Journal of Physics: Conference Series

PAPER • OPEN ACCESS

Spinning particles and higher spin field equations

To cite this article: Fiorenzo Bastianelli *et al* 2019 *J. Phys.: Conf. Ser.* **1208** 012006

IOP Conf. Series: Journal of Physics: Conf. Series **1208** (2019) 012006 doi:10.1088/1742-6596/1208/1/012006

Spinning particles and higher spin field equations

Fiorenzo Bastianelli[1], Roberto Bonezzi[1], Olindo Corradini[2,3] and Emanuele Latini[4]

[1] Dipartimento di Fisica ed Astronomia, Università di Bologna and
INFN, Sezione di Bologna, via Irnerio 46, I-40126 Bologna, Italy
[2] Facultad de Ciencias en Física y Matemáticas, Universidad Autónoma de Chiapas, Ciudad Universitaria, Tuxtla Gutiérrez 29050, México
[3] Dipartimento di Scienze Fisiche, Informatiche e Matematiche
Università di Modena e Reggio Emilia, Via Campi 213/A, I-41125 Modena, Italy
[4] Institut für Mathematik, Universität Zürich-Irchel, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

Abstract. Relativistic particles with higher spin can be described in first quantization using actions with local supersymmetry on the worldline. First, we present a brief review of these actions and their use in first quantization. In a Dirac quantization scheme the field equations emerge as Dirac constraints on the Hilbert space, and we outline how they lead to the description of higher spin fields in terms of the more standard Fronsdal-Labastida equations. Then, we describe how these actions can be extended so that the propagating particle is allowed to take different values of the spin, i.e. carry a reducible representation of the Poincaré group. This way one may identify a four dimensional model that carries the same degrees of freedom of the minimal Vasiliev's interacting higher spin field theory. Extensions to massive particles and to propagation on (A)dS spaces are also briefly commented upon.

1. Introduction

Spinning particle actions based on local supersymmetry on the worldline [1, 2] identify amusing systems that exemplify several theoretical constructions and form an arena where to test various methods and ideas. From this perspective they attracted the interest of Victor, who wanted to use them to test quantization methods whose development he had contributed to [3]. These models can be employed to study higher spin field theories in first quantization.

Higher spin theories have recently attracted much interest in a desire to understand better Vasiliev's constructions of interacting theories of higher spin fields [4, 5] and their use in AdS/CFT dualites [6, 7]. Vasiliev's theories are non-lagrangian, and it is at present unclear how to perform their quantization. A first quantized approach, similar to the one used in string theory, might be welcome to address the problem. With this perspective in mind, we have analyzed first quantization of the (higher) spinning particles in a series of paper [8, 9, 10, 11], whose content will be summarized in the following. After that we discuss how to modify the action to allow the particle to take different values of the spin, i.e. carry a reducible representation of the Poincarè group. Ideally, one would like to construct a system that carries the same degrees of freedoms of the Vasiliev's theories, be it a particle, string or more general mechanical system, as a first step towards a first quantized realization of interacting higher spin theories. Here we will find a particle model that carries the same degrees of freedom of a four

IOP Conf. Series: Journal of Physics: Conf. Series **1208** (2019) 012006 doi:10.1088/1742-6596/1208/1/012006

dimensional Vasiliev's theory, though the model is constructed in flat space and extension to AdS remains unclear.

2. Actions

The class of higher spin particles that we are going to discuss are singled out by actions with an $O(N)$-extended supersymmetry on the worldline. They are constructed as supersymmetric extensions of the usual relativistic scalar particle action. The latter has a geometrical interpretation: it is proportional to the length of the worldline. In natural units it is given by

$$S = -m \int ds \tag{1}$$

where $ds = \sqrt{-\dot{x}^\mu \dot{x}_\mu}$, with $x^\mu(\tau)$ the functions that describe the worldlines in a flat spacetime with cartesian coordinates x^μ. The parameter τ is largely arbitrary, as one can perform reparametrizations of the form

$$\tau \to \tau' = \tau'(\tau) \ , \qquad x^\mu(\tau) \to x'^\mu(\tau') = x^\mu(\tau) \tag{2}$$

as long as these transformations are invertible. This is a local symmetry that is crucial to keep Lorentz invariance manifest and recover unitarity of the quantum theory. Indeed one may use the freedom of selecting which parameter to use, and choose $\tau = x^0$ so that x^0 stops being a dynamical variable. Then, the action takes the following standard form without gauge symmetries

$$S[\mathbf{x}(t)] = -m \int dt \ \sqrt{1 - \dot{\mathbf{x}}(t) \cdot \dot{\mathbf{x}}(t)} \tag{3}$$

where $t \equiv x^0$ is the time and $\mathbf{x}$ the space coordinates. However, it is often preferable to keep a manifest Lorentz invariance, and accept a redundancy in the description of the system, by using all of the $x^\mu(\tau)$ as dynamical variables. The local symmetry in (2) manifests itself in the form of a first class constraint when considering the hamiltonian formulation, a preliminary step for canonical quantization. Computing the conjugate momenta p_μ from (1), one finds a constraint, the mass-shell constraint, and a vanishing canonical hamiltonian H_c

