On the Coupled, Flexural-Flexural-Torsional Vibrations of an Asymmetric Concrete Beam

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On the Coupled, Flexural-Flexural-Torsional Vibrations of an Asymmetric Concrete Beam.

John N Karadelis, BSc MPhil(Eng) PhD TEE
Faculty of Engineering, Environment and Computing, Coventry University, Coventry, CV1 5FB, UK.
ORCID ID: 0000-0001-7206-4650
Email: john.karadelis@coventry.ac.uk

Abstract.

This article presents the analytical solution of an L-shaped cross-section asymmetric beam (concrete terrace unit) undergoing *triple coupling* that is, flexural vibration in two mutually perpendicular planes (vertical and horizontal) plus torsional vibration about an axis passing through its shear centre, using the classical approach. Essentially, the procedure involved the development of three governing, coupled, partial differential equations based on Euler-Bernoulli theory for beams with isotropic material properties, from which the *exact* solution was extracted. The warping effect was considered in the torsional equation.

A comparison between the analytical solution and corresponding numerical and experimental results obtained earlier was attempted and similarity and accuracy were discussed.

It is reasonable to state that the analytical method in calculating the natural frequencies of a system is the most reliable, compared to experimental (needs skills and experience) and numerical (calibration, updating, validation, etc).

However, even the analytical solution may not be as accurate as expected, as it depends on several factors/parameters beyond the full control of the investigator. Some useful comments and conclusions are drawn.

Email: john.karadelis@coventry.ac.uk

1. Introduction & Significance:

Beams and other structural components/elements with asymmetric cross section are relatively common in the construction industry today. Yet, their dynamic analysis is not straight forward. This is because their centroid and shear centre do not coincide and therefore they undergo coupled, flexural and torsional vibrations under dynamic actions, if the line of application of these actions is not the shear centre axis.

The case of a beam with a single axis of symmetry through its cross-section, undergoing flexural vibrations in one plane coupled with torsional vibrations, the so called *double coupling*, has been studied by some researchers, mainly using Euler-Bernoulli theory although the warping stiffness has been neglected by most of them (Weaver et al., 1990; Documaci, 1987; Banerjee & Williams, 1992; Bercin & Tanaka, 1997; Bishop et al., 1989; Klausbruckner & Pryputniewicz, 1977; Hallauer & Liu, 1982). The case of *triple coupling* (flexural-flexural-torsional), all coupled, has been dealt by very few.

Yaman (1997) presented the complex problem of forced, coupled flexural-torsional vibrations of uniform, open section channels with one axis of symmetry using the wave propagation approach. Warping was also included in the analysis. The channel sections considered were assumed to be of the Euler-Bernoulli beam type, claiming that this method favours thin beams. One drawback of the analytical model presented was that it did not account for the effects of the cross-sectional distortion of the channels which can be substantial at high frequencies.

Tanaka & Bercin (1999) solved the governing equations of motion (EoM) for bending about two perpendicular axes coupled with torsional vibrations of a uniform thin-walled beam possessing no cross-sectional symmetry, and developed a code based on *Mathematica* (1991) to solve these equations

Work by Arpaci & Bozdag (2002) is associated with the triple coupled, free vibration of thin walled beams of non-symmetrical section, like channels and Z-purlins. They derived the governing EoM and even included terms like the product of inertia, hence amending previous studies by Tanaka and Bercin who conveniently left the term out. They stressed the importance of coupling stiffness and the risk of substantial errors if the latter is not included in the system of equations. However, they too left out the fact that, these very thin sections will distort substantially, especially at high frequencies.

Only recently, Wang (2013) studied the coupled free vibration of composite beams with asymmetric cross-sections based on Euler-Bernoulli beam theory. He included more coupling terms in the governing equations and developed his own algorithm to solve them because both, odd and even order spatial derivatives had to be considered. He concluded that coupled flexural-torsional vibrations occur due to the anisotropy of the composite material considered and that the asymmetric cross-section can cause substantial changes in the natural frequencies and modes of vibration of the beam.

All researchers used thin-walled, open cross-sections to study and verify their findings Essentially, in the case of beams having no plane of symmetry, Timoshenko and his co-workers stated that the problem becomes significantly more complex and involved (Weaver et al., 1990). They recommended coupled torsional and flexural vibrations in the two principal planes and talked about the development of three simultaneous differential equations instead of two and significantly more complex analysis.

Sparked by the above, Arpaci et al., (2003) produced an account of triply coupled vibrations with direct reference to thin-walled open cross-sections, with no axis of symmetry. This time they included both, warping effects and rotary inertia in an effort to come closer to the exact solution. They concluded that the effect of rotary inertia may alter the natural frequencies dramatically for certain boundary conditions and thicknesses and their solution could not be generalised. Surprisingly, they found that the relative error when not taking into consideration the rotary inertia reached 170% unless the warping effect was also excluded! They attributed this gross error to the fact that excluding warping, decreases the torsional rigidity of beams and consequently the natural frequencies and that some, originally bending modes, change to coupled, or even torsional modes. Hence, traces of uncertainty appeared in their approach.

The author has carried out experimental and numerical modal analyses in the past using concrete beams (Karadelis, 2009a, 2009b; Karadelis, 2012). He concluded that results are sensitive to boundary conditions and that certain modes of vibration have a tendency to change from predominantly flexural, to flexural-torsional, or even torsional, when, for instance, boundaries are altered or, reinforcement is taken into consideration. Likewise, assigning different thicknesses, will affect the stiffness of the section, and therefore the natural frequencies are expected to change.

As Timoshenko predicted, the exact solution has already become significantly complex and involved. Therefore, a simpler approach, as presented in the next few pages, may be of some value.

