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GEODESICS OF SURFACE OF REVOLUTION

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Wenli Chang

September 2011

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ABSTRACT

In this thesis, I study the differential geometry of curves and surfaces in three-dimensional Euclidean space. Some important concepts such as, Curvature, Fundamental Form, Second Fundamental Form, Christoffel symbols, and Geodesic Curvature and equations are explored.

I then investigate the geodesics on a surface of revolution through solving differential equations of geodesic. Main result are stated in Theorem (8.1).

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Chapter 1

Introduction

The study of geodesics is one of the main subjects in differential geometry. The shortest path on a surface joining two arbitrary points is represented by a geodesic. In this project, I focus on the study of geodesics on a surface of revolution. I first introduce some of the key concepts in differential geometry in the first 6 chapters. In chapter 7, I derive the differential equations for a curve being a geodesic.

A theorem on geodesics of a surface of revolution is proved in chapter 8.

Chapter 2

Curves

In this Chapter, we discuss the curves in 3-dimensional Euclidean space \mathbb{R}_3 .

2.1 What Is a Curve

Definition 2.1. A curve in \mathbb{R}^3 is a differentiable map $X : I \rightarrow \mathbb{R}^3$ of I into \mathbb{R}^3 . $I = [a, b]$ be an interval on the real line \mathbb{R}^1 . For each $t \in I$ we have

$$X(t) = (x_1(t), x_2(t), x_3(t)), \quad (2.1)$$

where x_1, x_2, x_3 are the Euclidean coordinate functions of X , and t is called the parameter of the curve X . $X(t)$ can be considered as the position vector of a moving point on the image set $X(I)$ of the curve X .

Example 1: Straight Line.

A straight line in space passing through the point $X(0) = \mathbf{a}$ and in \mathbf{b} position can be represented in the following parametric form:

$$X(t) = \mathbf{a} + \mathbf{b}t = (a_1 + b_1t, a_2 + b_2t, a_3 + b_3t) \quad (2.2)$$

where

$$\mathbf{a} = (a_1, a_2, a_3), \quad \mathbf{b} = (b_1, b_2, b_3).$$

a, b are two constant vectors.

Example 2: An ellipse with center at origin

An ellipse with center at origin is given by

$$X(t) = (a \cos t, b \sin t, 0) \quad (2.3)$$

in particular, if $a = b = r$ we have a circle of radius r as

$$X(t) = (r \cos t, r \sin t, 0). \quad (2.4)$$

Example 3: Helix

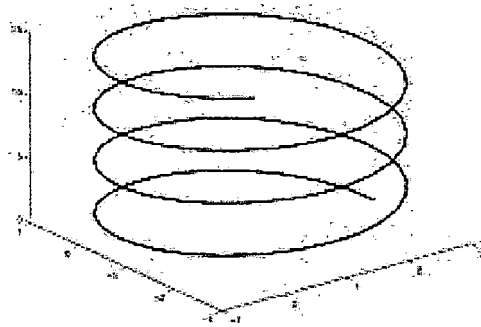


Figure 2.1: Helix

Helix is a curve given by

$$X(t) = (r \cos t, r \sin t, ct), \quad a > 0, c > 0, \quad (2.5)$$

Helix rises at a constant rate on the cylinder $x_1^2 + x_2^2 = r^2$, and it is said to be right winding, if $c > 0$, then the helix is said to be left winding; and the X_3 -axis is called the axis.

Example 4: Folium of Descartes

The folium of Descartes can be represented in parametric form as follows:

$$X(t) = \left(\frac{3t}{1+t^2}, \frac{3t^2}{1+t^3}, 0 \right). \quad (2.6)$$

This curve has a double point at $(x_1, x_2) = (0, 0)$.

Example 5: the limaçon

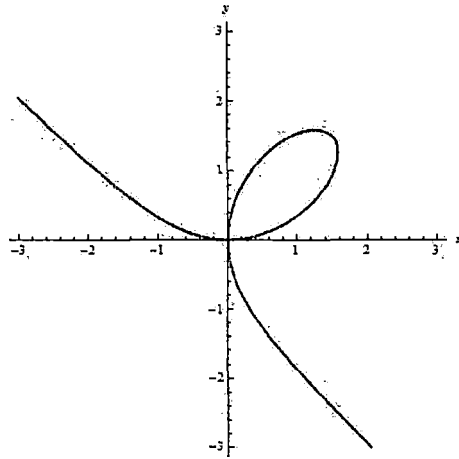


Figure 2.2: Folium Descartes

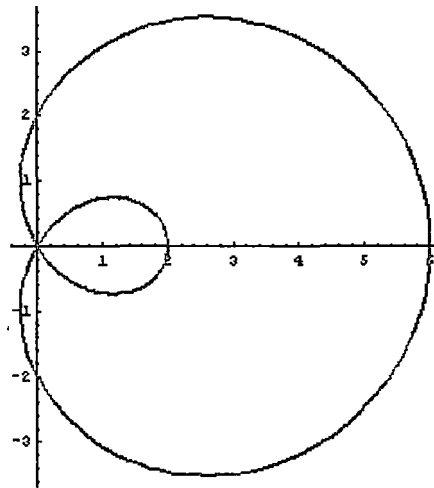


Figure 2.3: Limaçon

The *limaçon* is the parametrized curve:

$$X(t) = ((1 + 2 \cos t) \cos t, (1 + 2 \cos t) \sin t), \quad t \in \mathbb{R}. \quad (2.7)$$

Note that X has a self-intersection at the origin in the sense that $X(t) = 0$ for $t = \frac{2\pi}{3}$ and for $t = \frac{4\pi}{3}$.

2.2 Arc Length

Definition 2.2. Let $X(t)$ ($a \leq t \leq b$) be a curve, then the arc length of the curve $X(t)$ defined as

$$s(t) = \int_{t_0}^t |X'(t)| dt \quad (2.8)$$

where,

$$|X'(t)| dt = \sqrt{(x_1')^2 + (x_2')^2 + (x_3')^2} = \sqrt{X' \cdot X'} \quad (2.9)$$

is the length of the vector $X'(t)$.

Note that: $X'(t)$ is the tangent vector of the curve and $s(t)$ is a differentiable function of t .

Theorem 2.3. A curve is parametrized by the arc length if and only if

$$|X'(t)| = 1 \quad (2.10)$$

Proof. By the Fundamental Theorem of Calculus

$$ds = |X'| dt,$$

$$\left| \frac{ds}{dt} \right| = \frac{ds}{ds} = 1,$$

when $t=s$ is the arc length.

On the other hand, if for the parameter t , we have

$$|X'(t)| = 1 \Rightarrow \frac{ds}{dt} = 1.$$

Thus,

$$s(t) = \int_{t_0}^t dt = t - t_0.$$

Let

$$t_0 = 0,$$

t is the arc length.

□

Definition 2.4. *We will write*

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = dX \cdot dX \quad (2.11)$$

ds is called the element of arc or linear element of C .

Remark: In this paper, some notations are defined as:

$$\begin{aligned} \dot{X} &= \frac{dX}{ds} \\ \ddot{X} &= \frac{d^2X}{ds^2} \\ X' &= \frac{dX}{dt} \\ X'' &= \frac{d^2X}{dt^2}. \end{aligned}$$

Example 1: For the circular helix as in (2.5):

$$X(t) = (r \cos t, r \sin t, ct) \quad (c \neq 0)$$

then

$$\begin{aligned} X' &= (-r \sin t, r \cos t, c) \\ X' \cdot X' &= r^2 + c^2 \end{aligned}$$

and therefore

$$S(t) = t\sqrt{(r^2 + c^2)}$$

Let

$$w = \sqrt{(r^2 + c^2)}$$

Then,

$$X(S) = (r \cos(s/w) \quad r \sin(r/w) \quad cs/w).$$

is the circular helix with arc length as a parameter.

2.3 Tangent, Normal and Osculation Plane

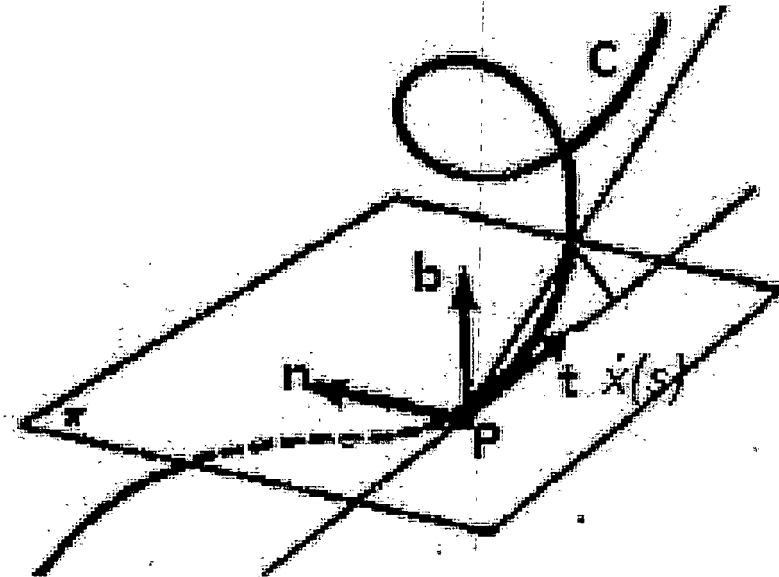


Figure 2.4: Tangent, Normal and Binormal Vector

Definition 2.5. Let C be an arbitrary curve in the space \mathbb{R}^3 , and $X(s)$ be a parametric representation of C with arc length s as parameter. The vector

$$t(s) = \frac{dX}{ds} = \dot{X}(s) \tag{2.12}$$

is called the unit tangent vector of the curve C at the point $X(s)$. This vector is a unit

vector, because

$$|t|^2 = t \cdot t = \dot{X} \cdot \dot{X} = \frac{dX}{ds} \cdot \frac{dX}{ds} = 1 \quad (2.13)$$

Definition 2.6. All the vectors pass through a point P of C and orthogonal to the corresponding unit tangent vector lie in a plane. This plane is called the normal plane to C at P .

Definition 2.7. The plane determined by the unit tangent $t(s)$ and $\ddot{X}(s)$ is called the osculating plane of the curve C at P .

Definition 2.8. The intersection of the osculating plane with the corresponding normal plane is called the principal normal.

2.4 Curvature

Curvature is a very important concept of a curve. Curvature measures that how curved a curve is.

Let $X(s)$ represent a curve C with arc length s as parameter.