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} \quad \to \quad p_\mu p^\mu + m^2 = 0 \ , \qquad H_c = p_\mu \dot{x}^\mu - L = 0 \ . \tag{4}$$

The nontrivial dynamics is fully contained in the constraint, traditionally called H and normalized as

$$H \equiv \frac{1}{2}(p_\mu p^\mu + m^2) = 0 \ . \tag{5}$$

It generates the gauge transformations (the reparametrizations) in phase space. The phase space action takes the form

$$S[x^\mu, p_\mu, e] = \int d\tau \ (p_\mu \dot{x}^\mu - eH) \tag{6}$$

where e is the Lagrange multiplier (the einbein) that implements the constraint $H = 0$. It is a gauge field, since under reparametrizations it transforms as the derivative of the infinitesimal gauge parameter ζ

$$\delta x^\mu = \zeta p^\mu \ , \qquad \delta p_\mu = 0 \ , \qquad \delta e = \dot{\zeta} \ . \tag{7}$$

Eliminating the momenta by their algebraic equations of motion $p^\mu = e^{-1}\dot{x}^\mu$, one finds the configuration space action

$$S[x^\mu, e] = \int d\tau \ \frac{1}{2}(e^{-1}\dot{x}^\mu \dot{x}_\mu - em^2) \tag{8}$$

IOP Conf. Series: Journal of Physics: Conf. Series **1208** (2019) 012006 doi:10.1088/1742-6596/1208/1/012006

which has the advantage over (1) of having a smooth massless limit, just as the phase space action. It can be put in an arbitrarily curved space, and in that form was used in [12] to develop a worldline approach to scalar fields coupled to background gravity.

The $N = 1$ supersymmetric extension of the scalar model produces an action for a spin 1/2 particle. One introduces fermionic partners ψ^μ to the bosonic coordinates x^μ, and gauges the supersymmetry that relates them. The action in phase space takes the form

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2}\psi_\mu \dot{\psi}^\mu - eH - i\chi Q \right) \tag{9}$$

where the first class constraints given by

$$H = \frac{1}{2}p^2 \,, \qquad Q = p_\mu \psi^\mu \tag{10}$$

are gauged by the einbein e and gravitino χ. Both the ψ^μ's and χ are real Grassmann valued variables, and Q is called the susy charge as it generates supersymmetry transformations on the worldline. The constraints realize the $N = 1$ susy algebra in one dimension through the Poisson brackets

$$\{Q, Q\} = -2iH \,. \tag{11}$$

Upon quantization the ψ^μ play the role of the gamma matrices, and the constraint $Q = 0$ becomes the massless Dirac equation in the Dirac quantization scheme. A mass term can be introduced by dimensional reduction. This action was formulated in [13], and its quantum mechanics analysed in [14]. It is used in [15, 16] for developing a worldline description of quantum Dirac fields coupled to background gravity.

The extension to $N = 2$ supersymmetries is quite instructive, as it contains additional elements useful for understanding the general case. One introduces complex fermionic parterns ψ^μ and $\bar{\psi}^\mu$ to the coordinates x^μ, and gauges the full $N = 2$ extended supersymmetry that relates them. The action in phase space takes the form

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + i\bar{\psi}_\mu \dot{\psi}^\mu - eH - i\bar{\chi}Q - i\chi\bar{Q} - a(J - c) \right) \tag{12}$$

where the first class constraints

$$H = \frac{1}{2}p_\mu p^\mu \,, \quad Q = p_\mu \psi^\mu \,, \quad \bar{Q} = p_\mu \bar{\psi}^\mu \,, \quad J = \bar{\psi}^\mu \psi_\mu \,. \tag{13}$$

are gauged by the variables $e, \bar{\chi}, \chi, a$. The constraints realize the $N = 2$ extended susy algebra

$$\{Q, \bar{Q}\} = -2iH \,, \quad \{J, Q\} = iQ \,, \quad \{J, \bar{Q}\} = -i\bar{Q} \tag{14}$$

(other Poisson brackets vanish). Note that there is a $U(1) \sim SO(2)$ group, the so-called R-symmetry group, generated by the charge J that is gauged by the $U(1)$ gauge field a. The latter is allowed to have an additional Chern-Simons coupling constant c, whose net effect is to modify the constraint implemented by a from $J = 0$ to $J - c = 0$. The action is manifestly Poincaré invariant in target space and thus identifies a relativistic model. The $H, Q, \bar{Q}$ constraints guarantee unitarity, as they can be used to eliminate the negative norm states generated by the variables $x^0, \psi^0, \bar{\psi}^0$, while the J constraint guarantees irreducibility of the model, i.e. it describes a particle that carries an irreducible representation of the Poincaré group of target space. It is seen that this model describes a spin 1 massless particle through the free Maxwell equations. Let us explain how this arises in some detail, considering a spacetime of dimension

