Mathematical modelling and the learning trajectory: tools to support the teaching of linear algebra

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In this article we present a didactic proposal for teaching linear algebra based on two compatible theoretical models: emergent models and mathematical modelling. This proposal begins with a problematic situation related to the creation and use of secure passwords, which leads students toward the construction of the concepts of spanning set and span. The objective is to evaluate this didactic proposal by determining the level of match between the hypothetical learning trajectory (HLT) designed in this study with the actual learning trajectory (ALT) in the second experimental cycle of an investigation design-based research more extensive. The results show a high level of match between the trajectories in more than half of the conjectures, which gives evidence that the HLT has supported, in many cases, the achievement of the learning objective, and that additionally, mathematical modelling contributes to the construction of these linear algebra concepts.

Keywords: hypothetical learning trajectory; actual learning trajectory; emergent models heuristic; mathematical modelling; design-based research; spanning set; span

1. Introduction

Linear algebra is difficult for students in both the cognitive and conceptual sense,[1] and it is for this reason that a number of innovations have been made in teaching this subject, including the use of mathematical modelling, and the application of instructional designs based on the emergent models heuristic.

Trigueros and Possani [2] affirm that mathematical modelling can be successful in the teaching of the concepts of linear algebra because it gives students the opportunity to use their prior knowledge and to confront new conceptual needs. For their part, Wawro et al. [3] posit that the use of the emergent models heuristic for creating instructional designs for linear algebra helps students to progress from informal mathematical reasoning toward more complex and formal reasoning.

To implement an innovation in teaching one must take into account the creation of a hypothetical learning trajectory (HLT). This is because, in agreement with Daro et al. [4], it is a tool that can help teachers rethink teaching, which enables them to have a general vision of the class before they start it.

According to Simon, [5] the HLT is a prediction of the trajectory that the learning process is likely to follow, and provides a basis for the design of the teaching itself. The HLT has three components: the learning objective, which defines the goals to be achieved, the learning activities, and a possible route of learning or cognitive process, which is a prediction of how the thinking and the understanding of the students will be developed in the context of the learning activities.

In contrast with the HLT, Leikin and Dinur [6] defined the actual learning trajectory (ALT) as the learning trajectory that effectively occurs, which is to say, the trajectory the students have followed in the context of the implementation of an instructional design. The ALT is inferred from the data collected, since it is not possible to directly measure the actual learning of the students.[7]

When the HLT has a high match with the ALT, Stylianides and Stylianides [8] determine that the HLT supported the realization of the learning objectives.

In specific instances, there are important gaps in understanding in the learning trajectories of various topics within mathematics, including linear algebra. It is for this reason that this investigation has constructed an HLT for concepts of the spanning set and span of linear algebra, which are included in a proposal for teaching based on the emergent models heuristic and mathematical modelling in the context of the creation of passwords.

The objective of this study is to evaluate this didactic proposal through the level of match between the HLT designed in this study and the ALT in relation to the construction of the concepts of spanning set and span.

2. Theoretical Framework

This section summarizes the conceptual framework that consists in the emergent models heuristic, which has been part of several recent studies in mathematics education at the university level,[9-11] because at other educational levels, it has been a powerful design heuristic.[12] In addition, we also consider mathematical modelling, which according to Alsina,[13] has been successful at the university level, noting that students learn better in context, either because it provides motivation and interest, or because they are involved in the resolution of real world problems. Therefore, the emphasis on applied problems or the mathematisation of reality can be a positive step toward success in learning, as confirmed, for example, by Dominguez et al. [14], in a recent study in linear algebra.