Since s is the arc length, then by (2.10) :

$$|X'(s)| = 1$$

i.e

$$\dot{X}(s) \cdot \dot{X}(s) = 1.$$

Differential above equation respect to s , we have

$$\dot{X}(s) \cdot \ddot{X}(s) = 0$$

thus

$$\ddot{X}(s) \perp \dot{X}(s)$$

If $\ddot{X}(s) \neq 0$, $\ddot{X}(s)$ is orthogonal to the unit tangent vector $\dot{X}(s)$ and lies in the normal plane to C .

Definition 2.9. The unit vector

$$\mathbf{P}(s) = \frac{\ddot{X}(s)}{|\ddot{X}(s)|} \quad (2.14)$$

is called the unit principal normal vector to the curve C at the point $X(s)$.

Definition 2.10. :

$$k(s) = |\dot{t}(s)| = \sqrt{\{\ddot{X}(s) \cdot \ddot{X}(s)\}} \quad (k \geq 0) \quad (2.15)$$

is called the curvature of the curve C at the point $X(s)$.

Definition 2.11. Set

$$K(s) = \dot{t}(s) \quad (2.16)$$

and it is called curvature vector.

Theorem 2.12. A curve is a straight line if and only if $k \equiv 0$.

Proof. : Part 1: If $C : X = X(s)$ is a straight line, then

$$X = X_0 + as$$

where X_0 and a are constant.

Therefore,

$$\dot{X} = a$$

$$k = |\ddot{X}(s)| \equiv 0$$

Part 2: If $k \equiv 0$, By $k = |\ddot{X}(s)|$, we have \dot{X} is constant, say a . Thus,

$$a = \dot{X}(s)$$

taking integral respect to s of the above equation,

$$X(s) = \int ads = as + X_0$$

It is a straight line. □

Note that:

$$\dot{t} = \ddot{X}$$

$$k(s) = |\dot{t}(s)|$$

$$P(s) = \frac{\dot{t}(s)}{|\dot{t}(s)|}$$

It is equivalent to say that

$$\dot{t}(s) = kP$$

Taking the cross product of both sides of this equation by t , we find:

$$t \times \dot{t} = \ddot{X} \times \ddot{X} = t \times kP$$

then,

$$|\dot{X} \times \ddot{X}| = |t \times kP| = k|t \times P|$$

Now, since t and P are unit and orthogonal, $|t \times P| = 1$ then we obtain that:

$$k = |\dot{X} \times \ddot{X}|.$$

With respect to any parameter t , we find that

$$\begin{aligned} \dot{X} &= \frac{dX}{ds} \\ &= \frac{dX}{dt} \cdot \frac{dt}{ds} \\ \ddot{X} &= \frac{d\dot{X}}{ds} \\ &= \frac{d}{ds} \left(\frac{dX}{dt} \cdot \frac{dt}{ds} \right) \\ &= \frac{d}{dt} \left(\frac{dX}{dt} \cdot \frac{dt}{ds} \right) \cdot \frac{dt}{ds} \\ &= \left(\frac{d^2X}{dt^2} \cdot \frac{dt}{ds} + \frac{dX}{dt} \cdot \frac{d^2t}{ds^2} \right) \cdot \frac{dt}{ds} \\ &= \frac{d^2X}{dt^2} \left(\frac{dt}{ds} \right)^2 + \frac{dX}{dt} \cdot \frac{d^2t}{ds^2} \cdot \frac{dt}{ds} \end{aligned}$$

then

$$\begin{aligned} \dot{X} \times \ddot{X} &= \left(\frac{dX}{dt} \cdot \frac{dt}{ds} \right) \times \left(\frac{d^2X}{dt^2} \left(\frac{dt}{ds} \right)^2 + \frac{dX}{dt} \cdot \frac{d^2t}{ds^2} \cdot \frac{dt}{ds} \right) \\ &= \left(\frac{dt}{ds} \right)^2 \frac{dX}{dt} \times \left(\frac{d^2X}{dt^2} \frac{dt}{ds} + \frac{dX}{dt} \frac{d^2t}{ds^2} \right) \\ &= \left(\frac{dt}{ds} \right)^2 \frac{dX}{dt} \times \left(\frac{dX}{dt} \frac{dt}{ds} \right)' \\ &= \left(\frac{dt}{ds} \right)^3 X' \times X''. \end{aligned}$$

That is

$$k = |\dot{X} \times \ddot{X}| = |X' \times X''| \left(\frac{dt}{ds} \right)^3$$

by definition (2.3), we have that

$$ds^3 = dX \cdot dX \cdot dX.$$

We can rewrite above equation as:

$$k = |\dot{X} \times \ddot{X}| = |X' \times X''| \left(\frac{dt}{dX} \right)^3$$

i.e

$$k = \frac{|X' \times X''|}{|X'|^3} \left(\frac{dX}{dt} = X' \right) \quad (2.17)$$

we can write (2.16) as:

$$k = \frac{(\sqrt{X' \times X''}) \cdot (X' \times X')}{(\sqrt{X' \cdot X'})^3}$$

Since by the *the Identity of Lagrange*:

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

where a, b, c, d are vectors.

It is equivalent to

$$k = \frac{\sqrt{\{(X' \cdot X')(X'' \cdot X'') - (X' \cdot X'')^2\}}}{(X' \cdot X')^{\frac{3}{2}}} \quad (2.18)$$

In particular $t = s$, (2.17) reduces to the form (2.15).

Examples: The Curvature of Some Curves

Example 1. When C is a circle represented by

$$X(t) = (r \cos t, r \sin t, 0)$$

then

$$X' = (-r \sin t, r \cos t, 0), \quad X'' = (-r \cos t, -r \sin t, 0).$$

By (2.17),

$$k = \frac{\sqrt{\{(r^2 \sin^2 t + r^2 \cos^2 t) \cdot (r^2 \cos^2 t + r^2 \sin^2 t) - (r^2 \sin t \cos t - r^2 \sin t \cos t)\}}}{(r^2 \sin^2 t + r^2 \cos^2 t)^{\frac{3}{2}}}$$

thus, for a circle

$$k = \frac{r^4}{(r^2)^{\frac{3}{2}}} = r.$$

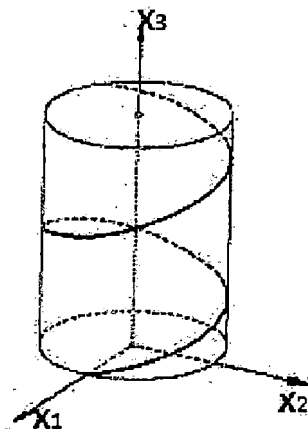


Figure 2.5: Cylinder Circular

Example 2: The circular helix

$$X(t) = (r \cos t, r \sin t, ct)$$

$$X' = (-r \sin t, r \cos t, c)$$

$$X'' = (-r \cos t, -r \sin t, 0)$$

Thus by (2.17), the curvature of the helix is:

$$k = \frac{\sqrt{\{(r^2 \sin^2 t + r^2 \cos^2 t + c^2) \cdot (r^2 \cos^2 t + r^2 \sin^2 t) - (r^2 \sin t \cos t - r^2 \sin t \cos t)\}}}{(r^2 \sin^2 t + r^2 \cos^2 t + c^2)^{\frac{3}{2}}}$$

$$k = \frac{\sqrt{(r^2 + c^2)(r^2)}}{(r^2 + c^2)^{\frac{3}{2}}}$$

Thus,

$$k = \frac{r}{r^2 + c^2}$$

So, the curvature of the circular helix is constant.

Example 3: The hyperbolic spiral

$$X(t) = (a \cosh t, a \sinh t, at), \quad (a > 0);$$

From above parametric function,

$$X' = (a \sinh t, a \cosh t, a),$$

And,

$$X'' = (a \cosh t, a \sinh t, 0),$$

Then,

$$X' \times X'' = (-a^2 \sinh t, a^2 \cosh t, -a^2),$$

$$|X' \times X''| = \sqrt{2}a \cosh t,$$

$$|X'| = \sqrt{2}a \cosh t$$

Thus, the curvature is :

$$k = \frac{|X' \times X''|}{|X'|^3}$$

i.e

$$k = \frac{\sqrt{2}a^2 \cosh t}{2\sqrt{2}a^3 \cosh^3 t}$$

$$k = \frac{1}{2a \cosh^2 t}$$

Chapter 3

Concepts of a Surface. First Fundamental Form

In this chapter, we discuss the surfaces in \mathbb{R}^3 .

3.1 Concept of a Surface in Differential Geometry

3.1.1 Parametric Representation of Surfaces

Definition 3.1. Let $D \subset \mathbb{R}^2$ is a open domain, (u, v) represent the points in \mathbb{R}^2 . There is 1-1 map: $r : D \rightarrow \mathbb{R}^3$, if (x_1, x_2, x_3) represent the Cartesian coordinate of the points on \mathbb{R}^3 , then the map can be presented as:

$$\begin{cases} x_1 = x_1(u, v) \\ x_2 = x_2(u, v) \\ x_3 = x_3(u, v) \end{cases} \quad (u, v) \in D$$

and $x_1(u, v), x_2(u, v), x_3(u, v)$ are all differentiable functions respect to u, v , under this map, the image of D form a surface S in \mathbb{R}^3 . (u, v) is called the parameters of surface S and the vector parametric function of S is

$$X(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)) \quad (3.1)$$

Remark: In Differential Geometry we use calculus to analyst surfaces. The functions on a surface must be differentiable. In order to be able to apply differential calculus to

geometric problems of $X(u, v)$ with respect to u, v . The following assumptions are made:

The Jacobin matrix

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} \end{pmatrix}$$

is of rank 2 in D .

It means the Jacobins $J_{x_1}, J_{x_2}, J_{x_3}$ are not all simultaneously zero, i.e.,

$$J_{x_1}^2 + J_{x_2}^2 + J_{x_3}^2 \neq 0$$

and the derivatives of X_1, X_2 and X_3 with respect to u and v are continuous.

The Jacobians are defined as:

$$J_{x_1} = \frac{\partial(x_2, x_3)}{\partial(u, v)}$$

$$J_{x_2} = \frac{\partial(x_3, x_1)}{\partial(u, v)}$$

$$J_{x_3} = \frac{\partial(x_1, x_2)}{\partial(u, v)}$$

Where,

$$\frac{\partial(x_2, x_3)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix}$$

$$\frac{\partial(x_1, x_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \end{vmatrix}$$

$$\frac{\partial(x_3, x_1)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u} \\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v} \end{vmatrix}$$

Remark: Rank $J = 2$ is equivalent to the condition that $\frac{\partial X}{\partial u}$ and $\frac{\partial X}{\partial v}$ are linearly independent.

