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MOORE-GIBSON-THOMPSON THERMOELASTICITY

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Abstract: We consider a thermoelastic theory where the heat conduction is described by the Moore-Gibson-Thompson equation. In fact, this equation can be obtained after the introduction of a relaxation parameter in the Green-Naghdi type III model. We analyse the one and the three dimensional cases. In three dimensions we obtain the well-posedness and the stability of solutions. In one dimension we get the exponential decay and the instability of the solutions depending on the conditions over the system of constitutive parameters. We also propose possible extensions for these theories.

Keywords: Moore-Gibson-Thompson equation, relaxation parameter, thermoelasticity, existence, stability.

1. Introduction

The Moore-Gibson-Thompson equation (MGT, for short) has deserved a big interest in the recent years and many papers has been directed to study and understand it (see among others [4, 12, 14, 21, 28, 29]). It arises from the fluids mechanics [38]. In this paper we approach the MGT equation from the viewpoint of thermoelasticity. We will see that it can be derived in a natural way from one of the Green-Naghdi heat conduction models. We will use it to propose a new system of equations for the thermoelasticity.

Let us recall that the classical theory of heat conduction based on the Fourier law jointly with the energy equation

$$q_{i,i} = c\dot{\theta}, \tag{1.1}$$

(where q_i os the flux vector, θ is the temperature and c is the thermal capacity) predicts the instantaneous propagation of the thermal waves. This fact contradicts the *causality principle*. For this reason many scientists have proposed alternative equations for the heat conduction. Most known theory is the one introduced by Maxwell and Cattaneo [1], which changes the Fourier law by a constitutive equation which containing a relaxation parameter in the following form

$$\tau\dot{q}_i + q_i = k\theta_{,i}, \tag{1.2}$$

where k is the thermal conductivity and τ is the relaxation parameter which is assumed to be positive. As usual, a dot over the variable means the time derivative, while the colon and subscript $_i$ means the spatial derivative with respect to the corresponding variable.

When one adjoins (1.2) with (1.1), a damped hyperbolic equation is obtained. In this case the thermal waves propagate with finite speed. A lot of attention has been paid to this equation. It

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has also been extended to the thermoelasticity situation as the Lord-Shulman theory [18]. This proposition has been deeply investigated (see among others [2, 9, 10, 11, 34, 35]). In fact the quantity of contributions for this theory is huge.

At the end of the last century Green and Naghdi [6, 7, 8] introduced three new alternative theories for the heat conduction based on a rational way that they called type I, II and III respectively.The linear version of the type I agrees with the one proposed by the Fourier law. The type II drives to another hyperbolic equation for the heat conduction. In this case there is no dissipation and the flux vector is obtained as a linear expression of the thermal displacement. That is

$$q_i = k^* \alpha_{,i}, \ \ \dot{\alpha} = \theta,$$

where k^* is the conductivity rate parameter. The type III theory is described by the constitutive equation

$$q_i = k^* \alpha_{,i} + k\theta_{,i}.$$

Notice that when $k^* = 0$ we recover the type I, meanwhile type II is obtained when $k = 0$. This theory has also deserved much attention in recent years (see among others [5, 15, 16, 13, 25, 20, 22, 23, 24, 26, 30, 32, 33, 37]). If k and k^* are both positive and we adjoin this last equation with (1.1) we obtain a generalization of the Fourier classical theory. The exponential decay of solutions has been obtained in this situation. However this theory has the same drawback as the usual Fourier theory and it also predicts the instantaneous propagation of the thermal waves (see for instance [36], page 39). Again the causality principle is not satisfied. Therefore, it is also natural to modify this proposal introducing a relaxation parameter in the constitutive equation to overcome this problem. That is, to consider the equation

$$\tau \dot{q}_i + q_i = k^* \alpha_{,i} + k\theta_{,i}. \tag{1.3}$$

This equation is the natural generalization of (1.2) and therefore the combination of it with (1.1) will be the natural extension of the Cattaneo-Maxwell heat equation.

It is remarkable that if we adjoin equation (1.3) to (1.1), we obtain the linear version of the Moore-Gibson-Thompson equation. When $k^* = 0$ we recover the Cattaneo-Maxwell heat equation.

Given a heat conduction theory a thermoelastic counterpart can be developed. Therefore from this new thermal equation we can propose a thermoelastic theory. This is the aim of this paper. We want to propose and analyse the basic properties of the new proposal where we see the MGT equation is considered as the heat equation.

In the next section we obtain the MGT-inhomogeneous equation and we describe the basic properties. Later we study a thermoelastic system where the heat equation is defined by the MGT-equation. Well-posedness of the problem is obtained as well as the stability of the solutions in the three dimensional case. In Section four we consider the one-dimensional situation and we prove the exponential decay of solutions as well as the instability when an inequality is satisfied by the parameters (Section five). Last section is devoted to propose several generalizations of the MGT-equation in the context of the heat conduction with memory.

