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# $SU(2)$ INVARIANTS OF SYMMETRIC QUBIT STATES

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**Abstract**

We express the density matrix for the $N$-qubit symmetric state or spin-$j$ state ($j = N/2$) in terms of the well-known Fano statistical tensor parameters. Employing the multi-axial representation, where the spin-$j$ density matrix is shown to be characterized by $j(2j + 1)$ axes and $2j$ real scalars, we enumerate the number of invariants constructed out of these axes and scalars. We calculate these invariants explicitly in the particular case of the pure and mixed spin-1 state.

**Keywords:** $SU(2)$ invariants, symmetric state, density matrix, quantum entanglement.

## 1. Introduction

The problem of enumeration of local invariants of the quantum state described by a density matrix $\rho$ is important in the context of quantum entanglement. Nonlocal correlations in quantum systems reflect entanglement between their parts (subsystems). Genuine nonlocal properties should be described in a form invariant under local unitary operations. Two $N$-qubit states are said to be locally equivalent, if one can be transformed into the other by local operations. i.e., $\rho' = U\rho U^\dagger$, where $U \in SU(2)^{\times N}$ and the two quantum states $\rho$ and $\rho'$ are said to be equally entangled.

A general prescription to identify the invariants associated with a multi-particle system has been outlined by in [1]. Well-known algebraic methods for generating invariants already exist in the literature [2–5]. A geometric approach for constructing the $SU(2)$ and $SL(2, C)$ invariants has been presented in [6]. A complete set of 18 local polynomial invariants of two-qubit mixed states was considered in [7], where the usefulness of these invariants to study entanglement was also demonstrated. As the number of subsystems increases, the problem of identifying and interpreting independent invariants rapidly becomes very complicated. It was shown in [8] that a set of 6 invariants which is a subset of a more general set of 18 invariants proposed in [7] is sufficient to characterize the nonlocal properties of a symmetric two qubit system. We concentrate on the symmetric two-qubit states since here the problem of identifying independent invariants becomes easier. Our approach makes use of the geometrical multi-axial representation of an arbitrary spin-$j$ density matrix [9] which is completely characterized by a set of $j(2j + 1)$ axes and $2j$ real positive scalars.

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The paper is organized as follows.

In Sec. 2, we present the decomposition of a density matrix in terms of the well-known Fano statistical tensor parameters. We discuss the multi-axial description of the density matrix using the Wigner-$D$ matrices and enumerate the invariants associated with $N$-qubit symmetric state in Sec. 3. In Sec. 4, we calculate explicitly the invariants of two-qubit symmetric mixed states as well as the most general pure state. To make our task easier, we consider the special Lakin frame widely used in nuclear reactions.

## 2. Symmetric Subspace

Here we are interested in the set of $N$-particle pure states that remain unchanged under permutations of the individual particles. The symmetric states offer an elegant mathematical analysis since the dimension of the Hilbert space reduces drastically from $2^N$ to $(N+1)$, when $N$ qubits respect the exchange symmetry. Such a Hilbert space is considered to be spanned by the eigenstates $\{|j,m\rangle; -j \leq m \leq +j\}$ of angular momentum operators $J^2$ and $J_z$, where $j = N/2$. Analyzing the general state of $N$-particle spin-1/2 system represented by the density matrix of dimension $2^N \times 2^N$ is difficult because the system's Hilbert space increases exponentially with the number of qubits $N$. Fortunately, a large number of experimentally relevant states possesses symmetry under the particle exchange, and this property allows us to reduce significantly the computational complexity.

Completely symmetric systems are experimentally interesting, largely because it is often easier to address nonselectively an entire ensemble of particles rather than individually address each member, and it is possible to express the dynamics of these systems using only the symmetry-preserving operators. The symmetric subspace therefore provides a convenient, computationally accessible class of spin states. Specifically, if we have $N$ two-level atoms, each atom may be represented as a spin-1/2 system, and theoretical analysis can be carried out in terms of the collective spin operator $\vec{J} = (1/2)\Sigma_{\alpha=1}^{N}\vec{\sigma}_\alpha$. Here $\vec{\sigma}_\alpha$ denotes the Pauli spin operator of the $\alpha$th qubit. The standard expression for the most general spin-$j$ density matrix in terms of the Fano statistical tensor parameters $t^k_q{}'s$ is given by

$$\rho(\vec{J}) = \frac{\mathrm{Tr}\,(\rho)}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} t^k_q \, \tau^{k^\dagger}_q(\vec{J}), \tag{1}$$

where $\tau^k_q$ (with $\tau^0_0 = I$, the identity operator) are irreducible tensor operators of rank $k$ in the $(2j+1)$-dimensional spin space with projection $q$ along the axis of quantization in the real three-dimensional space. The $\tau^k_q$ satisfy the orthogonality relations,

$$\mathrm{Tr}\,(\tau^{k^\dagger}_q \tau^{k'}_{q'}) = (2j+1)\delta_{kk'}\delta_{qq'}. \tag{2}$$

