A Relation-Algebraic Approach to L-Fuzzy Topology

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Abstract

Any science deals with the study of certain models of the real world. However, a model is always an abstraction resulting in some uncertainty, which must be considered. The theory of fuzzy sets is one way of formalizing one of the types of uncertainty that occurs when modeling real objects. Fuzzy sets have been applied in various real-world problems such as control system engineering, image processing, and weather forecasting systems.

This research focuses on applying the categorical framework of abstract L-fuzzy relations to L-fuzzy topology with ideas, concepts and methods of the theory of L-fuzzy sets. Since L-fuzzy sets were introduced to deal with the problem of approximate reasoning, $t-norm$ based operations are essential in the definition of L-fuzzy topologies. We use the abstract theory of arrow categories with additional $t-norm$ based connectives to define L-fuzzy topologies abstractly. In particular, this thesis will provide an abstract relational definition of an L-fuzzy topology, consider bases of topological spaces, continuous maps, and the first two separation axioms T_0 and T_1. The resulting theory of L-fuzzy topological spaces provides the foundation for applications and algorithms in areas such as digital topology, i.e., analyzing images using topological features.

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Chapter 1

Introduction

The concept of a fuzzy sets, introduced in 1985 by Zadeh [20], has aroused great interest among mathematicians, both theoretical and applied. It also inspired enthusiasm among some engineers, biologists, psychologists, economists and specialists in other fields who use mathematical ideas and methods in their research. General topology was one of the first areas of theoretical mathematics to which fuzzy sets became systematically involved. In 1968, three years after the appearance of the Zadeh's work, Chang introduced the concept of a fuzzy set to general topology [3]. This work was followed by others in which Chang's fuzzy spaces and other structures of topological type for systems of fuzzy sets were considered. Since then, the intensity of research in the field of fuzzy topology has increased dramatically, and currently there are many publications on this topic.

The purpose of this thesis is to construct and discuss a categorical framework of abstract L-fuzzy relations and apply it to L-fuzzy topology. The theory of L-fuzzy topology combines general topology and ideas, concepts and methods of the theory of L-fuzzy sets. An important characteristic of L-fuzzy topology is given by the fact that the axiom requiring that the intersection of two open sets is open is replaced by a modified intersection based on a t-norm like operation $*$ on the underlying lattice instead of the logical "and" in the Boolean case or the lattice meet in lattice case, respectively. The lack of theoretical investigation of $*$ operations in the context of abstract L-fuzzy relations requires additional intensive research. In this thesis we define L-fuzzy topologies abstractly using arrow categories with an additional $*$ operation.

The resulting theory of L-fuzzy topological spaces provides the foundation for applications and algorithms in areas such as digital topology, i.e., analyzing images using topological features. For example, concepts of topology are used to specify and develop important image analysis algorithms such as border tracing, surface tracing, detecting and counting of components, holes or tunnels, or region filling [9]. Since image data might be uncertain, using L-fuzzy sets as basic components allows these concepts and algorithms to be applied in a more general context [12].

We briefly describe the content of the thesis. Chapter 2 presents the minimum background information which is necessary for reading the main text of the work. The chapter is divided into three main sections, namely lattices, L-fuzzy relations and categories of relations. In the second section we define the new set of $*$ operations based on t-norm which plays key role in constructing of fuzzy categories in third section. Chapter 3 gives briefly definitions of general, fuzzy and relational topologies. This review of the classical approaches in literature also serves as a reference to compare the abstract relational approach taken in this thesis. Chapter 4, the main contribution of this thesis, presents a relation-algebraic approach to fuzzy topology. In this chapter we take a fresh look at many of the issues discussed earlier and cover important topics of fuzzy topology in terms of relations, such as bases, construction of topologies and separations axioms. The last chapter includes our concluding remarks and some ideas about future work.

The thesis has been proposed as a seminar in Foundations and Applications of Computational Topology and Information Processing conference and will be published in Relational and Algebraic Methods in Computer Science conference.

Chapter 2

Mathematical Preliminaries

2.1 Lattices

This chapter will introduce main concepts of lattice theory and provide some examples. For more details we refer to [4, 17].

One of the most researched concepts is partially-ordered set or poset.

Definition 2.1. *A partial-ordered set or poset is a set A on which a binary relation $\leq$ is defined that satisfies the following axioms for all $x, y, z \in A$*

$x \leq x$ (reflexive),

if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive),

if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric).

If $x \leq y$ or $y \leq x$ is true for all $x, y \in A$, then such a set is called a linearly ordered set.

Example 1. We can take some set of numbers $\{0, 1, 2, \cdots, 9\}$ and consider the regular order of smaller or equal. Visualization of the posets is done by Hasse diagrams, where vertices are the elements of the posets and edges are relationship by order. Please note that in a Hasse diagram elements that are printed higher are bigger than the elements below. Furthermore, a Hasse diagram only shows the essential information. Lines due to transitivity and reflexivity are usually not shown.

Similarly, we can draw a set of letters in alphabetical order. Figure 2.1 shows the Hasse diagram of natural numbers and alphabets in linear order.

Example 2. Consider the set $\{a, b\}$ and all its subsets, i.e., the so-called power set $\mathcal{P}(\{a, b\})$ of $\{a, b\}$. There are four of them: $\varnothing, \{a\}, \{b\}, \{a, b\}$. The order relation

Figure 2.1: Hasse diagrams of natural numbers and alphabets.

is the inclusion relation $\subseteq$. In this order there is the smallest element $\varnothing$, the largest element $\{a, b\}$ and two element $\{a\}$ and $\{b\}$ between them. These two intermediate elements are not comparable to each other (none of the sets $\{a\}$ and $\{b\}$ is a subset of the other).

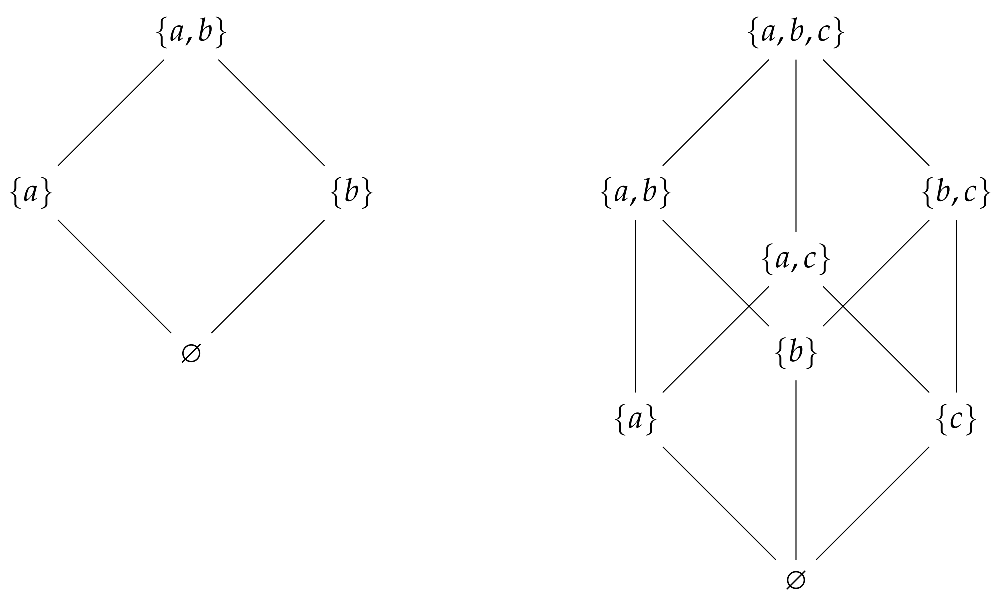

Figure 2.2: Hasse diagrams of the powersets of $\{a, b\}$ and $\{a, b, c\}$.

Similarly consider all subsets of the three-element set $\{a, b, c\}$ (there are eight of them) and also order them by inclusion. Figure 2.2 shows the Hasse diagram of the two powersets of $\{a, b\}$ and $\{a, b, c\}$.

Example 3. Let the set consist of some positive integers, and $x \leq y$ is understood as "x is a divisor of y". Each number is its divisor (reflexivity). If a is divisible by b, and at the same time b is divisible by a, then these numbers are equal (antisymmetry). Finally, if a divides b and b divides c, then $\frac{b}{a}$ and $\frac{c}{b}$ are integers, and their product equal to $\frac{c}{a}$ is also an integer, that is a divides c. So we get a partially ordered set.

Now lets consider a partially ordered set $(A, \leq)$. If M is a subset of A, then every element $a \in A$ (not necessarily $a \in M$) satisfying the condition $a \geq x$ for all $x \in M$ is called the *upper bound* of the subset M of the set A. The dual concept is called a *lower bound*. The greatest elements of all the lower bounds of M is called the *greatest lower bound* of the subset M. If there exists an greatest lower bound, then it is unique. The *least upper bound* of the subset M is defined similarly and they are the least elements of all the upper bounds of M (if it exists) [17].

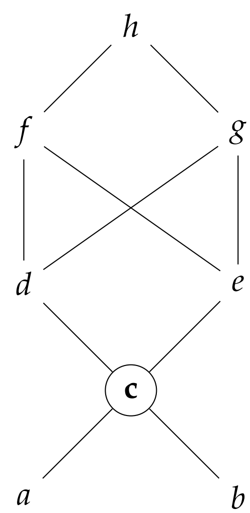

Figure 2.3: The greatest lower bound and the least upper bound of $A = \{a, b\}$ and $B = \{c, d, e\}$.

Figure 2.3 illustrates all least upper bounds and all greatest lower bounds of $A = \{a, b\}$ and $B = \{c, d, e\}$. Since A has no lower bounds, it has no greatest lower bound. However, elements $c - h$ are upper bounds of A, and c is the least upper bound. In case of B there are three lower bounds c, a and b, where c is greatest lower bound for B. The upper bounds of B are f, g and h. Since f and g are not comparable, B has no least upper bound.

Definition 2.2. *A partially ordered set L is called an upper semilattice if each pair of its elements x and y has a least upper bound or join, denoted by $x \vee y$; and a lower semilattice if each pair of elements has a greatest lower bound or meet, denoted by $x \wedge y$. Both upper and lower semilattice are called complete iff every subset M has a least upper bound denoted by $\bigvee M$ and a greatest lower bound denoted by $\bigwedge M$. L is called a lattice iff it is a lower* *and upper semilattice. Furthermore, it is called bounded iff it has a least element* 0 *and a greatest element* 1*.*

The elements 1 and 0 of upper and lower semilattices respectively are unique. If L is complete, then we have $1 = \bigvee L$ and $0 = \bigwedge L$.

Obviously, each pair of elements can have only one upper and lower bound (if they exist), so finding the bounds can be treated as the result of the meet and join operations. In this sense, the semilattice is equivalent to an algebra with a poset A and a signature $\{\vee, \wedge\}$.

For semilattice operations, the following rules are true:

$$\begin{array}{ccc} x \wedge x = x, & x \vee x = x & \textit{idempotency;} \\ x \wedge y = y \wedge x, & x \vee y = y \vee x & \textit{commutativity;} \\ x \wedge (y \wedge z) = (x \wedge y) \wedge z, & x \vee (y \vee z) = (x \vee y) \vee z & \textit{associativity;} \\ x \wedge y = x \iff x \leq y, & x \vee y = x \iff x \leq y & \textit{consistency;} \\ y \leq z \Rightarrow x \wedge y \leq x \wedge z, & y \leq z \Rightarrow x \vee y \leq x \vee z & \textit{monotonicity.} \end{array}$$

If the partially ordered set is a lattice, then we have in addition:

$$x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x \quad \textit{absorption.}$$

A proof of the properties above can be found in [6].

In particular, a complete lattice $(L, \leq)$ is a regular lattice, it necessarily contains a greatest and a least elements. As an example of a complete lattice, we can consider any powerset with $\bigcap$ and $\bigcup$ operations or the unit interval $[0, 1]$. The real numbers $(\mathbb{R}, \leq)$ is not a complete lattice. However the extension $(\mathbb{R} \cup \{-\infty, \infty\}, \leq)$ with $-\infty < x < +\infty$ for all $x \in R$ is a complete lattice.

A function f from a lattice M to a lattice L is called a *homomorphism* if

$$f(x \vee y) = f(x) \vee f(y), \quad f(x \wedge y) = f(x) \wedge f(y)$$

for all $x, y \in M$.

Similarly, a *complete lattice homomorphism* is a function f from L_1 to L_2 satisfying:

$$f(\bigvee M) = \bigvee_{x \in M} f(x), \quad f(\bigwedge M) = \bigwedge_{x \in M} f(x)$$

for all subsets M of L_1.

2.1.1 Distributive Lattices

There are many classes of lattices. One of the most important classes from the combinatorial point of view is formed by distributive lattices.

Lemma 2.3. *Let L be a lattice. Then the following inequalities are true*

(1) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$,

(2) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$.

Proof. Since $x \leq x \vee y$ and $x \leq x \vee z, x \leq (x \vee y) \wedge (x \vee z)$. Similarly, $y \wedge z \leq y \leq x \vee y$ and $y \wedge z \leq z \leq x \vee y$ imply $y \wedge z \leq (x \vee y) \wedge (z \vee y)$. Together, we obtain $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$. Analogously, we can show the second inequality is a dual of the first inequality. □

Now we can define distributive lattices, by replacing inequalities of the previous lemma to equalities.

Definition 2.4. *A lattice L is said to be distributive if one of the following laws is satisfied:*

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

for all $x, y, z \in L$.

It can be shown that any of these laws is equivalent to each other, and the following lemma will prove it.

Lemma 2.5. *For all lattices L, (1) and (2) are equivalent.*

Proof. We can prove equivalency of above axioms by using absorption, associativity and commutativity laws. Let us start with (1) is equivalent to (2):

$$
\begin{aligned}
x \vee (y \wedge z) &= \big(x \vee (x \wedge z)\big) \vee (y \wedge z) && \text{absorption} \\
&= x \vee \big((x \wedge z) \vee (y \wedge z)\big) && \text{associativity} \\
&= x \vee \big((z \wedge x) \vee (z \wedge y) && \text{commutativity} \\
&= x \vee \big(z \wedge (x \vee y)\big) && (1) \\
&= \big(x \wedge (x \vee y)\big) \vee \big(z \wedge (x \vee y)\big) && \text{absorption} \\
&= \big((x \vee y) \wedge x\big) \vee \big((x \vee y) \wedge z\big) && \text{commutativity} \\
&= (x \vee y) \wedge (x \vee z) && (1)
\end{aligned}
$$

Similarly (changing $\vee$ to $\wedge$ and vice versa), we can obtaine a proof of equivalency from (2) to (1). □

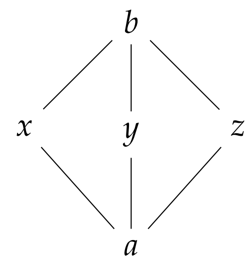

Figure 2.4: The example of nondistributive lattice.

