GREEN FUNCTIONS FOR BOUNDARY VALUE PROBLEMS ON PRODUCT NETWORKS

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1. Introduction

Green's functions on a connected network are closely related with self-adjoint boundary value problems for positive semidefinite Schrödinger operators (BVP in the sequel). Since the matrices associated with these class of operators are symmetric M-matrices, each Green function can be interpreted as the group inverse of such a matrices. There exists a very interesting variety of self-adjoint boundary value problems on a network, see for instance [2], that leads to interesting results in many areas including the properties of random walks, chip-firing games, analysis of online communities, machine learning, algorithms and load balancing in networks. In addition, we also can interpret these BVP as the discrete analogue of the corresponding problems for elliptic differential operators or even as the discretization of such a boundary value problems.

For sake of simplicity, we restrict ourselves here to analyze either the *Dirichlet Problem* or the *Poisson equation*. For product networks, these kind of boundary value problems have been studied by F. Chung, R. Ellis and S.T. Yau, see [6, 8, 9], considering the normalized Laplacian. However, since in general the normalized Laplacian of a product network is not expressible in separated variables involving the normalized Laplacian of the factor networks, in the above referred works the authors must consider only cartesian product of regular networks, that is also a regular network. We remark that in this case, the problem is reduced to the analysis of the combinatorial Laplacian, since that for regular networks the normalized Laplacian is a multiple of the combinatorial one.

As a motivation of our work, we consider $\mathsf{A} \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $\mathsf{B}\in \mathcal{M}_{m\times m}(\mathbb{R})$ two irreducibles and symmetric M-matrices. If $\mathsf{I}_n \in \mathcal{M}_{n\times n}(\mathbb{R})$ denotes the identity matrix and $\mathsf{B} = (b_{ij})$, we can consider the M–matrix $\mathsf{M} \in \mathcal{M}_{nm\times nm}(\mathbb{R})$ defined as

$$\mathsf{M} = \begin{bmatrix} \mathsf{A} + b_{11}\mathsf{I}_n & b_{12}\mathsf{I}_n & \cdots & b_{1m}\mathsf{I}_n \\ b_{21}\mathsf{I}_n & \mathsf{A} + b_{22}\mathsf{I}_m & \cdots & b_{2m}\mathsf{I}_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}\mathsf{I}_n & b_{m2}\mathsf{I} & \cdots & \mathsf{A} + b_{mm}\mathsf{I}_n \end{bmatrix}$$

and then we raised the following questions:

(i) Is the spectrum of M related with the spectra of A and B?
(ii) Is $\mathsf{M}^{\#}$ related with $(\mathsf{A} + z\mathsf{I})^{\#}$ and $(\mathsf{B} + w\mathsf{I})^{\#}$ for some (or many) $z, w \in \mathbb{C}$?

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Here, $\mathsf{K}^{\#}$ denotes the group inverse of K.

We will take advantage by considering A and B as operators on Finite Networks and M as an operator on the Product Network. Our treatment appears as the discrete version of the *Separation of Variables Method* for BVP for PDE.

2. Finite Networks and Schrödinger Operators

A finite network $\Gamma = (V, c)$, consists of a finite set V, called *vertex set* and a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$, called *conductance*, satisfying that $c(x, x) = 0$ for any $x \in V$. Two vertices $x, y \in V$ are *adjacent* iff $c(x, y) > 0$ and we always assume that Γ is *connected*.

In what follows $\mathcal{C}(V) = \mathcal{C}(V; \mathbb{R})$ and $\mathcal{C}(V; \mathbb{C})$ stand respectively for the spaces of real and complex functions defined on the vertex set V. Given $v \in \mathcal{C}(V; \mathbb{C})$, $\bar{v}$ denotes its conjugate and then, $\langle u, v\rangle = \sum\limits_{x\in V} u(x)\bar{v}(x)$ determines an inner product on $\mathcal{C}(V; \mathbb{C})$, whose associated norm is denoted by $||\cdot||$. Given $u \in \mathcal{C}(V, \mathbb{C})$, $u^{\perp}$ denotes the subespace of $\mathcal{C}(V, \mathbb{C})$ orthogonal to u. For any $x \in V$, ε_x is the *Dirac function* at x. Moreover, κ denotes the (generalized) *degree of* Γ; that is, the funcion defined as $\kappa(x) = \sum\limits_{y\in V} c(x, y)$, for any $x \in V$.