The object of this article is to include the effects of warping and rotational inertia in a distinctive triply coupled (free) vibration analysis over the natural frequencies of a solid, L-shaped concrete terrace unit, the asymmetric beam shown in Figure 1. The literature cited is employed as a “starting point” for this investigation. The latter will be compared with data obtained from carefully crafted experimentation and numerical analysis solutions.

The author was motivated by the fact that the number of studies allied to triple coupled, *flexural-flexural-torsional* vibrations are very limited, compared to those of double coupled, flexural-torsional vibrations. It is true that one can obtain these frequencies with the aid of the finite element method. However, the emphasis was placed on the development of a simple, efficient and accurate analytical technique, since it offers a hands-on, comprehensive perspective of the solution with all relevant parameters present and can even become a valuable teaching tool in advanced engineering courses, avoiding “*black-box”* style solutions. It is stressed that this study considers solid, asymmetric, as opposed to light, thin, open cross-sections. As this has not been attempted before, it is hoped that it will become a valuable add-on to the studies mentioned earlier.

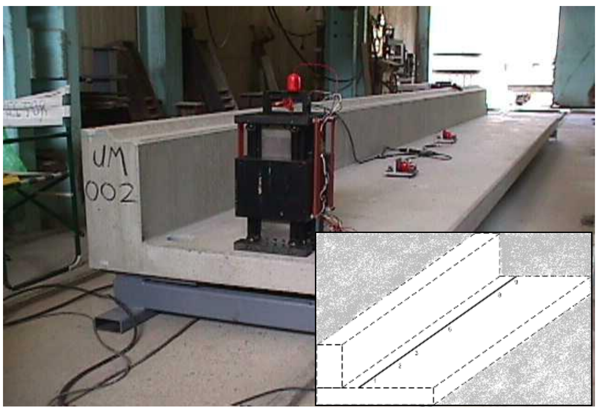


Figure 1. Typical asymmetric beam (concrete terrace unit). A dynamic shaker and several accelerometers are visible. Inset: lumped mass approximation model.

In terms of shear consideration, two main mathematical models exist, namely the shear un-deformable Euler-Bernoulli (E-B) model and the shear-deformable Timoshenko model. In this case it is argued that the thick concrete beam with the horizontal (tread) member would not undergo significant shear deformation compared to bending deformations, hence the E-B model is adopted and the deformation at a section is the rotation due to (double) bending only.

2. Methodology

Figure 1 shows a typical reinforced concrete terrace unit with asymmetric cross-section in the laboratory. The unit has been studied previously, both experimentally and numerically and the results have been published elsewhere (Karadelis, 2009a, 2009b). It should be interesting to study the same unit in an analytical manner and compare the findings with the results mentioned earlier. Some useful conclusions should then be drawn.

A Cartesian coordinate system x,y (lower case) is taken through the section's centroid, CC, as shown in Figure 2. In contrast, X,Y (upper case) represents a second orthogonal system of axes passing through the unit's shear centre, SC.

The deferential equations of flexure in the statics domain, for bending in a vertical and horizontal plane respectively and the torsion equation about the shear centre axis (SC) (Karadelis, 2012) (assuming torsion takes place about SC) should first be developed. The equations of motion (EoM) can then be formed for the frequency equation to be extracted. As there is no plane of symmetry present, the product moment of inertia is a non-zero quantity, ($I_{xy} \neq 0$) and therefore terms such as $\left[EI_{xy}\frac{d^4u}{dz^4}\right]$, allowing for coupling, should be included in the system of these equations.

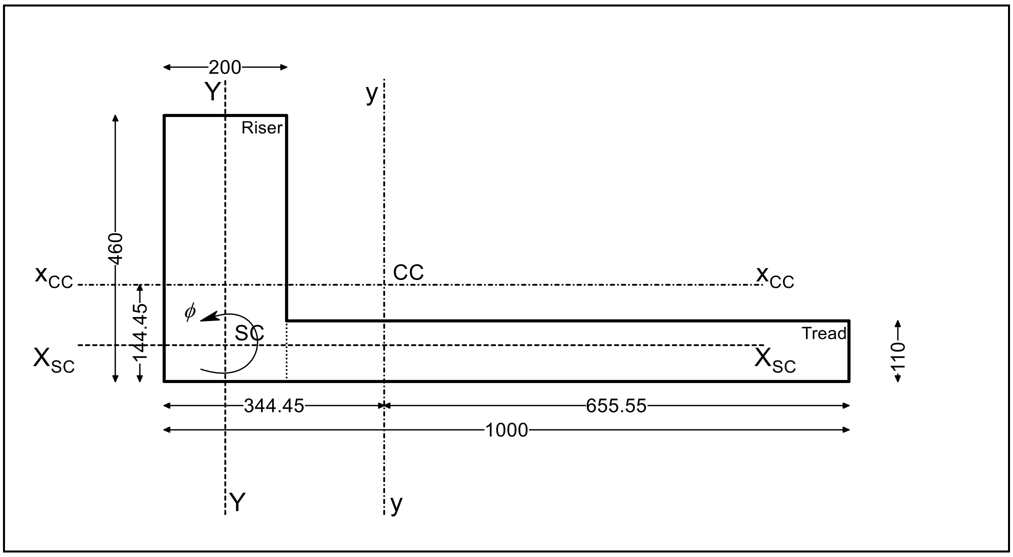


Figure 2. Annotated diagram of a cross-section of the terrace unit shown in Figure 1. All dimensions in mm.