2. Inhomogeneous MGT-equation

The general constitutive equation for the linear type III theory for centrosymmetric materials is given by

$$q_i(\mathbf{x}, t) = (k_{ij}(\mathbf{x})\theta_{,j}) + (k^*_{ij}(\mathbf{x})\alpha_{,j}), \tag{2.1}$$

where k_{ij} is the thermal conductivity tensor and k^*_{ij} is the conductivity rate tensor. The inhomogeneous version for the energy equation is

$$q_{i,i} = c(\mathbf{x})\dot{\theta}. \tag{2.2}$$

As we have pointed out before, if we adjoin these two equations we obtain

$$c(\mathbf{x})\dot{\theta} = (k^*_{ij}(\mathbf{x})\alpha_{,i})_{,j} + (k_{ij}(\mathbf{x})\theta_{,i})_{,j} \tag{2.3}$$

which allows the propagation of the thermal waves with infinite speed and violates the causality principle. To overcome this drawback we propose to introduce a relaxation parameter into equation (2.1) as in the Maxwell-Cattaneo case.

We obtain an expression of the form

$$\tau\dot{q}_i(\mathbf{x},t) + q_i(\mathbf{x},t) = (k_{ij}(\mathbf{x})\theta_{,j}) + (k^*_{ij}(\mathbf{x})\alpha_{,j}), \tag{2.4}$$

where $\tau > 0$. If we substitute relation (2.4) into (2.2), we obtain an inhomogeneous version of the MGT-equation which can be written as

$$\tau c(\mathbf{x})\dddot{\theta} + c(\mathbf{x})\ddot{\theta} = (k^*_{ij}(\mathbf{x})\theta_{,i})_{,j} + (k_{ij}(\mathbf{x})\dot{\theta}_{,i})_{,j}. \tag{2.5}$$

We consider this equation in a three-dimensional domain B whose boundary is smooth enough to apply the divergence theorem. To have a well-posed problem we need to introduce the initial conditions:

$$\theta(\mathbf{x},0) = \theta^0(\mathbf{x}),\ \ \dot{\theta}(\mathbf{x},0) = \phi^0(\mathbf{x})\ \ \ddot{\theta}(\mathbf{x},0) = \psi^0(\mathbf{x}),\ \ \mathbf{x} \in B, \tag{2.6}$$

and the homogeneous Dirichlet boundary conditions

$$\theta(\mathbf{x},t) = 0,\ \ \mathbf{x} \in \partial B. \tag{2.7}$$

To simplify the notation we will denote $K_{ij}(\mathbf{x}) = k_{ij}(\mathbf{x}) - \tau k^*_{ij}(\mathbf{x})$

From now on, we assume that the constitutive tensors are upper bounded and that:

(i) There exists a positive constant c_0 such that $c(\mathbf{x}) \geq c_0$.

(ii) There exists a positive constant k_0 such that

$$k^*_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i, \tag{2.8}$$

for every vector ξ_i.

(iii) There exists a positive constant k_1 such that

$$K_{ij}\xi_i\xi_j \geq k_1\xi_i\xi_i, \tag{2.9}$$

for every vector ξ_i.

It is worth noting that (ii) and (iii) imply that there exists a positive constant k_2 such that

$$k_{ij}\xi_i\xi_j \geq k_2\xi_i\xi_i, \tag{2.10}$$

for every vector ξ_i.

Condition (ii) is natural in the type II/III theories to guarantee the stability of the solutions. Inequality (2.10) is a consequence of the second principle (see [6]). Estimate (iii) is imposed to guarantee the stability of solutions and we will see later a instability result when this condition fails. The meaning of (i) is clear.

In view of the arguments and results proposed in [19] we can guarantee that under the assumptions imposed above there exists a quasi-contractive semigroup in $H^1_0(B)\times H^1_0(B)\times L^2(B)$[1] which

[1]Here we denote the usual Sobolev space.

generates the solutions to the problem determined by the equation and the initial and boundary conditions. However, under our assumptions it can be proved the existence of a contractive semigroup. In fact, in this case the conservation of the energy reads

$$E(t) + F(t) = E(0), \tag{2.11}$$

where

$$E(t) = \frac{1}{2}\int_B \left(c|\dot{\theta}|^2 + c\tau^2|\ddot{\theta}|^2 + 2c\tau\dot{\theta}\ddot{\theta} + k^*_{ij}\theta_{,i}\theta_{,j} + k_{ij}\tau\dot{\theta}_{,i}\dot{\theta}_{,j} + 2k^*_{ij}\tau\theta_{,i}\dot{\theta}_{,i} \right) dv \tag{2.12}$$

and

$$F(t) = \int_0^t \int_B K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j} dv. \tag{2.13}$$

In fact, the exponential decay of the solutions can be proved[2].

We note that this energy is equivalent to the norm induced by the usual inner product in $H^1_0(B) \times H^1_0(B) \times L^2(B)$. To see it, we can point out that

$$k^*_{ij}\theta_{,i}\theta_{,j} + k_{ij}\tau\dot{\theta}_{,i}\dot{\theta}_{,j} + 2k^*_{ij}\tau\theta_{,i}\dot{\theta}_{,i} = k^*_{ij}(\theta_{,i} + \tau\dot{\theta}_{,j})(\theta_{,j} + \tau\dot{\theta}_{,j}) + \tau K_{ij}\dot{\theta}_{,i}\dot{\theta}_{,j}. \tag{2.14}$$

It is worth noting that we could also obtain the MGT-heat equation in an alternative way in the context of the three-phase-lag theory [3]. If we consider the equation

$$q_i(\mathbf{x}, t + \tau_1) = (k_{ij}(\mathbf{x})\theta_{,j}(t + \tau_2) + (k^*_{ij}(\mathbf{x})\alpha_{,j}(t + \tau_3)), \tag{2.15}$$

and assume that $\tau_2 = \tau_3$ and $\tau = \tau_1 - \tau_2 > 0$ we will obtain the MGT-equation whenever we make the approximation $q_i(\mathbf{x}, t + \tau) \approx q_i(\mathbf{x}, t) + \tau\dot{q}_i(\mathbf{x}, t)$.

