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Homeomorphisms acting on Besov and Triebel-Lizorkin spaces of local regularity $\psi(t)$

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Abstract

The aim of this paper is to show that the integral and derivative operators defined by local regularities are homeomorphisms for generalized Besov and Triebel-Lizorkin spaces with local regularities. The underlying geometry is that of homogeneous type spaces and the functions defining local regularities belong to a larger class of growth functions than the potentials t^{α}, related to classical fractional integral and derivative operators and Besov and Triebel-Lizorkin spaces.

1. Introduction

The Besov and Triebel-Lizorkin spaces arise from the unified study of most of the classical function spaces, provided by the theory of Littlewood-Paley. By means of this theory, Lebesgue spaces, Hardy spaces, different kind of Lipschitz spaces and BMO are characterized through the action of an appropriate class of operators. There is a formula, due to Calderón, that allows to describe those spaces as special cases of the Besov and Triebel-Lizorkin spaces. (For an insight on these topics, see, for example [3], [15], [12], [13] and [14]).

In the more general setting of spaces of homogeneous type, in [1], David, Journé and Semmes established the Littlewood-Paley theory for L^p, $1 < p < \infty$. Han and Sawyer, in [6], defined discrete in the time variable versions of the homogeneous Besov and Triebel-Lizorkin spaces. To develop Littlewood-Paley characterizations on these

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spaces the authors proved a Calderón-type reproduction formula in terms of the difference operators $D_k = S_k - S_{k-1}$ where $\{S_k\}_{k\in Z}$ is an approximation to the identity. Deng and Han, in [2], proved a continuous version of that formula in terms of the derivative operators $Q_t = -t\frac{\partial S_t}{\partial t}$ of an approximation to the identity $\{S_t\}_{t>0}$ and defined time-continuous versions of the spaces equivalent to the ones given in [6].

In the context of $\mathbb{R}^n$ it is a known result that the fractional integral operator I_α is an isomorphism between the Besov spaces $\dot{B}_p^{\beta,q}$ and $\dot{B}_p^{\alpha+\beta,q}$ and between the Triebel-Lizorkin spaces $\dot{F}_p^{\beta,q}$ and $\dot{F}_p^{\alpha+\beta,q}$ and its inverse is the fractional derivative operator D_α. An application of this fact is the identification, with equivalence of norms, between the fractional Sobolev space $\dot{L}_\alpha^p$, of all tempered distributions f such that $D_\alpha f \in L^p$, and the space $\dot{F}_p^{\alpha,2}$, (see [3]).

We are interested in a larger class of Besov and Triebel-Lizorkin spaces, with the underlying structure of spaces of homogeneous type. For this class, defined in [7], local regularity is determined by more general '*moduli of continuity*' than the potentials t^α. Those are growth functions $\psi(t)$ as, for instance, $t^\beta \log(1+t)$ or $\max(t^\alpha, t^\beta)$. In the range of these spaces can be characterized, for example, the Lipschitz space $\dot{\Lambda}^\psi = \{f : X \to C, |f(x) - f(y)| \leq C\psi(\delta(x,y))\}$, where ψ belongs to a class (defined later in this work) of non-negative and quasi-increasing function such that $\lim_{t\to 0^+}\psi(t) = 0$ and $\lim_{t\to\infty}\psi(t) = \infty$.

In order to find the natural isomorphisms mapping one space of local regularity ψ_1, onto another of regularity ψ_2 we defined, in [8], operators named -in analogy with the classical ones- Integral and Derivative operators of *'functional order'* ϕ, I_ϕ and D_ϕ, following an idea of Gatto, Segovia and Vági in [4] for the definition of fractional integral and derivative operators on spaces of homogeneous type. We proved their continuity between generalized Lipschitz, Besov and Triebel-Lizorkin spaces, with the integral operator increasing regularity and the derivative operator decreasing it.

Seeking for the invertibility of these operators, we considered the composition $T_\phi = D_\phi \circ I_\phi$ and $S_\phi = I_\phi \circ D_\phi$. We proved in [9] that they are Calderón - Zygmund operators bounded on the generalized Besov and Triebel-Lizorkin spaces.

We prove in this work that the operator T_ϕ (and the same proof applies to S_ϕ) is invertible on the Besov and Triebel-Lizorkin spaces $\dot{B}_p^{\psi,q}$ and $\dot{F}_p^{\psi,q}$ respectively, over normal spaces of homogeneous type, for an adequate relation between the types of the functions ϕ and ψ. From the identification between L^2 and $\dot{F}_2^{0,2}$ we recover the results of invertibility on L^2 proved in [4] when $\phi(t) = t^\alpha$ and $\psi(t) = 1$.

Moreover, we also prove that T_ϕ^{-1} and S_ϕ^{-1} also are Calderón-Zygmund operators in the classical sense. In this way, even though T_ϕ and S_ϕ are not the identity, for an appropriate class of functions ϕ we show that they 'almost' are. From the above results follows that I_ϕ and D_ϕ are homeomorphisms.

This paper is organized in the following way: in Section 2 are defined growth functions (subsection 2.1), spaces of homogeneous type and approximations to the identity (subsection 2.2), Lipschitz, molecular, Besov and Triebel-Lizorkin spaces (subsection 2.3) and Calderón-Zygmund operators and $T1$-theorem stated and proved in [7] (subsection 2.4). Section 3 is devoted to the definition of the integral and derivative operators I_ϕ and D_ϕ and a survey of known results on them. Main theorems are stated in Section 4. Lemmas and proofs of the theorems are in Section 5 and, finally, an application of the results is in Section 6.

2. Previous definitions and known results

2.1. Quasi-increasing and quasi-decreasing functions

Let first define the class of functions related to the operators defined in this work.

A function $\phi(t)$ defined on $t > 0$ is said to be *quasi-increasing* if there is a positive constant C such that if $t_1 < t_2$ then $\phi(t_1) \leq C\phi(t_2)$.

Analogously, $\phi(t)$ is *quasi-decreasing* if there is a positive constant C such that if $t_1 < t_2$ then $\phi(t_2) \leq C\phi(t_1)$.

A non-negative function ϕ is said to be of *lower type* β, $0 \leq \beta$, if there is a constant $C > 0$ such that

$$\phi(uv) \leq Cu^\beta \phi(v) \text{ for } u < 1 \text{ and all } v > 0. \tag{2.1}$$

The condition (2.1) is equivalent to the condition

$$\phi(uv) \geq \frac{1}{C} u^\beta \phi(v), \text{ for } u \geq 1 \text{ and all } v > 0.$$

A non-negative function $\phi(t)$ is of *upper type* α, $0 \leq \alpha$ if there is a constant $C > 0$ such that

$$\phi(uv) \leq Cu^\alpha \phi(v) \text{ for } u \geq 1 \text{ and all } v > 0. \tag{2.2}$$

The condition (2.2) is equivalent to the condition

$$\phi(uv) \geq \frac{1}{C} u^\alpha \phi(v), \text{ for } u < 1, v > 0.$$

Clearly, the potential t^α, with $\alpha \geq 0$, is of lower and upper type α. The functions $\max(t^\alpha, t^\beta)$ and $\min(t^\alpha, t^\beta)$, with $\alpha < \beta$, are both of lower type α and upper type β. Also, $t^\beta(1 + \log^+ t)$, with $\beta \geq 0$, is of lower type β and of upper type $\beta + \epsilon$, for every $\epsilon > 0$.

A straightforward proof shows that if $\phi(t)$ is of both lower type β and upper type α then $\beta \leq \alpha$.

Also, if $\phi(t)$ is quasi-increasing then $\phi(t)$ is of lower-type 0 and, reciprocally, if $\phi(t)$ is of lower type $\beta \geq 0$ then it is quasi-increasing.