2.1 Equations from Statics:

$$EI_x \frac{d^4 v}{dz^4} = w_y \quad \text{(a)}$$

$$EI_y \frac{d^4 u}{dz^4} = w_x \quad \text{(b)}$$

$$T_{(SC)} = GJ \frac{d\varphi}{dz} - EI_w \frac{d^3 \varphi}{dz^3} \quad \text{(c)}$$

Differentiating eqn. (c) w.r.t z:

$$we_x = GJ \frac{d^2 \varphi}{dz^2} - EI_w \frac{d^4 \varphi}{dz^4} \quad \text{(d)}$$

where:

EI_x, EI_y= flexural rigidity in vertical and horizontal planes; v, u= displacements in vertical and horizontal directions; w_y, w_x = intensity of distributed load; z= longitudinal direction; GJ= torsional rigidity; EI_w= warping rigidity; φ = angle of twist, anti-clockwise positive; we_x= intensity of torque

2.2 Equations of Motion

Using d'Alembert's principle:

$$\left.\begin{aligned}
&EI_y\frac{\partial^4 u}{\partial z^4}+EI_{xy}\frac{\partial^4 v}{\partial z^4}+\rho A\frac{\partial^2 u}{\partial t^2}-\rho Ae_y\frac{\partial^2\varphi}{\partial t^2}=0\\
&EI_x\frac{\partial^4 v}{\partial z^4}+EI_{xy}\frac{\partial^4 u}{\partial z^4}+\rho A\frac{\partial^2 v}{\partial t^2}+\rho Ae_x\frac{\partial^2\varphi}{\partial t^2}=0\\
&EI_w\frac{\partial^4\varphi}{\partial z^4}-GJ\frac{\partial^2\varphi}{\partial z^2}+\rho Ae_x\frac{\partial^2 v}{\partial t^2}-\rho Ae_y\frac{\partial^2 u}{\partial t^2}+\rho I_{sc}\frac{\partial^2\varphi}{\partial t^2}=0
\end{aligned}\right\} \qquad (1)$$

where:

u, v = displacements of the shear centre, SC, in X, Y directions; ρ = mass density; A = cross-sectional area; e_y, e_x = distances from the centroid, CC, to X and Y (shear centre) axes respectively; I_{sc} = polar moment of inertia about the SC; EI_{xy} = coupling stiffness (rigidity); t = time

Note that terms such as:

$\left[\rho A\frac{\partial^2 u}{\partial t^2}\right]$, $\left[\rho A\frac{\partial^2 v}{\partial t^2}\right]$ and $\left[\rho Ae_y\frac{\partial^2\varphi}{\partial t^2}\right]$, $\left[\rho Ae_x\frac{\partial^2\varphi}{\partial t^2}\right]$ denote mass and inertia, associated with linear and angular acceleration respectively. Terms $\left[EI_{xy}\frac{d^4 u}{dz^4}\right]$ and $\left[EI_{xy}\frac{d^4 v}{dz^4}\right]$ indicate coupling effects. $\left[EI_w\frac{\partial^4\varphi}{\partial z^4}\right]$, $\left[GJ\frac{\partial^2\varphi}{\partial z^2}\right]$ are the warping and torsional terms respectively. Finally, terms $\left[\rho Ae_x\frac{\partial^2 v}{\partial t^2}\right]$, $\left[\rho Ae_y\frac{\partial^2 u}{\partial t^2}\right]$ represent inertial forces. Whereas term $\left[\rho I_{sc}\frac{\partial^2\varphi}{\partial t^2}\right]$ represents inertial torque.

Assuming the vibration is harmonic, the translational and rotational (due to torsion) displacements can be expressed as:

$$\left.\begin{aligned} u(z,t) &= U(z)sin\omega t \\ v(z,t) &= V(z)sin\omega t \\ \varphi(z,t) &= \Phi(z)sin\omega t \end{aligned}\right\} \quad (2)$$

where:

U, V, Φ = amplitudes, normal (modes) functions; ω = angular frequency of vibration

The following partial derivatives need to be formed from eqns (2):

$$\frac{\partial^4 u}{\partial z^4}, \quad \frac{\partial^4 v}{\partial z^4}, \quad \frac{\partial^4 \varphi}{\partial z^4}, \quad \frac{\partial^2 \varphi}{\partial z^2}, \quad \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial^2 v}{\partial t^2}, \quad \frac{\partial^2 \varphi}{\partial t^2}$$

$$\frac{\partial u}{\partial z} = U'(z)sin\omega t$$

$$\frac{\partial^2 u}{\partial z^2} = U''(z)sin\omega t$$

$$\frac{\partial^3 u}{\partial z^3} = U'''(z)sin\omega t$$

$$\left.\begin{aligned} \frac{\partial^4 u}{\partial z^4} &= U''''(z)sin\omega t \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{\partial^4 v}{\partial z^4} &= V''''(z)sin\omega t \\ \frac{\partial^2 \varphi}{\partial z^2} &= \Phi''(z)sin\omega t \\ \frac{\partial^4 \varphi}{\partial z^4} &= \Phi''''(z)sin\omega t \end{aligned}\right\} \quad (3)$$

Also,

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} = \omega U(z) cos\omega t \\
&\frac{\partial^2 u}{\partial t^2} = -\omega^2 U(z) sin\omega t \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\frac{\partial^2 v}{\partial t^2} = -\omega^2 V(z) sin\omega t \\
&\frac{\partial^2 \varphi}{\partial t^2} = -\omega^2 \Phi(z) sin\omega t
\end{aligned}\right\} \tag{4}$$

Substituting eqs. (3) and (4) into eqs. (1), and cancelling out $sin\omega t,$ $(\omega t \neq 0)$:

$$EI_y U''''(z) + EI_{xy} V''''(z) - \omega^2 \rho A U(z) + \omega^2 \rho A e_y \Phi(z) = 0 \tag{5}$$

$$EI_x V''''(z) + EI_{xy} U''''(z) - \omega^2 \rho A V(z) - \omega^2 \rho A e_x \Phi(z) = 0 \tag{6}$$

$$EI_w \Phi''''(z) - GJ\Phi''(z) - \omega^2 \rho A e_x V(z) + \omega^2 \rho A e_y U(z) - \omega^2 \rho I_{sc} \Phi(z) = 0 \tag{7}$$

Solutions must be found now for U(z), V(z) and Φ(z) that satisfy eqs. (5), (6) and (7) as well as the particular end conditions of the unit.