2.1. The emergent models heuristic

The emergent models heuristic is an alternative to instructional approaches that focus on teaching ready-made representations.[15] Its objective is to create a sequence of tasks that allow students to initially develop their informal mathematical activity, to later transform it into a more sophisticated form of mathematical reasoning.[12]

Emergent models are intermediaries for changing informal procedures into a more formal mathematical reasoning.[16] The ‘emergent’ label has a double meaning that refers to the process by which the models emerge, and to the process by which they support the emergence of formal mathematical knowledge.[15]

Emergent models adopt a dynamic view of learning that allows students to understand mathematics. Within this approach, symbols and mathematical models can be developed jointly. The idea is that the form of an informal model emerges when students are in the process of reorganizing an activity and looking for the solution to the problem on context. Later, this model can serve as a basis for the development of more formal mathematical knowledge, which is to say a model is first constituted within a specific context as the model operating in that situation, and then the model can be generalized to suit other situations. Thus, the model changes its character and becomes an entity that can function as a model for the formal mathematical reasoning. The change from model of to model for concurs with a change in the thinking of the students: thinking about the situation modelled to thinking about the mathematical relations.[15]

For the transition from model of to model for, one can distinguish four levels of activity that do not involve any strictly ordered hierarchy, known as: situational, referential, general and formal.[12] The situational activity involves students working toward the mathematical objectives through an experience that is real to them. The referential activity involves models of descriptions, concepts, and procedures that relate to the problem of the situational activity. The general activity involves models to explore, to reflect upon, and to generalize about what appeared at the previous level, but with a mathematical focus on strategies, without making any reference to the initial problem. The formal activity leads to students reflect the emergence of a new reality in mathematics, therefore, it involves working with procedures and conventional notations.

The emergent models heuristic does not specify to the instructional designer where to find the appropriate models, but does describe what an emergent model may resemble, what its features are, and how it works. In this way, this heuristic can help designers in the choice of models: thinking through them, elaborating upon them, and improving them.[16]

2.2. Mathematical Modelling

The possibility of introducing new concepts by means of the mathematical modelling in the classroom has received considerable attention in recent years. However, Possani et al. [17] point out that few studies have been conducted in undergraduate mathematics courses such as linear algebra.

The assumption behind the introduction of mathematical modelling in the classroom implies an expectation that when students face problematic situations of interest, they should be able to: explore ways to represent them in mathematical terms, explore the relationships that appear in these representations, and to manipulate and develop powerful ideas that can be channeled toward the mathematics will wish to teach.[18]

At present, there are various ways of approaching mathematical modelling in the classroom. This study adopts the educational perspective proposed by Kaiser and Schwarz,[19] that considers mathematical modelling as a vehicle or as a didactic technique. Which is to say, as posited by Julie and Mudaly,[20] it is a tool to help in the study of mathematics that motivates students and provides a basis for the development of mathematical content.

In addition, in order to guide students in the resolution of the task of modelling proposed here, this study uses the modelling cycle proposed by Blum and Leiss.[21] The relationship between the seven steps of this cycle and the task of modelling the didactic proposal is presented in Figure 1.

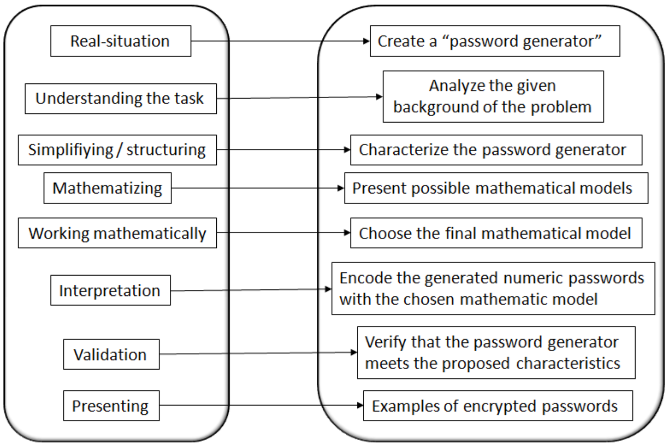

Figure 1. Relationship between modelling cycle by Blum and Leiss [21] and the modelling task of didactic proposal of Cárcamo et al. [22].

From these theoretical considerations, we design a didactic proposal that includes an HLT. In one sense, mathematical modelling is seen as a tool to introduce the concepts under study, and other, the emergent models heuristic can have the purpose of guiding the instructional design as well as offering students the opportunity to reflect on his the informal mathematical activity, and after will be generalize toward a more formal reasoning.