Finally, if $\phi(t)$ is of lower type β and $\xi(t)$ is of upper type $\lambda \leq \beta$ then $\phi(t)/\xi(t)$ is quasi-increasing.

We say that two functions $\psi(t)$ and $\phi(t)$ are *equivalent*, and we denote it $\psi \simeq \phi$, if there are positive constants C_1 and C_2 such that $C_1 \leq \phi/\psi \leq C_2$. Lower and upper types are clearly invariant by equivalence of functions. That is, if ϕ is of lower (upper) type δ and $\psi \simeq \phi$ then ψ is of lower (upper) type δ.

If $\phi(t)$ is of lower type $\beta > 0$ then $\phi(t)/t^\gamma$ is quasi-increasing for each $\gamma \leq \beta$, nevertheless, the function

$$\tilde{\phi}(t) = t^\gamma \sup_{s \leq t} \frac{\phi(s)}{s^\gamma}$$

is equivalent to ϕ and $\tilde{\phi}(t)/t^\gamma$ is nondecreasing. Moreover, if ϕ also is of upper type α and $\gamma < \beta$ then there exists a function ϕ_γ equivalent to ϕ which is differentiable and such that $\phi_\gamma(t)/t^\gamma$ is nondecreasing. More precisely, the function

$$\phi_\gamma(t) = t^\gamma \int_0^t \frac{\phi(u)}{u^{\gamma+1}} du \tag{2.3}$$

satisfies

$$\phi_\gamma(t) = \int_0^1 \frac{\phi(tu)}{u^{\gamma+1}} du \leq C_1 \phi(t) \int_0^1 u^{\beta-\gamma} \frac{du}{u} = C_1 \frac{\phi(t)}{\beta-\gamma}$$

and also

$$\phi_\gamma(t) \geq C_2 \phi(t) \int_0^1 u^{\alpha-\gamma} \frac{du}{u} = C_2 \frac{\phi(t)}{\alpha-\gamma}.$$

Analogously, if ϕ is of upper type α then $\phi(t)/t^\delta$ is quasi-decreasing for $\delta \geq \alpha$. Nevertheless,

$$\bar{\phi}_\delta(t) = t^\delta \sup_{s \geq t} \frac{\phi(s)}{s^\delta} \tag{2.4}$$

is equivalent to ϕ and $\bar{\phi}_\delta(t)/t^\delta$ is non-increasing. If, in addition, ϕ is of lower type β and $\delta > \alpha$ then,

$$\check{\phi}(t) = t^\delta \int_t^\infty \frac{\phi(s)}{s^{\delta+1}} ds$$

is a differentiable function, equivalent to ϕ such that $\check{\phi}(t)/t^\delta$ is non-increasing.

Let denote $\mathcal{C}$ the class of all non-negative functions ϕ of positive lower type and upper type lower than 1. Let also denote $\mathcal{A}$ the class of functions $\phi(t)$ defined on $t > 0$ such that

$$\phi(t) = \phi(1) e^{\int_1^t (\eta(s))/s \, ds} \tag{2.5}$$

where $\eta(t)$ is a measurable function defined on $t > 0$ and $\beta \leq \eta(t) \leq \alpha$ for some $0 < \beta \leq \alpha < 1$. □

The following lemma, proved in [10], shows that there is an identification between the classes $\mathcal{A}$ and $\mathcal{C}$.

Lemma 2.1

The class $\mathcal{A}$ is included in the class $\mathcal{C}$ and for every function in $\mathcal{C}$ there is an equivalent function in $\mathcal{A}$.

Moreover, if $\phi(t) = \phi(1) e^{\int_1^t (\eta(s))/s \, ds}$ and $\beta \leq \eta(t) \leq \alpha$, then for $s < 1$

$$\phi(t) s^\alpha \leq \phi(st) \leq \phi(t) s^\beta, \tag{2.6}$$

and, for $s > 1$,

$$\phi(t) s^\beta \leq \phi(st) \leq \phi(t) s^\alpha. \tag{2.7}$$

Proof. Indeed, if $\phi(t) \in \mathcal{A}$ then, for $s < 1$ we have

$$\begin{aligned} \phi(st) &= \phi(1) e^{\int_1^{st} (\eta(\tau))/\tau \, d\tau} = \phi(1) e^{\int_1^t (\eta(\tau))/\tau \, d\tau - \int_{st}^t (\eta(\tau))/\tau \, d\tau} \\ &= \phi(t) e^{-\int_{st}^t (\eta(\tau))/\tau \, d\tau} \leq \phi(t) e^{\beta \ln s} = \phi(t) s^\beta. \end{aligned}$$

A similar proof shows that $\phi(st) \geq \phi(t) s^\alpha$. For $s > 1$, by splitting $t = s^{-1}(st)$ and using (2.6), follows (2.7).

On the other hand, if ϕ is of lower type $\beta > 0$ and of upper type $\alpha < 1$ then, by choosing the function $\bar{\phi}_\alpha$ equivalent to ϕ as in (2.4), we can suppose that $\phi(t)/t^\alpha$ is non-increasing. Then, taking $\phi_\gamma(t)$ as in (2.3), equivalent to ϕ if $\gamma < \beta$, we have that

$$\phi_\gamma(t) = t^\gamma \int_0^t \frac{\phi(s)}{s^{\alpha+1-\alpha+\gamma}} ds \geq t^\gamma \frac{\phi(t)}{t^\alpha} \int_0^t \frac{dt}{t^{1-\alpha+\gamma}} = \frac{\phi(t)}{\alpha - \gamma}. \tag{2.8}$$

Then

$$\begin{aligned} \phi'_\gamma(t) = & \quad \gamma t^{\gamma-1} \textstyle\int_0^t \frac{\phi(s)}{s^{1+\gamma}} ds + \frac{\phi(t)}{t} & = \gamma \frac{\phi_\gamma(t)}{t} + \frac{\phi(t)}{t} \\ \leq & \quad \gamma \tfrac{\phi_\gamma(t)}{t} + (\alpha - \gamma)\tfrac{\phi_\gamma(t)}{t} & = \alpha \frac{\phi_\gamma(t)}{t} \end{aligned}$$

and also,

$$\phi'_\gamma(t) \geq \gamma \frac{\phi_\gamma(t)}{t}.$$

Hence, ϕ_γ is a function equivalent to ϕ which belongs to $\mathcal{A}$ since

$$\gamma \leq \eta(t) = \frac{t\phi'_\gamma(t)}{\phi_\gamma(t)} \leq \alpha. \ \Box$$

2.2. Spaces of homogeneous type and approximations to the identity

Let define the structure of space of homogeneous type, which is the underlying geometry for the test functions we shall consider. Given a set X and a real valued function $\delta(x, y)$ defined on $X \times X$, we say that δ is a quasi-distance on X if there exists a positive constant A such that for all $x, y, z \in X$ it verifies:

$$\begin{aligned} &\delta(x, y) \geq 0 \text{ and } \delta(x, y) = 0 \text{ if and only if } \quad x = y \\ &\delta(x, y) = \delta(y, x) \\ &\delta(x, y) \leq A[\delta(x, z) + \delta(z, y)]. \end{aligned}$$

In a set X endowed with a quasi-distance $\delta(x, y)$, the balls $B_\delta(x, r) = \{y : \delta(x, y) < r\}$ form a basis of neighborhoods of x for the topology induced by the uniform structure on X.

