $G_{xy} = K_1$, then by Theorem A, $G/\pi \in \mathcal{CL} \subseteq \mathcal{SL}$, contrary to $G/\pi \notin \mathcal{SL}$.

If $G_{xy} = K_2$, then G has two subgraphs H_1 and H_2 such that $\{u, z\} \subseteq V(H_1)$ and $E(H_1) = E(H_x)$ and $V(H_1) = (V(H_x) - \{x\}) \cup \{u, z\}$, and $\{v, w\} \subseteq V(H_2)$ and $E(H_2) = E(H_y)$ and $V(H_2) = (V(H_y) - \{y\}) \cup \{v, w\}$. Therefore, $|E(G)| = |E(H_1)| + |E(H_2)| + |E(H_0)| = |E(H_1)| + |E(H_2)| + 4$ and $|V(G)| = |V(H_1)| + |V(H_2)|$. Then

$$\begin{aligned} F(H_1) + F(H_2) &= (2|V(H_1)| - |E(H_1)| - 2) + (2|V(H_2)| - |E(H_2)| - 2) \\ &= 2(|V(H_1)| + |V(H_2)|) - (|E(H_1)| + |E(H_2)| + 4) \\ &= (2|V(G)| - |E(G)| - 2) + 2 = F(G) + 2 = 5. \end{aligned}$$

We may assume $F(H_1) \leq 2$. Since H_1 is reduced, by Theorem E(c), $H_1 \in \{K_1, K_2, K_{2,t}\}$. Since H_x is a nontrivial maximal collapsible subgraph in G/π and G is reduced, $H_1 \notin \{K_1, K_2\}$. Hence $H_1 = K_{2,t}$. Then H_1 has a degree two vertex $v_0 \notin \{u, z\}$. Then $d_H(v_0) = d_G(v_0) = 2$, contrary to $\delta(G) \geq 3$. Thus, $G_{xy} = K_2$ is impossible.

If $G_{xy} = K_{2,t}$, then since $\delta(G/\pi) \geq 3$ and $K_{2,t}$ $(t \geq 1)$ has at least 3 vertices with degree less than 3, G/π has at least 3 nontrivial maximal collapsible subgraphs, a contradiction. Theorem 3.1 is proved. □

Lemma 3.2. Let G be a connected reduced graph of order n. Let H be a spanning bipartite subgraph of G with bipartition $\{X, Y\}$ where $|Y| \geq |X|$ and $d_H(v) \geq 3$ for any $v \in Y$. If $|X| \leq \frac{n+5}{3}$, then $G = H$ and $F(G) = 3$.
Proof. Since $|Y| \geq |X|$ and $d_H(v) \geq 3$ for any $v \in Y$, $|E(H)| \geq 3|Y|$ and $|X| \geq 3$. Hence $H \notin \{K_1, K_2, K_{2,t}\}$ and so $G \notin \{K_1, K_2, K_{2,t}\}$ $(t \geq 1)$. By Theorem E(c), $F(G) \geq 3$. Since $E(H)$, $E(G[X])$ and $E(G[Y])$ are pairwise disjoint subsets of $E(G)$,

$$\begin{aligned} |E(G)| &\geq |E(H)| + |E(G[X])| + |E(G[Y])| \\ &\geq 3|Y| + |E(G[X])| + |E(G[Y])|. \end{aligned} \tag{4}$$

By Theorem E(b), (4), $|Y| \leq n - |X|$ and $|X| \leq \frac{n+5}{3}$,

$$\begin{aligned} 3 \leq F(G) &= 2|V(G)| - |E(G)| - 2 \\ &\leq 2(|X| + |Y|) - 3|Y| - (|E(G[X])| + |E(G[Y])|) - 2 \end{aligned}$$

$$\begin{aligned} &= 3|X| - n - 2 - (|E(G[X])| + |E(G[Y])|) \\ &\leq 3(\frac{n+5}{3}) - n - 2 - (|E(G[X])| + |E(G[Y])|) \\ &= 3 - (|E(G[X])| + |E(G[Y])|). \end{aligned}$$

Thus, $|E(G[X])| + |E(G[Y])| = 0$ and so $G = H$ and $F(G) = 3$. Lemma 3.2 is proved. $\Box$

Several properties on reduced bipartite graphs are given in the following.
Theorem 3.3. Let G be a 3-edge-connected reduced graph. Let H be a connected reduced bipartite graph with bipartition $\{X, Y\}$ where $|X| \leq 7$, $|Y| \geq |X|$ and $d_H(v) \geq 3$ for any $v \in Y$.