$D = 4$ for simplicity. Wave functions can be seen as depending on the generalized coordinates x^μ and ψ^μ

$$\phi(x,\psi) = F(x) + F_\mu(x)\psi^\mu + \frac{1}{2}F_{\mu\nu}(x)\psi^\mu\psi^\nu + \frac{1}{3!}F_{\mu\nu\rho}(x)\psi^\mu\psi^\nu\psi^\rho + \frac{1}{4!}F_{\mu\nu\rho\sigma}(x)\psi^\mu\psi^\nu\psi^\rho\psi^\sigma \quad (15)$$

whose Taylor expansion in the ψ's stops, as the latter are Grassmann variables. The momenta $p_\mu = -i\frac{\partial}{\partial x^\mu}$ and $\bar\psi_\mu = \frac{\partial_L}{\partial \psi^\mu}$ act as derivatives (we use left derivatives for the Grassmann variables, meaning that we remove the increment from the left). Now, the classical constraints C become differential operators $\hat C$ that select physical wave functions by requiring $\hat C \phi_{phys}(x,\psi) = 0$. The J constraint suffers from quantum ordering ambiguities, so that using the antisymmetric ordering $J = \frac{1}{2}(\bar\psi^\mu\psi_\mu - \psi_\mu\bar\psi^\mu)$ one finds the differential operator $\hat J = 2 - \psi^\mu\frac{\partial_L}{\partial\psi^\mu}$. Choosing a vanishing Chern-Simons coupling, one finds the constraint $\hat J\phi_{phys}(x,\psi) = 0$ which is solved by

$$\phi_{phys}(x,\psi) = \frac{1}{2}F_{\mu\nu}(x)\psi^\mu\psi^\nu \ . \quad (16)$$

Then, the constraints $\hat Q\phi_{phys} = 0$ gives integrability conditions on the surviving tensor $F_{\mu\nu}(x)$ (Bianchi identities upon the introduction of a gauge potential)

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \quad (17)$$

and the constraint $\hat Q^\dagger\phi_{phys} = 0$, arising form quantizing $\bar Q$, produces the remaining free Maxwell equations

$$\partial^\mu F_{\mu\nu} = 0 \ . \quad (18)$$

Thus we see how the standard description of a free spin 1 massless particle emerges in first quantization. More generally, one can use different values of the quantized Chern-Simons coupling to describe differential p-forms satisfying generalized Maxwell equations in arbitrary dimensions [17]. This description was used in [18, 19] to treat spin 1 and antisymmetric tensor fields coupled to gravity in first quantization.

The general case, where one introduces N real fermionic partners ψ_i^μ $(i = 1, ..., N)$ associated to the bosonic coordinates x^μ, and gauges the resulting $O(N)$-extended supersymmetry present on the worldline, was discussed in [1, 2], and describes a particle of spin $s = \frac{N}{2}$. The action in phase space takes the form

$$S \ = \ \int d\tau \left(p_\mu\dot x^\mu + \frac{i}{2}\psi_{i\mu}\dot\psi_i^\mu - eH - i\chi_i Q_i - \frac{1}{2}a_{ij}J_{ij}\right) \quad (19)$$

where the first class constraints

$$H = \frac{1}{2}p_\mu p^\mu \ , \quad Q_i = p_\mu\psi_i^\mu \ , \quad J_{ij} = i\psi_i^\mu\psi_{j\mu} \quad (20)$$

are gauged by the fields e, χ_i, a_{ij}. The Poisson bracket algebra of the constraints is indeed that of the $O(N)$-extended supersymmetry

$$\begin{aligned} \{Q_i, Q_j\} \ &= \ -2i\delta_{ij}H \ , \qquad \{J_{ij}, Q_k\} = \delta_{jk}Q_i - \delta_{ik}Q_j \\ \{J_{ij}, J_{kl}\} \ &= \ \delta_{jk}J_{il} - \delta_{ik}J_{jl} - \delta_{jl}J_{ik} + \delta_{il}J_{jk} \ . \end{aligned} \quad (21)$$

Quantization à la Dirac shows that the model describes a massless particle with spin $s = \frac{N}{2}$ in terms of the Bargmann-Wigner equations [20]. Let us review briefly the analysis for integer spin s. In this case one can form complex combinations of the fermionic variables

$$\psi_I^\mu = \frac{1}{\sqrt 2}(\psi_i^\mu + i\psi_{i+s}^\mu) \ , \qquad \bar\psi^{\mu I} = \frac{1}{\sqrt 2}(\psi_i^\mu - i\psi_{i+s}^\mu) \ , \qquad I = i = 1, .., s \quad (22)$$