Let μ be a positive measure on a σ- algebra of subsets of X which contains the open set and the balls $B_\delta(x, r)$. The triple $X := (X, \delta, \mu)$ is a *space of homogeneous type* if there exists a finite constant $A' > 0$ such that

$$\mu(B_\delta(x, 2r)) \leq A'\mu(B_\delta(x, r))$$

for all $x \in X$ and $r > 0$. Macías and Segovia in [11], showed that it is always possible to find a quasi-distance $d(x, y)$ equivalent to $\delta(x, y)$ and $0 < \theta \leq 1$, such that

$$|d(x, y) - d(x', y)| \leq C r^{1-\theta} d(x, x')^\theta \tag{2.9}$$

holds whenever $d(x, y) < r$ and $d(x', y) < r$. If δ satisfies (2.9) then X is said to be *of order* θ. Furthermore, X is a *normal space* if $A_1 r \leq \mu(B_\delta(x, r)) \leq A_2 r$ for every $x \in X$ and $r > 0$ and some positive constants A_1 and A_2.

In this work $X := (X, \delta, \mu)$ *means a normal space of homogeneous type of order* θ *and* A *denotes the constant of the triangular inequality associated to* δ.

Let now define an approximation to the identity of Coifman type, as in [4]:
A family $\{S_t\}_{t>0}$ of operators is an *approximation to the identity* if there exist $\epsilon \leq \theta$ and C, C_1 and $C_2 < \infty$ such that for all $t > 0$ and all x, x', y and $y' \in X$, the kernel $s_t(x, y)$ of S_t, are functions from $X \times X$ into $\mathbb{R}$ satisfying:

$$s_t(x,y) = 0 \text{ for } \delta(x,y) > b_1 t, \tag{2.10}$$

$$\frac{C_2}{t} < s_t(x,y) \text{ if } \delta(x,y) < b_2 t;, \tag{2.11}$$

$$|s_t(x,y)| \leq \frac{C_1}{t}, \tag{2.12}$$

$$|s_t(x,y) - s_t(x',y)| + |s_t(y,x) - s_t(y,x')| \leq C\frac{\delta(x,x')^\epsilon}{t^{1+\epsilon}}, \tag{2.13}$$

$$\int s_t(x,y)d\mu(y) = \int s_t(x,y)d\mu(x) = 1, \quad \text{and} \tag{2.14}$$

$$s_t(x,y) \text{ is continuously differentiable in } \quad t. \tag{2.15}$$

In addition we also require the kernels to be positive and symmetric, that is, for all $t > 0$, x and $y \in X$

$$s_t(x,y) \geq 0; \tag{2.16}$$

$$s_t(x,y) = s_t(y,x). \tag{2.17}$$

Denote

$$q_t(x,y) = -t\frac{\partial}{\partial t}s_t(x,y) \tag{2.18}$$

and Q_t be the operator defined by

$$Q_t f(x) = \int_X q_t(x,y)f(y)d\mu(y),$$

for $f \in L^1_{\text{loc}}$ and $t > 0$. The family $\{Q_t\}_{t>0}$ satisfies the condition $\int_0^\infty Q_t \frac{dt}{t} = I$ in L^2 in the sense that

$$\lim_{\epsilon\to 0, R\to\infty} \Big\| \int_\epsilon^R Q_t f \frac{dt}{t} - f \Big\|_2 = 0.$$

This collection of operators will perform a crucial roll in the undergoing definitions and statements.

2.3. Lipschitz functions, molecules, Besov and Triebel-Lizorkin spaces

Let now consider a positive and quasi-increasing function $\eta(t)$ defined on $t > 0$ such that $\lim_{t\to 0} \eta(t) = 0$. The Lipschitz space Λ^η is the class of all functions $f : X \to \mathbb{C}$ such that

$$|f|_\eta = \sup_{x\neq y} \frac{|f(x) - f(y)|}{\eta(\delta(x,y))} < \infty.$$

The quantity $|f|_\eta$ defines a semi-norm on Λ^η, since $|f|_\eta = 0$ for all constants functions f. Given a ball B in X, $\Lambda^\eta(B)$ denotes the set of functions $f \in \Lambda^\eta$ with support in B. Since a function belonging to this space is bounded, the number

$$\|f\|_\eta = \|f\|_\infty + |f|_\eta,$$

defines a norm that gives a Banach structure to $\Lambda^\eta(B)$. We say that a function f belongs to Λ^η_0 iff $f \in \Lambda^\eta(B)$ for some ball B. The space Λ^η_0 is the inductive limit of the Banach spaces $\Lambda^\eta(B)$. Finally, $(\Lambda_0^\eta)'$ will stand for the dual space of Λ_0^η.

Another suitable class of test functions was defined in [6].

Let $0 < \beta \leq 1$, $\gamma > 0$ and $x_0 \in X$ be fix. A function f is a *smooth molecule of type* (β, γ) *of width* d *centered in* x_0, if there exists a constant $C > 0$ such that

$$|f(x)| \leq C \frac{d^\gamma}{(d + \delta(x, x_0))^{1+\gamma}}, \tag{2.19}$$

$$|f(x) - f(x')| \leq C \frac{\delta(x, x')^\beta}{d^\beta} \left(\frac{d^\gamma}{(d + \delta(x, x_0))^{1+\gamma}} + \frac{d^\gamma}{(d + \delta(x', x_0))^{1+\gamma}} \right), \tag{2.20}$$

$$\int f(x) d\mu(x) = 0, \tag{2.21}$$

hold for every $x \in X$.

If the norm $\|f\|_{(\beta,\gamma)}$ is defined by the lowest of the constants appearing in (2.19) and (2.20), the set $\mathcal{M}^{(\beta,\gamma,x_0,d)}$ of all smooth molecules of type (β, γ) of width d centered in x_0 is a Banach space. By fixing $x_0 \in X$ and $d = 1$, that space will be named $\mathcal{M}^{(\beta,\gamma)}$, and its dual space, $(\mathcal{M}^{(\beta,\gamma)})'$. Along this work $< h, f >$ denotes the natural application of $h \in (\mathcal{M}^{(\beta,\gamma)})'$ to $f \in \mathcal{M}^{(\beta,\gamma)}$.

Local regularity of Besov and Triebel-Lizorkin spaces is associated to a function ψ being represented as the quotient of two quasi-increasing functions.

More precisely, *in the sequel* $\psi = \psi_1/\psi_2$, *where* $\psi_1(t)$ *and* $\psi_2(t)$ *are quasi-increasing functions of upper types* $s_1 < \epsilon$ *and* $s_2 < \epsilon$, *respectively*, where $\epsilon \leq \theta$ is the order of regularity of $\{S_t\}_{t>0}$. (With this election of ψ dual spaces are enclosed in the definitions and also are recovered the range of spaces of order α with $-\epsilon < \alpha < \epsilon$, as defined in [2]).

For $f \in (\mathcal{M}^{(\beta,\gamma)})'$, with $0 < \beta, \gamma < \epsilon$, a norm is defined by

$$\|f\|_{\dot{B}_p^{\psi,q}} = \left(\int_0^\infty \left(\frac{1}{\psi(t)} \|Q_t f\|_p \right)^q \frac{dt}{t} \right)^{1/q} \quad \text{if } \ 1 \leq p \leq \infty, 1 \leq q \leq \infty, \tag{2.22}$$

with the obvious change for the case $q = \infty$. By interchanging the order of the norms in L^p and l^q it is also defined the norm

$$\|f\|_{\dot{F}_p^{\psi,q}} = \left\| \left(\int_0^\infty \left(\frac{1}{\psi(t)} |Q_t f| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p}, \ \text{if } \ 1 < p, q < \infty. \tag{2.23}$$

The Besov space of order ψ, $\dot{B}_p^{\psi,q}$, for $1 \leq p, q \leq \infty$, is the set of all $f \in \left(\mathcal{M}^{(\beta,\gamma)}\right)'$, with $\beta > s_1$ and $\gamma > s_2$, such that

$$\|f\|_{\dot{B}_p^{\psi,q}} < \infty \ \text{and} \ \ |\langle f, h \rangle| \leq C \|f\|_{\dot{B}_p^{\psi,q}} \|h\|_{(\beta,\gamma)},$$

for all $h \in \mathcal{M}^{(\beta,\gamma)}$.