2.3 Boundary Conditions

For a simply supported unit the following boundary conditions exist:

$$\forall z \in \mathbb{R}^{+}:\ z = 0, or\ z = \ell\ (ie: span) \begin{cases} u = 0, & \dfrac{\partial^2 u}{\partial z^2} = 0 \\ v = 0, & \dfrac{\partial^2 v}{\partial z^2} = 0 \\ \varphi = 0, & \dfrac{\partial^2 \varphi}{\partial z^2} = 0 \end{cases} \tag{8}$$

(rem: φ= '*torsional*' rotation about the Z-axis.)

The above conditions are satisfied by the following relationships where C_k, D_k, H_k are constants:

$$\left.\begin{aligned} U_k &= C_k \sin\frac{k\pi}{\ell} z \\ V_k &= D_k \sin\frac{k\pi}{\ell} z \\ \Phi_k &= H_k \sin\frac{k\pi}{\ell} z \end{aligned}\right\} \qquad \forall k \in \mathbb{N}:\ \ k = 1, 2, 3, \ldots \tag{9}$$

Eqs. (9) must be substituted into eqs. (5), (6) and (7) after their second and fourth derivatives w.r.t. z are obtained:

$$\left.\begin{aligned} U''_k &= -C_k \left(\frac{k\pi}{\ell}\right)^2 \sin\frac{k\pi}{\ell} z \\ U''''_k &= C_k \left(\frac{k\pi}{\ell}\right)^4 \sin\frac{k\pi}{\ell} z \\ V''_k &= -D_k \left(\frac{k\pi}{\ell}\right)^2 \sin\frac{k\pi}{\ell} z \\ V''''_k &= D_k \left(\frac{k\pi}{\ell}\right)^4 \sin\frac{k\pi}{\ell} z \\ \Phi''_k &= -H_k \left(\frac{k\pi}{\ell}\right)^2 \sin\frac{k\pi}{\ell} z \\ \Phi''''_k &= H_k \left(\frac{k\pi}{\ell}\right)^4 \sin\frac{k\pi}{\ell} z \end{aligned}\right\} \tag{10}$$

Substituting the corresponding derivatives from Eqn. (10) into eqs. (5), (6) and (7) while cancelling out term $sin\frac{k\pi}{\ell}z$, as before, and re-arranging common factors for C_k, D_k, H_k:

$$EI_y U''''(z) + EI_{xy} V''''(z) - \omega^2 \rho A U(z) + \omega^2 \rho A e_y \Phi(z) = 0$$

$$\left(EI_y \cdot \frac{k^4\pi^4}{\ell^4} - \omega^2 \rho A\right) C_k + \left(EI_{xy} \cdot \frac{k^4\pi^4}{\ell^4}\right) D_k + \left(\omega^2 \rho A e_y\right) H_k = 0 \tag{11}$$

$$EI_x V''''(z) + EI_{xy} U''''(z) - \omega^2 \rho A V(z) - \omega^2 \rho A e_x \Phi(z) = 0$$

$$\left(EI_{xy} \cdot \frac{k^4\pi^4}{\ell^4}\right) C_k + \left(EI_x \cdot \frac{k^4\pi^4}{\ell^4} - \omega^2 \rho A\right) D_k - (\omega^2 \rho A e_x) H_k = 0 \tag{12}$$

$$EI_w \Phi''''(z) - GJ\Phi''(z) - \omega^2 \rho A e_x V(z) + \omega^2 \rho A e_y U(z) - \omega^2 \rho I_{sc} \Phi(z) = 0$$

$$\left(\omega^2 \rho A e_y\right) C_k - (\omega^2 \rho A e_x) D_k + \left(EI_w \cdot \frac{k^4\pi^4}{\ell^4} + GJ\frac{k^2\pi^2}{\ell^2} - \omega^2 \rho I_{sc}\right) H_k = 0 \tag{13}$$

2.4 The Frequency Equation and its Coefficients

Equations (11), (12) and (13) can provide non-trivial (other than zero) solutions for C_k, D_k, and H_k if, and only if, the determinant of their coefficients vanishes (det= 0).

$$\begin{vmatrix} \left(EI_y \cdot \frac{k^4\pi^4}{\ell^4} - \omega^2\rho A\right) & \left(EI_{xy} \cdot \frac{k^4\pi^4}{\ell^4}\right) & \left(\omega^2\rho A e_y\right) \\ \left(EI_{xy} \cdot \frac{k^4\pi^4}{\ell^4}\right) & \left(EI_x \cdot \frac{k^4\pi^4}{\ell^4} - \omega^2\rho A\right) & -(\omega^2\rho A e_x) \\ \left(\omega^2\rho A e_y\right) & -(\omega^2\rho A e_x) & \left(EI_w \cdot \frac{k^4\pi^4}{\ell^4} + GJ\frac{k^2\pi^2}{\ell^2} - \omega^2\rho I_{sc}\right) \end{vmatrix} = 0 \tag{14}$$