A real-valued function $\omega \in \mathcal{C}(V)$ is called *weight* if $\omega(x) > 0$ for any $x \in V$ and in addition $||\omega|| = 1$. The sets of weights on V is denoted by $\Omega(V)$ or simply by Ω when it does not lead to confusion.

Given $F \subset V$ a nonempty subset, F^c denotes its complementary and $\mathcal{C}(F)$ and $\mathcal{C}(F; \mathbb{C})$ are the subspaces of real and complex functions vanishing on F^c. It is clear that $\mathcal{C}(F)$ and $\mathcal{C}(F; \mathbb{C})$ can be identified respectively with the space of real or complex functions defined on F. Moreover, the set

$$\delta(F) = \big\{z \in F^c : c(z, y) > 0 \text{ for some } y \in F\big\}$$

is called the *boundary of* F and then, $\bar{F} = F \cup \delta(F)$ is the *closure* F, see Figure 1. Clearly, $\delta(F) = \emptyset$, or equivalently $F = \bar{F}$, iff $F = V$.

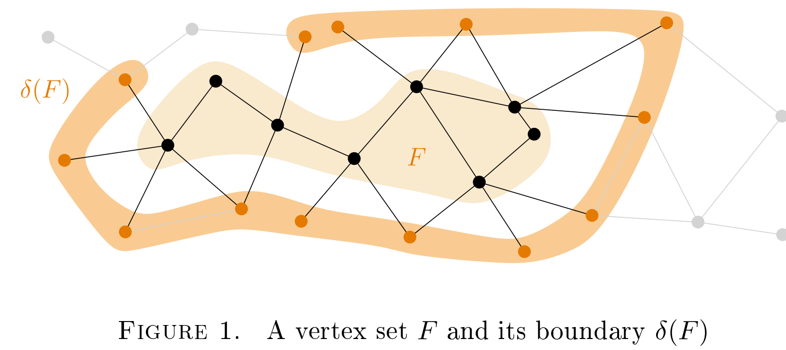

Figure 1. A vertex set F and its boundary $\delta(F)$

The *combinatorial Laplacian* of Γ, or simply the *Laplacian of* Γ, is the linear operator $\mathcal{L}\colon \mathcal{C}(V; \mathbb{C}) \longrightarrow \mathcal{C}(V; \mathbb{C})$ that assigns to any $u \in \mathcal{C}(V; \mathbb{C})$ the function $\mathcal{L}(u)$ defined as

$$\mathcal{L}(u)(x) = \sum_{y\in V} c(x, y)\,\big(u(x) - u(y)\big), \quad x \in V.$$

More generally, given $q \in \mathcal{C}(V;\mathbb{C})$, the *Schrödinger* operator with *potential* q is $\mathcal{L}_q\colon \mathcal{C}(V;\mathbb{C}) \longrightarrow \mathcal{C}(V;\mathbb{C})$ defined as $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$ for any $u \in \mathcal{C}(V;\mathbb{C})$. The Schrödinger operator whose potential is the conjugate of q; that is, $\mathcal{L}_{\bar{q}}$, is called *the adjoint of* $\mathcal{L}_q$ since it satisfies that $\langle \mathcal{L}_q(u), v\rangle = \langle u, \mathcal{L}_{\bar{q}}(v)\rangle$ for any $u, v \in \mathcal{C}(V;\mathbb{C})$.

For a given nonempty subset $F \subset V$ and a given potential $q \in \mathcal{C}(V;\mathbb{C})$ we consider the following *Boundary Value Problem*:

Given $f \in \mathcal{C}(F;\mathbb{C})$ and $g \in \mathcal{C}(\delta(F);\mathbb{C})$, find $u \in \mathcal{C}(\bar{F};\mathbb{C})$ such that

$$\mathcal{L}_q(u) = f \text{ on } F, \quad u = g, \text{ on } \delta(F). \tag{1}$$

When $F \neq V$, this problem is known as *Dirichlet Problem on F*, whereas when $F = V$ it is called *Poisson equation on V*. In this last case the data g has no sense, since then $\delta(F) = \emptyset$.