(a) If $|Y| \geq |X|$, then either H has a 4-cycle with a vertex of degree at least 4 in X or $|Y| = |X|$ and $H \in \mathcal{SL}$.

(b) If $|Y| = |X|$ and $|X| \leq 6$, then H has a 4-cycle.

(c) If G has such a bipartite graph H as a spanning subgraph, then $G \in \mathcal{SL}$.

Proof. (a) If $|Y| > |X|$, then since H is a bipartite graph and $d_H(v) \geq 3$ for any $v \in Y$, there is at least one vertex (say x) in X such that $d_H(x) \geq 4$. Let $N_H(x) = \{y_1, y_2, y_3, y_4, \cdots\}$. Since H is a bipartite graph, $\cup_{i=1}^4 N_H(y_i) \subseteq X$. Since $|N_H(y_i)| \geq 3$ $(1 \leq i \leq 4)$ and $|X| \leq 7$, there are at least two vertices (say y_1 and y_2) in $\{y_1, y_2, y_3, y_4\}$ such that $(N_H(y_1) - \{x\}) \cap (N_H(y_2) - \{x\}) \neq \emptyset$. Let x_1 be a vertex in $(N_H(y_1) - \{x\}) \cap (N_H(y_2) - \{x\})$. Then $xy_1x_1y_2x$ is a 4-cycle in H with $d_H(x) \geq 4$. Theorem 3.3(a) is proved for this case.

Next, we consider the case $|Y| = |X|$.

We may assume $H \notin \mathcal{SL}$. Since $|X| \leq 7$, $|V(H)| = |X| + |Y| \leq 14$.

If $\delta(H) \leq 2$, then similar to the argument above, H has a 4-cycle with the stated properties. We are done if $\delta(H) \leq 2$. Thus, in the following we assume $\delta(H) \geq 3$.

If $\kappa'(H) \geq 3$, then by Theorem F(b), either $H \in \mathcal{SL}$, contrary to $H \notin \mathcal{SL}$, or $H \in \{P, P_{14}\}$, contrary to that H is a bipartite graph. Thus $\kappa'(H) \leq 2$.

Let E_1 be a minimum edge-cut of H with $|E_1| \leq 2$. Let H_1 and H_2 be the two components of $H - E_1$ and $|V(H_1)| \leq |V(H_2)|$. Since $\delta(H) \geq 3$ and $|V(H)| \leq 14$, no matter whether $|E_1| = 1$ or 2, we have $\delta(H_1) \geq 2$ with $|D_2(H_1)| \leq 2$ and $1 < |V(H_1)| \leq 7$. By Theorem F(a), H_1 is not reduced, contrary to that H is reduced. Theorem 3.3(a) is proved.

(b). If $\delta(H) \leq 2$, then similar to the argument above, H has a 4-cycle with a vertex of degree at least 4 in X. We are done for this case.

If $\delta(H) \geq 3$, then let x_0 be a vertex in X. Let y_1, y_2 and y_3 be three distinct vertices in $N(x_0)$. Since H is a connected bipartite graph, $\cup_{i=1}^{3}(N_H(y_i) - \{x_0\}) \subseteq X - \{x_0\}$ and so $| \cup_{i=1}^{3} (N_H(y_i) - \{x_0\})| \leq |X| - 1 = 5$. Since $d_H(y_i) \geq 3$ $(1 \leq i \leq 3)$, $|N_H(y_i) - \{x_0\}| \geq 2$. Thus, $\sum_{i=1}^{3} |N_H(y_i) - \{x_0\})| \geq 6 > 5 \geq | \cup_{i=1}^{3} (N_H(y_i) - \{x_0\})|$. Hence, there are some $i, j \in \{1, 2, 3\}$ $(i \neq j)$ such that $(N_H(y_i) - \{x_0\}) \cap (N_H(y_j) - \{x_0\}) \neq \emptyset$, and so H has a 4-cycle. Theorem 3.3(b) is proved.

(c). Suppose $G \notin \mathcal{SL}$. Let $n = |V(G)|$. If $n \geq 16$, then $\frac{n+5}{3} \geq 7 \geq |X|$ and $|Y| \geq 9 > |X|$. By Lemma 3.2, $G = H$ and $F(G) = 3$. By Theorem 3.1, G has no 4-cycles. But by (a) above, G has a 4-cycle, a contradiction. Thus $G \in \mathcal{SL}$ if $n \geq 16$.

If $n \leq 14$, then since $\kappa'(G) \geq 3$ and $G \notin \mathcal{SL}$, by Theorem F(b), $G \in \{P, P_{14}\}$. However, P and P_{14} have no spanning bipartite subgraphs with the stated properties. This is impossible.