Examples of surface

Example 1: A sphere of radius r with center at $X = (0, 0, 0)$

A *sphere* is the set of points of \mathbb{R}^3 that are a fixed distance (the *radius*) from a fixed point (its *center*). So the sphere of radius r with center at $X = (0, 0, 0)$ can be represent as:

$$x_1^2 + x_2^2 + x_3^2 = r^2$$

from this we can obtain a representation of the form:

$$x_3 = \pm \sqrt{(r^2 - x_1^2 - x_2^2)}.$$

It represents the two hemispheres when $x_3 \geq 0$ or $x_3 \leq 0$.

The parametric representation of the sphere can be given as :

$$X(u, v) = (r \cos v \cos u, r \cos v \sin u, r \sin v) \quad (3.2)$$

$$0 \leq u < 2\pi, -\frac{1}{2}\pi \leq v < \frac{1}{2}\pi$$

that is,

$$x_1 = r \cos v \cos u$$

$$x_2 = r \cos v \sin u$$

$$x_3 = r \sin v$$

As Figure(3.1), u is the longitude and v is the latitude.

Example 2: Let C be a curve in the x_1x_2 -plane defined by

$$X(x_1, x_2) = c, \quad x_3 = 0$$

Then the cylinder S (Figure(3.2)) generated by the line L perpendicule to the x_1x_2 -plane along the curve C is given by

$$X(x_1, x_2) = c. \quad (3.3)$$

If C is not a closed curve, and $H(u) = (h_1(u), h_2(u), 0)$ is a parametrization of C , then

$$X(u, v) = (h_1(u), h_2(u), v) \quad (3.4)$$

is a parametrization of the cylinder S .

A *circular cylinder* is the set of points of \mathbb{R}^3 that are at a fixed distance (the *radius* of the cylinder) from a fixed straight line (its *axis*).

So C (from above example 2) is a circle with a radius a , the parametrization equation of C is:

$$H(u) = (a \cos u, a \sin u, 0).$$

Thus,

$$X(u, v) = (a \cos u, a \sin u, v) \quad (3.5)$$

is a representation of the cylinder of revolution which has radius a and the x_3 axis as axis of revolution.

The corresponding matrix

$$\mathbf{J} = \begin{pmatrix} -a \sin u & 0 \\ a \cos u & 0 \\ 0 & 1 \end{pmatrix}$$

is of rank 2.

Example 3: A cone of revolution with apex at $X = (0, 0, 0)$ and with x_3 -axis as axis of revolution can be represented as:

$$X(u, v) = (u \cos v, u \sin v, au). \quad (3.6)$$

The curves $u = \text{const}$ are circles parallel to the x_1x_2 - plane while the curves $v = \text{const}$ are the generating straight lines of the cone.

Example 4: Torus is rotating a circle C in a plane Π around a straight line l in Π that does not intersect C . If Π to be the xz -plane, l to be the z -axis, $a > 0$ the distance of the center of C from l , and $b < a$ the distance of C , then the torus is a smooth surface with parametrization.

$$X(u, v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u)$$

3.2 Curve on a Surface , Tangent Plane to a Surface

Any curve $X(u, v)$ on a surface S can be determined by a parametric function:

$$\begin{cases} u = u(t) \\ v = v(t) \end{cases} \quad a \leq t \leq b$$

thus, the parametric equation of C can be written as:

$$X = X(u(t), v(t))$$

the tangent vector of C on a surface S is:

$$\begin{aligned} X' &= \frac{dX}{dt} = \frac{\partial X}{\partial u} \frac{du}{dt} + \frac{\partial X}{\partial v} \frac{dv}{dt} \\ &= X_u u' + X_v v', \quad (X_u = \frac{\partial X}{\partial u}, X_v = \frac{\partial X}{\partial v}). \end{aligned} \quad (3.7)$$

This means that the tangent vector of C on a surface S is linear combination of the vector X_u and X_v . We assume that $X_u \times X_v \neq 0$, X_u, X_v are linearly independent.

Definition 3.2. The vectors X_u, X_v in (3.7) spans a plane $E(P)$ called the tangent plane at P to the surface S . $E(P)$ contains the tangent to any curve on S at P passing through the point P .

Remark: There is a only one tangent direction pass a point at a curve, but there are infinite tangent directions pass a point on a surface, these tangent vectors form the tangent plane.

Definition 3.3.

$$\mathbf{n} = \frac{X_u \times X_v}{|X_u \times X_v|} \quad (3.8)$$

is called an unit normal vector to S at D , where $X_u = \frac{\partial X}{\partial u}$ and $X_v = \frac{\partial X}{\partial v}$.

Example 1: Tangent vector and Normal vector of a Circular cylinder

The parametric function of a circular cylinder as (3.5)

$$X(u, v) = (a \cos u, a \sin u, v)$$

and $0 < u < 2\pi$, $-\infty < v < \infty$, its tangent vector is:

$$X_1 = X_u = (-a \sin u, a \cos u, 0)$$

$$X_2 = X_v = (0, 0, 1)$$

its unit normal vector is:

$$\mathbf{n} = \frac{X_u \times X_v}{|X_u \times X_v|} = (\cos u, \sin u, 0)$$

Example 2: Catenary

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u).$$

Calculate partial derivative:

$$X_1 = (\sinh u \cos v, \sinh u \sin v, 1)$$

$$X_2 = (-\cosh u \sin v, \cosh u \cos v, 0)$$

then

$$X_1 \times X_2 = (-\cosh u \cos v, -\cosh u \sin v, 0)$$

$$\begin{aligned} |X_1 \times X_2| &= \sqrt{\cosh^2 u \cos^2 v + \cosh^2 u \sin^2 v + \sinh^2 u \cosh^2 u} \\ &= \sqrt{\cosh^2 u (1 + \sinh^2 u)} = \cosh^2 u \end{aligned}$$

and,

$$\mathbf{n} = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{1}{\cosh u} (-\cos v, -\sin v, \sinh u)$$

3.3 First Fundamental Form

Let S be a surface defined by a parametric equation $X(u, v)$, as we pointed out in the preceding section, any curve on surface S can be represented in the form.

$$u = u(t), v = v(t)$$

the parametric equation of the curve is:

$$X(u, v) = X(u(t), v(t)).$$

From (3.7), we have:

$$X' = \frac{dX}{dt} = \frac{\partial X}{\partial u} \frac{du}{dt} + \frac{\partial X}{\partial v} \frac{dv}{dt} = X_u u' + X_v (v)'$$

Multiply dt on both sides of the above equation:

$$dX = \left(X_u \frac{du}{dt} + X_v \frac{dv}{dt} \right) dt$$

i.e;

$$dX = X_u du + X_v dv.$$

Now, from (2.11)

$$ds^2 = dX \cdot dX$$

$$ds^2 = dX \cdot dX = (X_u du + X_v dv)(X_u du + X_v dv)$$

thus,

$$ds^2 = X_u X_u (du)^2 + 2X_u X_v dudv + X_v X_v (dv)^2. \quad (3.9)$$

We set

$$X_u \cdot X_u = g_{11} \quad (3.10)$$

$$X_u \cdot X_v = g_{12} \quad (3.11)$$

$$X_v \cdot X_u = g_{21} \quad (3.12)$$

$$X_v \cdot X_v = g_{22} \quad (3.13)$$

since $g_{12} = g_{21}$, then (3.9) is written as:

$$ds^2 = g_{11}(du)^2 + 2g_{12}dudv + g_{22}(dv)^2 \quad (3.14)$$

3.14 is called *the First Fundamental Form* of the surface S . It is also expressed as

$$Edu^2 + 2Fdudv + Gdv^2.$$

That is:

$$X_u \cdot X_u = g_{11} = E,$$

$$X_u \cdot X_v = X_v \cdot X_u = g_{12} = g_{21} = F,$$

$$X_v \cdot X_v = g_{22} = G$$

Example 1: Plane

$$X = (u, v, 0)$$

then

$$X_u = \frac{dX}{du} = 1, \quad X_v = \frac{dX}{dv} = 1$$

$$X_u \cdot X_u = 1 = E, \quad X_v \cdot X_v = 1 = G$$

$$X_u \cdot X_v = |X_u||X_v| \cos \alpha = 0 = F$$

α is the angle between x and y , is right angle.

Thus, the First Fundamental Form for a plane is

$$ds^2 = dx^2 + dy^2.$$

Example 2: Circular cylindrical surface

$$X(u, v) = (a \cos u, a \sin u, v)$$

then

$$X_u = (-a \sin u, a \cos u, 0)$$

$$X_v = (0, 0, 1)$$

$$X_u \cdot X_u = \left(\sqrt{(a^2 \sin^2 u + a^2 \cos^2 u + 0)} \right)^2 = a^2$$

$$X_v \cdot X_v = (\sqrt{0 + 0 + 1})^2 = 1$$

$$X_u \cdot X_v = 0.$$

Thus, the First Fundamental Form of a circular cylinder is

$$ds^2 = a^2 du^2 + dv^2$$

Example 3: Hyperbolic paraboloid:

$$X = \{a(u + v), b(u - v), 2uv\}$$

$$X_u = (a, b, 2v), \quad X_v = (a, -b, 2u)$$

Then

$$E = X_u \cdot X_u = a^2 + b^2 + 4v^2$$

$$F = X_u \cdot X_v = a^2 - b^2 + 4uv$$

$$G = X_v \cdot X_v = a^2 + b^2 + 4u^2$$

thus, the First Fundamental Form of Hyperbolic paraboloid is:

$$ds^2 = (a^2 + b^2 + 4v^2)du^2 + 2(a^2 - b^2 + 4uv)dudv + (a^2 + b^2 + 4u^2)dv^2$$

Example 4: Sphere

$$X(u, v) = (a \cos u \cos v, a \cos u \sin v, a \sin u)$$

$$X_u = (-a \cos v \sin u, -a \sin v \sin u, a \cos u)$$

$$X_v = (-a \cos u \sin v, a \cos u \cos v, 0).$$

then

$$\begin{aligned} E = X_u \cdot X_u &= \left(\sqrt{a^2 \cos^2 v \sin^2 u + a^2 \sin^2 v \sin^2 u + a^2 \cos^2 u} \right)^2 \\ &= \left(\sqrt{a^2 \sin^2 u (\sin^2 v + \cos^2 v)} + a^2 \cos^2 u \right)^2 = a^2 \end{aligned}$$

$$\begin{aligned} G = X_v \cdot X_v &= \left(\sqrt{a^2 \cos^2 u \sin^2 v + a^2 \cos^2 u \cos^2 v + 0} \right)^2 \\ &= a^2 \cos^2 u \end{aligned}$$

$$X_u \cdot X_v = 0.$$

The first fundamental form of Sphere is:

$$ds^2 = a^2(du^2 + \cos^2 u dv^2)$$

Example 5: Catenary

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$$

$$X_u = (\sinh u \cos v, \sinh u \sin v, 1)$$

$$X_v = (-\cosh u \sin v, \cosh u \cos v, 0)$$

$$\begin{aligned} X_u \cdot X_u &= \left(\sqrt{\sinh^2 u \cos^2 v + \sinh^2 u \sin^2 v + 1} \right)^2 \\ &= \sqrt{\sinh^2 u + 1} = \cosh^2 u \end{aligned}$$

$$\begin{aligned} X_v \cdot X_v &= \left(\sqrt{\cosh^2 u \sin^2 v + \cosh^2 u \cos^2 v + 0} \right)^2 \\ &= \cosh^2 u \end{aligned}$$

and,

$$F = X_u \cdot X_v = 0$$

thus, The first fundamental form of Catenary is:

$$ds^2 = \cosh^2 u du^2 + \cosh^2 dv^2$$

Chapter 4

The Second Fundamental Form. Christoffel Symbols

Consider an arbitrary surface $S : X(u, v)$ of class $r \geq 2$ and an arbitrary curve C on S :

$$u = u(s), \quad v = v(s)$$

Where s is the arc length of C .