When $F \neq V$, each Dirichlet problem on F is equivalent to a semihomogenoeus Dirichlet problem. Specifically, $u \in \mathcal{C}(\bar{F};\mathbb{C})$ is a solution of Problem (1) iff $v = u - g$ is a solution of the Dirichlet problem

$$\mathcal{L}_q(u) = f - \mathcal{L}(g) \text{ on } F, \quad u = 0, \text{ on } \delta(F). \tag{2}$$

Therefore, to analyze the existence and uniqueness of solution of the boundary value problem for any $f \in \mathcal{C}(F;\mathbb{C})$ is equivalent to analyze the same topics for the following problem:

$$\text{Given } f \in \mathcal{C}(F;\mathbb{C}), \text{ find } u \in \mathcal{C}(F;\mathbb{C}) \text{ such that } \mathcal{L}_q(u) = f \text{ on } F. \tag{3}$$

This formulation encompasses both, Dirichlet problems and Poisson equations; the last ones appear when $F = V$.

For any weight $\omega \in \Omega$, we call the function $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$ the *Doob potential asociated with* ω. Therefore,

$$q_\omega(x) = -\kappa(x) + \omega(x)^{-1}\sum_{y \in V} c(x,y)\omega(y) > -\kappa(x), \quad \text{for any } x \in V.$$

Although in a first glance, Doob transforms could seem a bit strange and Doob potentials a very specific kind of potentials, they play a main role among real-valued potentials. In fact, as a consequence of the *Perron-Frobenius Theory*, given a real-valued potential $q \in \mathcal{C}(V)$ there exist an unique unitary weight $\omega \in \Omega$ and a unique real value $\lambda \in \mathbb{R}$ such that $q = q_\omega + \lambda$, see [1].

The variational characterization of the solutions for the boundary value problems (3) is described in the following result, see [3, Proposition 3.5] for its proof.

Proposition 2.1 (Dirichlet Principle)**.** *Let $F \subset V$ be a non empty subset, $\omega \in \Omega$, $\lambda \geq 0$ and the potential $q = q_\omega + \lambda$. Given $f \in \mathcal{C}(F)$ consider the quadratic functional : $\mathcal{C}(V) \longrightarrow \mathbb{R}$ given by*

$$(u) = \mathcal{E}_q(u) - 2\langle f, u\rangle.$$

Then $u \in \mathcal{C}(F)$ satisfies that $\mathcal{L}_q(u) = f$ on F iff it minimizes on $\mathcal{C}(F)$. Moreover has a unique minimum except when $F = V$ and $\lambda = 0$ simultaneously. In this case, has a minimum iff $f \in \omega^\perp$ and moreover there exists a unique minimum belonging to $\omega^\perp$.

3. Green functions, eigenvalues and eigenfuctions

In this section we consider fixed the finite and connected network $\Gamma = (V, c)$, a weight $\omega \in \Omega$, a non-negative value $\lambda \geq 0$, the real-valued potential $q = q_\omega + \lambda$ and its corresponding Schrödinger operator $\mathcal{L}_q$. Under these hypotheses, for any proper subset $F \subset V$ and any $f \in \mathcal{C}(F)$ the Dirichlet Problem (3) has a unique solution; that is there exists a unique $u \in \mathcal{C}(F)$ such that $\mathcal{L}_q(u) = f$ on F. Moreover, when $\lambda > 0$ for any $f \in \mathcal{C}(V)$ the Poisson equation (3) has a unique solution; that is there exists a unique $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f$ on V.