Or, in terms of its cofactors:

$$\left(EI_y \cdot \frac{k^4\pi^4}{\ell^4} - \omega^2\rho A\right) \begin{vmatrix} \left(EI_x \cdot \frac{k^4\pi^4}{\ell^4} - \omega^2\rho A\right) & -(\omega^2\rho A e_x) \\ -(\omega^2\rho A e_x) & \left(EI_w \cdot \frac{k^4\pi^4}{\ell^4} + GJ\frac{k^2\pi^2}{\ell^2} - \omega^2\rho I_{sc}\right) \end{vmatrix} -$$

$$-\left(EI_{xy} \cdot \frac{k^4\pi^4}{\ell^4}\right) \begin{vmatrix} \left(EI_{xy} \cdot \frac{k^4\pi^4}{\ell^4}\right) & -(\omega^2\rho A e_x) \\ \left(\omega^2\rho A e_y\right) & \left(EI_w \cdot \frac{k^4\pi^4}{\ell^4} + GJ\frac{k^2\pi^2}{\ell^2} - \omega^2\rho I_{sc}\right) \end{vmatrix} +$$

$$+\left(\omega^2\rho A e_y\right) \begin{vmatrix} \left(EI_{xy} \cdot \frac{k^4\pi^4}{\ell^4}\right) & \left(EI_x \cdot \frac{k^4\pi^4}{\ell^4} - \omega^2\rho A\right) \\ \left(\omega^2\rho A e_y\right) & -(\omega^2\rho A e_x) \end{vmatrix} = 0 \tag{15}$$

Computations to evaluate eqn. (15) are lengthy but straight forward and will be omitted. They can be available on request. The final form of the **frequency equation** expressed in terms of descending powers of *ω,* is shown below.

$$\omega^6\rho^3A^2\{A(e_x^2+e_y^2)-I_{sc}\}+$$

$$+\omega^4\rho^2\left\{AEI_{sc}\frac{k^4\pi^4}{\ell^4}(I_y+I_x)-A^2E\frac{k^4\pi^4}{\ell^4}(I_ye_x^2+I_xe_y^2)+A^2EI_w\frac{k^4\pi^4}{\ell^4}-2A^2EI_{xy}e_xe_y\frac{k^4\pi^4}{\ell^4}+\right.$$

$$\left.+A^2GJ\frac{k^2\pi^2}{\ell^2}\right\}-$$

$$-\omega^2\rho\left\{E^2I_{sc}\frac{k^8\pi^8}{\ell^8}(I_xI_y-I_{xy}^2)+AE^2I_w\frac{k^8\pi^8}{\ell^8}(I_y+I_x)+AEGJ\frac{k^6\pi^6}{\ell^6}(I_y+I_x)\right\}+$$

$$+E^3I_w\frac{k^{12}\pi^{12}}{\ell^{12}}(I_xI_y-I_{xy}^2)+E^2GJ\frac{k^{10}\pi^{10}}{\ell^{10}}(I_xI_y-I_{xy}^2)=0 \tag{20}$$

Table 1 presents all quantities (constants) needed, to compute the roots of eqn. (20).

Table 1. Quantities used for the evaluation of coefficients shown in equation 22.

Quantity	Definition	SI units
E_{con}	Modulus of elasticity of concrete	29×10^9 Pa
G_{con}	Modulus of rigidity of concrete	13.6×10^9 Pa
$I_{y\text{-}y}$	Second moment of area about y-y axis	16244.44×10^{-6} m^4
$I_{x\text{-}x}$	Second moment of area about x-x axis	3088.44×10^{-6} m^4
I_{xy}	Product moment of inertia	$3935.89\text{x}10^{-6}$ m^4
I_{sc}	Polar moment of inertia about the shear centre, SC. $I_{SC}=I_Y+I_X$.	$30148.5\text{x}10^{-6}$ m^4
J	St Venant's torsional constant.	$2579.685\text{x}10^{-6}$ m^4
I_w	Warping torsional constant	$16445.61\text{x}10^{-18}$ m^6
A	Cross sectional area of unit.	0.18 m^2
ρ	Mass per unit volume of specimen	2230 kgm^{-3}
e_x	$e_x=A\tilde{x}/A$	0.244 m
e_y	$e_y=A\tilde{y}/A$	0.08945 m
ℓ	Span	7 m

Hence, eqn. (20) was re-written with coefficients C_1, C_2, C_3, … C_{11}, as follows:

$$\boldsymbol{\omega}^{\mathbf{6}}\rho^{3}A^{2}\{C_{1}\}+\boldsymbol{\omega}^{\mathbf{4}}\rho^{2}\{C_{2}-C_{3}+C_{4}-C_{5}+C_{6}\}-\boldsymbol{\omega}^{\mathbf{2}}\rho\{C_{7}+C_{8}+C_{9}\}+\{C_{10}+C_{11}\}=0 \tag{21}$$

with:

$C_1 = \{(Ae_x^2 + Ae_y^2) - I_{sc}\} = 0.18 \times 0.244^2 + 0.18 \times 0.08945^2 - 30148.5 \times 10^{-6} =$

$-0.018\, m^4$

$C_2 = EI_{sc}A\frac{k^4\pi^4}{\ell^4}(I_y + I_x) = (29 \times 10^9) \times (30148.5 \times 10^{-6}) \times (0.18) \times \left(\frac{1^4\pi^4}{7^4}\right) \times$

$(16244.44 \times 10^{-6} + 3088.44 \times 10^{-6}) = 103.20 \times 10^3\ \mathrm{Nm}^4$

and so on…

Table 2 presents all these coefficients and their values in SI units for *k*= 1. Each *k*-value provides a different version of the frequency equation. It is tedious to report on all six versions of the latter due to the vast amount of computations needed. The final form of the frequency equation for *k*= 1, is:

$$\boldsymbol{\omega}^{\mathbf{6}} - 0.2 \times 10^6\, \boldsymbol{\omega}^{\mathbf{4}} + 7.1187 \times 10^9\, \boldsymbol{\omega}^{\mathbf{2}} - 31.27 \times 10^{12} = 0 \tag{22}$$

Table 2. Coefficients for angular frequency, ω, and their values in SI units for k= 1