Let γ be the angle between the unit principal vector \mathbf{p} to C and \mathbf{n} be the unit normal vector to S . Since \mathbf{p} and \mathbf{n} are unit vectors, we have:

$$\mathbf{p} \cdot \mathbf{n} = |\mathbf{p}| |\mathbf{n}| \cos \gamma \Rightarrow$$

$$\cos \gamma = \frac{\mathbf{p} \cdot \mathbf{n}}{|\mathbf{p}| \cdot |\mathbf{n}|} \Rightarrow$$

$$\cos \gamma = \mathbf{p} \cdot \mathbf{n}$$

since, $\mathbf{p} = \frac{\dot{X}}{|\dot{X}|} = \frac{\dot{X}}{k}$, we have

$$k \cos \gamma = \dot{X} \cdot \mathbf{n}.$$

By chain rule

$$\dot{X} = \frac{dX}{du} \frac{du}{ds} + \frac{dX}{dv} \frac{dv}{ds} = X_u \dot{u} + X_v \dot{v} \quad (4.1)$$

and

$$\ddot{X} = X_{uu} \dot{u} \dot{u} + X_{uv} \dot{u} \dot{v} + X_{vu} \dot{v} \dot{u} + X_{vv} \dot{v} \dot{v} + X_u \ddot{u} + X_v \ddot{v}. \quad (4.2)$$

Recall the notations:

$$\begin{aligned} X_u &= \frac{dX}{du}, & X_v &= \frac{dX}{dv}, \\ X_{uv} &= \frac{d^2X}{dvdu}, & X_{vu} &= \frac{d^2X}{dvdu}, \\ X_{uu} &= \frac{d^2X}{dudv}, & X_{vv} &= \frac{d^2X}{dvdv} \end{aligned}$$

Let \mathbf{n} be the *unit normal vector* of the surface, then \mathbf{n} is orthogonal to both X_u and X_v .

$$X_u \cdot \mathbf{n} = 0, \quad X_v \cdot \mathbf{n} = 0$$

therefore, scalar product of \mathbf{n} to (4.2)

$$\ddot{X} \cdot \mathbf{n} = (X_{uu}\dot{u}\dot{u} + X_{uv}\dot{u}\dot{v} + X_{vu}\dot{v}\dot{u} + X_{vv}\dot{v}\dot{v}) \cdot \mathbf{n}.$$

Now, we set notation as

$$\begin{aligned} b_{11} &= X_{uu} \cdot \mathbf{n} & b_{12} &= X_{uv} \cdot \mathbf{n} \\ b_{22} &= X_{vv} \cdot \mathbf{n} & b_{21} &= X_{vu} \cdot \mathbf{n} \end{aligned}$$

since,

$$X_{uv} = X_{vu}$$

thus,

$$b_{12} = b_{21}$$

and

$$\ddot{X} \cdot \mathbf{n} = b_{11}(\dot{u})^2 + 2b_{12}\dot{u}\dot{v} + b_{22}(\dot{v})^2$$

i.e

$$\frac{d^2X}{ds^2} \cdot \mathbf{n} = b_{11} \left(\frac{du}{ds} \right)^2 + 2b_{12} \frac{du}{ds} \frac{dv}{ds} + b_{22} \left(\frac{dv}{ds} \right)^2 \quad (4.3)$$

We get :

$$d^2X \cdot \mathbf{n} = b_{11}(du)^2 + 2b_{12}dudv + b_{22}(dv)^2.$$

The quadratic form,

$$b_{11}(du)^2 + 2b_{12}dudv + b_{22}(dv)^2 \quad (4.4)$$

(4.4) is called the *the Second fundamental form* of the surface S . We are going to rewrite it as:

$$L(du)^2 + 2Mdudv + N(dv)^2 \quad (4.5)$$

Where

$$\begin{aligned} b_{11} &= L = X_{uu} \cdot \mathbf{n}, \\ b_{12} &= b_{21} = M = X_{uv} \cdot \mathbf{n} = X_{21} \cdot \mathbf{n}, \\ b_{22} &= N = X_{vv} \cdot \mathbf{n} \end{aligned}$$

The second fundamental form of some surfaces are calculated here.

Case 1:Plane

From previous chapter we have:

$$X_u = (-a \sin u, a \cos u, 0)$$

$$X_v = (0, 0, 1)$$

$$X_u = 1, \quad X_v = 1$$

so,

$$X_{uu} = X_{vv} = 0$$

thus, The second fundamental form of a plane is: 0.

Case 2: Circular Cylinder

$$X = (a \cos t, a \sin t, ct)$$

$$X_u = (-a \sin u, a \cos u, 0)$$

$$X_v = (0, 0, 1)$$

$$E = a^2, \quad F = 0, \quad G = 1$$

$$X_{uu} = (-a \cos u, -a \sin u, 0)$$

$$X_{vv} = 0$$

$$X_{uv} = 0$$

$$\mathbf{n} = (\cos u, \sin u, 0)$$

and,

$$L = X_{uu} \cdot \mathbf{n} = -a$$

$$M = 0 \quad N = 0.$$

Thus, the second fundamental form of Circular Cylinder is:

$$-ad^2u$$

Case 3: Catenary

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, 1)$$

From above chapter we have:

$$X_u = (\sinh u \cos v, \sinh u \sin v, 1)$$

$$X_v = (-\cosh u \sin v, \cosh u \cos v, 0)$$

$$E = \cosh^2 u$$

$$G = \cosh^2 u$$

$$F = 0$$

Now,

$$X_{uu} = (\cosh u \cos v, \cosh u \sin v, 0)$$

$$X_{vv} = (-\cosh u \cos v, \cosh u \sin v, 0)$$

And from previous chapter:

$$\mathbf{n} = \frac{1}{\cosh u} (-\cos v, -\sin v, \sinh u)$$

Thus,

$$L = X_{uu} \cdot \mathbf{n} = -1$$

$$M = X_{uv} \cdot \mathbf{n} = 0$$

$$N = X_{vv} \cdot \mathbf{n} = 1$$

The second fundamental form of Catenary is:

$$-d^2u + d^2v$$

4.1 The Christoffel Symbols

Christoffel symbols are shorthand notations for various functions associated with quadratic differential forms. Each Christoffel symbol is essentially a triplet of three indices, i, j and k , where each index can assume values from 1 to 2 for the case of two variables.

Let $X(u, v)$ be a surface with first and second fundamental forms

$$Edu^2 + 2Fdudv + Gdv^2$$

$$Ldu^2 + 2Mdudv + Ndv^2$$

we now consider the partial derivatives

$$X_{uu} = \frac{\partial^2 X}{\partial u \partial u}, \quad X_{uv} = \frac{\partial^2 X}{\partial u \partial v}$$

$$X_{vu} = \frac{\partial^2 X}{\partial v \partial u}, \quad X_{vv} = \frac{\partial^2 X}{\partial v \partial v}$$

of the vectors X_u and X_v .

Theorem 4.1.

$$X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + Ln \tag{4.6}$$

$$X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + Mn \tag{4.7}$$

$$X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + Nn \tag{4.8}$$

Where

$$\begin{aligned} \Gamma_{11}^1 &= \frac{GE_u - 2F_u + FE_v}{2(EG - F^2)} & \Gamma_{11}^2 &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)} \\ \Gamma_{12}^1 &= \frac{GE_v - FG_u}{2(EG - F^2)} & \Gamma_{12}^2 &= \frac{EG_u - FE_v}{2(EG - F^2)} \\ \Gamma_{22}^1 &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} & \Gamma_{22}^2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)} \end{aligned}$$

The six Γ coefficients in these formulas are called Christoffel symbols.

Proof. Since $\{X_u, X_v, \mathbf{n}\}$ is a basis of \mathbb{R}^3 , the second order derivatives of X should be a linear combination of them.

We will write

$$X_{uu} = \alpha_1 X_u + \alpha_2 X_v + \alpha_3 \mathbf{n}, \quad (4.9)$$

$$X_{uv} = \beta_1 X_u + \beta_2 X_v + \beta_3 \mathbf{n}, \quad (4.10)$$

$$X_{vu} = \gamma_1 X_u + \gamma_2 X_v + \gamma_3 \mathbf{n}, \quad (4.11)$$

Taking the dot product of each above equation with the *unit normal vector* \mathbf{n} , we have

$$X_{uu} \cdot \mathbf{n} = \alpha_1 X_u \cdot \mathbf{n} + \alpha_2 X_v \cdot \mathbf{n} + \alpha_3 \mathbf{n} \cdot \mathbf{n},$$

$$L = X_{uu} \cdot \mathbf{n} = 0 + 0 + \alpha_3 = \alpha_3$$

similarly,

$$M = \beta_3, \quad N = \gamma_3$$

Recall that L , M and N are the coefficient of Second Fundamental Form, now, taking dot product of 4.9,4.10,4.11 with X_u , X_v ,

$$\begin{aligned} X_{uu} \cdot X_u &= \alpha_1 X_u \cdot X_u + \alpha_2 X_v \cdot X_u + \alpha_3 \mathbf{n} \cdot X_u \\ &= E\alpha_1 + F\alpha_2 + 0 \end{aligned}$$

and

$$X_{uu} \cdot X_v = F\alpha_1 + G\alpha_2$$

Since $E = X_u \cdot X_u$, by differentiability with respect to u and v , we have

$$E_u = X_{uu}X_u + X_{uu}X_u$$

$$= 2X_{uu} \cdot X_u.$$

$$E_v = 2X_{uv} \cdot X_u$$

On the other hand,

$$X_{uu} \cdot X_v = (X_u \cdot X_v)_u - X_u \cdot X_{uv} = F_u - \frac{1}{2}E_v. \quad (4.12)$$