IOP Conf. Series: Journal of Physics: Conf. Series **1208** (2019) 012006 doi:10.1088/1742-6596/1208/1/012006

and have s pairs (indexed by I) of fermionic creation/annihilation operators (of course, each pair has an additional spacetime index acting as a spectator). In this basis only the subgroup $U(s) \subset SO(2s)$ is manifest. As in the $N = 2$ case, one can take the ψ_I^μ as fermionic coordinates, on which a generic wave function may depend, and $\bar{\psi}_\mu^I = \frac{\partial}{\partial \psi_I^\mu}$ as corresponding momenta, realized as left derivatives with respect to the coordinates. Then it follow that the generic wave function $R(x, \psi)$ contains all possible tensors having s blocks of antisymmetric indices, as the indices of each block arise from the Taylor expansion of the same type of fermion, (i.e. a fermion with the same internal index I), in a way similar to what seen in eq. (15). Then one must impose the quantum constraints on the wave function. The constraints due to the $SO(N)$ charges J_{ij} select the tensor with s blocks of $d = \frac{D}{2}$ indices, implying that a nontrivial solution is present for even spacetime dimensions D only

$$R_{\mu_1^1..\mu_d^1,...,\mu_1^s..\mu_d^s} \; . \tag{23}$$

In addition, the J_{ij} constraints require this tensor to be totally traceless and with the symmetries of a Young tableau with d rows and s columns. The susy charges can also be split in pairs of complex conjugate charges, just like the fermions. The contraints from the susy charges $Q_I = p_\mu \psi_I^\mu$ imply integrability conditions of the form

$$\partial_{[\mu} R_{\mu_1^1..\mu_d^1],...,\mu_1^s..\mu_d^s} = 0 \tag{24}$$

for each block (interpreted as Bianchi identities once solved). The other half of susy charges $\bar{Q}_I = p_\mu \bar{\psi}_I^\mu$ produce "Maxwell equations" of the form

$$\partial^\mu R_{\mu\mu_2^1..\mu_d^1,...,\mu_1^s..\mu_d^s} = 0 \; . \tag{25}$$

The H constraint is satisfied identically as consequence of the algebra. These are the geometric equations that describe a free field of spin s, equivalent to the massless Bargmann-Wigner equations, usually given with the wave function in a multispinor basis [20]. They are called geometric as the tensors R can be interpreted as linearized curvatures. A proper analysis of these equations, showing in particular that they are equivalent to the Fronsdal-Labastida ones and that they propagate the correct degrees of freedom, has been carried out in [21, 22, 23], also reviewed in [24]. The above equations have also the property of being conformal invariant [25, 26]. They can be related to the more standard description in terms of the Fronsdal-Labastida equations [27, 28], as we are going to describe next.

3. Fronsdal-Labastida equations

In this section we discuss how the Fronsdal-Labastida equations, describing the propagation of a massless particle of integer spin $s = \frac{N}{2}$, emerge from solving some of the constraints of the canonical analysis of the previous section. In analogy with the case of the electromagnetism, we introduce the gauge potential

$$\varphi(x, \psi) := \varphi(x)_{\mu_1^1..\mu_{d-1}^1,...,\mu_1^s..\mu_{d-1}^s} \psi_1^{\mu_1^1}..\psi_1^{\mu_{d-1}^1} \cdots \psi_s^{\mu_1^s}..\psi_s^{\mu_{d-1}^s} \tag{26}$$

with the symmetries of a rectangular Young tableau with $d-1$ rows and s columns, and we solve the condition (24) by setting

$$R(x, \psi) = Q_1 \cdots Q_s \varphi(x, \psi) \tag{27}$$

where we have used the complex charges Q_I with $I = 1, ...s$. A generalisation of the Poincaré lemma assures that this solution is unique, modulo the equivalence relation, or gauge symmetry, of the form $\varphi(x,\psi) \sim \varphi(x,\psi) + Q_I \xi^I(x,\psi)$. Observe now that the curvature must vanish when we trace over the sectors I and J, making the tensor in (23) totally traceless: this condition is encoded into the $SO(N)$ constraint J^{IJ} and produces the following equation

$$\frac{\partial}{\partial \psi_I^\rho} \frac{\partial}{\partial \psi_{J\rho}} R(x,\psi) = 0 \qquad \rightarrow \qquad Q_1..Q_{I-1}Q_{I+1}..Q_{J-1}Q_{J+1}..Q_s(\mathbf{G}\varphi(x,\psi)) = 0 \tag{28}$$

where the charges Q_I and Q_J are missing, and where we have introduced the Fronsdal-Labastida operator defined by

$$\mathbf{G} := -2H + Q_I \bar{Q}^I + \frac{1}{2} Q_I Q_J J^{IJ} \, . \tag{29}$$

The next task is to get rid of the operator on the left of the Fronsdal-Labastida operator in (28). This can be achieved by introducing a new independent field, called the compensator

$$\rho^{IJK}(x,\psi) := w^{\mu\nu\delta}(x) \frac{\partial}{\partial \psi_I^\mu} \frac{\partial}{\partial \psi_J^\nu} \frac{\partial}{\partial \psi_K^\delta} \rho(x,\psi) \tag{30}$$

where $\rho(x,\psi)$ has the same symmetries and expansion of the gauge potential given in (26). The compensator parametrizes elements in the kernel of the operator in (28) acting on $\mathbf{G}\varphi(x,\psi)$. Thus, one may rewrite eq. (28) as

$$\mathbf{G}\varphi(x,\psi) = Q_I Q_J Q_K \rho^{IJK}(x,\psi) \, . \tag{31}$$

This is the Fronsdal-Labastida equation with compensator. The latter was introduced in this context in [29, 30]. One can also use part of the gauge symmetry to set the compensator to zero, and recover the original Fronsdal-Labastida system

$$\mathbf{G}\, \varphi(x,\psi) = 0 \tag{32}$$

that enjoys the residual gauge invariance with a traceless gauge parameter

$$\delta\varphi = Q_I \xi^I(x,\psi) \, , \quad \text{with} \quad J^{[IJ} \xi^{K]}(x,\psi) = 0 \, . \tag{33}$$