3. Methodology

3.1. *Participants and context*

The study participants were 45 first-year students of engineering at a Spanish university who had not previously worked with mathematical modelling, nor had studied the concepts of spanning set and span.

The students participated in the second experimentation cycle of a teaching proposal based on the emergent models heuristic and mathematical modelling, in which they solved a set of tasks aimed at the construction of the concepts of spanning set and span. This experiment was carried out over 5 hours, divided into 3 class sessions, in which they worked in 14 groups (of 3 to 4 students), and then individually, in the final task.

3.2. *Data and focus research*

The methodology of this study is the design-based research that is characterized by the design of innovative educational environments that are intertwined with the experimentation and the development of the theory.[23] In this research, a didactic proposal is designed and evaluated for the construction of the concepts of spanning set and span.

The data collected in this experiment were as follows: audio and video recordings of group work, the written responses of the students to the tasks proposed in the HLT, and an individual interview at the end of the experimentation.

Regarding the analysis of the data, following the ideas of Dierdorp et al. [7], we compared the HLT with the ALT, looking for background to support or rebut the conjectures of the HTL, and then we used a data analysis matrix for comparing the HLT and the ALT as shown in Table 1. The left side of the matrix summarizes the HLT, and the right side synthesizes the ALT: through the written responses, or excerpts from the transcripts, and a description of results by the investigator, as well as a quantitative impression of the level of match between the HLT and the ALT. It was considered that there was a high match between the two (+ sign) when the evidence suggested that the conjecture had been confirmed by at least two thirds of the groups or the students, or a moderate match (± sign) when the evidence suggested that the conjecture had been confirmed by more than a third but less than two thirds of the groups or the students, and a low match (- sign) in other case.

Table 1. Data analysis matrix for comparing the HLT and the ALT.

Hypothetical learning trajectory			Actual learning trajectory		Match between HLT and ALT
Task	Description of the task	Conjecture of how students would respond	Extract of written or oral response	Result	Quantitative impression of how well the conjecture and actual learning matched (expressed as: -,±,+)

3.3. *The HLT*

The objective of the HLT was to support students in the construction of the concepts of spanning set and span. The HLT was composed of four tasks. Table 2 presents a summary of the connections between: the major task features presented in this study, the major conjectures of the HLT, and the activity levels proposed by Gravemeijer.[12]

Table 2: Summary of the connection between: the major task features presented in this study, the major conjectures of the HLT, and the activity levels proposed by Gravemeijer.[12].

Major task features	Major conjectures of the HLT	Activity level
Task 1: (a) In task 1, students are presented a brief reading regarding secure passwords. (b) Students are asked to create a secure password generator based on a mathematical model that involves vectors.	(a) Students read information from the secure passwords; (b) Students created a generator password by following the steps of the modelling cycle and using their previous knowledge of vectors and passwords.	Situational
Task 2 – Students are asked to make an analogy table between their password generator and the concepts of spanning set and span.	(a) Students properly related the concept of spanning set with their password generator set, which contains the vectors that, after creating the linear combination is obtained by the vector for each numeric password with them; (b) Students properly related the concept of span with their password generator set which has all the vectors that allow the generate numerical passwords to be	Referential

	generated. (c) Students generate an analogy table where the following are linked: the concepts of spanning set and span, with two sets of the password generator, and the name that they receive in the password generator.	
Task 3: Students are asked determine if the sets A, B and C generate to R^2,that is to say, if they are spanning sets of this space.	(a) Students determine and relate the characteristics of a spanning set for space R^2 with the sets: A, B and C. (b) Students deduce that the set C generates a R^2 while giving a coherent justification. (c) Students deduce that the sets A and B cannot generate R^2 while giving a coherent justification.	General
Task 4a: Given a span of R^4, determine a spanning set for it	(a) The student relates the span of R^4 with the concept of spanning set; (b) The student observes the characteristics of the span of R^4 to obtain a spanning set for it; (c) The student proposes a spanning set appropriate for the given subspace of R^4.	Formal
Task 4b: Given a span for R^4, determine if two vectors belong to it.	(a) The student performs an appropriate process to determine if each vector belongs to the subspace of R^4 or not; (b) The student determines that the first vector (1,5,0,0) does not belong and that the second vector (5,0,5,0) does belong.	

