

NEW CHARACTERIZATIONS OF SOBOLEV METRIC SPACES

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ABSTRACT. We provide new characterizations of Sobolev and BV spaces in doubling and Poincaré metric spaces in the spirit of the Bourgain-Brezis-Mironescu and Nguyen limit formulas holding in domains of \mathbb{R}^N .

1. INTRODUCTION

1.1. **Overview.** Around 2001, J. Bourgain, H. Brezis and P. Mironescu, investigated [5, 6, 8] the asymptotic behaviour of a class of nonlocal functionals on a domain $\Omega \subset \mathbb{R}^N$, including those related to the norms of the fractional Sobolev space $W^{s,p}(\Omega)$, as $s \nearrow 1$. More precisely, if $p \geq 1$ and $u \in W^{1,p}(\Omega)$, then

$$\lim_{s \nearrow 1} (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} d\mathcal{L}^N(x) d\mathcal{L}^N(y) = K_{p,N} \int_{\Omega} |\nabla u|^p d\mathcal{L}^N(x),$$

where $|\cdot|$ denotes the Euclidean norm, \mathcal{L}^N the Lebesgue measure on \mathbb{R}^N and

$$K_{p,N} = \frac{1}{p} \int_{\mathbb{S}^{N-1}} |\omega \cdot x|^p d\mathcal{H}^{N-1},$$

being $\omega \in \mathbb{S}^{N-1}$ arbitrary. By replacing the Euclidean distance $|x - y|$ with a distance $d_K(x, y) = \|x - y\|_K$, where K denotes the unit ball for $\|\cdot\|_K$, it was proved in [23, 24, 30] that, if $u \in W^{1,p}(\Omega)$

$$\lim_{s \nearrow 1} (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{d_K(x, y)^{N+ps}} d\mathcal{L}^N(x) d\mathcal{L}^N(y) = \int_{\Omega} \|\nabla u\|_{Z_p^* K}^p d\mathcal{L}^N(x),$$

where we have set

$$\|\xi\|_{Z_p^* K} := \left(\frac{N+p}{p} \int_K |\xi \cdot x|_p^p d\mathcal{L}^N(x) \right)^{1/p}, \quad \xi \in \mathbb{R}^N.$$

Similar results hold for BV spaces [16, 32] and for magnetic Sobolev spaces [31] and criteria for recognizing constants among measurable functions can be obtained [8]. The nonlocal norms thus converge in the limit as $s \nearrow 1$ to a Dirichlet type energy which depends on p, N and on the distance d_K . More in general, it is natural to wonder if similar characterizations may hold for some classes of BV and Sobolev spaces on a complete metric measure space (X, d, μ) in place of \mathbb{R}^N , at least in the case where some structural assumption is assumed on the measure μ acting on X . The general definition of Sobolev and BV space will be given in Section 2.1, and for every Sobolev function u it is defined a weak upper gradient $|\nabla u|_w \in L^p(X)$, along with its *Cheeger energy* (introduced by Cheeger in [14])

$$\text{Ch}_p(u) := \int_X |\nabla u|_w^p d\mu.$$

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In particular Ch_p will be l.s.c. with respect to the strong convergence in L^p and so is a good generalization of the Dirichlet energy in an Euclidean context, where the two notions coincide. Moreover $W^{1,p}(X, d, \mu)$ is a Banach space with the norm $\|u\|_{1,p} = (\|u\|_{L^p}^p + \text{Ch}_p(u))^{1/p}$.

We recall here (using [20, 22]), that whenever μ is doubling and it satisfies a $(1, p)$ -Poincaré inequality, $W^{1,p}(X, \mu, d)$ coincides with the *Hajlasz-Sobolev* space, that is the space of $u \in L^p(X, \mu)$ such that there exists $g \in L^p(X, \mu)$ with

$$(1.1) \quad |u(x) - u(y)| \leq d(x, y)(g(x) + g(y)), \quad \text{for } \mu \text{ a.e. } x, y \in X.$$

Moreover, we can choose g such that $\|g\|_p^p \leq C \cdot \text{Ch}(u)$.

Furthermore, if $p > 1$ and $\Omega \subset \mathbb{R}^N$ is an extension domain, then $W^{1,p}(\Omega, \mathcal{L}^N, d)$ coincides with the usual space $W^{1,p}(\Omega)$ and the norms are equivalent.

For any $p \geq 1$ and $0 < s < 1$, the fractional space $\mathcal{H}^{s,p}(X, \mu, d)$ can be defined as the space of $u \in L^p(X, \mu)$ such that the Gagliardo seminorm $[u]_{\mathcal{H}^{s,p}(X)}$ is finite, where

$$[u]_{\mathcal{H}^{s,p}(X)} := \left(\int_X \int_X \frac{|u(x) - u(y)|^p}{d(x, y)^{ps} \rho(x, y)} d\mu(x) d\mu(y) \right)^{1/p},$$

and ρ is a doubling kernel for μ (see Definition 1.2). A *fractional counterpart* of the Hajlasz-Sobolev spaces can also be introduced as follows. For $0 < s < 1$ we define $W^{s,p}(X, \mu, d)$ as the spaces of $u \in L^p(X, \mu)$ such that there is a function $g \in L^p(X, \mu)$ with

$$|u(x) - u(y)| \leq d^s(x, y)(g(x) + g(y)),$$

for almost any $x, y \in X$. When the measure is N -Ahlfors it follows (see [18]) that

$$\mathcal{H}^{s,p}(X, \mu, d) \hookrightarrow W^{s,p}(X, \mu, d) \hookrightarrow \mathcal{H}^{s-\varepsilon,p}(X, \mu, d),$$

for all $\varepsilon \in (0, s)$, so that the two spaces are comparable.

The main goal of this paper is to provide a proof of the connection between

$$\limsup_{s \nearrow 1} (1-s) \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x, y)^{ps} \rho(x, y)} d\mu(x) d\mu(y) < +\infty$$

and $u \in W^{1,p}(X)$ for $p > 1$ or $u \in BV(X)$ for $p = 1$. A second characterization we want to provide is in terms of the family of nonlocal integrals

$$u \mapsto \iint_{\{|u(x)-u(y)|>\delta\}} \frac{\delta^p}{d(x, y)^p \rho(x, y)} d\mu(x) d\mu(y).$$

In the Euclidean case $X = \mathbb{R}^N$, Nguyen [26–29] (see also the recent works [9–13] by Brezis and Nguyen) proved that, if $p > 1$, then $u \in W^{1,p}(\mathbb{R}^N)$ if and only if $u \in L^p(\mathbb{R}^N)$ and

$$\sup_{0 < \delta < 1} \iint_{\{|u(x)-u(y)|>\delta\}} \frac{\delta^p}{|x-y|^{N+p}} d\mathcal{L}^N(x) d\mathcal{L}^N(y) < +\infty,$$

in which case

$$\lim_{\delta \searrow 0} \iint_{\{|u(x)-u(y)|>\delta\}} \frac{\delta^p}{|x-y|^{N+p}} d\mathcal{L}^N(x) d\mathcal{L}^N(y) = K_{p,N} \int_{\mathbb{R}^N} |\nabla u|^p d\mathcal{L}^N(x).$$

In the case $p = 1$ this property fails, in general [13].