If $n = 15$, then by Theorem F(c), G has girth at least 5. Since $|X| \leq 7$, $|Y| \geq 8 > |X|$. By (a) again, G has a 4-cycle, a contradiction. Theorem 3.3(c) is proved. $\Box$

Using Theorems 3.1 and 3.3, we prove the following result, Theorem 3.4, for the size of maximum matchings in reduced graphs, which is an improvement of a result in [20].

Let $q(G)$ denote the number of odd components of G.

Theorem H (Berge [2], Tutte [24]). Let G be a graph of order n. Then $\alpha'(G) = (n - t)/2$, where $t = \max_{S \subset V(G)}\{q(G - S) - |S|\}$. $\Box$

Theorem 3.4. Let G be a 3-edge-connected reduced graph of order n and $G \notin \mathcal{SL}$. If $n \leq 17$, then $\alpha'(G) \geq (n - 1)/2$.

Proof. By Theorem F(b), if $n \leq 14$, then $G \in \{P, P_{14}\}$ and so G has a perfect matching. We are done for $n \leq 14$. Thus, we may assume $n \geq 15$.

Let t be the integer defined in Theorem H. By way of contradiction, suppose $t \geq 2$. Let $S \subset V(G)$ be chosen such that $t = q(G - S) - |S|$. Let $m = q(G - S)$ and let G_1, G_2, $\cdots$, G_m be the odd components of $G - S$. We may assume that

$$|V(G_1)| \leq |V(G_2)| \leq \cdots \leq |V(G_m)|.$$

For each odd integer i, let $\mathcal{R}_i$ be the collection of components of $G - S$ consisting of exactly i vertices, and let $r_i = |\mathcal{R}_i|$. Let $S_i = \cup_{H \in \mathcal{R}_i} V(H)$. Then $|S_i| = ir_i$ $(i = 1, 3, \cdots)$. For each component H of $G - S$, let $\partial(H)$ be the set of edges in which every edge incident with at least one vertex in $V(H)$. Then

$$n \quad \geq \quad |S| + \Sigma_{i=1}^{m} |V(G_i)| = |S| + r_1 + 3r_3 + 5r_5 + \cdots; \tag{5}$$

$$m = |S| + t = q(G - S) = r_1 + r_3 + r_5 + \cdots. \quad (6)$$

We have

$$n \geq |S| + (r_1 + r_3 + r_5 + \cdots) + (2r_3 + 4r_5 + \cdots);$$
$$n \geq |S| + m + 2(r_3 + 2r_5 + \cdots) = 2|S| + t + 2(r_3 + 2r_5 + \cdots). \quad (7)$$

By (7), $t \geq 2$ and $n \leq 17$, $2|S| \leq 17 - t \leq 15$ and so $|S| \leq 7$. Furthermore, if $|S| = 7$, then by (7) again, $2(r_3 + 2r_5 + \cdots) = n - 2|S| - t \leq 1$ and so $r_i = 0$ $(i = 3, 5, \cdots)$. Thus, $V(G) = S \cup S_1$. Since $n \geq 15$, $|S_1| = r_1 = n - |S| \geq 8 > |S|$.

Let H be the bipartite graph induced by the edges between S and S_1. Since $\delta(G) \geq 3$ and each vertex v in S_1 is only adjacent to the vertices in S, $d_H(v) \geq 3$ for any $v \in S_1$. Therefore, G has a spanning bipartite subgraph H with the properties stated in Theorem 3.3. By Theorem 3.3(c), $G \in \mathcal{SL}$, a contradiction.

In the following, we assume that $|S| \leq 6$.

Case 1. $r_1 + r_3 = 0$.

Let $i \geq 5$ be the smallest integer such that $r_i \neq 0$. Then by (5), $m = |S| + t$ and $t \geq 2$,

$$n \geq |S| + im \geq |S| + 5m = 6|S| + 5t \geq 6|S| + 10.$$

Therefore, since $n \leq 17$, $|S| \leq \frac{n-10}{6} \leq \frac{7}{6}$ and so $|S| = 1$ and $i = 5$.

Hence, $|V(G_1)| = 5$. Let $H = G[S \cup V(G_1)]$. Since G is reduced, H is reduced. Since $|S| = 1$ and G is 3-edge-connected, H is a graph with $|V(H)| = |V(G_1)| + |S| = 6$ and $\delta(H) \geq 3$. By Theorem F(a), H is not reduced, a contradiction. Case 1 is proved.

Case 2. $r_1 + r_3 \neq 0$.