Note that, for consistency, the gauge field has to be doubly traceless $J^{IJ} J^{KL} \varphi(x,\psi) = 0$. This condition is a consequence of (32). In four dimensions the gauge field is a completely symmetric tensor $\varphi_{\mu_1 \cdots \mu_s}$, and eq. (32) translates into the Fronsdal equation

$$\partial^\nu \partial_\nu \varphi_{\mu_1 \cdots \mu_s} - (\partial_{\mu_1} \partial^\nu \varphi_{\nu\mu_2 \cdots \mu_s} + \cdots) + (\partial_{\mu_1} \partial_{\mu_2} \varphi^\nu{}_{\nu\mu_3 \cdots \mu_s} + \cdots) = 0 \tag{34}$$

where the brackets contain s and $\frac{1}{2}s(s-1)$ terms, respectively, needed for symmetrizing the μ indices. Of course, the gauge field must be doubly traceless, i.e. $\varphi^{\nu\delta}{}_{\nu\delta\mu_5 \cdots \mu_s} = 0$.

We have presented the analysis for integer spins only, but a similar program can be carried out for the case of half-integer spins as well [31]. The equivalent BRST quantization for this model is described in [32, 33]. In particular, in [33] one uses it to construct second quantized actions for any spin in flat spaces of arbitrary dimensions. Quantized point particles of any spin have been treated also in [34].

4. Other gaugings and the case of $U(s)$

We have seen that the constraints of the $O(N)$ spinning particle allow to recover a unitary irrep of the Poincaré group, corresponding to a massless particle with fixed spin (helicity). Unitarity is guaranteed by the hamiltonian constraint H and the susy constraints Q_i, as they can be used to remove the dangerous polarizations in the wave function generated by the x^0

and ψ_i^0 quantum variables. The additional constraints due to the $O(N)$ charges J_{ij} can be relaxed without destroying unitarity of the quantum theory, as they serve the only purpose of selecting an irreducible representation of the Poincaré group. In fact, one may actually prefer to describe the propagation of a multiplet of states, especially if one imagines that they could be made interacting somehow. One example was treated in [35] for the $O(4)$ particle in $D = 4$. As $O(4) \sim SU(2) \otimes SU(2)$, Pashnev and Sorokin gauged only a $SU(2)$ factor, finding that the emerging model propagates a graviton and three scalars. In general, one may investigate the consequences of gauging different subgroups of the $O(N)$ symmetry group. For example, one may not gauge anything at all, and thus find the propagation of a maximum number of states with different spins. A more refined option is to gauge the $U(1)^s$ subgroup of $O(N)$ (we consider even $N = 2s$, restricting ourselves to bosonic particles). Each factor $U(1)$ may have an additional independent Chern-Simons coupling, useful to project to the subsector of the wave functions containing a fixed number of antisymmetric indices of each favour (different I's indicate different flavours). This projection was briefly discussed earlier for the $N = 2$ case, and constitutes a useful method to project to a desired subsector of the Hilbert space. It has been used often in worldline applications, as in [36] and [37].

In this section we are going to analyze, as a particular example, the gauging of a $U(s)$ subgroup of the full R-symmetry group $O(2s)$. In the previous sections the $O(2s)$ generators were split in $U(s)$ covariant form as $J_{ij} = (J_{IJ}, J^{IJ}, J^I_J)$, where J_{IJ} and J^{IJ} insert the metric tensor and compute a trace in the IJ family of indices, respectively, while J^I_J is the $U(s)$ generator that performs anti-symmetrization of indices between the aforementioned families (we recall that an upper index J is equivalent to a lower index $\bar{J}$). By gauging only the J^I_J generators one gains the freedom of adding a Chern-Simons coupling c, as in the $N = 2$ model, see eq. (12). The phase space action will thus read

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + i\bar{\psi}^I_\mu \dot{\psi}^\mu_I - eH - i\bar{\chi}^I Q_I - i\chi_I \bar{Q}^I - a^I_J (J^J_I - c\,\delta^J_I) \right) . \tag{35}$$

The presence of the Chern-Simons coupling allows to set the eigenvalue of the number operators $N_I := J^I_I$ with fixed I (i.e. no summation) to any desired value. In particular, by setting $c = p+1-D/2$ one obtains curvature tensors described by rectangular Young tableaux with $p+1$ rows and s columns in any spacetime dimension D. Indeed, the covariant Dirac quantization of the model gives as physical field a curvature R, which forms an irreducible $GL(D)$ tensor with the symmetries of the rectangular Young tableau described above, obeying the Maxwell-like equations

$$Q_I\, R = \bar{Q}^I\, R = 0 \, . \tag{36}$$

These Maxwell-like fields propagate a multiplet of single-particle states, as the trace constraint is not imposed. This is analogous to the case of the $U(N)$ spinning particles, introduced in [38] and analyzed in [39], modelling particles in a euclidean complex space, which do not have any natural trace operator (see refs. [40, 41] for further analysis of these amusing systems). As field theory models, these Maxwell-like fields were introduced at the level of gauge potentials in [42] following ideas developed in [43], while their curvature description was studied in [44, 45].

