Discrete Yamabe problem for polyhedral surfaces

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Abstract

We introduce a new discretization of the Gaussian curvature on piecewise flat surfaces. As the prime new feature the curvature is scaled by the factor $1/r^2$ upon scaling the metric globally with the factor r. We develop a variational principle to tackle the corresponding discrete uniformisation theorem – we show that each piecewise flat surface is discrete conformally equivalent to one with constant discrete Gaussian curvature. This surface is in general not unique. We demonstrate uniqueness for particular cases and disprove it in general by providing explicit counterexamples. Special attention is paid to dealing with change of combinatorics.

1 Introduction

The celebrated Poincaré-Koebe uniformisation theorem states that any (closed, oriented) Riemannian surface is conformally equivalent to one with constant Gaussian curvature $K \in \{-1, 0, 1\}$. This surface is unique if $K = -1$, unique up to global scale factor if $K = 0$, and unique up to Möbius transformations if $K = 1$.

In order to discretize the uniformisation theorem, reasonable definitions of discrete *surface and metric*, *conformal equivalence* and *Gaussian curvature* need to be established. In the past decades, various definitions of these notions have been developed. We briefly mention the work of William Thurston [14] and Bennet Chow and Feng Luo [5] on circle packing metrics.

We discretize the uniformisation theorem in the realm of piecewise flat surfaces. A *piecewise flat* surface is a (closed) oriented topological surface S, together with a finite set $V \subseteq S$ of *marked points*, and a metric d on S such that a neigbourhood of every point $p \in S$ is isometric to the Euclidean plane if $p \notin V$, and isometric to the tip of a Euclidean cone, with cone angle $\alpha_p > 0$, possibly being equal to 2π, if $p \in V$. Such a metric d is called a *PL-metric.*

The *angle defect* is the map

$$W : V \to \mathbb{R}, \qquad p \mapsto W_p := 2\pi - \alpha_p.$$

The angle defect is widely used as the discretization of the Gaussian curvature. However, it does not mirror the following property of its continuous counterpart: if a Riemannian metric is globally scaled by a factor r, the Gaussian curvature is scaled by the factor $1/r^2$. Upon global rescaling of a PL-metric, the angle defect stays the same. We introduce a new discretization of the Gaussian curvature, that mirrors this property.

Definition 1.1. *The* ***discrete Gaussian curvature*** *at vertex $i \in V$ is the quotient of the angle defect W_i and the area A_i of the Voronoi cell V_i:*

$$K : V \to \mathbb{R}, \qquad i \mapsto K_i := \frac{W_i}{A_i}.$$

With the definition of conformal equivalence pioneered by Luo [10], and developed by Bobenko, Pinkall, Springborn [3], we obtain the following theorem.

Theorem 1.2 (Discrete uniformisation theorem)**.** *For every PL-metric d on a marked surface (S, V), there exists a conformally equivalent PL-metric $\tilde{d}$ such that the surface $(S, V, \tilde{d})$ has constant discrete Gaussian curvature. The PL-metric $\tilde{d}$ is, in general, not unique.*

Proof. The existence of metrics with constant discrete Gaussian curvature in every conformal class for surfaces of genus $g = 0$ and $g > 2$ follows from the variational principle Theorem 4.13 and Corollary 5.2. For surfaces of genus one, metrics with constant discrete Gaussian curvature are critical points of the function $\mathbb{E}$ (see Definition 4.3 and Theorem 4.6) without constraints. Existence of critical points of the function $\mathbb{E}$ was demonstrated by Springborn in [13], Theorem 11.2. □

Remark. *The metric $\tilde{d}$ is unique up to global scaling in the following three cases:*
(1) S is of genus zero and $|V| = 3$,
(2) S is of genus one,
(3) S is of genus larger than one and $|V| = 1$.

Uniqueness of metrics with constant discrete Gaussian curvature for surfaces of type (1) *follows from [9], Theorem 7.2.1. For surfaces of type* (2)*, uniqueness follows from the local convexity of the function* $\mathbb{E}$*. For an explicit proof see Proposition 7.12 in [13]. The same results were also shown in [7] and [6]. Uniqueness of PL-metrics for surfaces of type* (3) *follows from the fact that, in this case, two PL-metrics are discrete conformally equivalent only if they differ by global scaling. See also [9], Chapter 7 and 9.1.*

Counterexamples for uniqueness are presented in Section 3.

2 Preliminaries

The majority of the notions in this chapter is well-known and thus the proofs are omitted. An excellent summary can be found in [13], Chapters 4 and 10. Delaunay triangulations are also described in [2].

2.1 Special triangulations

Let (S, V) be a marked surface. A *triangulation of* (S, V) is any triangulation of S with the vertex set equal to V. We denote it by Δ. The set of edges and faces of Δ is denoted by E_Δ and F_Δ, respectively. Let d be a PL-metric, or a complete finite area hyperbolic metric on $S\backslash V$, with cusps in V. A *geodesic triangulation of* (S, V, d) is any triangulation of (S, V) where the edges are geodesics with respect to the metric d.

Every piecewise flat surface (S, V, d) posesses a unique *Voronoi tessellation*. Let $p \in S$ and let $d(p, V) := \min_{i \in V} d(p, i)$. Consider the set $\Gamma_V(p)$ of all geodesics realizing the distance between the point p and the set V. The (open) 2-cells of the Voronoi tessellation of (S, V, d) are the connected components of

$$\mathcal{C}_2 = \{p \in S \mid |\Gamma_V(p)| = 1\}.$$

The 1-cells and 0-cells of the Voronoi tessellation are the connected components of

$$\mathcal{C}_1 = \{p \in S \mid |\Gamma_V(p)| = 2\} \quad \text{and} \quad \mathcal{C}_0 = \{p \in S \mid |\Gamma_V(p)| \geq 3\},$$

respectively.

For a point $i \in V$, the (closed) *Voronoi cell* V_i is the closure of the 2-cell of the Voronoi tessellation containing i. We denote the area of V_i by A_i.

Delaunay triangulation of piecewise flat surfaces A *Delaunay triangulation* of a piecewise flat surface (S, V, d) arises from the Delaunay tessellation by adding edges to triangulate the non-triangular faces.

Let d be a PL-metric on (S, V), and let Δ be any geodesic triangulation of (S, V, d). Let ijk, ijl be two triangles in F_Δ. The edge $ij \in E_\Delta$ is called a *Delaunay edge* if it satisfies the *empty circumcircle property*: the vertex l is not contained in the interior of the circumcircle of ijk.

Fact 2.1. *Let α_k, α_l be the angles opposite of the edge ij in the triangles ijk and ijl, respectively. The edge ij is Delaunay if one of the following equivalent* ***Delaunay conditions*** *hold:*
a) $\cot \alpha_k + \cot \alpha_l \geq 0$,
b) $\alpha_k + \alpha_l \leq \pi$,
c) $\cos \alpha_k + \cos \alpha_l \geq 0$.

Proposition 2.2. *A geodesic triangulation of a piecewise flat surface is Delaunay if and only if each of its edges is Delaunay.*

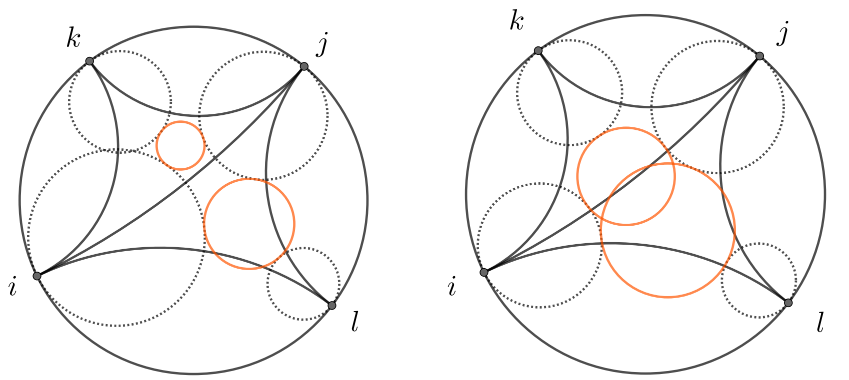
Ideal Delaunay triangulation Consider an ideal hyperbolic polyhedron (S, V, d_{hyp}), decorated with a horocycle at each cusp, small enough such that the horocycles bound disjoint cusp neighbourhoods.

Definition 2.3. *An* ***ideal Delaunay decomposition*** *of (S, V, d_{hyp}) is an ideal cell decomposition of (S, V, d_{hyp}), such that for each face f of the lift of (S, V, d_{hyp}) to the hyperbolic plane H^2 via an isometry of the universal cover, the following condition is satisfied: there exists a circle that touches all lifted horocycles centred at the vertices of f externally and does not meet any other lifted horocycles.*

An ***ideal Delaunay triangulation*** *is any refinement of an ideal Delaunay decomposition by decomposing the non-triangular faces into ideal triangles by adding geodesic edges.*

Theorem 2.4. *For each decorated ideal hyperbolic polyhedron with at least one cusp, there exists a unique ideal Delaunay decomposition.*

Definition 2.5. *Let (S, V, d_{hyp}) be a decorated ideal hyperbolic polyhedron with a geodesic triangulation Δ. Consider an edge $ij \in E_\Delta$ and its neigbouring two triangles $ijk, ijl \in F_\Delta$. The edge ij is called* ***Delaunay*** *if the two circles touching the horocycles at vertices i, j, k and i, j, l are externally disjoint or externally tangent (see Figure 2.1).*



(a) The two orange circles are disjoint, the edge ij is Delaunay.

(b) The two orange circles intersect, the edge ij is not Delaunay.

Figure 2.1: Delaunay and non-Delaunay edge, in the Poincaré disc model of hyperbolic geometry.

Proposition 2.6. *An ideal geodesic triangulation of a decorated ideal hyperbolic polyhedron is Delaunay if and only if each of its edges is Delaunay.*

2.2 Discrete metric and Penner coordinates

Let Δ be a triangulation of a marked surface (S, V).

Definition 2.7. *A* ***discrete metric*** *on* (S, V, Δ) *is a function*

$$\ell : E_\Delta \to \mathbb{R}_{>0}, \qquad \ell(ij) = \ell_{ij},$$

such that for every triangle $ijk \in F_\Delta$*, the (sharp) triangle inequalities are satisfied, that is,*

$$\ell_{ij} + \ell_{jk} > \ell_{ki}, \quad \ell_{jk} + \ell_{ki} > \ell_{ij}, \quad \ell_{ki} + \ell_{ij} > \ell_{jk}.$$

The logarithm of this function,

$$\lambda_{ij} = 2 \log \ell_{ij}, \tag{1}$$

is called the ***logarithmic lengths****.*

Fact 2.8. *Let* d *be a PL-metric on a marked surface* (S, V)*, and let* Δ *be a geodesic triangulation of* (S, V, d)*. Then* d *induces a discrete metric* ℓ_d *on* (S, V, Δ) *by measuring the lengths of the edges in* E_Δ*.*

Vice versa, each discrete metric ℓ *on a marked triangulated surface* (S, V, Δ) *induces a PL-metric on* (S, V)*, which we denote by* d_ℓ*.*

Indeed, ℓ imposes a Euclidean metric on each triangle $ijk \in F_\Delta$ by transforming it into a Euclidean triangle with edge lengths $\ell_{ij}, \ell_{jk}, \ell_{ki}$. The metrics fit isometrically along the edges of the triangulation. Thus, by patching up the triangles along the edges we equip the marked surface with a PL-metric.