Since G is K_3-free and $\delta(G) \geq 3$,

$$|\partial(H_0)| \geq 3 \text{ for each } H_0 \in \mathcal{R}_1\text{; and } |\partial(H_1)| \geq 7 \text{ for each } H_1 \in \mathcal{R}_3. \quad (8)$$

Let $G_0 = G[S_0 \cup S_1 \cup S_3]$ where S_0 is the largest subset of S such that G_0 is connected. Then $|S_0| \leq |S|$ and $E(G_0) = \cup_{H \in \mathcal{R}_1 \cup \mathcal{R}_2} E(G[V(H) \cup S_0])$. By Theorem E(a), G_0 is a reduced graph with

$$|V(G_0)| = |S_0| + |S_1| + |S_3| = |S_0| + r_1 + 3r_3. \quad (9)$$

Since for any two H_1, $H_2 \in \mathcal{R}_1 \cup \mathcal{R}_3$ with $H_1 \neq H_2$, $\partial(H_1) \cap \partial(H_2) = \emptyset$, $|E(G_0)| = \Sigma_{H \in \mathcal{R}_1 \cup \mathcal{R}_2} |\partial(H)| + |E(G[S_0])|$. By (8)

$$|E(G_0)| \geq 3r_1 + 7r_3. \quad (10)$$

Claim 1. $G_0 \notin \{K_1, K_2, K_{2,s}\}$ $(s \geq 1)$.

Since each vertex $v \in S_1$ is only adjacent to the vertices in S and each vertex $v \in S_3$ is only adjacent to vertices in $S \cup S_3$, and since $\delta(G) \geq 3$, $d_H(v) = d(v) \geq 3$ for any $v \in S_1 \cup S_3$, and so $|S| \geq 3$. Thus $G_0 \notin \{K_1, K_2\}$. Next we will show $G_0 \neq K_{2,s}$.

Suppose that $G_0 = K_{2,s}$ $(s \geq 1)$. Then G_0 has at most two vertices of degree greater than 2. Thus $r_3 = |S_3| = 0$ and $r_1 = |S_1| \leq 2$. By (5), (6), $t \geq 2$ and $m = |S| + t$,

$$n \geq |S| + r_1 + 5(m - r_1) = |S| + 5m - 4r_1 = 6|S| + 5t - 4r_1 \geq 6|S| + 2.$$

Since $n \leq 17$, $6|S| \leq n - 2 \leq 15$. Thus, $|S| \leq 2$, contrary to $|S| \geq 3$. Claim 1 is proved.

Since $G_0 \notin \{K_1, K_2, K_{2,s}\}$ $(s \geq 1)$, by Theorem E(c), $F(G_0) \geq 3$. By Theorem E(b), $|E(G_0)| \leq 2|V(G_0)| - 5$. By (9) and (10),

$$\begin{aligned} 3r_1 + 7r_3 &\leq |E(G_0)| \leq 2|V(G_0)| - 5 = 2(|S_0| + r_1 + 3r_3) - 5, \\ r_1 + r_3 &\leq 2|S_0| - 5 \leq 2|S| - 5. \end{aligned} \tag{11}$$

By (5), (6), (11), $n \leq 17$ and $t \geq 2$,

$$\begin{aligned} n &\geq |S| + r_1 + 3r_3 + 5(m - r_1 - r_3) \geq 6|S| + 5t - 2(r_1 + r_3) - 2r_1; \\ 2r_1 &\geq 6|S| + 5t - 2(r_1 + r_3) - n \geq 6|S| - 2(r_1 + r_3) - 7. \\ 2r_1 &\geq 6|S| - 2(2|S| - 5) - 7 = 2|S| + 3 \end{aligned}$$

Therefore, $r_1 \geq |S| + 2$. By (11) and $|S| \leq 6$,

$$\begin{aligned} |S| + 2 + r_3 &\leq r_1 + r_3 \leq 2|S| - 5; \\ r_3 &\leq |S| - 7 \leq -1, \end{aligned}$$

contrary to $r_3 \geq 0$. Theorem 3.4 is proved. □

4. Degree condition of adjacent vertices for supereulerian graphs

With the theorems on the properties of reduced graphs proved in section 3, we are able to prove a new result for 3-edge-connected graph G that satisfies $\overline{\sigma}_2(G) \geq \frac{2n}{p} - 2$.

Different from the study on Ore-type degree sum conditions of non-adjacent vertices for hamiltonian graphs, Brualdi and Shaney [5] studied degree-sum conditions of adjacent vertices to obtain a result on Hamiltonian line graphs.