In the following we perform a light-cone analyses of the particle model in order to count the physical degrees of freedom. We define light-cone coordinates in target space as $x^\pm := \frac{1}{\sqrt{2}}(x^1 \mp x^0)$, while x^a denote transverse coordinates, such that $A \cdot B = A_+B_- + A_-B_+ + A_aB_a$. By using worldline reparametrizations one can gauge fix $x_+ = \tau$ and solve the corresponding constraint $H = 0$ by setting $p_- = -\frac{p_a p_a}{2p_+}$, since p_+ is assumed to be invertible in light-cone analysis. Now one can use the local supersymmetries to gauge fix $\psi_{+I} = \bar{\psi}^I_+ = 0$, and solve

$Q_I = \bar{Q}^I = 0$ with $\psi_{-I} = -\frac{p_a \psi_{aI}}{p_+}$ and $\bar{\psi}^I_- = -\frac{p_a \bar{\psi}^I_a}{p_+}$. The action thus reduces to

$$S = \int d\tau \left(p_+ \dot{x}^+ + p_a \dot{x}^a - \frac{p_a p_a}{2p_+} + i\bar{\psi}^I_a \dot{\psi}^a_I - a^I_J (\hat{J}^J_I - c\,\delta^J_I) \right) , \tag{37}$$

where we have defined the reduced $U(s)$ generator as $\hat{J}^I_J = \psi^a_J\,\bar{\psi}^I_a$. At the quantum level this generator suffers from an ordering ambiguity, for which we choose again the anti-symmetric ordering, such that

$$\hat{J}^I_J = \frac{1}{2}[\psi^a_J, \bar{\psi}^I_a] = \psi^a_J\,\bar{\psi}^I_a - \delta^I_J\,\frac{D-2}{2} ,$$

and the quantum constraint $\hat{J}^I_J - c\,\delta^I_J$ acts as the differential operator $(\psi^a_I \frac{\partial}{\partial \psi^a_J} - p\,\delta^J_I)$, for $c = p + 1 - \frac{D}{2}$. One can see then that the states are tensors of $GL(D-2)$, and the constraint $\hat{J}^I_J - c\,\delta^I_J$ provides $GL(D-2)$ irreducibility by fixing the length of the columns to be p, and by gluing the columns together in a rectangular Young tableau with p rows and s columns. The physical state consists thus of a massless irreducible tensor of $GL(D-2)$ with the symmetries spelled above. Its particle spectrum is provided by the branching of the rectangular Young tableau of $GL(D-2)$ into traceless $SO(D-2)$ representations. In particular, for $p = 1$ we have a symmetric $GL(D-2)$ tensor of rank s, and the spectrum is given by massless particles of spin $s, s-2, s-4, ..$ down to spin zero or one, according to if s is even or odd, respectively. It is rather interesting to note that taking even spin s and sending $s \to \infty$ one finds the spectrum of the Vasiliev's minimal bosonic models, that contain even spins ranging from zero to infinity [5].

To summarize this section, we have discussed how a unitary spectrum is maintained by relaxing the gauging of the full $O(N)$ R-symmetry group to a subgroup $G \subset O(N)$. There are many options to choose from, and we have indicated only a few without the pretention of being complete. As a final example to check further how relaxing the gauging increases the unitary spectrum, one may consider the simple case of $O(4)$ in $D = 4$ dimensions. Gauging the full $O(4)$ group produces the spectrum of a pure spin $s = 2$ (2 dof (degrees of freedom)), gauging a $U(2) \subset O(4)$ with $p = 1$, as discussed above, produces the Maxwell-like spectrum containing a spin 2 and a spin 0 (3 dof), gauging a $SU_L(2) \subset SU_L(2) \otimes SU_R(2) = O(4)$ produces the Pashnev-Sorokin spectrum of a graviton plus 3 scalars (5 dof), while gauging nothing at all gives the spectrum with a maximum number of propagating states consisting in a graviton, four vectors, and six scalars (16 dof). As final curiosity, one may check that gauging the subgroup $U(1)^2$, as indicated at the beginning of the section, one obtains the unitary propagation of massless spins $s = 2$, $s = 1$, and $s = 0$ (5 dof), that includes a vector on top of the Maxwell-like spectrum. This is achieved by fixing the two independent Chern-Simons couplings, allowed in the present situation, to have physical states (in light-cone quantization) with occupation number $p = 1$ for each flavor of worldline fermions.