The professor in this teaching experiment tried to maintain a balance between his (minimal) guidance and the (maximal) independence of students, mainly making strategic interventions.[24]

The tasks of the HLT were based on the emergent models heuristic and mathematical modelling. Mathematical modelling has been considered to be a teaching tool that can start the construction of the concepts under study through a problem, in this case, by means of creating a password generator using vectors. Likewise, as pointed out by Gravemeijer and Stephan [16], the emergent models heuristic served to help design and structure the tasks, in such a way as to motivate students in their transition from a informal mathematical reasoning of their activity toward a more formal type of mathematical reasoning.

4. Results

The results are presented for each of the tasks proposed in Table 2 with the objective of evaluating the didactic proposal of this study through the level of match between the HLT and the ALT, in relation to the construction of the concepts of spanning set and span. This section is organized into four parts that correspond with each of the tasks making up the HLT.

4.1. *Task 1*

The context to start this teaching experience was the creation of passwords. Students were presented the problem in Figure 2.

To prevent strangers from entering our accounts from different web pages (e-mail, Facebook, Twitter, etc.) it is necessary to have secure passwords with letters, numbers, and symbols. However, these passwords are not effective if users choose the same one for all sites, because in that case, if a stranger has decrypted the password for one of these sites, that person now has the freedom to access all our personal information. So what can you do? One alternative is to use an Excel spreadsheet to create your own secure passwords and to store them there as well.

With this information in mind, develop a generator for secure passwords based on a mathematical model that involves vectors. Explain the process used to create the encrypted passwords.

Figure 2. Proposed Task 1 for students to work in groups.

The groups selected their mathematical model for generating passwords, following in the footsteps of the mathematical modelling cycle proposed by Blum and Leiss [21], as seen in Table 3, in the written replies and in some dialogs held by the three students that formed group 3.

Table 3. Written answers from Task 1 and some dialogs held by students of group 3 and their correspondence with the steps of the cycle of mathematical modeling.

Steps 1 and 2 of the modelling cycle. Understand the situation, structure and simplify: characteristics of your password generator **Translated** *"Generates the password from the component 2, x "* Genera la contraseña a partir de la componente 2, x.
Step 3 of the modelling cycle. Mathematisation: mathematical models proposed by the members of the group $(\frac{1}{2}x, 2x, x)$, $(4x, x, 5x)$, $(x, 2x, 3x)$
Step 4 of the modelling cycle. Working mathematically: choice of the mathematical model to generate passwords **Translated** *"We chose the model (4x,x,5x) since it works with very large numbers and therefore is more difficult to decipher"* Elegimos el modelo (4x, x, 5x) ya que se trabaja con números mas grandes y por lo tanto mas dificil de decifrar.
Step 5 of the modelling cycle. Interpretation: coding to use to generate passwords 0 → f, 1 → x, 2 → d, 3 → 8, 4 → 4, 5 → K, 6 → j, 7 → %, 8 → &, 9 → # S7: Are you going to put *a*, *b*, *c*? S8: No, look, we can put some numbers to be misleading. … S7: 4, it's 4 ¿no?

S9: 4, 4.
S8: Yes, because this misleads them.
S7: Okay, letters, okay.
S9: 5.
S7: I would put a Japanese kanji.
S9: Yes, yes, put a character there.
S8: No, a *k*, a *k* (writes). And 6?
S9: I said a *j*.
S8: 6 (writes).

Step 6 and 7 of the modelling cycle. Validation and presentation of the solution: Example of an encrypted password created by your password generator

$(4\cdot 2, 2, 5\cdot 2)$
$(8, 2, 10)$
$8\,2\,1\,0$
$\downarrow\downarrow\downarrow\downarrow$
&dxf

S8: What number is *x*?
S9: It's 2. 8, 2, 10.
S8: (Typing the answer) 8 is an ampersand, 2 is the letter *d*, 1 the letter *x* and 0 is the *f*. Now we've got the password.
S7: It's very difficult to break, that is, it's really safe.