1.2. **Main results.** In the following, (X, d, μ) denotes a complete and separable metric measure space with measure μ .

Definition 1.1 (Doubling). We say that μ is a doubling measure if there exists a constant c_D such that

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r)), \quad \text{for all } x \in \text{supp}(\mu) \text{ and any } r > 0.$$

Definition 1.2 (Doubling kernel). Let (X, d, μ) be a metric space with μ doubling and let us denote $\rho_1(x, y) = \mu(B(x, d(x, y)))$ and $\rho_2(x, y) = \mu(B(y, d(x, y)))$. We say $\rho : X \times X \rightarrow \mathbb{R}$ is a *doubling kernel* if there exists a constant $C_\rho > 0$ such that

$$\frac{1}{C_\rho} \rho_1(x, y) \leq \rho(x, y) \leq C_\rho \rho_1(x, y), \quad \text{for all } x, y \in \text{supp}(\mu).$$

There are several examples of doubling kernels used in the literature; here we list a few

$$\rho_1, \quad \rho_2, \quad \rho_1 + \rho_2, \quad \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}, \quad \sqrt{\rho_1 \rho_2},$$

and in general $f(\rho_1, \rho_2)$ where $\min\{t, s\} \leq f(t, s) \leq \max\{t, s\}$. In the special case when μ is N -ahlfors, also $d(x, y)^N$ is a doubling kernel.

Definition 1.3 (Poincaré inequality). We say that μ satisfies a $(1, p)$ -Poincaré inequality if there is $c_P > 0$ such that for any ball $B \subset X$ of radius $t > 0$

$$\begin{aligned} \int_B |u_B - u(x)|^p d\mu(x) &\leq t^p c_P \int_B g^p(x) d\mu(x), \quad u \in W^{1,p}(X), \text{ (Sobolev case),} \\ \int_B |u_B - u(x)| d\mu(x) &\leq t c_P |Du|(B), \quad u \in BV(X), \text{ (BV case).} \end{aligned}$$

Notice that this definition is a bit different and less general than the usual one, that allows the integral on the right hand side to be performed over a larger ball $B(x, \tau r)$, for some $\tau \geq 1$. We prefer to stick to this version since the proof becomes clearer, but of course modifications can be done in order to fit the more general definition.

The main results of the paper are the following.

Theorem 1.4 (BBM type characterization). *Let $p \geq 1$. Assume that (X, d, μ) is a complete and separable metric measure space and μ is doubling and satisfies a $(1, p)$ -Poincaré inequality. Let ρ be a doubling kernel: then there exist $C_U > 0$ and $C_L > 0$ depending on p, N, C_ρ, c_P, c_D such that for every $u \in L^p(X)$ we have:*

$$\begin{aligned} \limsup_{s \nearrow 1} (1-s) \int_X \int_X \frac{|u(x) - u(y)|^p}{\rho(x, y) d(x, y)^{ps}} d\mu(x) d\mu(y) &\leq C_U \text{Ch}_p(u), \\ \liminf_{s \nearrow 1} (1-s) \int_X \int_X \frac{|u(x) - u(y)|^p}{\rho(x, y) d(x, y)^{ps}} d\mu(x) d\mu(y) &\geq C_L \text{Ch}_p(u). \end{aligned}$$

In the case $p > 1$, Theorem 1.4 was already obtained in [25] with a different and more involved technique, while the BV case, to the best of our knowledge, was open. The details in [25] are present only for Ahlfors measures, in which case an upper bound is firstly obtained on balls by exploiting the definition (1.1) and

$$\sup_{0 < s < 1} (1-s) \int_{B(y, r)} \frac{1}{d(x, y)^{N-p(1-s)}} d\mu(x) < +\infty, \quad \text{for } \mu \text{ a.e. } y \in X \text{ and all } r > 0,$$

which essentially follows from the fact that the measure of the balls of radius t grows N -polynomially. On the contrary, the lower bound in [25] is extremely involved and based, among other tools, upon

some deep differentiation result contained in [15], which says that every Lipschitz map from X into a Banach space with the Radon-Nikodym Property is almost everywhere differentiable.

The following result is instead new in metric spaces, up to our knowledge.

Theorem 1.5 (Nguyen type characterization). *Let $p > 1$. Assume that (X, d, μ) is a complete and separable metric measure space and μ is doubling and satisfies a $(1, p)$ -Poincaré inequality. Let ρ be a doubling kernel: then there exist $C_U > 0$ and $C_L > 0$ depending on p, N, C_ρ, c_P, c_D such that for every $u \in L^p(X)$ we have:*

$$C_L \text{Ch}_p(u) \leq \limsup_{\delta \searrow 0} \int_X \int_{\substack{X \\ \{|u(x)-u(y)|>\delta\}}} \frac{\delta^p}{\rho(x, y) d(x, y)^p} d\mu(x) d\mu(y) \leq C_U \text{Ch}_p(u).$$

We will now outline the proof of the results.

In the case of the BBM type characterization the key tool is a clever use of Fubini theorem that let us compare the quantity we want to estimate with

$$\mathcal{S}_t := \int_X \frac{1}{\mu(B(x', t))^2} \iint_{B(x', t) \times B(x', t)} |u(x) - u(y)|^p d\mu(x) d\mu(y) d\mu(x'),$$

which is very reminiscent of the Korevaar and Shoen definition of Sobolev functions [21]. As a result of this estimate we show that, in order to conclude, it is sufficient to have a good bound on the liminf/limsup of $\frac{\mathcal{S}_t}{t^p}$ as $t \rightarrow 0$. Then an easy application of the Poincaré inequality will give us the upper bound while for the lower bound we use Lemma 2.6, and the fact that $\frac{\mathcal{S}_t}{t^p}$ can be seen as the energy of g_t , which, up to a constant, is an upper gradient up to scale $t/2$ of the function u^t , that in turn is an approximation of u .

As for the Nguyen-type characterization, for the upper bound we use the Hajlasz-Sobolev characterization of Sobolev functions, while for the lower bound we again use cleverly Fubini (as done by Nguyen in its original work [26]), and then we use again Lemma 2.6, but this time the proof is more involved because the estimate is not so direct.

Remark 1.6. We work with the assumption that (X, d) is complete but the results in the Euclidean spaces are true also if restricted to an open domain Ω . In order to cover this case one should first adapt the definition of Sobolev spaces; we believe that Definition 2.4 is still the correct one, but the equivalence with other definitions is yet to be proven. If this is the case then the results of Theorem 1.4 and the lower bound in Theorem 1.5 are still valid. However, in order to deduce the upper bound in Theorem 1.5, one has to use a different proof since we relied on Proposition 2.9, which in turn relies on the self-improving property of the Poincaré inequality, which is not true neither for a general $\Omega \subseteq \mathbb{R}^n$. Of course an easy solution is to suppose that a $(1, q)$ -Poincaré inequality is valid, for some $q < p$, otherwise one should use an integralo-geometric interpretation of $\iint_{B \times B}$ in the spirit of the original proof, which is yet to be proven in a general setting.