When either $F \subset V$ is a proper subset or $\lambda > 0$, the *Green Function of F for the potential q* is $G_q^F \colon F \times F \longrightarrow \mathbb{R}$ such that for any $y \in F$, $G_q^F(\cdot, y)$ is the unique solution of the Dirichlet Problem $\mathcal{L}_q(u) = \varepsilon_y$ on F, $u = 0$ en $\delta(F)$, when F is proper, or the Poisson equation $\mathcal{L}_q(u) = \varepsilon_y$ on V when $F = V$ but $\lambda > 0$.

The *Green operator of F for the potential q* is $\mathcal{G}_q^F \colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$ defined for any $f \in \mathcal{C}(F)$ as $\mathcal{G}_q^F(f)(x) = \sum\limits_{y \in F} G_q^F(x,y) f(y)$, $x \in F$. Then $\mathcal{G}_q^F$ is self-adjoint and for any $f \in \mathcal{C}(F)$, the function $u = \mathcal{G}_q^F(f) \in \mathcal{C}(F)$ satisfies that $\mathcal{L}_q(u) = f$ on F. The self–adjointness of $\mathcal{G}_q^F$ implies that G_q^F is a symmetric function, see for instance [3].

When $\lambda = 0$, then $q = q_\omega$ and the Poisson equation $\mathcal{L}_q(u) = f$ on V is solvable only if $f \in \omega^\perp$ and in this case, there exists a unique solution belonging to $\omega^\perp$. The *Green Function of V for the potential q* is $G_q^V \colon V \times V \longrightarrow \mathbb{R}$ such that for any $y \in V$, $G_q^V(\cdot, y)$ is the unique solution of the Poisson equation $\mathcal{L}_q(u) = \varepsilon_y - \omega(y)\omega$ belonging to $\omega^\perp$.

The *Green operator of V for the potential q* is $\mathcal{G}_q^V \colon \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ defined for any $f \in \mathcal{C}(V)$ as $\mathcal{G}_\omega(f)(x) = \sum\limits_{y \in V} G_q^V(x,y) f(y)$, $x \in V$. Then for any $f \in \mathcal{C}(V)$, $\mathcal{G}_q^V(f) = \mathcal{G}_q^V(f - \langle \omega, f \rangle \omega)$, $\mathcal{G}_q^V$ is self-adjoint and the function $u = \mathcal{G}_q^V(f) \in \mathcal{C}(V)$ is the unique function in $\omega^\perp$ satisfying that $\mathcal{L}_q(u) = f - \langle \omega, f \rangle \omega$. Newly, the self–adjointness of $\mathcal{G}_q^V$ implies that G_q^V is a symmetric function.

On the other hand, if we label the vertices of Γ, say $V = \{x_1, \ldots, x_n\}$ where $n = |V|$, then each endomorphism of $\mathcal{C}(F)$ can be interpreted as a matrix of order $|F|$. So $\mathcal{L}_q$ is identified with the matrix L_q^V whose diagonal entries are $\kappa(x_j) + q(x_j)$ and whose off-diagonal entries are $-c(x_i, x_j)$, $i, j = 1, \ldots, n$. Moreover if for a proper subset $F \subset V$, we interpret $\mathcal{L}_q$ as an endomorphism of $\mathcal{C}(F)$, then it can be identified with the matrix L_q^F obtained from L_q^V by eliminating the rows and the columns corresponding to the vertices in F^c. Notice that, as the potential are real-valued, all the above matrices are real-valued and symmetric.

We also denoted by G_q^F the matrix identified with the Green operator $\mathcal{G}_q^F$ defined above. With these identifications, $\mathsf{G}_q^F = \left(\mathsf{L}_q^F\right)^{-1}$ when either F is a proper subset of V or $\lambda > 0$. Moreover, when $\lambda = 0$, then $\mathsf{G}_q^V = \left(\mathsf{L}_q^V\right)^{\#}$, the *Group Inverse of* L_q^V. Since the group inverse coincides with the inverse when the matrix is invertible, we have that $\mathsf{G}_q^F = \left(\mathsf{L}_q^F\right)^{\#}$ for any non-empty subset $F \subset V$ and any $\lambda \geq 0$

Given a non-empty subset $F \subset V$, an *eigenvalue of the boundary problem* (3) is $z \in \mathbb{C}$ such that the Schrödinder operator $\mathcal{L}_{q-z}$ is singular on $\mathcal{C}(F; \mathbb{C})$. Equivalently,

$z \in \mathbb{C}$ is an eigenvalue of the boundary problem (3) if there exists $u \in \mathcal{C}(F;\mathbb{C})$ non-null and such that $\mathcal{L}_q(u) = z\,u$ on F. Each $u \in \mathcal{C}(F;\mathbb{C})$ satisfying the above identity is called *eigenfunction of the boundary problem* (3) *associated with* z.