Analogously, the Triebel–Lizorkin space of order ψ, $\dot{F}_p^{\psi,q}$, with $1 < p, q < \infty$, is the set of all $f \in \left(\mathcal{M}^{(\beta,\gamma)}\right)'$, with $\beta > s_1$ and $\gamma > s_2$, such that

$$\|f\|_{\dot{F}_p^{\psi,q}} < \infty, \ \text{and} \ \ |\langle f, h \rangle| \leq \|f\|_{\dot{F}_p^{\psi,q}} \|h\|_{(\beta,\gamma)},$$

for all $h \in \mathcal{M}^{(\beta,\gamma)}$. A similar proof to that in [2] shows that these norms are equivalent to those given in [8] or [9] in terms of the differences of a discrete approximation to the identity $\{S_k\}_{k\in Z}$.

When $\psi(t) = t^\alpha$ we recover the spaces of order α, $\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$ with defined in [6] and [2].

By using the properties of the function ψ, and in the way it is proved in [6], follows that the classes $\dot{B}_p^{\psi,q}$, $1 \le p, q < \infty$ and $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$ are Banach spaces and their dual spaces are $\dot{B}_{p'}^{1/\psi,q'}$ and $\dot{F}_{p'}^{1/\psi,q'}$ respectively, with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

Moreover, the molecular space $\mathcal{M}^{(\beta,\gamma)}$ is embedded in $\dot{B}_p^{\psi,q}$ and $\dot{F}_p^{\psi,q}$ when $s_1 < \beta$ and $s_2 < \gamma$ and $\mathcal{M}^{(\epsilon',\epsilon')}$ is dense in $\dot{B}_p^{\psi,q}$ and $\dot{F}_p^{\psi,q}$ for all ϵ', such that $\max(s_1, s_2) < \epsilon' < \epsilon$.

2.4. Singular integral operators and $T1$-theorem

A continuous complex-valued function $K(x, y)$ defined on $\Omega = \{(x, y) \in X \times X : x \neq y\}$ is called a *standard kernel* if there exist $0 < \epsilon \le \theta$, and $C < \infty$ such that for all $x, y \in X$ with $x \neq y$,

$$|K(x, y)| \le C\delta(x, y)^{-1}, \quad \text{for every} \quad x \neq y; \tag{2.24}$$

and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \le C\delta(x, x')^\epsilon / \delta(x, y)^{1+\epsilon}, \tag{2.25}$$

for $\delta(x, y) > 2A\delta(x, x')$.

A continuous linear operator $T : \Lambda_0^\beta \to (\Lambda_0^\beta)'$ is a singular integral operator if there is a standard kernel K such that

$$< Tf, g >= \int\int K(x, y)f(y)g(x)d\mu(y)d\mu(x)$$

for all $f, g \in \Lambda_0^\beta$ with supp $f \cap$ supp $g = \emptyset$. We then write $T \in CZK(\epsilon)$.

A singular integral operator T is a Calderón-Zygmund operator if it can be extended to a bounded operator on L^2. For such operators T we write $T \in CZ0$.

To state a $T1$-theorem on spaces we need to define the weak boundedness property of an operator: A singular integral operator T is *weakly bounded* if there exist β, $0 < \beta \le 1$, and $C < \infty$ such that

$$|\langle Tf, g\rangle| \le C\mu(B)^{1+2\beta}\|f\|_\beta\|g\|_\beta, \tag{2.26}$$

for all f and g in $\Lambda^\beta(B)$ and each ball $B \subseteq X$.

The following is a version, for $CZK(\epsilon)$ operators, of the $T1$-theorems proved in [7] needed to establish our results. In that work the theorem was proved under weaker conditions of smoothness and size on the kernel.

Theorem 2.2 [7]

Let T be a $CZK(\epsilon')$ operator for some $0 < \epsilon' \leq \epsilon$, which is weakly bounded. If $T1 = T^*1 = 0$ then T can be extended to a bounded linear operator on $\dot{B}_p^{\psi_1/\psi_2,q}$, $1 \leq p, q < \infty$, and on $\dot{F}_p^{\psi_1/\psi_2,q}$, $1 < p, q < \infty$, if $\max(s_1, s_2) < \epsilon'$.

Since the theory of Littlewood-Paley on spaces of homogeneous type proved in [6] provides, the identification between L^2 and $\dot{F}_2^{t^0,2}$, it follows as a particular case that T is a CZO.

It is worthy mention that the constant of continuity obtained in the above theorem only depends of the constants appearing in (2.24), (2.25) and (2.26).

3. Integral and derivative operators of order ϕ

The fractional integral and derivative operators I_α and D_α defined on the context of R^n are associated to the kernels $|x-y|^{\alpha-n}$, and $|x-y|^{-\alpha-n}$, respectively, for $0 < \alpha < n$. To set the general idea of what follows later, set $n = 1$. Then, one can feature the kernel $|x-y|^{\alpha-1}$ in terms of an approximation to the identity in the following way: for $x \neq y$ and $-\infty < \alpha < 1$,

$$\begin{aligned} |x-y|^{\alpha-1} &= (\alpha-1)\int_{|x-y|}^{\infty} t^{\alpha-2}dt \\ &= (\alpha-1)\int_0^{\infty} t^{\alpha-1}t^{-1}\chi_{B(0,t)}(|x-y|)dt \\ &= 2(\alpha-1)\int_0^{\infty} t^{\alpha-1}s_t(x-y)dt, \end{aligned} \tag{3.27}$$

where the family of kernels $s_t(x-y) = (2t)^{-1}\chi_{B(0,t)}(|x-y|)$, $t > 0$ determines an approximation to the identity, namely, $\lim_{t\to 0^+} s_t * f = f$ and $\lim_{t\to\infty} s_t * f = 0$, both limits in $L^2(R)$.

In the setting of spaces of homogeneous type, Gatto, Segovia and Vági, in [4], defined kernels $\delta_\alpha(x,y)^{\alpha-1}$ with similar features, by means of an approximation to the identity.

If we consider a function $\phi(t)$ instead of the potential t^α, ($\alpha > 0$), we aim to define the kernel associated to the integral operator of order ϕ resembling $\phi(\delta(x,y))/\delta(x,y)$, as well as the kernel of the derivative operator should resemble $(\phi(\delta(x,y))\delta(x,y))^{-1}$. On the other hand, it makes sense to replace the function $t^{\alpha-1}$ or $t^{-\alpha-1}$ on the right side of (3.27) by $\frac{d}{dt}\phi(t)$ or $\frac{d}{dt}(1/\phi(t))$. In the fashion of [4] and for $\phi \in \mathcal{C}$, the kernels associated to our operators were defined in [8]. To prove the invertibility results we need to restrict our attention to the class $\mathcal{A}$.

Let, then, consider a positive and symmetric approximation to the identity of order $\epsilon \leq \theta$, $\{S_t\}_{t>0}$ associated to the family of kernels $s_t(x,y)$, $t > 0$, as defined in Section 2.2.

Let also consider $\phi \in \mathcal{A}$, that is

$$\phi(t) = \phi(1)e^{\int_1^t (\eta(s))/s\,ds}, \tag{3.28}$$

with η a measurable function such that $0 < \beta \leq \eta \leq \alpha < 1$.

Let define

$$s(\phi) = \sup_{t>0} \eta(t) = \sup_{t>0} \frac{t\phi'(t)}{\phi(t)},$$

and

$$i(\phi) = \inf_{t>0} \eta(t).$$

It follows from (2.6) and (2.7) that ϕ is of lower-type $i(\phi)$ and upper-type $s(\phi)$.