Penner coordinates were introduced by Robert Penner in [12] to study the Decorated Teichmüller space.

Definition 2.9. *Let i and j be two ideal points of the hyperbolic plane. Let $\mathcal{H}_i$ and $\mathcal{H}_j$ be two horocycles, anchored at the points i and j, respectively. Let ℓ be the hyperbolic line connecting i and j, and let $p_i = \ell \cap \mathcal{H}_i$, and $p_j = \ell \cap \mathcal{H}_j$. The* ***signed horocycle distance*** *between $\mathcal{H}_i$ and $\mathcal{H}_j$ is the hyperbolic distance between the points p_i and p_j, taken negative if $\mathcal{H}_i$ and $\mathcal{H}_j$ intersect.*

Definition 2.10. ***Penner coordinates*** *is a pair consisting of a triangulation Δ of (S, V) and a map*

$$\lambda : E_\Delta \to \mathbb{R}, \qquad ij \mapsto \lambda_{ij}.$$

Fact 2.11. *Penner coordinates (Δ, λ) define a complete area hyperbolic metric with cusps on (S, V), together with a decoration of horocycle at each cusp, such that the signed distance between the horocycles $\mathcal{H}_i$ and $\mathcal{H}_j$, with $ij \in E_\Delta$, is λ_{ij}. We denote it by d_λ^{hyp}.*

Vice versa, let Δ be a geodesic triangulation of a decorated ideal hyperbolic polyhedron (S, V, d_{hyp}). Then (S, V, d_{hyp}) induces Penner coordinates (Δ, λ) by measuring the signed horocycle distance between horocycles $\mathcal{H}_i$ and $\mathcal{H}_j$, with $ij \in E_\Delta$.

Penner coordinates are illustrated in Figure 2.2. The coordinate λ_{ij} is negative, whereas the coordinates λ_{jk} and λ_{ki} are positive.

Proposition 2.12. *Let (S, V, d) be a surface with a PL-metric, let Δ be a Delaunay triangulation of (S, V, d) and let ℓ_d be the discrete metric induced by d. Let $\lambda : E_\Delta \to \mathbb{R}$ be defined via Equation (1). Then the hyperbolic metric d_λ^{hyp}, induced by the Penner coordinates (Δ, λ) as described in Fact 2.11, is isometric to the hyperbolic metric d_{hyp} induced by the PL-metric d via the formula (2).*

The following theorem links the Euclidean Delaunay and ideal Delaunay triangulations.

Theorem 2.13. *Let (S, V) be a marked surface with a triangulation Δ.*

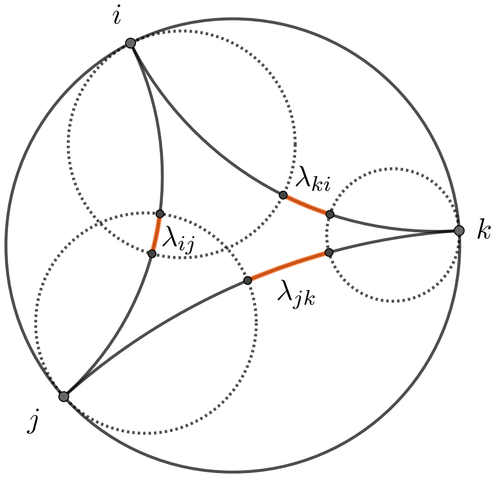


Figure 2.2: Penner coordinates of a decorated ideal hyperbolic triangle ijk, in the Poincaré disc model of hyperbolic geometry.

a) Let $\ell : E_\Delta \to \mathbb{R}_{\geq 0}$ *be a discrete metric, such that* Δ *is a Delaunay triangulation of the piecewise flat surface* (S, V, d_ℓ)*. Let* λ *be the logarithmic lengths of* ℓ *(see Equation* (1)*). Then* Δ *is an ideal Delaunay triangulation of the decorated ideal hyperbolic polyhedron defined by Penner coordinates* (Δ, λ)*.*

b) Vice versa, let (Δ, λ) *be Penner coordinates on* (S, V)*, such that* Δ *is an ideal Delaunay triangulation of the decorated ideal hyperbolic polyhedron defined by* (Δ, λ)*. Then the map* $\ell : E_\Delta \to \mathbb{R}_{\geq 0}$*, defined by Equation* (1)*, is a discrete metric on* (S, V, Δ)*, and* Δ *is a Delaunay triangulation of the piecewise flat surface* (S, V, d_ℓ)*.*

2.3 Discrete conformal classes

The *Delaunay tessellation* of (S, V, d) is the dual of the Voronoi tessellation. It possesses the *empty (immersed) disc property:* Let $C \subseteq S$ be a closed 2-cell of the Delaunay tessellation. Then there exists an open disc $D_C \subseteq \mathbb{R}^2$ and a local isometry $\varphi_C : \bar{D}_C \to S$ such that $\varphi_C^{-1}(C)$ is a cyclic polygon with circumcircle ∂D_C and vertices $\varphi_C^{-1}(C \cap V)$.

The PL-metric d on a marked surface (S, V) induces a hyperbolic metric d_{hyp} on the set $S \backslash V$. Indeed, for $x, y \in C$, let $a, b \in$

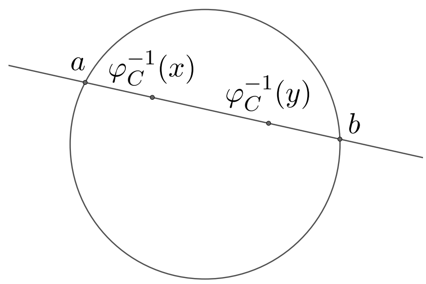


Figure 2.3

D_C be the intersection points of the line through $\varphi_C^{-1}(x)$ and $\varphi_C^{-1}(y)$ and the circumcircle ∂D_C, as illustrated in Figure 2.3. The formula

$$d_{hyp}(x,y) = \frac{1}{2}\log\left(\frac{\|\varphi_C^{-1}(x) - b\|\|\varphi_C^{-1}(y) - a\|}{\|a - \varphi_C^{-1}(x)\|\|b - \varphi_C^{-1}(y)\|}\right) \tag{2}$$

induces a hyperbolic metric with cusps on each Delaunay cell C. These metrics fit along the edges. The triple (S, V, d_{hyp}) is a complete finite area hyperbolic surface with $|V|$ punctures. We call it an *ideal hyperbolic polyhedron*.

Definition 2.14. *Two PL-metrics $d, \tilde{d}$ on a marked surface (S, V) are* ***discrete conformally equivalent*** *if the induced hyperbolic metrics d_{hyp} and $\tilde{d}_{hyp}$ are isometric. This notion induces an equivalence relation, splitting the space of all PL-metrics on (S, V) into equivalence classes called* ***conformal classes****.*

The notion of discrete conformal equivalence for PL-metrics with prescribed fixed combinatorics was introduced by Feng Luo in [10]. The equivalence of Luo's and our definition for surfaces with prescribed fixed combinatorics has been demonstrated by Boris Springborn, Ulrich Pinkall and Alexander Bobenko in [3], Proposition 5.1.2. Luo's definition has been generalized by Xianfeng Gu, Feng Luo, Jian Sun and Tianqui Wu in [7], [6], with the aim of comparing metrics with varying combinatorics. Springborn proved that this generalization is also equivalent to Definition 2.14, see [13], Chapter 10.

Definition 2.15. *Let d and $\tilde{d}$ be two conformally equivalent PL-metrics on (S, V). Let Δ and $\tilde{\Delta}$ be Delaunay triangulations of the piecewise flat surfaces (S, V, d) and $(S, V, \tilde{d})$, and let (Δ, λ) and $(\tilde{\Delta}, \tilde{\lambda})$ be the induced Penner coordinates. Let $i \in V$. Denote by $\mathcal{H}_i$ the horocycle anchored at i of (Δ, λ), and by $\tilde{\mathcal{H}}_i$ the horocycle anchored at i of $(\tilde{\Delta}, \tilde{\lambda})$. Let u_i be the signed distance from the horocycle $\mathcal{H}_i$ to the horocycle $\tilde{\mathcal{H}}_i$, measured in the direction of the cusp. The map $u : V \to \mathbb{R}$ is called a* ***conformal change****, or a* ***conformal factor****. The PL-metric $\tilde{d}$ and Penner coordinates $(\tilde{\Delta}, \tilde{\lambda})$, in dependence of the PL-metric d and the conformal change u, are denoted by $d(u)$, and $(\Delta(u), \lambda(u))$, respectively.*

The signed distance from the horocycle $\mathcal{H}_i$ to the horocycle $\tilde{\mathcal{H}}_i$ is illustrated in Figure 2.4. The following proposition provides a direct link between the conformal change and the discrete metrics of two discrete conformally equivalent PL-metrics.

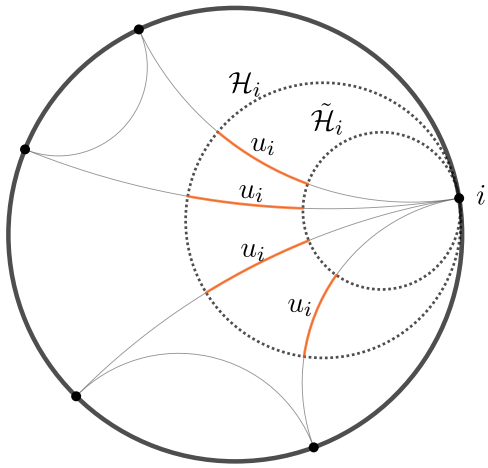


Figure 2.4: The distance between two horocycles.

Proposition 2.16. *Let d and $\tilde{d}$ be two conformally equivalent PL-metrics on a marked surface (S, V), related by the conformal factor $u : V \to \mathbb{R}$, and let Δ be a geodesic triangulation of the surface (S, V, d), as well as the surface $(S, V, \tilde{d})$. Then the induced discrete metrics ℓ_d and $\ell_{\tilde{d}}$ satisfy*

$$\ell_{\tilde{d}}(jk) = \ell_d(jk) e^{\frac{u_j + u_k}{2}}$$

for every edge $jk \in E_\Delta$.

Remark. *Proposition 2.16 is the original definition of discrete conformal equivalence, due to Luo [10].*

Proposition 2.17. *Let (S, V, d) be a piecewise flat surface. The conformal class of the PL-metric d is parametrised by the vector space*

$$\mathbb{R}^V := \{u : V \to \mathbb{R}\} \cong \mathbb{R}^{|V|}.$$

Proof. Definition 2.15 implies, that the conformal class of the PL-metric d can be parametrised by a subset of the space $\mathbb{R}^V$. Let (Δ, λ) be Penner coordinates of the decorated ideal hyperbolic polyhedron induced by d. We show that every conformal factor $u \in \mathbb{R}^V$ defines a PL-metric $d(u)$, discrete conformally equivalent to d.

Let $u \in \mathbb{R}^V$ be a conformal factor. Let $\mathcal{H}_i(u)$ be the horocycle anchored at the cusp i, at the distance u_i from the horocycle at i of (Δ, λ), and let $\lambda_{ij}(u)$ measure the distance between the horocycles $\mathcal{H}_i(u)$ and $\mathcal{H}_j(u)$. Let $\Delta(u)$ be an ideal Delaunay triangulation of the ideal hyperbolic polyhedron (Δ, λ), decorated by $\mathcal{H}_i(u)$. This exists due to Theorem 2.4. Then the map

$$\lambda(u) : E_{\Delta(u)} \to \mathbb{R}, ij \mapsto \lambda_{ij}(u)$$

defines Penner coordinates $(\Delta(u), \lambda(u))$. This induces a PL-metric $d(u)$, due to Theorem 2.13. □

Furthermore, the vector space $\mathbb{R}^V$ can be partitioned as follows.