Figure 2.4 illustrates the example of nondistributive lattice. Obviously, this Hasse diagram defines a lattice and that $x \vee (y \wedge z) = x \vee a = x$, but $(x \vee y) \wedge (x \vee z) = b \wedge b = b \neq x$.

Definition 2.6. *A complete lattice L is said to satisfy the first infinite distribution law iff*

$$x \wedge \bigvee M = \bigvee_{y \in M} (x \wedge y)$$

for all $x \in L$ and $M \subseteq L$.

Obviously, every complete lattice that satisfies the first infinite distribution law is also distributive.

2.1.2 Complete Heyting Algebra

This class of lattices is interesting by providing a notion of complement or negation. This is also known as a *relative pseudo-complement*.

Definition 2.7. *Let L be a bounded lattice. L is called Heyting algebra if for every two elements x and y there exists a pseudo-complement x relative to y $(x \rightarrow y)$, satisfying:*

$$z \leq y \rightarrow x \Longleftrightarrow x \wedge z \leq y$$

for all $z \in L$.

In every Heyting Algebra we have $x \rightarrow x = 1$. The pseudo-complement of x relative to 0 is called the pseudo-complement of x.

A Heyting Algebra homomorphism is a homomorphism of bounded lattices f that preserves relative pseudo-complementation, i.e, $f(x \to x) = f(x) \to f(y)$ for all x, y.

Now we can show the distributive property of Heyting Algebra.

Theorem 2.8. *Every Heyting Algebra is distributive.*

Proof. From the distributive law found in entry distributive inequalities (Lemma 2.5), we only need to show that

$$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z).$$

Since $x \wedge y \leq (x \wedge y) \vee (x \wedge z)$, so $y \leq x \colon \big((x \wedge y) \vee (x \wedge z)\big)$. Analogously, $z \leq x \to \big((x \wedge y) \vee (x \wedge z)\big)$. Then $y \vee z \leq x \to \big((x \wedge y) \vee (x \wedge z)\big)$, or $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. □

From Definition 2.6 and Theorem 2.8 we can conclude the following:

Definition 2.9. *If L is a complete lattice, then the following are equivalent:*

(1) L satisfies the first infinite distribution law.

(2) L is a Heyting algebra.

2.2 L-Fuzzy Relations

L-Fuzzy relations play a fundamental role in the theory of fuzzy systems. The apparatus of the theory of L-fuzzy relations is used in modeling the structure of complex systems, analyzing decision-making processes, modeling managing technological processes, etc..

The theory of L-fuzzy relations is also used in problems in which the theory of regular relations is traditionally applied. The apparatus of the theory of regular relations is used for a qualitative analysis of the relations between objects of the system under investigation. However, when methods of quantitative analysis of relations for some reasons cannot be applied, L-fuzzy relations come in handy.

The regular $n - ary$ relation R_r is defined as a subset of the Cartesian product of n sets

$$R_r \subseteq X_1 \times X_2 \times \cdots \times X_n.$$

Like a fuzzy set, fuzzy relations can be specified using its membership function

$$f(R_r) : X_1 \times X_2 \times \cdots \times X_n \to [0, 1],$$

where $[0,1]$ is the unit interval of the real numbers. However, in the theory of L-fuzzy relations it is often convenient to take a more general structure than the interval $[0,1]$. As a L-fuzzy relation we understand a function

$$R: X_1 \times X_2 \times \cdots \times X_n \to L,$$

which maps the Cartesian product of the sets $X_1 \times X_2 \times \cdots \times X_n$ into the complete Heyting algebra L.

Notice that L-fuzzy relations generalizes the notion of fuzzy relations since the unit interval is a complete Heyting algebra.

For finite relations it is often convenient to represent them as matrices. For example, if $R : X \times Y \to L$ with $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$ is an L-fuzzy relation, then the matrix representation of R is shown in Figure 2.5. The d in the second row of the matrix means that the second element of X is in relation R to the second element of Y by degree d.

$$R = \begin{pmatrix} 1 & a & b & 1 & 0 \\ c & d & 0 & 0 & 1 \\ 0 & 1 & c & a & b \\ 1 & 0 & a & c & d \end{pmatrix}$$

$L =$

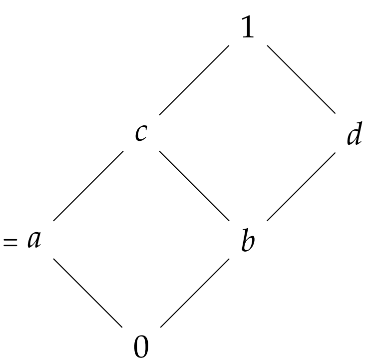

Figure 2.5: Matrix representation of L-fuzzy relations

2.2.1 Basic Operations and Properties

The known operations on regular relations can be generalized to L-fuzzy relations. Let $Q, R: A \to B$, $S: B \to C$ and $P : D \to B$ be L-fuzzy relations [17]. Then we can define the following operations:

The intersection between relations Q and R is defined as

$$(Q \cap R)(x,y) := Q(x,y) \wedge R(x,y).$$

Union between those relations is defined as

$$(Q \cup R)(x,y) := Q(x,y) \vee R(x,y).$$

The conversion of relation Q is defined as

$$Q^T(x,y) := Q(y,x).$$

The composition of relations Q and S is defined as

$$(Q \circ S)(x,z) := \bigvee_{y \in B} \big((Q(x,y) \wedge S(y,z)\big).$$

The inclusion of relations Q and R is defined as

$$Q \subseteq R \iff \forall x \in A, y \in B : Q(x,y) \leq R(x,y).$$

The least and greatest relations are defined as

$$⫫_{AB}(x,y) := 0, \top\!\!\top_{AB}(x,y) := 1.$$

The identity relation on the set A is defined as

$$\mathbb{I}_A(x_1,x_2) := \begin{cases} 1 & \text{iff } x_1 = x_2, \\ 0 & \text{otherwise.} \end{cases}$$

The left residual with $\to$ relative pseudocomplement is defined as

$$(Q/P)(x,u) := \bigwedge_y P(u,y) \to Q(x,y).$$

Let us give an example for each of the operations above. Consider the following matrix representations of relations Q, R, and S mentioned earlier, with the additional relation $P : A \to B$, where $A = \{x_1, x_2, x_3\}, B = \{y_1, y_2, y_3, y_4\}$ and $C = \{z_1, z_2, z_3, z_4\}$.

$$Q = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.3 & 0.2 & 1 & 0 \\ 0.8 & 1 & 0 & 0.2 \\ 0.5 & 0 & 0.4 & 0 \end{pmatrix} \end{array}, \quad R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.3 & 0 & 0.7 & 0 \\ 0.1 & 0.8 & 1 & 1 \\ 0.6 & 0.9 & 0.3 & 0.2 \end{pmatrix} \end{array},$$

$$P = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.4 & 0.9 & 1 & 0.2 \\ 1 & 1 & 0.5 & 0.6 \\ 0.8 & 0.2 & 0.7 & 0 \end{pmatrix} \end{array}, \quad S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.9 & 0 & 0.3 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \\ 0.8 & 0 & 0.7 & 1 \\ 0.4 & 0.2 & 0.3 & 0 \end{pmatrix} \end{array}.$$

Then the intersection and union between Q and R performed component-wise using join $\wedge$ and meet $\vee$ operations respectively, or in other words, minimum and maximum functions. The 0.7 and 1 in the first row of the both matrices below means $Q(x_1, y_3) \wedge R(x_1, y_3) = 1 \wedge 0.7 = 0.7$ and $Q(x_1, y_3) \vee R(x_1, y_3) = 1 \vee 0.7 = 1$ respectively.

$$Q \cap R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.3 & 0 & 0.7 & 0 \\ 0.1 & 0.8 & 0 & 0.2 \\ 0.5 & 0 & 0.3 & 0 \end{pmatrix} \end{array}, \quad Q \cup R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0.3 & 0.2 & 1 & 0 \\ 0.8 & 1 & 1 & 1 \\ 0.6 & 0.9 & 0.4 & 0.2 \end{pmatrix} \end{array}.$$

The composition (or product) of L-fuzzy relations is important in the theory of fuzzy sets, and it is done similarly to matrix multiplication in linear algebra. However, instead of multiplication and summing, we use join $\wedge$ and meet $\vee$ operations. The 0.5 in the third row of the composition matrix below is computed the following way:

$$
\begin{aligned}
(Q \circ S)(x_3, z_1) &= \bigvee_y \big(Q(x_3, y) \wedge S(y, z_1)\big) \\
&= (Q(x_3, y_1) \wedge S(y_1, z_1)) \vee (Q(x_3, y_2) \wedge S(y_1, z_2)) \\
&\quad \vee (Q(x_3, y_3) \wedge S(y_3, z_2)) \vee (Q(x_3, y_4) \wedge S(y_1, z_4)) \\
&= 0.5 \vee 0 \vee 0.4 \vee 0 \\
&= 0.5
\end{aligned}
$$

The conversion of relation R and composition of relations Q and S are shown below

$$
R^T = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array}
\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 0.3 & 0.1 & 0.6 \\ 0 & 0.8 & 0.9 \\ 0.7 & 1 & 0.3 \\ 0 & 1 & 0.2 \end{pmatrix} \end{array}, \quad
Q \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}
\begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{pmatrix} 0.8 & 0.2 & 0.7 & 1 \\ 0.8 & 1 & 0.8 & 0.4 \\ 0.5 & 0 & 0.4 & 0.4 \end{pmatrix} \end{array}.
$$

The least $\perp\!\!\!\perp_{AB}$ and greatest $\top\!\!\!\top_{AB}$ relations are as follows

$$
\perp\!\!\!\perp_{AB}(x, y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}
\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad
\top\!\!\!\top_{AB}(x, y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}
\begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array},
$$

Since each element of relation Q is less than each element of relation P, or $Q(x, y) \leq P(x, y)$ for all $x \in A, y \in B$, it is obvious that relation P includes relation Q, or $Q \subseteq P$.

The following properties of L-fuzzy relations and their proofs are adopted from [17].

Theorem 2.10. *Let L be a complete Heyting Algebra, then the following properties are true for all L-fuzzy relations $Q, Q', Q_i : A \to B, R, R_i : B \to C$, and $S : C \to D$ for $i \in \mathbb{I}$.*

(1) $Q \circ \mathbb{I}_B = Q$ *and* $\mathbb{I}_B \circ R = R$,

(2) $(Q \circ R) \circ S = Q \circ (R \circ S)$,

(3) $(Q \cap Q')^T = Q^T \cap Q'^T$,

(4) $(Q \circ R)^T = R^T \circ Q^T$,

(5) $(Q^T)^T = Q$,

(6) $Q \circ (\bigcap_{i \in \mathbb{I}} R_i) \subseteq \bigcap_{i \in \mathbb{I}} (Q \circ R_i)$ *and* $(\bigcap_{i \in \mathbb{I}} Q_i) \circ R \subseteq \bigcap_{i \in \mathbb{I}} (Q_i \circ R)$,

(7) $Q \circ R \cap T \subseteq Q \circ (R \cap Q^T \circ T)$,

(8) $Q \circ \perp\!\!\perp_{(BC)} = \perp\!\!\perp_{AC}$,

(9) $Q \circ (\bigcup_{i \in \mathbb{I}} R_i) = \bigcup_{i \in \mathbb{I}} (Q \circ R_i)$ *and* $(\bigcup_{i \in \mathbb{I}} Q_i) \circ R = \bigcup_{i \in \mathbb{I}} (Q_i \circ R)$,

Proof. (1) Let us breakdown the definitions of composition and identity relations

$$\begin{aligned}
(Q \circ \mathbb{I}_B)(x,z) &= \bigvee_{y \in B} \big(Q(x,y) \wedge \mathbb{I}_B(y,z)\big) \\
&= Q(x,z) \wedge 1 \\
&= Q(x,z).
\end{aligned}$$

(2) Since L is a complete Heyting Algebra, it also satisfies the first infinite distribution law. Then we can use the first equation from Definition 1.8 and show the following

$$\begin{aligned}
((Q \circ R) \circ S)(x,u) &= \bigvee_{z \in C} \big((Q \circ R)(x,z) \wedge S(z,u)\big) \\
&= \bigvee_{z \in C} \left(\Big(\bigvee_{y \in B} \big(Q(x,y) \wedge R(y,z)\big)\Big) \wedge S(z,u)\right) \\
&= \bigvee_{z \in C} \bigvee_{y \in B} \big(Q(x,y) \wedge R(y,z) \wedge S(z,u)\big) \\
&= \bigvee_{y \in B} \bigvee_{z \in C} \big(Q(x,y) \wedge R(y,z) \wedge S(z,u)\big) \\
&= \bigvee_{y \in B} \left(Q(x,y) \wedge \Big(\bigvee_{z \in C} \big(R(y,z) \wedge s(z,u)\big)\Big)\right) \\
&= \bigvee_{y \in B} \big(Q(x,y) \wedge (R \circ S)(y,u)\big) \\
&= \big(Q \circ (R \circ S)(x,u)\big).
\end{aligned}$$

(3) From the basic intersection and conversion operations on relations mentioned earlier, the next derivative is true.

$$\begin{aligned}(Q \cap Q')^T(x,y) &= (Q \cap Q')(y,x)\\ &= Q(y,x) \wedge Q'(y,x)\\ &= Q^T(x,y) \wedge Q'^T(x,y)\\ &= (Q^T \cap Q'^T)(x,y).\end{aligned}$$