3. Thermoelasticity

We can extend the comments and the problems proposed for the heat equation to a thermoelastic situation. We need the equations

$$t_{ij} = C_{ijkl}u_{k,l} - \beta_{ij}\theta, \quad \eta = c\theta + \beta_{ij}u_{i,j}. \tag{3.1}$$

Here (u_i) is the displacement vector, t_{ij} is the stress tensor, η is the entropy, C_{ijkl} is the elasticity tensor which satisfies the symmetry condition

$$C_{ijkl} = C_{klij}, \tag{3.2}$$

and β_{ij} is the coupling tensor.

We adjoin the evolution equations

$$\rho\ddot{u}_i = t_{ij,j}, \quad T_0\dot{\eta} = q_{i,i} \tag{3.3}$$

where T_0 is the reference temperature which is assumed uniform and $\rho(\mathbf{x})$ is the mass density.

After substitution of the constitutive equations into the evolution equations we obtain the system

$$\rho\ddot{u}_i = (C_{ijkl}u_{k,l} - \beta_{ij}\theta)_{,j} \tag{3.4}$$

$$c\ddot{\theta} + c\tau\dddot{\theta} = -\beta_{ij}(\ddot{u}_{i,j} + \tau\dddot{u}_{i,j}) + (k_{ij}(\mathbf{x})\dot{\theta}_{,j})_{,i} + (k^*_{ij}(\mathbf{x})\theta_{,j})_{,i} \tag{3.5}$$

where we have assumed $T_0 = 1$ to simplify the calculations.

[2]We will give later an argument which can be applied to this easier case. We don't do it here to not repeat the arguments.

If we denote $\hat{f} = f + \tau\dot{f}$ and $v_i = \dot{u}_i$ our system implies that

$$\rho\ddot{\hat{v}}_i = (C_{ijkl}\hat{v}_{k,l} - \beta_{ij}(\dot{\theta} + \tau\ddot{\theta}))_{,j} \tag{3.6}$$

$$c\ddot{\theta} + c\tau\dddot{\theta} = -\beta_{ij}\dot{\hat{v}}_{i,j} + (k_{ij}(\mathbf{x})\dot{\theta}_{,j})_{,i} + (k^*_{ij}(\mathbf{x})\theta_{,j})_{,i}, \tag{3.7}$$

It is clear that the solutions for this last system generates the solutions for our primitive system (3.4)-(3.5). Therefore, from now on, we will omit the hats.

Apart from the assumptions proposed before, here and from now on, we assume that the constitutive tensors are upper boundeded and

(iv) There exists a positive constant ρ_0 such that $\rho(\mathbf{x}) \geq \rho_0$.
(v) There exists a positive constant C_0 such that

$$\int_B C_{ijkl}u_{i,j}u_{k,l}dv \geq C_0 \int_B u_{i,j}u_{i,j}dv, \tag{3.8}$$

for every vector (u_i) vanishing at the boundary of the domain.

The meaning of condition (iv) is clear. Condition (v) can be understood with the help of the elastic stability theory.

To the initial and boundary conditions proposed above for the thermal variables, we also adjoin that

$$v_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \;\; \dot{v}_i(\mathbf{x}, 0) = w_i^0(\mathbf{x}), \;\; x \in B, \tag{3.9}$$

and the boundary conditions

$$v(\mathbf{x}, t) = 0, \;\; \mathbf{x} \in \partial B. \tag{3.10}$$

Again the existence and uniqueness of solutions is guaranteed by the result obtained in [19]. However in that contribution the authors prove the existence of a quasi-contractive semigroup. We here want to give a new step and to prove that in our case the semigroup is contractive and therefore the stability of solutions will be guaranteed.

We now transform our initial boundary value problem into a Cauchy problem in a suitable Hilbert space. We consider

$$\mathcal{H} = \mathbf{H}_0^1(B) \times \mathbf{L}^2(B) \times H_0^1(B) \times H_0^1(B) \times L^2(B). \tag{3.11}$$

If $U = (\mathbf{v}, \mathbf{w}, \theta, \phi, \psi)$ and $U^* = (\mathbf{v}^*, \mathbf{w}^*, \theta^*, \phi^*, \psi^*)$ we define the inner product
(3.12)

$$\langle U, U^*\rangle = \frac{1}{2}\int_B \left(\rho w_i\bar{w}_i^* + C_{ijkl}v_{i,j}\bar{v}^*_{k,l} + c(\phi + \tau\psi)(\bar{\phi}^* + \tau\bar{\psi}^*) + k^*_{ij}(\theta_{,i} + \tau\phi_{,i})(\bar{\theta}^*_{,j} + \tau\bar{\phi}^*_{,j}) + \tau K_{ij}\phi_{,i}\bar{\phi}^*_{,j}\right) dv.$$

As usual, the bar over a variable means the conjugated complex. It is worth noting that under the assumptions proposed previously the norm defined by this inner product is equivalent to the usual one in $\mathcal{H}$.