Table 3 shows that the students in group 3 defined the generic vector (4x, x, 5x) as its mathematical model to generate passwords, and then to validate this, they gave as an example the password &dxf. That is to say, they passed from the mathematical world to the everyday world and vice versa. In addition, the dialogs held by group 3 revealed that all its members participated in the construction of its password generator. Here, we infer that the students participated actively in the resolution of the problem, and that this could serve as the basis for start the construction of the concepts of spanning set and span.

With regard to the mathematical model used to create passwords, the groups presented two types: a generic vector and a linear combination of vectors. Those who chose the first option, proposed a generic vector of R^3, R^4 or R^6, while others opted for a linear combination of R^3. In Table 4, we can see that group 12 proposed a generic vector of R^6 with three variables, and group 5 indicated two vectors each represented by a linear combination of three vectors and also with three variables. This gives evidence that the students triggered their previous knowledge of vectors in the context of creating secure passwords and of task 1. This is evidence that students activated their previous knowledge of vectors in the context of creating secure passwords, placing them in the situational activity level [12] in this task.

Table 4. Mathematical models proposed by groups 12 and 5 in Task 1.

Mathematical model proposed by group 12
$P = (x, 2x, y, 3y, z, 4z)$
Mathematical model proposed by group 5
Vector 1: $a(3,-3,4)+b(-1,1,2)+c(0,4,2)$
Vector 2: $a(1,2,-3)+b(-4,4,2)+c(2,0,-7)$

In the analysis of the written responses of the groups relating to this task, along with the audio tracks of the group work, it was noted that the ALT had a high match with the HLT (100% of the groups).

4.2. *Task 2*

After the professor introduced the concepts of spanning set and span as related to the context of passwords, the students made a table in which they established an analogy between two sets associated with their password generator and the concepts of spanning set and span.

In the analogy between the concept of span and a set associated with the password generator (which possesses all the vectors that allowed the generation of numeric passwords) we observed two types of responses: groups wrote the set correctly using mathematical notation and related it to the name of the span (43% of the groups) and those who lacked rigorousness in the notation of the set that should have been related with the name of span, because they lacked a parenthesis, a sign or the superscript of R^n or had not used the parentheses appropriate for the set that related with the span. Figure 3 shows that group 1 wrote a set in correct mathematical notation of span, but it lacks a parenthesis in the second vector that wrote.

Moreover, regarding the analogy between the concept of spanning set and a set associated with their password generator (that contain vectors that, after making the linear combination with them, the vector was obtained for each numerical password) we observed two main types of responses: those who wrote the set correctly in mathematical notation and related to the name of the spanning set (57% of the groups) and those that did not do so, instead writing a mathematical expression that included a generic vector or a linear combination, as seen in the example in Figure 3 where group 1 wrote a vector of R^2 as a linear combination of the two vectors of R^4. This difficulty suggests that some students have not yet internalized the notion that only the numeric vectors that make up the linear combination form the spanning set, but not the linear combination itself.

Translated

Name given in your password generator	*How it is written in mathematical language*	*Mathematical name for this concept*
Set that contains the vectors to generate numeric passwords	$\langle (2,0,1,0),(0,1,0,3) \rangle$	*Span*
Set that doing the linear combination with their vectors is obtained each vector that generates a password	$(x,y) = \alpha \cdot (2,0,1,0) + \beta \cdot (0,1,0,3)$	*Spanning Set*

Nombre que recibe en tu generador de contraseñas	Cómo se escribe en lenguaje matemático	Nombre que recibe en matemática
conjunto que contiene los vectores para generar contraseñas numéricas	$\langle (2,0,1,0),(0,1,0,3 \rangle$	Espacio generado
Conjunto que al hacer combinación lineal con sus vectores se obtiene cada vector que genera una contraseña	$(x,y) = \alpha (2,0,1,0) + \beta (0,1,0,3)$	conjunto generador

Figure 3. Example of a response by group 1 of analogy table between the generator of

passwords and the concepts of spanning set and span.