(4) From the composition and conversion operations on relations, the next derivative is clear

$$\begin{aligned}(Q \circ R)^T(x,z) &= (Q \circ R)(z,x)\\ &= \bigvee_{y \in B} \big(Q(z,y) \wedge R(y,x)\big)\\ &= \bigvee_{y \in B} \big(R^T(x,y) \wedge Q^T(y,z)\big)\\ &= (R^T \circ Q^T)(x,z).\end{aligned}$$

(5) The following from the conversion definition.

$$\begin{aligned}(Q^T)^T(x,y) &= Q^T(y,x)\\ &= Q(x,y)\end{aligned}$$

(6) Again, using the composition operation on relations, we can show the following is true.

$$\begin{aligned}\Big(Q \circ \big(\bigcap_{i \in \mathbb{I}} R_i\big)\Big)(x,z) &= \bigvee_{y \in B} \Big(Q(x,y) \wedge \big(\bigcap_{i \in \mathbb{I}} R_i\big)(y,z)\Big)\\ &= \bigvee_{y \in B} \big(Q(x,y) \wedge \bigwedge_{i \in \mathbb{I}} R_i(y,z)\big)\\ &= \bigvee_{y \in B} \bigwedge_{i \in \mathbb{I}} \big(Q(x,y) \wedge R_i(y,z)\big)\\ &\leq \bigwedge_{i \in \mathbb{I}} \bigvee_{y \in B} \big(Q(x,y) \wedge R_i(y,z)\big)\\ &= \bigwedge_{i \in \mathbb{I}} (Q \circ R_i)(x,z)\\ &= \big(\bigcap_{i \in \mathbb{I}} (Q \circ R_i)\big)(x,z)\end{aligned}$$

(7) Since L is complete upward-distributive, and using composition and intersec-

tion operations, the following computation is true.

$$\begin{aligned}
(Q \circ R \cap T)(x,z) &= (Q \circ R)(x,z) \wedge T(x,z) \\
&= \Big(\bigvee_{y \in B} \big(Q(x,y) \wedge R(y,z)\big)\Big) \wedge T(x,z) \\
&= \bigvee_{y \in B} \big(Q(x,y) \wedge R(y,z) \wedge T(x,z)\big) \\
&= \bigvee_{y \in B} \big(Q(x,y) \wedge R(y,z) \wedge Q(x,y) \wedge T(x,z)\big) \\
&\le \bigvee_{y \in B} \Bigg(Q(x,y) \wedge R(y,z) \wedge \Big(\bigvee_{x' \in A} \big(Q(x',y) \wedge T(x',z)\big)\Big)\Bigg) \\
&= \bigvee_{y \in B} \big(Q(x,y) \wedge R(y,z) \wedge (Q^T \circ T)(y,z)\big) \\
&= \bigvee_{y \in B} \big(Q(x,y) \wedge (R \cap Q^T \circ T)(y,z)\big) \\
&= \big(Q \circ (R \cap Q^T \circ T)\big)(y,z).
\end{aligned}$$

(8) The next proof comes from the composition and least element $⫫_{BC}(x,z) = 0$

$$\begin{aligned}
(Q \circ ⫫_{BC})(x,z) &= \bigvee_{y \in B} \big(Q(x,y) \wedge ⫫_{BC}(y,z)\big) \\
&= 0 \\
&= ⫫_{AC}(x,z).
\end{aligned}$$

(9) The proof can be found analogously to (6), in which we use the union instead of the intersection operation. Also, it means that $\circ$ is a lower adjoint of a triple of residuated operations. The upper left and lower right adjoints are denoted in [17] by S/R and $Q\backslash S$.
(10) Proof of the definition of the residuals can be found in [17]. □

2.2.2 Crispness and Arrow Operations

As mentioned earlier, L-fuzzy relations are a generalized form of regular relations which includes least element 0 and greatest element 1. We call an L-fuzzy relation $0-1$ crisp if it uses only those elements. In other words, an L-fuzzy relation Q is called $0-1$ crisp iff $Q(x,y) = 0$ or $Q(x,y) = 1$ for all x and y. The crisp relation may be identified with regular relations, i.e. with relations over the truth values $\mathbb{B}$. It is clear that the set of $0-1$ relations is closed under all operations defined above [17].

Another important class of relations are scalar relations. Scalar relations were first introduced in [5] and [8]. Scalar relations are relations $R\colon A \to B$ fulfilling $\top_{AA};R;\top_{BB} = R$ [17]. Figure 2.6 shows two scalar relations as matrices in which u is an element of the underlying lattice L. They can be characterized as diagonal matrices with one element from L on the diagonal.

$$\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Scalar relation $\qquad$ $\perp_{AA}$ as a scalar relation

Figure 2.6: Two scalar relations.

Definition 2.11. *For some $u \in L$ and $\forall x, y \in A$, a scalar $\alpha\colon A \to A$ is*

$$\alpha_A^u(x,y) = \begin{cases} u & \textit{iff } x = y, \\ 0 & \textit{otherwise} \end{cases}$$

The scalar relation on the left of Figure 2.6 is the scalar α_A^u.

L-fuzzy relations can be decomposed as a regular relations, and conversely, from regular relations we can synthesize L-fuzzy relations. This decomposition and synthesis can be done by utilizing the α-cuts of L-fuzzy relations.

Definition 2.12. *Let $\alpha \in L$ and $R : A \to B$. Then the α-cut of an L-fuzzy relations R is a* $0-1$ *crisp relation defined by the following.*

$$R_\alpha(x,y) := \begin{cases} 1 & \textit{iff } R(x,y) \geq \alpha, \\ 0 & \textit{otherwise} \end{cases}$$

If we choose $\alpha = 0.6$ in the example of Section 2.2.1, then we obtain $R_{0.6}$ as the following matrix.

$$R_{0.6}(x,y) = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

where all pairs of relation R that are greater or equal to 0.6 are increased to 1, and others decreased to 0. From a theoretical point of view it is sufficient to introduce just two additional operations in order to obtain all α-cut operations. These new operations are called *up-arrow* (support) and *down-arrow* (kernel). They are defined as follows.

$$R^{\uparrow}(x,y) := \begin{cases} 1 & \text{iff } R(x,y) \neq 0, \\ 0 & \text{otherwise} \end{cases} \qquad R^{\downarrow}(x,y) := \begin{cases} 1 & \text{iff } R(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Arrow operations are dual to each other. The up-arrow operation increases all membership degrees that are higher than 0, and the down-arrow operation decreases those less than 1. Both operations form $0-1$ crisp relations, which are least and greatest $0-1$ crisp relations respectively. For the same example above, arrow operations on relation R will be the following matrices.

$$R^{\uparrow}(x,y) = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}, \quad R^{\downarrow}(x,y) = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Lastly we can introduce the definition of arbitrary α-cuts.

Definition 2.13. *Let $R : A \to A$ be a relation and $\alpha^u : A \to A$ be a scalar with some $u \in L$. Then the relational expression $(\alpha^u \backslash R)^{\downarrow}$ computes the α-cut of R.*

As an example to above definition consider arbitrary relation R in Figure 2.7 with the scalar α^u, the α-cut of R is as follows.

$$(\alpha \backslash R)^{\downarrow} = \left(\begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \backslash \begin{pmatrix} 1 & c & 0 \\ u & u & 0 \\ 0 & b & 1 \end{pmatrix} \right)^{\downarrow} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & b & 1 \end{pmatrix}^{\downarrow} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & c & 0 \\ u & u & 0 \\ 0 & b & 1 \end{pmatrix}$$

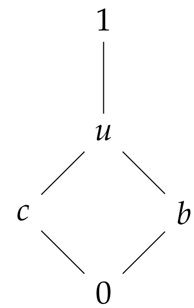

Figure 2.7: Arbitrary relation R.

The above operations have their own properties, and they are grouped in the next lemma adopted from [17].

Lemma 2.14. *Let L be a complete Heyting Algebra. Then the following properties are true for all L-fuzzy relations $Q, R\colon A \to B$ and $S\colon B \to C$.*

(1) Q *is* $0-1$ *crisp iff* $Q^{\uparrow} = Q$ *iff* $Q^{\downarrow} = Q$,

(2) $(R^T \circ S^{\downarrow})^{\uparrow} = R^{\uparrow T} \circ S^{\downarrow}$,

(3) $(Q \cap R^{\downarrow})^{\uparrow} = Q^{\uparrow} \cap R^{\downarrow}$,

(4) *if* $u \neq 0$, *then* $\alpha_A^{u\uparrow} = \mathbb{I}_A$,

(5) $Q_u = (\alpha_A^u \backslash Q)^{\downarrow}$.

Proofs for above properties of operations can be found in [17].

2.2.3 T-Norm Based Operations

The definition of t-norm was first used in the study of probabilistic metric spaces [15] where triangular inequalities were enlarged using the theory of t-norm. Then its area of use extended to the fuzzy set theory, where t-norm was used for the intersection and union of fuzzy sets [7] .

In this section we will introduce new arbitrary operations $*$, $\overset{*}{;}$ and a set of axioms, so that operations $*$ and $\overset{*}{;}$ will became abstract versions of the meet and composition operations based on the t-norm we mentioned earlier.

Before we go any further with the set of axioms, let us give some basic definitions of generalized t-norms between $[0,1]$ to arbitrary complete lattices [19].

Definition 2.15. *If L is complete lattice, then* $\langle L, *, 1\rangle$ *is called an Abelian monoid iff*

(1) $*$ is associative and commutative with neutral element 1,

(2) $*$ is monotonic in both parameters.

The weakest t-norm with monoid operation $\circledast$ is given by

$$x \circledast y := \begin{cases} x & \text{iff } y = 1, \\ y & \text{iff } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.16. *Let $\langle L, \subseteq, *, 1\rangle$ be an Abelian monoid. Then the following is true for all $x, y \in L$.*

(1) $x * y \subseteq x$ and $x * y \subseteq y$,

(2) $x * 0 = 0 * x = 0$,

(3) $x * (y \wedge z) \subseteq x * y \wedge x * z$ and $(x \wedge y) * z \subseteq x * z \wedge y * z$,

(4) $x \circledast y \subseteq x * y \subseteq x \wedge y$,

(5) $* = \wedge$ iff $*$ is idempotent.

In this work, we use the same $*$ sign to denote the operation on scalars and $*$ based meet operation on relations. In the case of L-relations, it is clear from the next component-wise definition of the meet and composition operators based on $*$ on scalar relations.

Definition 2.17. *Let $Q, R : A \to B$ and $S : B \to C$ are L-relations. Then we define*

$$(Q * R)(x, y) = Q(x, y) * R(x, y),$$

$$(Q \overset{*}{;} S)(x, z) = \bigvee_{y \in B} Q(x, y) * S(y, z).$$

As we mentioned earlier, $*$ operators from the left side of the above equations are not the same as $*$ operators from the right side.

If the $*$ operation distributes over arbitrary unions, then we call $*$ continuous. The continuous condition, among other things, ensures a good behavior of the residual derived from the meet operation. Continuous t-norms $*$ have a unique residuum, which is a binary operation $\overset{*}{\twoheadrightarrow}$ such that for all x, y and z in $[0, 1]$, $x * y \leq z$ iff $x \leq y \overset{*}{\twoheadrightarrow} z$ [19].

Let us start with the $*$ operation. The following two lemmas adopted from [19], and their proofs will be discussed in the next chapter.

Lemma 2.18. *Let $Q, R : A \to B$ and $S : B \to C$ are L-relations. Then the following is true.*

(1) $*$ *is associative and commutative,*

(2) $*$ *is continuous,monotonic and* $Q * R \sqsubseteq S \Leftrightarrow Q \sqsubseteq R \mathrel{-\!\!*\!\!\twoheadrightarrow} S$,

(3) $(Q * R)^T = Q^T * R^T$,

(4) $Q * R^{\downarrow} = Q \cap R^{\downarrow}$.

Now, let us consider $\overset{*}{⨟}$ operation, which we assume binds tighter than $*$ operation.

Lemma 2.19. *Let $P, Q : A \to B$ and $R, S : B \to C$ and $O : A \to C$ are L-relations. Then the following is true.*

(1) $\overset{*}{⨟}$ *is associative,*

(2) $\overset{*}{⨟}$ *is continuous, monotonic, and* $Q \overset{*}{⨟} R \sqsubseteq S \Leftrightarrow Q \sqsubseteq S \mathrel{\overset{*}{/}} R$,

(3) $(Q \overset{*}{⨟} R)^T = Q^T \overset{*}{⨟} R^T$,

(4) $Q \overset{*}{⨟} R^{\downarrow} = Q; R^{\downarrow}$

(5) $(P * Q) \overset{*}{⨟} (R * S) \sqsubseteq P \overset{*}{⨟} R * Q \overset{*}{⨟} S$,

(6) $Q; R * O \sqsubseteq Q; (R \cap Q^T \overset{*}{⨟} S)$,

(7) $Q \overset{*}{⨟} R * O \sqsubseteq Q; (R * Q^T \overset{*}{⨟} O)$,

(8) $(P * Q) \overset{*}{⨟} R * O \sqsubseteq P \overset{*}{⨟} (R * Q^T \overset{*}{⨟} O)$.

where ; is composition that binds tighter than lattice operations.

2.3 Categories of Relations

In this section, we will discuss the various categorical paths that requires the complete Heyting algebra to formalize binary relations between two different sets.

2.3.1 Categories

Category theory offers a framework that is sufficiently expressive for many concepts, constructions and theories in mathematics, which often makes certain hidden aspects of the theory explicit and allows new viewpoints.

The following definitions are basic notions from category theory as defined in [17]. An extensive theory can be found in [1].

Definition 2.20. *Category $\mathcal{C}$ consists of*

(1) a class of objects $\mathrm{Obj}_{\mathcal{C}}$*,*

(2) for every pair of objects A and B a class of morphisms $\mathcal{C}[A,B]$,

(3) an associative binary operation ; mapping each pair of morphisms f in $\mathcal{C}[A,B]$ and g in $\mathcal{C}[B,C]$ to a morphism $f;g$ in $\mathcal{C}[A,C]$,

(4) for every object A a morphism $\mathbb{I}_A$ such that for all f in $\mathcal{C}[A,B]$ and g in $\mathcal{C}[C,A]$ we have $\mathbb{I}_A;f = f$ and $g;\mathbb{I}_A = g$.

Let us give some common categories with its objects and morphisms.