We define several operators

$$A_i(\mathbf{v}) = \rho^{-1}(C_{ijkl}v_{k,l})_{,j}, \quad \mathbf{A} = (A_i)$$
$$B_i(\phi) = -\rho^{-1}(\beta_{ij}\phi)_{,j}, \quad \mathbf{B} = (B_i)$$
$$C_i(\psi) = -\rho^{-1}(\beta_{ij}\psi)_{,j}, \quad \mathbf{C} = (C_i)$$
$$D^*(\mathbf{w}) = -(c\tau)^{-1}\beta_{ij}w_{i,j}, \quad E(\theta) = (c\tau)^{-1}(k^*_{ij}\theta_{,j})_{,i}$$

$$F(\phi) = (c\tau)^{-1}(k_{ij}\phi_{,j})_{,i}, \quad G(\psi) = -\tau^{-1}\psi$$

and the matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \mathbf{A} & 0 & 0 & \mathbf{B} & \mathbf{C} \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & D^* & E & F & G \end{pmatrix}. \tag{3.13}$$

Our initial boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = U^0, \tag{3.14}$$

where

$$U^0 = (\mathbf{v}^0, \mathbf{w}^0, \theta^0, \phi^0, \psi^0).$$

It is clear that the domain of the operator $\mathcal{D}(\mathcal{A})$ is a dense subspace on the Hilbert space $\mathcal{H}$.

Lemma 3.1. *For every $U \in \mathcal{D}(\mathcal{A})$, $Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq 0$.*

Proof. We note that

$$\text{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_B K_{ij}\phi_{,i}\phi_{,j}dv.$$

In view of the assumptions our lemma is proved. □

Lemma 3.2. *0 belongs to the resolvent of $\mathcal{A}$ (in short, $0 \in \rho(\mathcal{A})$).*

Proof. Let $(\mathbf{f}_1, \mathbf{f}_2, f_3, f_4, f_5) \in \mathcal{H}$. We have to prove the existence of solutions to the system

$$\mathbf{w} = \mathbf{f}_1, \ \phi = f_3, \ \psi = f_4 \tag{3.15}$$

$$\mathbf{A}\mathbf{v} + \mathbf{B}\phi + \mathbf{C}\psi = \mathbf{f}_2, \quad D^*\mathbf{w} + E\theta + F\phi + G\psi = f_5. \tag{3.16}$$

We have the desired solutions for $\mathbf{w}, \phi$ and ψ. To obtain the solutions for $\mathbf{v}$ and θ we have to solve the system

$$\mathbf{A}\mathbf{v} = \mathbf{f}_2 - \mathbf{B}f_3 - \mathbf{C}f_4, \quad E\theta = f_5 - D^*\mathbf{f}_1 - Ff_3 - Gf_4. \tag{3.17}$$

In view of the properties of the elliptic systems we can obtain the solution for the last two unknowns and the proof is complete. □

Theorem 3.3. *The operator $\mathcal{A}$ generates a contraction C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ in $\mathcal{H}$.*

Proof. The theorem is a consequence of the two previous lemmas and the use of the Lumer-Phillips corollary to the Hille–Yosida theorem (see, e.g., [27]). □

As we have a contractive semigroup we know that

$$||S(t)U|| \leq ||U||,$$

for every $t \geq 0$.

Finally, as a consequence, we state the main result of this section.

Theorem 3.4. *Assume that $U^0 \in \mathcal{D}(\mathcal{A})$. Then, there exists a unique solution $U(t) \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), \mathcal{D}(\mathcal{A}))$ to problem (3.14).*

4. Exponential stability in the one dimensional case

If we consider a one-dimensional homogeneous material we deal with the system

$$\rho\ddot{v} = \mu v_{xx} - \beta(\dot{\theta}_x + \tau\ddot{\theta}_x) \tag{4.1}$$

$$c\ddot{\theta} + c\tau\dddot{\theta} = -\beta\dot{v}_x + k\dot{\theta}_{xx} + k^*\theta_{xx} \tag{4.2}$$

In this section we will study the exponential stability for the problem determined by this system with the initial conditions:
(4.3)
$$v(x,0) = v_0(x),\ \ \dot{v}(x,0) = w_0(x),\ \ \theta(x,0) = \theta_0(x),\ \ \dot{\theta}(x,0) = \phi(x)\ \ \ddot{\theta}(x,0) = \psi(x),\ \ x \in (0,l),$$

and the boundary conditions

$$v(0,t) = v(l,t) = \theta_x(0,t) = \theta_x(l,t) = 0. \tag{4.4}$$

That is, we assume Dirichlet boundary conditions for the mechanical variable and Neumann conditions for the thermal component. It is worth saying that the boundary conditions are proposed in such a way that allows to simplify the mathematical analysis.

The parameters that appear in the system are related with the properties of the material and have to satisfy some thermomechanical restrictions. In particular, we assume that

$$c > 0,\ \rho > 0,\ \mu > 0,\ k^* > 0, k > k^*\tau,\ \beta \neq 0, \tau > 0. \tag{4.5}$$

Our assumptions are in agreement with the thermomechanical axioms and the empirical experiments. The assumptions concerning the mass density and the thermal capacity are obvious. The condition on μ can be understood with the help of the elastic stability. The conditions on the k, k^* and τ are the natural ones to have dissipation. The assumption on β is strictly needed to guarantee the coupling between the mechanical and the thermal parts.