Here the normalization has been chosen to be in agreement with the Madison convention [10]. The spherical tensor parameters $t^k_q{}'s$, which characterize the given system, are the average expectation values $t^k_q = \mathrm{Tr}\,(\rho\,\tau^k_q)/\mathrm{Tr}\,\rho$. Since $\rho$ is Hermitian and $\tau^{k^\dagger}_q = (-1)^q \tau^k_{-q}$, $t^k_q$ satisfy the condition

$$t^{k^*}_q = (-1)^q \, t^k_{-q}. \tag{3}$$

The spherical tensor parameters $t^k_q{}'s$ have simple transformation properties under coordinate rotation in the three-dimensional space. In the rotated frame, $t_q^k{}'s$ are given by

$$(t_q^k)^R = \sum_{q'=-k}^{+k} D_{q'q}^k(\phi,\theta,\psi)\, t_{q'}^k, \tag{4}$$

where $D_{q'\,q}^k(\phi,\theta,\psi)$ denote the Wigner-$D$ rotation matrix described by the Euler angles $(\phi,\theta,\psi)$.

## 3. Multiaxial Description of Density Matrix

It has already been shown [9] that a spin-$j$ density matrix is characterized by $j(2j+1)$ axes and $2j$ real positive scalars. For the sake of completeness, we reproduce it here. In general, $t_{\pm k}^k$ can be made zero for any $k$ by suitable rotation, i.e.,

$$(t_{\pm k}^k)^R = 0 = \sum_{q'=-k}^{+k} D_{q',\pm k}^k(\phi,\theta,\psi) t_{q'}^k. \tag{5}$$

Using the well-known Wigner expression for the rotation matrix $D^k$, we can write the above equation as follows:

$$\left(t_{\pm k}^k\right)^R = 0 = \left[\pm_{\cos}^{\sin}(\theta/2)\right]^{2k} \exp[i(\phi+\psi)] \sum_{r=0}^{2k} C_r Z^r, \tag{6}$$

where the complex variable $Z = \cot(\theta/2)e^{-i\phi}$ in the case of $(t_{+k}^k)^R = 0$ and $Z = \tan(\theta/2)e^{-i(\phi+\pi)}$ in the case of $(t_{-k}^k)^R = 0$. The expansion coefficients $C_r$ in the polynomial are the same in both cases and are given by $C_r = \begin{pmatrix} 2k \\ k+q \end{pmatrix}^{1/2} t_q^k = \begin{pmatrix} 2k \\ r \end{pmatrix}^{1/2} t_{r-k}^k$. By solving the above polynomial equation, one can obtain, in general, two sets of $k$-coordinate frames, in which $(t_{\pm k}^k) = 0$. Explicitly, if $t_k^k = 0$ in the coordinate system where the $\hat{Z}$ axis is directed along $(\theta,\phi)$ in the laboratory, $t_{-k}^k = 0$ in the coordinate system where the $\hat{Z}$ axis is directed along $(\pi-\theta,\phi+\pi)$. One set is obtained by the other by inverting the $\hat{Z}$ axis. Therefore, it is sufficient to enumerate the $k$ independent solutions $\hat{Q}_i(\theta_i,\phi_i)$, $i = 1,2,\ldots k$ which constitute any arbitrary $t_q^k$ as a spherical tensor product of the form

$$t_q^k = r_k \left(\cdots(\hat{Q}_1 \otimes \hat{Q}_2)^2 \otimes \hat{Q}_3)^3 \otimes \cdots\right)^{k-1} \otimes \hat{Q}_k)_q^k, \tag{7}$$

where

$$(\hat{Q}_1 \otimes \hat{Q}_2)_q^2 = \sum_{q_1} C(11k; q_1q_2q)(\hat{Q}_1)_{q_1}(\hat{Q}_2)_{q_2}, \qquad (\hat{Q})_q = \sqrt{4\pi/3}\, Y_{1q}(\theta,\phi). \tag{8}$$