Theorem I (Brualdi [5]). Let G be a graph of order $n \geq 4$. If for any edge $uv \in E(G)$, $\overline{\sigma}_2(G) \geq n$, then G contains a dominating Eulerian subgraph, hence $L(G)$ is hamiltonian.

Since then, many results had been found on the degree-sum conditions of adjacent vertices for spanning and dominating Eulerian subgraphs of graphs (see [1, 14, 19, 23, 25]). The following was proved by Veldman [25].
Theorem J (Veldman [25]). Let G be a 2-edge-connected simple graph of order n. If for any $uv \in E(G)$, $\overline{\sigma}_2(G) > \frac{2n}{5} - 2$, then for n sufficiently large, $L(G)$ is Hamiltonian.

For 3-edge-connected graphs, the degree-sum condition in Theorem J can be lower.
Theorem K (Chen and Lai [19] and Veldman [25]). Let G be a 3-edge-connected simple graph of order n. If n is large and $\overline{\sigma}_2(G) \geq \frac{n}{5} - 2$, then either $G \in \mathcal{SL}$ or $n = 10s$ $(s > 0)$ and $G' = P$ with the preimage of each vertex in P is a K_s or $K_s - e$ for some $e \in E(K_s)$.

Here we show how such graphs change when p is increased to 15.
Theorem 4.1. Let G be a 3-edge-connected simple graph of order n. If n is sufficiently large and

$$\overline{\sigma}_2(G) > 2(\frac{n}{15} - 1), \tag{12}$$

then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$. Furthermore, if $\overline{\sigma}_2(G) \geq 2(\frac{n}{14} - 1)$ and $G' = P_{14}$, then $n = 14s$ and each vertex in P_{14} is obtained by contracting a K_s or $K_s - e$ for some $e \in E(K_s)$.

We prove the following lemma first:
Lemma 4.2. Let G be a 3-edge-connected graph of order n with $\overline{\sigma}_2(G) \geq \frac{2n}{p} - 2$, where p is a given positive number. Let G' be the reduction of G. Let v be a vertex in G' and $H(v)$ be the preimage of v. Then when n is large, each of the following holds:
(a) If $|V(H(v))| = 1$, then for any $x \in N_{G'}(v)$, $|V(H(x))| \geq \overline{\sigma}_2(G) + 1 - d_{G'}(v) - d_{G'}(x)$.
(b) If $|V(H(v))| > 1$, then $|V(H(v))| \geq \frac{\overline{\sigma}_2(G)}{2} + 1$.
Proof. For a vertex $y \in V(G)$, let $i(y)$ be the number of edges in G' incident with y in G. If $y \in V(H(v))$ where $H(v)$ is the preimage of $v \in V(G')$, then

$$d_G(y) \leq i(y) + |V(H(v))| - 1. \tag{13}$$

By Theorem D, $|V(G')| \leq 3p - 4$. Then

$$\Delta(G') \leq |V(G')| - 1 \leq 3p - 5. \tag{14}$$

(a) Since $|V(H(v))| = 1$, v is a trivial contraction. Then $d_{G'}(v) = d_G(v)$. For any $x \in N_{G'}(v)$, there is a vertex x_0 in G such that $e = xy = x_0v$.

Then $d_G(x_0) \le d_{G'}(x) + |V(H(x))| - 1$. Since $d_G(v) + d_G(x_0) \ge \overline{\sigma}_2(G)$,

$$\overline{\sigma}_2(G) \le d_G(v) + d_G(x_0) \le d_{G'}(v) + d_{G'}(x) + |V(H(x))| - 1.$$

Lemma 4.2(a) is proved.

(b). Since $|V(H(v))| > 1$, $E(H(v)) \ne \emptyset$. Let xy be an edge in $E(H(v))$. There are at most $d_{G'}(v)$ number of edges in $E(G')$ incident with x and y and so $i(x)+i(y) \le d_{G'}(v) \le \Delta(G')$. Since $d_G(x)+d_G(y) \ge \overline{\sigma}_2(G) \ge \frac{2n}{p}-2$, by (13) and (14),

$$\begin{aligned}
\overline{\sigma}_2(G) &\le d_G(x) + d_G(y) \\
&\le (i(x) + |V(H(v))| - 1) + (i(y) + |V(H(v))| - 1); \\
\overline{\sigma}_2(G) &\le i(x) + i(y) + 2|V(H(v))| - 2; \qquad (15) \\
\frac{2n}{p} - (3p-5) &\le \overline{\sigma}_2(G) - (i(x) + i(y)) + 2 \le |V(H(v))|.
\end{aligned}$$