Coefficient	**Relationship**	**Value (SI units)**
C1	$\{(Ae_x^2 + Ae_y^2) - I_{sc}\}$	-0.018 m^4
C2	$EI_{sc}A\frac{k^4\pi^4}{\ell^4}(I_y + I_x)$	103.20x10^3 Nm^4
C3	$EA^2\frac{k^4\pi^4}{\ell^4}(I_y e_x^2 + I_x e_y^2)$	31639.70 Nm^2
C4	$A^2EI_w\frac{k^4\pi^4}{\ell^4}$	5.25x10^{-7} N
C5	$2A^2EI_{xy}e_xe_y\frac{k^4\pi^4}{\ell^4}$	5.46x10^3 Nm^4
C6	$A^2GJ\frac{k^2\pi^2}{\ell^2}$	195.05x10^3 Nm^4

C7	$E^2 I_{sc} \dfrac{k^8\pi^8}{\ell^8}\left(I_x I_y - I_{xy}^2\right)$	35.55x10^9 N^2
C8	$AE^2 I_w \dfrac{k^8\pi^8}{\ell^8}\left(I_y + I_x\right)$	1.946 N^2
C9	$AEGJ \dfrac{k^6\pi^6}{\ell^6}\left(I_y + I_x\right)$	20.61x10^{12} N^2
C10	$E^3 I_w \dfrac{k^{12}\pi^{12}}{\ell^{12}}\left(I_x I_y - I_{xy}^2\right)$	543.30x10^6 N^3m^{-4}
C11	$E^2 GJ \dfrac{k^{10}\pi^{10}}{\ell^{10}}\left(I_x I_y - I_{xy}^2\right)$	202.25x10^{18} N^3m^{-4}

The roots of eqn. 22 (frequency equation) were evaluated by employing MatLab (2000) and are listed in Table 3, below. MatLab computes the roots using Newton's Method. Briefly, if the first root converges to a complex number, it prints out its conjugate as a second root; if it converges to a real number, it divides throughout that root to form a *quintic* equation and then obtains the second root by running Newton's Method on the latter. After the two roots are found, it divides out again to obtain a quartic and so on. The results obtained for *k*= 1 produced six real roots of which only three were positive. The lowest can be regarded as the fundamental (natural) frequency. For more roots to be extracted the procedure has to be repeated for *k*= 2, 3, … and so on, hence it can be extremely laborious.

Table 3. Roots, ω, ($rads^{-1}$) of the sextic frequency equation as obtained by MatLab.

ω x 10^3 ($rads^{-1}$)	*f* (Hz)
-394.36	
394.36	62.80
-198.43	
198.43	31.60
-71.46	
71.46	11.38

3. Earlier Experimental and Numerical Studies.

It is of interest to look at the results obtained from an earlier finite element modal analysis and those retrieved from a parallel experimental investigation (Karadelis, 2016). Any conclusions drawn, and uncertainties brought to surface, may be used as a guide for future researchers and practitioners.

The objective of the experimental study was to quantify the response of a structure to a known (measured) excitation force. The terrace unit shown in Figure 1 was dividing into a suitable test grid (*lumped masses)* shown in the same figure as an inset. Masses were used as data collection (reference) points (RPs) shown as numbers in the inset. The properties of the structure were determined by measuring the FRF (Frequency Response Function) at each of the RPs (Ewins, 2000; Maia et al., 1997). A summary of the main data acquisition parameters is given in Table 4. Typical excitation and response time histories are shown in Figure 3, whereas a typical FRF, is presented in Figure 4, after a FFT is passed over the time history results. Modal parameter estimation was attained by using ICATS (1997) software. Table 5 displays all experimental, numerical and analytical results.

Table 4. Main data acquisition parameters and their values.

Parameter	Setting/Value
Acquisition Bandwidth (Sampling Rate):	80 Hz (325.5 Hz)
Acquisition Duration:	25.166 seconds
Frequency Resolution:	0.0397 Hz
No. of Frequency Domain Averages:	4
Exponential Window Time Constant:	0
Excitation Type:	Chirp
Excitation Duration:	18.87 seconds
Excitation Frequency 'Span':	1 – 79 Hz

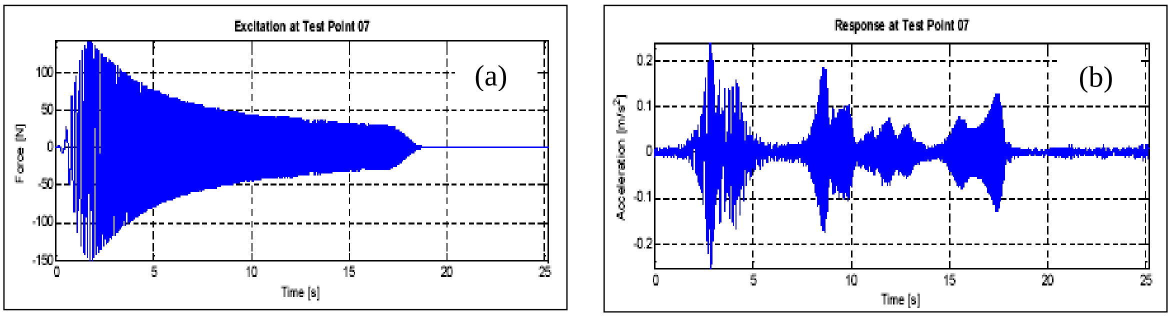


Figure 3. Typical (a) excitation and (b) response signals on a grandstand terrace unit.