Remark 1.7. Concerning the case $p = 1$ in the previous Theorem 1.5, in general, already in the Euclidean case, the assertion cannot hold true, in the sense that examples can be found [7, 13] of functions u in $W^{1,1}(\Omega)$ such that

$$\lim_{\delta \searrow 0} \int_{\substack{\Omega \\ \{|u(x)-u(y)|>\delta\}}} \int_{\Omega} \frac{\delta}{|x-y|^{N+1}} d\mathcal{L}^N(x) d\mathcal{L}^N(y) = +\infty.$$

Moreover, it is desirable to have a lower bound of the liminf as in Theorem 1.4, but this is more difficult and in the Euclidean context it was solved in [7].

Open problem 1.8. Let $u \in L^1(X)$. Let $\{\delta_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$ with $\delta_n \rightarrow 0$. Assume that

$$\liminf_{\delta_n \rightarrow 0} \int_X \int_X \frac{\delta_n}{\rho(x, y) d(x, y)} d\mu(x) d\mu(y) < +\infty.$$

Then $u \in BV(X)$ and there exists a positive constant C such that

$$\liminf_{\delta_n \rightarrow 0} \int_X \int_X \frac{\delta_n}{\rho(x, y) d(x, y)} d\mu(x) d\mu(y) \geq C \text{Ch}_1(X)$$

This rather subtle assertion was proved in the Euclidean case in [7] (see also [13]).

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2. PRELIMINARIES

In this section we will introduce the well established theory of Sobolev spaces in metric measure spaces, as well as some technical results that will be needed in the proofs.

2.1. Sobolev spaces in metric measure spaces. Several equivalent definition of $W^{1,p}(X, \mu, d)$ and BV are available in the literature: we refer to [2–4, 17, 19, 33] as general references. We will use the definition of Sobolev spaces given in [3] (and in [2] for BV spaces), where it is also proved to be equivalent to the more common definition of Newtonian spaces $N^{1,p}$, defined for example in [33]. In the sequel p will be the Sobolev exponent and q is its dual exponent, namely $1/p + 1/q = 1$.

We will denote by $AC([0, 1]; X)$ the space of absolutely continuous curves $\gamma : [0, 1] \rightarrow X$, for which it is defined the metric derivative $|\gamma'|$ almost everywhere. Moreover, we set $e_t : AC([0, 1]; X) \rightarrow X$ as the evaluation of γ at time t , namely $e_t(\gamma) = \gamma(t)$. Another useful definition is that of push forward: given a Borel function $f : X \rightarrow Y$ and a measure μ on X we define $\nu = f_{\#}\mu$ as the measure on Y such that $\nu(A) = \mu(f^{-1}(A))$.

A key useful concept for Sobolev Spaces is the upper gradient.

Definition 2.1 (Upper gradients). *Let $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow [0, \infty]$. We say that g is an upper gradient for f if for every curve $\gamma \in AC([0, 1]; X)$ we have the so called upper gradient inequality*

$$(2.1) \quad |f(\gamma(1)) - f(\gamma(0))| \leq \int_0^1 g(\gamma(t)) |\gamma'| (t) dt.$$

We will often substitute the right hand side with the shorter notation $\int_{\gamma} g$. Moreover, we say that g is an upper gradient of f up to scale δ if (2.1) is satisfied for every γ such that $\ell(\gamma) > \delta$.

We will need one more class object in order to define the Sobolev Spaces: the p -plans.

Definition 2.2 (p -plans). *Let π be a probability measure on $C([0, 1]; X)$. We say π is a p -plan if*

- *there exists $C > 0$ such that $(e_t)_{\#}\pi \leq C\mu$ for every $0 \leq t \leq 1$;*
- *there exists $b_{\pi} \in L^q(X, \mu)$, called barycenter of π , such that*

$$\int_{AC} \left(\int_{\gamma} g \right) d\pi = \int_X g \cdot b_{\pi} d\mu, \quad \forall g \in C_b(X, d).$$

We will say that a property on AC is true for p -almost every curve if it is true for π -almost every curve, for every p -plan π . Conversely a set of curves Γ is said to be p -null or p -negligible if $\pi(\Gamma) = 0$ for every p -plan π .

With this notion of p -almost every curve, we can relax the notion of upper gradient, and with this relaxed notion we can define the Sobolev Space.

Definition 2.3 (p -weak upper gradient). *A function $g \in L^p(X, \mu)$ is a p -weak upper gradient for $f \in L^p(X, \mu)$ if for p -almost every curve γ we have that $f \circ \gamma$ is $W^{1,1}$ and moreover*

$$\left| \frac{d}{dt} f \circ \gamma(t) \right| \leq g(\gamma(t)) |\gamma'(t)|,$$

for almost every $t \in [0, 1]$.

Definition 2.4 (Sobolev space). *Let $p \geq 1$. A function $f \in L^p(X, \mu)$ belongs to $W^{1,p}(X, d, \mu)$ if equivalently*

- (a) f has a p -weak upper gradient; then there exists a minimal weak upper gradient (in the μ -a.e. sense), denoted by $|\nabla f|_w$.
- (b) (only if $p > 1$) there exists a constant C such that for every p -plan π we have

$$\int_{AC} |f(\gamma(0)) - f(\gamma(1))| d\pi \leq \|b_\pi\|_q \cdot C^{1/p}.$$

- (c) there exists $g \in L^p(X, \mu)$ such that for every p -plan π we have

$$\int_{AC} |f(\gamma(0)) - f(\gamma(1))| d\pi \leq \int_X g \cdot b_\pi d\mu.$$

Moreover, the least constant C in (b) is equal to $\int_X |\nabla f|_w^p d\mu$ and the minimal g that satisfies (c) is again $|\nabla f|_w$.

Definition 2.5 (BV space). *A function $f \in L^1(X, \mu)$ belongs to $BV(X, d, \mu)$ if equivalently*

- (a) $f \circ \gamma$ is BV for p -almost every curve and there exists a finite measure ν such that for every 1-plan π we have

$$\int_{AC} \gamma_\# |D(f \circ \gamma)|(A) d\pi \leq \|b_\pi\|_\infty \cdot \nu(A) \quad \forall A \subseteq X \text{ open set.}$$

- (b) there exists a constant C such that for every 1-plan π we have

$$\int_{AC} |f(\gamma(0)) - f(\gamma(1))| d\pi \leq \|b_\pi\|_\infty \cdot C.$$

- (c) there exists a finite measure ν such that for every 1-plan π we have

$$\int_{AC} |f(\gamma(0)) - f(\gamma(1))| d\pi \leq \int_X b_\pi^* d\nu,$$

where b_π^* denotes the upper semicontinuous relaxation of b_π .