The aim of this section is to determine the behaviour of the solutions (with respect to the time) to the problem. In fact, we want to prove that the solutions decay in an exponential way when the appropriate damping mechanisms are considered in the system.

We note that there are solutions (uniform in the variable x) that do not decay. To avoid these cases, we will also assume that

$$\int_0^l \theta_0(x)\,dx = \int_0^l \phi_0(x)\,dx = \int_0^l \psi_0(x)\,dx = 0. \tag{4.6}$$

We consider the Hilbert space

$$\mathcal{H} = \{(v, w, \theta, \phi, \psi) \in H_0^1 \times L^2 \times H_*^1 \times H_*^1 \times L_*^2\}, \tag{4.7}$$

where

$$L_*^2 = \{f \in L^2, \int_0^\pi f(x)dx = 0\} \text{ and } H_*^1 = L_*^2 \cap H^1.$$

Taking into account that $\dot v = w$, $\dot\theta = \phi$ and $\dot\phi = \psi$ and writing $D = \frac{d}{dx}$, we can restate our system in the following way:

$$
\begin{cases}
\dot v = w \\
\dot w = \dfrac{1}{\rho}\big(\mu D^2 u - \beta D\phi - \tau\beta D\psi\big) \\
\dot\theta = \phi \\
\dot\phi = \psi \\
\dot\psi = \dfrac{1}{c\tau}\big(kD^2\phi - \beta Dw + k^* D^2\theta - \tau\psi\big)
\end{cases}
\tag{4.8}
$$

Moreover, if $U = (v, w, \theta, \phi, \psi)$, then our initial-boundary value problem can be written as

$$
\frac{dU}{dt} = \mathcal{A}U, \quad U^0 = (v^0, w^0, \theta^0, \phi^0, \psi^0),
\tag{4.9}
$$

where $\mathcal{A}$ is the following 5×5-matrix

$$
\mathcal{A} = \begin{pmatrix}
0 & I & 0 & 0 & 0 \\
\frac{\mu}{\rho}D^2 & 0 & 0 & \frac{\beta}{\rho}D & \frac{\beta\tau}{\rho}D \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I \\
0 & -\frac{\beta}{c\tau}D & \frac{k^*}{c\tau}D^2 & \frac{k}{c\tau}D^2 & -1/c
\end{pmatrix}
\tag{4.10}
$$

and I is the identity operator. We note that the domain of $\mathcal{A}$, that we will denote (again) by $\mathcal{D}(\mathcal{A})$, is dense in $\mathcal{H}$.

We define an inner product in $\mathcal{H}$. If $U^* = (v^*, w^*, \theta^*, \phi^*, \psi^*)$, then

$$
\begin{aligned}
\langle U, U^*\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^\pi \Big(&\rho w\bar w^* + c\phi\bar\phi^* + c\tau^2\psi\bar\psi^* + c\tau(\phi\bar\psi^* + \bar\phi^*\psi) + \mu v_x \bar v_x^* \\
&+ k\tau\phi_x\bar\phi_x^* + k\theta\bar\theta^* + k^*\tau(\theta\bar\phi^* + \bar\theta^*\phi)\Big)dx.
\end{aligned}
\tag{4.11}
$$

It is worth mentioning that this product is equivalent to the usual product in the Hilbert space $\mathcal{H}$. In fact, it is the natural restriction of the one proposed previously to this section.

Lemma 4.1. *For every $U \in \mathcal{D}(\mathcal{A})$, $Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le 0$.*

Proof. Direct computation gives

$$
\mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{k - k^*\tau}{2}\int_0^l |\phi_x|^2 dx.
$$

As we are assuming that $k^*\tau < k$ the lemma is proved. □

Lemma 4.2. *0 belongs to the resolvent of $\mathcal{A}$ (in short, $0 \in \rho(\mathcal{A})$).*

Proof. The proof can be obtained in a similar way to the one proposed in the previous section. □

In view of these two lemmas and recalling the fact that the domain of the operator is dense, we can use the Lumer-Phillips corollary to the Hille-Yosida theorem to obtain the following result.

Theorem 4.3. *The operator given by matrix $\mathcal{A}$ generates a contraction C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\ge 0}$ in $\mathcal{H}$.*

Now, we will show the exponential decay of the solutions for our problem.

To prove the exponential decay, we recall the characterization stated in the book of Liu and Zheng [17].