Definition 2.18. *Let (S, V, d) be a piecewise flat surface, and let $\mathfrak{T}(S, V)$ denote the set of all triangulations of (S, V). For a triangulation $\Delta \in \mathfrak{T}(S, V)$, define the* ***Penner cell*** *of (S, V, d) as*

$$\mathcal{A}_\Delta = \{u \in \mathbb{R}^V \mid \Delta \text{ is a Delaunay triangulation of } (S, V, d(u))\}.$$

Theorem 2.19. *Let (S, V, d) be a piecewise flat surface and let $\mathfrak{D}(S, V) \subseteq \mathfrak{T}(S, V)$ be the set of all triangulations $\Delta \in \mathfrak{T}(S, V)$, such that the Penner cell $\mathcal{A}_\Delta$ is non-empty. Then the set $\mathfrak{D}(S, V)$ is finite.*

This theorem was proved by Hirotaka Akiyoshi [1]. In particular,

$$\mathbb{R}^V = \bigcup_{\Delta \in \mathfrak{D}(S,V)} \mathcal{A}_\Delta,$$

and the Penner cells $\mathcal{A}_\Delta$ are closed top-dimensional cells in $\mathbb{R}^V$.

Definition 2.20. *Let (Δ, λ) define a decorated ideal hyperbolic polyhedron. Let $ij \in E_\Delta$, and consider the quadrilateral built by the neighbouring ideal triangles $ijk, ijl \in F_\Delta$. The* ***Ptolemy flip*** *of ij consists of replacing the diagonal ij of the quadrilateral by the hyperbolic line connecting k to l.*

The Ptolemy flip has the following properties:

- Any edge can be Ptolemy-flipped.
- ([13], Proposition 3.6) Let $\lambda_a, \ldots, \lambda_d, \lambda_e, \lambda_f$ be the Penner coordinates of the four edges and two diagonals, respectively, of an ideal hyperbolic quadrilateral. If $\ell_x := e^{\frac{\lambda_x}{2}}$, then the lengths ℓ_x satisfy the *Ptolemy relation*

$$\ell_e \ell_f = \ell_a \ell_c + \ell_b \ell_d.$$

- If the original edge is not Delaunay, the flipped edge is.

The following theorem suggests that the Ptolemy flip can be used to compute ideal Delaunay triangulations.

Theorem 2.21. *[Jeffrey Weeks [15]] Let (S, V, d_{hyp}) be a decorated ideal hyperbolic polyhedron, and let Δ be a geodesic triangulation on (S, V, d_{hyp}). Flip a non-Delaunay edge in E_Δ using the Ptolemy flip and update the triangulation. Repeat until all edges are Delaunay. This algorithm terminates after finitely many flips.*

Fact 2.22. *If a pair of Penner cells has non-empty intersection, $u \in \mathcal{A}_\Delta \cap \mathcal{A}_{\tilde{\Delta}}$, there exists a sequence $\Delta_0, \ldots, \Delta_m$ of ideal triangulations such that*
- $u \in \mathcal{A}_{\Delta_0} \cap \cdots \cap \mathcal{A}_{\Delta_m}$,
- $\Delta_0 = \Delta, \Delta_m = \tilde{\Delta}$, *and*
- Δ_i *differs from* Δ_{i+1} *by a Ptolemy flip of one edge.*

3 Counterexamples

There exist piecewise flat surfaces whose conformal classes contain more than one metric with constant discrete Gaussian curvature. To demonstrate this, we first construct a so-called *symmetric* piecewise flat surface – a surface where all faces are congruent triangles. Due to the symmetry, this surface has constant discrete Gaussian curvature. Then we investigate a particular one-parameter family of PL-metrics in the conformal class of this surface to find another PL-metric with constant discrete Gaussian curvature.

A *tetrahedron* is a sphere with four marked points (S, V), a triangulation Δ of (S, V) that has the combinatorics of a tetrahedron, and a discrete metric ℓ on (S, V, Δ). The pairs of edges that are not adjacent one to another are called the *opposite edges*. We call a surface of genus two a *double torus*.

Definition 3.1. *An* ***almost symmetric tetrahedron*** *is a tetrahedron with two pairs of opposite edges of equal length. Its faces are two copies of a triangle with edge lengths a, b, c, and two copies of a triangle with edge lengths $\bar{a}, b, c$.*

Let S be a double torus, and let $V \subseteq S$ consist of two points, which we denote by w for white, and b for black. An ***almost symmetric double torus*** *is a piecewise flat surface (S, V, d) with a triangulation Δ, that consists of four copies of two triangles, with edge lengths a, b, c and $\bar{a}, b, c$, respectively. The edges are glued together according to the scheme represented in Figure 3.1.*

In both cases, we use the notation introduced in Figure 3.1 to label the angles.

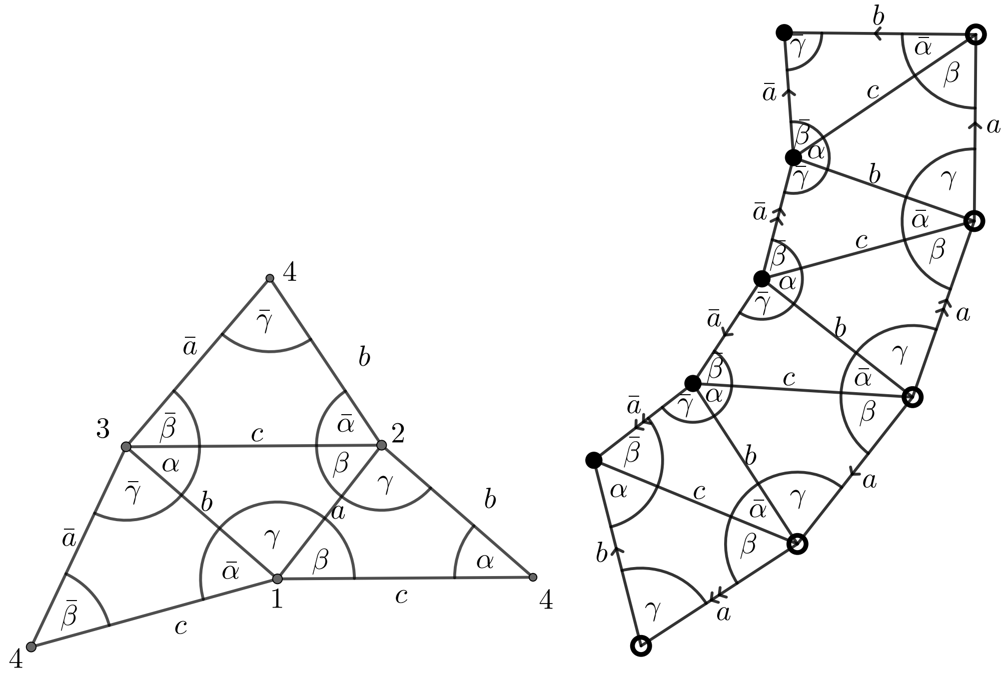


Figure 3.1: An almost symmetric tetrahedron (left) and an almost symmetric double torus (right).

Proposition 3.2. *The discrete Gaussian curvature of an almost symmetric tetrahedron with Delaunay edges satisfies*

$$K_1 = K_2, \qquad \textit{and} \qquad K_3 = K_4.$$

Proof. Follows from the fact that $W_1 = W_2, W_3 = W_4, K_1 = K_2$ and $K_3 = K_4$. □

Definition 3.3. *An almost symmetric tetrahedron or double torus is called* ***symmetric*** *if all its faces are congruent. We call this face* ***the defining triangle*** *of the tetrahedron.*

Fact 3.4. *A symmetric tetrahedron or double torus*

- *is defined by the edge lengths a, b, c of the defining triangle. This definition is unique up to the relabeling of the edges.*
- *has constant discrete Gaussian curvature.*
- *has Delaunay edges if and only if the defining triangle is acute.*

An almost symmetric tetrahedron or double torus with edges $a, b, c, \bar{a}, b, c$ is symmetric if and only if $a = \bar{a}$.

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Consider the symmetric tetrahedron or double torus $\mathfrak{t}$ with defining triangle with edges a_0, b_0, c_0, satisfying the triangle inequalities, such that the edge lengths satisfy

$$a_0 = 1, \qquad a_0 \leq b_0 \leq c_0.$$

Then $\mathfrak{t}$ has Delaunay edges if and only if

$$c_0^2 \leq a_0^2 + b_0^2 = 1 + b_0^2. \tag{3}$$

We apply the following conformal change to the discrete metric ℓ of $\mathfrak{t}$.

Lemma 3.5. *Let*

$$\mathcal{S}_{(b_0,c_0)} := \left[-\log(b_0^2 + c_0^2), \log(b_0^2 + c_0^2)\right].$$

Let further

$$u(v) = (u_1, u_2, u_3, u_4)(v) := (0, 0, v, v), \qquad v \in \mathbb{R}.$$

The tetrahedron given by the discrete metric $\ell(u(v))$ is almost symmetric. It has Delaunay edges if $v \in \mathcal{S}_{(b_0,c_0)}$.

Let

$$u(v) = (u_w, u_b)(v) := (0, v), \qquad v \in \mathbb{R}.$$

The double torus given by the discrete metric $\ell(u(v))$ is almost symmetric. It has Delaunay edges if $v \in \mathcal{S}_{(b_0,c_0)}$.

Proof. For each $v \in \mathbb{R}$, the tetrahedron or double torus with discrete metric $\ell(u(v))$ has edge lengths

$$a = 1, \qquad b = b_0 e^{v/2}, \qquad c = c_0 e^{v/2}, \qquad \bar{a} = e^v,$$

and is thus almost symmetric. The minimal and maximal value of the parameter v follow from Equation (3) and the Delaunay properties (Fact 2.1). $\square$

Corollary 3.6. *The symmetric tetrahedron or double torus with the defining triangle with edge lengths a_0, b_0, c_0 is given by the discrete metric $\ell(u(0))$.*

Remark. *The conformal class of a symmetric double torus contains only almost symmetric double tori.*

Let A and $\bar{A}$ denote the area of the triangles with side lengths a, b, c and $\bar{a}, b, c$, respectively, and let $F_a, \ldots, F_{\bar{c}}$ denote the area components as in Figure 3.2.