The analogy table, in spite of the difficulties it presented with the mathematical language, enabled students to link two sets in mathematical notation with their assigned names in the context of their password generator, but also with their denominations of the mathematics of spanning set and span. Which is to say, they expanded the vision of these concepts in a real context. This could help students better recall the characteristics of each of these concepts, and avoid confusion.

This task placed the students at a referential activity level, [12] because in order to solve it, they had to bear in mind the initial task of generating passwords.

In the analysis of task 2, we observed a moderate level of match between the HLT and the ALT (43% of the groups), since some groups did not present an analogy table that was totally correct, mainly due to difficulties with the mathematical language. This result suggests a need to modify this task to improve this level of match in any forthcoming experimentation.

4.3. *Task 3*

In Task 3, students conjectured with respect to the properties associated with the concepts of spanning set and span, but in this case, this took place outside the context of passwords. As seen in the question in Figure 4, they were requested to identify whether sets A, B and C corresponded to space R^2.

Determines whether the following sets generate to R^2, that is to say, if without generating sets of this space

Set	Justification of your determination
$A=\{(0,-3)\}$	
$B=\{(5,0),(7,0)\}$	
$C=\{(1,0),(1,-1)\}$	

Figure 4. Example of a question in Task 3, worked on in a group.

All the groups claimed that set C generated the set R^2, and six of them argued that the vectors were linearly independent (or that none was a linear combination of other). Others suggested an argument that was more intuitive, and among these, that C generated R^2 because it had two components, or two vectors. One group did not substantiate its response. Table 5 illustrates the responses of groups 1 and 14.

Table 5. Examples of responses that set C={(1,0),(1,-1)}if they generates the R^2 space in Task 3

Example of response by group 1 **Translated** *"Yes, because it has 2 vectors, and none of those vectors is a linear combination of the other"* Si porque tiene 2 vectores y ninguno es combinación lineal del otro
Example of response by group 14

Translated

"Yes, because it has 2 components"

Sí, porque tiene 2 componentes.

With regard to sets A and B, the majority of the groups indicated that they did not generate R^2 (with the exception of two groups who indicated that B generated the R^2 space). The main argument was that to generate R^2, the Assembly had to contain two vectors. Among the justifications noting that B does not generate R2, one stated that the vectors of this assembly were linearly dependent.

From the results obtained in this task, it is inferred that the groups of students progressed toward a general activity level [12] by deepening their grasp of the concepts of spanning set and span when performing conjectures in problems related to these concepts, but without making reference to the situation of the passwords. Likewise, in the analysis of task 3 through the written responses of the students, there was evidence that the ALT showed a high match with the HLT (86% of the groups).

4.4. *Task 4*

In Task 4, students individually applied the concepts of spanning set and span in a purely mathematical context. A question that was asked of them was, given a span $W=\{(x,y,z,w) \in R^4/x=z,y=w=0\}$, to determine: (a) a spanning set W and (b) if the vectors (1,5,0,0) and (5,0,5,0) belonged to W.

With respect to establishing a spanning set for W, 59% of the students responded correctly. Table 6 presents, as an example, the written and oral response of student 35, who shows evidence of having progressed toward a more formal reasoning of the concepts in the study because the student identified the characteristics of span W, and immediately, as he explains in an interview, applied a procedure to obtain a suitable spanning set and wrote it using the relevant mathematical notation. Among those students who did not respond correctly are those that used the span parentheses instead of the spanning set, as shown in the response of student 41 (Table 6) who, in addition, writes two linearly dependent vectors.

Table 6. Examples of responses to question 4a of Task 4.

Student 35	$W=\{(x,0,x,0)\} \rightarrow x\cdot(1,0,1,0)$ $G=\{(1,0,1,0)\}$ I: How do I get the spanning set? S35: From the space, I replaced each variable according to the equivalent and it gave me this (indicating the vector (*x*, 0, *x*, 0). That would leave (*x*, 0, *x*, 0) or the same would be, (*z*, 0, z, 0) and from there you take what would be the variable, which is the vector (1, 0, 1, 0) and this is (indicating his response).
Student 41	$\langle(4,0,4,0),(2,0,2,0)\rangle$

In regard to the question of determining if two vectors belonged to W, 90% of the students responded adequately. The main justification was to check whether each vector complied with the conditions of the span given. Other students based their attempt on doing a linear combination of the vector of the spanning set of the span, given as seen in Figure 5.