Here $C(11k;q_1q_2q)$ is the Clebsch–Gordan coefficient and $Y_{1q}(\theta,\phi)$ are the well-known spherical harmonics. If one of the $\hat{Q}_i's$ is inverted, the sign of Eq. (7) is changed. Hence it is possible to choose $k$ axes $\hat{Q}_i's$, $i = 1,2,\ldots k$ in such a way that $r_k$ is always positive. Each axis requires two independent parameters $(\theta,\phi)$ to characterize it, hence the $k$ axes together with the overall multiplicative factor account for exactly $(2k+1)$ real parameters needed to specify a spherical tensor $t_q^k$ satisfying Eq. (3). Thus, any spherical tensor of rank $k$ can be represented geometrically by a set of $k$ vectors $\hat{Q}_i$ on the surface of a sphere of radius $r$. Consequently, the state of a spin-$j$ assembly can be represented geometrically by a set of $2j$ spheres, one corresponding to each value of $k$ ($k = 1, \ldots, 2j$), the $k$th sphere having $k$ vectors specified on its surface.

Since $(\hat{Q}_i(\theta_i, \phi_i) \otimes \hat{Q}_j(\theta_j, \phi_j))^0_0$ is an invariant ($i \neq j$), one can construct, in general, $\begin{pmatrix} j(2j+1) \\ 2 \end{pmatrix}$ invariants from $j(2j+1)$ axes. Together with $2j$ real positive scalars, there are $\begin{pmatrix} j(2j+1) \\ 2 \end{pmatrix} + 2j$ invariants characterizing spin-$j$ density matrix. Thus using this multiaxial parametrization of the density matrix, we enumerate the total number of $SU(2)$ invariants characterizing a spin-$j$ density matrix. Let us consider the example of two-qubit symmetric state for a detailed discussion.

## 4. Invariants of Two-Qubit Symmetric State or Spin-1 State

### 4.1. Pure Spin-1 State

Consider the direct product $|\psi_1\rangle \otimes |\psi_2\rangle$ of two spinors in the qubit basis as

$$|\psi_{12}\rangle = \begin{pmatrix} \cos(\theta_1/2) \\ \sin(\theta_1/2)\, e^{i\phi_1} \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta_2/2) \\ \sin(\theta_2/2)\, e^{i\phi_2} \end{pmatrix} = \begin{pmatrix} \cos(\theta_1/2)\cos(\theta_2/2) \\ \cos(\theta_1/2)\sin(\theta_2/2)\, e^{i\phi_2} \\ \sin(\theta_1/2)\cos(\theta_2/2)\, e^{i\phi_1} \\ \sin(\theta_1/2)\sin(\theta_2/2)\, e^{i(\phi_1+\phi_2)} \end{pmatrix}, \tag{9}$$

where $0 \leq \theta_{1,2} \leq \pi$ and $0 \leq \phi_{1,2} \leq 2\pi$. In the symmetric angular momentum subspace $|11\rangle$, $|10\rangle$, and $|1-1\rangle$, the combined state has the form

$$|\psi_{12}\rangle_{sym} = \begin{pmatrix} \cos(\theta_1/2)\cos(\theta_2/2) \\ \frac{1}{\sqrt{2}}\left[\cos(\theta_1/2)\sin(\theta_2/2)\, e^{i\phi_2} + \sin(\theta_1/2)\cos(\theta_2/2)\, e^{i\phi_1}\right] \\ \sin(\theta_1/2)\sin(\theta_2/2)\, e^{i(\phi_1+\phi_2)} \end{pmatrix}. \tag{10}$$

Since the two directions $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$ associated with the above two spinors define a plane, we choose this to be the $xz$ plane with respect to a frame $x_0y_0z_0$, with $\hat{z}_0$ being the bisector of the above two directions. Thus, the azimuths of the above two directions $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$ with respect to $x_0$ are, respectively, 0 and $\pi$. If the angular separation between the two directions is $2\theta$, the state $|\psi\rangle$ has the explicit form

$$|\psi\rangle = \frac{\sqrt{2}}{\sqrt{1+\cos^2\theta}}\left[\cos^2(\theta/2)|11\rangle_{\hat{Z}_0} - \sin^2(\theta/2)|1-1\rangle_{\hat{Z}_0}\right]. \tag{11}$$

The density matrix corresponding to the above state is given by

$$\rho_s = \frac{2}{(1+\cos^2\theta)} \begin{pmatrix} \cos^4(\theta/2) & 0 & -\sin^2(\theta/2)\cos^2(\theta/2) \\ 0 & 0 & 0 \\ -\sin^2(\theta/2)\cos^2(\theta/2) & 0 & \sin^4(\theta/2) \end{pmatrix}. \tag{12}$$

