DISCRETE TRACE FUNCTION AND POINCARÉ INEQUALITY FOR THE STUDY OF SOME LINEAR SYSTEMS

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In this work we study linear systems whose coefficient matrix is irreducible and has the following block structure

$$A = \begin{pmatrix} L & -C \\ -C^T & D \end{pmatrix}$$

where D is a diagonal matrix with positive diagonal entries, $C \geq 0$ and L is a symmetric Z-matrix. This class of linear systems appears in relation to self-adjoint Dirichlet-Robin boundary value problems associated with a Schrödinger operator on a finite network. Hence, we take advantage of this relation and then we can analyze the linear systems by using the common tools in the context of boundary value problems. Specifically, we first define the discrete version of the trace function between Sobolev spaces. The trace allows us to consider null Robin boundary conditions and to reduce the dimension of the problem by incorporating the boundary conditions to the Schrödinger operator. Therefore, we can characterize when the Energy is positive semi-definite on the subspace of functions that vanish the boundary conditions. For that the description of admissible potentials throughout Doob potentials that also verify a discrete Poincaré type inequality is essential.

1. Discrete trace function and admissible potentials

In this work we study mixed boundary value problems for Schrödinger operators. Specifically, we are interested in obtaining necessary and sufficient conditions for the existence and uniqueness of solution of such a problems.

Throughout the paper, we consider $F \subset V$ a proper subset, $\emptyset \neq F_N \subset \delta(F)$ and $F_D = \delta(F) \setminus F_N$; that is, $\delta(F) = F_D \cup F_N$ is a partition of $\delta(F)$, where F_D can be the empty set. In addition, we always assume that $F \cup F_N$ is a connected subset on the network $\Gamma(F)$; that is, it is connected with respect to the conductance c_F. Then, we define the *outer degree of F, with respect to F_D*, as the function $p_F \in \mathcal{C}(F)$ given by

$$p_F(x) = \sum_{y \in F_D} c(x,y) = \sum_{y \in F_D} c_F(x,y) \quad \text{for any } x \in F. \tag{1}$$

Therefore, $p_F \in \mathcal{C}^+(F)$ and moreover $p_F = 0$ when $F_D = \emptyset$, whereas $\emptyset \neq \mathsf{supp}(p_F) \subseteq \delta(V \setminus F)$ when $F_D \neq \emptyset$.

Our aim is to study self-adjoint boundary value problems associated with the Schrödinger operator with potential $q \in \mathcal{C}(F \cup F_N)$. Specifically, for any $f \in \mathcal{C}(F)$,

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$g \in \mathcal{C}(F_{\scriptscriptstyle N})$ and $h \in \mathcal{C}(F_{\scriptscriptstyle D})$ the *boundary value problem on F with data f, g, h*, BVP in the sequel, consists on finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = f \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = g \quad \text{on} \quad F_{\scriptscriptstyle N} \qquad \text{and} \qquad u = h \quad \text{on} \quad F_{\scriptscriptstyle D}. \tag{2}$$

Any $u \in \mathcal{C}(\bar{F})$ satisfying the above identities is called *solution of the BVP*. When $g = 0$ and $h = 0$, then Problem (2) becomes

$$\mathcal{L}_q(u) = f \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = 0 \quad \text{on} \quad F_{\scriptscriptstyle N} \qquad \text{and} \qquad u = 0 \quad \text{on} \quad F_{\scriptscriptstyle D} \tag{3}$$

and it is called *semi-homogeneous boundary value problem with data f*. The *associated homogeneous boundary value problem* consists in finding $u \in \mathcal{C}(\bar{F})$ such that $\mathcal{L}_q(u) = 0$ on F, $\dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = 0$ on $F_{\scriptscriptstyle N}$ and $u = 0$ on $F_{\scriptscriptstyle D}$. The space of solutions of the homogeneous boundary value problem is denoted by $\mathcal{V}_q^H$, whereas the subspace of $\mathcal{C}(\bar{F})$ formed by the functions vanishing the boundary conditions is denoted by $\mathcal{V}_q$. Therefore,

$$\mathcal{V}_q = \left\{ u \in \mathcal{C}(\bar{F}) : \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = 0 \text{ on } F_{\scriptscriptstyle N} \text{ and } u = 0 \text{ on } F_{\scriptscriptstyle D} \right\}$$

and clearly, we have $\mathcal{V}_q^H \subset \mathcal{V}_q \subset \mathcal{C}(F \cup F_{\scriptscriptstyle N})$. The boundary value problem is called *regular* when $\mathcal{V}_q^H = \{0\}$. Observe that for any $u \in \mathcal{C}(F \cup F_{\scriptscriptstyle N})$, we have that

$$\mathcal{E}_q^F(u,u) = \frac{1}{2} \int_{F \cup F_N} \int_{F \cup F_N} c_{\scriptscriptstyle F}(x,y) \big(u(x) - u(y)\big)^2 dx\, dy + \int_{F \cup F_N} (q + p_{\scriptscriptstyle F})\, u^2. \tag{4}$$

Moreover, if $P_{\scriptscriptstyle F} = \min\limits_{x \in \mathsf{supp}(p_F)} \{p_{\scriptscriptstyle F}(x)\}$, then for any $q \geq -P_{\scriptscriptstyle F} \chi_{\mathsf{supp}(p_F)}$, it is satisfied that

$$\mathcal{E}_q^F(u,u) \geq 0$$

and hence the energy is positive semi-definite. We can interpret the value $P_{\scriptscriptstyle F}$ as a discrete version of the Poincaré constant.