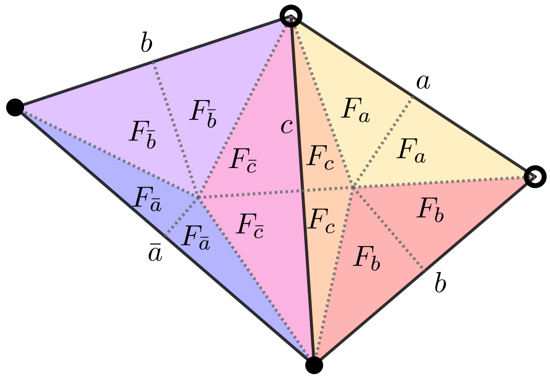


Figure 3.2

Lemma 3.7. *Let u be defined as in Lemma 3.5. A tetrahedron given by the discrete metric $\ell(u(v))$, with $v \in \mathcal{S}_{(b_0,c_0)}$, has constant discrete Gaussian curvature if and only if v is a zero of the map*

$$g_{(b_0,c_0)} : \mathcal{S}_{(b_0,c_0)} \to \mathbb{R}, \qquad v \mapsto 2\pi(F_{\bar{a}} - F_a) + (\alpha - \bar{\alpha})(A + \bar{A}).$$

A double torus with discrete metric $\ell(u(v))$, with $v \in \mathcal{S}_{(b_0,c_0)}$ has constant discrete Gaussian curvature if and only if v is a zero of the map

$$h_{(b_0,c_0)} : \mathcal{S}_{(b_0,c_0)} \to \mathbb{R}, \qquad u \mapsto \pi(F_{\bar{a}} - F_a) + (\bar{\alpha} - \alpha)(A + \bar{A}).$$

Proof. From Lemma 3.5 and Proposition 3.2 we know that the discrete Gaussian curvature satisfies $K_1 = K_2$ and $K_3 = K_4$. The equality of the values of the discrete Gaussian curvature at vertices 1 and 3 is equivalent to the expression:

$$W_1 A_3 = W_3 A_1 \iff 2\pi(F_{\bar{a}} - F_a) = (\bar{\alpha} - \alpha)(A + \bar{A}).$$

The discrete Gaussian curvatures at vertices b and w are equal if and only if

$$W_w A_b = W_b A_w \iff \pi(F_{\bar{a}} - F_a) = (\alpha - \bar{\alpha})(A + \bar{A}).$$

□

The number of critical points of g and h varies depending on the choice of (b_0, c_0). Figure 3.3 illustrates the graphs of g and h for various values of (b_0, c_0). The red and green curves correspond to conformal classes with more than one metric with constant discrete Gaussian curvature.

The structure of conformal classes of the sphere with four marked points, and the double torus are studied more thoroughly in [9], Chapters 8 and 9.

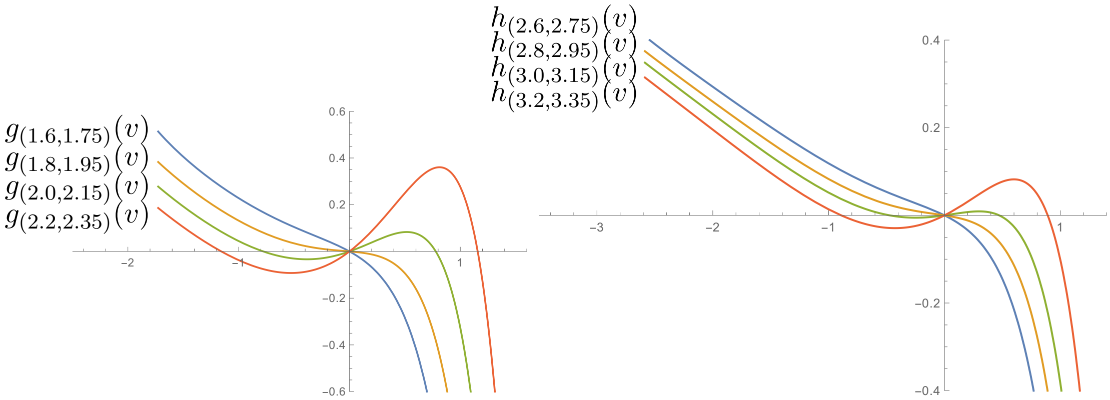


Figure 3.3: Graphs of the functions g (left) and h (right) for various values of b_0 and c_0.

4 Variational principles

4.1 Two essential building blocks

The function $\mathbb{E}$, which we will introduce shortly, was defined by Bobenko, Pinkall and Springborn in [3]. Its building block is a peculiar function f.

Definition 4.1. *Consider a Euclidean triangle with edge lengths a, b, c and angles α, β, γ, opposite to edges a, b, c, respectively. Let*

$$x = \log a, \qquad y = \log b, \qquad z = \log c,$$

as illustrated in Figure 4.1a. Let $\mathfrak{A}$ be the set of all triples $(x, y, z) \in \mathbb{R}^3$, such that (a, b, c) satisfy the triangle inequalities:

$$\mathfrak{A} = \{(x, y, z) \in \mathbb{R}^3 \mid a + b - c > 0, a - b + c > 0, -a + b + c > 0\}.$$

The function f is defined as follows:

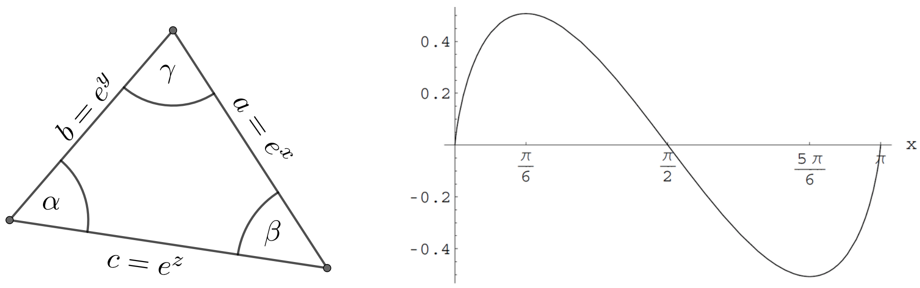
$$f : \mathfrak{A} \to \mathbb{R}, \quad f(x, y, z) = \alpha x + \beta y + \gamma z + \mathbb{L}(\alpha) + \mathbb{L}(\beta) + \mathbb{L}(\gamma),$$

where

$$\mathbb{L}(\alpha) = -\int_0^\alpha \log|2\sin(t)|\, dt$$

is Milnor's Lobachevsky function, introduced by Milnor in [11].

Fact 4.2. *Milnor's Lobachevsky function $\mathbb{L}(x)$ is odd, 2π-periodic, and smooth except at $x \in \pi\mathbb{Z}$.*



(a) Logarithmic edge lengths of a triangle.

(b) Graph of Milnor's Lobachevsky function, $y = \mathbb{L}(x)$.

Figure 4.1

We first define the function $\mathbb{E}$ on the Penner cells.

Definition 4.3. *Let (S, V, d) be a piecewise flat surface, and let $\Delta \in \mathfrak{D}(S, V)$. On the Penner cell $\mathcal{A}_\Delta$, the function $\mathbb{E}_\Delta$ is defined as follows:*

$$\mathbb{E}_\Delta : \mathcal{A}_\Delta \to \mathbb{R},$$

$$\mathbb{E}_\Delta(u) = \sum_{ijk \in F_\Delta} \left(2f\left(\frac{\tilde{\lambda}_{ij}}{2}, \frac{\tilde{\lambda}_{jk}}{2}, \frac{\tilde{\lambda}_{ki}}{2} \right) - \frac{\pi}{2}(\tilde{\lambda}_{ij} + \tilde{\lambda}_{jk} + \tilde{\lambda}_{ki}) \right) + 2\pi \sum_{i \in V} u_i,$$

where $\tilde{\lambda}_{ij}$ are the logarithmic lengths of the discrete metric induced by the PL-metric $d(u)$.

Lemma 4.4. *The partial derivatives of the function $\mathbb{E}_\Delta$ satisfy the equation*

$$\frac{\partial \mathbb{E}_\Delta}{\partial u_i} = W_i, \tag{4}$$

where W_i is the angle defect at vertex i of the piecewise flat surface $(S, V, d(u))$.

Proof. Follows from Proposition 4.1.2. in [3]. □

The functions f and $\mathbb{E}_\Delta$ have the following properties:

Proposition 4.5 (Properties of f and $\mathbb{E}_\Delta$). *The functions f and $\mathbb{E}_\Delta$ are analytic and locally convex on $\mathfrak{A}$ and $\mathcal{A}_\Delta$, respectively. Their second derivatives are positive semidefinite quadratic forms with one-dimensional kernels, spanned by $(1, 1, 1) \in \mathfrak{A}$, $(1, \ldots, 1) \in \mathcal{A}_\Delta$, respectively. Further,*

$$f(x+t, y+t, z+t) = f(x, y, z) + \pi t \qquad \text{for all } (x, y, z) \in \mathfrak{A},$$
$$\mathbb{E}(u + t(1, \ldots, 1)) = \mathbb{E}(u) + 2\pi\chi(S)t \qquad \text{for all } u \in \mathcal{A}_\Delta,$$

where $\chi(S)$ denotes the Euler characteristic of the surface S.

Proof. See [3], Equation (4-5) or [13], Propositions 7.2 and 7.7. □

Theorem 4.6 (Extension)**.** *For a conformal factor $u \in \mathbb{R}^V$, let $\Delta(u)$ be a Delaunay triangulation of the surface $(S, V, d(u))$. The map*

$$\mathbb{E} : \mathbb{R}^V \to \mathbb{R}, \qquad u \mapsto \mathbb{E}_{\Delta(u)}(u),$$

is well-defined and twice continuously differentiable. Its second derivative is a positive semidefinite quadratic form with one-dimensional kernel, spanned by $(1, \dots, 1) \in \mathbb{R}^V$. Explicitly,

$$d^2\mathbb{E} = \frac{1}{4} \sum_{ij \in E} (\cot \alpha_k^{ij} + \cot \alpha_l^{ij})(du_i - du_j)^2.$$

Proof. Follows from [3], Proposition 4.1.6, and [13], Section 7 and 8. □

***

Definition 4.7. *Let (S, V, d) be a piecewise flat surface, and let $\Delta \in \mathfrak{D}(S, V)$. On the Penner cell $\mathcal{A}_\Delta$, the function A_{tot}^Δ is defined as follows:*

$$A_{tot}^\Delta : \mathcal{A}_\Delta \to \mathbb{R}, \qquad A_{tot}^\Delta(u) = \sum_{ijk \in F_\Delta} A_{ijk}(u),$$

where $A_{ijk}(u)$ is the area of the triangle with vertices $i, j, k \in V$ on the piecewise flat surface $(S, V, d(u))$.

Notation 4.8. *Consider a triangle ijk, with edges and angles labeled as in Figure 4.2. Let A_{jk}^i denote half of the area of the isosceles triangle with base length ℓ_{jk} and legs of length R_{ijk}. Then*

$$A_{jk}^i = \frac{\ell_{jk}^2}{8} \cot \alpha_i^{jk}.$$

The area of the Voronoi cell V_i of a piecewise flat surface (S, V, d) satisfies the equation

$$A_i = \sum_{jk | ijk \in F_\Delta} 2A_{ki}^j + 2A_{ij}^k.$$

Lemma 4.9. *The function A_{tot}^Δ is analytic on each non-empty Penner cell $\mathcal{A}_\Delta$. Its partial derivatives satisfy the equation*

$$\frac{\partial A_{tot}^\Delta}{\partial u_i} = 2A_i. \tag{5}$$

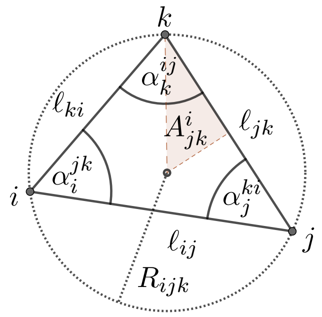


Figure 4.2

Proof. Deploying Notation 4.8,

$$\frac{\partial A_{ijk}}{\partial u_i} = 2A^j_{ki} + 2A^k_{ij} - R^2_{ijk}\frac{\partial}{\partial u_i}(\underbrace{\alpha_i^{jk} + \alpha_j^{ki} + \alpha_k^{ij}}_{=\pi}) = 2A^j_{ki} + 2A^k_{ij}.$$

Due to the linearity of the area function,

$$\frac{\partial A^{\Delta}_{tot}}{\partial u_i} = \sum_{jk|ijk\in F_\Delta} 2A^j_{ki} + 2A^k_{ij} = 2A_i.$$

□

Theorem 4.10 (Extension)**.** *For a conformal factor $u \in \mathbb{R}^V$, let $\Delta(u)$ be a Delaunay triangulation of the surface $(S, V, d(u))$. The map*

$$A_{tot} : \mathbb{R}^V \to \mathbb{R}, \qquad u \mapsto A^{\Delta(u)}_{tot}(u),$$

is well-defined and twice continuously differentiable. Its second derivative is

$$d^2 A_{tot} = \sum_{ij\in E} 2A_{ij}(du_i + du_j)^2 - \frac{1}{2}\sum_{ij\in E}(R^2_{ijk}\cot\alpha^{ij}_k + R^2_{ijl}\cot\alpha^{ij}_l)(du_i - du_j)^2,$$

where the vertices $k, l \in V$ are such that the triangles ijk, ijl lie in $F_{\Delta(u)}$, and $A_{ij} = A^k_{ij} + A^l_{ij}$.