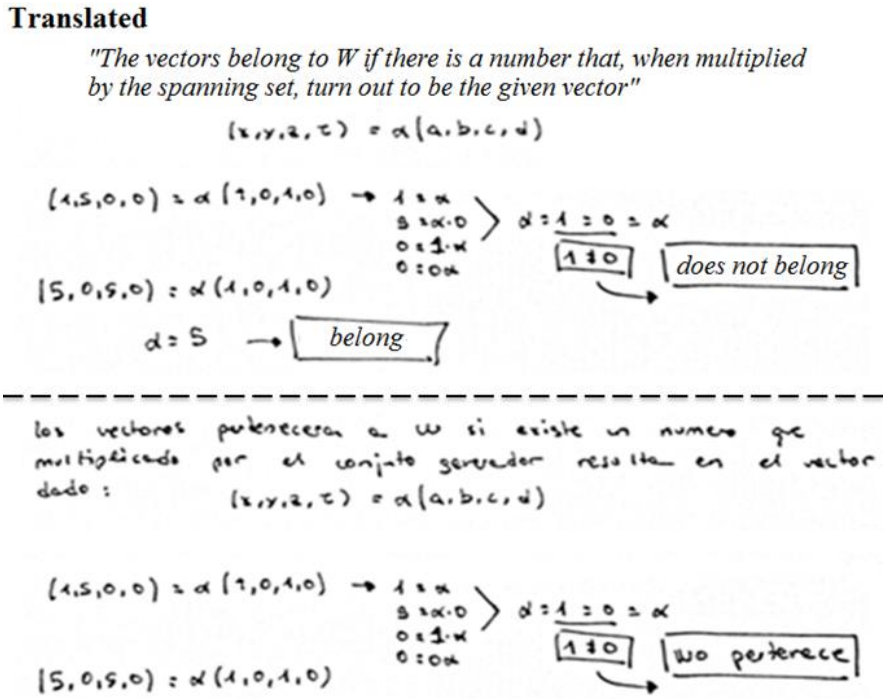

Figure 5. Written answer of a student to question 4b of Task 4.

In Figure 5, it is noted that the student correctly used the vector of spanning set W to determine if the two vectors belonged to this span. It follows that he assumed that all vectors of W can be expressed as a linear combination of the element of the spanning set from this span.

In this task, the students worked with procedures and conventional notations in mathematics related to the concepts of spanning set and span, which shows that they indeed progressed toward a formal activity level.[12]

In the analysis of Task 4, we observed in one case moderate match (59% of the students in question 4a) and, in other case high match (90% of the students in question 4b) between the learning trajectories, which allows us conclude that the tasks worked on in groups influenced the progress of several students concerning the construction of the concepts in study.

The results obtained in the implementation of this teaching innovation based on the emergent models heuristic and mathematical modelling show the potential that this proposal offers to contribute to the construction of the concepts of spanning set and span, because, as noted in the responses to the tasks, the groups progressed through different activity levels, from the informal level that began with the problem of creating a secure passwords generator to the formal level. It was found that when they worked with the procedures and conventional notations, their work concerned both spanning set and span. Table 7 presents a synthesis of the level of match between the HLT and the ALT in the tasks given to the students.

Table 7. Synthesis of the level of match between the HLT and the ALT in the proposed tasks for students.