Theorem 4.4. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if the following two conditions are satisfied:*

(i) $i\mathbb{R} \subset \rho(\mathcal{A})$,
(ii) $\overline{\lim\limits_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Lemma 4.5. *The operator $\mathcal{A}$ defined in (4.10) satisfies $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. We here follow the arguments given in the book of Liu and Zheng ([17], page 25). Let us assume that the intersection of the imaginary axis and the spectrum is non-empty. Therefore, there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors $U_n = (v_n, w_n, \theta_n, \phi_n, \psi_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \to 0. \tag{4.12}$$

In our case, writing this condition term by term we get

$$i\lambda_n v_n - w_n \to 0 \text{ in } H^1, \tag{4.13}$$

$$i\lambda_n w_n - \frac{1}{\rho}\left(\mu D^2 v_n - \beta D\phi_n - \beta\tau D\psi_n\right) \to 0 \text{ in } L^2, \tag{4.14}$$

$$i\lambda_n \theta_n - \phi_n \to 0 \text{ in } H^1, \tag{4.15}$$

$$i\lambda_n \phi_n - \psi_n \to 0 \text{ in } H^1, \tag{4.16}$$

$$i\lambda_n \psi_n - \frac{1}{c\tau}\left(-\beta D w_n + kD^2\phi_n + k^* D^2\theta_n - \tau\psi_n\right) \to 0 \text{ in } L^2. \tag{4.17}$$

In view of the dissipative term for the operator, we see that

$$D\phi_n \to 0 \text{ in } L^2. \tag{4.18}$$

From (4.15) we also see that $D\theta_n \to 0$ in L^2. We now want to see that ψ_n tends to zero in L^2. To this end we multiply (4.16) by ψ_n to see that

$$i\langle\phi_n, \lambda_n\psi_n\rangle - ||\psi_n||^2 \to 0 \tag{4.19}$$

The convergence of ψ_n will be guaranteed whenever we show that $\langle\phi_n, \lambda_n\psi_n\rangle \to 0$. From (4.17) we see

$$ic\tau\langle\phi_n, \lambda_n\psi_n\rangle = \langle\phi_n, -\beta D w_n + kD^2\phi_n + k^*D^2\theta_n - \tau\psi_n\rangle \tag{4.20}$$

After integration by parts and taking into account the convergences previously noted we see that the right hand side tends to zero and therefore the convergence of ψ_n to zero follows in L^2. We now prove that v_n tends to zero in H^1. We see from (4.17) that

$$-\beta D v_n + \lambda_n^{-1} kD^2\phi_n + \lambda_n^{-1} k^* D^2\theta_n \to 0 \text{ in } L^2. \tag{4.21}$$

After multiplication by Dv_n we see

$$\beta||Dv_n||^2 + k\langle D\phi_n, \lambda_n^{-1}D^2 v_n\rangle + k^*\langle D\theta_n, \lambda_n^{-1}D^2 v_n\rangle \to 0. \tag{4.22}$$

But from (4.14) we see that $\lambda_n^{-1}D^2v_n$ is bounded. Therefore we see that $\langle D\phi_n, \lambda_n^{-1}D^2v_n\rangle$ and $\langle D\theta_n, \lambda_n^{-1}D^2v_n\rangle$ tend to zero and therefore we also obtain that Dv_n converges to zero. The convergence of w_n is obtained after multiply (4.14) by v_n. This contradicts the fact that the sequence has unit norm and the lemma is proved. □

Lemma 4.6. *The operator $\mathcal{A}$ satisfies*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. The proof also follows a contradiction argument. Suppose that the thesis is not true. It follows the existence of a sequence of real numbers λ_n such that $|\lambda_n| \to \infty$ and a sequence of unit vectors in $\mathcal{D}(\mathcal{A})$ in such a way that (4.12) holds. Therefore, conditions (4.13)–(4.17) still hold. Now we can use a similar argument to the one used in the proof of the previous lemma because the key point is that λ_n does no tend to zero. □

The two previous lemmas give rise to the following result.

Theorem 4.7. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)U\| \leq M\|U\|e^{-\alpha t}$.*

Proof. The proof is a direct consequence of the two previous Lemmas and the Theorem 4.4. □

5. Instability of solutions

The aim of this section is to prove that when $k < k^*\tau$ we can find a solution of the problem which is unstable. To make the calculations easier we assume that $l = \pi$. Our analysis is similar to the one proposed for the dual-phase-lag equation [31].

The solutions in this case will be combinations of functions of the form

$$v = A\exp(\omega t)\sin nx, \theta = B\exp(\omega t)\cos nx. \tag{5.1}$$

We obtain the following homogeneous system with the unknowns A and B:

$$A(\rho\omega^2 + \mu n^2) - B\beta n(\omega + \tau\omega^2) = 0 \tag{5.2}$$

$$A\beta n\omega + B((c\omega^2 + c\tau\omega^3 + k\omega n^2 + k^* n^2)) = 0 \tag{5.3}$$

Our aim is to obtain a nontrivial solution. To this end, we impose that the determinant of the system above is zero. Here ω must be a solution of the equation

$$\begin{aligned}\rho c\tau x^5 + \rho c x^4 + (\rho k n^2 + \beta^2 n^2\tau + \mu n^2 c\tau)x^3 \\ (\rho k^* n^2 + \mu n^2 c + \beta^2 n^2)x^2 + \mu n^4 k x + \mu n^4 k^* = 0.\end{aligned} \tag{5.4}$$