Set

$$K_\phi(x,y) = \int_0^\infty \frac{\phi(t)\eta(t)}{t} s_t(x,y)dt \text{ for } x \neq y. \tag{3.29}$$

The application K_ϕ has the representation in terms of a quasi-metric we are seeking for. Indeed, since $\eta(t) \leq \alpha < 1$ then $\phi(t)/t$ is continuous, strictly decreasing and invertible on R^+ and, since the integral in (3.29) is positive then it is possible to define a nonnegative application $\delta_\phi(x,y)$ as the unique solution of the equation

$$\frac{\phi(\delta_\phi(x,y))}{\delta_\phi(x,y)} = K_\phi(x,y) \text{ for } x \neq y, \text{ and}$$
$$\delta_\phi(x,y) = 0 \text{ for } x = y.$$

In fact, it follows from the properties of $\phi(t)/t$ and s_t that the $\delta_\phi(x,y)$ defines a quasi-metric equivalent to the natural quasi-metric δ of X and, also, $\phi(\delta_\phi(x,y))/\delta_\phi(x,y)$ is equivalent to $\phi(\delta(x,y))/\delta(x,y)$.

The integral operator I_ϕ will be defined for $f \in \Lambda^\xi \cap L^1$, where ξ is a quasi-increasing function of upper-type $s > 0$, as follows:

$$I_\phi f(x) = \int_X K_\phi(x,y)f(y)d\mu(y). \tag{3.30}$$

There is also an extension of this operator to the entire space Λ^ξ: If $\alpha + s < \theta$ and $f \in \Lambda^\xi$ then

$$\tilde{I}_\phi f(x) := \int_X (K_\phi(x,y) - K_\phi(x_0,y))f(y)d\mu(y), \tag{3.31}$$

for every $x \in X$ and an arbitrary fix $x_0 \in X$.

We now define the kernel associated to the derivative operator of order ϕ as:

$$K_{1/\phi}(x,y) = \int_0^\infty \frac{\eta(t)}{\phi(t)t} s_t(x,y)dt \text{ for } x \neq y.$$

Reasoning as before, there is a quasi-distance $\delta_{1/\phi}(x,y)$, equivalent to δ, defined as the unique solution of

$$\big(\phi(\delta_{1/\phi}(x,y))\delta_{1/\phi}(x,y)\big)^{-1} = K_{1/\phi}(x,y) \text{ for } x \neq y \text{ and}$$
$$\delta_{1/\phi}(x,y) = 0 \text{ for } x = y.$$

Moreover, $K_{1/\phi}(x,y)$ is equivalent to $(\phi(\delta(x,y))\delta(x,y))^{-1}$.

Thus, for $f \in \Lambda^\xi \cap L^\infty$, where ξ is a function of positive lower-type ι and upper-type s, and $\alpha < \iota$, we define the derivative operator by

$$D_\phi f(x) = \int_X K_{1/\phi}(x,y)(f(y) - f(x))d\mu(y). \tag{3.32}$$

Its extension to Λ^{ξ} is given by

$$\tilde{D}_{\phi}f(x) = \int_X (K_{1/\phi}(x,y)(f(y)-f(x)) \\ - K_{1/\phi}(x_0,y)(f(y)-f(x_0)))d\mu(y) \tag{3.33}$$

for each $x \in X$ and an arbitrary, but fix, $x_0 \in X$.

To study the action of these operators on Lipschitz spaces, and on Besov and Triebel-Lizorkin spaces, we need to know regularity conditions of their kernels. These properties, and also a cancellation one for K_{ϕ}, are obtained from conditions (2.10) to (2.17), of s_t, and properties of ϕ, given in Section 2.

Since the molecules are integrable and bounded Lipschitz functions and, furthermore, they are dense on Besov and Triebel-Lizorkin spaces, we are able to extend the definitions and to study the action of both operators on those spaces. The results obtained in [8] in this direction, with $\phi \in \mathcal{C}$, are the following ones:

Theorem 3.1

Let ϕ be of lower type $\beta > 0$ and upper type $\alpha < \epsilon \leq \theta$. If $s_1 + \alpha < \epsilon$ and $s_2 + \alpha - \beta < \epsilon$, then I_{ϕ} is a linear continuous operator from $\dot{B}_p^{\psi,q}$ to $\dot{B}_p^{\phi\psi,q}$ and from $\dot{F}_p^{\psi,q}$ to $\dot{F}_p^{\phi\psi,q}$.

Also, if $s_1 < \epsilon$ and $s_2 + \alpha < \epsilon$ then D_{ϕ} maps with continuity $\dot{B}_p^{\psi,q}$ on $\dot{B}_p^{\psi/\phi,q}$, and from $\dot{F}_p^{\psi,q}$ to $\dot{F}_p^{\psi/\phi,q}$ s

In studying the composition $T_{\phi} = D_{\phi} \circ I_{\phi}$, the following result was obtained in [9]:

Theorem 3.2

Let ϕ be of positive lower type and of upper type α such that $\alpha < \epsilon$.
If $\max(s_1, s_2) + \alpha < \epsilon$ *then $T_{\phi} = D_{\phi} \circ I_{\phi}$ is a Calderón-Zygmund operator bounded on $\dot{F}_p^{\psi,q}$* and *$\dot{B}_p^{\psi,q}$.*

Since I_{ϕ} and D_{ϕ} are self-adjoint, (see [8]) then $S_{\phi} = I_{\phi} \circ D_{\phi}$ is the adjoint operator of T_{ϕ} and the statement of Theorem 3.2 is also true for S_{ϕ}.

4. Main theorems

Let denote $\psi = \psi_1/\psi_2$, where ψ_1 and ψ_2 are quasi-increasing functions of upper types s_1 and s_2 respectively and denote

$$T_{\phi} = D_{\phi} \circ I_{\phi}.$$

Our main results are the following.

Theorem 4.1

Let $\phi \in \mathcal{A}$ and denote $\alpha = s(\phi)$. If $\max(s_1, s_2) + \alpha < \epsilon$ then the inverse operator T_ϕ^{-1} exists and is bounded on $\dot{B}_p^{\psi,q}$, $1 \leq p, q < \infty$ and $\dot{F}_p^{\psi,q}$, $1 < p, q < \infty$ for $\alpha > 0$ small enough.

Theorem 4.2

Let $\phi \in \mathcal{A}$ and denote $\alpha = s(\phi)$. If $\max(s_1, s_2) + \alpha < \epsilon$ then T_ϕ^{-1} is a singular integral operator, for α small enough. More precisely, $T_\phi^{-1} \in CZO(\epsilon')$ for every $0 < \epsilon' < \epsilon$.

From the above theorems follows that T_ϕ^{-1} is a Calderón–Zygmund operator bounded on Besov and Triebel-Lizorkin of order ψ. The assertions in Theorem 4.1 and Theorem 4.2 follow in the same way for $S_\phi = I_\phi \circ D_\phi$.

Furthermore, it also follows that I_ϕ and D_ϕ are both invertible and their inverses, given by

$$I_\phi^{-1} = T_\phi^{-1} \circ D_\phi = D_\phi \circ S_\phi^{-1}$$

and

$$D_\phi^{-1} = S_\phi^{-1} \circ I_\phi = I_\phi \circ T_\phi^{-1},$$

are continuous linear operators. From the previous result and Theorem 3.1, it follows that.