Moreover the minimal ν in either (a) or (c) is denoted by $|Df|$ and the least constant C in (b) is equal to $|Df|(X)$.

In the following we will denote

$$(2.2) \quad \text{Ch}_p(f) := \int_X |\nabla f|_w^p d\mu, \quad \text{for } p > 1, \quad \text{Ch}_1(f) := |Df|(X).$$

For the next lemma in the case $p > 1$ we refer the reader to [1].

Lemma 2.6 (Semicontinuity). *Let $p \geq 1$ and let $f_n, g_n \in L^p_{\text{loc}}(X, \mu)$ be functions such that g_n is an upper gradient up to scale δ_n of f_n . Suppose that $\delta_n \downarrow 0$, $f_n \rightarrow f$ in $L^p(X, \mu)$ and $g_n \rightarrow g$*

weakly in $L^p_{\text{loc}}(X, \mu)$ (respectively in the sense of measure). Then g is a p -weak upper gradient for f (respectively we have $|Df| \leq g$). In particular we have also

$$\liminf_{n \rightarrow \infty} \int_X g_n^p d\mu \geq \sup_{R > 0} \liminf_{n \rightarrow \infty} \int_{B(x_0, R)} g_n^p d\mu \geq \text{Ch}_p(f).$$

Proof. For every $M > 0$, let us denote by $\mathcal{A}_M \subseteq AC([0, 1]; X)$ the set

$$\mathcal{A}_M := \{\ell(\gamma) \geq \frac{1}{M}\} \cap \{\gamma([0, 1]) \subseteq B_M(x_0)\}.$$

If $\ell(\gamma) > 0$ we have that $\gamma \in \mathcal{A}_M$ for $M = \sup\{\ell(\gamma)^{-1}, d(\gamma(0), x) + \ell(\gamma)\}$, so, in particular

$$AC([0, 1]; X) = \{\ell(\gamma) = 0\} \cup \bigcup_{n=1}^{\infty} \mathcal{A}_n.$$

We can now define $\mathcal{B}_n = \{\ell(\gamma) = 0\} \cup \mathcal{A}_n$. Let us consider $\pi_n := \pi|_{\mathcal{B}_n}$ and compute

$$\int_{AC} |f(\gamma_1) - f(\gamma_0)| d\pi_n \leq \int_{AC} |f_m(\gamma_0) - f_m(\gamma_1)| d\pi_n + 2C \int_{B(x_0, n)} |f_m - f| d\mu,$$

where we used the triangle inequality and the first property of p -plans. Then we take m big enough such that $\delta_m \leq \frac{1}{n}$ and in this way we can use the upper gradient property π_n -almost everywhere (notice also that if $\ell(\gamma) = 0$ the upper gradient property is trivial) to get

$$\begin{aligned} \int_{AC} |f(\gamma_1) - f(\gamma_0)| d\pi_n &\leq \int_{AC} \left(\int_{\gamma} g_m \right) d\pi_n + 2C \int_{B(x_0, n)} |f_m - f| d\mu \\ &\leq \int_{B(x_0, n)} g_m \cdot b_{\pi} d\mu + 2C \int_{B(x_0, n)} |f_m - f| d\mu \end{aligned}$$

Taking the limit as $m \rightarrow \infty$ (using $b_{\pi} \in L^q$ and the weak convergence of g_m to g), and then taking $n \rightarrow \infty$ we get precisely Definition 2.4 (respectively 2.5) (c), and so we can conclude. \square

2.2. Preliminaries on doubling spaces equipped with Poincaré inequality. Let us define a regularization operator $M_t : L^p(X) \rightarrow L^p(X)$

$$M_t f(x) = \frac{1}{\mu(B(x, t))} \int_{B(x, t)} f(y) d\mu(y).$$

We state its main properties

Lemma 2.7 (Boundedness of M_t). *Let μ be a doubling measure with doubling constant c_D . Then M_t is a linear bounded operator from $L^p(X)$ to itself, in particular*

$$\|M_t f\|_p \leq c_D \|f\|_p, \quad \text{for every } f \in L^p(X).$$

Moreover we have $\|M_t f - f\|_p \rightarrow 0$ as $t \rightarrow 0$ for every $f \in L^p(X)$.

Proof. For the first part we use first Jensen inequality

$$\int_X \left(\frac{1}{\mu(B(x, t))} \int_{B(x, t)} f(y) d\mu(y) \right)^p d\mu(x) \leq \int_X \frac{1}{\mu(B(x, t))} \int_{B(x, t)} f^p(y) d\mu(y) d\mu(x),$$

and then Fubini to obtain

$$\int_X \frac{1}{\mu(B(x, t))} \int_{B(x, t)} f^p(y) d\mu(y) d\mu(x) = \int_X f^p(y) g_t(y) d\mu(y),$$

where $g_t(y) = \int_{B(y,t)} \frac{1}{\mu(B(x,t))} d\mu(x)$. Using the doubling property we get

$$g_t(y) \leq \int_{B(y,t)} \frac{c_D}{\mu(B(x,2t))} d\mu \leq \int_{B(y,t)} \frac{c_D}{\mu(B(y,t))} d\mu = c_D.$$

The convergence of $M_t f$ to f is obvious for Lipschitz functions with bounded support and then we can conclude using the boundedness of M_t and the density of Lipschitz functions in $L^p(X)$. \square

Lemma 2.8. *If μ is doubling, there exist $C > 0$ such that for every $x \in X$, $r > 0$, we have*

$$\int_{\{d(x,y) \geq r\}} \frac{1}{\rho(x,y)d(x,y)^p} d\mu(y) \leq \frac{C}{r^p}.$$

Proof. We consider the annuli $A_i(x) = \{2^i r \leq d(x,y) < 2^{i+1} r\}$. Now, whenever $y \in A_i$ we have $d(x,y) \geq 2^i r$, but also $\rho(x,y) \geq \frac{1}{C} \mu(B(x,2^i r))$, since μ is doubling and $\rho(x,y)$ is comparable to $\mu(B(x,d(x,y)))$. We thus estimate

$$\int_{\{d(x,y) \geq r\}} \frac{1}{\rho(x,y)d(x,y)^p} d\mu(y) = \sum_{i=0}^{\infty} \int_{A_i} \frac{1}{\rho(x,y)d(x,y)^p} d\mu(y) \leq \sum_{i=0}^{\infty} C \frac{\mu(A_i)}{\mu(B(x,2^i r))r^p 2^{ip+p}}.$$

In the end we use $\mu(A_i) \leq \mu(B(x,2^{i+1}r))$ and then the doubling condition again to get

$$\int_{\{d(x,y) \geq r\}} \frac{1}{\rho(x,y)d(x,y)^p} d\mu(y) \leq \frac{C}{r^p} \frac{1}{2^p - 1} \leq \frac{C}{r^p},$$

which concludes the proof. \square

In the spirit of the Hajlasz-Sobolev space we then state the following

Proposition 2.9. *Let $p > 1$, μ be a doubling measure that satisfies a $(1,p)$ -Poincaré inequality. Then for every $r > 0$ there exists a constant C_r such that for every $u \in W^{1,p}(X,d,\mu)$ there exists $g \in L^p$ such that $\|g\|_p^p \leq C_r \cdot \text{Ch}_p(X)$ and*

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \quad \forall x, y \in X, d(x,y) \leq r.$$

Proof. It is sufficient to combine the results from [22] and [20], along with the boundedness of the maximal function operator in doubling spaces. \square

3. PROOF OF THEOREM 1.4

We prove separately the upper and the lower bound.