We use the Hurwitz theorem that says that the necessary and sufficient condition to guarantee that the solutions of the equation

$$x^5 + q_1x^4 + q_2x^3 + q_3x^2 + q_4x + q_5 = 0, \tag{5.5}$$

have negative real part is:

$$\Lambda_1 = q_1 > 0, \Lambda_2 = \det\begin{pmatrix} q_1 & 1 \\ q_3 & q_2\end{pmatrix} > 0, \Lambda_3 = \det\begin{pmatrix} q_1 & 1 & 0 \\ q_3 & q_2 & q_1 \\ q_5 & q_4 & q_3\end{pmatrix} > 0, \tag{5.6}$$

$$\Lambda_4 = \det \begin{pmatrix} q_1 & 1 & 0 & 0 \\ q_3 & q_2 & q_1 & 1 \\ q_5 & q_4 & q_3 & q_2 \\ 0 & 0 & q_5 & q_4 \end{pmatrix} > 0, \text{ and } \Lambda_5 = \det \begin{pmatrix} q_1 & 1 & 0 & 0 & 0 \\ q_3 & q_2 & q_1 & 1 & 0 \\ q_5 & q_4 & q_3 & q_2 & 1 \\ 0 & 0 & q_5 & q_4 & q_3 \\ 0 & 0 & 0 & 0 & q_5 \end{pmatrix} > 0. \tag{5.7}$$

We have that in our case

$$\Lambda_2 = \frac{n^2}{c}(\tau^{-1}k - k^*) \tag{5.8}$$

which is less than zero whenever $k < k^*\tau$. Therefore the instability of solutions follows.

Again the analysis does not depend on the length of the interval and our argument can be extended without troubles to intervals with arbitrary but bounded length.

6. Further comments

The equations studied previously can be extended to consider problems depending on the history.

We assume that

$$\lim_{t\to-\infty} q_i(\mathbf{x},t) \text{ is bounded }.$$

We have that

$$\tau\dot{q}_i + q_i = \tau(\dot{q}_i + \tau^{-1}q_i) = \tau\exp(-\tau^{-1}t)\frac{d}{dt}(q_i\exp(\tau^{-1}t)). \tag{6.1}$$

In view of (2.4) we see

$$\frac{d}{dt}(q_i\exp(\tau^{-1}t)) = \tau^{-1}\exp(\tau^{-1}t)\left(k_{ij}(\mathbf{x})\theta_{,j} + k^*_{ij}(\mathbf{x})\alpha_{,j}\right) \tag{6.2}$$

and

$$q_i\exp(\tau^{-1}t) = \int_{-\infty}^{t}\exp(\tau^{-1}s)\left(\tau^{-1}k_{ij}(\mathbf{x})\theta_{,j}(s) + \tau^{-1}k^*_{ij}(\mathbf{x})\alpha_{,j}(s)\right)ds. \tag{6.3}$$

Then

$$q_i = \int_{-\infty}^{t}\exp(-\tau^{-1}(t-s))\left(\tau^{-1}k_{ij}(\mathbf{x})\theta_{,j}(s) + \tau^{-1}k^*_{ij}(\mathbf{x})\alpha_{,j}(s)\right)ds. \tag{6.4}$$

This relation can be generalized in the following form

$$q_i = \int_{-\infty}^{t} h(t-s)\left(k_{ij}(\mathbf{x})\theta_{,j}(s) + k^*_{ij}(\mathbf{x})\alpha_{,j}(s)\right)ds, \tag{6.5}$$

where $h(s)$ is a non-increasing function.

Moreover we can think in the expression

$$q_i = \int_{-\infty}^{t}(h_{ij}(\mathbf{x},t-s)\theta_{,j}(s) + h^*_{ij}(\mathbf{x},t-s)\alpha_{,j}(s))ds, \tag{6.6}$$

where $h_{ij}(s)$ and $h^*_{ij}(s)$ are non-increasing functions.

The juxtaposition of these equations with equation (2.1) determines a possible extension of the MGT-equation to the context of the heat conduction. Therefore our equation becomes

$$c\dot{\theta} = \int_{-\infty}^{t} (h_{ij}(\mathbf{x}, t-s)\theta_{,j}(s) + h^*_{ij}(\mathbf{x}, t-s)\alpha_{,j}(s))_{,i} ds. \tag{6.7}$$

As far as the author knows this equation has not deserved much attention in the literature yet.

The same happens if we consider (3.1) and (3.3) and the proposed equations for the heat flux vector. Therefore we can determine a new system of equations for the thermoelasticity.

Of course to determine the problems we need to impose initial and boundary conditions, but this is a natural task.