Theorem 4.3

Let $\phi \in \mathcal{A}$ and denote $\alpha = s(\phi)$ and suppose that $\max(s_1, s_2) + \alpha < \epsilon$. There is $\alpha_0 > 0$ such that if $0 < \alpha < \alpha_0$ then I_ϕ is an homeomorphism from $\dot{B}_p^{\psi,q}$ onto $\dot{B}_p^{\phi\psi,q}$ and from $\dot{F}_p^{\psi,q}$ onto $\dot{F}_p^{\phi\psi,q}$. Analogously, D_ϕ is an homeomorphism from $\dot{B}_p^{\phi\psi,q}$ onto $\dot{B}_p^{\psi,q}$ and from $\dot{F}_p^{\phi\psi,q}$ onto $\dot{F}_p^{\psi,q}$.

5. Main lemmas and proofs of Theorem and Theorem

Let $\phi \in \mathcal{A}$ and denote $\alpha = s(\phi)$ and $\beta = i(\phi)$.

Let also $\{S_t\}_{t>0}$ be an approximation to the identity of order ϵ.

In the next lemmas we obtain representation formulas, as in [4], for I_ϕ, D_ϕ and T_ϕ in terms of the family of operators $Q_t = -t\frac{\partial}{\partial t}S_t$, $(t > 0)$, defined in 2.2.

The letter C will denote a constant that may change from step to step.

Lemma 5.1

Let f be a smooth molecule of type (η, γ), $\gamma > 0$. The following formulas hold punctually and in the weak sense:
if $\alpha + \eta \leq \epsilon$ then

$$I_\phi f(x) = \int_0^\infty \phi(t) Q_t f(x) \frac{dt}{t}; \tag{5.34}$$

if $0 < \alpha < \eta \leq \epsilon$ then

$$-D_\phi f(x) = \int_0^\infty \frac{1}{\phi(t)} Q_t f(x) \frac{dt}{t}; \tag{5.35}$$

and if $\eta \leq \epsilon$ *then*

$$f(x) = \int_0^\infty Q_t f(x) \frac{dt}{t}. \tag{5.36}$$

Proof. In fact, by Definition 3.30 and Fubini's theorem we have that

$$\begin{aligned} I_\phi f(x) &= \int_X \int_0^\infty \frac{\phi(t)\eta(t)}{t} s_t(x,y) f(y) dt d\mu(y) \\ &= \int_0^\infty \frac{\phi(t)\eta(t)}{t} u(x,t) dt, \end{aligned}$$

where $u(x,t) := S_t f(x) = \int s_t(x,y) f(y) dy$.

If we denote $v(x,t) = -t\frac{\partial}{\partial t} u(x,t)$, since $\frac{\phi(t)\eta(t)}{t} = \frac{d}{dt}\phi(t)$ then integration by parts gives

$$I_\phi f(x) = \lim_{a\to 0, b\to\infty} (\phi(b)u(x,b) - \phi(a)u(x,a)) + \int_0^\infty \phi(t) v(x,t) \frac{dt}{t}.$$

But, by (2.12) and the fact that ϕ is of upper type $\alpha < 1$ we have

$$\lim_{b\to\infty} \phi(b)|u(x,b)| \leq \lim_{b\to\infty} \frac{\phi(b)}{b} \|f\|_1 = 0 \tag{5.37}$$

and, on the other hand, $\lim_{a\to 0} \phi(a) = 0$ and $u(x,a)$ is bounded since, by (2.10) and (2.12)

$$\begin{aligned} |u(x,a) - f(x)| &\leq \int |s_a(x,y)||f(y) - f(x)| d\mu(y) \\ &\leq C\|f\|_{(\eta,\gamma)} \int |s_a(x,y)| \delta(x,y)^\eta d\mu(y) \\ &\leq C\|f\|_{(\eta,\gamma)} a^\eta. \end{aligned} \tag{5.38}$$

Thus, since $v(x,t) = Q_t f(x)$ we get equation (5.34).

Analogously, since $\frac{\eta(t)}{t\phi(t)} = \frac{d}{dt}(\frac{-1}{\phi(t)})$ then

$$\begin{aligned} D_\phi f(x) &= \int_0^\infty \frac{\eta(t)}{t\phi(t)} (u(x,t) - f(x)) dt \\ &= -\int_0^\infty \frac{1}{\phi(t)} v(x,t) \frac{dt}{t}, \end{aligned}$$

by (2.12), (5.38) and the fact that $a^\eta/\phi(a) \leq C a^{\eta-\alpha}$ for $a < 1$.

Finally equation (5.36) is the known identity of Coifman.

The identities in the weak sense

$$\lim_{a\to 0, b\to\infty} \int_a^b \phi(t) < Q_t f, g > \frac{dt}{t} =< I_\phi f, g >$$

and

$$\lim_{a\to 0, b\to\infty} \int_a^b (\phi(t))^{-1} < Q_t f, g > \frac{dt}{t} =< D_\phi f, g >$$

follow from observing that the double integrals

$$\int_0^\infty \int \phi(t) Q_t f(x) g(x) d\mu(x) \frac{dt}{t}$$

and

$$\int_0^\infty \int (\phi(t))^{-1} Q_t f(x) g(x) d\mu(x) \frac{dt}{t} \, \Box$$

converge absolutely for all $g \in \mathcal{M}^{(\rho,\gamma)}$. $\Box$

In the sequel, we use the notation

$$\chi(E) := \chi_E(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in E \\ 0 & \text{if} \quad (x,y) \notin E, \end{cases}$$

and $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$.

In view of (2.10) to (2.17), the kernel $q_t(x,y)$ of Q_t, defined in (2.18), is symmetric and also satisfy

$$q_t(x,y) = 0 \text{ for } \ \delta(x,y) > b_1 t, \tag{5.39}$$

$$|q_t(x,y)| \leq \frac{C}{t} \text{ for all } \ (x,y), \tag{5.40}$$

$$|q_t(x,y) - q_t(x',y)| + |q_t(y,x) - q_t(y,x')| \leq C \frac{\delta(x,x')^\epsilon}{t^{1+\epsilon}}, \tag{5.41}$$

$$\int q_t(x,y) d\mu(y) = \int q_t(y,x) d\mu(y) = 0 \text{ for all } \ x \in X \text{ and } \ t > 0. \tag{5.42}$$

The operator $Q_s Q_t$ has kernel

$$q_s q_t(x,y) = \int_X q_s(x,z) q_t(z,y) d\mu(z).$$

Given $m = 1, 2, \ldots$ the operator $Q_{u_1} Q_{u_1 v_1} \circ \cdots \circ Q_{u_m} Q_{u_m v_m}$ has associated the kernel

$$q_{u_1} q_{u_1 v_1} \ldots q_{u_m} q_{u_m v_m}(x,y) =$$
$$\int \cdots \int q_{u_1} q_{u_1 v_1}(x, z_1) q_{u_2} q_{u_2 v_2}(z_1, z_2) \ldots q_{u_m} q_{u_m v_m}(z_{m-1}, y) d\mu(z_1) \ldots d\mu(z_{m-1}).$$

The following estimates are continuous versions of those given in [6] smoothness), in page 17 –for size– and page 25 –for smoothness–, of [6].

Lemma 5.2

There is a constant $C > 0$ such that for every $0 < \gamma < \epsilon$

$$\begin{aligned} &\int_0^\infty |q_{u_1} q_{u_1 v_1} \ldots q_{u_m} q_{u_m v_m}(x,y)| \frac{du_1}{u_1} \cdots \frac{du_m}{u_m} \\ &\leq \ C^{2m} m^2 A^m \left(\left(v_1 \wedge \frac{1}{v_1}\right) \ldots \left(v_m \wedge \frac{1}{v_m}\right) \right)^\gamma \delta(x,y)^{-1} \end{aligned} \tag{5.43}$$

for $x \neq y$, and $v_1 > 0, \ldots, v_m > 0$.