3.1. Upper bound of (doubling) Theorem 1.4. For every ball $B = B(x',t)$, denoting by

$$u_B := \frac{1}{\mu(B)} \int_B u d\mu,$$

we have

$$(3.1) \quad \begin{aligned} \mu(B) \int_B |u_B - u(x)|^p d\mu(x) &\leq \int_B \int_B |u(x) - u(y)|^p d\mu(x) d\mu(y) \\ &\leq 2^p \mu(B) \int_B |u_B - u(x)|^p d\mu(x). \end{aligned}$$

The first inequality follows by Hölder inequality, while the second one follows from the elementary inequality $|a+b|^p \leq 2^{p-1}|a|^p + |b|^p$ applied with $a = u(x) - u_B$ and $b = u_B - u(y)$. We now write

$$\frac{1}{d(x,y)^{ps}} = ps \int_{d(x,y)}^{\infty} \frac{1}{t^{ps+1}} dt.$$

Then we apply the Fubini-Tonelli Theorem and get in turn

(3.2)

$$\begin{aligned} \int_{X \times X} \frac{|u(x) - u(y)|^p}{\rho(x, y) d(x, y)^{ps}} d\mu(x) d\mu(y) &= ps \int_X \int_X \int_{d(x, y)}^{\infty} \frac{|u(x) - u(y)|^p}{\rho(x, y) t^{ps+1}} dt d\mu(x) d\mu(y) \\ &= ps \int_0^{\infty} \frac{1}{t^{ps+1}} \left(\iint_{\{d(x, y) \leq t\}} \frac{|u(x) - u(y)|^p}{\rho(x, y)} d\mu(x) d\mu(y) \right) dt. \end{aligned}$$

Now, let us define the quantities

$$\begin{aligned} \mathcal{K}_t &:= \iint_{\{d(x, y) \leq t\}} \frac{|u(x) - u(y)|^p}{\rho(x, y)} d\mu(x) d\mu(y), \\ \mathcal{H}_t &:= \iint_{\{d(x, y) \leq t\}} \frac{|u(x) - u(y)|^p}{\sqrt{\mu(B(x, t))\mu(B(y, t))}} d\mu(x) d\mu(y), \\ \mathcal{S}_t &:= \int \frac{1}{\mu(B(x', t))^2} \iint_{B(x', t) \times B(x', t)} |u(x) - u(y)|^p d\mu(x) d\mu(y) d\mu(x'). \end{aligned}$$

We will prove a lemma that deals with relations between these quantities, and then an estimate from above of \mathcal{S}_t .

Lemma 3.1. *There exist $0 < c < C < \infty$, depending only on the doubling and Poincaré constants, and possibly the constant C_p of the doubling kernel, such that for every $t > 0$ we have*

- (i) $c\mathcal{H}_t \leq \mathcal{K}_t \leq C \sum_{k=0}^{\infty} \mathcal{H}_{t/2^k}$;
- (ii) $c\mathcal{H}_{t/2} \leq \mathcal{S}_t \leq C\mathcal{H}_{2t}$;
- (iii) $\mathcal{S}_t \leq C t^p \text{Ch}_p(u)$.
- (iv) if $t \geq 1$ then $\mathcal{K}_t \leq \mathcal{K}_1 + C \log_2(2t) \int_X u^p d\mu$

Before proving Lemma 3.1 we use it to deduce the upper bound: first of all we have

$$(3.3) \quad \mathcal{K}_t \leq \frac{C}{c} \sum_{k=0}^{\infty} \mathcal{S}_{2t/2^k} \leq \frac{C^2}{c} \text{Ch}_p(X) \sum_{k=0}^{\infty} \left(\frac{4t}{2^k}\right)^p \leq C t^p \text{Ch}_p(X).$$

Now we can use (3.2) in order to find

$$(1-s) \int_{X \times X} \frac{|u(x) - u(y)|^p}{\rho(x, y) d(x, y)^{ps}} d\mu(x) d\mu(y) = (1-s)ps \int_0^{\infty} \frac{\mathcal{K}_t}{t^{ps+1}} dt.$$

Splitting the last integral in $t \leq 1$ and $t > 1$ will let us conclude using (3.3) in the first part and Lemma 3.1 (iv) for the second part:

$$\begin{aligned} (1-s)ps \int_0^{\infty} \frac{\mathcal{K}_t}{t^{ps+1}} dt &\leq (1-s)psC \int_0^1 \frac{\text{Ch}_p(X)}{t^{p(s-1)+1}} dt + (1-s)ps \|u\|_p^p C \int_1^{\infty} \frac{\log_2(2t)}{t^{ps+1}} dt \\ &\leq Cs \cdot \text{Ch}_p(X) + (1-s)C \|u\|_p^p \int_0^{\infty} \left(1 + \frac{\tau}{ps}\right) e^{-\tau} d\tau \\ &= Cs \cdot \text{Ch}_p(X) + (1-s)C \|u\|_p^p \left(1 + \frac{1}{ps}\right). \end{aligned}$$

In particular, letting $s \rightarrow 1$ we obtain the upper bound.

Proof of Lemma 3.1. Every constant inside this proof will depend on c_D, c_P, C_p , and possibly p .

- (i) Since both ρ and $\sqrt{\rho_1 \rho_2}$ (see Definition 1.2) are both doubling kernels we have, thanks to the doubling property, that $c\sqrt{\rho_1 \rho_2} \leq \rho \leq \frac{1}{c}\sqrt{\rho_1 \rho_2}$ for some $c > 0$. In particular, up to losing a multiplicative constant, we can assume $\rho = \sqrt{\rho_1 \rho_2}$. In this case the inequality