1. **Set** - category of sets, where sets are objects, and functions between them are morphisms.

2. **Rel** - category of relations, where sets are objects and relations are morphisms.

3. L**-Rel** - category of L -fuzzy relations, where the objects are non-empty sets and the morphisms are L -fuzzy relations.

4. **Vect** - category of vector spaces and linear mapping as objects and morphisms respectively.

5. **PO** - category of posets with objects and morphisms as posets and monotone functions respectively.

In this work, the morphism f in $\mathcal{C}[A,B]$ is denoted by $f : A \to B$, similar to a relation $R : A \to B$, where relations will be morphisms.

Most literature in category theory use diagrams to visualize equations in categories. The nodes of the graph is the objects of a category, and an arrow between nodes represents a morphism between the corresponding objects. It is understood that a path in the graph represents the composition of the arrows along the path. Please note that this representation already uses the associativity of composition, since a path of length more than 2 has more than one potential interpretation as a sequence of compositions. All of those different interpretations are equivalent due to the associativity of composition.

Any two paths with the same start and end nodes are understood as an equality of the two compositions along the two paths. For example, the left diagram in Figure 2.8 shows that the composition of f and g is equal to h, i.e. $f;g = h$. The right diagram shows the identity law of categories.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "\mathbb{I}_B"] \\ & B \end{tikzcd}$$

Figure 2.8: Visual representation of categories.

However, we will often use diagrams to indicate the typing, i.e., the source and target, of relations appearing in a relational term. In this case no equation is implied by the corresponding diagram.

The following Lemma defines L **-Rel**, the category of L-fuzzy relations we mentioned earlier as an example of categories.

Definition 2.21. *Let L be a complete Heyting algebra. Then the structure satisfying that*

(1) The objects are nonempty sets,

(2) A relation $R : A \to B$ is a function $A \times B \to L$,

together with composition of L-relations and the identity relation forms a category.

Another important notion in category theory is the notion of a *functor*. Functors are homomorphisms between categories.

Definition 2.22. *Functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a mapping that converts objects to objects, and morphisms to morphisms, denoted by pair of functions $(F_{\mathrm{Obj}}, F_{\mathrm{Mor}})$ where*

(1) $F_{\mathrm{Mor}}(f);F_{\mathrm{Mor}}(g) = F_{\mathrm{Mor}}(f;g)$ *for all morphisms* $f : A \to B$ *and* $g : B \to C$ *and objects* $A, B,$ *and* C *in* $\mathcal{C}_1$,

(2) $F_{\mathrm{Mor}}(\mathbb{I}_A) = \mathbb{I}_{F_{\mathrm{Obj}}(A)}$ *for all objects* A *in* $\mathcal{C}_1$.

A functor F is called *injective* iff F_{Mor} is injective, and *full* iff for all objects A and B and morphisms $g : F_{\mathrm{Mor}}(A) \to F_{\mathrm{Mor}}(B)$ there is a morphism $f : A \to B$ such that $F_{\mathrm{Mor}}(f) = g$.

The Simplest example of functors is the identity functor $\mathbb{I}_{\mathcal{C}_1} : \mathcal{C}_1 \to \mathcal{C}_1$, which assigns to each object of the category itself this object and to each morphism itself this morphism.

2.3.2 Allegories

In theory of categories, an *allegory* is a category with some structural elements of the category of sets and binary relations between them. Allegories can be used as an abstraction of the categories of relations. The following is a general definition of allegories as defined in [17].

Definition 2.23. *An allegory* $\mathcal{R}$ *is a category that satisfies the following.*

(1) *For all objects* A *and* B*, the class of morphisms* $\mathcal{R}[A, B]$ *is a lower semilattice. The meet and the induced ordering are denoted by* $\sqcap, \sqsubseteq$ *respectively. The elements in* $\mathcal{R}[A, B]$ *are called relations.*

(2) *There is a monotone operation* $\breve{}$ *(converse operation) such that for all relations* $Q, R : A \to B$ *and* $S : B \to C$ *the following is true:*

$$(Q;S)\breve{} = S\breve{};Q\breve{}\,and(Q\breve{})\breve{} = Q.$$

(3 *For all relations* $Q : A \to B$ *and* $R, S : B \to C$ *the following is true:*

$$Q;(R \sqcap S) \sqsubseteq Q;R \sqcap Q;S.$$

(4) *For all relations* $Q : A \to B$*,* $R : B \to C$ *and* $S : A \to C$ *the following is true (modular law):*

$$Q;R \sqcap S \sqsubseteq Q;(R \sqcap Q\breve{};S).$$

The following definition provides an important class of relations which is given by mappings [17].

Definition 2.24. *Let $\mathcal{R}$ be an allegory and relation $Q : A \to B$. Then we call*

(1) Q univalent iff $Q^{\smile}; Q \sqsubseteq \mathbb{I}_B$,

(2) Q total iff $\mathbb{I}_A \sqsubseteq Q; Q^{\smile}$,

(3) Q a map iff Q is univalent and total,

(4) Q injective iff $Q^{\smile}$ is univalent,

(5) Q surjective iff $Q^{\smile}$ is total,

(6) Q bijective iff $Q^{\smile}$ is a map,

(7) Q a bijective iff Q is a bijective map.

The next lemmas show some properties of allegories and univalent relations as described in [17], and their proofs can be found in the same reference.

Lemma 2.25. *Let $\mathcal{R}$ be an allegory, $Q : A \to B, R : A \to C, S : D \to B$ be relations, and $f : B \to C$ and $g : A \to D$ be mappings. Then the followings are true:*

(1) $Q; f \sqsubseteq R$ iff $Q \sqsubseteq R; f^{\smile}$,

(2) $g^{\smile}; Q \sqsubseteq$ iff $Q \sqsubseteq g; S$.

Lemma 2.26. *Let $\mathcal{R}$ be an allegory, relation $Q : A \to B$ be univalent, and $R, S : B \to C, T : C \to A, U : C \to B$. Then we have*

(1) $Q; (R \sqcap S) = Q; R \sqcap Q; S$,

(2) $T; Q \sqcap U = (T \sqcap U; Q^{\smile}); Q$.

The first example of an allegory is category **Rel** of sets and relations. Recall that objects of this allegory are sets, and the morphism $A \to B$ is a binary relation between A and B.

Another example is the category of L-fuzzy relations L**-Rel** with meet and converse operations. If we define a distributive structure by replacing lower semilattices in allegories to distributive lattices as the order, then the allegory L**-Rel** of L-fuzzy relations becomes a distributive allegory. The following definition of distributive allegory is adopted from [17].

Definition 2.27. *A distributive allegory $\mathcal{R}$ is an allegory, and for all relations $Q : A \to B$, $R, S : B \to C$*

(1) *The classes* $\mathcal{R}[A, B]$ *are distributive lattices with a least element. Union and the least element are denoted by* $\sqcup$ *and* $\perp\!\!\!\perp_{AB}$*,*

(2) $Q; \perp\!\!\!\perp_{BC} = \perp\!\!\!\perp_{AC}$*,*

(3) $Q; (R \sqcup S) = Q; R \sqcup Q; S$*.*

By adding the residual operation of a relation algebra to a distributive structure, we can get a division allegory as defined in [17].

Definition 2.28. *A division allegory* $\mathcal{R}$ *is a distributive allegory such that composition* ; *has an upper left adjoint, i.e., for all relations* $R : B \to C$ *and* $S : A \to C$ *there is a relation* $S/R : A \to B$ *(called the left residual of* S *and* R*) such that for all* $Q : A \to B$ *the following holds*

$$Q; R \sqsubseteq S \iff Q \sqsubseteq S/R.$$

From the component-wise definition of residuals (Theorem 2.10 (10)), we can say L-**Rel** is a division allegory with S/R as residual.

We can also define an upper right adjoint $(S^{\smile}/Q^{\smile})^{\smile}$ for ;, and it will be denoted by $Q \backslash S$. The symmetric version of the residuals defined as $syq(Q, R) := (Q \backslash R) \sqcap (Q^{\smile}/R^{\smile})$ and is called the *symmetric quotient* in [17].

The next lemma provides some basic properties of the symmetric quotient, and all the proofs can be found in [17].

Lemma 2.29. *Let* $\mathcal{R}$ *be a division allegory,* $Q : A \to B, R : A \to C, S : A \to D$ *be relations, and* $f : D \to A$ *be mapping. Then the following are true*

(1) $f; syq(Q, R) = syq(Q; f^{\smile}, R)$*,*

(2) $syq(Q, R)^{\smile} = syq(R, Q)$*,*

(3) $syq(Q, R); syq(R, S) \sqsubseteq syq(Q, S)$*.*

2.3.3 Dedekind Categories

In this section we will discuss an important categorical structure which requires the order structure to be complete. Basically, Dedekind category is a division allegory, where distributive lattices $\mathcal{R}[A, B]$ are complete Heyting algebras for all objects A and B. The full definition of Dedekind categories as defined in [19] is as follows.

Definition 2.30. *A Dedekind category $\mathcal{R}$ is a category satisfying the following for all relations $Q : A \to B$, $R : B \to C$ and $S : A \to C$*

(1) For all objects A and B the collection $\mathcal{R}[a, B]$ is a complete Heyting Algebra. Meet, join, the inducing ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq$ $, \perp\!\!\!\perp_{AB}$ and π_{AB} respectively.

(2) There is a monotone operation $\smile$ (converse operation) mapping a relation $Q : A \to B$ to $Q^\smile : B \to A$ such that the following are true

$$(Q; R)^\smile = R^\smile; Q^\smile,$$

$$(Q^\smile)^\smile = Q.$$

(3) Holds the following modular law

$$(Q; R) \sqcap S \sqsubseteq Q; (R \sqcap (Q^\smile; S)).$$

(4) There is a relation $S/R : A \to B$ (the left residual of S and R) such that for all $X : A \to B$ the following is true

$$X; R \sqsubseteq S \Longleftrightarrow X \sqsubseteq S/R.$$

As defined in (4), it is possible to formulate the right residual of R and S by $R\backslash S = (S/R)^\smile$, and it is identified as $R; X \sqsubseteq S \longleftrightarrow X \sqsubseteq R\backslash S$.

Next, we need to state some properties of Dedekind categories. Let us start with properties of the greatest elements defined as in [17].

Lemma 2.31. *Let $\mathcal{R}$ be a Dedekind category. Then for all objects A and B in $\mathcal{R}$ the following are true.*

(1) $\pi_{AB}^\smile = \pi_{BA}$,

(2) $\pi_{AA}; \pi_{AB} = \pi_{AB}; \pi_{BB} = \pi_{AB}$,

(3) $\pi_{AB} = \pi_{AB}; \pi_{BA}; \pi_{AB}$.

The proof of above lemma is an easy exercise and can be found in [17].

The next lemma gives a few other properties valid in Dedekind categories adopted from [17, 19].

Lemma 2.32. *Let $\mathcal{R}$ be a Dedekind category, $Q : A \to B, R : B \to C, S : A \to D$, and $T : D \to C$. Then the following are true*

(1) $(Q \sqcap S;\pi_{DB});R = Q;R \sqcap S;\pi_{DC}$,

(2) $\mathbb{I}_A \sqcap Q;Q^{\smile} = \mathbb{I}_A \sqcap Q;\pi_{BA} = \mathbb{I}_A \sqcap \pi_{AB};Q^{\smile}$.

The first formula has its dual equation by reversing the direction of composition and displays as follows $Q;(R \sqcap \pi_{BD};T) = Q;R \sqcap \pi_{AD};T$. For the full proof of the above lemma see [17], or it can be done similarly to properties of L-fuzzy relations in the Section 2.1.

Other important properties of Dedekind categories are related to partial identities, and they are shown in next lemma [19].

Lemma 2.33. *Let $\mathcal{R}$ be a Dedekind category, $S : A \to A$ partial identities, and $R : C \to A, U : A \to B$. Then the following are true*

(1) $S = \mathbb{I}_A \sqcap S;\pi_{AA} = \mathbb{I}_A \sqcap \pi_{AA};S$,

(2) $R;S = R \sqcap \pi_{CA};S$ *and* $S;U = U \sqcap S;\pi_{AB}$,

(3) $S^{\smile} = S$.

The proof of the above lemma is an easy exercise, and can be found in [17, 19].

Since L-**Rel**$[A,B]$ is a complete Heyting algebra, L-**Rel** forms a Dedekind category. However, this category cannot satisfy one of the important properties of L-fuzzy relations, particularly, the $0-1$ crispness. Therefore, by the definition of categories, the construction of new categories from a given one, we need to define a new category that covers the crispness property of L-fuzzy relations. Adding two arrow operations, the up-arrow $\uparrow$ and the down-arrow $\downarrow$, gives us a new structure called the *Arrow category* [18].

2.3.4 Arrow Categories

This section will introduce an extended Dedekind category which defines a suitable algebraic theory of L-fuzzy relations. It is done by adding two operations, up-arrow $\uparrow$ and down-arrow $\downarrow$, mentioned earlier.

The abstract definition of arrow categories is as follows [17].

Definition 2.34. *An arrow category $\mathcal{A}$ is a Dedekind category with $\pi_{AB} \neq \perp\!\!\!\perp_{AB}$ for all A, B and operations $\uparrow, \downarrow$ satisfying the following. For all $Q, R : A \to B, S : B \to A$, and $T : B \to C$*

(1) $R^{\uparrow}, R^{\downarrow} : A \to B$,

(2) $(\uparrow, \downarrow)$ *is a Galois correspondence, i.e.,* $Q^{\uparrow} \sqsubseteq R$ *iff* $Q \sqsubseteq R^{\downarrow}$,

(3) $(S^{\smile}; T^{\downarrow})^{\uparrow} = S^{\uparrow\smile}; T^{\downarrow}$,

(4) $(Q \sqcap R^{\downarrow})^{\uparrow} = Q^{\uparrow} \sqcap R^{\downarrow}$,

(5) If $\alpha_A \neq \perp\!\!\!\perp_{AA}$ *is a non-zero scalar then* $\alpha_A^{\uparrow} = \mathbb{I}_A$.

Now we can show the algebraic theory of L-fuzzy relations using the next lemma as defined in [17].