Comparing Eq. (12) with the standard representation of the density matrix

$$\rho_s = \frac{\mathrm{Tr}\,(\rho)}{3}\begin{pmatrix} 1+\sqrt{3/2}\,t^1_0 + t^2_0/\sqrt{2} & (\sqrt{3}/2)\,(t^1_{-1}+t^2_{-1}) & \sqrt{3}\,t^2_{-2} \\ -(\sqrt{3}/2)\,(t^1_1+t^2_1) & 1-\sqrt{2}\,t^2_0 & (\sqrt{3}/2)\,(t^1_{-1}-t^2_{-1}) \\ \sqrt{3}\,t^2_2 & -(\sqrt{3}/2)\,(t^1_1-t^2_1) & 1-(\sqrt{3/2})\,t^1_0+t^2_0/\sqrt{2} \end{pmatrix}, \tag{13}$$

we obtain the nonzero $t^{k\prime}_q s$ to be

$$t^1_0 = \frac{\sqrt{6}\cos\theta}{1+\cos^2\theta}, \qquad t^2_0 = \frac{1}{\sqrt{2}}, \qquad t^2_2 = t^2_{-2} = \frac{\sqrt{3}\sin^2\theta}{2(1+\cos^2\theta)}.$$

Since $t^1_{\pm1}=0$, $\hat{z}_0$ itself is the axis $(\hat{Q}_1)$ associated with $t^1$. As $t^1_0 = r_1(\hat{Q}_1)^1_0$,

$$r_1 = \frac{t^1_0}{(\hat{Q}_1)^1_0}. \tag{14}$$

Solving the polynomial equation (6) for $t^2$, we obtain $(\hat{Q}_2)^1_q = \sqrt{4\pi/3}Y^1_q(\theta,0)$ and $(\hat{Q}_3)^1_q = \sqrt{4\pi/3}Y^1_q(\theta,\pi)$. Hence

$$r_2 = \frac{t^2_0}{(\hat{Q}_2\otimes\hat{Q}_3)^2_0} = \frac{t^2_2}{(\hat{Q}_2\otimes\hat{Q}_3)^2_2}. \tag{15}$$

The invariants associated with the most general pure spin-1 state are

$$I_1 = r_1, \qquad I_2 = r_2, \qquad I_3 = (\hat{Q}_1\otimes\hat{Q}_2)^0_0, \qquad I_4 = (\hat{Q}_1\otimes\hat{Q}_3)^0_0, \qquad I_5 = (\hat{Q}_2\otimes\hat{Q}_3)^0_0. \tag{16}$$

Explicitly,

$$I_1 = \frac{\sqrt{6}|\cos\theta|}{1+\cos^2\theta}, \qquad I_2 = \frac{\sqrt{3}}{1+\cos^2\theta}, \qquad I_3 = I_4 = -\frac{\cos\theta}{\sqrt{3}}, \qquad I_5 = -\frac{\cos 2\theta}{\sqrt{3}}. \tag{17}$$

It is clear from Eq. (11) that the state $|\psi\rangle$ is separable for $\theta=0$ and $\pi$. Hence the invariants in the case of pure spin-1 separable states are

$$I_1 = \sqrt{3/2}, \qquad I_2 = \sqrt{3}/2, \qquad I_3 = I_4 = \mp 1/\sqrt{3}, \qquad I_5 = -1/\sqrt{3}.$$

### 4.2. Mixed Spin-1 State

Consider the example of a channel spin-1 system, which plays an important role in nuclear physics experiments like hadron scattering and reaction processes [11–15]. A beam of nucleons colliding with a proton target provides such an example. If both the beam and the target are prepared to be in mixed states, the corresponding density matrices are given by

$$\rho(i) = \frac{1}{2}[I+\vec{\sigma}(i)\cdot\vec{p}(i)] = \frac{1}{2}\sum_{k,q} t^k_q(i)\tau^{k\dagger}_q(i), \qquad i=1,2, \tag{18}$$

where $\vec{p}(i)$ are the polarization vectors and $\vec{\sigma}(i)$ are the Pauli spin matrices.