5. Massive particles and couplings to AdS

A class of actions for massive particles of higher spins can be obtained by dimensionally reducing the massless model through the Scherk-Schwarz mechanism [46]. The massive particle lives in odd dimensions, as the original massless model with the fully gauged $O(N)$-extended supersymmetry lives in even dimensions. Further, taking the massless limit produces a model propagating several helicities. This procedure gives another method for generating actions for the propagation of a multiplet of particle states [11]. In addition, one might add the option of gauging a smaller subgroup of the R-symmetry group, thus producing an extended range of possibilities. We are not going to analyze them further here.

A different option to construct worldline models for higher spinning particles is that of gauging some part of the symmetry algebra of the $Sp(2N)$ quantum mechanics described in [47], where instead of anticommuting variables ψ_i^μ one uses complex commuting variables z_i^μ and $\bar{z}_i^\mu$. This approach has been analyzed in [48].

A key problem is to study allowed interactions of massless particles of higher spin. As already mentioned, a set of consistent interacting models has been produced by Vasiliev, that constructed suitable nonlinear field equations on (A)dS backgrounds. At the first quantized level, an initial step it to study how the $O(N)$ spinning particles could be coupled to a nontrivial background. For a while it was thought that no coupling could be allowed at all, but eventually a way to couple them to the (A)dS spaces [49], and more generally to conformally flat spaces [9], was found. In [9] the method employed was that of covariantizing the constraint algebra in phase space, and check if the algebra could be still of first class. On (A)dS the algebra becomes quadratic, with the only modification with respect to the flat space case (21) sitting in the Poisson brackets of the susy charges

$$\{Q_i, Q_j\} \;=\; -2i\delta_{ij}H + ib\Big(J_{ik}J_{jk} - \frac{1}{2}\delta_{ij}J_{kl}J_{kl}\Big) \tag{38}$$

where the constant b is related to the (A)dS curvature scalar by $b = \frac{R}{D(D-1)}$. On conformally flat backgrounds, the algebra acquires more complicated structure functions. The model can be quantized, and in [10] we have used it to study the one-loop effective action on (A)dS spaces, computing the first few Seeley-DeWitt coefficients, namely the ones that are related to the diverging terms in four dimensions. A similar calculation presumably could be carried out for the massive models introduced in [11], though the gauge fixing procedure may be more difficult because of the more complicated structure functions. We expect that restricting the calculation of the effective action to the (A)dS spaces from the start (in [10] we have kept the background arbitrary, and restricted to the (A)dS case only at the end) one might push the calculation to the next perturbative order (a three loop calculation on the worldline). The conterterms to be used for such a calculation are already available [50]. On an arbitrary geometry a three loop calculation is quite demanding [51], but the simplications due to restricting to (A)dS spaces make it much more manageable [52]. We plan to report on that task in the near future.

Acknowledgments

The work of OC was partly supported by the UCMEXUS-CONACYT grant CN-12-564. EL acknowledges partial support from SNF Grant No. 200020-149150/1.

References

[1] Gershun V and Tkach V 1979 *JETP Lett.* **29** 288–291
[2] Howe P S, Penati S, Pernici M and Townsend P K 1988 *Phys.Lett.* **B215** 555
[3] Villanueva V M, Govaerts J and Lucio-Martinez J L 2000 *J.Phys.* **A33** 4183–4202 (*Preprint* `hep-th/9909033`)
[4] Vasiliev M A 1990 *Phys.Lett.* **B243** 378–382
[5] Vasiliev M 2003 *Phys.Lett.* **B567** 139–151 (*Preprint* `hep-th/0304049`)
[6] Sezgin E and Sundell P 2002 *Nucl.Phys.* **B644** 303–370 (*Preprint* `hep-th/0205131`)
[7] Klebanov I and Polyakov A 2002 *Phys.Lett.* **B550** 213–219 (*Preprint* `hep-th/0210114`)
[8] Bastianelli F, Corradini O and Latini E 2007 *JHEP* **0702** 072 (*Preprint* `hep-th/0701055`)
[9] Bastianelli F, Corradini O and Latini E 2008 *JHEP* **0811** 054 (*Preprint* `0810.0188`)
[10] Bastianelli F, Bonezzi R, Corradini O and Latini E 2012 *JHEP* **1212** 113 (*Preprint* `1210.4649`)
[11] Bastianelli F, Bonezzi R, Corradini O and Latini E 2014 *JHEP* **1409** 158 (*Preprint* `1407.4950`)
[12] Bastianelli F and Zirotti A 2002 *Nucl.Phys.* **B642** 372–388 (*Preprint* `hep-th/0205182`)
[13] Brink L, Deser S, Zumino B, Di Vecchia P and Howe P S 1976 *Phys.Lett.* **B64** 435
[14] Henneaux M and Teitelboim C 1982 *Annals Phys.* **143** 127
[15] Bastianelli F, Corradini O and Zirotti A 2003 *Phys.Rev.* **D67** 104009 (*Preprint* `hep-th/0211134`)
[16] Bastianelli F, Corradini O and Zirotti A 2004 *JHEP* **0401** 023 (*Preprint* `hep-th/0312064`)
[17] Howe P S, Penati S, Pernici M and Townsend P K 1989 *Class.Quant.Grav.* **6** 1125
[18] Bastianelli F, Benincasa P and Giombi S 2005 *JHEP* **0504** 010 (*Preprint* `hep-th/0503155`)