Lemma 1.1 (Self-adjointness)**.** *The boundary value problem* (2) *is self-adjoint; that is,*

$$\int_F v\, \mathcal{L}_q(u) = \int_F u\, \mathcal{L}_q(v) = \mathcal{E}_q^F(u,v), \qquad \text{for all } u, v \in \mathcal{V}_q.$$

Problem (2) is generically known as a *Mixed Dirichlet-Robin problem* and, in particular, includes the following boundary value problems:

(i) *Neumann problem*: $q = 0$ on $F_{\scriptscriptstyle N}$, $F_{\scriptscriptstyle N} = \delta(F)$ and hence $F_{\scriptscriptstyle D} = \emptyset$.

(ii) *Robin problem*: $q \neq 0$ on $F_{\scriptscriptstyle N}$, $F_{\scriptscriptstyle N} = \delta(F)$ and hence $F_{\scriptscriptstyle D} = \emptyset$.

(iii) *Mixed Dirichlet-Neumann problem*: $F_{\scriptscriptstyle N}, F_{\scriptscriptstyle D} \neq \emptyset$ and $q = 0$ on $F_{\scriptscriptstyle N}$.

The boundary value problem (2) when $q \in \mathcal{C}^+(\bar{F})$ has been extensively treated in the literature, see for instance [2, 4, 7, 13, 18] where the existence and uniqueness of solutions was established. It corresponds to linear systems with *d.d.* M-matrices as coefficient matrix, see [5]. The analysis for Dirichlet Problem ($F_{\scriptscriptstyle N} = \emptyset$) and Poisson equation ($F = V$) for more general potentials has been analyzed in [3, 4] and correspond to linear systems with a general symmetric M-matrix as coefficient matrix. As we are assuming that $F_{\scriptscriptstyle N} \neq \emptyset$, we are not considering here neither Dirichlet nor Poisson problems.

1.1. **The trace map.** We first establish that under a simple condition, for any data the BVP (2) can be transformed into a semi-homogeneous one; that is, a BVP in which the data is supported by F, or equivalently in which the boundary conditions are null. Recall that we are assuming $F_{N} \neq \emptyset$ and $F \cup F_{N}$ is connected in $\Gamma(F)$.

Lemma 1.2. *Suppose that* $q(x)+\kappa_{F}(x) \neq 0$ *for any* $x \in F_{N}$*. If given* $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(F_{N})$ *and* $h \in \mathcal{C}(F_{D})$*, we consider* $u_g \in \mathcal{C}\big(F_{N}\big)$ *and* $f_{g,h} \in \mathcal{C}(F)$ *defined respectively as*

$$u_g = \frac{g}{q+\kappa_{F}} \qquad \textit{and} \qquad f_{g,h} = \mathcal{L}(u_g + h)\cdot \chi_{F},$$

then u *is a solution of BVP* (2) *with data* f,g,h *iff* $u = v + u_g + h$ *where* v *is a solution of the semi-homogeneous BVP* (3) *with data* $f - f_{g,h}$*.*

Observe that for any $x \in F$ we have

$$\mathcal{L}(u_g)(x) = -\sum_{y\in F_{N}} \frac{c(x,y)g(y)}{q(y)+\kappa_{F}(y)} \quad \text{and} \quad \mathcal{L}(h)(x) = -\sum_{y\in F_{D}} c(x,y)h(y),$$

and hence $f_{g,h}$ depends only on the values of the data g and h.

Next we prove that under the same hypothesis than before, the values on F_{N} of any solution of BVP (2), depend only on its values on F.

Proposition 1.3. *Suppose that* $q(x)+\kappa_{F}(x) \neq 0$ *for any* $x \in F_{N}$*. Then, for any* $u \in \mathcal{C}(F)$ *there exists a unique extension of* u *to* $F \cup F_{N}$, $\gamma(u)$*, such that* $\gamma(u) \in \mathcal{V}_q$*. Moreover,* $\gamma\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F\cup F_{N})$ *is given by*

$$\gamma(u)(x) = \frac{1}{q(x)+\kappa_{F}(x)} \sum_{y\in F} c(x,y)u(y), \quad x \in F_{N},$$

and establishes an isomorphism onto $\mathcal{V}_q$*. Therefore,* $\dim \mathcal{V}_q = |F|$ *and when* $\kappa_{F} + q > 0$ *on* F_{N}*, then* $\omega \in \Omega(F)$ *iff* $\gamma(\omega) \in \Omega(F\cup F_{N})$*.*

According with its continuous analogue, the map γ defined in the above proposition, will be named *trace map for* $F \cup F_{N}$.