The combined density matrix is the direct product of the individual density matrices

$$\rho_c = \rho(1)\otimes\rho(2). \tag{19}$$

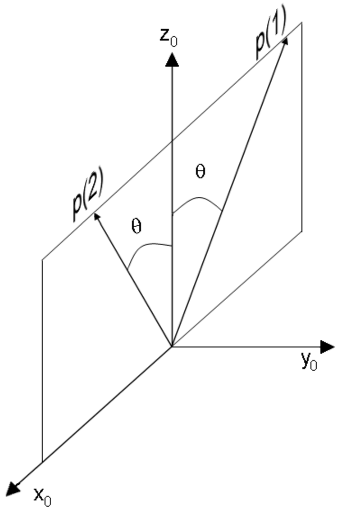


**Fig. 1.** $x_0y_0z_0$ frame with mean spin direction $\hat{z}_0$ as the bisector of two directions $\vec{p}(1)$ and $\vec{p}(2)$.

Even though the combined density matrix is a direct product of individual matrices, in this case entanglement appears due to the projection of the combined density matrix onto the desired spin-1 space. While solving this problem, the special Lakin frame (SLF) which is widely used in studying nuclear reactions is considered: Choose $\hat{z}_0$ to be along $\vec{p}(1)+\vec{p}(2)$. Since $\vec{p}(1)$ and $\vec{p}(2)$ together define a plane in any general situation, we choose $\hat{x}_0$ to be in this plane such that the azimuths of $\vec{p}(1)$ and $\vec{p}(2)$ with respect to $\hat{x}_0$ are, respectively, 0 and $\pi$. The $\hat{y}_0$ axis is then chosen to be along $\hat{z}_0 \times \hat{x}_0$. The frame so chosen is indeed the special Lakin frame (SLF), since here $t^1_{\pm 1} = 0$ and $t^2_2 = t^2_{-2}$, see Fig. 1. Choose a simple case of $|\vec{p}(1)| = |\vec{p}(2)| = p$; then we obtain $t^2_{\pm 1} = 0$ in SLF.

The density matrix so obtained for the spin-1 mixed system in the symmetric subspace $|11\rangle$, $|10\rangle$, and $|1-1\rangle$ is

$$\rho_s = \frac{1}{(3+p^2\cos 2\theta)}\begin{pmatrix}(1+p\cos\theta)^2 & 0 & -p^2\sin^2\theta \\ 0 & 1-p^2 & 0 \\ -p^2\sin^2\theta & 0 & (1-p\cos\theta)^2\end{pmatrix}. \quad (20)$$

Observe that, when $p = 1$, the mixed-state density matrix is exactly the same as that of the pure-state density matrix given by Eq. (12). Comparing the above density matrix with the standard form [Eq. (13), we get the nonzero $t^k_q{}'s$ as

$$t^1_0 = \frac{2\sqrt{6}p\cos\theta}{(3+p^2\cos 2\theta)}, \qquad t^2_0 = \frac{\sqrt{2}p^2(1+\cos^2\theta)}{(3+p^2\cos 2\theta)}, \qquad t^2_2 = \frac{\sqrt{3}p^2\sin^2\theta}{(3+p^2\cos 2\theta)}.$$

Solving the polynomial equation (6) for $t^1$ and $t^2$, we obtain $\hat{Q}_1 = \hat{z}_0$, $\hat{Q}_2 = \vec{p}(1)$, and $\hat{Q}_3 = \vec{p}(2)$. Thus, the invariants associated with the most general mixed spin-1 state are found to be

$$I_1 = \frac{2\sqrt{6}p|\cos\theta|}{(3+p^2\cos 2\theta)}, \qquad I_2 = \frac{2\sqrt{3}p^2}{(3+p^2\cos 2\theta)}, \qquad I_3 = I_4 = \cos\theta, \qquad I_5 = -\frac{\cos 2\theta}{\sqrt{3}}. \quad (21)$$

Note that in both the pure and the mixed states, $I_3 = I_4 = -\cos\theta/\sqrt{3}$ and $I_5 = -\cos 2\theta/\sqrt{3}$. For $p = 1$ and $\theta = 0, \pi$, the state is separable as in the case of pure state. For $p < 1$, the state is separable for a range of values of $\theta$. It is observed that as $p$ decreases, the region of $\theta$ for which entanglement appears also decreases [16].

## 5. Conclusions

We have considered symmetric $N$-qubit density matrix expressed in terms of the Fano statistical-tensor parameters. Making use of the well-known multiaxial decomposition of the density matrix, we have enumerated $SU(2)$ invariants of the most general symmetric state. Considering the special case of two-qubit symmetric state, we have explicitly computed five invariants which form a complete set. Our framework can be applied to enumerate a complete set of invariants of any qudit state. The study of the relationship between various measures of entanglement and our complete set of invariants is in progress.

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