Since p is a fixed, when n is large (say $n > p(3p-5)$), $H(v)$ has an edge xy such that $i(x) = i(y) = 0$. Thus, by (15), $|V(H(v))| \ge \frac{\overline{\sigma}_2(G)}{2} + 1$. Lemma 4.2(b) is proved. □

Proof of Theorem 4.1. Suppose that $G \notin \mathcal{SL}$. Let G' be the reduction of G. By Theorem A, $G' \notin \mathcal{SL}$. Since $\kappa'(G) \ge 3$, $\kappa'(G') \ge 3$. By Theorem D with $p = 15$, $\alpha'(G') \le 15/2$ and so $\alpha'(G') \le 7$. By Theorem E(d), $|V(G')| \le 3\alpha'(G') - 4 = 17$. Thus, by Theorem 3.4, $\alpha'(G) \ge (|V(G')|-1)/2$ and so $|V(G')| \le 15$. If $|V(G')| \le 14$, then by Theorem F(b) and $G' \notin \mathcal{SL}$, $G' \in \{P, P_{14}\}$. We are done for this case.

Next, we show that $|V(G')| = 15$ is impossible.

If $|V(G')| = 15$, then by Theorem F(c), G' has girth at least 5 and $V(G') = D_3(G') \cup D_4(G')$ where $D_4(G')$ is an independent set. Hence, for any $xy \in V(G')$,

$$d_{G'}(x) + d_{G'}(y) \le 7. \qquad (16)$$

Let $Y_0 = \{v \in V(G') \mid |V(H(v))| = 1\}$. Let $X = \cup_{v \in Y_0} N_{G'}(v)$. Let $Z = V(G') - X - Y_0$.

For each $v \in Z$, $|V(H(v))| > 1$. By Lemma 4.2(b) and $\overline{\sigma}_2(G) > 2(\frac{n}{15} - 1)$,

$$|V(H(v))| \ge \frac{\overline{\sigma}_2(G)}{2} + 1 > \frac{n}{15}. \qquad (17)$$

For any $x \in X$, by Lemma 4.2(a), (16) and (12), $|V(H(x))| \ge \overline{\sigma}_2(G) + 1 - 7 > \frac{2n}{15} - 8$. Since $\cup_{x \in X} V(H(x)) \subseteq V(G)$,

$$n = |V(G)| \ge \Sigma_{x \in X} |V(H(x))| \ge |X|(\frac{2n}{15} - 8). \qquad (18)$$

Thus, when n is large, $|X| \leq 7$.

Case 1. $|Z| \leq 1$. Let $Y = Y_0 \cup Z$. Note that if $|Z| = 1$, then by the definitions of Z the vertex in Z is only adjacent to vertices in X. Thus, the edges between X and Y forms a spanning bipartite subgraph H_a of G such that $d_{H_a}(v) = d(v) \geq 3$ for any $v \in Y$. Since $|X| \leq 7$ and $|X| + |Y| = |V(H_a)| = |V(G')| = 15$, $|X| < |Y|$. Thus, H_a is a bipartite graph with the properties stated in Theorem 3.3(a), and so H_a has 4-cycle, contrary to that G' has girth at least 5. Case 1 is proved.

Case 2. $|Z| \geq 2$. Then $|X| + |Y_0| \leq 13$. Let H_b be the bipartite subgraph formed by the edges between X and Y_0. Since $\kappa'(G') \geq 3$, $d_{H_b}(v) = d(v) \geq 3$ for any $v \in Y_0$. Since $V(G) = \cup_{x\in X} V(H(x)) \cup Y_0 \cup_{v\in Z} V(H(v))$ and $|Z| = 15 - (|X| + |Y_0|)$, by (17) and (18)

$$
\begin{aligned}
n = |V(G)| &\geq |X|(\frac{2n}{15} - 8) + |Y_0| + |Z|\frac{n}{15} \qquad (19)\\
&= |X|\frac{2n}{15} - 8|X| + |Y_0| + (15 - |X| - |Y_0|)\frac{n}{15}\\
&\geq n + \frac{|X| - |Y_0|}{15} - 8|X| + |Y_0|.
\end{aligned}
$$

Therefore, when n is large, $|X| \leq |Y_0|$. Since $|X| + |Y_0| \leq 13$, $6 \geq |X|$. H_b is a bipartite graph with the properties stated in Theorem 3.3. By Theorem 3.3(a) and (b), H_b has a 4-cycle, a contradiction. This shows that $|V(G')| = 15$ is impossible.