[19] Bastianelli F, Benincasa P and Giombi S 2005 *JHEP* **0510** 114 (*Preprint* `hep-th/0510010`)
[20] Bargmann V and Wigner E P 1948 *Proc.Nat.Acad.Sci.* **34** 211
[21] Bekaert X and Boulanger N 2004 *Commun.Math.Phys.* **245** 27–67 (*Preprint* `hep-th/0208058`)
[22] Bekaert X and Boulanger N 2003 *Phys.Lett.* **B561** 183–190 (*Preprint* `hep-th/0301243`)
[23] Bekaert X and Boulanger N 2003 *Mixed symmetry gauge fields in a flat background, Proceedings of the 5th International Seminar on Supersymmetries and Quantum Symmetries (SQS'03), Dubna, Russia* (Dubna: JINR, 2004) (*Preprint* `hep-th/0310209`)
[24] Bekaert X and Boulanger N 2007 *Commun.Math.Phys.* **271** 723–773 (*Preprint* `hep-th/0606198`)
[25] Siegel W 1988 *Int.J.Mod.Phys.* **A3** 2713–2718
[26] Siegel W 1989 *Int.J.Mod.Phys.* **A4** 2015
[27] Fronsdal C 1978 *Phys.Rev.* **D18** 3624
[28] Labastida J 1989 *Nucl.Phys.* **B322** 185
[29] Francia D and Sagnotti A 2002 *Phys. Lett.* **B543** 303–310 (*Preprint* `hep-th/0207002`)
[30] Francia D and Sagnotti A 2003 *Class.Quant.Grav.* **20** S473–S486 (*Preprint* `hep-th/0212185`)
[31] Corradini O 2010 *JHEP* **1009** 113 (*Preprint* `1006.4452`)
[32] Marnelius R and Martensson U 1989 *Nucl.Phys.* **B321** 185
[33] Siegel W 1999 *Fields, Chapter XII* (*Preprint* `hep-th/9912205`)
[34] Henneaux M and Teitelboim C 1987 *First and second quantized point particles of any spin, Santiago 1987 Proceedings, Quantum mechanics of fundamental systems 2, 113-152* (Plenum Press, 1989)
[35] Pashnev A and Sorokin D P 1991 *Phys.Lett.* **B253** 301–305
[36] Bastianelli F, Bonezzi R, Corradini O and Latini E 2013 *JHEP* **1310** 098 (*Preprint* `1309.1608`)
[37] Bastianelli F and Bonezzi R 2013 *JHEP* **1307** 016 (*Preprint* `1304.7135`)
[38] Marcus N 1995 *Nucl.Phys.* **B439** 583–596 (*Preprint* `hep-th/9409175`)
[39] Bastianelli F and Bonezzi R 2009 *JHEP* **0903** 063 (*Preprint* `0901.2311`)
[40] Bastianelli F and Bonezzi R 2011 *JHEP* **1109** 018 (*Preprint* `1107.3661`)
[41] Bastianelli F, Bonezzi R and Iazeolla C 2012 *JHEP* **1208** 045 (*Preprint* `1204.5954`)
[42] Campoleoni A and Francia D 2013 *JHEP* **1303** 168 (*Preprint* `1206.5877`)
[43] Francia D 2010 *Phys. Lett.* **B690** 90–95 (*Preprint* `1001.5003`)
[44] Francia D 2012 *Class. Quant. Grav.* **29** 245003 (*Preprint* `1209.4885`)
[45] Bekaert X, Boulanger N and Francia D 2015 *J. Phys.* **A48** 225401 (*Preprint* `1501.02462`)
[46] Scherk J and Schwarz J H 1979 *Nucl.Phys.* **B153** 61–88
[47] Hallowell K and Waldron A 2007 *SIGMA* **3** 089 (*Preprint* `0707.3164`)
[48] Bastianelli F, Corradini O and Waldron A 2009 *JHEP* **0905** 017 (*Preprint* `0902.0530`)
[49] Kuzenko S and Yarevskaya Z 1996 *Mod.Phys.Lett.* **A11** 1653–1664 (*Preprint* `hep-th/9512115`)
[50] Bastianelli F, Bonezzi R, Corradini O and Latini E 2011 *JHEP* **1106** 023 (*Preprint* `1103.3993`)
[51] Bastianelli F and Corradini O 2001 *Phys.Rev.* **D63** 065005 (*Preprint* `hep-th/0010118`)
[52] Bastianelli F and Hari Dass N 2001 *Phys.Rev.* **D64** 047701 (*Preprint* `hep-th/0104234`)