For every $0 < \epsilon' < \epsilon$, there exist γ, with $0 < \gamma < \epsilon$ such that

$$\int_0^\infty |q_{u_1}q_{u_1v_1}\dots q_{u_m}q_{u_mv_m}(x,y) - q_{u_1}q_{u_1v_1}\dots q_{u_m}q_{u_mv_m}(x,y')|$$
$$+|q_{u_1}q_{u_1v_1}\dots q_{u_m}q_{u_mv_m}(y,x) - q_{u_1}q_{u_1v_1}\dots q_{u_m}q_{u_mv_m}(y',x)|\frac{du_1}{u_1}\dots\frac{du_m}{u_m}$$
$$\leq C^{2m}m^M A^m\left(\left(v_1\wedge\frac{1}{v_1}\right)\dots\left(v_m\wedge\frac{1}{v_m}\right)\right)^\gamma \delta(y,y')^{\epsilon'}\delta(x,y)^{-(1+\epsilon')}, \tag{5.44}$$

for $\delta(x,y) > 2A\delta(y,y')$, some $M > 0$ and $v_1 > 0,\dots,v_m > 0$.

We now give a representation formula for T_ϕ, analogous to that given in [4], applying Lemma 5.1 and Theorem 2.2.

Lemma 5.3

Let $\phi \in \mathcal{A}$ and denote $\alpha = s(\phi)$, then the following formulae hold for all $f \in \mathcal{M}^{(\eta,\rho)}$, with $\alpha + \eta < \epsilon$ and $\rho > 0$ and all $g \in \mathcal{M}^{(\eta',\rho')}$, $0 < \eta' < \epsilon$ and $\rho' > 0$:

$$< T_\phi f, g >= -\int_0^\infty\int_0^\infty \frac{\phi(t)}{\phi(s)} < Q_sQ_tf, g > \frac{dt}{t}\frac{ds}{s} \tag{5.45}$$

and, for all $0 < \eta < \epsilon$

$$< f, g >= \int_0^\infty\int_0^\infty < Q_sQ_tf, g > \frac{dt}{t}\frac{ds}{s}. \tag{5.46}$$

Proof. Let first see that the integral in the right side of (5.45) converges absolutely for $f \in \dot{B}_p^{\psi,q}$ $(\dot{F}_p^{\psi,q})$ and $g \in \dot{B}_{p'}^{1/\psi,q'}$ $(\dot{F}_{p'}^{1/\psi,q'})$. Making the change of variable $s = u$ and $t = uv$ in (5.45) and (5.46), we obtain

$$\int_0^\infty\left(\int_0^\infty \frac{\phi(uv)}{\phi(u)} < Q_uQ_{uv}f, g > \frac{du}{u}\right)\frac{dv}{v} \tag{5.47}$$

and

$$\int_0^\infty\left(\int_0^\infty < Q_uQ_{uv}f, g > \frac{du}{u}\right)\frac{dv}{v}. \tag{5.48}$$

The estimates given in Lemma 5.2 for the case $m = 1$, the weak boundary property -which follows from continuous versions of inequalities (2.52) and (2.53) in [6]-, the fact that $Q_uQ_{uv}1 = 0$ and $(Q_uQ_{uv})^*1 = Q_{uv}Q_u1 = 0$, and Theorem 2.2 show that:

For every $0 < \epsilon' < \epsilon$ such that $\max(s_1, s_2) < \epsilon'$ there exists γ, with $0 < \gamma < \epsilon$, and a constant $C > 0$ such that, for all $v > 0$,

$$\int_0^\infty | < Q_uQ_{uv}f, g > |\frac{du}{u} \leq C\left(v\wedge\frac{1}{v}\right)^\gamma \|f\|_V\|g\|_{V'}, \tag{5.49}$$

where $V = \dot{B}_p^{\psi,q}$ $(\dot{F}_p^{\psi,q})$ and $V' = \dot{B}_{p'}^{1/\psi,q'}$ $(\dot{F}_{p'}^{1/\psi,q'})$.

Now using estimates (2.6) and (2.7), and denoting $\alpha = s(\phi)$, $\beta = i(\phi)$, we have that

$$\int_0^\infty \int_0^\infty \frac{\phi(uv)}{\phi(u)} | < Q_u Q_{uv} f, g > | \frac{du}{u}\frac{dv}{v}$$
$$\leq C \int_0^\infty (v^\beta \vee v^\alpha) \left(v \wedge \frac{1}{v} \right)^\gamma \frac{dv}{v} \|f\|_V \|g\|_{V'},$$

the last integral being finite if $0 \leq \beta \leq \alpha < \gamma$. Thus, (5.47) and the integral in (5.45) are absolutely convergent. Also, (5.49) gives absolute convergence for (5.48) and thus, for (5.46).

The identities (5.45) and (5.46) now follow easily using Lemma 5.1, from the fact that $< Q_s Q_t f, g > = < Q_t f, Q_s g >$.

5.1. Proof of Theorem 4.1

By a density argument it is enough to show that for every $0 < r < 1$ there exists $\alpha_0 < \epsilon$ such that

$$\|(I + T_\phi)f\| < r$$

for every $\alpha = s(\phi) < \alpha_0$ and $f \in \mathcal{M}^{(\epsilon', \epsilon')}$ such that $\alpha + \max(s_1, s_2) < \epsilon' < \epsilon$, where the norm is taken in either any of our spaces and I denotes the identity operator. From identities (5.47) and (5.48) we can write

$$(I + T_\phi)f = \int_0^\infty \int_0^\infty \left(1 - \frac{\phi(uv)}{\phi(v)}\right) Q_u Q_{uv} f(x) \frac{du}{u}\frac{dv}{v}. \tag{5.50}$$

Notice first that from (2.6) and (2.7) follows that,

$$\left|1 - \frac{\phi(uv)}{\phi(u)}\right| \leq |1 - v^\alpha| . \tag{5.51}$$

Hence, we have

$$\|(I + T_\phi)f\| \leq C \int_0^\infty |1 - v^\alpha| \left(v \wedge \frac{1}{v} \right)^\gamma \frac{dv}{v} \|f\| . \tag{5.52}$$

But it is proved in [4] that there exists $\alpha_0 < \gamma$ such that for every $\alpha < \alpha_0$

$$C \int_0^\infty |1 - v^\alpha| \left(v \wedge \frac{1}{v} \right)^\gamma \frac{dv}{v} < r. \tag{5.53}$$

Indeed, there exists N big enough such that

$$\int_0^{1/N} (1 - v^\alpha) v^\gamma \frac{dv}{v} < \frac{r}{3C}$$

uniformly in α. On the other hand for all $\alpha \leq \alpha_1 < \gamma$,

$$\int_N^\infty (v^\alpha - 1) v^{-\gamma} \frac{dv}{v} \leq \int_N^\infty (v^{\alpha_1} - 1) v^{-\gamma} \frac{dv}{v} \leq \int_N^\infty v^{\alpha_1 - \gamma} \frac{dv}{v} < \frac{r}{3C}$$

for $N \geq N_0$. Once N is fixed , we can choose $\alpha_0 < \gamma$ such that if $\alpha < \alpha_0$ then

$$\int_{1/N}^1 (1 - v^\alpha) v^\gamma \frac{dv}{v} + \int_1^N (v^\alpha - 1) v^{-\gamma} \frac{dv}{v} = \int_1^N (v^\alpha - v^{-\alpha}) v^{-\gamma} \frac{dv}{v}$$
$$\leq (N^\alpha - N^{-\alpha}) \int_1^\infty v^{-\gamma} \frac{dv}{v} < \frac{r}{3C}.$$