Lemma 2.35. *Let L be a complete Heyting algebra with* $0 \neq 1$*. Then* $L - \mathrm{Rel}$ *together with* $\uparrow$ *and* $\downarrow$ *is an arrow category.*

The proof of the above lemma is an easy exercise. The formula (1) is obvious, and the formulas (2) – (5) were already shown in Lemma 2.14 (1) – (4).

From Lemma 2.14 we can define crispness in an arbitrary arrow category as follows [17].

Definition 2.36. *A relation* $R : A \to B$ *of an arrow category* $\mathcal{A}$ *is called crisp iff* $R^{\uparrow} = R$*. The crisp fragment* $\mathcal{A}^{\uparrow}$ *of* $\mathcal{A}$ *is defined as the collection of all crisp relations.*

The following lemma summarizes some basic properties of arrow categories as shown in [17].

Lemma 2.37. *Let* $\mathcal{A}$ *be an arrow category and* $Q, R : A \to B, S : B \to C, T : A \to C$*. Then the following are true*

(1) $\mathbb{I}_A^{\uparrow} = \mathbb{I}_A \neq \perp\!\!\!\perp_{AA}$,

(2) $R^{\downarrow\uparrow} = R^{\downarrow}$,

(3) $R^{\uparrow\downarrow} = R^{\uparrow}$,

(4) $\uparrow, \downarrow$ *are closure and kernel operations respectively,*

(5) $R = R^{\uparrow}$ *iff* $R^{\downarrow} = R^{\uparrow}$ *iff* $R^{\downarrow} = R$,

(6) $\perp\!\!\!\perp_{AB}^{\uparrow} = \perp\!\!\!\perp_{AB}$ *and* $\top\!\!\!\top_{AB}^{\downarrow} = \top\!\!\!\top_{AB}$,

(7) $(R^{\smile}; S^{\uparrow})^{\uparrow} = R^{\uparrow\smile}; S^{\uparrow}$,

(8) $R^{\smile\uparrow} = R^{\uparrow\smile}$ *and* $R^{\smile\downarrow} = R^{\downarrow\smile}$,

(9) $(R;S^{\downarrow})^{\uparrow} = R^{\uparrow};S^{\downarrow}$ *and* $(R^{\downarrow};S)^{\uparrow} = R^{\downarrow};S^{\uparrow}$,

(10) $(R;S^{\uparrow})^{\uparrow} = R^{\uparrow};S^{\uparrow}$ *and* $(R^{\uparrow};S)^{\uparrow} = R^{\uparrow};S^{\uparrow}$,

(11) $(Q \sqcap R^{\uparrow})^{\uparrow} = Q^{\uparrow} \sqcap R^{\uparrow}$,

(12) *For all nonzero ideal relations* $J^{\uparrow} = \pi_{AB}$ *holds,*

(13) $R^{\downarrow} \to Q^{\uparrow} = (R \to Q^{\uparrow})^{\downarrow} \sqsubseteq (R \to Q)^{\downarrow}$ *and* $(R \to Q)^{\uparrow} \sqsubseteq R^{\uparrow} \to Q^{\downarrow}$,

(14) $Q^{\uparrow}\backslash T^{\downarrow} = (Q^{\uparrow}\backslash T)^{\downarrow} \sqsubseteq (Q\backslash T)^{\downarrow}$ *and* $(Q\backslash T)^{\uparrow} \sqsubseteq Q^{\downarrow}\backslash T^{\uparrow}$,

(15) $T^{\downarrow}/S^{\uparrow} = (T/S^{\uparrow})^{\downarrow} \sqsubseteq (T/S)^{\downarrow}$ *and* $(T/S)^{\uparrow} \sqsubseteq T^{\uparrow}/S^{\downarrow}$.

The full proof of above lemma can be found in [17].

The next lemma shows a collection of closure properties of the class of crisp relations, and the proof can be found in [17].

Lemma 2.38. *Let $\mathcal{A}$ be an arrow category and $Q_i, Q, T : A \to B$ for $i \in I, R : A \to C$, and $S : B \to C$ crisp relations. Then the following are true*

(1) $\bigsqcup_{i \in I} Q_i$ *and* $\bigsqcap_{i \in I} Q_i$ *are crisp,*

(2) $Q^{\smile}$ *is crisp,*

(3) $Q;S$ *is crisp,*

(4) R/S *and* $Q\backslash R$ *are crisp,*

(5) $Q \to T$ *is crisp.*

Lemma 2.38 (3) gives the following inclusion axiom.

Lemma 2.39. *Let $\mathcal{A}$ be an arrow category and $Q : A \to B$ and $R : B \to C$ crisp relations. Then the following holds $Q^{\downarrow};R^{\downarrow} \sqsubseteq (Q;R)^{\downarrow}$.*

Proof. It follows from above lemma.

$$\begin{aligned} Q^{\downarrow};R^{\downarrow} &= (Q^{\downarrow};R^{\downarrow})^{\downarrow} && \text{Lemma 2.38} \\ &\sqsubseteq (Q;R)^{\downarrow} \end{aligned}$$

□

There are some other properties of an arrow categories and they are collected in the following lemma.

Lemma 2.40. *Let $\mathcal{A}$ be an arrow category and $Q : A \to B$ and $R : C \to B$ crisp relations. $f : A \to C$ is a crisp map. Then the following holds:*

(1) $f;Q^{\downarrow} = (f;Q)^{\downarrow}$,

(2) $f;syq(Q,R)^{\downarrow} = syq(Q;f^{\smile},R)^{\downarrow}$,

(3) $f;(Q\backslash R)^{\downarrow} = (Q;f^{\smile}\backslash R)^{\downarrow}$,

(4) $(Q\backslash R)^{\downarrow};f^{\smile} = (Q\backslash R;f^{\smile})^{\downarrow}$.

Proof. (1) Since f is crisp we obtain

$$\begin{aligned} f^{\smile};(f;Q)^{\downarrow} &\sqsubseteq (f^{\smile};f;Q)^{\downarrow} && \text{Lemma 2.39 and } f^{\smile} \text{ is crisp} \\ &\sqsubseteq Q^{\downarrow} && f \text{ is univalent} \end{aligned}$$

This implies

$$\begin{aligned} (f;Q)^{\downarrow} &\sqsubseteq f;f^{\smile};(f;Q)^{\downarrow} && f \text{ is total} \\ &\sqsubseteq f;Q^{\downarrow}. \end{aligned}$$

(2) Consider the following computation:

$$\begin{aligned} f;syq(Q,R)^{\downarrow} &= (f;syq(Q,R))^{\downarrow} && \text{from (1)} \\ &= syq(Q;f^{\smile},R)^{\downarrow}. && \text{Lemma 2.29} \end{aligned}$$

(3) We have

$$\begin{aligned} & X \sqsubseteq f;(Q\backslash R) \\ \iff\ & f^{\smile};X \sqsubseteq Q\backslash R && \text{Lemma 2.25(2)} \\ \iff\ & Q;f^{\smile};X \sqsubseteq R \\ \iff\ & X \sqsubseteq Q;f^{\smile}\backslash R, \end{aligned}$$

which immediately implies $f;(Q\backslash R) = Q;f^{\smile}\backslash R$.

(4) Also solves similar to (2)

$$
\begin{aligned}
& X \sqsubseteq (Q\backslash R); f^{\smile} \\
\Leftrightarrow\ & X; f \sqsubseteq Q\backslash R && \text{Lemma 2.25 (1)} \\
\Leftrightarrow\ & Q; X; f \sqsubseteq R \\
\Leftrightarrow\ & Q; X \sqsubseteq R; f^{\smile} && \text{Lemma 2.25 (1)} \\
\Leftrightarrow\ & X \sqsubseteq Q\backslash R; f^{\smile}.
\end{aligned}
$$

which immediately implies $(Q\backslash R); f^{\smile} = (Q\backslash R); f^{\smile}$. □

The next lemma shows another important property of symmetric quotients in arrow categories that will be used later in later sections of the thesis.

Lemma 2.41. *Let $\mathcal{A}$ be an arrow category and $Q : A \to B$ and $R : C \to B$ crisp relations. If $syq(Q,R)^{\downarrow}$ is surjective, then the following holds:*

$$Q; syq(Q,R)^{\downarrow} = R.$$

Proof. First we obtain the inclusion $\sqsubseteq$ by

$$Q; syq(Q,R)^{\downarrow} \sqsubseteq Q; syq(Q,R) \sqsubseteq Q; (Q\backslash R) \sqsubseteq R.$$

Then we have

$$
\begin{aligned}
R &\sqsubseteq R; syq(Q,R)^{\downarrow\smile}; syq(Q,R)^{\downarrow} && \text{surjectivity} \\
&= R; syq(R,Q)^{\downarrow}; syq(Q,R)^{\downarrow} && \text{Lemma 2.29(2) and Lemma 2.37} \\
&\sqsubseteq Q; syq(Q,R)^{\downarrow}. && \text{similar to} \sqsubseteq \text{above}
\end{aligned}
$$

□

2.3.5 Fuzzy Categories

In this section we will extend an arrow category further by adding t-norm based operations mentioned in Section 2.2.3. t-norm based operations play an important role in fuzzy theory. For example, we can replace binary meets with t-norm based meet operations in order to relax the property that open sets are closed under forming meets in the definition [19].

Now let us define the fuzzy category.

Definition 2.42. *A fuzzy category $\mathcal{F}$ is an arrow category with t-norm based meet, composition, pseudo-complement, and residual operators $*, \overset{*}{;}, \mathrel{-\!\!*\!\!\!\rightarrow}, \mathrel{*\!\!\!/}$ respectively satisfying*

(1) $*$ *is associative and commutative,*

(2) $Q * P \sqsubseteq S \Leftrightarrow Q \sqsubseteq P \mathrel{-\!\!*\!\!\!\rightarrow} S$,

(3) $(Q * P)^{\smile} = Q^{\smile} * P^{\smile}$,

(4) $Q * P^{\downarrow} = Q \sqcap P^{\downarrow}$,

(5) $\overset{*}{;}$ *is associative,*

(6) $Q \overset{*}{;} R \sqsubseteq S \Leftrightarrow Q \sqsubseteq S \mathrel{*\!\!\!/} R$,

(7) $(Q \overset{*}{;} R)^{\smile} = Q^{\smile} \overset{*}{;} R^{\smile}$,

(8) $Q \overset{*}{;} R^{\downarrow} = Q;R^{\downarrow}$,

(9) $(P * Q) \overset{*}{;} (R * S) \sqsubseteq P \overset{*}{;} R * Q \overset{*}{;} S$,

(10) $Q;R * T \sqsubseteq Q;(R \sqcap Q^{\smile} \overset{*}{;} S)$,

(11) $Q \overset{*}{;} R * T \sqsubseteq Q;(R * Q^{\smile} \overset{*}{;} T)$,

(12) $(P * Q) \overset{*}{;} R * T \sqsubseteq P \overset{*}{;} (R * Q^{\smile} \overset{*}{;} T)$

for all relations $P, Q : A \to B$ and $R, S : B \to C$ and $T : A \to C$.

Now we can show the algebraic theory of L-fuzzy relations using next the lemma and provide proofs as in [19].

Lemma 2.43. *Let L be a complete Heyting algebra. Then $L-\mathrm{Rel}$ together with t-norm based operators is a fuzzy category.*

Proof. The properties (1) – (8) are easy to prove by component-wise definition of $*$, where (4) and (8) use $x * 1 = x = x \wedge 1$ and $x * 0 = 0 = x \wedge 0$ for all $x \in L$.

(9) We immediately derive

$$\begin{aligned}
((P * Q)\overset{*}{;}(R * S))(x,z) &= \bigvee_{y\in B} (P * Q)(x,y) * (R * S)(y,z) \\
&= \bigvee_{y\in B} P(x,y) * Q(x,y) * R(y,z) * S(y,z) \\
&= \bigvee_{y\in B} P(x,y) * R(y,z) * Q(x,y) * S(y,z) \\
&\subseteq (\bigvee_{y\in B} P(x,y) * R(y,z)) * (\bigvee_{y\in B} Q(x,y) * S(y,z)) \\
&= (P\overset{*}{;}R)(x,z) * (Q\overset{*}{;}S)(x,z) \\
&= (P\overset{*}{;}R * Q\overset{*}{;}S)(x,z)
\end{aligned}$$

(10) We immediately derive

$$\begin{aligned}
(Q;R * O)(x,z) &= (Q;R)(x,z) * O(x,z) \\
&= (\bigvee_{y\in B} Q(x,y) \wedge R(y,z)) * O(x,z) \\
&= \bigvee_{y\in B} (Q(x,y) \wedge R(y,z)) * O(x,z) \\
&\subseteq \bigvee_{y\in B} Q(x,y) * O(x,z) \wedge R(y,z) * O(x,z) \\
&\subseteq \bigvee_{y\in B} Q(x,y) \wedge Q(x,y) * O(x,z) \wedge R(y,z) \\
&\subseteq \bigvee_{y\in B} Q(x,y) \wedge (\bigvee_{x\in A} Q(x,y) * O(x,z)) \wedge R(y,z) \\
&= \bigvee_{y\in B} Q(x,y) \wedge (Q^T\overset{*}{;}O)(y,z) \wedge R(y,z) \\
&= \bigvee_{y\in B} Q(x,y) \wedge (R \wedge Q^T\overset{*}{;}O)(y,z) \\
&= (Q;(R \wedge Q^T\overset{*}{;}O))(x,z).
\end{aligned}$$

(11) We immediately derive

$$
\begin{aligned}
(Q \overset{*}{;} R * O)(x,z) &= (Q \overset{*}{;} R)(x,z) * O(x,z) \\
&= (\bigvee_{y \in B} Q(x,y) * R(y,z)) * O(x,z) \\
&= \bigvee_{y \in B} Q(x,y) * R(y,z) * O(x,z) \\
&= \bigvee_{y \in B} Q(x,y) \wedge Q(x,y) * R(y,z) * O(x,z) \\
&\subseteq \bigvee_{y \in B} Q(x,y) \wedge R(y,z) * (\bigvee_{x \in A} Q(x,y) * O(x,z)) \\
&= \bigvee_{y \in B} Q(x,y) \wedge R(y,z) * (Q^T \overset{*}{;} O)(y,z) \\
&= \bigvee_{y \in B} Q(x,y) \wedge (R * Q^T \overset{*}{;} O)(y,z) \\
&= (Q; (R * Q^T \overset{*}{;} O))(x,z).
\end{aligned}
$$

(12) We immediately derive

$$
\begin{aligned}
((P * Q) \overset{*}{;} R * O)(x,z) &= ((P * Q) \overset{*}{;} R)(x,z) * O(x,z) \\
&= (\bigvee_{y \in B} (P * Q)(x,y) * R(y,z)) * O(x,z) \\
&= (\bigvee_{y \in B} P(x,y) * Q(x,y) * R(y,z)) * O(x,z) \\
&= \bigvee_{y \in B} P(x,y) * Q(x,y) * R(y,z) * O(x,z) \\
&\subseteq \bigvee_{y \in B} P(x,y) * R(y,z) * (\bigvee_{x \in A} Q(x,y) * O(x,z)) \\
&= \bigvee_{y \in B} P(x,y) * R(y,z) * (Q^{\smile} \overset{*}{;} O)(y,z) \\
&= \bigvee_{y \in B} P(x,y) * (R * Q^{\smile} \overset{*}{;} O)(y,z) \\
&= (P \overset{*}{;} (R * Q^{\smile} \overset{*}{;} O))(x,z).
\end{aligned}
$$

□

The next few lemmas show important properties of fuzzy categories.