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References

[1] Cattaneo, C. *On a form of heat equation which eliminates the paradox of instantaneous propagation*, C. R. Acad. Sci. Paris, 247, 431–433, 1958.
[2] Chandrasekharaiah, D. S. *Hyperbolic thermoelasticity: a review of recent literature*, Appl. Mech. Rev. 51, 705–729, 1998.
[3] Choudhuri, S. K. R. *On a thermoelastic three-phase-lag model*, J. Thermal Stresses, 30, 231-238, 2007.
[4] Conejero, J. A., Lizama C. and Ródenas, F. *Chaotic behaviour of the solutions of the Moore- Gibson-Thompson equation*, Applied Mathematics and Information Sciences, 9, 2233-2238, 2015.
[5] Giorgi, C., Grandi, D., Pata, V. *On the Green-Naghdi type III heat conduction model.* Discrete and Continuous Dynamical Systems B, 19, pp. 2133–2143, 2014.
[6] Green, A. E., Naghdi, P. M. *On undamped heat waves in an elastic solid*, J. Thermal Stresses, 15, 253–264, 1992.
[7] Green, A. E., Naghdi, P. M. *Thermoelasticity without energy dissipation*, J. Elasticity, 31, 189–208, 1993.
[8] Green, A. E., Naghdi, P. M. *A unified procedure for contruction of theories of deformable media. I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua.* Proc. Royal Society London A 448, 335–356, 357–377, 379–388, 1995.
[9] Hetnarski, R. B. and Ignaczak, J. *Generalized thermoelasticity*, J. Thermal Stresses, 22, 451-470, 1999.
[10] Hetnarski, R. B. and Ignaczak, J. *Nonclassical dynamical thermoelasticity*, International J. Solids Structures, 37, 215-224, 2000.
[11] Ignaczak, J. and Ostoja-Starzewski, M. *Thermoelasticity with Finite Wave Speeds*, Oxford: Oxford Mathematical Monographs, (2010).
[12] Kaltenbacher, B. Lasiecka, I. and Marchand, R. *Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Control Cybernet, 40, 971-988, 2011.
[13] Kothari, S. and Mukhopadhyay, S. *Study of harmonic plane waves in rotating thermoelastic media of tyep III* Math. Mech. solids, 17, 824-839, 2012
[14] Lasiecka, I and Wang, X. *Moore-Gibson-Thompson equation with memory, part II: General decay of energy*, J. Differential Equations, 259, 7610-7635, 2015.
[15] Liu, Y. Y. Lin, Y. and Li, Y. *Convergence result for the thermoelasticity of type III* Appl. Math. Letters 26, 97-102, 2013.
[16] Liu, Z., Quintanilla, R. *Analiticity of solutions in type III thermoelastic plates.* [Republished article.] IMA Journal of Applied Mathematics, 75, 637–646, 2010.
[17] Liu, Z., Zheng, S. *Semigroups associated with dissipative systems*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 398, Chapman & Hall/CRC, Boca Raton, FL, (1999).
[18] Lord, H. W., Shulman, Y. *A generalized dynamical theory of thermoelasticity.* Journal of the Mechanics and Physics of Solids, 15, 299–309, 1967.

[19] Magaña, A., Quintanilla, R. *On the existence and uniqueness in phase-lag thermoelasticity* Meccanica 53, 125–14, 2018.

[20] Magaña, A., Quintanilla, R. *Exponential stability in type III thermoelasticity with microtemperatures.* Zeitschrift fur Angewandte Mathematik und Physik (ZAMP) 69(5), 129(1)–129(8), 2018.

[21] Marchand, R. McDevitt, T. and Triggiani, R. *An abstract semigroup approach to the third order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability*, Math. Methods Appl. Sci., 35, 1896-1929, 2012.

[22] Miranville, A. and Quintanilla, R. *Some generalizations of the Caginalp phase-field system*, Applicable Analysis, 88, 877-894, 2009.

[23] Miranville, A. and Quintanilla, R. *A type III phase-field system with logarithmic potential*, Appl. Math. Letters, 24, 1003-1008, 2011.

[24] Miranville, A., Quintanilla, R.*Exponential stability in type III thermoelasticity with voids.* Appl. Math. Letters 94, 30-37, 2019.

[25] Mukhopadhyay, S. and Kumar, R. *A problem of thermoelastic interactions in an infinite medium with a cylindrical hole in generalized thermoelasticity III*, Jour. therm. Stres., 31, 455-475, 2008.

[26] Mukhopadhyay, S. and Prasad, R. and Kumar, R. *Variational and reciprocal principles in linear theory of type-III thermoelasticity.* Math. and Mechanics of Solids 16, 435–444, 2011.

[27] Pazy, A. *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, (1983).

[28] Pellicer, M. and Said-Houari, B. *Wellposedness and decay rates for the Cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Appl. Math. Optimization, 1-32, 2017.

[29] Pellicer, M. and Sola-Morales, J. *Optimal scalar products in the Moore-Gibson-Thompson equation*, Evolution Equations and control Theory, 8, 203-220, 2019.

[30] Quintanilla, R. *Structural stability and continuous dependence of solutions in thermoelasticity of type III.* Discrete and Continuous Dynamical Systems, B, 1, 463–470, 2001.

[31] Quintanilla R. *A condition on the delay parameters in the one-dimensional dual-phase-lag thermoelastic theory*, Journal of Thermal Stresses. 26, 713–721, 2003.

[32] Quintanilla, R. *Convergence and structural stability in thermoelasticity*, Applied Mathematics and Computation, 135, 287–300, 2003.

[33] Quintanilla, R., Straughan, B. *Growth and uniqueness in thermoelasticity.* Proc. Roy. Soc. London A, 455, 1429–1429, 2000.

[34] Racke, R. *Asymptotic behavior of solutions in linear 2- or 3-D thermoelasticity with second sound* Quart. Appl. Math. 61, 315-328, 2003.

[35] Racke, R. *Thermoelasticity with second sound—exponential stability in linear and non-linear 1-d* , Math.l Meth. Appl. Sci. 25, 409-441, 2002.

[36] Renardy, M. Hrusa W. J. and Nohel, J. A. *Mathematical Problems in Viscoelasticity*, Longman Scientific and Technical, (1987).

[37] Straughan, B. *Heat Waves*, Appl. Math. Sci. **177**, Springer-Verlag, Berlin, (2011).

[38] Thompson, P. A. *Compressible-Fluid Dynamics*, McGraw-Hill, (1972).