Notice that if $q(x) \leq -\kappa_{F}(x)$ for some $x \in F_{N}$, then $\Omega(F\cup F_{N}) \cap \mathcal{V}_q = \emptyset$ and hence, no weight on F can be extend to a weight on $F \cup F_{N}$ satisfying the boundary conditions of the BVP (2).

On the other hand, for any $g \in \mathcal{C}(F_{N})$ and for any $u \in \mathcal{C}(F)$, the function $v = \gamma(u) + u_g$ is the unique extension of u to $F\cup F_{N}$ such that $\dfrac{\partial v}{\partial \mathsf{n}_{F}} + qv = g$ on F_{N}.

Next, we use the trace function to transform the mixed BVP into a Poisson Equation on a network without boundary. Thus, we reduce the dimension of the problem and moreover, this new formulation will allow us to tackle the spectral analysis in the next section.

Suppose that $q + \kappa_{F} > 0$ on F_{N} and consider the network $\widehat{\Gamma}(F) = (F, \widehat{c})$ and the potential $\widehat{q} \in \mathcal{C}(F)$, where

$$\widehat{c}(x,y) = c(x,y) + \sum_{z\in F_{N}} \frac{c(x,z)c(z,y)}{q(z)+\kappa_{F}(z)}, \qquad \widehat{q}(x) = q(x) + \sum_{z\in F_{N}} \frac{c(x,z)q(z)}{q(z)+\kappa_{F}(z)},$$

for any $x, y \in F$, see Figure 1.

Compare the conductance of the above network with the expressions considered in [16, pg. 581] and [22, pg. 2169].

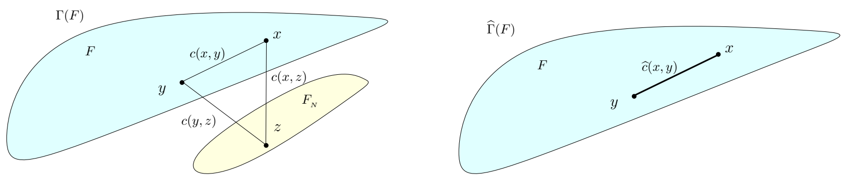

FIGURE 1. Example of a network $\Gamma(F)$ and its associated $\widehat{\Gamma}(F)$.

Proposition 1.4. *The network $\widehat{\Gamma}(F)$ is connected and for any $u \in \mathcal{C}(F)$*

$$\widehat{\mathcal{L}}_{\widehat{q}}(u) = \mathcal{L}_q\big(\gamma(u)\big) \quad on\ F.$$

Moreover, $\mathcal{L}_q$ is positive semi-definite (definite) on $\mathcal{V}_q$ iff $\widehat{\mathcal{L}}_{\widehat{q}}$ is positive semi-definite (definite) on $\mathcal{C}(F)$ and given $v \in \mathcal{V}_q$, it is satisfied $\mathcal{L}_q(v) = 0$ on F iff $\widehat{\mathcal{L}}_{\widehat{q}}(v \cdot \chi_{_F}) = 0$ on F.

We remark that the above proposition transforms a mixed boundary value problem on $F \cup F_{_N}$ into a Poisson equation on F with respect to a new Schrödinger operator. This is due to the fact that the values of a function verifying the boundary condition on $F_{_N}$ are uniquely determined by the values of the function on F. A particular version of this technique was use in [16, 22] in the context of Neumann boundary value problems for the combinatorial Laplacian. Therefore, given $f \in \mathcal{C}(F)$, then $\widehat{\mathcal{L}}_{\widehat{q}}(u) = f$ iff $\gamma(u)$ is a solution of the semihomogeneus BVP (3) with data f. Moreover, for any semi-homogeneous mixed boundary value problem the Fredholm Alternative is in force: Given $f \in \mathcal{C}(F)$, Problem (3) has a solution iff $\langle v, f\rangle_{_F} = 0$, for any $v \in \mathcal{V}_q^H$. In particular, Problem (3) has a unique solution iff $\mathcal{V}_q^H$ is the trivial subspace.

Observe that the procedure we have just described is the operational version of the Schur complement method to solve linear systems. In this case,

$$\widehat{\mathsf{L}}_{\widehat{q}} = \mathsf{L}_q/\mathsf{D} = \mathsf{L}_q(F;F) - \mathsf{C}(F;F_{_N})\mathsf{D}^{-1}\mathsf{C}(F_{_N};F),$$

where D is the diagonal matrix whose diagonal elements are $k_{_F}(x) + q(x)$, $x \in F_{_N}$. Moreover, the matrix associated with γ is

$$\begin{bmatrix} \mathsf{I} \\ -\mathsf{D}^{-1}\mathsf{C}(F_{_N};F) \end{bmatrix}.$$

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