Therefore, we have

$$E\alpha_1 + F\alpha_2 = \frac{1}{2}E_u$$

$$F\alpha_1 + G\alpha_2 = F_u - \frac{1}{2}E_v$$

solving the equations gives

$$\alpha_1 = \frac{GE_u - 2F_u + FEv}{2(EG - F^2)}$$

$$\alpha_2 = \frac{2EF_u - EEv - FE_u}{2(EG - F^2)}$$

similarly,

$$\beta_1 = \Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)}$$

$$\beta_2 = \Gamma_{12}^2 = \frac{EG_u - FEv}{2(EG - F^2)}$$

$$\gamma_1 = \Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\gamma_2 = \Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$

The coefficient $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2 here are called Christoffel symbols Γ_{**}^* ,

$$\Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)}$$

$$\Gamma_{12}^2 = \frac{EG_u - FEv}{2(EG - F^2)}$$

$$\Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

□

Chapter 5

Normal and Geodesic Curvature

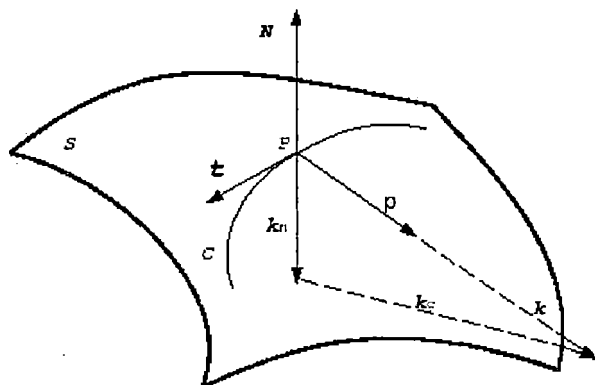


Figure 5.1: Normal and Geodesic Curvature

Let $C : X(s)$ (s is arc length) is a curve on surface S , then \dot{X} is a unit vector and a tangent vector to S . Hence, \dot{X} is perpendicular to the unit normal \mathbf{n} of S , so, \dot{X} , \mathbf{n} and $\mathbf{n} \times \dot{X}$ are mutually perpendicular unit vectors. Again since \ddot{X} is perpendicular to \dot{X} , and hence is a linear combination of \mathbf{n} and $\mathbf{n} \times \dot{X}$:

$$\ddot{X} = \kappa_n \mathbf{n} + \kappa_g \mathbf{n} \times \dot{X} \quad (5.1)$$

Definition 5.1. *The scalars κ_n and κ_g in equation 6.1 are called the normal curvature and the geodesic curvature of C , respectively.*

Proposition 5.2. *With above notations (see figure), we have*

$$\kappa_n = \ddot{X} \cdot \mathbf{n}, \quad (5.2)$$

$$\kappa_g = \ddot{X} \cdot (\mathbf{n} \times \dot{X}), \quad (5.3)$$

$$\kappa_n = \kappa \cos \phi, \quad (5.4)$$

$$\kappa_g = \pm \kappa \sin \phi \quad (5.5)$$

Where κ is the curvature of C and ϕ is the angle between \mathbf{n} and the unit principal normal \mathbf{p} of C .

Theorem 5.3. *The geodesic curvature of κ_g of a curve C on a surface S depends on the first fundamental form of S only.*

Proof. : Note that

$$\dot{X} = X_u \dot{u} + X_v \dot{v}$$

Respect to the arc length S of C , we have

$$\begin{aligned} \ddot{X} &= X_{uu} \dot{u} + X_{uv} \dot{u} \dot{v} + X_{vu} \dot{v} \dot{u} \\ &\quad + X_{vv} \dot{v} \dot{v} + X_u \ddot{u} + X_v \ddot{v} \end{aligned}$$

Thus,

$$\begin{aligned} \dot{X} \times \ddot{X} &= [X_u \dot{u} + X_v \dot{v}] \times [X_{uu} \dot{u} \dot{u} + X_{uv} \dot{u} \dot{v} \\ &\quad + X_{vu} \dot{v} \dot{u} + X_{vv} \dot{v} \dot{v} + X_u \ddot{u} + X_v \ddot{v}] \end{aligned}$$

By previous section, we have

$$\mathbf{n} = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{X_u \times X_v}{\sqrt{g}}$$

$$X_u \times X_v = \sqrt{g} \mathbf{n},$$

$$X_v \times X_u = -\sqrt{g} \mathbf{n},$$

Where, $G = EG - F^2$ since

$$\begin{aligned} X_u \times X_u &= 0 & X_v \times X_v &= 0 \\ \mathbf{n} \times \mathbf{n} &= 0 & \mathbf{n} \cdot \mathbf{n} &= 1 \\ X_u \times X_v &= \sqrt{g}\mathbf{n} & X_v \times X_u &= -\sqrt{g}\mathbf{n} \end{aligned}$$

then

$$\begin{aligned} (\dot{X} \times \ddot{X}) \cdot \mathbf{n} &= [(X_u \dot{u} + X_v \dot{v}) \times (X_{uu} \dot{u} \dot{u} + X_{uv} \dot{u} \dot{v} \\ &\quad + X_{vu} \dot{v} \dot{u} + X_{vv} \dot{v} \dot{v} + X_u \ddot{u} + X_v \ddot{v})] \cdot \mathbf{n} \end{aligned}$$

Note that, in terms of Christoffel symbols,

$$\begin{aligned} X_{uu} &= \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + b_{11} \mathbf{n} \\ X_{uv} &= \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + b_{12} \mathbf{n} \\ X_{vu} &= \Gamma_{21}^1 X_u + \Gamma_{21}^2 X_v + b_{21} \mathbf{n} \\ X_{vv} &= \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + b_{22} \mathbf{n} \end{aligned}$$

then,

$$\begin{aligned} (\dot{X} \times \ddot{X}) &= (X_u \dot{u} + X_v \dot{v}) \times (X_{uu} \dot{u} \dot{u} + X_{uv} \dot{u} \dot{v} \\ &\quad + X_{vu} \dot{v} \dot{u} + X_{vv} \dot{v} \dot{v} + X_u \ddot{u} + X_v \ddot{v}) \\ &= X_u \times X_{uu} (\dot{u})^3 + X_u \times X_{uv} (\dot{u})^2 \dot{v} \\ &\quad + X_u \times X_{vu} (\dot{u})^2 \dot{v} + X_u \times X_{vv} \dot{u} (\dot{v})^2 \\ &\quad + X_u \times X_u \dot{u} \ddot{u} + X_u \times X_v \dot{u} \ddot{v} + X_v \times X_{uu} \dot{v} (\dot{u})^2 \\ &\quad + X_v \times X_{uv} \dot{u} (\dot{v})^2 + X_v \times X_{vu} \dot{u} (\dot{v})^2 \\ &\quad + X_v \times X_{vv} (\dot{v})^3 + X_v \times X_u \dot{v} \ddot{u} + X_v \times X_v \dot{v} \ddot{v} \\ &= X_u \times X_v \dot{u} \ddot{v} - X_u \times X_v \dot{v} \ddot{u} + X_u \times X_{uu} (\dot{u})^3 + X_v \times X_{vv} (\dot{v})^3 \\ &\quad + X_u \times X_{uv} (\dot{u})^2 \dot{v} + X_v \times X_{uu} \dot{v} (\dot{u})^2 \\ &\quad + X_v \times X_{uv} \dot{u} (\dot{v})^2 + X_v \times X_{vv} (\dot{v})^3 \end{aligned}$$

$$\begin{aligned}
& X_u \times X_v(\dot{u}\ddot{v}) - X_u \times X_v\dot{v}\ddot{u} + X_u \times (\Gamma_{11}^1 X_u \\
& + \Gamma_{11}^2 X_v + b_{11}\mathbf{n})(\dot{u})^3 + X_v \times (\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + b_{22}\mathbf{n})(\dot{v})^3 \\
& + X_u \times (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + b_{12}\mathbf{n})(\dot{u})^2\dot{v} \\
& + X_v \times (\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + b_{11}\mathbf{n})\dot{v}(\dot{u})^2 \\
& + X_v \times (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + b_{12}\mathbf{n})\dot{u}(\dot{v})^2 \\
& + X_v \times (\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + b_{22}\mathbf{n})(\dot{v})^3 \\
= & X_u \times X_v\dot{u}\ddot{v} - X_u \times X_v\dot{v}\ddot{u} + X_u X_v(\dot{u})^3\Gamma_{11}^2 + X_u \times b_{11}(\dot{u})^3\mathbf{n} \\
& + X_v \times X_u(\dot{v})^3\Gamma_{22}^1 + X_v \times b_{22}(\dot{v})^3\mathbf{n} + X_u \times X_v(\dot{u})^2\dot{v}\Gamma_{22}^2 \\
& + X_u \times (\dot{u})^2\dot{v}b_{22}\mathbf{n} + X_v \times X_u\dot{v}(\dot{u})^2\Gamma_{11}^1 + X_v \times b_{11}\dot{v}(\dot{u})^2\mathbf{n} \\
& + X_u \times X_v\dot{v}\ddot{u} + X_u \times X_v(\dot{u})^3\Gamma_{11}^2 \\
& + X_u \times b_{11}(\dot{u})^3\mathbf{n} + X_v \times X_u(\dot{v})^3\Gamma_{22}^1 \\
& + X_v \times b_{22}(\dot{v})^3\mathbf{n} + X_u b_{12}(\dot{u}^2\dot{v}\mathbf{n} + X_v \times X_u(\dot{v})(\dot{u})^2\Gamma_{11}^1 \\
& + X_v \times b_{11}\dot{v}(\dot{u})^2\mathbf{n} + X_v \times X_u\dot{u}(\dot{v})^2\Gamma_{12}^1 + X_v \times b_{12}\dot{u}(\dot{v})^2\mathbf{n} \\
& + X_v \times X_u(\dot{v})^3\Gamma_{22}^1 + X_v \times b_{22}(\dot{v})^3\mathbf{n}
\end{aligned}$$

now, since

$$\begin{aligned}
X_u \times X_u &= 0, & X_v \times X_v &= 0 \\
X_u \times X_v &= \sqrt{g}\mathbf{n}, & X_v \times X_u &= -\sqrt{g}\mathbf{n}
\end{aligned}$$

simplify above, we have

$$\begin{aligned}
\dot{X} \times \ddot{X} &= \sqrt{g}\mathbf{n}\dot{u}\ddot{u} + \sqrt{g}\mathbf{n}\dot{u}\Gamma_{12}^2\dot{u}\dot{v} \\
& + \sqrt{g}\mathbf{n}\dot{u}\Gamma_{21}^2\dot{v}\dot{u} + \sqrt{g}\mathbf{n}\dot{u}\Gamma_{22}^2\dot{v}\dot{v} - \sqrt{g}\mathbf{n}\dot{v}\Gamma_{11}^1\dot{u}\dot{u} \\
& - \sqrt{g}\mathbf{n}\dot{v}\Gamma_{12}^1\dot{u}\dot{v} - \sqrt{g}\mathbf{n}\dot{v}\Gamma_{21}^1\dot{v}\dot{u} - \sqrt{g}\mathbf{n}\dot{v}\Gamma_{22}^1\dot{v}\dot{v} \\
& + \sqrt{g}\mathbf{n}\dot{u}\dot{v} - \sqrt{g}\mathbf{n}\dot{v}\ddot{u} + (X_u\dot{u}b_{11}\dot{u}\dot{u} \\
& + X_u\dot{u}b_{21}\dot{v}\dot{u} + X_u\dot{u}b_{22}\dot{v}\dot{v} + X_v\dot{v}b_{11}\dot{u}\dot{u} \\
& + X_v\dot{v}b_{12}\dot{u}\dot{v} + X_v\dot{v}b_{21}\dot{v}\dot{u} + X_v\dot{v}b_{22}\dot{v}\dot{v}) \times \mathbf{n}
\end{aligned}$$