Thus the theorem is proved. □

5.2. Proof of Theorem 4.2

In view of Theorem 4.1, since $\|I+T_\phi\|<1$ for $\alpha=s(\phi)$ small enough, we consider the von Neumann representation of the operator T_ϕ^{-1} as

$$-T_\phi^{-1}=\sum_{m=0}^{\infty}(I+T_\phi)^m. \tag{5.54}$$

Denote

$$T_v f=\int_0^\infty\left(1-\frac{\phi(uv)}{\phi(v)}\right)Q_uQ_{uv}f\,\frac{du}{u},\quad (v>0). \tag{5.55}$$

From (5.50) and (5.55), for each $m\in\mathbb{N}$ we have,

$$(I+T_\phi)^m = \int_0^\infty\cdots\int_0^\infty T_{v_1}\circ\cdots\circ T_{v_m}\frac{dv_1}{v_1}\cdots\frac{dv_m}{v_m} \tag{5.56}$$

where

$$T_{v_1}\circ\cdots\circ T_{v_m}= \int_0^\infty\cdots\int_0^\infty\left(1-\frac{\phi(u_1v_1)}{\phi(u_1)}\right)\cdots\left(1-\frac{\phi(u_mv_m)}{\phi(u_m)}\right)Q_{u_1}Q_{u_1v_1}\circ\cdots\circ Q_{u_m}Q_{u_mv_m} \frac{du_1}{u_1}\cdots\frac{du_m}{u_m}. \tag{5.57}$$

Let denote

$$K_{v_1\dots v_m}(x,y)= \int_0^\infty\cdots\int_0^\infty\left(1-\frac{\phi(u_1v_1)}{\phi(u_1)}\right)\cdots\left(1-\frac{\phi(u_mv_m)}{\phi(u_m)}\right)q_{u_1}q_{u_1v_1}\dots q_{u_m}q_{u_mv_m}(x,y) \frac{du_1}{u_1}\cdots\frac{du_m}{u_m}, \tag{5.58}$$

$$K^m(x,y) = \int_0^\infty\cdots\int_0^\infty K_{v_1\dots v_m}(x,y)\frac{dv_1}{v_1}\cdots\frac{dv_m}{v_m}, \tag{5.59}$$

and

$$K_\phi^{-1}(x,y)=-\sum_{m=0}^{\infty}K^m(x,y)\,. \tag{5.60}$$

Applying inequalities (5.51) and (5.43), from (5.59) and (5.58), for every $0<\gamma<\epsilon$ we have that

$$\begin{aligned}|K^m(x,y)|&\\ \le C^{2m}&m^2A^m\delta(x,y)^{-1}\\ &\times\int_0^\infty\cdots\int_0^\infty|1-v_1^\alpha|\dots|1-v_m^\alpha|\left(\left(v_1\wedge\frac{1}{v_1}\right)\cdots\left(v_m\wedge\frac{1}{v_m}\right)\right)^\gamma\frac{dv_1}{v_1}\cdots\frac{dv_m}{v_m}\\ =C^{2m}&m^2A^m\left(\int_0^\infty|1-v^\alpha|\left(v\wedge\frac{1}{v}\right)^\gamma\frac{dv}{v}\right)^m\delta(x,y)^{-1}.\end{aligned} \tag{5.61}$$

Analogously, by (5.51) and (5.44), for all $\epsilon' < \epsilon$ there exists $\gamma < \epsilon$ such that

$$\begin{aligned}|K^m(x,y) - K^m(x,y')| \\ + |K^m(y,x) - K^m(y',x)| \\ \le C^{2m} m^M A^m \left(\int_0^\infty |1 - v^\alpha| \Big(v \wedge \frac{1}{v}\Big)^\gamma \frac{dv}{v}\right)^m \delta(y,y')^{\epsilon'} \delta(x,y)^{-(1+\epsilon')}; \qquad (5.62)\end{aligned}$$

for $\delta(x,y) > 2A\delta(y,y')$.

Choosing a constant $\bar{C}$ such that $\max(m^2, m^M) \le \bar{C}^m$, denoting

$$\mu = \int_0^\infty |1 - v^\alpha| \Big(v \wedge \frac{1}{v}\Big)^\gamma \frac{dv}{v}$$

and reasoning as in Theorem 4.1, there exists $\alpha_0 < \epsilon$ such that $r = C^2 \bar{C} A \mu < 1$, for each $\alpha < \alpha_0$.

Hence, for $\alpha < \alpha_0$

$$|K^m(x,y)| \le r^m \delta(x,y)^{-1}, \ x \ne y. \qquad (5.63)$$

and,

$$\begin{aligned}|K^m(x,y) - K^m(x,y')| + |K^m(y,x) - K^m(y',x)| \\ \le r^m \delta(x,y)^{-(1+\epsilon')} \delta(y,y')^{\epsilon'}, \qquad (5.64)\end{aligned}$$

for $\delta(x,y) > 2A\delta(y,y')$. From (5.60), adding on m in (5.63) and (5.64) we get the assertion of the theorem. □

6. Application

In this section we obtain an identification between the Sobolev-type space $L^{p,\phi}$ and the inhomogeneous Triebel-Lizorkin space $F_p^{\phi,2}$ in the context of normal spaces of homogeneous type.

This result was first prove in [5] for the case $\phi(t) = t^\alpha$. Our proof is shorter after using Theorem 4.3.

Let X be a normal space of homogeneous type and $\phi \in \mathcal{A}$, $0 < \alpha = s(\phi) < \theta$, satisfying (2.6). We extend the definition of D_ϕ to functions in L^p, $1 < p < \infty$, in the following way.

DEFINITION 6.1. Let $f \in L^p$, $1 < p < \infty$, if there exists a function $g \in L^p$ such that $< f, D_\phi h > = < g, h >$ for all $h \in \Lambda_0^\beta$, $\theta \ge \beta > \alpha$, then we define $g = D_\phi f$.

DEFINITION 6.2. Let $1 < p < \infty$ and $\phi \in \mathcal{A}$. The space $L^{p,\phi}$ is the set of all $f \in L^p$ with $D_\phi f$ in L^p, with the norm

$$\|f\|_{L^{p,\phi}} = \|f\|_{L^p} + \|D_\phi f\|_{L^p}.$$

DEFINITION 6.3. Let $1 < p < \infty$ and $\phi \in \mathcal{A}$. The space $F_p^{\phi,2}$ is the set of functions $f \in L^p$ for which $\|f\|_{\dot{F}_p^{\phi,2}} < \infty$, with the norm

$$\|f\|_{F_p^{\phi,2}} = \|f\|_{L^p} + \|f\|_{\dot{F}_p^{\phi,2}}.$$

Theorem 6.1.

For $1 < p < \infty$ and $\phi \in \mathcal{A}$, there exists $\alpha_0 > 0$ such that for $s(\phi) < \alpha_0$, $L^{p,\phi} = F_p^{\phi,2}$ and the corresponding norms are equivalent.

Proof. Since $L^{p,\phi}$ and $F_p^{\phi,2}$ both are Banach spaces and the space Λ_0^{β} is dense in them for $s(\phi) < \beta \le \theta$, it is enough to show that for $f \in \Lambda_0^{\beta}$,

$$\|D_\phi f\|_{L^p} \simeq \|f\|_{\dot{F}_p^{\phi,2}}. \tag{6.65}$$

But by the Littlewood-Paley theory developed in [6] and in view of Theorem 3.1 we have

$$\|D_\phi f\|_{L^p} \simeq \|D_\phi f\|_{\dot{F}_p^{0,2}} \simeq \|f\|_{\dot{F}_p^{\phi,2}}, \tag{6.66}$$

since D_ϕ is an homeomorphism from $\dot{F}_p^{\phi,2}$ onto $\dot{F}_p^{02}$, for s_ϕ small enough. □

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