Next, we assume that $\overline{\sigma}_2(G) \geq \frac{2n}{14} - 2$ and $G' = P_{14}$.

Claim 1. $Y_0 = \emptyset$.

Suppose $Y_0 \neq \emptyset$. Then $X \neq \emptyset$. By Lemma 4.2, for each $v \in Z$, $|V(H(v))| \geq \frac{n}{14}$, and for each $x \in X$, $|V(H(x))| \geq \frac{2n}{14} - 7$. Replacing $\frac{n}{15}$ by $\frac{n}{14}$ and replacing $|X|(\frac{2n}{15} - 8)$ by $|X|(\frac{2n}{14} - 7)$ and using $|Z| = 14 - |X| - |Y_0|$ in (19), we have $|X| \leq |Y_0|$ when n is sufficiently large.

Let H_b be the bipartite subgraph defined in Case 2 above. Since $d_{H_b}(v) \geq 3$ for any $v \in Y_0$, $|X| \geq 3$. Since $|Y_0| \geq |X|$ and G' has no $K_{3,3}$, $|Y_0| \geq 4$. By Lemma 4.2(a), Y_0 is an independent set. However, by observation on P_{14}, $|X| = |\cup_{v\in Y_0} N_{G'}(v)| \geq 7$ for any independent set Y_0 with $|Y_0| \geq 4$. P_{14} has no such bipartite subgraph H_b. Claim 1 is proved.

Therefore, $Z = V(P_{14})$. Then by $|V(H(v))| \geq \frac{n}{14}$ for each $v \in Z$,

$$n = |V(G)| = \Sigma_{v\in Z}|V(H(v))| \geq |Z|\frac{n}{14} = n. \qquad (20)$$

Thus the equality of (20) holds and so $|V(H(v))| = \frac{n}{14}$ for any $v \in V(P_{14})$. Let $s = |V(H(v))| = \frac{n}{14}$. Since for any $uv \in E(G)$, $d(u) + d(v) \geq \overline{\sigma}_2(G) \geq$

$\frac{2n}{14} - 2$, $H(v)$ is either K_s or $K_s - e$ for some $e \in E(K_s)$. (See G_b in Figure 4.1(b) for such a graph). $\Box$

Remark: From Theorem 4.1 and Theorem K, we can see that for a 3-edge-connected graph G of order n with $\overline{\sigma}_2(G) \geq \frac{2n}{p} - 2$, the structures of G change when p is increased:

(a) if $\overline{\sigma}_2(G) > \frac{2n}{10} - 2$ then $G \in \mathcal{SL}$;

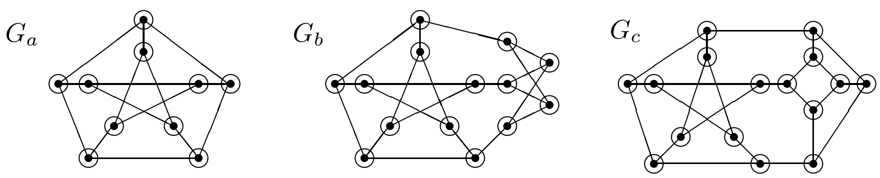
(b) if $\overline{\sigma}_2(G) \geq \frac{2n}{10} - 2$ then $G \in \mathcal{SL}$ or $G = G_a$ as shown in Figure 4.1(a) where $n = 10s$ and each circle is a K_s or a $K_s - e$;

(c) if $\overline{\sigma}_2(G) > \frac{2n}{14} - 2$, then $G \in \mathcal{SL}$ or $G' = P$;

(d) if $\overline{\sigma}_2(G) \geq \frac{2n}{14} - 2$, then $G \in \mathcal{SL}$ or $G' = P$ or $G = G_b$ as shown in Figure 4.1(b) where $n = 14s$ and each circle is a K_s or $K_s - e$;

(e) if $\overline{\sigma}_2(G) > \frac{2n}{15} - 2$, then $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$.

Graphs G_a and G_b in Figure 4.1 are the extremal graphs with the boundary value on $p = 10$ and 14 for $\overline{\sigma}_2(G) \geq \frac{2n}{p} - 2$, while G_c is the next possible extremal graph for $p = 16$.