Task	Synthesis of conjecture of the HLT	ALT	Match between HLT and ALT

1	The students created a password generator by following the steps of the modelling cycle and using their preconceptions of vectors and passwords	100% of the groups created a generator of passwords using vectors, either by presenting a mathematical model of generic vector of R^n or a linear combination of vectors of R^3	+
2	Students are given an analogy table where they related the concepts of spanning set and span with two sets of their password generator and mentioned the name which they receive in their password generator.	57% of the groups scored the mathematical notation for spanning set correctly and assigned it that name. 43% of the groups scored the mathematical notation of span correctly and assigned it that name. The other groups had difficulty with the mathematical language.	±
3	Students determine that sets A and B do not generate a $R^{2,}$ giving a coherent justification, and determine that the set C generates a R^2, giving a coherent justification.	100% of the groups gave a coherent justification of why A does not generate. 86% of the groups gave a coherent justification of why B does not generate and 93% of gave a coherent justification of why C does generate.	+
4a	The student proposes a correct spanning set for the sub R^4 given.	59% of students propose a correct spanning set, of which 20% of students determined the whole was equal to the subspace (W). Others write a set of vectors with parentheses < >.	±
4b	The students uses an appropriate process to determine if each vector belongs to the subspace R^4 or does not, and determines that the first vector (1,5,0,0) does not belong and that the second vector (5,0,5,0) does belong.	90% of students determine that the first vector (1,5,0,0) does not belong and that the second vector (5,0,5,0) does belong, justifying properly by the conditions of subspace R^4 or making a linear combination with the vector of spanning set.	+

5. Discussion and Conclusion

This study presented a didactic proposal based on the emergent models heuristic and mathematical modelling with the objective of evaluating it, by determining the level of match between the HLT and the ALT in relation to the construction of the concepts of spanning set and span. As a result, it was found that this proposal allowed students to begin to construct new concepts in linear algebra starting from an informal mathematical activity (through a situation that involves mathematical modelling) and moving toward more formal knowledge by means of emergent models heuristic as proposed by Gravemeijer.[12]

The results show a high level of match between the HLT and the ALT in at least half of the tasks, which according to Stylianides and Stylianides[8] suggests that the instructional design supported the learning objective, which is to say that it helped the students to build the concepts of spanning set and span. This is not a coincidence, but is the result of what has been developed up to this point in design-based research that aims to develop a theory of local statements for the construction of the concepts of spanning set and span.

From the analysis of the results, it was observed that the following characteristics of the HLT tasks supported the construction of the concepts of spanning set and span:

- Task 1, which was to create a password generator with vectors, allowed students to activate their previous knowledge of vectors that helped them in later tasks.
- Task 2, through the analogy table, contributed to the students ability to distinguish the concepts of spanning set and span, both in a real context as mathematical, which offers them the opportunity to avoid confusing them when they are presented in a concrete situation.
- Task 3 offered students the ability to move toward a general level of content to explore conjectures involving the concepts of spanning set and span.
- Task 4 allowed students to progress toward a formal activity level of the concepts, to resolve activities with conventional mathematical notation, involving spanning set and span.

The difficulties that were observed in the ALT were mainly associated with the mathematical notation of the sets in study and the procedure of obtaining a spanning set. It is therefore proposed that the next HLT should have an emphasis on the mathematical notation of both spanning set as well as span, and incorporate a task to work in groups that approached the formal level of the emergent models, because it is considered that this would help each student to achieve a better understanding of the concepts.

In addition, the results of this study reveal that many students, as proposed by Gravemeijer,[12] went from model of to model for within formal mathematical reasoning with spanning set and span, progressing from a task of the situational activity level, that only required the use of their previous knowledge, both of vectors and passwords (Table 3), toward a formal activity level task, that required the application of these concepts (Table 6 and Figure 5).

We agree with Alsina [13] that mathematical modelling can be a positive step toward success in learning, because the problem of creating passwords provided support for students to continue toward learning spanning set and span since in this task, they linked the context of passwords with the concepts under study, and used mathematical notation through an analogy table (Figure 3).

In accordance with what we have presented, this study shows that this didactic proposal, based on the emergent models heuristic and mathematical modelling, favors the construction of the concepts of spanning set and span. Thus, the main contribution of this research is a teaching innovation for these linear algebra concepts, which had been poorly explored in the field of mathematics education.

Finally, it must be emphasise that the results of this experiment provide a general outlook for both the benefits and the limitations of the HLT proposal, which will be useful for redesigning and applying it again in a new cycle of experimentation. In addition, it is important to note that implementing this specifically designed teaching proposal to another student population could derive similar results.

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