Proof. The function A_{tot} is well-defined and analytic in the interior of every Penner cell, since the area of each triangle $ijk \in F_\Delta$, A_{ijk}, is an analytic function with respect to the conformal factors. If $\mathcal{A}_\Delta \cap \mathcal{A}_{\tilde\Delta} \neq \emptyset$, the triangulations Δ and $\tilde\Delta$ differ by finitely many Ptolemy flips, and the functions A^{Δ}_{tot} and $A^{\tilde\Delta}_{tot}$ agree on $\mathcal{A}_\Delta \cap \mathcal{A}_{\tilde\Delta}$ up to second derivative. Due to Fact 2.22, we may assume that Δ and $\tilde\Delta$ differ only by one Ptolemy flip, flipping edge $ij \in E_\Delta$

to the edge $kl \in E_{\tilde{\Delta}}$. Thus, it suffices to prove that the functions $A_{ijk} + A_{ijl}$ and $A_{ikl} + A_{jkl}$ agree up to second derivative on $\mathcal{A}_\Delta \cap \mathcal{A}_{\tilde{\Delta}}$. The proof of this consists of a largely technical unilluminating calculation which can be found in [9], Chapter 10.

In the upcoming calculations we use the following formula, which was proved in [3], Equation (4-8).

Lemma. *Let a, b, c be edge lengths of a triangle, α, β, γ angles opposite of a, b, c, respectively, and let $\lambda_a, \lambda_b, \lambda_c$ be the logarithmic lengths. Then*

$$2d\alpha = (\cot\beta + \cot\gamma)d\lambda_a - \cot\gamma d\lambda_b - \cot\beta d\lambda_c.$$

We use Notation 4.8. Since

$$\frac{\partial A_{ki}^j}{\partial u_i} = A_{ki}^j - \frac{1}{2}R_{ijk}^2 \frac{\partial \alpha_j^{ki}}{\partial u_i} = A_{ki}^j - \frac{1}{4}R_{ijk}^2 \cot\alpha_k^{ij},$$

we obtain

$$\frac{\partial^2 A_{tot}}{\partial u_i^2} = 2A_i - \frac{1}{2}\sum_{jk|ijk\in F_\Delta} R_{ijk}^2(\cot\alpha_k^{ij} + \cot\alpha_j^{ki}).$$

Let $i, j \in V$ be two vertices. If j is not adjacent to i,

$$\frac{\partial^2 A_{tot}}{\partial u_i \partial u_j} = 0.$$

If j is adjacent to i, let $k, l \in V$ be the two vertices such that $ijk, ijl \in F_\Delta$. Since

$$\frac{\partial A_{jk}^i}{\partial u_i} = -\frac{1}{2}R_{ijk}^2 \frac{\partial \alpha_i^{jk}}{\partial u_i} = \frac{1}{4}R_{ijk}^2(\cot\alpha_j^{ki} + \cot\alpha_k^{ij}),$$

the mixed partial derivative equals

$$\frac{\partial^2 A_{tot}}{\partial u_i \partial u_j} = \underbrace{2A_{ij}^k + 2A_{ij}^l}_{=2A_{ij}} + \frac{1}{2}(R_{ijk}^2 \cot\alpha_k^{ij} + R_{ijl}^2 \cot\alpha_l^{ij}).$$

Thus,

$$d^2 A_{tot} = \sum_{ij\in E_\Delta} 2A_{ij}(du_i + du_j)^2 - \frac{1}{2}\sum_{ij\in E_\Delta}(R_{ijk}^2 \cot\alpha_k^{ij} + R_{ijl}^2 \cot\alpha_l^{ij})(du_i - du_j)^2.$$

□

4.2 The variational principles

Theorem 4.11 (Variational principle with equality constraints)**.** *Let (S, V, d) be a piecewise flat surface. The PL-metrics in the conformal class of the metric d are in one-to-one correspondence with the critical points of the function*

$$\mathbb{E} : \mathbb{R}^V \to \mathbb{R}, \qquad u \mapsto \mathbb{E}(u),$$

under the constraint

$$A_{tot}(u) = 1.$$

Proof. We use the method of Lagrange multipliers. A conformal factor $u \in \mathbb{R}^V$ is a critical point of the function $\mathbb{E}$ under the constraint $A_{tot} = 1$ if and only if there exists a Lagrange multiplier $\lambda \in \mathbb{R}$, such that

$$0 = \frac{\partial(\mathbb{E} - \lambda A_{tot})}{\partial u_i} \stackrel{(4),(5)}{=} W_i - 2\lambda A_i.$$

This holds if and only if

$$K_i := \frac{W_i}{A_i} = 2\lambda = const.$$

$\square$

Proposition 4.12. *Let (S, V, d) be a piecewise flat surface with constant discrete Gaussian curvature K_{av} at every vertex. Denote the total area of the surface by A_{tot}, and the Euler characteristics of S by $\chi(S)$. Then,*

$$K_{av} = \frac{2\pi\chi(S)}{A_{tot}}.$$

Proof. The claim follows from the discrete Gauss–Bonnet theorem. Let Δ be a geodesic triangulation of the surface (S, V, d). Then:

$$2\pi\chi(S) = 2\pi|V| - \pi|F_\Delta| = \sum_{i\in V}\left(2\pi - \sum_{jk|ijk\in F_\Delta}\alpha_i^{jk}\right) = \sum_{i\in V} W_i = \sum_{i\in V} A_i K_i.$$

$\square$

Theorem 4.13 (Variational principle with inequality constraints)**.** *Let (S, V, d) be a piecewise flat surface. The existence of PL-metrics in the conformal class of the metric d follows from the existence of minima of the function $\mathbb{E}$ under the following inequality constraints:*

- *if the Euler characteristic of S satisfies $\chi(S) = 2$, the inequality constraint is*
$$A_{tot} \geq 1,$$
- *if the Euler characteristic of S satisfies $\chi(S) < 0$, the inequality constraint is*
$$A_{tot} \leq 1.$$

Proof. Proposition 4.15 shows that, if $u \in \mathbb{R}^V$ is a minimum of the function $\mathbb{E}$ under one of these constraints, then $A_{tot}(u) = 1$. Since a minimum is a critical point, the claim follows from Theorem 4.11. $\square$

Proposition 4.14. *Let*

$$\mathcal{A}_+ = \{u \in \mathbb{R}^V \mid A_{tot}(u) \geq 1\}, \qquad \mathcal{A}_- = \{u \in \mathbb{R}^V \mid A_{tot}(u) \leq 1\},$$

be two sets in the vector space $\mathbb{R}^V$. The sets $\mathcal{A}_+$ and $\mathcal{A}_-$ have the following properties:

a) The sets $\mathcal{A}_+$ and $\mathcal{A}_-$ are closed subsets of $\mathbb{R}^V$.

b) Let $\mathbb{I} = (1, \ldots, 1) \in \mathbb{R}^V$, and let $u \in \mathbb{R}^V$ be a conformal factor. Then the rays

$$\mathcal{R}_u^+ = \left\{ u + c\,\mathbb{I} \mid c \geq -\frac{1}{2} \log A_{tot}(u) \right\}, \; \mathcal{R}_u^- = \left\{ u + c\,\mathbb{I} \mid c \leq -\frac{1}{2} \log A_{tot}(u) \right\}$$

are completely contained in the sets $\mathcal{A}_+$ and $\mathcal{A}_-$, respectively.

c) The sets $\mathcal{A}_+$ and $\mathcal{A}_-$ are unbounded.

Proof. a) The proof follows from the fact that the sets $\mathcal{A}_+$ and $\mathcal{A}_-$ satisfy the equation

$$\mathcal{A}_+ = A_{tot}^{-1}([1, \infty)), \qquad \mathcal{A}_- = A_{tot}^{-1}([0, 1]).$$

b) The statement follows from the fact that

$$A_{tot}(u + c\,\mathbb{I})) = A_{tot}(u) \exp(2c).$$

c) The statement follows from claim b).

$\square$

Proposition 4.15. *Let (S, V, d) be a piecewise flat surface. If*

- *the Euler characteristic of S satisfies $\chi(S) = 2$ and the function $\mathbb{E}$ attains a minimum in the set $\mathcal{A}_+$, or*
- *the Euler characteristic of S satisfies $\chi(S) < 0$ and the function $\mathbb{E}$ attains a minimum in the set $\mathcal{A}_-$,*

the minimum lies at the boundary of the sets,

$$\partial\mathcal{A}_+ = \partial\mathcal{A}_- = \left\{u \in \mathbb{R}^V \mid A_{tot}(u) = 1\right\}.$$

Proof. Let $\chi(S) = 2$, let $u \in \mathcal{A}_+$ be a minimum of the function $\mathbb{E}$ in $\mathcal{A}_+$ and let $u + c : \mathbb{I} \in \mathcal{R}_u^+$. Then

$$\mathbb{E}(u) \leq \mathbb{E}(u + c\,\mathbb{I}) = \mathbb{E}(u) + 4\pi c,$$

with equality if and only if $-\frac{1}{2}\log A_{tot}(u) = c = 0$.

For surfaces with $\chi(S) < 0$ the proof is analogous. □

5 Existence

We show, that the function $\mathbb{E}$ possesses minima in the sets $\mathcal{A}_+$ and $\mathcal{A}_-$. The following theorem is a classical theorem from calculus. We omit the proof.

Theorem 5.1. *Let $A \subseteq \mathbb{R}^m$ be a closed set and let $f : A \to \mathbb{R}$ a continuous function. If every unbounded sequence $(x_n)_{n\in\mathbb{N}}$ in A has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} f(x_{n_k}) = +\infty,$$

then f attains a minimum in A.

To obtain minima in the sets $\mathcal{A}_+$ and $\mathcal{A}_-$, it suffices to apply the aforementioned Theorem 5.1 and to prove the following Proposition:

Proposition (Proposition 5.15)**.** *a) Let $\chi(S) < 0$ and let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_-$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} \mathbb{E}(u_{n_k}) = +\infty.$$

b) Let $\chi(S) = 2$ and let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_+$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathbb{E}(u_{n_k}) = +\infty.$$

Corollary 5.2. *If $\chi(S) < 0$, the function $\mathbb{E}$ possesses a minimum in the set $\mathcal{A}_-$. If $\chi(S) = 2$, the function $\mathbb{E}$ possesses a minimum in the set $\mathcal{A}_+$.*

Proof. Follows from Proposition 5.15, Proposition 4.14 and Theorem 5.1. □

The goal is to extract a suitable subsequence from any given unbounded sequence in the sets $\mathcal{A}_+$ and $\mathcal{A}_-$.