Since q is a real-valued potential, any eigenvalue must be real, the eigenfunctions are real functions and eigenfunctions corresponding to different eigenvalues must be orthogonal each other.

If $z \in \mathbb{C}$ is not an eigenvalue of the boundary value problem (3), then $\mathcal{L}_{q-z}$ is an automorphism of $\mathcal{C}(F;\mathbb{C})$ and then we denote by its inverse by $\mathcal{G}^F_{q-z}$. Moreover, if $G^F_{q-z}\colon F \times F \longrightarrow \mathbb{R}$ is given for any $y \in V$ as $G^F_{q-z}(\cdot, y)$, the unique solution of the equation $\mathcal{L}_q(u) = \varepsilon_y$ on F, then $\mathcal{G}^F_{q-z}(f)(x) = \sum\limits_{y\in F} G^F_{q-z}(x,y)f(y)$, for any $f \in \mathcal{C}(F,\mathbb{C})$ and any $x \in F$.

The following result is the discrete version of the well-known *Spectral Theorem*, see [1].

Theorem 3.1 (Spectral Theorem). *For any non-empty subset $F \subset V$, there exist real values $\mu^F_1 \le \cdots \le \mu^F_{|F|}$ and an orthonormal basis $\{v^F_j\}_{j=1}^{|F|} \subset \mathcal{C}(F)$ satisfying the following properties:*

(i) $\mathcal{L}_q(v^F_j) = \mu^F_j v^F_j$ *on F, $j = 1, \dots, |F|$. Moreover, if $z \in \mathbb{R}$ is an eigenvalue of the boundary value problem* (3)*, then $z = \mu^F_j$ for some $j = 1, \dots, |F|$.*

(ii) $\lambda \le \mu^F_1 < \mu^F_2$ *and $v^F_1(x) > 0$ for any $x \in F$. Moreover, $\mu^F_1 = \lambda$ iff $F = V$ and then $v^F_1 = \omega$. In particular, $\mu^F_1 > 0$, except when $F = V$ and $\lambda = 0$, simultaneously.*

(iii) *For any $u \in \mathcal{C}(F;\mathbb{C})$ then* $\mathcal{L}_q(u)(x) = \sum\limits_{j=1}^{|F|} \mu^F_j \langle u, v^F_j\rangle v^F_j(x)$ *for any $x \in F$.*

The Spectral Theorem has as a very nice consequence, that we can also obtain the expression for the Green function of F for the potential q in terms of the eigenvalues and eigenfunctions. Prior to do this, for any $a \in \mathbb{C}$ we define $a^{\#}$ as a^{-1} when $a \neq 0$ or $a^{\#} = 0$ when $a = 0$.

Theorem 3.2 (Mercer Theorem). *Given a non-empty subset $F \subset V$, then*

$$G^F_q(x,y) = \sum_{j=1}^{n} (\mu^F_j)^{\#} v^F_j(x) v^F_j(y), \quad x, y \in V.$$

Moreover, if $z \in \mathbb{C} \setminus \{\mu^F_1 \le \cdots \le \mu^F_{|F|}\}$, then

$$G^F_{q-z}(x,y) = \sum_{j=1}^{n} (\mu^F_j - z)^{-1} v^F_j(x) v^F_j(y), \quad x, y \in V.$$

4. Product Networks

Obtaining eigenvalues and eigenfunctions and hence the corresponding Green functions in a given network is, in general, a very difficult task. In fact, there exist explicit expressions for these issues, only in a few cases corresponding to highly structured networks. Let us consider two different connected networks (Γ_1, c_1) and (Γ_2, c_2) with vertex sets V_1 and V_2.