Note: all $\dot{u}b_{11}(\dot{u})(\dot{u}) \dots$ are constants. Since

$$\mathbf{n} \times \mathbf{n} = 0$$

$$b_{11} = X_{11} \cdot \mathbf{n}, \quad b_{12} = X_{12} \cdot \mathbf{n}$$

$$b_{21} = X_{21} \cdot \mathbf{n}, \quad b_{22} = X_{22} \cdot \mathbf{n}$$

so in above expression, all terms cross multiply with \mathbf{n} become zero. Thus,

$$\begin{aligned} \dot{X} \times \ddot{X} &= \sqrt{g}n\dot{u}\dot{u} + \sqrt{g}n\dot{u}\tau_{12}^2\dot{u}\dot{v} + \sqrt{g}n\dot{u}\tau_{21}^2\dot{v}\dot{u} + \sqrt{g}n\dot{u}\tau_{22}^2\dot{v}\dot{v} \\ &\quad - \sqrt{g}n\dot{v}\tau_{11}^1\dot{u}\dot{u} - \sqrt{g}n\dot{v}\tau_{12}^1\dot{u}\dot{v} - \sqrt{g}n\dot{v}\tau_{21}^1\dot{v}\dot{u} \\ &\quad - \sqrt{g}n\dot{v}\tau_{22}^1\dot{v}\dot{v} + \sqrt{g}n\dot{u}\ddot{v} - \sqrt{g}n\dot{v}\ddot{u} \end{aligned}$$

Since $\mathbf{n} \cdot \mathbf{n} = 1$, then, we can have

$$\begin{aligned} (\dot{X} \times \ddot{X}) \cdot \mathbf{n} &= \sqrt{g}\dot{u}\dot{u} + \sqrt{g}\dot{u}\Gamma_{12}^2\dot{u}\dot{v} + \sqrt{g}\dot{u}\Gamma_{21}^2\dot{v}\dot{u} \\ &\quad + \sqrt{g}\dot{u}\Gamma_{22}^2\dot{v}\dot{v} - \sqrt{g}\dot{v}\Gamma_{11}^1\dot{u}\dot{u} - \sqrt{g}\dot{v}\Gamma_{12}^1\dot{u}\dot{v} \\ &\quad - \sqrt{g}\dot{v}\Gamma_{21}^1\dot{v}\dot{u} - \sqrt{g}\dot{v}\Gamma_{22}^1\dot{v}\dot{v} + \sqrt{g}\dot{u}\ddot{v} - \sqrt{g}\dot{v}\ddot{u} \end{aligned}$$

since

$$\Gamma_{12}^2 = \Gamma_{21}^2, \quad \Gamma_{12}^1 = \Gamma_{21}^1$$

then,

$$\begin{aligned} \kappa_g &= (\dot{X} \times \ddot{X}) \cdot \mathbf{n} \\ &= \sqrt{g}[\Gamma_{11}^2(\dot{u})^3 + \Gamma_{12}^2(\dot{u})^2\dot{v} + \Gamma_{21}^2(\dot{u})^2\dot{v} \\ &\quad + \Gamma_{22}^2(\dot{v})^2\dot{u} - \Gamma_{11}^1(\dot{u})^2\dot{v} - \Gamma_{12}^1(\dot{u})(\dot{v})^2\Gamma_{21}^1(\dot{v})^2\dot{u} \\ &\quad - \Gamma_{22}^1(\dot{v})^3 + \dot{u}\ddot{v} - \dot{v}\ddot{u}] \end{aligned}$$

so

$$\kappa_g = \sqrt{g}[\Gamma_{11}^2(\dot{u})^3 + (2\Gamma_{12}^2 - \Gamma_{11}^1)(\dot{u})^2\dot{v} - (2\Gamma_{12}^1 - \Gamma_{22}^2)\dot{u}(\dot{v})^2 - \Gamma_{22}^1(\dot{v})^3 + \dot{u}\ddot{v} - \dot{v}\ddot{u}]. \quad (5.6)$$

Thus, κ_g is only depend on *the first fundamental form* since the Christoffel symbols is only depend on *the first fundamental form*.

□

Ge

Chapter 6

Geodesic and Geodesic Equations

6.1 Definition and Basic Properties

Definition 6.1. *A curve C on a surface S is called a geodesic if its geodesic curvature is zero everywhere.*

Proposition 6.2. *Every straight line is a geodesic.*

Proof. Assume that a straight line in S

$$X(s) = p + sq$$

Which the arc length as parameter.

Then $\ddot{X} = 0$,

$$\kappa_g = \ddot{X} \cdot (\mathbf{n} \times \dot{X}) = 0$$

So it follows that X is a geodesic. □

6.2 Derive the Geodesic Equations

Recall

$$K = \ddot{X} = \kappa_n \mathbf{n} + \kappa_g (\mathbf{n} \times \dot{X})$$

$$\kappa_g = 0$$

$$\ddot{X} \perp \dot{X}, \quad \dot{X} \cdot \ddot{X} = 0.$$

If C is geodesic, then $\kappa_g = 0$. It implies

$$\ddot{X} = \kappa_n \mathbf{n}.$$

i.e., \ddot{X} is parallel to the normal vector of the surface.

Therefore, C is a geodesic, if and only if \ddot{X} is perpendicular to X_u and X_v . i.e.,

$$\kappa_g = (\dot{X} \times \dot{X}) \cdot \mathbf{n} = 0.$$

Let

$$\dot{X} = X_u \dot{u} + X_v \dot{v}$$

then

$$\begin{aligned} \ddot{X} &= \frac{d}{ds}(X_u \dot{u} + X_v \dot{v}) \\ &= X_{uu} \dot{u} \dot{u} + X_{uv} \dot{u} \dot{v} + X_u \ddot{u} + X_{vu} \dot{u} \dot{v} + X_{vv} \dot{v} \dot{v} + X_v \ddot{v} \\ &= \ddot{u} X_u + (\dot{u})^2 X_{uu} + 2\dot{u} \dot{v} X_{uv} + \ddot{v} X_v + (\dot{v})^2 X_{vv} \end{aligned} \quad (6.1)$$

By (4.6), (4.7), (4.8)

$$\begin{aligned} X_{uu} &= \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L \mathbf{n} \\ X_{uv} &= \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + M \mathbf{n} \\ X_{vv} &= \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + N \mathbf{n} \end{aligned}$$

Apply them to equation (6.1)

$$\begin{aligned} \ddot{X} &= \ddot{u} X_u + (\dot{u})^2 (\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L \mathbf{n}) + 2\dot{u} \dot{v} (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + M \mathbf{n}) \\ &\quad + \ddot{v} X_v + (\dot{v})^2 (\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + N \mathbf{n}) \end{aligned}$$

Recall that

$$\begin{aligned} L &= b_{11} = X_{uu} \cdot \mathbf{n} \\ M &= b_{12} = b_{21} = X_{uv} \cdot \mathbf{n} = X_{vu} \cdot \mathbf{n} \\ N &= b_{22} = X_{vv} \cdot \mathbf{n}. \end{aligned}$$

Then

$$\begin{aligned} \ddot{X} &= \ddot{u} X_u + (\dot{u})^2 (\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + X_{uu} \mathbf{n} \cdot \mathbf{n}) \\ &\quad + 2\dot{u} \dot{v} (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + X_{uv} \mathbf{n} \cdot \mathbf{n}) \\ &\quad + \ddot{v} X_v + (\dot{v})^2 (\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + X_{vv} \mathbf{n} \cdot \mathbf{n}) \\ &= f_1 X_u + f_2 X_v + f \cdot \mathbf{n} \end{aligned}$$

where

$$f_1 = \ddot{u} + (\dot{u})^2 \Gamma_{11}^1 + 2\dot{u}\dot{v} \Gamma_{12}^1 + (\dot{v})^2 \Gamma_{22}^1 \quad (6.2)$$

$$f_2 = \ddot{v} + (\dot{u})^2 \Gamma_{11}^2 + 2\dot{u}\dot{v} \Gamma_{12}^2 + (\dot{v})^2 \Gamma_{22}^2 \quad (6.3)$$

$$f = (\dot{u})^2 X_{uu} \cdot \mathbf{n} + 2\dot{u}\dot{v} X_{uv} \cdot \mathbf{n} + (\dot{v})^2 X_{22} \cdot \mathbf{n} \quad (6.4)$$

then

$$\begin{aligned} \dot{X} \times \ddot{X} &= (\dot{u}X_u + \dot{v}X_v) \times (f_1X_u + f_2X_v + f\mathbf{n}) \\ &= (\dot{u}X_u) \times f_1X_u + (\dot{u}X_u) \times f_2X_v + (\dot{u}X_u) \times f\mathbf{n} \\ &\quad + \dot{v}X_v \times f_1X_u + \dot{v}X_v \times f_2X_v + \dot{v}X_v \times f\mathbf{n} \end{aligned}$$

now,

$$\kappa_g = (\dot{X} \times \ddot{X}) \cdot \mathbf{n}$$

and recall

$$X_u \times X_u = 0, \quad X_v \times X_v = 0$$

$$(X_u \times \mathbf{n}) \cdot \mathbf{n} = 0$$

$$(X_v \times \mathbf{n}) \cdot \mathbf{n} = 0$$

$$X_u \times X_v = \sqrt{g}\mathbf{n}$$

$$X_v \times X_u = -\sqrt{g}\mathbf{n}.$$

So, combining above, we have

$$\begin{aligned} \kappa_g &= (\dot{X} \times \ddot{X}) \cdot \mathbf{n} \\ &= [(\dot{u}X_u) \times f_1X_u] \cdot \mathbf{n} + [(\dot{u}X_u) \times f_2X_v] \cdot \mathbf{n} \\ &\quad + [(\dot{u}X_u) \times f \cdot \mathbf{n}] \cdot \mathbf{n} \\ &\quad + [\dot{v}X_v \times f_1X_u] \cdot \mathbf{n} + [\dot{v}X_v \times f_2X_v] \cdot \mathbf{n} \\ &\quad + [\dot{v}X_v \times f \cdot \mathbf{n}] \cdot \mathbf{n} \\ &= \dot{u}f_2[(X_u \times X_v) \cdot \mathbf{n}] + \dot{v}f_1[(X_v \times X_u) \cdot \mathbf{n}] \\ &= (\dot{u}f_2 - \dot{v}f_1)[(X_u \times X_v) \cdot \mathbf{n}]. \end{aligned}$$