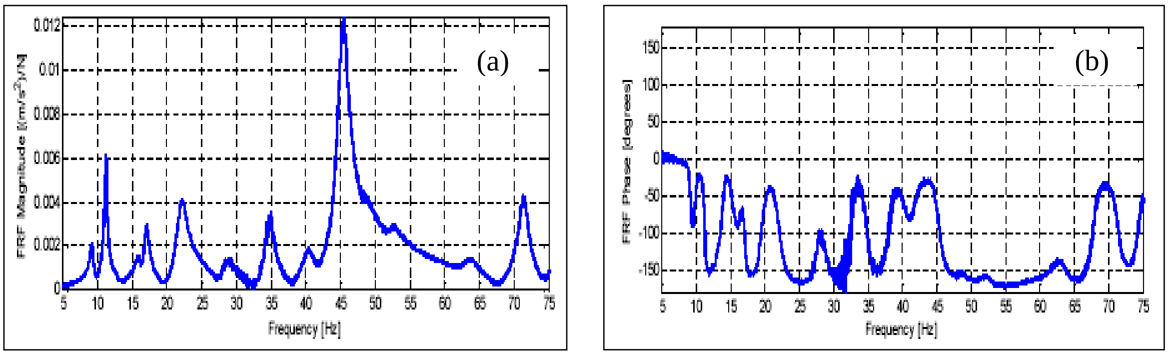
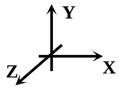
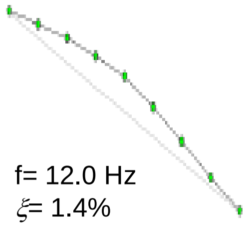
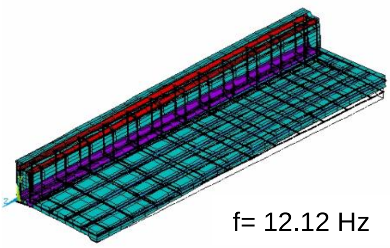
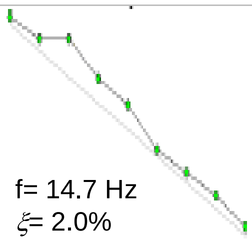
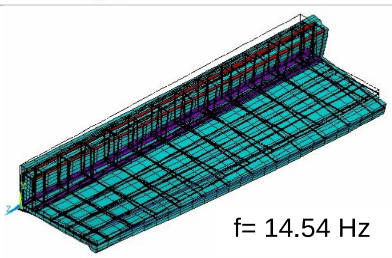
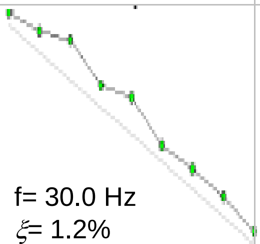
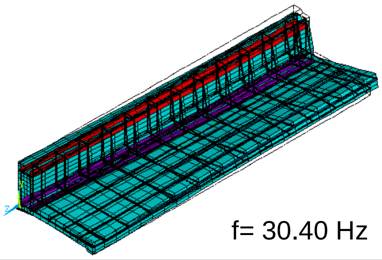
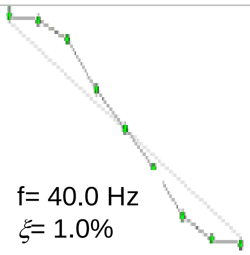
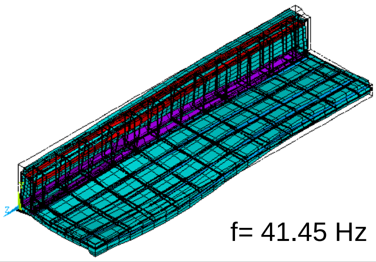
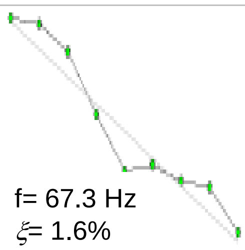
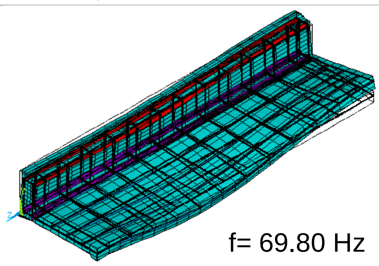
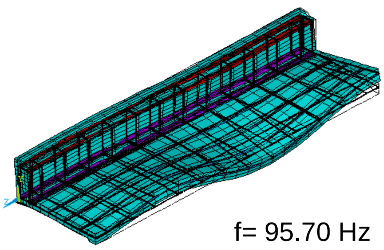


Figure 4. Typical FRF peaks. (a): Natural frequencies corresponding to specific modes of vibration; (b): their characteristic phase angles.

Table 5. Measured (experimental), predicted (computational) and calculated (analytical) natural frequencies and mode shapes. (Note: Symmetry has been used in the FE model).

Mode No.	**Measured** Natural Frequencies, Mode Shapes %-age Damping	**Predicted** (FE Modal Analysis). Nat. Frequencies & Mode Shapes	**Calculated** Natural Frequencies	**Comments** 
1	 f= 12.0 Hz ξ= 1.4%	 f= 12.12 Hz	11.38 Hz	Fundamental, bending mode of vibration.
2	 f= 14.7 Hz ξ= 2.0%	 f= 14.54 Hz		Predominantly torsional mode. Also showing small amounts of bending.
3	 f= 30.0 Hz ξ= 1.2%	 f= 30.40 Hz	31.60 Hz	Similar to Mode 2.
4	 f= 40.0 Hz ξ= 1.0%	 f= 41.45 Hz		Second flexural mode of vibration.
5	 f= 67.3 Hz ξ= 1.6%	 f= 69.80 Hz	62.80 Hz	Third flexural mode Also inhibiting a small amount of torsion.
6	?	 f= 95.70 Hz		Predominantly flexural mode

4. A comparison between Experimental, Numerical and Analytical results

Table 6 presents all the results in a synoptic way. The standard deviation (STDEV) per mode shape (how far the values of a set of data are dispersed from their mean value) of the measured (lab), predicted (FEA) and calculated (hand) natural frequencies is given in the last column. It is noticed that the value of STDEV is very small (0.324) for the first mode, and increases with mode shape (Mode 5, STDEV= 2.89).

Table 6. Measured Predicted and Calculated Natural Frequencies.