Lemma 2.44. *Let $\mathcal{A}$ be a fuzzy category and $Q : A \to B$ and $R, S : B \to C$ are L-relations.*

If Q is crisp, then the following holds:

$$Q;(R*S) \sqsubseteq Q;R*Q;S.$$

Proof. Consider the following derivatives:

$$\begin{aligned}
Q;(R*S) &= Q\overset{*}{;}(R*S) && \text{Lemma 2.19(4)}\\
&= (Q\cap Q)\overset{*}{;}(R*S) && \\
&= (Q*Q)\overset{*}{;}(R*S) && \text{Lemma 2.18(4)}\\
&\sqsubseteq Q\overset{*}{;}R*Q\overset{*}{;}S && \text{Lemma 2.19(5)}\\
&= Q;R*Q;S. && \text{Lemma 2.19(4)}
\end{aligned}$$

□

Lemma 2.45. *Suppose $Q : A \to B$ and $R, S : B \to C$ are L-relations. If Q is crisp and univalent, then the following holds:*

$$Q;(R*S) = Q;R*Q;S.$$

Proof. The inclusion $\sqsubseteq$ was already shown in Lemma 2.44. For the converse inclusion consider the following derivation

$$\begin{aligned}
Q;R*Q;S &= Q\overset{*}{;}R*Q;S && \text{Lemma 2.19(4)}\\
&\sqsubseteq Q;(R*Q^{\smile}\overset{*}{;}(Q;S)) && \text{Lemma 2.19(7)}\\
&= Q;(R*Q^{\smile};Q;S) && \text{Lemma 2.19(4)}\\
&\sqsubseteq Q;(R*S) && \text{Q is univalent}
\end{aligned}$$

□

Chapter 3

Topology

This chapter will briefly introduce definitions of general topology and fuzzy topology and provide relation-algebraic formulations for them.

3.1 General Topology

The term *topology* was first introduced by German mathematician J.B. Listing in 1847 and for a long period was termed *analysis situs* [10]. Since then, topology has been given many definitions, such as considerations of neighborhoods, open sets, and closed sets. This work will only focus on the definition based on open sets.

A special place among the areas of topology is general topology. At present, general topology has reached the most natural level of generality, which allows us to present the topological principles, concepts and constructions with the greatest transparency, and at the same time ensure the widest applicability in other branches of mathematics. The term general topology refers to the topology used by most mathematicians. It teaches us to speak clearly and accurately about things related to the idea of continuity.

The following definition of the topology is given using open sets.

Definition 3.1. *A topology τ on a nonempty set X is a collection of subsets of X, called open sets, such that*

(1) The empty set $\varnothing$ and the set X are open,

(2) The union of an arbitrary collection of open sets is open set,

(3) The intersection of a finite number of open sets is open.

The pair (X, τ) is a *topological space.*

The simplest example of a topology is the discrete topology where τ is a collection of all subsets of X. Indiscrete topology is the opposite example, in which the topological structure is the most modest, and it consists of only X and $\varnothing$. The standard topology on $\mathbb{R}$ is the most common example.

Often, a topological structure is defined by describing some part of it, sufficient to restore the entire structure. The *bases* of a topology is a set of open sets, such that every non-empty open set is representable as a union of sets from this set.

Definition 3.2. *The collection of $\mathcal{B}$ of open sets X is called base of τ, if and only if every element from τ is a union of elements of $\mathcal{B}$.*

Let $X = \mathbb{R}$ - the set of all real numbers. Then the collection of all open intervals of the form (a, b) defines the base of the topology on R. It should be noted that not every collection of sets can serve as a base for some topology.

Now that we have defined the basic structure of a topological space, we can describe functions between them. One of the important functions between spaces is *continuous maps*.

Definition 3.3. *Let X and Y be topological spaces. Mapping $f : X \to Y$ is called continuous if for each open set $V \subseteq Y$, the inverse image $f^{-1}(V)$ is open in X.*

The inverse image of V in the Definition 3.3 under the function f is defined to be the set $f^{-1}(V) = \{x \in X : f(x) \in V\}$ containing elements $x \in X$ such that $f(x) \in V$. Using a base of a topology, it is enough to check that the inverse image of every basis element in Y is open [11].

3.2 Fuzzy Topology

Fuzzification of general topology is done by replacing subsets in the definition of general topology by fuzzy sets, and can be formulated as in the following definition.

Definition 3.4. *The fuzzy topology on a set X is a family τ of fuzzy subsets which satisfies the following three axioms*

(1) The empty set $\varnothing$ and the set X are open ($\varnothing(x) = 0$ and $X(x) = 1$),

(2) $(A * B)(x) = A(x) * B(x)$,

(3) if $\{A_j : j \in J\} \sqsubseteq \tau$, *then* $\bigvee_j A_j \in \tau$.

Since the component-wise empty set and the whole set are least and greatest elements respectively, the first axiom remains the same as in the case of general topology. For the second axiom, instead of the union, the operation $*$ is used, i.e., the intersection of two sets is computed based on $*$ instead of $\wedge$.

3.3 Relational Topology

Recently, a relation-algebraic approach to topology became a new and separate approach to this research area. Considering topology algebraically means to deal with algebraic rules of the topology itself instead of component-wise reasoning. All the conversions between several concepts of the topology can be formulated by relation-algebraic rules.

Before we define the relation algebraic approach to topology, a few new operators and constructions mentioned earlier need to be introduced.

In addition to the basic operators in Chapter 2, *symmetric quotient* mentioned in Section 2.3.2 is needed in order to compare two relations column-wise. First, we want to demonstrate how residuals are computed component-wise in the case of L-fuzzy relations. Therefore, consider two relations

$$Q : A \to B \text{ and } R : A \to C.$$

Then we have

$$(Q\backslash R)(y,z) = \bigwedge_{x \in A} Q(x,y) \to R(x,z),$$

i.e., y is related to z by a degree u iff u is the greatest value so that $Q(x,y) \wedge u \leq R(x,z)$. In other words, u is the greatest degree so that every entry in the column related to y meet u is smaller or equal to the corresponding entry in the column related to z. This could also be phrased as the column of y is included in the column of z with a degree of u. Please note that, in the case of crisp relations, this simply means that $(Q\backslash R)(y,z)$ iff the column of y is included in the column of z. By definition of the symmetric quotient, this implies that $syq(Q,R)(y,z) = u$ iff u is the greatest value so that $Q(x,y) \wedge u = R(x,z) \wedge u$. Following the wording above, this could be phrased as the column of y is similar (or equal) to of the column of z by degree u. Again, note that in the case of crisp relations, we have $syq(Q,R)(y,z)$ iff the column of y is equal to the column of z.

Now we would like to interpret a relation $Q : A \to B$ as follows. Each column of Q, i.e., each element in B, can be seen as a L-fuzzy set of elements of A. Concretely, if $y \in B$, then y represents the L-fuzzy subset of A defined by the membership function $f(x) = Q(x, y)$. Now a family of sets in this interpretation is a relation in which any two different columns have different entries, i.e., $syq(Q, Q)^{\downarrow} = \mathbb{I}$. In other words, we call relation Q a *family of sets* over A.

Earlier we have shown that a topology can be generated from bases. Using bases instead of topologies themselves helps to avoid defining all of the open sets in the space. However, we have to be able to compare bases in order to determine whether they produce the same topology.

$$\begin{tikzcd} & B \\ A \arrow[ur, "R"] \arrow[dr, "S"'] & \\ & C \arrow[uu, "f"'] \end{tikzcd}$$

Figure 3.1: Base comparison.

Now we might have multiple families of subsets over A, e.g, a family of sets S over A as shown in Figure 3.1. Those sets might be different because they can consist of different amounts of subsets. The comparison of a family of sets is done by the injective function $f = syq(R, S)^{\downarrow}$. The following example illustrates the injective function from Figure 3.1.

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = syq(R, S)^{\downarrow} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Definition 3.5. *Let τ_1 and τ_2 be families of sets. Then we define $\tau_1 \leq \tau_2$ iff there is a crisp injective function so that $\tau_2 ; f^{\smile} = \tau_1$.*

Definition 3.6. *Let τ_1 and τ_2 be families of sets. Then τ_1 and τ_2 are said to be equivalent if $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_1$.*

The above definition basically means B and C are the same, however, their subsets ordering might different. From the above definitions we can obtain the next lemma.

Lemma 3.7. *Suppose $\tau_1 \leq \tau_2$. i.e., there is a function f with $\tau_2;f^{=}\tau_1$. Then we have:*

(1) $f = syq(\tau_1,\tau_2)^{\downarrow}$,

(2) $syq(\tau_1,\tau_2)^{\downarrow}$ is crisp and bijective.

Proof. (1) First we show that

$$f \sqsubseteq syq(\tau_1,\tau_2)^{\downarrow}.$$

From the assumption, we obtain $f;\tau_2^{\smile} = (\tau_2;f^{\smile})^{\smile} = \tau_1^{\smile}$ so that $f = f^{\downarrow} \sqsubseteq (\tau_2^{\smile}/\tau_1^{\smile})^{\downarrow}$ follows. On the other hand,

$$\tau_1;f = \tau_2;f^{\smile};f \sqsubseteq \tau_2$$

implies $f = f^{\downarrow} \sqsubseteq syq(\tau_1,\tau_2)^{\downarrow}$. Together this shows $f \sqsubseteq syq(\tau_1,\tau_2)^{\downarrow}$. The opposite inclusion simply follows from the fact that f is total and $syq(\tau_1,\tau_2)^{\downarrow}$ is univalent because τ_2 is a family of sets.

(2) The assertion follows immediately from (1) and the fact that τ_1 is a family of sets. □

Next step, we are interested in the family of all subsets. The following definition describes the relation between the set and its direct power as in [14].

Definition 3.8. *Let A be an object. An object $\mathcal{P}(A)$ together with a relation $\varepsilon : A \longrightarrow \mathcal{P}(A)$ is called a relational power iff*

(1) $syq(\varepsilon,\varepsilon)^{\downarrow} \sqsubseteq \mathbb{I}$,

(2) $syq(Q,\varepsilon)^{\downarrow}$ is total for every $Q : A \to B$.

The matrix representation of this relationship can be illustrated as follows:

$$\varepsilon = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{pmatrix} \{\} & \{a\} & \{b\} & \{a,b\} & \{c\} & \{a,c\} & \{b,c\} & \{a,b,c\} \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The next step is to define *direct product, strict fork and meet operators*. The full definitions from [14] is follows.

Definition 3.9. *The product of two objects A and B is an object $A \times B$ together with projection relations $\pi : A \times B \to A$ and $\rho = A \times B \to B$ if the following axioms hold*

(1) π, ρ *are crisp,*

(2) $\pi^{\smile};\pi \sqsubseteq \mathbb{I}$,

(3) $\rho^{\smile};\rho \sqsubseteq \mathbb{I}$,

(4) $\pi;\pi^{\smile} \sqcap \rho;\rho^{\smile} = \mathbb{I}$,

(5) $\pi^{\smile};\rho = \mathbb{T}_{AB}$.

Now if any direct products is given by the following projections $\pi : A \times B \to A$ and $\rho = A \times B \to B$, we can define a new operator as an operation for relations. Let R be relation $C \to A$ and S be relation $C \to B$, then the *fork* operator is

$$R \olessthan S := R;\pi^{\smile} \sqcap S;\rho^{\smile}.$$

The fork operator is illustrated in Figure 3.2.

$$\begin{tikzcd}
 & A \\
C \arrow[ur, "R"] \arrow[r, "R \olessthan S"] \arrow[dr, "S"'] & A \times B \arrow[u] \arrow[d] \\
 & B
\end{tikzcd}$$

Figure 3.2: Visual illustration of the fork operator.

Next, we will define the meet relation of a powerset ordering $\mathcal{P}(A)$ via membership deletion. The binary meet relation $\mathfrak{M} : P(A) \times P(A) \to P(A)$ is defined by

$$\mathfrak{M} := syq((\varepsilon \olessthan \varepsilon), \varepsilon).$$

Intuitively, this relation takes a pair of sets , computes the elements that are common in both sets, and relates the original pair with the set that contains exactly those elements.

Another valuable construction is a combination of forming equivalence classes and moving to a sub-object. If $Q : A \to A$ is an equivalence relation on a subset B of A, then we can form the set of equivalence classes of B. The following definition captures this concept abstractly as in [17].

Definition 3.10. *Let $Q : A \to A$ be a crisp symmetric idempotent relation, i.e. $Q^{\downarrow} = Q$, $Q^{\smile} = Q$ and $Q;Q = Q$. Then an object B together with a crisp relation $R : B \to A$ satisfying $R;R^{\smile} = \mathbb{I}_B$ and $R^{\smile};R = Q$ is called a splitting of Q.*

The following example illustrates relations Q and R from the above definition. Let A and B be sets $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2\}$ respectively.

$$Q = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}, \quad R = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

Chapter 4

Relation-Algebraic Formulation of Fuzzy Topology

This chapter introduces a relation-algebraic approach to fuzzy topology. In this chapter we take a fresh look at many of the issues discussed earlier and cover important topics of fuzzy topology in terms of relations, such as bases, continuous map, and separations axioms.