Then, $\kappa_g = 0$ if and only if

$$\dot{u}f_2 - \dot{v}f_1 = 0$$

thus

$$\dot{u}f_2 = \dot{v}f_1$$

$$f_2 = \frac{\dot{v}f_1}{\dot{u}} \quad \text{or} \quad f_1 = \frac{\dot{u}f_2}{\dot{v}}.$$

Now, considering $\dot{X} \cdot \ddot{X}$

$$\begin{aligned} \dot{X} \cdot \ddot{X} &= \dot{u}f_1(X_u X_u) + \dot{u}f_2(X_u X_v) + \dot{v}f_2(X_v X_v) + \dot{v}f_1(X_v X_u) \\ &= \dot{u}f_1(X_u X_u) + \dot{u}(X_u X_v) + \dot{v}f_2(X_v X_v) + \dot{v}f_1(X_v X_u) \end{aligned}$$

then

$$\begin{aligned} \dot{u} \cdot (\dot{X} \cdot \ddot{X}) &= (\dot{u})^2 f_1(X_u X_u) + \dot{u} \frac{\dot{v}f_1}{\dot{u}}(X_u X_v) + \dot{v} \frac{\dot{v}f_1}{\dot{u}}(X_v X_v) + \dot{v}f_1(X_v X_u) \\ &= f_1[(\dot{u})^2(X_u X_u) + \dot{v}f_1(X_u X_v) + (\dot{v})^2(X_v X_v) + \dot{v}(X_v X_u)] \\ &= f_1(\dot{u}X_u + \dot{v}X_v)(\dot{u}X_u + \dot{v}X_v) \\ &= f_1(\dot{X} \cdot \dot{X}) \\ &= f_1 \end{aligned}$$

thus

$$\dot{X} \cdot \dot{X} = 1$$

$$\dot{u}(\dot{X} \cdot \ddot{X}) = f_1$$

similarly,

$$\dot{v}(\dot{X} \cdot \ddot{X}) = f_2$$

therefore,

$$\kappa_g = 0$$

\iff

$$\dot{u}f_2 = \dot{v}f_1$$

\iff

$$f_1 = \dot{u}(\dot{X} \cdot \ddot{X})$$

$$f_2 = \dot{v}(\dot{X} \cdot \ddot{X})$$

Since,

$$\dot{X} \cdot \ddot{X} = 0$$

so

$$\kappa_g = 0 \iff f_1 = 0 \quad \text{and} \quad f_2 = 0$$

i.e.,

$$\ddot{u} + (\dot{u})^2 \Gamma_{11}^1 + 2\dot{u}\dot{v} \Gamma_{12}^1 + (\dot{v})^2 \Gamma_{22}^1 = 0 \tag{6.5}$$

$$\ddot{v} + (\dot{v})^2 \Gamma_{11}^2 + 2\dot{u}\dot{v} \Gamma_{12}^2 + (\dot{u})^2 \Gamma_{22}^2 = 0 \tag{6.6}$$

We have proved the following theorem on geodesics.

Theorem 6.3. *Let $C : Xs = X(u(s), v(s))$ be a curve on the surface with parametric equation $X(u, v)$. Then C is a geodesic if and only if the equations 6.5 and 6.6 hold.*

(Therefore, (6.5) and (6.6) are called the geodesic equations).

Chapter 7

A Surface of Revolution

Definition 7.1. : A surface S generated by a given plane curve C rotating about a fixed straight line A is called a surface of revolution. A is called the axis of S . C is called the profile curve.

7.1 The Parametric Representation of a Surface of Revolution

Taking the axis of rotation to be the Z -axis, the plane to be the xz - plane. By the definition of the *Surface of Revolution*, we have that any point, say \mathbf{p} of the surface is obtained by rotation some \mathbf{q} of the profile curve through an angle say u around the z -axis. Now, if

$$C = (f(v), 0, g(v))$$

is a parametrization of the profile curve containing \mathbf{q} , then \mathbf{q} is out of the form (see figure)

$$X_{(u,v)} = (f(v) \cos u, f(v) \sin u, h(v))$$

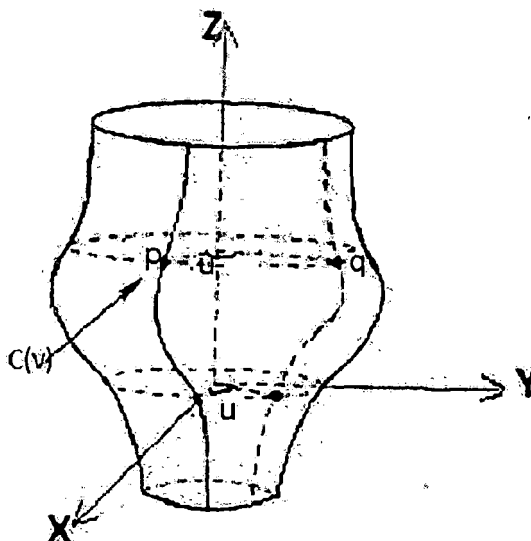


Figure 7.1: Surface of Revolution

7.2 First and Second Fundamental Form of a Surface of Revolution

7.2.1 The First Fundamental Form of a Surface of Revolution

Recall: The first fundamental form of a surface is,

$$Edu^2 + 2Fdudv + Gdv^2$$

where

$$E = X_u \cdot X_u, \quad F = X_u \cdot X_v, \quad G = X_v \cdot X_v$$

let

$$X(u, v) = (f(v) \cos u, f(v) \sin u, h(v)) \quad (7.1)$$

be a surface of revolution.

Then

$$X_u = (-f \sin u, f \cos u, 0)$$

$$X_v = (f' \cos u, f' \sin u, h') \quad (h' = \frac{dh(v)}{dv})$$

Where f' denoting $\frac{d}{dv}$

$$\begin{aligned} E = X_u \cdot X_u &= ((-f \sin u)(-f \sin u) + (f \cos u)(f \cos u) + 0) \\ &= (f^2 \sin^2 u + f^2 \cos^2 u + 0) \\ &= f^2 \end{aligned}$$

and,

$$F = X_u \cdot X_v = (-f \sin u \cos u + f \sin u \cos u + 0) = 0$$

$$G = X_v \cdot X_v = (f'^2 \cos^2 u + f'^2 \sin^2 u + h'^2) = f'^2 + h'^2.$$

So, for a surface of revolution, the first fundamental form has coefficients:

$$\begin{aligned} E &= f^2, \\ F &= 0, \\ G &= f'^2 + h'^2 \end{aligned} \tag{7.2}$$

7.2.2 The Second Fundamental Form of a Surface of Revolution

Recall: The second fundamental form of a surface:

$$Ldu^2 + 2Mdudv + Ndv^2$$

where,

$$\begin{aligned} L &= X_{uu} \cdot n, \\ M &= X_{uv} \cdot n, \\ N &= X_{vv} \cdot n, \\ n &= \frac{X_u \times X_v}{|X_u \times X_v|}, \\ |X_u \times X_v|^2 &= EG - F^2. \end{aligned}$$

Now, for the surface of revolution

$$X_{uu} = (-f \cos u, -f \sin u, 0) \quad (7.3)$$

$$X_{uv} = (-f' \sin u, f' \cos u, 0) \quad (7.4)$$

$$X_{vv} = (f'' \cos u, f'' \sin u, h'') \quad (7.5)$$

and,

$$\begin{aligned} X_u \times X_v &= \begin{vmatrix} i & j & k \\ -f \sin u & f \cos u & 0 \\ f' \cos u & f' \sin u & h' \end{vmatrix} \\ &= \begin{vmatrix} f \cos u & 0 \\ f' \sin u & h' \end{vmatrix} i - \begin{vmatrix} -f \sin u & 0 \\ f' \sin u & h' \end{vmatrix} j + \begin{vmatrix} -f \sin u & f \cos u \\ f' \cos u & f' \sin u \end{vmatrix} k \\ &= (fh' \cos u)i + (fh' \sin u)j + (-ff' \sin^2 u - ff' \cos^2 u)k \\ &= (fh' \cos u, fh' \sin u, -ff') \end{aligned}$$

We assume that $f(v) > 0$ and that the profile curve $v \mapsto (f(v), 0, h(v))$ is unit speed, i.e., $f'^2 + h'^2 = 1$. Then

$$\begin{aligned} |X_u \times X_v| &= \\ &= \sqrt{(fh' \cos u)^2 + (fh' \sin u)^2 + (-ff')^2} \\ &= \sqrt{f^2 h'^2 + (ff')^2} \\ &= f \sqrt{f'^2 + h'^2} \\ &= f \end{aligned}$$

$$\mathbf{n} = \frac{(fh' \cos u, fh' \sin u, -ff')}{f} = (h' \cos u, h' \sin u, f')$$

So, the second fundamental form of a surface of revolution is:

$$L = X_{uv} \cdot \mathbf{n} = (-f \cos u, -f \sin u, 0) \cdot (h' \cos v, h' \sin v, f')$$

$$= -fh'.$$

Samilaly,

$$\begin{aligned} M = X_{uv} \cdot \mathbf{n} &= (-f' \sin u, f' \cos u, 0) \cdot (h' \cos v, h' \sin v, f') \\ &= (-f' h' \sin u \cos u + f' h' \sin u \cos u + 0) = 0 \end{aligned}$$

$$\begin{aligned} N = X_{vv} \cdot \mathbf{n} &= (f'' \cos u, f'' \sin u, h'')(h' \cos u, h' \sin u, f') \\ &= f' h'' - f'' h' \end{aligned}$$

thus

$$(f' h'' - f'' h') du^2 + fh' dv^2$$

is the second fundamental form of surface of revolution.