$\mathcal{H}_t \leq \mathcal{K}_t$ is trivial thanks to the monotonicity of measure of balls. The other inequality comes from the fact that

$$\mathcal{K}_t = \sum_{k=0}^{\infty} \iint_{\{\frac{t}{2^{k+1}} \leq d(x,y) \leq \frac{t}{2^k}\}} \frac{|u(x) - u(y)|^p}{\rho(x,y)} d\mu(x) d\mu(y);$$

then in every term we have

$$\rho(x,y) \geq \sqrt{\mu(B(x, \frac{t}{2^{k+1}}))\mu(B(y, \frac{t}{2^{k+1}}))} \geq \frac{1}{c_D} \sqrt{\mu(B(x, \frac{t}{2^k}))\mu(B(y, \frac{t}{2^k}))},$$

and so we have

$$\mathcal{K}_t \leq c_D \sum_{k=0}^{\infty} \mathcal{H}_{t/2^k}.$$

(ii) Let us begin by writing more explicitly \mathcal{S}_t , by doing the integration in x' first, which yields

$$\mathcal{S}_t = \int_X \int_X |u(x) - u(y)|^p f_t(x,y) d\mu(x) d\mu(y),$$

where we have set

$$f_t(x,y) := \int_{B(x,t) \cap B(y,t)} \frac{1}{\mu(B(x',t))^2} d\mu(x').$$

Thus, it is sufficient to prove that

$$\frac{c}{\sqrt{\mu(B(x,t))\mu(B(y,t))}} \chi_{\{d(x,y) \leq t/2\}} \leq f_t(x,y) \leq \frac{C}{\sqrt{\mu(B(x,2t))\mu(B(y,2t))}} \chi_{\{d(x,y) \leq 2t\}}.$$

For the second inequality, if $d(x,y) > 2t$ we have $f_t(x,y) = 0$, since $B(x,t) \cap B(y,t) = \emptyset$. Moreover we can bound from above using $\mu(B(x,t)) \leq \mu(B(x',2t)) \leq c_D \mu(B(x',t))$ and the same for y :

$$f_t(x,y) \leq c_D^2 \frac{\mu(B(x,t) \cap B(y,t))}{\sqrt{\mu(B(x,t))\mu(B(y,t))\mu(B(x,t))}} \leq \frac{c_D^3}{\sqrt{\mu(B(x,2t))\mu(B(y,2t))}}.$$

For the first inequality we need only to check that if $d(x,y) \leq t/2$ then f_t is bounded from below. But in this case we have $B(x',t) \subseteq B(x,2t)$ and so $\mu(B(x',t)) \leq \mu(B(x,2t))$ and the same is true for y . In particular, since this time $B(x,t/2) \subset B(x,t) \cap B(y,t)$, we get

$$f_t(x,y) \geq \frac{\mu(B(x,t/2))}{\sqrt{\mu(B(x,2t))\mu(B(y,2t))\mu(B(x,2t))}} \geq \frac{1}{c_D^4 \sqrt{\mu(B(x,t))\mu(B(y,t))}}.$$

(iii) We use the Poincaré inequality in the form (remember that B is a ball of radius t)

$$\begin{aligned} \int_B |u_B - u(x)|^p d\mu(x) &\leq t^p c_P \int_B g^p(x) d\mu(x), \\ \int_B |u_B - u(x)| d\mu(x) &\leq t c_P |Du|(B). \end{aligned}$$

In the spirit of treating the Sobolev and the BV case together, we can write

$$\nu(E) = \int_E g^p d\mu, \quad \nu(E) = |Du|(E),$$

respectively. We then have, using Equation (3.1) and Poincaré inequality

$$\begin{aligned} \mathcal{S}_t &\leq 2^p \int_X \frac{1}{\mu(B(x, t))} \int_{B(x, t)} |u_{B(x, t)} - u(y)|^p d\mu(y) d\mu(x) \\ &\leq c_P 2^{pt} \int_X \frac{\nu(B(x, t))}{\mu(B(x, t))} d\mu(x) = c_P (2t)^p \int_{X \times X} \frac{\chi_{\{d(x, y) \leq t\}}(x, y)}{\mu(B(x, t))} d\mu \otimes \nu(x, y). \end{aligned}$$

Notice now that if $d(x, y) \leq t$ then we have $B(y, t) \subseteq B(x, 2t)$ and in particular, using the doubling condition, $\mu(B(y, t)) \leq \mu(B(x, 2t)) \leq c_D \mu(B(x, t))$. Then we deduce that

$$\frac{\chi_{\{d(x, y) \leq t\}}(x, y)}{\mu(B(x, t))} \leq c_D \frac{\chi_{\{d(x, y) \leq t\}}(x, y)}{\mu(B(y, t))}.$$

Using Fubini-Tonelli we then get $\mathcal{S}_t \leq c_P c_D (2t)^p \nu(X)$.

- (iv) In this case, we want to control also the part where $d(x, y) \geq 1$; there we will use the triangle inequality $|u(x) - u(y)|^p \leq 2^{p-1}(|u(x)|^p + |u(y)|^p)$ and also $\rho(x, y) \geq C\mu(B(x, d(x, y)))$ and $\rho(x, y) \geq C\mu(B(y, d(x, y)))$ (from the property of being a doubling kernel), to get

$$\begin{aligned} \mathcal{K}_t &= \iint_{\{d(x, y) \leq t\}} \frac{|u(x) - u(y)|^p}{\rho(x, y)} d\mu(x) d\mu(y), \\ &= \mathcal{K}_1 + \iint_{\{1 \leq d(x, y) \leq t\}} \frac{|u(x) - u(y)|^p}{\rho(x, y)} d\mu(x) d\mu(y) \\ &\leq \mathcal{K}_1 + \frac{2^p}{C} \iint_{\{1 \leq d(x, y) \leq t\}} \frac{|u(x)|^p}{\mu(B(x, d(x, y)))} d\mu(x) d\mu(y) \\ &= \mathcal{K}_1 + \frac{2^p}{C} \int_X |u(x)|^p \int_{\{1 \leq d(x, y) \leq t\}} \frac{1}{\mu(B(x, d(x, y)))} d\mu(y) d\mu(x). \end{aligned}$$

In order to estimate the last integral we divide in shells $S_k = \{y : 2^k \leq d(x, y) \leq 2^{k+1}\}$ and then we have

$$\begin{aligned} \int_{\{1 \leq d(x, y) \leq t\}} \frac{1}{\mu(B(x, d(x, y)))} d\mu(y) &\leq \sum_{k=0}^{\lfloor \log_2(t) \rfloor} \int_{S_k} \frac{1}{\mu(B(x, 2^k))} d\mu(y) \\ &= \sum_{k=0}^{\lfloor \log_2(t) \rfloor} \frac{\mu(S_k)}{\mu(B(x, 2^k))} \leq \lfloor \log_2(2t) \rfloor \cdot (c_D - 1), \end{aligned}$$

which concludes the proof. \square

3.2. Lower bound of Theorem 1.4.

We first recall the following

Definition 3.2 (Upper gradient). Given a function $f \in L^1 + L^\infty$ and a function $g \geq 0$, we say that g is an upper gradient up to scale δ of f if for every curve γ of length $\geq \delta$ we have

$$(3.4) \quad |f(\gamma_a) - f(\gamma_b)| \leq \int_\gamma g.$$

Let us define $u^t = M_t u$ and

$$g_t(x') := \frac{1}{\mu(B(x', t))^2} \iint_{B(x', t) \times B(x', t)} \left| \frac{u(x) - u(y)}{t} \right| d\mu(x) d\mu(y).$$

In this way we have, by Jensen,

$$\mathcal{S}_t \geq t^p \int g_t(x')^p d\mu(x').$$

Now, the idea is that for some $C > 0$, we have that Cg_{2t} is an upper gradient up to scale $t/2$ of the function u^t . This is significant thanks to Lemma 2.6 and Lemma 2.7.