We define the *Product network* as the network $\Gamma = \Gamma_1 \times \Gamma_2 = (V, c)$ where $V = V_1 \times V_2$ and the conductance is given by

$$c\big((x_1, y_1), (x_2, y_2)\big) = \begin{cases} c_1(x_1, x_2), & \text{if } x_2 = y_2, \\ c_2(y_1, y_2), & \text{if } x_1 = y_1, \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Clearly $\Gamma_1 \times \Gamma_2$ is also connected.

Given $u \in \mathcal{C}(V_1 \times V_2)$ for any $(x, y) \in V_1 \times V_2$, $u_y \in \mathcal{C}(V_1)$, $u^x \in \mathcal{C}(V_2)$ denote the functions defined as $u_y(z) = u(z, y)$ for any $z \in V_1$ and by $u^x(z) = u(x, z)$ for any $z \in V_2$.

Given $u \in \mathcal{C}(V_1)$ and $v \in \mathcal{C}(V_2)$ the *tensor product of* u *and* v is $u \otimes v \in \mathcal{C}(V_1 \times V_2)$ defined as $(u \otimes v)(x, y) = u(x)v(y)$ for any $(x, y) \in V_1 \times V_2$. Notice that given two weights $\omega_i \in \Omega(V_i)$, $i = 1, 2$, then $\omega_1 \otimes \omega_2 \in \Omega(V_1 \times V_2)$. Moreover, given $x \in V_1$ and $y \in V_2$ we have $\varepsilon_{(x,y)} = \varepsilon_x \otimes \varepsilon_y$.

We denote by $\mathcal{L}^i$ the combinatorial Laplacian of the network Γ_i, $i = 1, 2$ and by $\mathcal{L}$ the combinatorial Laplacian of the product network $\Gamma_1 \times \Gamma_2$. The following result establishes that the combinatorial Laplacian of a product network is expressed in separable variables when applies on a tensor product function. This property justifies the name of *separable variables* for the solution technique of boundary value problems on product networks.

Proposition 4.1. *Given* $u_i \in \mathcal{C}(V_i)$, $i = 1, 2$ *then*

$$\mathcal{L}(u_1 \otimes u_2) = \mathcal{L}^1(u_1) \otimes u_2 + u_1 \otimes \mathcal{L}^2(u_2).$$

In particular, if $\omega_i \in \Omega(V_i)$, $i = 1, 2$, *then* $q_{\omega_1 \otimes \omega_2} = q_{\omega_1} + q_{\omega_2}$ *and hence, for any* $u \in \mathcal{C}(V_1 \times V_2)$ *we have*

$$\mathcal{L}_{q_{\omega_1 \otimes \omega_2}}(u)(x, y) = \mathcal{L}^1_{q_{\omega_1}}(u_y)(x) + \mathcal{L}^2_{q_{\omega_2}}(u^x)(y), \quad (x, y) \in V_1 \times V_2.$$

Proof. Given $u \in \mathcal{C}(V_1 \times V_2)$ for any $(x, y) \in V_1 \times V_2$ we have that

$$\begin{aligned} \mathcal{L}(u)(x, y) &= \sum_{\substack{z \in V_1 \\ w \in V_2}} c\big((x, y), (z, w)\big)\big(u(x, y) - u(z, w)\big) \\ &= \sum_{z \in V_1} c_1(x, z)\big(u(x, y) - u(z, y)\big) + \sum_{w \in V_2} c_2(y, w)\big(u(x, y) - u(x, w)\big) \\ &= \mathcal{L}^1(u_y)(x) + \mathcal{L}^2(u^x)(y). \end{aligned}$$

On the other hand, since $(u_1 \otimes u_2)_y = u_1 u_2(y)$ and $(u_1 \otimes u_2)^x = u_1(x) u_2$ we obtain that

$$\mathcal{L}(u_1 \otimes u_2)(x, y) = u_2(y)\mathcal{L}^1(u_1)(x) + u_1(x)\mathcal{L}^2(u_2)(y).$$