In Section 5.1 we study the behaviour of triangulations under sequences of conformal factors. We show that, up to a global scaling, a Delaunay triangulation of a closed, piecewise flat surface can exhibit only two types of behaviour. In Section 5.2 we investigate the behaviour of the function $\mathbb{E}$ under the sequences of conformal factors.

We use the following standard definition from calculus.

Definition 5.3. *A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is called **properly divergent to** $+\infty$ if, for each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$, such that $x_n > M$ for all $n > N$.*

*A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is called **properly divergent to** $-\infty$ if, for each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$, such that $x_n < M$ for all $n > N$.*

5.1 Sequences of conformal factors

Definition 5.4. *A sequence of conformal factors $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^V$ is called **consistently ordered convergent** if*

a) it lies in a Penner cell $\mathcal{A}_\Delta$ of $\mathbb{R}^V$,

b) there exists a vertex $i^* \in V$ such that, for all $j \in V$ and $n \in \mathbb{N}$, $u_{i^*,n} \leq u_{j,n}$, and the sequence $(u_{i^*,n})_{n\in\mathbb{N}}$ either converges, diverges properly to $+\infty$ or diverges properly to $-\infty$,

c) for every triangle $t \in F_\Delta$, there exists a labeling of the vertices i, j, l of t such that

i) for all $n \in \mathbb{N}$, $u_{i,n} \geq u_{j,n} \geq u_{l,n}$,

ii) the sequences $(u_{j,n} - u_{i^*,n})_{n\in\mathbb{N}}$ and $(u_{l,n} - u_{i^*,n})_{n\in\mathbb{N}}$ converge, and the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ either converges or diverges properly to $+\infty$.

Let (S, V, d) be a piecewise flat surface, let $\mathcal{A}_\Delta$ be a non-empty Penner cell, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_\Delta$. Denote the evolution of the discrete metric $\ell_d : E_\Delta \to \mathbb{R}$ along $(u_n)_{n\in\mathbb{N}}$ by

$$\ell_{ij}^n := \ell_{ij} \exp\left(\frac{u_{i,n} + u_{j,n}}{2}\right). \tag{6}$$

This, in particular, implies that the edge lengths of every triangle of the surface (S, V, Δ) with the discrete metric ℓ^n satisfy the triangle inequalities.

Proposition 5.5. *Each sequence of conformal factors in $\mathbb{R}^V$ possesses a consistently ordered convergent subsequence.*

Corollary 5.6. *Let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathbb{R}^V$.*

a) If there exists a vertex $i \in V$, such that the sequence $(u_{i,n})_{n\in\mathbb{N}}$ is not bounded from below, the sequence $(u_n)_{n\in\mathbb{N}}$ possesses a consistently ordered convergent subsequence $(u_{n_k})_{k\in\mathbb{N}}$, such that the sequence $(u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $-\infty$.

b) If, for all vertices $i \in V$, the sequences $(u_{i,n})_{n\in\mathbb{N}}$ are bounded from below, the sequence $(u_n)_{n\in\mathbb{N}}$ possesses a consistently ordered convergent subsequence $(u_{n_k})_{k\in\mathbb{N}}$ with at least one vertex $j \in V$, such that the sequence $(u_{j,n_k})_{k\in\mathbb{N}}$ diverges properly to $+\infty$.

Proof of Proposition 5.5. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of conformal factors in $\mathbb{R}^V$. The existence of a subsequence of $(u_n)_{n\in\mathbb{N}}$, satisfying the condition *a)* of Definition 5.4 follows from Akiyoshi's Theorem 2.19. The existence of a further subsequence, satisfying the conditions *b)* and *c) i)* follows from the fact, that the number of vertices $|V|$ is finite.

Assume that the sequence $(u_n)_{n\in\mathbb{N}}$ already satisfies properties $a) - c)\ i)$ from Definition 5.4. Let $t_0 \in F_\Delta$ be any triangle with vertex i^*. Label the vertices of t_0 by i, j, l, as in Definition 5.4 $c)\ i)$. Due to Definition 5.4 $b)$, $u_{l,n} - u_{i^*,n} = 0$ for all $n \in \mathbb{N}$, and the sequences $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ and $(u_{j,n} - u_{i^*,n})_{n\in\mathbb{N}}$ are bounded from below by zero. If the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ is bounded, then so is the sequence $(u_{j,n} - u_{i^*,n})_{n\in\mathbb{N}}$, and there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that the sequences $(u_{i,n_k} - u_{i^*,n_k})_{k\in\mathbb{N}}$ and $(u_{j,n_k} - u_{i^*,n_k})_{k\in\mathbb{N}}$ converge. If the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ is unbounded, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that the sequence $(u_{i,n_k} - u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $+\infty$. The sequence $(u_{j,n_k} - u_{i^*,n_k})_{k\in\mathbb{N}}$ must be bounded due to the following lemma:

Lemma 5.7. *Let $\ell_{12}, \ell_{23}, \ell_{31}$ be edge lengths of a Euclidean triangle and let $(u_n)_{n\in\mathbb{N}} = (u_{1,n}, u_{2,n}, u_{3,n})_{n\in\mathbb{N}}$ be a sequence of conformal factors in $\mathbb{R}^3$. If*
- *the sequence $(u_{1,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$,*
- *the sequence $(u_{2,n})_{n\in\mathbb{N}}$ is not bounded from above and*
- *the sequence $(u_{3,n})_{n\in\mathbb{N}}$ is bounded,*

there exists an $n \in \mathbb{N}$ such that the triangle inequalities do not hold for $\ell_{12}^n, \ell_{23}^n, \ell_{31}^n$.

There exists a further subsequence of the sequence $(u_{n_k})_{k\in\mathbb{N}}$, along which the sequence $(u_{j,n_k} - u_{i^*,n_k})_{k\in\mathbb{N}}$ converges.

Assume the sequence $(u_n)_{n\in\mathbb{N}}$ already satisfies properties $a) - c)$ $i)$, and the property $c)$ $ii)$ on triangle t_0. Let $t_1 \in F_\Delta \backslash \{t_0\}$ be a triangle sharing an edge $e = vw$ with the triangle t_0, and label the vertex in the triangle t_1 and not in the triangle t_0 by m. If the sequences $(u_{v,n} - u_{i^*,n})_{n\in\mathbb{N}}$ and $(u_{w,n} - u_{i^*,n})_{n\in\mathbb{N}}$ converge, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that the sequence $(u_{m,n_k} - u_{i^*,n_k})_{k\in\mathbb{N}}$ either converges (if $(u_{m,n} - u_{i^*,n})_{n\in\mathbb{N}}$ is bounded) or diverges properly to $+\infty$ (if $(u_{m,n} - u_{i^*,n})_{n\in\mathbb{N}}$ is unbounded). If one of the sequences $(u_{v,n} - u_{i^*,n})_{n\in\mathbb{N}}, (u_{w,n} - u_{i^*,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$, the sequence $(u_{m,n} - u_{i^*,n})_{n\in\mathbb{N}}$ is bounded due to Lemma 5.7, and there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that the sequence $(u_{m,n_k} - u_{i^*,n_k})_{k\in\mathbb{N}}$ converges. ◻

Proof of Lemma 5.7. Without loss of generality we may assume, that both $(u_{1,n})_{n\in\mathbb{N}}$ and $(u_{2,n})_{n\in\mathbb{N}}$ diverge properly to $+\infty$, and that $u_{1,n} \leq u_{2,n}$ for all $n \in \mathbb{N}$. Assume that, for all $n \in \mathbb{N}$, the triangle inequality

$$\ell_{12}^n \leq \ell_{23}^n + \ell_{31}^n$$

holds. Then

$$\begin{aligned}
0 < \ell_{12} &= \exp\left(\frac{-u_{1,n} - u_{2,n}}{2}\right) \ell_{12}^n \\
&\leq \exp\left(\frac{-u_{1,n} - u_{2,n}}{2}\right) (\ell_{23}^n + \ell_{31}^n) \\
&= \exp\left(\frac{u_{3,n} - u_{1,n}}{2}\right) \left(\ell_{23} + \ell_{31} \underbrace{\exp\left(\frac{u_{1,n} - u_{2,n}}{2}\right)}_{\leq 1}\right) \\
&\leq \exp\left(\frac{u_{3,n} - u_{1,n}}{2}\right) (\ell_{23} + \ell_{31}) \xrightarrow{n\to\infty} 0.
\end{aligned}$$

This is a contradiction. ◻

Behaviour around a vertex star Let $(u_n)_{n\in\mathbb{N}}$ be a consistently ordered convergent sequence of conformal factors in $\mathcal{A}_\Delta$ and let $i \in V$ be a vertex such that the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$. We explore the behaviour of the triangles around i.

Definition 5.8. *Let $i \in V$ be a vertex. A **vertex star around vertex** i is the subset of the triangles $F_\Delta^i \subseteq F_\Delta$ that contain the vertex i. At a vertex star we use the following labeling: Let $s = \deg i$. We label the vertex i by 0* *and the vertices adjacent to i by $1, \ldots, s$, such that, for each $j \in \{1, \ldots, s\}$, the vertices $0, j$ and $j+1$ belong to a triangle. Whenever necessary, we use the convention $1-1=s$.*

Lemma 5.9. *Let $(u_n)_{n\in\mathbb{N}} = (u_{1,n}, u_{2,n}, u_{3,n})_{n\in\mathbb{N}}$ be a sequence of conformal factors in $\mathbb{R}^3$. Assume that, for all $n \in \mathbb{N}$, the edge lengths $\ell_{12}^n, \ell_{23}^n, \ell_{31}^n$ satisfy the triangle inequalities. Suppose that the sequence $(u_{1,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$, and the sequences $(u_{2,n})_{n\in\mathbb{N}}$ and $(u_{3,n})_{n\in\mathbb{N}}$ converge. Then*

$$\frac{\ell_{12}^n}{\ell_{31}^n} \xrightarrow{n\to\infty} 1,$$

and the sequence of angles α_1^n, opposite to the edge 23 in the triangle with edge lengths $\ell_{12}^n, \ell_{23}^n, \ell_{31}^n$, satisfies

$$\alpha_1^n \xrightarrow{n\to\infty} 0.$$

Proof. We use the triangle inequality $\ell_{31}^n \leq \ell_{23}^n + \ell_{12}^n$. Dividing both sides by ℓ_{31}^n yields the inequality

$$1 \leq \frac{\ell_{23}^n}{\ell_{31}^n} + \frac{\ell_{12}^n}{\ell_{31}^n} = \frac{\ell_{23}}{\ell_{31}} \exp\left(\frac{1}{2}(u_{2,n} - u_{1,n})\right) + \frac{\ell_{12}^n}{\ell_{31}^n}.$$

Dividing both sides of the triangle inequality $\ell_{12}^n \leq \ell_{23}^n + \ell_{31}^n$ by ℓ_{12} yields the inequality

$$1 \leq \frac{\ell_{23}}{\ell_{12}} \exp\left(\frac{1}{2}(u_{3,n} - u_{1,n})\right) + \frac{\ell_{31}^n}{\ell_{12}^n}.$$

Since, for $i = 2, 3, \exp\left(\frac{1}{2}(u_{i,n} - u_{1,n})\right) \xrightarrow{n\to\infty} 0$, we obtain

$$\frac{\ell_{23}^n}{\ell_{31}^n} \xrightarrow{n\to\infty} 0, \qquad \frac{\ell_{23}^n}{\ell_{12}^n} \xrightarrow{n\to\infty} 0.$$

The convergence of the fraction $\frac{\ell_{12}^n}{\ell_{31}^n}$ follows from the inequalities

$$1 \leq \lim_{n\to\infty} \frac{\ell_{12}^n}{\ell_{31}^n} \leq 1.$$

From the cosine rule we obtain the convergence

$$2\cos\alpha_1^n = \frac{\ell_{12}^n}{\ell_{31}^n} + \frac{\ell_{31}^n}{\ell_{12}^n} - \frac{(\ell_{23}^n)^2}{\ell_{31}^n \ell_{12}^n} \xrightarrow{n\to\infty} 2,$$

and thus $\alpha_1^n \xrightarrow{n\to\infty} 0$. $\square$

In the following we drop the index n labeling the sequences when we talk about angles. Figure 5.1 illustrates the notation used at a vertex star.