(a) $\overline{\sigma}(G_a) = \frac{2n}{10} - 2$ (b) $\overline{\sigma}(G_b) = \frac{2n}{14} - 2$ (c) $\overline{\sigma}(G) = \frac{2n}{16} - 2$

Figure 4.1

Let G be the graph of order n defined in Figure 4.1(c) in which each circle is a $K_{n/16}$. Then $\overline{\sigma}_2(G) \geq 2(\frac{n}{16} - 1)$ and $G' = P_{16}$. Thus, (12) in Theorem 4.1 cannot be replaced by $\overline{\sigma}_2(G) \geq \frac{2n}{16} - 2$. But if Conjecture B is true, then we can have $\overline{\sigma}_2(G) \geq \frac{2n}{16} - 2$ for (12) with the conclusion that either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}, P_{16}\}$ and when $G' = P_{16}$, $G = G_c$.

As we can see that P_{14} and P_{16} can be contracted to P by contracting a subgraph to a vertex in P. If we relax the conclusion of Theorem 4.1 from "the reduction of G is in $\{P, P_{14}\}$" to "G can be contracted to P", the degree condition (12) may be lower. It was conjectured in [20] that for any 3-edge-connected graph G of order n if $\overline{\sigma}_2(G) > n/9 - 2$, then when n is large either $G \in \mathcal{SL}$ or G can be contracted to P.

References

[1] A. Benhocine, L. Clark, N. Köhler and H. J. Veldman, On circuits and pancyclic line graphs, J. Graph Theory 10 (1986), 411 - 425.

[2] C. Berge, Sur le couplage maximum d'un graphs. CR Acad. Sci. Paris 247 (1958) 258–259.

[3] E.T. Boesch, C. Suffel, R. Tindell, The spanning subgraphs of Eulerian graphs, J. Graph Theory 1(1977) 79-84

[4] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications". American Elsevier, New York (1976).

[5] R. A. Brualdi, R. F. Shanny, Hamiltonian line graphs. J. Graph Theorey 5 (1981), 3, 307-314

[6] P. A. Catlin, A reduction method to find spanning eulerian subgraphs. J. Graph Theory 12 (1988) 29-44.

[7] P. A. Catlin, Supereulerian graphs, collapsible graphs, and four-cycles. Congressus Numerantium 58 (1987) 233-246.

[8] P. A. Catlin, Contractions of graphs with non spanning Eulerian subgraphs, Combinatorica 8 (4) 1988 313-321.

[9] P. A. Catlin, Double cycle covers and the Petersen graph, J. Graph Theory 13 (1989) 465–483.

[10] P. A. Catlin, Double cycle covers and the Petersen graph, II. Congressus Numerantium 76(1990) 173-181

[11] P. A. Catlin, Supereulerian Graphs: A Survey, J. Graph Theory 16 (1992) 177-196.

[12] P. A. Catlin, Z. Han, and H.-J. Lai, Graphs without spanning eulerian trails. Discrete Math. 160 (1996) 81-91.

[13] Z.-H. Chen, Supereulerian graphs and the Petersen graph, J. of Combinatorial Math. and Combinatorial Computing. Vol 9 (1991) 79-89.

[14] Z.-H. Chen, Spanning Closed trails in graphs, Discrete Math. 117(1993) 57-71

[15] Z.H. Chen, Supereulerian graphs, independent sets, and degree-sum conditions, Discrete Math.179 (1998) 73-87.

[16] W.-G. Chen and Z.-H. Chen, Spanning Eulerian subgraphs and Catlin's reduced graphs, J. of Combinatorial Math. and Combinatorial Computing, (accepted, 2014).

[17] W,-G. Chen, Z.-H. Chen, Fan-Type conditions for spanning Eulerian subgraphs, Graphs and Combinatorics, (accepted 2014).

[18] W.-G. Chen, Z.-H. Chen, Lai's degree conditions for spanning and dominating closed trails, (submitted)

[19] Z.-H. Chen and H.-J. Lai, Collapsible graphs and Matchings. J. of Graph Theory, Vol 17, No. 5, 597-605 (1993).

[20] Z.-H. Chen and H.-J. Lai, Supereulerian graphs and the Petersen graph II. Ars Combinatoria 48 (1998) PP. 271-282.

[21] H.-J. Lai, Y. Shao, M. Zhan, Hamiltonicity in 3-connected claw-free graphs, J. Combin. Theory Ser. B 96 (2006) 493-504.

[22] H.-J. Lai, Eulerian subgraphs containing given vertices and Hamiltonian line graphs, Discrete Math. 178 (1998), 93-107.

[23] D. Li, H.-J. Lai, Z. Zhan, Eulerian subgraphs and hamiltonian connected line graphs, Discrete Applied Math 145 (2005) 422-428.

[24] W. T. Tutte, The factorization of linear graphs. J. London Math. Soc. 22 (1947) 107-111.

[25] H. J. Veldman, On dominating and spanning circuits in graphs, Discrete Math., 124 (1994) 229-239.