In particular $\mathcal{L}(\omega_1 \otimes \omega_2) = \mathcal{L}^1(\omega_1) \otimes \omega_2 + \omega_1 \otimes \mathcal{L}^2(\omega_2)$ and hence,

$$q_{\omega_1 \otimes \omega_2} = -(\omega_1 \otimes \omega_2)^{-1}\mathcal{L}(\omega_1 \otimes \omega_2) = -\omega_1 \mathcal{L}^1(\omega_1) - \omega_2 \mathcal{L}^2(\omega_2) = q_{\omega_1} + q_{\omega_2}.$$

From all above identities we finally obtain that

$$\begin{aligned} \mathcal{L}_{q_{\omega_1 \otimes \omega_2}}(u)(x, y) &= \mathcal{L}(u)(x, y) + q_{\omega_1 \otimes \omega_2}(x, y)u(x, y) \\ &= \mathcal{L}^1(u_y)(x) + \mathcal{L}^2(u^x)(y) + \big(q_{\omega_1}(x) + q_{\omega_2}(y)\big)u(x, y) \\ &= \mathcal{L}^1(u_y)(x) + \mathcal{L}^2(u^x)(y) + q_{\omega_1}(x)u_y(x) + q_{\omega_2}(y)u^x(y) \\ &= \mathcal{L}^1_{q_{\omega_1}}(u_y)(x) + \mathcal{L}^2_{q_{\omega_2}(y)}(u^x)(y). \end{aligned}$$

$\square$

The boundary value problems we analyze in $\Gamma_1 \times \Gamma_2$, refers to subsets that are also expressed as cartesian products. So, given non empty subsets $F_i \subset V_i$, $i = 1,2$ we consider $F = F_1 \times F_2 \subset V_1 \times V_2$. Then, it is satisfied that

$$\delta(F_1 \times F_2) = \big(F_1 \times \delta(F_2)\big) \cup \big(\delta(F_1) \times F_2\big), \tag{5}$$

where we allow that $F_i = V_i$ in which case $\delta(F_i) = \emptyset$, $i = 1,2$.

Given $\omega_i \in \Omega(V_i)$, $i = 1,2$ and $\lambda \geq 0$, we consider the real-valued potential $q = q_{\omega_1 \otimes \omega_2} + \lambda$. We are interested in study the boundary value problem (3) on $F = F_1 \times F_2$ and also in compute the corresponding Green function $\mathcal{G}_q^F$. To do this, we first split λ as $\lambda_1 + \lambda_2$ where $\lambda_1, \lambda_2 \geq 0$ and then apply the Spectral Theorem to each boundary value problems $\mathcal{L}_{q_i}(u_i) = f_i$ on F_i, where $q_i = q_{\omega_i} + \lambda_i$ and $f_i, u_i \in \mathcal{C}(F_i)$, $i = 1,2$. Specifically, let $\mu_1^{F_i} \leq \cdots \leq \mu_{|F_i|}^{F_i}$ the eigenvalues of the boundary value problem $\mathcal{L}_{q_i}(u_i) = f_i$ on F_i, $i = 1,2$ and $\{v_j^{F_i}\}_{j=1}^{|F_i|} \subset \mathcal{C}(F_i)$ a corresponding orthonormal system of eigenfunctions.

Remember that always $\mu_1^{F_i}$ is simple and moreover $v_1^{F_i} > 0$ on F_i, $i = 1,2$. In addition, $\mu_1^{F_i} = \lambda_i$ iff $F_i = V_i$ and then $v_1^{F_i} = \omega_i$. Therefore, $\mu_1^{F_i} > 0$, except when $F_i = V_i$ and $\lambda_i = 0$, simultaneously.

The main result in product networks is that the eigenvalues and the eigenfunctions for the boundary value problem (3) in product subsets, is completely characterized in terms of the eigenvalues and eigenfunctions of each factor.