Proposition 5.10. *Let $i \in V$ be a vertex and let F_Δ^i be a vertex star around i. Label the vertices adjacent to i as in Definition 5.8. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of conformal factors in $\mathcal{A}_\Delta$, such that the sequence $(u_{0,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$, and, for all $j = 1, \ldots, s$, the sequences $(u_{j,n})_{n\in\mathbb{N}}$ converge. Then the sequences of angles in the triangles in F_Δ^i satisfy*

$$\lim_{n\to\infty} \alpha_0^{j,j+1} = 0, \qquad \lim_{n\to\infty} \alpha_{j+1}^{j,0} = \lim_{n\to\infty} \alpha_j^{j+1,0} = \pi/2, \qquad j \in \{1, \ldots, s\}.$$

Proof. Denote the limit of a sequence of angles $\alpha_k^{i,j}$ along $(u_n)_{n\in\mathbb{N}}$ by $\bar{\alpha}_k^{i,j}$. Due to Lemma 5.9,

$$\bar{\alpha}_0^{j,j+1} = 0,$$

and thus, for all $j = 1, \ldots, s$,

$$\bar{\alpha}_j^{0,j+1} + \bar{\alpha}_{j+1}^{0,j} = \pi.$$

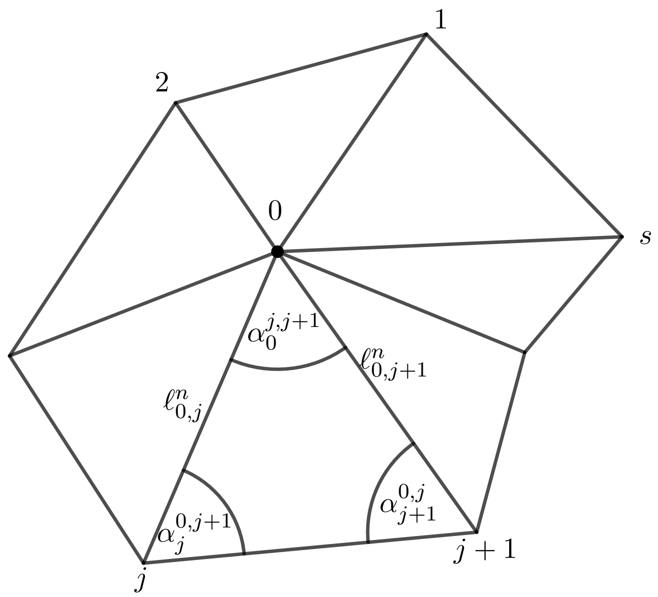


Figure 5.1

Since the edges $0j$ are Delaunay, the Delaunay inequality

$$\bar{\alpha}_{j-1}^{0,j} + \bar{\alpha}_{j+1}^{0,j} \leq \pi$$

is satisfied for each $j \in \{1, \dots, s\}$. Summing up the Delaunay inequalities for edges $01, \dots, 0s$, we obtain

$$\pi s \geq \sum_{j=1}^{s} (\bar{\alpha}_{j-1}^{0,j} + \bar{\alpha}_{j+1}^{0,j}) = \sum_{j=1}^{s} (\bar{\alpha}_{j}^{0,j+1} + \bar{\alpha}_{j+1}^{0,j}) = \pi s.$$

We deduce, that each Delaunay inequality is actually an equality. Let $\varphi := \bar{\alpha}_1^{0,2}$. Then

$$\bar{\alpha}_{j-1}^{0,j} = \varphi, \qquad \bar{\alpha}_{j}^{0,j-1} = \pi - \varphi,$$

for all $j \in \{1, \dots, s\}$.

To show that $\varphi = \pi/2$, we apply the following equation:

Consider a triangle with sides a, b, c, and opposite angles α, β, γ. Then

$$b - a = c \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\gamma}{2}\right)}. \tag{7}$$

Denote the limit of the lengths of edges $j, j+1$ by $\lim_{n\to\infty} \ell_{j,j+1}^n = \bar{\ell}_{j,j+1}$. Since, for all $n \in \mathbb{N}$, holds

$$\sum_{j=1}^{s} \left(\ell_{0,j+1}^n - \ell_{0,j}^n\right) = 0,$$

in the limit

$$0 = \lim_{n\to\infty} \sum_{j=1}^{s} \left(\ell_{0,j+1}^n - \ell_{0,j}^n\right) \overset{(7)}{=} \sin\left(\frac{\pi - 2\varphi}{2}\right) \sum_{j=1}^{s} \bar{\ell}_{j,j+1}.$$

Since, for all $j = 1, \dots, s$, the sequences of conformal factors $(u_{j,n})_{n\in\mathbb{N}}$ converge,

$$\sum_{j=1}^{s} \bar{\ell}_{j,j+1} > 0.$$

We deduce that

$$\sin\left(\frac{\pi - 2\varphi}{2}\right) = 0,$$

and thus $\varphi = \pi/2$. $\square$

The last property we need to explore is the behaviour of the area of the triangles under sequences of conformal factors.

Lemma 5.11. *Let $(u_n)_{n\in\mathbb{N}}$ be a consistently ordered convergent sequence in $\mathcal{A}_\Delta$ and let $ijl \in F_\Delta$ be a triangle, such that $u_{i,n} \geq u_{j,n} \geq u_{l,n}$ for all $n \in \mathbb{N}$. Denote by A^n_{ijl} the area of the triangle with edge lengths $\ell^n_{ij}, \ell^n_{jk}, \ell^n_{ki}$. Then the behaviour of the sequence $(\log A^n_{ijl})_{n\in\mathbb{N}}$ is governed by the behaviour of the sequence $(u_{i^*,n})_{n\in\mathbb{N}}$.*

a) If the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ converges, there exists a convergent sequence of real numbers $(C_n)_{n\in\mathbb{N}}$, such that the area of the triangle ijl satisfies

$$\log A^n_{ijl} = C_n + 2u_{i^*,n},$$

b) if the sequence $(u_{i,n} - u_{i^*,n})_{n\in\mathbb{N}}$ diverges to $+\infty$, there exists a convergent sequence of real numbers $(C_n)_{n\in\mathbb{N}}$, such that the area of the triangle ijl satisfies

$$\log A^n_{ijl} = C_n + \frac{1}{2}(u_{i,n} + 3u_{i^*,n}).$$

Proof. The proof follows from the continuity of the area function and from the fact, that the sequence $(u_n)_{n\in\mathbb{N}}$ is consistently ordered convergent. □

We now have all ingredients to extract the correct subsequence from an unbounded sequence in the sets $\mathcal{A}_+$ and $\mathcal{A}_-$.

Theorem 5.12. *a) Let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_-$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that $(u_{n_k})_{k\in\mathbb{N}}$ is consistently ordered convergent and the sequence $(u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $-\infty$.*

b) Let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_+$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that $(u_{n_k})_{k\in\mathbb{N}}$ is consistently ordered convergent and there exists at least one vertex $j \in V$ such that the sequence $(u_{j,n_k})_{k\in\mathbb{N}}$ diverges properly to $+\infty$. If the sequence $(u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $-\infty$, there exists at least one vertex $j \in V$, such that the sequence $(u_{j,n_k} + 3u_{i^*,n_k})_{k\in\mathbb{N}}$ is bounded from below.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence of conformal factors in $\mathbb{R}^V$. Apply Corollary 5.6. If all sequences $(u_{i,n})_{n\in\mathbb{N}}$ are bounded from below, there exists a consistently ordered convergent subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ and a vertex $j \in V$, such that the sequence $(u_{j,n_k})_{k\in\mathbb{N}}$ diverges properly to $+\infty$. Due to Lemma 5.11, we obtain

$$\lim_{k\to\infty} A_{tot}(u_{n_k}) = \infty.$$

Thus, if the sequence $(u_n)_{n\in\mathbb{N}}$ lies in $\mathcal{A}_-$, it must be of type $a)$ in Corollary 5.6.

Suppose that the sequence $(u_n)_{n\in\mathbb{N}}$ lies in $\mathcal{A}_+$ and is of type $a)$ in Corollary 5.6. Then there exists a consistently ordered convergent subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that the sequence $(u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $-\infty$. The area of at least one of the triangles of the piecewise flat surface $(S, V, d(u_{n_k}))$ has to be bounded away from zero. Due to Lemma 5.11, this is the case only if there exists a vertex $j \in V$, such that the sequence $(u_{j,n_k} + 3u_{i^*,n_k})_{k\in\mathbb{N}}$ is bounded from below. $\square$

5.2 Behaviour of the function $\mathbb{E}$ along sequences of conformal factors

Let (S, V, d) be a piecewise flat surface, with a Delaunay triangulation Δ. Consider a triangle in F_Δ, with vertices labeled by $1, 2, 3 \in V$ and an initial discrete metric $\ell_{12}, \ell_{23}, \ell_{31}$, uniquely determined by d. Define

$$\mathcal{A}_{123} := \{(u_1, u_2, u_3) \mid u \in \mathcal{A}_\Delta\}.$$

Recall the function f from Definition 4.1. Let

$$h : \mathcal{A}_{123} \to \mathbb{R}, \qquad h(u_1, u_2, u_3) := 2f\left(\frac{\tilde{\lambda}_{12}}{2}, \frac{\tilde{\lambda}_{23}}{2}, \frac{\tilde{\lambda}_{31}}{2}\right) - \frac{\pi}{2}(\tilde{\lambda}_{12} + \tilde{\lambda}_{23} + \tilde{\lambda}_{31}).$$

Proposition 5.13. *Let $(u_n)_{n\in\mathbb{N}} = (u_{1,n}, u_{2,n}, u_{3,n})_{n\in\mathbb{N}}$ be a sequence of conformal factors in $\mathcal{A}_{123}$. Suppose, that the sequence $(u_{1,n})_{n\in\mathbb{N}}$ diverges properly to $+\infty$, and the sequences $(u_{2,n})_{n\in\mathbb{N}}$ and $(u_{3,n})_{n\in\mathbb{N}}$ converge to $\overline{u_2}, \overline{u_3}$, respectively. Then*

$$\lim_{n\to\infty} h(u_{1,n}, u_{2,n}, u_{3,n}) = -\pi\left(\log \ell_{23} + \frac{1}{2}(\overline{u_2} + \overline{u_3})\right).$$