7.3 The Christoffel Symbols for a Surface of Revolution

Recall the formulas for Christoffel Symbols from the previous chapter:

$$\begin{aligned} \Gamma_{11}^1 &= \frac{GE_u - 2F_u + FE_v}{2(EG - F^2)} & \Gamma_{11}^2 &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)} \\ \Gamma_{12}^1 &= \frac{GE_v - FG_u}{2(EG - F^2)} & \Gamma_{12}^2 &= \frac{EG_u - FE_v}{2(EG - F^2)} \\ \Gamma_{22}^1 &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} & \Gamma_{22}^2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)} \end{aligned}$$

note that the Christoffel symbols only depend on the coefficients of the first fundamental form. For a surface of revolution, the coefficients of first fundamental form are the following:

$$E = f^2, \quad F = 0, \quad G = f'^2 + h'^2 = 1$$

Replace these first fundamental form of surface of revolution into the previous Christoffel symbols formula:

$$\Gamma_{11}^1 = \frac{0 - 0 + 0}{2f^2} = 0$$

$$\Gamma_{12}^1 = \frac{(f^2)'}{2f^2} = \frac{1}{f}$$

$$\Gamma_{22}^1 = \frac{0 - 0 - 0}{2f^2} = 0$$

$$\Gamma_{11}^2 = \frac{0 - f^2 2f - 0}{2f^2} = -f$$

$$\Gamma_{22}^2 = \frac{f^2(0 + 2h'h'') - 0 + 0}{2f^2} = h'h''$$

$$\Gamma_{12}^2 = 0$$

thus

$$\Gamma_{11}^1 = 0$$

$$\Gamma_{12}^1 = \frac{1}{f}$$

$$\Gamma_{22}^1 = 0$$

$$\Gamma_{11}^2 = -f$$

$$\Gamma_{22}^2 = h'h''$$

$$\Gamma_{12}^2 = 0$$

are the Christoffel symbols of the surface of revolution.

Chapter 8

Geodesic of a Surface of Revolution

In this chapter, we are going to study the following question: what are the geodesics on a surface of revolution.

Recall that a surface of revolution obtained by revolving a curve C in XZ -plane about Z -axis can be parametrized by an equation $X(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$, where $(f(v), g(v))$ is the parametric equation for the curve C in xz -plane.

In the following, we will assume that C is given by a function of x in xz -plane, namely $Z = h(x)$.

Then, the parametric equation for the surface can be written as $X(u, v) = (v \cos u, v \sin u, h(v))$.

8.1 Geodesic Equations of a Surface of Revolution

Now, let S be a surface of revolution with the parametric equation

$$X(u, v) = (v \cos u, v \sin u, h(v))$$

We find out

$$E = v^2, \quad F = 0, \quad G = 1 + h''$$

$$L = \frac{-vh'}{\sqrt{1+h'^2}}, \quad M = 0, \quad N = \frac{-h''}{\sqrt{1+h'^2}}$$

$$\begin{aligned}\Gamma_{11}^1 &= 0, & \Gamma_{12}^1 &= \frac{1}{v}, \\ \Gamma_{22}^1 &= 0, & \Gamma_{11}^2 &= -v, \\ \Gamma_{22}^2 &= \frac{h'}{1+h'^2}\end{aligned}$$

Let $\alpha(s) = (v(s) \cos u(s), v(s) \sin u(s), h(v(s)))$ be a curve on S .

By the theorem (6.3), $\alpha(s)$ is a geodesic if and only if

$$\begin{aligned}\ddot{v} - \frac{v}{1+h'^2}(\dot{u})^2 + 2 \cdot 0 + \frac{h'h''}{1+h'^2}(\dot{v})^2 &= 0 \\ \ddot{u} + 0 + 2 \cdot \frac{1}{v} \dot{u} \cdot \dot{v} + 0 &= 0\end{aligned}$$

$$\begin{aligned}\ddot{u} + \frac{2}{v} \dot{u} \dot{v} &= 0 \\ \ddot{v} - \frac{v \cdot (\dot{u})^2}{1+h'^2} + \frac{h'h''(\dot{v})^2}{1+h'^2} &= 0\end{aligned}$$

i.e., Therefore, the geodesic equations for a surface of revolution curve;

$$\ddot{u} + \frac{2}{v} \dot{u} \dot{v} = 0 \tag{8.1}$$

$$\ddot{v} - \frac{v \cdot (\dot{u})^2}{1+h'^2} + \frac{h'h''(\dot{v})^2}{1+h'^2} = 0 \tag{8.2}$$

if $u(s) = \text{constant}$, then $\dot{u} = 0$. The equation (8.1) will obviously holds.

The equation (8.2) will be studied in the following.

First for a curve, on any surface in general:

$$X(s) = X(u(s), v(s))$$

we have,

$$\dot{X} \cdot \dot{X} = 1$$

where

$$\dot{X} = X_u \dot{u} + X_v \dot{v}$$

therefore,

$$\begin{aligned}\dot{X} \cdot \dot{X} &= (X_u \dot{u} + X_v \dot{v})(X_u \dot{u} + X_v \dot{v}) \\ &= (X_u \cdot X_u)(\dot{u})^2 + 2(X_u \cdot X_v)\dot{u}\dot{v} + (X_v \cdot X_v)(\dot{v})^2\end{aligned}$$

recall that

$$X_u \cdot X_u = E, \quad X_u \cdot X_v = F, \quad X_v \cdot X_v = G$$

are the coefficients of the first fundamental form of a surface of revolution, And, for the surface of revolution,

$$E = (v)^2, \quad F = 0, \quad G = 1 + h'^2 \quad (h' = \frac{dh}{dv})$$

then , we have that:

$$\dot{X} \cdot \dot{X} = (v)^2 \cdot (\dot{u})^2 + (1 + h'^2)(\dot{v})^2 = 1. \quad (8.3)$$

When $u(s)$ is constant,

$$(v)^2 \cdot (\dot{u})^2 = 0$$

then (8.4) implies that:

$$(1 + h'^2)(\dot{v})^2 = 1 \quad (8.4)$$

that is

$$(1 + h'^2)(\dot{v})^2 = (\dot{v})^2 + (\dot{v})^2 \cdot h'^2 = 1$$

differentiating it with respect to the arc length s , we have the following,

$$2\dot{v}(\ddot{v} + h'h''\dot{v} + h'^2\ddot{v}) = 0$$

Since $v(s)$ here can not be constant for an actual curve,

then the equation (8.5) implies:

$$\ddot{v} + h'h''\dot{v} + h'^2\ddot{v} = 0$$

or

$$\ddot{v}(1 + h'^2) + h'h''\dot{v} = 0. \quad (8.5)$$

Now, considering the second geodesic equation (8.2) of a surface of revolution

$$\ddot{v} - \frac{v \cdot (\dot{u})^2}{1 + h'^2} + \frac{h'h''(\dot{v})^2}{1 + h'^2} = 0$$

when u is constant, it becomes,

$$\ddot{v} - 0 + \frac{h' h'' (v)^2}{1 + h'^2} = 0$$

multiplying both sides by $1 + h'^2$, we have that

$$\ddot{v}(1 + h'^2) + h' h'' (v)^2 = 0$$

therefore, we have just proved the following theorem.

Theorem 8.1. *On a surface of revolution $S : X(u, v) = (v \cos u, v \sin u, h(v))$ all the v -curves (i.e., $u = \text{constant}$) are geodesic.*

A v -curve on S is also called a meridian which is basically a rotation of the profile curve C about z -axis. In the mean time, a u -curve is called a parallel and all parallel are circles.

Therefore, the theorem 8.1 states that all meridians on a surface of revolution are geodesic.

On the other hand, if $v(s)$ is constant for a curve $\alpha(s)$, then the geodesic equations (8.1) and (8.2) imply that $\dot{u} = 0$, i.e., $u(s)$ must be constant. Hence, a parallel on a surface of a revolution is not a geodesic.

8.2 Examples of Geodesic of a Surface of Revolution

In the following, we take a close look at some examples of surface of revolution.

Example 1 Geodesics on a Sphere

We consider the upper half sphere of radius r centered at $(0, 0, 0)$ as a surface of revolution by revolving a quarter of circle in xz -plane by z -axis.

Then a meridian is part of a great circle on the sphere. Therefore, by the theorem 8.1, all the great circle on a sphere are geodesic.

If $c = 0 \neq a$, then X is a circle around the z - axis.

Example 3 Geodesic on right circular Cone

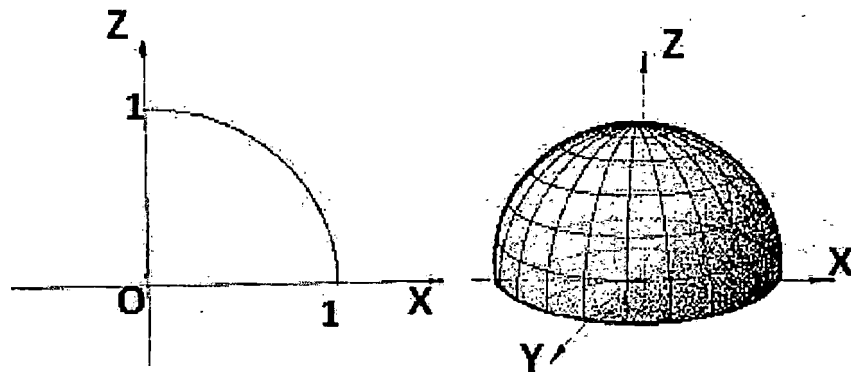


Figure 8.1: Geodesics on Sphere

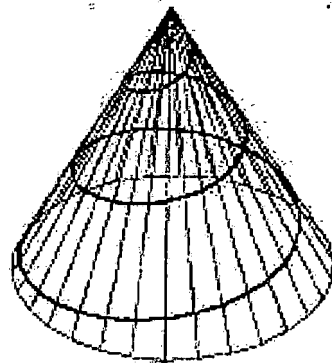


Figure 8.2: Geodesic on Cone

A right circular Cone can be realized as a surface of revolution by revolving a half line $z = ax$ in xz -plane about z -axis, $x > 0$, therefore, it has a parametric equation of $X(u, v) = (v \cos u, v \sin u, av)$.

(That is, $h(v) = av$ in our general notation.)

Meridians on a cone are those straight edges which are the geodesics, by the theorem (8.1).

Parallels are the circles around the cone which are not geodesics.

We also observe that if one cuts the cone along its edge, the cone unwrap into a sector of the Euclidean plane. Therefore, the geodesics on the come should yield straight line segments in the sector.

It is clear that the unwrapping of a parallel (a circle) on the cone is not a straight line segment in the sector. It shows that parallels on a cone are not geodesics.

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