Theorem 4.2. *For any $j = 1,\dots,|F_1|$ and any $k = 1,\dots,|F_2|$ we have that*

$$\mathcal{L}_q(v_j^{F_1} \otimes v_k^{F_2}) = (\mu_j^{F_1} + \mu_k^{F_2}) v_j^{F_1} \otimes v_k^{F_2} \quad \text{on } F_1 \times F_2.$$

Moreover $\big\{\mu_j^{F_1} + \mu_k^{F_2}\big\}_{\substack{1\leq j\leq |F_1|\\ 1\leq k\leq |F_2|}}$ *determines all eigenvalues and* $\big\{v_j^{F_1} \otimes v_k^{F_2}\big\}_{\substack{1\leq j\leq |F_1|\\ 1\leq k\leq |F_2|}}$ *is an orthonormal basis in $\mathcal{C}(F_1 \times F_2)$.*

The fundamental consequence of Theorem 4.2 is that we can compute the Green function for product networks in terms of the eigenvalues and the eigenfunctions of each factor by applying the Mercer Theorem.

Corollary 4.3. *In the above conditions for any $(x_1,y_1),(x_2,y_2) \in V_1 \times V_2$, we have that*

$$G_q^{F_1 \times F_2}\big((x_1,y_1),(x_2,y_2)\big) = \sum_{j=1}^{|F_1|} \sum_{k=1}^{|F_2|} (\mu_j^{F_1} + \mu_k^{F_2})^{\#} v_j^{F_1}(x_1) v_j^{F_1}(x_2) v_k^{F_2}(y_1) v_k^{F_2}(y_2).$$

The above formula requires the knowledge of eigenvalues and eigenfunctions for the two factors. Therefore, except for structured networks, the application of the above method is very restricted. We finish this paper showing a technique that only requires the computation of eigenvalues and eigenfunctions for one of the factor networks and also the computation of a finite family of Green functions corresponding to the other product network. In fact this method is nothing else but the discrete version of the well-known *Separation of Variables Method* to solve boundary value problems in PDE.

The key issue to apply the Separation of Variables Method lies on use an adequate expression for functions in $\mathcal{C}(F_1 \times F_2)$. With the above notations, for any Given $h \in \mathcal{C}(F_1 \times F_2)$, for any $j = 1,\dots,|F_1|$ and any $k = 1,\dots,|F_2|$ we consider the functions $\alpha_j(h) \in \mathcal{C}(F_2)$ and $\beta_k(h) \in \mathcal{C}(F_1)$ defined as

$$\alpha_j(h)(y) = \langle h_y, v_j^{|F_1|}\rangle = \sum_{z\in V_1} h(z,y)v_j^{|F_1|}(z) = \sum_{z\in F_1} h(z,y)v_j^{|F_1|}(z), \quad x \in V_1,$$
$$\beta_k(h)(x) = \langle h^x, v_k^{|F_2|}\rangle = \sum_{z\in V_2} h(x,z)v_k^{|F_2|}(z) = \sum_{z\in F_2} h(x,z)v_k^{|F_2|}(z), \quad y \in V_2.$$

Theorem 4.4. *Under the conditions and notations in this section, for* $i = 1, 2$ *consider the real-valued potentials* $p_k^1 = q_1 + \mu_k^{|F_2|} = q_{\omega_1} + \lambda_1 + \mu_k^{|F_2|} \in \mathcal{C}(F_1)$, $k = 1, \dots, |F_2|$ *and* $p_j^2 = q_2 + \mu_j^{|F_1|} = q_{\omega_2} + \lambda_2 + \mu_j^{|1|} \in \mathcal{C}(F_2)$, $j = 1, \dots, |F_2|$. *Then,*

$$\begin{aligned} G_q^{F_1\times F_2}\big((x_1,y_1),(x_2,y_2)\big) &= \sum_{k=1}^{|F_2|} G_{p_k^1}^{|F_1|}(x_1,x_2)v_k^{F_2}(y_1)v_k^{F_2}(y_2) \\ &= \sum_{j=1}^{|F_1|} G_{p_j^2}^{|F_2|}(y_1,y_2)v_j^{F_1}(x_1)v_j^{F_1}(x_2). \end{aligned}$$

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