Proof. Consider the notation as in Figure 5.2. Then,

$$\begin{aligned}\frac{1}{2}h(u_{1,n}, u_{2,n}, u_{3,n}) =& \alpha_n x_n + \beta_n y_n + \gamma_n z_n + \mathbb{L}(\alpha_n) + \mathbb{L}(\beta_n) + \mathbb{L}(\gamma_n) \\ &- \frac{\pi}{2}(x_n + y_n + z_n).\end{aligned}$$

In the limit, the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ satisfy

$$\lim_{n\to\infty} x_n = \log \ell_{23} + \frac{1}{2}(\overline{u_2} + \overline{u_3}) =: \overline{x}, \qquad \lim_{n\to\infty} y_n = +\infty, \qquad \lim_{n\to\infty} z_n = +\infty,$$

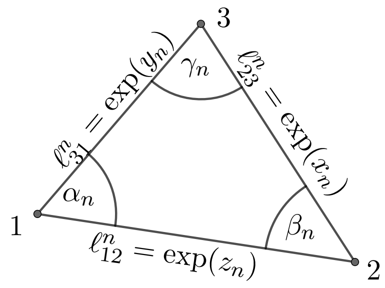


Figure 5.2

and, due to Proposition 5.10,

$$\lim_{n\to\infty} (\alpha_n, \beta_n, \gamma_n) = \left(0, \frac{\pi}{2}, \frac{\pi}{2}\right).$$

Thus,

$$\lim_{n\to\infty} \alpha_n x_n = 0,$$

and, since the Lobachevsky function is continuous and satisfies the equations $\mathbb{L}(0) = \mathbb{L}\left(\frac{\pi}{2}\right) = 0$, in the limit we obtain

$$\lim_{n\to\infty} (\mathbb{L}(\alpha_n) + \mathbb{L}(\beta_n) + \mathbb{L}(\gamma_n)) = 0.$$

In summary,

$$\lim_{n\to\infty} h(u_{1,n}, u_{2,n}, u_{3,n}) = 2 \lim_{n\to\infty} \left[\left(\beta_n - \frac{\pi}{2}\right) y_n + \left(\gamma_n - \frac{\pi}{2}\right) z_n\right] - \pi\overline{x}.$$

We rearrange the expression $\left(\beta_n - \frac{\pi}{2}\right) y_n + \left(\gamma_n - \frac{\pi}{2}\right) z_n$ to obtain

$$\left(\beta_n - \frac{\pi}{2}\right) y_n + \left(\gamma_n - \frac{\pi}{2}\right) z_n = -\frac{1}{2}\alpha_n(y_n + z_n) + \frac{1}{2}(\beta_n - \gamma_n)(y_n - z_n).$$

In the limit, $\lim_{n\to\infty}(\beta_n - \gamma_n) = 0$, and

$$\lim_{n\to\infty} (y_n - z_n) = \log \ell_{31} - \log \ell_{12} + \frac{1}{2}(\overline{u_3} - \overline{u_2}).$$

Thus,

$$\lim_{n\to\infty} \frac{1}{2}(\beta_n - \gamma_n)(y_n - z_n) = 0.$$

It is left to determine the limit

$$\lim_{n\to\infty} \alpha_n(y_n + z_n) = \lim_{n\to\infty} \alpha_n \log \ell_{31}^n + \lim_{n\to\infty} \alpha_n \log \ell_{12}^n.$$

We apply the sine rule and the L'Hospital's rule to obtain the expression

$$\begin{aligned}\lim_{n\to\infty} \alpha_n \log \ell_{31}^n &= \lim_{n\to\infty} (\alpha_n \log \ell_{23}^n + \alpha_n \log \sin \beta_n - \alpha_n \log \sin \alpha_n) \\ &= - \lim_{n\to\infty} \alpha_n \log \sin \alpha_n = 0.\end{aligned}$$

Similarly, $\lim_{n\to\infty} \alpha_n \log \ell_{12}^n = 0$.

Altogether, we see that

$$\lim_{n\to\infty} h(u_{1,n}, u_{2,n}, u_{3,n}) = -\pi\overline{x}.$$

□

Lemma 5.14. *For any real number $v \in \mathbb{R}$, the function h satisfies the equation*

$$h((u_1, u_2, u_3) + v(1, 1, 1)) = h(u_1, u_2, u_3) - \pi v.$$

Proof. Follows from the property of the function f from Proposition 4.5. □

Proposition 5.15. *a) Let $\chi(S) < 0$ and let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_-$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that*

$$\lim_{k\to\infty} \mathbb{E}(u_{n_k}) = +\infty.$$

b) Let $\chi(S) = 2$ and let $(u_n)_{n\in\mathbb{N}}$ be an unbounded sequence in $\mathcal{A}_+$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, such that

$$\lim_{k\to\infty} \mathbb{E}(u_{n_k}) = +\infty.$$

Proof. Due to the Euler formula, $2|V| - |F_\Delta| = 2\chi(S)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of conformal factors in $\mathbb{R}^V$, and let $(u_{n_k})_{k\in\mathbb{N}}$ be a consistently ordered convergent subsequence of $(u_n)_{n\in\mathbb{N}}$ in a Penner cell $\mathcal{A}_\Delta$. Applying Lemma 5.14 we obtain the equality

$$\begin{aligned}\mathbb{E}(u_{n_k}) &= \sum_{ijl\in F_\Delta} h(u_{i,n_k}, u_{j,n_k}, u_{l,n_k}) + 2\pi \sum_{j\in V} u_{j,n_k} \\ &= \underbrace{\sum_{ijl\in F_\Delta} h((u_{i,n_k}, u_{j,n_k}, u_{l,n_k}) - u_{i^*,n_k}(1,1,1))}_{=:C_{n_k}} - \pi|F_\Delta| u_{i^*,n_k} + 2\pi \sum_{j\in V} u_{j,n_k} \\ &= C_{n_k} + 2\pi \left(u_{i^*,n_k}\chi(S) + \sum_{j\in V}(u_{j,n_k} - u_{i^*,n_k}) \right).\end{aligned}$$

The sequence $(C_{n_k})_{k\in\mathbb{N}}$ converges due to Proposition 5.13 and due to the fact that the sequence $(u_{n_k})_{k\in\mathbb{N}}$ is consistently ordered convergent. Thus, the convergence of $\mathbb{E}(u_{n_k})$ is governed by the convergence of the expression

$$u_{i^*,n_k}\chi(S) + \sum_{j\in V}(u_{j,n_k} - u_{i^*,n_k}). \tag{8}$$

The sequence

$$\left(\sum_{j\in V}(u_{j,n_k} - u_{i^*,n_k})\right)_{k\in\mathbb{N}}$$

is bounded from below by zero and either converges or diverges properly to $+\infty$.

Proof of a): Let $\chi(S) < 0$ and consider an unbounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_-$. Due to Theorem 5.12, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ that is consistently ordered convergent, and the sequence $(u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $-\infty$. Since the Euler characteristic of S is negative, the sequence $\chi(S)(u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $+\infty$. Thus, the expression (8) diverges to $+\infty$ and we deduce, that

$$\lim_{k\to\infty}\mathbb{E}(u_{n_k}) = +\infty.$$

Proof of b): Let $\chi(S) = 2$ and consider an unbounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_+$. Due to Theorem 5.12, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ that is consistently ordered convergent, and there exists at least one vertex $j \in V$, such that the sequence $(u_{j,n_k})_{k\in\mathbb{N}}$ diverges properly to $+\infty$.

At first assume, that the sequence $(u_{i^*,n})_{n\in\mathbb{N}}$ is bounded from below. Then the sequence $\chi(S)(u_{i^*,n_k})_{k\in\mathbb{N}}$ either converges or diverges properly to $+\infty$. If it diverges, the expression (8) diverges to $+\infty$. If it converges, the sequence $(u_{j,n_k} - u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $+\infty$. Thus,

$$\lim_{k\to\infty}\mathbb{E}(u_{n_k}) = +\infty.$$

If the sequence $(u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $-\infty$, there exists a vertex $p \in V$, such that the sequence $(u_{p,n_k} + 3u_{i^*,n_k})_{k\in\mathbb{N}}$ is bounded from below. Rearranging the members of the sum in expression (8) we obtain

$$2u_{i^*,n_k} + \sum_{l\in V}(u_{l,n_k} - u_{i^*,n_k}) = -2u_{i^*,n_k} + (u_{p,n_k} + 3u_{i^*,n_k}) + \sum_{l\in V, l\neq p}(u_{l,n_k} - u_{i^*,n_k}).$$

Since both sequences

$$\left(\sum_{l\in V, l\neq p}(u_{l,n_k}-u_{i^*,n_k})\right)_{k\in\mathbb{N}} \qquad \text{and} \qquad (u_{p,n_k}+3u_{i^*,n_k})_{k\in\mathbb{N}}$$

are bounded from below, and the sequence $(-2u_{i^*,n_k})_{k\in\mathbb{N}}$ diverges properly to $+\infty$,

$$\lim_{k\to\infty}\mathbb{E}(u_{n_k})=+\infty.$$

□

References

[1] H. Akiyoshi. Finiteness of polyhedral decompositions of cusped hyperbolic manifolds obtained by the Epstein-Penner's method. *Proc. Amer. Math. Soc.*, 129(8):2431–2439, 2001.

[2] F. Aurenhammer, R. Klein, and D.-T. Lee. *Voronoi diagrams and Delaunay triangulations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[3] A. I. Bobenko, U. Pinkall, and B. A. Springborn. Discrete conformal maps and ideal hyperbolic polyhedra. *Geom. Topol.*, 19(4):2155–2215, 2015.

[4] A. I. Bobenko and B. A. Springborn. A discrete Laplace-Beltrami operator for simplicial surfaces. *Discrete Comput. Geom.*, 38(4):740–756, 2007.

[5] B. Chow and F. Luo. Combinatorial Ricci flows on surfaces. *J. Differential Geom.*, 63(1):97–129, 2003.

[6] X. Gu, R. Guo, F. Luo, J. Sun, and T. Wu. A discrete uniformization theorem for polyhedral surfaces II. *J. Differential Geom.*, 109(3):431–466, 2018.

[7] X. D. Gu, F. Luo, J. Sun, and T. Wu. A discrete uniformization theorem for polyhedral surfaces. *J. Differential Geom.*, 109(2):223–256, 2018.

[8] C. Indermitte, T. M. Liebling, M. Troyanov, and H. Clémençon. Voronoi diagrams on piecewise flat surfaces and an application to biological growth. *Theoret. Comput. Sci.*, 263(1-2):263–274, 2001. Combinatorics and computer science (Palaiseau, 1997).

[9] H. Kourimska. the thesis.

[10] F. Luo. Combinatorial Yamabe flow on surfaces. *Commun. Contemp. Math.*, 6(5):765–780, 2004.

[11] J. Milnor. Hyperbolic geometry: the first 150 years. *Bull. Amer. Math. Soc. (N.S.)*, 6(1):9–24, 1982.

[12] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113(2):299–339, 1987.

[13] B. Springborn. Hyperbolic polyhedra and discrete uniformization.

[14] W. Thurston. Geometry and topology of three-manifolds (lecture notes). 1978.

[15] J. R. Weeks. Convex hulls and isometries of cusped hyperbolic 3-manifolds. *Topology Appl.*, 52(2):127–149, 1993.