Mode No.	Measured f (Hz)	FEA f (Hz)	Calculated f (Hz)	STDEV
1	12.00	12.12	11.38	0.324
2	14.70	14.54		0.080
3	30.00	30.40	31.60	0.680
4	40.00	41.45		0.725
5	67.30	69.80	62.80	2.896
6		95.70		0

Figure 5 shows natural frequencies per mode shape for measured, predicted and calculated values. General exponential trends (trendlines) have been added to assist with comparison and forecast frequencies beyond mode shape 6. It is apparent that calculated, measured and predicted trendlines are quite close, signifying good agreement. A tendency for better agreement between experimental and numerical trendlines is also noticed.

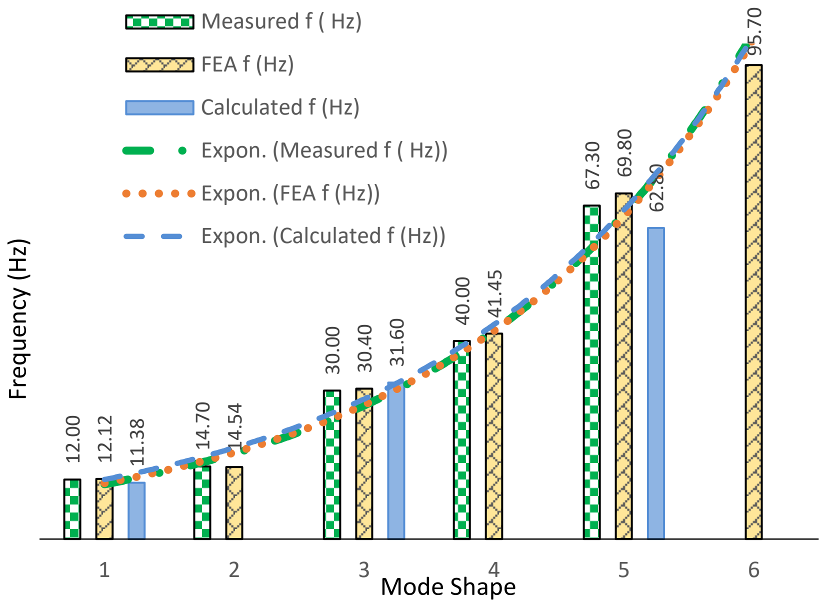


Figure 5. Frequencies per mode shape and their exponential trendlines.

It can be concluded with some confidence that it is possible to perform triply coupled modal analysis of an asymmetric thick beam by three different ways and expect reasonably good correlation.

Problems with more precise correlation between theoretical and experimental results have been reported by Arpaci & Bozdag, (2002) and a recommendation to allow for other than isotropic conditions as well as more accurate boundary conditions has been suggested by Wang (2013).

Inevitably, the equations developed depend on a series of constants such as: E_{con}, G_{con}, I_{y-y}, I_{x-x}, I_{xy}, I_{sc}, J, I_w, ρ, e_x and e_y, as well as their products and sums (EI_{sc}, EI_w, EI_{xy}, GJ, $I_x.I_y.I_{sc}$, $(I_x + I_y)$, $(I_xI_y - I^2_{xy})$). No doubt, some of them can be very specific. Some can be "borrowed" from the corresponding experimental investigation. The rest can only be estimated in an approximated manner. These terms appear in the final frequency equation and are raised to a high power. Hence, the initial error is exaggerated, and the accuracy compromised.

Continuously rounding up the decimal points appears to be another way of deviating from the "exact" solution.

It has been demonstrated (Wang, 2013; Karadelis, 2012) that accurate representation of supports (boundary conditions) is important for more successful numerical modal analyses, pointing towards a methodology involving micro-scale level simulations, at least for the latter. However, it is clear in this article that the level of agreement between the three different approaches is satisfactory and underpins the theory that the small discrepancies reported, are due to support conditions.

5. Concluding Remarks

This article makes a contribution to related literature by considering the effects of warping and rotational inertia in a unique, triply coupled vibration analysis of a thick, L-shaped concrete terrace unit (asymmetric beam). The analytical procedure presented, involved the development of three partial differential equations from which the *exact* solution was extracted. The analysis itself fitted into the continuous, classical theory without the employment of unfamiliar, or highly convoluted techniques. The results were compared with similar experimental and numerical studies carried out earlier by the same author. The following are reported:

It is pointed out that modes (*amplitudes*) U, V, Φ are sinusoidal functions, of the form: $\sin\frac{k\pi}{l}z$, in which k stands for the k_{th} order natural frequency. When a solution is obtained, for k=1, some roots may not be real. In this case, all roots were real but only three were positive. One of these roots provided the first order frequency, corresponding to experimental and numerical results. The remaining two should be near other frequencies obtained from the tests, or the numerical analysis. The procedure must be repeated for k =2, to extract a new set of roots containing the second order frequency; and so on, for k= 3, 4, 5, 6. This way, the first six natural frequencies can be obtained. This, of course, is a very lengthy and painstaking procedure.

It has been demonstrated that by including warping effects, torsional rigidity (stiffness) EI_w and coupling stiffness EI_{xy}, in the analysis, one can reduce errors reported by others (see Introduction) while computing the natural frequencies of beams.

In contrast, inadequate support conditions (simply supported) intensifies the errors, perhaps stressing the importance of a more rigorous representation of the supports.

Nevertheless, the analytical method in calculating the natural frequencies of a system is the most reliable, compared to experimental (needs skills and experience) and numerical (calibration, updating, validation, etc).

Finally, the idea of a strictly exact solution is probably academic. However, depending on the particular engineering circumstances, one may adopt one of the three as the “best” and use it as a benchmark to assess the remaining two.

Dedicated:

To my mother, Kyriaki, who passed away recently, during the revision of this article.

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