4.1 The fuzzy meet $\mathfrak{M}$ and fork $\otimes_*$ operators

The definition of fuzzy topology mentioned in Section 3.1 can be also similarly formulated in terms of relations by changing meet $\mathfrak{M}$ and strict fork $\otimes$ operators using $*$ as in Definition 3.4.

$$\mathfrak{M}_* = syq((\varepsilon;\pi^\smile * \varepsilon;\rho^\smile),\varepsilon)^\downarrow,$$

$$R \otimes_* S := R;\pi^\smile * S;\rho^\smile.$$

Before we give a definition of fuzzy topology let us provide necessary properties of the fuzzy meet $\mathfrak{M}_*$ and strict fork $\otimes_*$ operations.

Lemma 4.1. *Let $Q : C \to A, R : C \to B$ be relations, and f, g be crisp maps. Then the following holds:*

$$(Q \otimes_* R);(f^\smile \otimes g^\smile) = Q;f^\smile \otimes_* R;g^\smile.$$

Proof. Consider the following derivations:

$$
\begin{aligned}
&(Q \otimes_* R);(f^\smile \otimes g^\smile) = (Q;\pi^\smile * R;\rho^\smile);(f^\smile \otimes g^\smile) && \\
&= Q;\pi^\smile;(f^\smile \otimes g^\smile) * R;\rho^\smile;(f^\smile \otimes g^\smile) && \text{Lemma 2.45 is dual} \\
&= Q;(f^\smile;\pi^\smile \sqcap \pi^\smile;\rho;g^\smile;\rho^\smile) * R;(\rho^\smile;\pi;f^\smile;\pi^\smile \sqcap g^\smile;\rho^\smile) && \text{Lemma 2.26 (1)} \\
&= Q;(f^\smile;\pi^\smile \sqcap \mathbb{T};g^\smile;\rho^\smile) * R;(\mathbb{T};f^\smile;\pi^\smile \sqcap g^\smile;\rho^\smile) && \\
&= Q;f^\smile;\pi^\smile * R;g^\smile;\rho^\smile && f,g,\pi,\rho \text{ is total} \\
&= Q;f^\smile \otimes_* R;g^\smile. &&
\end{aligned}
$$

□

As mentioned earlier, the definition of topology can be formulated many ways. We would like to propose a new relation-algebraic formulation of fuzzy topology and give a proof for it.

Definition 4.2. *A relation $\tau : A \to B$ is called a topology iff*

(0) τ is total,

(1) $syq(\tau,\tau)^\downarrow \sqsubseteq \mathbb{I}$,

(2) $syq(\tau \otimes_* \tau,\tau)^\downarrow$ is total,

(3) $syq(\tau;R,\tau)^\downarrow$ total for every crisp R.

We call the target B of τ the set of open sets of τ. The first axiom requires that every element be contained in an open set. The second axiom requires τ to be a family of sets. The third axiom requires that, given two open sets, the set of common elements with degree computed as the $*$ of the degrees in the two sets is also an open set. Lastly, the arbitrary union of open sets is open.

4.2 Bases

As we mentioned in Chapter 3.2, a topological structure can be defined by describing some part of it, sufficient to restore the entire structure. Lets formulate the definition of base of a fuzzy topology in terms of relations, and give illustration of it as in Figure 4.1.

Definition 4.3. *Let $\tau : X \to C$ be topology. A relation $b : X \to B$ is called a base of topology τ iff the following holds:*

$$\begin{tikzcd} & C \\ X \arrow[ur, "\tau"] \arrow[dr, "b"'] & \\ & B \arrow[uu] \end{tikzcd}$$

Figure 4.1: Base of topology.

(1) $syq(b,b)^{\downarrow} \sqsubseteq \mathbb{I}$,

(2) $syq(b,\tau)^{\downarrow}$,

(3) $syq(b;syq(b,\tau)^{\downarrow};(\tau\backslash\tau)^{\downarrow},\tau)^{\downarrow} = \mathbb{I}$.

Axiom (1) requires that b is a family of sets that is by Axiom (2) a subfamily of τ. Axiom (3) requires that every open set in τ is the union of sets in b, i.e., for every open set A in τ, the union of all sets from b that are included in A is A again. The last axiom can be equivalently stated as shown in the lemma below. Since the two versions are equivalent, we will mainly use the simpler version in the remainder of the thesis.

Lemma 4.4. *Let τ be topology, and b a relation satisfying Axioms (1) and (2) of a base of τ. Then Axiom 4.3 (3) is equivalent to:*

$$\tau = b;(b\backslash\tau)^{\downarrow}.$$

Proof. First we show that Axiom (3) implies the equation above. Consider the following computation:

$$
\begin{aligned}
\tau &= \tau;\mathbb{I} \\
&= \tau;\mathbb{I}^{\smile} && \mathbb{I}^{\smile} = \mathbb{I} \\
&= \tau;syq(b;syq(b,\tau)^{\downarrow};(\tau\backslash\tau)^{\downarrow},\tau)^{\downarrow} && \text{Definition 4.3(3)} \\
&= \tau;syq(\tau,b;syq(b,\tau)^{\downarrow};(\tau\backslash\tau)^{\downarrow})^{\downarrow} && \mathbb{I}^{\smile} = \mathbb{I} \text{ and Lemma 2.29} \\
&= b;syq(b,\tau)^{\downarrow};(\tau\backslash\tau)^{\downarrow} && \mathbb{I} \text{ surjective} \\
&= b;(\tau;syq(b,\tau)^{\downarrow\smile}\backslash\tau)^{\downarrow} && \text{Lemma 2.40(3)} \\
&= b;(\tau;syq(\tau,b)^{\downarrow}\backslash\tau)^{\downarrow} && \text{Lemmas 2.29 and 2.37(8)} \\
&= b;(b\backslash\tau)^{\downarrow}. && \text{Lemma 2.41}
\end{aligned}
$$

Now, we assume the equation above is valid and we show Axiom (3) as follows:

$$
\begin{aligned}
syq(b;syq(b,\tau)^{\downarrow};(\tau,\tau)^{\downarrow},\tau)^{\downarrow} &= syq(b;(b\backslash\tau)^{\downarrow},\tau)^{\downarrow} && \text{as above proof} \\
&= syq(\tau,\tau)^{\downarrow} && \text{by assumption} \\
&= \mathbb{I}. && \text{Lemma 4.3(1)}
\end{aligned}
$$

□

The previous definition of base defines b as a base of the given topology. On the other hand, we want to define a relation b as base without a previous given topology. First we show that a base of a topology will always satisfy the following properties.

Lemma 4.5. *If b is a base of topology τ, then the followings hold:*

(1) b is total,

(2) $b \otimes_* b \sqsubseteq b;(b\backslash(b \otimes_* b))^{\downarrow}$.

Proof. (1) as follows:

$$
\begin{aligned}
b;\pi &= b;syq(b,\tau)^{\downarrow};(\tau,\tau)^{\downarrow};\pi && \mathbb{I} \sqsubseteq (\tau\backslash\tau)^{\downarrow} \text{ and } syq(b,\tau)^{\downarrow} \text{ is total} \\
&= b;(b\backslash\tau)^{\downarrow};\pi && \text{Lemma 4.4} \\
&= \tau;\pi && \text{Lemma 4.4} \\
&= \pi. && \text{Lemma 3.2}
\end{aligned}
$$

(2) Consider the following derivations:

$$
\begin{aligned}
syq(\tau \otimes_* \tau, \tau)^{\downarrow} &= syq(\tau \otimes_* \tau, \tau)^{\downarrow}; syq(b; (b\backslash\tau)^{\downarrow}, \tau)^{\downarrow} && \text{Lemma 4.4 and } \tau \text{ axiom} \\
&= syq(b; (b\backslash\tau)^{\downarrow}; syq(\tau \otimes_* \tau, \tau)^{\downarrow\smile}, \tau)^{\downarrow} && \text{Lemma 2.40(2) and } \tau \text{ axiom} \\
&= syq(b; (b\backslash\tau)^{\downarrow}; syq(\tau, \tau \otimes_* \tau)^{\downarrow}, \tau)^{\downarrow} && \text{Lemma 2.29} \\
&= syq(b; (b\backslash(\tau; syq(\tau, \tau \otimes_* \tau)^{\downarrow})), \tau)^{\downarrow} && \text{Lemma 2.40(4)} \\
&= syq(b; (b\backslash(\tau \otimes_* \tau))^{\downarrow}, \tau)^{\downarrow}. && \text{Lemma 2.41}
\end{aligned}
$$

This implies:

$$
\begin{aligned}
\tau \otimes_* \tau &= \tau; syq(\tau, \tau \otimes_* \tau)^{\downarrow} && \text{Lemma 2.41} \\
&= \tau; syq(\tau, b; (b\backslash(\tau \otimes_* \tau))^{\downarrow})^{\downarrow} \\
&= b; (b\backslash(\tau \otimes_* \tau))^{\downarrow}. && \text{Lemma 2.41}
\end{aligned}
$$

And finally, using above derivations we can prove given axiom as follows:

$$
\begin{aligned}
b \otimes_* b &= \tau; syq(\tau, b)^{\downarrow} \otimes_* \tau; syq(\tau, b)^{\downarrow} && \text{Lemma 2.41} \\
&= (\tau \otimes_* \tau); (syq(\tau, b)^{\downarrow} \otimes syq(\tau, b)^{\downarrow}) && \text{Lemma 4.1} \\
&= b; (b\backslash(\tau \otimes_* \tau))^{\downarrow}; (syq(\tau, b)^{\downarrow} \otimes syq(\tau, b)^{\downarrow}) \\
&= b; (b\backslash((\tau \otimes_* \tau); (syq(\tau, b)^{\downarrow} \otimes syq(\tau, b)^{\downarrow})))^{\downarrow} && \text{Lemma 2.40(4)} \\
&= b; (b\backslash(\tau; syq(\tau, b)^{\downarrow} \otimes_* \tau; syq(\tau, b)^{\downarrow}))^{\downarrow} && \text{Lemma 4.1} \\
&= b; (b\backslash(b \otimes_* b))^{\downarrow}. && \text{Lemma 2.41}
\end{aligned}
$$

□

Now we can provide a definition of base without any given topology as mentioned earlier. In other words, we are providing a definition of a general base in terms of relations.

Definition 4.6. *A relation $b : X \to B$ is called a base iff b is a family of sets and it satisfies:*

(1) b is total,

(2) $b \otimes_* b \sqsubseteq b; (b\backslash b \otimes_* b)^{\downarrow}$.

Please note that the inclusion in Axiom (2) is, in fact, an equation since the inclusion $\sqsupseteq$ is always true.

Every base gives rise to a topology by using a splitting of the direct power induced by the base as shown in Figure 4.2.

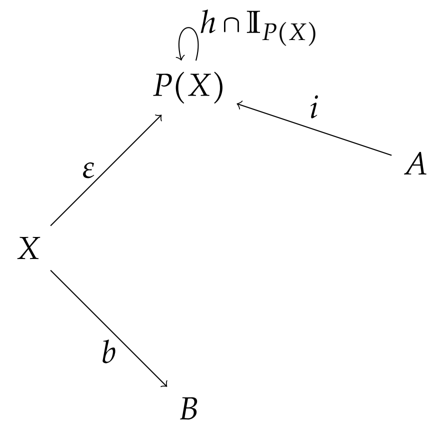

Figure 4.2: Topology generated by a base.

Definition 4.7. *Let $b : X \to B$ be a base and $i : A \to P(X)$ the splitting of $h \cap \mathbf{I}_{P(X)}$. Then $\varepsilon;i^{\smile}$ is called a topology generated by b.*

In order to justify our notation of the previous definition, we have to show that $\varepsilon;i^{\smile}$ is a topology.

Lemma 4.8. *With the notation of Definition 4.7, the relation $e;i^{\smile}$ is a topology with base b.*

(1) $\varepsilon;i^{\smile}$ *is total,*

(2) $syq(\varepsilon;i^{\smile},\varepsilon;i^{\smile})^{\downarrow} \sqsubseteq \mathbb{I}$*,*

(3) $syq(\varepsilon;i^{\smile};R^{\smile},\varepsilon;i^{\smile})^{\downarrow}$ *is total,*

(4) $syq(\varepsilon;i^{\smile} \otimes_{*} \varepsilon;i^{\smile},\varepsilon;i^{\smile})^{\downarrow}$ *is total.*

Proof. (1) The assertion is shown as follows:

$$
\begin{array}{rll}
syq(\pi,\varepsilon)^{\smile};h & = syq(b;(b\backslash\varepsilon)^{\downarrow};syq(\pi,\varepsilon)^{\downarrow\smile},\varepsilon)^{\downarrow} & \text{Lemma 2.40 (2)} \\
 & = syq(b;(b\backslash\varepsilon)^{\downarrow};syq(\varepsilon,\pi)^{\downarrow},\varepsilon)^{\downarrow} & \text{Lemma 2.29(1)} \\
 & = syq(b;(b\backslash\varepsilon;syq(\varepsilon,\pi)^{\downarrow}),\varepsilon)^{\downarrow} & \text{Lemma 2.40(4)} \\
 & = syq(b;(b\backslash\pi)^{\downarrow},\varepsilon)^{\downarrow} & \text{Lemma 2.41} \\
 & = syq(b;\pi,\varepsilon)^{\downarrow} & \text{definition of } \backslash \\
 & = syq(\pi,\varepsilon)^{\downarrow} & b \text{ is total} \\
syq(\pi,\varepsilon)^{\downarrow};(h\sqcap\mathbb{I}) & = syq(\pi,\varepsilon)^{\downarrow};h\sqcap syq(\pi,\varepsilon)^{\downarrow} & \text{Lemma 2.26(1)} \\
 & = syq(\pi,\varepsilon)^{\downarrow} & \text{as above} \\
syq(\pi,\varepsilon)^{\downarrow\smile};syq(\pi,\varepsilon)^{\downarrow} & = syq(\pi,\varepsilon)^{\downarrow\smile};syq(\pi,\varepsilon)^{\downarrow};(h\sqcap\mathbb{I}) & \text{above} \\
 & \sqsubseteq h\sqcap\mathbb{I} & syq(\pi,\varepsilon)^{\downarrow} \text{ is univalent} \\
\varepsilon;i^{\smile};\pi & = \varepsilon;i^{\smile};i;\pi & i \text{ total} \\
 & = \varepsilon;(h\sqcap\mathbb{I});\pi & \text{definition of } i \\
 & \sqsupseteq \varepsilon;syq(\pi,\varepsilon)^{\downarrow\smile};syq(\pi,\varepsilon)^{\downarrow};\pi & \text{above} \\
 & = \varepsilon;syq(\varepsilon,\pi)^{\downarrow};syq(\pi,\varepsilon)^{\downarrow};\pi & \text{Lemma 2.29(1)} \\
 & = \pi;syq(\pi,\varepsilon)^{\downarrow};\pi & \text{Lemma 2.41} \\
 & = \pi;\pi & syq(\pi,\varepsilon)^{\downarrow} \text{ is total} \\
 & = \pi & \text{total.}
\end{array}
$$