The proof that Cg_{2t} is an upper gradient up to scale $t/2$ of u^t is as follows: it is sufficient to check Equation (3.4) only on curves that have length between $t/2$ and t , and then use the triangle inequality. So let us consider $\gamma : [a, b] \rightarrow X$ with length between $t/2$ and t . Then for every $c \in (a, b)$ we have $d(\gamma_c, \gamma_a) \leq t$ and $d(\gamma_c, \gamma_b) \leq t$. In particular $B(\gamma_a, t) \subseteq B(\gamma_c, 2t) \subseteq B(\gamma_a, 4t)$ and so

$$\begin{aligned} |u^t(\gamma_a) - u^t(\gamma_b)| &\leq \frac{1}{\mu(B(\gamma_a, t))\mu(B(\gamma_b, t))} \int_{B(\gamma_a, t) \times B(\gamma_b, t)} |u(x) - u(y)| d\mu(x) d\mu(y) \\ &\leq \frac{1}{\mu(B(\gamma_a, t))\mu(B(\gamma_b, t))} \int_{B(\gamma_c, 2t) \times B(\gamma_c, 2t)} |u(x) - u(y)| d\mu(x) d\mu(y) \\ &\leq \frac{c_D^4}{\mu(B(\gamma_a, 4t))\mu(B(\gamma_b, 4t))} \int_{B(\gamma_c, 2t) \times B(\gamma_c, 2t)} |u(x) - u(y)| d\mu(x) d\mu(y) \\ &\leq 2tc_D^4 \cdot g_{2t}(\gamma_c). \end{aligned}$$

In particular we have that, taking $h_t = 4c_D^4 g_{2t}$,

$$\int_{\gamma} h_t \geq \int_{\gamma} \frac{2}{t} |u^t(\gamma_a) - u^t(\gamma_b)| = \frac{2l(\gamma)}{t} |u^t(\gamma_a) - u^t(\gamma_b)| \geq |u^t(\gamma_a) - u^t(\gamma_b)|.$$

Using Lemma 2.6 and 2.7 we get $\liminf_{t \rightarrow 0} \frac{\mathcal{S}_t}{t^p} \geq C \cdot \text{Ch}_p(X)$. Then we are done using $\mathcal{K}_t \geq c\mathcal{S}_{t/2}$

$$\begin{aligned} \liminf_{s \rightarrow 1} (1-s)ps \int_0^\infty \frac{\mathcal{K}_t}{t^{ps+1}} dt &\geq \liminf_{s \rightarrow 1} (1-s)ps \int_0^1 \frac{\mathcal{K}_t}{t^{ps+1}} dt \geq \liminf_{s \rightarrow 1} c \int_0^1 \frac{\mathcal{S}_{t/2}}{t^p} d\nu_{s,p} \\ &\geq c \liminf_{t \rightarrow 0} \frac{\mathcal{S}_{t/2}}{t^p} \geq C \cdot \text{Ch}_p(X), \end{aligned}$$

where we used that $\nu_{s,p} = \frac{(1-s)p}{tp^{(s-1)+1}}$ is a probability measure on $[0, 1]$ that goes weakly to δ_0 .

4. PROOF OF THEOREM 1.5

4.1. **Upper bound of Theorem 1.5.** Consider the quantities

$$\begin{aligned} A_\delta &:= \int_X \int_X \frac{\delta^p}{\rho(x, y)d(x, y)^p} d\mu(x) d\mu(y), \\ &\quad \{ |u(x) - u(y)| > \delta \} \\ B_{\delta, r} &:= \int_X \int_X \frac{\delta^p}{\rho(x, y)d(x, y)^p} d\mu(x) d\mu(y), \\ &\quad \{ |u(x) - u(y)| > \delta, d(x, y) \leq r \} \end{aligned}$$

Notice now that if $|u(x) - u(y)| > \delta$ we have $\delta^p \leq 2^{p-1}(\delta^p \wedge |u(x)|^p + \delta^p \wedge |u(y)|^p)$. Using this inequality and Lemma 2.8 we get

$$B_{\delta, r} \leq A_\delta \leq \frac{C}{r^p} \int_X (|u(x)| \wedge \delta)^p d\mu + B_{\delta, r}.$$

In particular, thanks to dominated convergence, we deduce that for every $r > 0$ the limit points as $\delta \rightarrow 0$ of $B_{\delta, r}$ and A_δ are the same. Now let us assume that $u \in W^{1,p}(X)$. In particular, by Proposition 2.9 there exists a function $g \in L^p$ such that $\int g^p d\mu \leq C \cdot \text{Ch}_p(u)$ and for any x, y with $d(x, y) \leq r$ we have

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)).$$

But then, the triangle inequality lets us conclude that in the subset $\{d(x, y) \leq r\}$ we have:

$$\{|u(x) - u(y)| > \delta\} \subseteq \{Cd(x, y) \cdot g(x) \geq \delta/2\} \cup \{Cd(x, y) \cdot g(y) \geq \delta/2\}.$$

By symmetry then we can estimate

$$\begin{aligned} B_{\delta, r} &\leq 2 \int_X \int_X \frac{\delta^p}{\rho(x, y)d(x, y)^p} d\mu(x) d\mu(y) \\ &\quad \{Cd(x, y) \cdot g(x) \geq \delta/2, d(x, y) \leq r\} \\ &= 2 \int_X \int_{d(x, y) \geq r_1(x)} \frac{\delta^p}{\rho(x, y)d(x, y)^p} d\mu(y) d\mu(x), \end{aligned}$$

where $r_1(x) = \frac{\delta}{2Cg(x)}$. Using again Lemma 2.8 we get

$$B_{\delta, r} \leq \tilde{C} \int g(x)^p d\mu(x) \leq \tilde{C} \cdot C \cdot \text{Ch}_p(u).$$

4.2. Lower bound of Theorem 1.5. In this case we suppose, without loss of generality, that

$$\sup_{0 < \delta < 1} A_\delta \leq C.$$

Notice that then for every $r \leq 1$ and for every $0 < \varepsilon < 1$ we have

$$C \geq \int_0^r \varepsilon \delta^{\varepsilon-1} A_\delta d\delta = \frac{\varepsilon}{p + \varepsilon} \int_X \int_X \frac{\inf\{|u(x) - u(y)|, r\}^{p+\varepsilon}}{\rho(x, y)d(x, y)^p} d\mu(x) d\mu(y).$$