(2) Consider the following

$$
\begin{array}{rll}
syq(\varepsilon;i^{\smile},\varepsilon;i^{\smile})^{\downarrow} & = i;syq(\varepsilon,\varepsilon)^{\downarrow};i^{\smile} & \text{Lemma 2.29(2) is dual} \\
 & = i;i^{\smile} & \text{definition of } \varepsilon \\
 & = \mathbb{I} & \text{definition of } i
\end{array}
$$

(3) The assertion is shown as follows

$$\begin{aligned}
\varepsilon;i^{\smile};R^{\smile} &= \varepsilon;i^{\smile};i;i^{\smile};R^{\smile} && \text{definition of } i\\
&= \varepsilon;(h\sqcap\mathbb{I});i^{\smile};R^{\smile} && \text{definition of } i\\
&= \varepsilon;(h^{\smile}\sqcap\mathbb{I});i^{\smile};R^{\smile} && \text{Lemma 2.33(3)}\\
&\sqsubseteq \varepsilon;h^{\smile};i^{\smile};R^{\smile} &&\\
&= b;(b\backslash\varepsilon)^{\downarrow};i^{\smile};R^{\smile} && \text{Lemma 2.41}\\
&\sqsubseteq b;((b\backslash\varepsilon);i^{\smile};R^{\smile})^{\downarrow} && \text{Lemma 2.39 and } h,R \text{ crisp}\\
&\sqsubseteq b;(b\backslash(\varepsilon;i^{\smile};R^{\smile}))^{\downarrow} &&
\end{aligned}$$

This implies $\varepsilon;i^{\smile};R^{\smile} = b;(b\backslash\varepsilon;i^{\smile};R^{\smile})^{\downarrow}$ since the inclusion $\sqsupseteq$ is alway true. Now, we immediately obtain

$$\begin{aligned}
syq(\varepsilon;i^{\smile};R^{\smile},\varepsilon)^{\downarrow};h &= syq(b;(b\backslash\varepsilon)^{\downarrow};syq(\varepsilon,\varepsilon;i^{\smile};R^{\smile})^{\downarrow},\varepsilon)^{\downarrow} && \text{Lemma 2.40(2)}\\
&= syq(b;(b\backslash\varepsilon;syq(\varepsilon,\varepsilon;i^{\smile};R^{\smile})^{\downarrow})^{\downarrow},\varepsilon)^{\downarrow} && \text{Lemma 2.40(4)}\\
&= syq(b;(b\backslash\varepsilon;i^{\smile};R^{\smile})^{\downarrow},\varepsilon)^{\downarrow} && \text{Lemma 2.41}\\
&= syq(\varepsilon;i^{\smile};R^{\smile},\varepsilon)^{\downarrow} && \text{above}
\end{aligned}$$

Last but not least, this implies

$$\begin{aligned}
&syq(\varepsilon;i^{\smile};R^{\smile},\varepsilon;i^{\smile})^{\downarrow};\mathbb{T} = syq(\varepsilon;i^{\smile};R^{\smile},\varepsilon)^{\downarrow};i^{\smile};\mathbb{T} &&\\
&= syq(\varepsilon;i^{\smile};R^{\smile},\varepsilon)^{\downarrow};i^{\smile};\mathbb{T} && i \text{ total and definition of } i\\
&= [syq(\varepsilon;i^{\smile};R^{\smile},\varepsilon)^{\downarrow};h\sqcap syq(\epsilon;i^{\smile};R^{\smile},\varepsilon)^{\downarrow}];\mathbb{T} && \text{Lemma 2.26(1)}\\
&= syq(\varepsilon;i^{\smile};R^{\smile},\varepsilon)^{\downarrow};\mathbb{T} && \text{above}\\
&= \mathbb{T} &&
\end{aligned}$$

(4) First we have

$$\begin{aligned}
b;(b\backslash\varepsilon)^{\downarrow}\oslash_{*}b;(b\backslash\varepsilon)^{\downarrow} &= (b\oslash_{*}b);((b\backslash\varepsilon)^{\downarrow}\otimes(b\backslash\varepsilon)^{\downarrow}) &&\\
&\sqsubseteq b;(b\backslash b\oslash_{*}b)^{\downarrow};((b\backslash\varepsilon)^{\downarrow}\otimes(b\backslash\varepsilon)^{\downarrow}) &&\\
&\sqsubseteq b;((b\backslash b\oslash_{*}b);((b\backslash\varepsilon)^{\downarrow}\otimes(b\backslash\varepsilon)^{\downarrow}))^{\downarrow} && \text{Lemma 2.39}\\
&\sqsubseteq b;(b\backslash(b\oslash_{*}b);((b\backslash\varepsilon)^{\downarrow}\otimes(b\backslash\varepsilon)^{\downarrow}))^{\downarrow} &&\\
&= b;(b\backslash b;(b\backslash\varepsilon)^{\downarrow}\oslash_{*}b;(b\backslash\varepsilon)^{\downarrow})^{\downarrow} &&\\
&\sqsubseteq b;(b\backslash\varepsilon\oslash_{*}\varepsilon)^{\downarrow} &&
\end{aligned}$$

Now, we are able to conclude

$$\begin{aligned}
\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile} &= \varepsilon;i^{\smile};i;i^{\smile} \oslash_* \varepsilon;i^{\smile};i;i^{\smile} && \text{definition of } i\\
&= \varepsilon;(h \sqcap \mathbb{I});i^{\smile} \oslash_* \varepsilon;(h \sqcap \mathbb{I});i^{\smile} && \text{definition of } i\\
&\sqsubseteq \varepsilon;h;i^{\smile} \oslash_* \varepsilon;h;i^{\smile} &&\\
&= (\varepsilon;h \oslash_* \varepsilon;h);(i^{\smile} \otimes i^{\smile}) && \text{Lemma 4.1}\\
&= (b;(b\backslash\varepsilon)^{\downarrow} \oslash_* b;(b\backslash\varepsilon)^{\downarrow});(i^{\smile} \otimes i^{\smile}) && \text{Lemma 2.41}\\
&\sqsubseteq b;(b\backslash\varepsilon \oslash_* \varepsilon)^{\downarrow};(i^{\smile} \otimes i^{\smile}) && \text{above}\\
&= b;(b\backslash(\varepsilon \oslash_* \varepsilon);(i^{\smile} \otimes i^{\smile}))^{\downarrow} && \text{Lemma 2.40(4)}\\
&= b;(b\backslash\varepsilon;i^{\smile} \oslash \varepsilon;i^{\smile})^{\downarrow} && \text{Lemma 4.1}
\end{aligned}$$

Form the following computation

$$\begin{aligned}
syq(\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile},\varepsilon)^{\smile};h &= syq(b(b\backslash\varepsilon)^{\downarrow};syq(\varepsilon,\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile})^{\downarrow},\varepsilon)^{\downarrow} && \text{Lemma 2.40(2)}\\
&= syq(b;(b\backslash\varepsilon;syq(\varepsilon,\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile})^{\downarrow})^{\downarrow},\varepsilon)^{\downarrow} && \text{Lemma 2.40(4)}\\
&= syq(b(b\backslash\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile})^{\downarrow},\varepsilon)^{\downarrow} && \text{Lemma 2.41}\\
&= syq(\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile},\varepsilon)^{\downarrow} &&
\end{aligned}$$

We now obtain

$$\begin{aligned}
&syq(\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile},\varepsilon;i^{\smile})^{\downarrow};\pi = syq(\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile},\varepsilon)^{\downarrow};i^{\smile};\pi &&\\
&= syq(\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile},\varepsilon)^{\downarrow};(h \sqcap \mathbb{I});\pi && i \text{ total}\\
&= [syq(\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile},\varepsilon)^{\downarrow};h \sqcap syq(\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile},\varepsilon)^{\downarrow}];\pi && \text{Lemma 2.26(1)}\\
&= syq(\varepsilon;i^{\smile} \oslash_* \varepsilon;i^{\smile},\varepsilon)^{\downarrow};\pi && \text{above}\\
&= \pi &&
\end{aligned}$$

□

4.3 Continuous maps

We have given an abstract notion of continuous maps between topological spaces earlier in Definition 3.3. Now we can give its relational-algebraic formulation. Let A and B be base sets, and f be a map between them as shown in Figure 4.3. Our task is to show that f is continuous. A general definition of a continuous map states that the inverse image of an open set is also open. Similarly we can define a continuous map between A and B.

$$\begin{tikzcd}
\tau_1 & \tau_2 \arrow[l, "syq(f;\tau_2,\tau_1)^{\downarrow}"'] \\
A \arrow[u, "\tau_1"'] \arrow[r, "f"'] & B \arrow[u, "\tau_2"']
\end{tikzcd}$$

Figure 4.3: Continuous maps.

Definition 4.9. *Let τ_1 be a topology on A, τ_2 be a topology on B, and $f : A \to B$ a map. Then f is called continuous if $syq(f;\tau_2,\tau_1)^{\downarrow}$ is total.*

Now the composition of continuous maps should be also continuous. Consider the following example in Figure 4.4.

$$\begin{tikzcd}
\tau_1 & \tau_2 \arrow[l, "syq(f;\tau_2,\tau_1)^{\downarrow}"'] & \tau_3 \arrow[l, "syq(g;\tau_3,\tau_2)^{\downarrow}"'] \\
A \arrow[u, "\tau_1"'] \arrow[r, "f"'] & B \arrow[u, "\tau_2"'] \arrow[r, "g"'] & C \arrow[u, "\tau_3"']
\end{tikzcd}$$

Figure 4.4: Composition of continuous maps.

The continuity of the composition of continuous maps f and g is an easy exercise, and can be shown as follows:

$$\begin{aligned}
syq(g;\tau_3,\tau_2)^{\downarrow};syq(f;\tau_2,\tau_1)^{\downarrow} &= syq(f;\tau_2;syq(\tau_2,g;\tau_3)^{\downarrow},\tau_1)^{\downarrow} \\
&= syq(f;g;\tau_3,\tau 1)
\end{aligned}$$

4.4 Separation axioms

The definition of topological spaces is very general and sometimes we need to be more specific. Separation axioms helps to define topological spaces with more restricted properties. First, we will introduce these axioms in their traditional forms [2].

- *A topological space is said to satisfy the separation axiom T_0 if for two different points of this space, at least for one of them there exists an open set not containing the other point. Spaces satisfying the separation axiom T_0 are called T_0 of a space or a Kolmogorov space.*

- *A topological space is said to satisfy the separation axiom T_1 if for each of two different points of this space there exists an open set that does not contain the other point. Spaces satisfying the separation axiom T_1 are called T_1 space.*

Now we can define them in terms of relational algebra and using the open set definition of topology.

Definition 4.10. *Let τ be a topology. Then the following axioms are called a*

(1) T_0-space or Kolmogorov space if $\quad syq(\tau^{\smile}, \tau^{\smile}) \sqsubseteq \mathbb{I}$,

(2) T_1-space if $\quad \tau^{\smile} \backslash \tau^{\smile} \sqsubseteq \mathbb{I}$.

Inclusion in above definition is also equality. A topological space fulfilling the axiom (2) also fulfils the axiom (1), since $T_1 \Rightarrow T_0$ which can easily be proved observing

$$
\begin{aligned}
syq(\tau^{\smile}, \tau^{\smile}) &\sqsubseteq \tau^{\smile} \backslash \tau^{\smile} && \text{definition } syq \\
&\sqsubseteq \mathbb{I}. && \text{axiom (2)}
\end{aligned}
$$

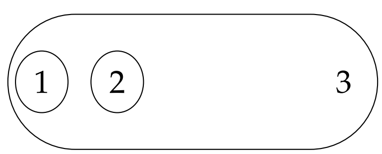

Figure 4.5: Topology τ.

Let τ be a topology as shown in Figure 4.5, then we have

$$
\tau = \begin{array}{c c}
 & \begin{array}{cccccccc} \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}
\end{array}
$$

Separation axioms T_0 and T_1 respectively:

$$syq(\tau^{\smile}, \tau^{\smile}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sqsubseteq \mathbb{I} \qquad \tau^{\smile} \backslash \tau^{\smile} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sqsubseteq \mathbb{I}$$

Chapter 5

Conclusion

This chapter covers a summary of this thesis followed by a discussion of the work planned in the future.

In this thesis we have presented an application of categorical framework of abstract L-fuzzy relations to L-fuzzy topology. The proposed categorical framework is an extension of an arrow category with additional new $*$ operations based on t-norms explained in Chapter 2. We were able to give a new set of definitions of fuzzy topology in terms of relational algebra, and it can be found in Chapter 4. In addition, we have covered related topics such as construction of topologies, set of separation axioms and their proofs.

Throughout the work, we sought to trace the connection between the fuzzy topology on the one hand and the general topology and some other areas of mathematics on the other. Fuzzy topology allows us to take a fresh look at many facts of general topology, on the role of classical logic in general topology.

Speaking about the application of ideas, methods and results of fuzzy topology in applied problems, it has been growing from automata theory to analysing images using topological features. Among the works in which fuzzy topology is used, we note Rosenfeld [12] (discussion of the use of fuzzy topology in image analysis) and others in the field of theory of automata [16, 13]. We hope that our work will contribute to the popularization of ideas of fuzzy topology and, possibly, expanding the list of its applications in applied problems.

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