Now let us define (φ is a 1-Lipschitz function)

$$\begin{aligned} g_t(x') &:= \frac{1}{\mu(B(x', t))^2} \iint_{B(x', t) \times B(x', t)} \frac{\inf\{|u(x) - u(y)|, r\}}{t} d\mu(x) d\mu(y), \\ g_{\varphi, t}(x') &:= \frac{1}{\mu(B(x', t))^2} \iint_{B(x', t) \times B(x', t)} \frac{|\varphi(u(x)) - \varphi(u(y))|}{t} d\mu(x) d\mu(y), \end{aligned}$$

and, with the same argument as in the proof of the lower bound in Theorem 1.4, we can estimate

$$C \geq \frac{\varepsilon}{p + \varepsilon} \int_X \int_X \frac{\inf\{|u(x) - u(y)|, r\}^{p+\varepsilon}}{\rho(x, y)d(x, y)^p} d\mu(x) d\mu(y) \geq c \int_0^1 \frac{\int_X g_t^{p+\varepsilon} d\mu}{t^{p+\varepsilon}} d\eta_\varepsilon.$$

Here $\eta_\varepsilon = \nu_{\frac{p}{p+\varepsilon}, p+\varepsilon} = \frac{\varepsilon}{t^{1-\varepsilon}}$ and so $\eta_\varepsilon \rightarrow 0$. In particular, using this inequality for $\varepsilon \rightarrow 0$, we deduce that there exists a sequence $t_\varepsilon \rightarrow 0$ such that

$$\lim_{\varepsilon \rightarrow 0} \int_X \left(\frac{g_{t_\varepsilon}}{t_\varepsilon} \right)^{p+\varepsilon} d\mu \leq C/c.$$

In particular, up to a subsequence we have $g_{t_\varepsilon}/t_\varepsilon \rightarrow h$ in $L^p_{\text{loc}}(X, \mu)$ and $\int_X h^p d\mu \leq C/c$.

Let us consider the class $\mathcal{L}_r \subset \text{Lip}(\mathbb{R})$ of 1-Lipschitz functions that have values in $[0, r]$; notice that for $\varphi \in \mathcal{L}_r$ we have $|\varphi(t) - \varphi(s)| \leq |t - s|$ and $|\varphi(t) - \varphi(s)| \leq r$. In particular we have $g_{\varphi, t} \leq g_t$ for $\varphi \in \mathcal{L}_r$; moreover we already know that, up to constants, $g_{\varphi, t}$ is a weak upper gradient at scale $2t$ for $M_t(\varphi \circ u)$. This implies that g_t is also a weak upper gradient at scale $2t$ for $M_t(\varphi \circ u)$ and using Lemma 2.6 and 2.7 we find that h is a p -weak upper gradient for $\varphi \circ u$ for every $\varphi \in \mathcal{L}_r$. Now we want to prove that h is a p -weak upper gradient also for u . Thanks to Definition 2.3, we have that for every φ and every p -plan π , there exists a set \mathcal{N}_φ that is π negligible, such that for $\gamma \notin \mathcal{N}_\varphi$ we have $\varphi \circ (f \circ \gamma) \in W^{1,1}(0, 1)$ and $|(\varphi \circ (u \circ \gamma))'(t)| \leq h \circ \gamma(t) |\gamma'(t)|$. In particular, we can take a countable dense set $\mathcal{S} \subset \mathcal{L}_r$ and, denoting by

$$\mathcal{N} = \bigcup_{\varphi \in \mathcal{S}} \mathcal{N}_\varphi,$$

if $\gamma \notin \mathcal{N}$ we have that $f = u \circ \gamma$ and $g = h \circ \gamma|\gamma'|$ satisfy the hypothesis of Lemma 4.1, and in particular we have

$$\forall \gamma \notin \mathcal{N} \quad u \circ \gamma \in W^{1,1}(0,1) \quad \text{and} \quad |u \circ \gamma'(t)| \leq g \circ \gamma(t)|\gamma'| \quad \text{for a.e. } t \in [0,1].$$

Since \mathcal{N} is a union of countably many π -negligible sets, it is itself π -negligible. Thanks to the arbitrariness of π , using again Definition 2.3 we conclude that h is indeed a p -weak upper gradient for u .

Lemma 4.1. *Let us consider $f : [0,1] \rightarrow \mathbb{R}$. Suppose there exists $g \in L^1(0,1)$ such that for every φ belonging to a dense subset of \mathcal{L}_r we have $\varphi \circ f \in W^{1,1}(0,1)$ and $|(\varphi \circ f)'(t)| \leq g(t)$ for \mathcal{L} -almost every $t \in [0,1]$. Then $f \in W^{1,1}(0,1)$ and $|f'| \leq g$.*

Proof. First of all let us observe that if the hypothesis is true for a dense subset of φ then it is true for every $\varphi \in \mathcal{L}_r$ since it is equivalent to require

$$|\varphi(f(x)) - \varphi(f(y))| \leq \int_x^y g(t) dt \quad \text{for almost every } x < y \in [0,1],$$

which is a condition stable for uniform convergence of φ .

Let us consider, for every $n \in \mathbb{N}$

$$\varphi_n(t) = \begin{cases} 0 & \text{if } t < rn \\ t - rn & \text{if } rn \leq t < r(n+1) \\ r & \text{if } t \geq r(n+1); \end{cases}$$

we also define $\varphi_{-n}(t) = -\varphi_n(-t)$. Then clearly we have $\varphi_n \in \mathcal{L}_r$; moreover

$$\sum_{n \in \mathbb{Z}} \varphi_n(t) = t.$$

Considering then $f_n = \varphi_n \circ f$ we have

$$(4.1) \quad \sum_{n \in \mathbb{Z}} f_n(x) = f(x) \quad \text{for every } x \in [0,1].$$

By hypothesis we have $f_n \in W^{1,1}$ and $|f'_n| \leq g$; however we have $f'_n = 0$ almost everywhere in $\{f_n = 0\} \cup \{f_n = r\}$ thanks to standard Sobolev theory. In particular denoting with $A_n = \{rn \leq f < r(n+1)\}$ we have more precisely $|f'_n| \leq g\chi_{A_n}$. Let us consider N big enough such that $\{f \leq Nr\}$ is not negligible. Then we have that $\{f_n = 0\}$ is not negligible for $n \geq N$ and then we have $\|f_n\|_\infty \leq \|g\chi_{A_n}\|_1$ thanks to the fact that there exists x_0 such that $f_n(x_0) = 0$ and the estimate

$$|f_n(x)| = |f_n(x) - f_n(x_0)| = \left| \int_{x_0}^x f'_n(y) dy \right| \leq \int_0^1 |f'_n|(y) dy \leq \|g\chi_{A_n}\|_1.$$

A similar argument can be used for n very negative. Now we have that $\|g\chi_{A_n}\|_1$ is summable and adds up to $\|g\|_1$. In particular this proves that $\sum_{|n| \leq N} f_n$ converges in L^∞ to some function \tilde{f} which will coincide with f almost everywhere thanks to (4.1). We will in particular have that

$$\sum_{|n| \leq N} f_n \xrightarrow{L^1} f \quad \sum_{|n| \leq N} f'_n \xrightarrow{L^1} \bar{g}$$

where $\bar{g} = f'_n$ in A_n . In particular we have $f \in W^{1,1}$ and $f' = \bar{g}$; in particular $|f'| = |\bar{g}| \leq g$. \square

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